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An Inverse Eigenvalue Problem for the Schrödinger Equation on the Unit Ball of R^3

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An Inverse Eigenvalue Problem for the Schrödinger Equation on the Unit Ball of $\mathbb{R}^3$

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Maryam Al Ghafli
Lexington, Kentucky

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2019

ABSTRACT OF DISSERTATION

An Inverse Eigenvalue Problem for the Schrödinger Equation on the Unit Ball of $\mathbb{R}^3$

The inverse eigenvalue problem for a given operator is to determine the coefficients by using knowledge of its eigenfunctions and eigenvalues. These are determined by the behavior of the solutions on the domain boundaries. In our problem, the Schrödinger operator acting on functions defined on the unit ball of $\mathbb{R}^3$ has a radial potential taken from $L^2_{\mathbb{R}}[0,1]$. Hence the set of the eigenvalues of this problem is the union of the eigenvalues of infinitely many Sturm-Liouville operators on $[0,1]$ with the Dirichlet boundary conditions. Each Sturm-Liouville operator corresponds to an angular momentum $l = 0, 1, 2.....$ In this research we focus on the uniqueness property. This is, if two potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ have the same set of eigenvalues then $p = q$. An early result of Pöschel and Trubowitz is that the uniqueness of the potential holds when the potentials are restricted to the subspace of the even functions of $L^2_{\mathbb{R}}[0,1]$ in the $l = 0$ case. Similarly when $l = 0$, by using their method we proved that two potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ are equal if their even extension on $[-1,1]$ have the same eigenvalues. Also we expect to prove the uniqueness if p and q have the same eigenvalues for finitely many l. For this idea we handle the problem by focusing on some geometric properties of the isospectral sets and trying to use these properties to prove the uniqueness of the radial potential by using finitely many of the angular momentum.

KEYWORDS: Schrödinger operator, potential, eigenvalue, eigenfunction, uniqueness, angular momentum

Author's signature: Maryam Al Ghafli

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An Inverse Eigenvalue Problem for the Schrödinger Equation on the Unit Ball of $\mathbb{R}^3$

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Date: December 18, 2019

Dedicated to my parents, Ali and Jamilah Al-Ghafli and my husband Mahdi Al-Ghafli.

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Chapter 1 Background: An Overview of The Direct Eigenvalue Problem

1.1 Introduction

In this dissertation we are interested to work in the field of inverse problem. Usually in the direct problem we have a system of equation and boundary conditions. The equation describes an physical phenomena appearing in some experiment. The equation coefficients which present some media properties of that experiment are well known. And the boundary conditions give an idea about the initial state of the particles in that phenomena. In these problem we try to find the exact solutions of the system and their properties like uniqueness and existences either locally or globally. In the inverse problem the system contains an equation with unknown coefficients and boundary conditions. We assume that we know the solutions of this system and our task is to discover the coefficients of the equation.

Specifically, we work on the inverse eigenvalue problem related to the Schrödinger equation. We have data including the solutions of the Schrödinger equation, which are called wave functions, Dirichlet boundary conditions, and the Dirichlet eigenvalues of this quantum system. Our goal is to discover the coefficient of this equation which is a radial potential. Hence we begin by setting

$$H_q = -\Delta + q(|X|),$$

for the Schrödinger operator where q is a radial potential taken from $L^2_{\mathbb{R}}[0,1]$ and Δ is the Laplacian operator on three dimension. H_q acts on functions defined on the unit ball of $\mathbb{R}^3$. The main eigenvalue problem in this paper contains the following equation

$$H_q\psi(X) = \lambda\psi(X), \qquad |X| \le 1, \quad \lambda \in \mathbb{C} \tag{1.1}$$

with the Dirichlet boundary condition

$$\psi(X) = 0, \qquad |X| = 1. \tag{1.2}$$

All the complex numbers λ when the boundary value problem (1.1) and (1.2) can be solved are called eigenvalues of q which we will denote by $\sigma(H_q)$. The corresponding nontrivial solutions are called eigenfunctions of q for λ. We will see later that the set of the eigenvalues $\sigma(H_q)$ is the union of the eigenvalues of infinitely many Sturm-Liouville operators on the unit interval $[0,1]$ with the Dirichlet boundary conditions. Each Sturm-Liouville operator

$$H_q^l = -\frac{d^2}{dx^2} + q + \frac{l(l+1)}{x^2}$$

corresponds to an angular momentum $l = 0, 1, 2, \ldots..$ So by denoting $\sigma(H_q^l)$ to the eigenvalues of the operator H_q^l, we have $\sigma(H_q) = \bigcup\limits_{l\ge 0} \sigma(H_q^l)$.

The main interesting question concerns about the uniqueness of the potential: if two potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ have the same set of eigenvalues, that is $\sigma(H_p) = \sigma(H_q)$, then $p = q$. Notice that if for a fixed angular momenta l we have $\sigma(H^l_p) = \sigma(H^l_q)$ implies $p = q$, then $\sigma(H_p) = \sigma(H_q)$ implies $p = q$. For that, many works focused on discovering the uniqueness and construction of the potentials by using the spectral data related to a single angular momenta l or finitely many angular momentum $l_1, l_2, ..., l_n$. Some of the research gave a description of the isospectral sets and their tangent spaces.

Begin with the zero angular momenta, in 1929, Ambartsumyan [12, Page 163] considered the following problem on the interval $[0,1]$,

$$H^0_q\psi(x) = \lambda\psi(x) \quad x \in (0,1) \tag{1.3}$$

with Neumann boundary conditions

$$\psi^{'}(0,\lambda,q) = 0, \quad \psi^{'}(1,\lambda,q) = 0 \tag{1.4}$$

When $q = 0$, the eigenvalues set of of this problem is $\{(\frac{n\pi}{2})^2\}_{n\geq 0}$. He showed the following result,

Theorem 1.1.1. *(Ambartsumyan's Theorem) For $q \in L^2_{\mathbb{R}}[0,1]$, if the eigenvalues of the problem* (1.3), (1.4) *are $\lambda_n = (\frac{n\pi}{2})^2$, $n \geq 0$, then $q = 0$ almost every where on* $(0,1)$.

The proof of this result can be found in [12]. Then in 1945, Borg in [13] found that the one set of the eigenvalues associated with a given boundary conditions is not enough data to determine the uniqueness of the potential. Hence in one method, He showed that adding another set of eigenvalues associated with second boundary conditions proves the uniqueness of the potential. So he consider $\{\lambda_n\}_{n\geq 0}$ to be the eigenvalues for (1.3) with the boundary conditions,

$$\psi'(0) + a_1\psi(0) = 0, \quad \psi'(1) + b\psi(1) = 0,$$

and $\{\mu_n\}_{n\geq 0}$ to be the eigenvalues for (1.3) with the boundary conditions,

$$\psi'(0) + a_2\psi(0) = 0, \quad \psi'(1) + b\psi(1) = 0$$

where $a_1 \neq a_2, a_3$ are real numbers. Then he showed that the two sets $\{\lambda_n\}_{n\geq 0}$ and $\{\mu_n\}_{n\geq 0}$ are uniquely determine a_1, a_2, b and q. In 1949, Levinson [18] proved the same result with different method. By another method, Borg considered (1.3) with the following initial condition

$$\psi(0,\lambda,q) = 0, \quad \psi^{'}(0,\lambda,q) = 0 \tag{1.5}$$

Beside to the eigenvalues $\{\lambda_n(q)\}_{n\geq 0}$ which is determined by $\psi(1,\lambda_n) = 0$, he defined additional spectral data consisting norming constants c_n which are given by

$$c_n(q) = \int_0^1 \psi^2(x,\lambda_n(q),q)dx.$$

where $\psi(x,\lambda_n(q),q)$ is the n^{th} eigenfunction for $\lambda_n(q)$. For problem (1.3) and (1.5), he proved

Theorem 1.1.2. *Suppose that* $q, p \in L^{\infty}_{\mathbb{R}}(0,1)$, *for all* n, $\lambda_n(q) = \lambda_n(p)$ *and* $c_n(q) = c_n(p)$ *then* $q = p$.

Also Pöschel and Trubowitz [1] worked on the direct and inverse eigenvalue problems of the operator H^0_q on the interval $[0,1]$. In the direct problem, they provided good estimates of the eigenfunctions and eigenvalues of the following Dirichlet problem

$$H^0_q \psi(x) = \lambda \psi(x) \qquad x \in (0,1)$$

and

$$\psi(0, \lambda, q) = 0 \qquad \text{and} \quad \psi(1, \lambda, q) = 0.$$

The basic estimate of the n^{th} eigenvalue $\lambda_{0,n}(q)$ is given by

$$\lambda_{0,n}(q) = n^2\pi^2 + [q] + \widetilde{\lambda}_{0,n}(q),$$

where $[q] = \displaystyle\int_0^1 q(x)dx$ and $\{\widetilde{\lambda}_{0,n}(q)\}_{n\geq 1} \in l^2$. The estimate of the n^{th} normalized eigenfunction is given by

$$\begin{aligned} g_n(x,q) &= \frac{\psi(x, \lambda_{0,n}(q), q)}{\|\psi(., \lambda_{0,n}(q), q)\|} \\ &= \sqrt{2}\ \sin(\pi n x) + O(\frac{1}{n}). \end{aligned}$$

In case the potential q is even with respect to $x = \frac{1}{2}$, that is, $q(x) = q(1-x)$. They proved that $g_n(x,q) = (-1)^{n+1} g_n(1-x,q)$. By using this property of g_n, they started their work on the inverse problem of the operator H^0_q by proving the following theorem,

Theorem 1.1.3. *For* $p, q \in \mathcal{E} = \{q \in L^2_{\mathbb{R}}[0,1] : q(x) = q(1-x)\}$, *suppose* $\lambda_{0,n}(p) = \lambda_{0,n}(q)$ *for all* $n \geq 1$ *then* $p = q$.

To prove their result, they considered the map λ from $L^2_{\mathbb{R}}[0,1]$ to $\mathbb{R} \times l^2$ such that $\lambda(q) = ([q], \{\widetilde{\lambda}_{0,n}(q)\}_{n\geq 1})$. They proved that this map is one to one if its domain is restricted to $\mathcal{E}$. The additional data in Theorem 1.1.2 was a good motivation to prove that map is one to one in the whole space by seeking another equivalent data to be added to the set of the eigenvalues. Thus they introduced a sequence of norming constants related to the terminal velocities of the eigenfunctions. These constants are defined by

$$\kappa_{0,n}(q) = \log|\psi'(1, \lambda_{0,n}(q), q)| \qquad n \geq 1,$$

where $\psi'(x, \lambda_{0,n}(q), q)$ is the derivative of n^{th} eigenfunction of the operator H^0_q with respect to x. The sequence $\{\kappa_{0,n}(q)\}_{n\geq 1}$ is in the Hilbert space l^2_1 of all real sequences $\mu = (\mu_1, \mu_2, ...)$ satisfying $\sum_{n\geq 1} n^2 \mu_n^2 < \infty$. Combining the sequence of the eigenvalues and the sequence of the norming constants they proved,

Theorem 1.1.4. *For* $p, q \in L^2_{\mathbb{R}}[0,1]$, *suppose* $\lambda_{0,n}(p) = \lambda_{0,n}(q)$ *and* $\kappa_{0,n}(p) = \kappa_{0,n}(q)$ *for all* $n \geq 1$ *then* $p = q$.

From studying the following map

$$\lambda \times \kappa : L^2_{\mathbb{R}}[0,1] \to \mathbb{R} \times l^2 \times l^2_1$$

$$\lambda \times \kappa(q) = \Big([q], \{\widetilde{\lambda}_{0,n}(q)\}_{n\geq 1}, \{\kappa_{0,n}(q)\}_{n\geq 1}\Big)$$

they showed that for any potential q, the isospectral set $M^0(q) = \{p : \lambda_{0,n}(p) = \lambda_{0,n}(q) \text{ for all } n\}$ is a real analytic submanifold of $L^2_{\mathbb{R}}[0,1]$, lying in the hyperplane of all functions with mean $[q]$.

In [3], Guillot and Ralston proved Theorem 1.1.4 for $l = 1$. Also Carlson [11] generalized Theorem 1.1.4 for any angular momentum $l \geq -\frac{1}{2}$. He considered another spectral data related to different boundary conditions at $x = 1$ as what Borg did in case $l = 0$. Instead of having a single sequence of eigenvalues, he considered two sequences of eigenvalues corresponding to two different boundary conditions. So by considering the following system,

$$H^l_q\psi(x) = \lambda\psi(x) \qquad x \in (0,1)$$

and

$$\psi(0) = 0, \qquad a\psi(1) + b\psi^{'}(1) = 0, \quad a,b \in \mathbb{R}.$$

and denoting the n^{th} eigenvalue of this system by $\lambda_{l,n}(q,a,b)$, he proved the following result,

Theorem 1.1.5. *Suppose that for all $n \geq 1$, we have $\lambda_{l,n}(p,a_j,b_j) = \lambda_{l,n}(q,a_j,b_j)$ for $j = 1,2$ and for linearly independent vectors (a_1,b_1) and (a_2,b_2). Then $p = q$.*

In [8], Carlson proved that for any $l \geq 0$ the isospectral set $M^l(q) = \{p : \lambda_{l,n}(p) = \lambda_{l,n}(q) \text{ for all } n\}$ is a real analytic submanifold of $L^2_{\mathbb{R}}[0,1]$ of infinite dimension and infinite codimension. This result emphasises that proving the uniqueness of the potential q of the operator H_q can not be done by using a spectrum of a single angular momuntum. By different method Serier in [4] proved Theorem 1.1.4 for any $l \geq 1$.

Another interesting way to study the uniqueness was by considering spectral data consisting two sequences of eigenvalues $\{\lambda_{l_i,n}\}_{n\geq 1}$ for two distinct choice of angular momentum l_1 and l_2. Rundell and Sacks, in [5], tried to prove the uniqueness of small potentials by using two sequences of eigenvalues corresponding to $l_1 = l$ and $l_2 = l+1$ for any $l = 0,1,2,.....$ Also they work in the case $l_1 = 0$ and $l_2 = 2$. A part of their work was by using numerical methods to prove their conjectures.

The last interesting way to work in this problem is by analyzing the geometric properties of the isospectral set $M^l(q)$. In [7], Carlson and Shubin by considering $T_pM^l(p)$ to be the tangent space of $M^l(p)$ at p proved the following geometric properties of the isospectral set $M^l(p)$.

Theorem 1.1.6. *If $l_1 + l_2 = 1$ mod 2, then*

$$dim(T_pM^{l_1}(p) \cap T_pM^{l_2}(p)) < \infty.$$

and $T_pM^{l_1}(p) + T_pM^{l_2}(p)$ is a closed subspace of finite codimension in $L^2_{\mathbb{R}}[0,1]$.

Theorem 1.1.7. *If $l_1 + l_2 = 1$ mod 2, then $M^{l_1}(p) \cap M^{l_2}(p)$ is locally a subset of finite dimensional manifold.*

In their work they considered two angular momentum with different parity. The results in Theorem 1.1.6 were proved before by Shubin in [6] for $l_1 = 0$ and $l_2 = 1$. Also she proved the following theorem

Theorem 1.1.8. *For each $p \in L^2_{\mathbb{R}}[0,1]$, $M^0(p) \cap M^1(p)$ is locally compact.*

In this research, we are interested to handle the problem by several ways. The first way concerns potentials that are not even in case $l = 0$. The method we will use for that case is inspired from the proof of the uniqueness for the even potential of H^0_q by Pöschel and Trubowitz [1]. Thus in the second chapter, we will consider two potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ with different parity. For each potential $q \in L^2_{\mathbb{R}}[0,1]$, we will extend the domain of the solutions to be $[-1,1]$ and consider the following problem

$$H^0_{\widetilde{q}}\psi(x) = \lambda\psi(x) \qquad x \in (-1,1)$$

and

$$\psi(-1,\lambda,q) = 0 \qquad \text{and} \quad \psi(1,\lambda,q) \ = 0.$$

where $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ is the even extension of q. We will prove that the set of the eigenvalues of the new system for the even extended potentials $\widetilde{p}, \widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ is enough data to prove $p = q$. Then in the third chapter our question is about if we can show the uniqueness by focusing on eigenvalues corresponding to finitely many angular momentum $l_1, l_2, ..., l_n$. We will see that assuming $\lambda_{l,n}(p) = \lambda_{l,n}(q) = \lambda_{l,n}$ for any fixed angular momentum l leads to

$$\int_0^1 (p(x) - q(x))\ \psi(x,\lambda_{l,n},p)\ \psi(x,\lambda_{l,n},q)\ dx = 0, \qquad \text{for each } n.$$

Hence, showing the set $\{\psi(x,\lambda_{l,n},p)\ \psi(x,\lambda_{l,n},q)\}_{n\geq 1}$ is complete in $L^2_{\mathbb{R}}[0,1]$ proves that $p = q$. In chapter four and five we work on the geometric properties of the isospectral sets. We proved the same result of Theorem 1.1.6 for $l_1 = 0$ and l_2 any positive integer number. In our work, we consider the tangent space to be at any point $q \in M^{l_1}(p) \cap M^{l_2}(p)$. Then we try to find links between the uniqueness of the potential and these properties of the isospectral sets. In the last chapter, we list some conjectures for this problem as open problems. Some of them we have worked on but we didn't get results, it may there are other good methods to solve them.

Since the work on this inverse eigenvalue problem depends on knowing the results of the direct eigenvalue problem. We will start by giving a background about the set of the eigenvalues of this operator and its solutions and their asymptotic. From these knowledge we will start to discover the uniqueness of the potential.

1.2 Solutions of The Direct Problem and Their Properties

In this section we present known results of the direct eigenvalue problem of the following Schrödinger equation in the unit ball of $\mathbb{R}^3$,

$$-\Delta U + q(|X|)U = \lambda U \qquad |X| < 1,\ \lambda \in \mathbb{C}, \tag{1.6}$$

with the boundary conditions,

$$U(X) = 0 \qquad |X| = 1. \tag{1.7}$$

Since $q \in L^2_{\mathbb{R}}[0,1]$ is a radial potential, we can convert the system (1.6) and (1.7) to an easier system in one dimensional space by considering solutions in the separated form

$$U(x,\theta,\phi) = \frac{\psi(x)}{x} Y_l^m(\theta,\phi)$$

where $X = (x,\theta,\phi)$ are spherical coordinates in $\mathbb{R}^3$ and Y_l^m is a spherical harmonic function. Since we will consider the spherical coordinates then we write the Laplacian operator in the following form

$$\Delta = \frac{1}{x^2}\frac{\partial}{\partial x}(x^2\frac{\partial}{\partial x}) + \frac{1}{x^2}L_{\theta,\phi}$$

where $L_{\theta,\phi} = \Big(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\big(\sin(\theta)\frac{\partial}{\partial\theta}\big) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\Big)$. Using this form with the separated form of U in (1.6) leads to

$$\frac{x^2}{\psi(x)}\psi''(x) + x^2(\lambda - q(x)) = \frac{-1}{Y_l^m(\theta,\phi)}L_{\theta,\phi}Y_l^m(\theta,\phi)$$

Since that holds for all $X = (x,\theta,\phi)$ in the unit ball of $\mathbb{R}^3$, then by separation of variables method, both sides are equal to a constant C. By solving the equation associated to the operator $L_{\theta,\phi}$, $Y_l^m(\theta,\phi)$ exists when $C = l(l+1)$, where $l = 0,1,2,...$ These l are called angular momenta. Hence for each angular momentum $l = 0,1,2,...$ we have the following Sturm-Liouville ordinary differential equation for ψ,

$$\Big(-\frac{d^2}{dx^2} + \; q(x) \; + \frac{l(l+1)}{x^2}\Big)\psi(x) = \lambda\psi(x), \qquad x \in (0,1),\; \lambda \in \mathbb{C} \tag{1.8}$$

with the Dirichlet boundary conditions

$$\psi(0,\lambda,q) = 0 \qquad \text{and} \quad \psi(1,\lambda,q) \; = 0. \tag{1.9}$$

1.2.1 Solutions of the Sturm-Liouville Operator Corresponding to $l \geq 1$

When $q = 0$, by writing $\psi(x) = xR(x)$, equation (1.8) becomes

$$x^2\frac{d^2R}{dx^2} + 2x\frac{dR}{dx} + ((\omega x)^2 - l(l+1))R(x) = 0$$

where $\omega = \sqrt{\lambda}$. The last equation has the following two linearly independent solutions

$$R_1(x) = j_l(\omega x), \qquad R_2(x) = \eta_l(\omega x)$$

where j_l and η_l are the spherical Bessel functions of the first and second type of order l. Hence the two linearly independent solutions of (1.8) when $q = 0$ are

$$u(x,\lambda) = \frac{1}{\omega^{l+1}}\; xj_l(\omega x) \qquad \text{and} \qquad v(x,\lambda) = -\omega^l x\eta_l(\omega x)$$

with Wronskian $W(u,v) = -1$.

In case $q \neq 0$ the solutions of (1.8) are constructed by Picard's iteration method, see [3] for $l = 1$ and [4] for $l \geq 1$. The following two linearly independent solutions ψ and ϕ are defined by

$$\psi(x,\lambda,q) = \sum_{k=0}^{\infty} \psi_k(x,\lambda,q) \qquad \phi(x,\lambda,q) = \sum_{k=0}^{\infty} \phi_k(x,\lambda,q)$$

with
$\psi_0(x,\lambda,q) = u(x,\lambda)$,

$\psi_{k+1}(x,\lambda,q) = \int_0^x G(x,t,\lambda)\ q(t)\ \psi_k(t,\lambda,q)\ dt$, for each $k \in \mathbb{N}$
and
$\phi_0(x,\lambda,q) = v(x,\lambda)$,

$\phi_{k+1}(x,\lambda,q) = -\int_x^1 G(x,t,\lambda)\ q(t)\ \phi_k(t,\lambda,q)\ dt$, for each $k \in \mathbb{N}$

where the Green's function G is given by,

$$G(x,t,\lambda) = v(x,\lambda)u(t,\lambda) - u(x,\lambda)v(t,\lambda).$$

Theorem 1.2.1. *The series* $\sum_{k=0}^{\infty} \psi_k(x,\lambda,q)$ *and* $\sum_{k=0}^{\infty} \phi_k(x,\lambda,q)$ *converge uniformly on bounded subsets of* $[0,1] \times \mathbb{C} \times L^2_{\mathbb{R}}[0,1]$ *towards solutions of equation* (1.8) *and satisfy the integral equations*

$$\psi(x,\lambda,q) = u(x,\lambda) + \int_0^x G(x,t,\lambda)\ q(t)\ \psi(t,\lambda,q)\ dt$$

$$\phi(x,\lambda,q) = v(x,\lambda) - \int_x^1 G(x,t,\lambda)\ q(t)\ \phi(t,\lambda,q)\ dt$$

and the estimates

$$|\psi(x,\lambda,q)| \leq \left(\frac{x}{1+|\omega|x}\right)^{l+1} e^{(|Im\omega|)x}\ e^{C\|q\|\sqrt{x}}$$

$$|\phi(x,\lambda,q)| \leq \left(\frac{1+|\omega|x}{x}\right)^{l} e^{(|Im\omega|)(1-x)} e^{C\|q\|\sqrt{x}}$$

where C is given in (15).

Proof. From (8), we have

$$|u(x,\lambda)| \leq \left(\frac{x}{1+|\omega|x}\right)^{l+1} e^{(|Im\omega|)x}$$

By using (8) and (10), we get

$$
\begin{aligned}
|\psi_1(x,\lambda,q)| &\le C\int_0^x \Big(\frac{|x|}{1+|\omega x|}\Big)^{l+1}\Big(\frac{1+|\omega t|}{|t|}\Big)^{l} e^{|Im\omega|(x-t)}e^{|Im\omega| t}\Big(\frac{|t|}{1+|\omega t|}\Big)^{l+1}|q(t)|dt \\
&\le C\Big(\frac{|x|}{1+|\omega x|}\Big)^{l+1} e^{|Im\omega| x}\int_0^x \frac{t|q(t)|}{1+|\omega| t}dt \\
&\le \frac{C}{\omega}\Big(\frac{|x|}{1+|\omega x|}\Big)^{l+1} e^{|Im\omega| x}\int_0^x |q(t)|dt
\end{aligned}
$$

Proceeding by induction we get,

$$
|\psi_n(x,\lambda,q)| \le \frac{1}{\omega}\Big(\frac{|x|}{1+|\omega x|}\Big)^{l+1} e^{|Im\omega| x}\frac{C^n}{n!}\Big(\int_0^x |q(t)|dt\Big)^n \tag{1.10}
$$

This shows the uniform convergence on bounded subset of $[0,1]\times\mathbb{C}\times L^2_{\mathbb{C}}[0,1]$ for ψ. By taking the sum over n for (1.10) we get

$$
|\psi(x,\lambda,q)| \le \Big(\frac{x}{1+|\omega| x}\Big)^{l+1} e^{(|Im\omega|)x}\ e^{C\|q\|\sqrt{x}}
$$

Similarly, the proof of the ϕ estimate. □

Recall that any complex number λ such that the boundary value problem (1.8) and (1.9) can be solved is called an eigenvalue of q. From definition of ψ, we have $\psi(0,\lambda,q)=0$ for all λ. Hence the eigenvalues will be determined by the second boundary condition, thus the eigenvalues are solutions of $\psi(1,\lambda,q)=0$. The corresponding nontrivial solution is called an eigenfunction of q for λ. We will denote for the n^{th} eigenvalue by $\lambda_{l,n}(q)$ and for the n^{th} eigenfunction by $\psi(x,\lambda_{l,n}(q),q)$. It is important to have asymptotic and basic properties of the normalized eigenfunctions and the product of ψ and ϕ. For any angular momentum l and each $q\in L^2_{\mathbb{R}}[0,1]$, let

$$
g_{l,n}(x,q) = \frac{\psi(x,\lambda_{l,n}(q),q)}{||\psi(.,\lambda_{l,n}(q),q)||_{L^2_{\mathbb{R}}[0,1]}}, \quad n\ge 1,
$$

to be the sequence of the normalized eigenfunctions. Also let $a_{l,n}(x,q)$ to be

$$
a_{l,n}(x,q) = \psi(x,\lambda_{l,n}(q),q)\phi(x,\lambda_{l,n}(q),q).
$$

Theorem 1.2.2. *For any angular momentum l and potential $q\in L^2_{\mathbb{R}}[0,1]$, the sequence of the normalized eigenfunctions $\{g_{l,n}(x,q)\}_{n\ge1}$ is an orthonormal basis for $L^2_{\mathbb{R}}[0,1]$. The asymptotic estimate of $g_{l,n}$*

$$
g_{l,n}(x,q) = \sqrt{2}j_l(\omega_{l,n}(q)x) + O(\frac{1}{n}).
$$

holds uniformly on bounded subsets of $[0,1]\times L^2_{\mathbb{R}}[0,1]$.

Theorem 1.2.3. *For any angular momentum l and potential $q\in L^2_{\mathbb{R}}[0,1]$, we have*

- *The vectors* $\{1, \{g_{l,n}^2 - 1\}_{n\geq 1}\}$ *are linearly independent, as well as* $\{\frac{d}{dx} g_{l,n}^2\}_{n\geq 1}$.
- *For all* $(n,m) \in \mathbb{R}^2$, *we have,*

  (i) $\langle g_{l,n}^2, \frac{d}{dx} g_{l,m}^2 \rangle = 0,$
  (ii) $\langle a_{l,n}, \frac{d}{dx} g_{l,m}^2 \rangle = \frac{1}{2}\delta_{m,n},$
  (iii) $\langle a_{l,n},, \frac{d}{dx} a_{l,m} \rangle = 0$

In case $l = 0$, the eigenfunctions have special properties. Pöschel and Trubowitz [1] gave detailed results of these eigenfunctions and their properties and estimates. In the next subsection we will present the most important of these results that will be needed later.

1.2.2 Solutions of the Sturm-Liouville Operator Corresponding to $l = 0$

When $l = 0$ the eigenvalue problem is

$$-\frac{d^2}{dx^2}\psi(x) + q(x)\psi(x) = \lambda\psi(x), \qquad x \in (0,1), \ \lambda \in \mathbb{C} \tag{1.11}$$

with the Dirichlet boundary conditions

$$\psi(0, \lambda, q) = 0 \qquad \text{and} \quad \psi(1, \lambda, q) \ = 0. \tag{1.12}$$

When $q = 0$, (1.11) corresponds to the spherical Bessel equation of order 0. Hence its solutions form by $j_0(x) = \dfrac{\sin(x)}{x}$ and $\eta_0(x) = -\dfrac{\cos(x)}{x}$. So the two linearly independent solutions of (1.11) are

$$u(x, \lambda) = \frac{\sin(\omega x)}{\omega} \quad \text{and} \quad v(x, \lambda) = -\cos(\omega x)$$

with Wronskian $W(u, v) = 1$. Since the eigenvalues are determined by the value of the solution at the boundary, then the eigenvalues are the zeros of $\sin(\omega) = 0$. Hence $\{\lambda_{0,n}(0)\}_{n\geq 1} = \{(n\pi)^2\}_{n\geq 1}$.

For $q \neq 0$, the two linearly independent solutions of equation (1.11) are defined by

$$\psi(x, \lambda, q) = \frac{\sin(\omega x)}{\omega} + \int_0^x G(x, t, \lambda)\ q(t)\ \psi(t, \lambda, q)\ dt, \qquad x \in [0, 1]$$
$$\phi(x, \lambda, q) = -\cos(\omega x) + \int_0^x G(x, t, \lambda)\ q(t)\ \phi(t, \lambda, q)\ dt, \qquad x \in [0, 1]\,,$$

where the Green function is giving by

$$G(x, t, \lambda) = \frac{\sin(\omega(x-t))}{\omega}$$

The basic estimates on these solutions following in the next theorem and for more details about these solutions, see the first chapter in [1].

Theorem 1.2.4. *The basic estimates for ψ and ϕ on $[0,1]\times\mathbb{C}\times L^2_{\mathbb{C}}$ are given by,*

$$\Big|\psi(x,\lambda,q)-\frac{\sin(\omega x)}{\omega}\Big|\leq\frac{1}{|\lambda|}exp(|Im\omega|x+\|q\|\sqrt{x})$$
$$\Big|\phi(x,\lambda,q)-\cos(\omega x)\Big|\leq\frac{1}{|\omega|}exp(|Im\omega|x+\|q\|\sqrt{x})$$

and

$$\Big|\psi(x,\lambda,q)\Big|\leq\frac{1}{|\omega|}exp(|Im\omega|x+\|q\|\sqrt{x})$$
$$\Big|\phi(x,\lambda,q)\Big|\leq exp(|Im\omega|x+\|q\|\sqrt{x}).$$

Theorem 1.2.5. *For any potential $q\in L^2_{\mathbb{R}}[0,1]$, the subspaces*

$$\Big\{\eta_0+\sum_{n\geq1}\eta_n(g^2_{0,n}-1):(\eta_n)\in l^2\Big\}\ \text{and}\ \Big\{\sum_{n\geq1}\xi_n\frac{d}{dx}g^2_{0,n}:\ (\xi_n)\in l^2_1\Big\}$$

are perpendicular and closed. In particular when q is even, they are the even and odd subspaces respectively.

Next theorem illustrates the product expansion of ψ at $x=1$. This representation follows from the product expansion of the sin function which forms the leading term of ψ,

$$\frac{\sin(\sqrt{\lambda})}{\sqrt{\lambda}}=\prod_{m\geq1}\frac{m^2\pi^2-\lambda}{m^2\pi^2}$$

The proof of this product expansion was given by Pöschel and Trubowitz in [1].

Theorem 1.2.6. *For $q\in L^2_{\mathbb{R}}[0,1]$,*

$$\psi(1,\lambda,q)=\prod_{m\geq1}=\frac{\lambda_m(q)-\lambda}{m^2\pi^2}$$

Hence,

- $\dot{\psi}(1,\lambda_{0,n}(q),q)=\dfrac{-1}{(n\pi)^2}\displaystyle\prod_{m\neq n}\frac{\lambda_{0,m}(q)-\lambda_{0,n}(q)}{(m\pi)^2}=\dfrac{(-1)^n}{2n^2\pi^2}\Big(1+O(\frac{\log n}{n})\Big)$
- $sgn\ \dot{\psi}(1,\lambda_{0,n}(q),q)=(-1)^n=sgn\ \psi'(1,\lambda_{0,n}(q),q)$

By this product expansion, Pöschel and Trubowitz proved specific property for the eigenfunction when the potential is even.

Theorem 1.2.7. *If $q\in\mathcal{E}$, then $g_{0,n}$ is even when n is odd and odd when n is even.*

1.3 Eigenvalues of the Main Operator H_q

For each angular momentum l, let $H_q^l = -d^2/dx^2 + q + (l(l+1))/x^2$ be the operator in (1.8), and $\sigma(H_q^l)$ be the spectrum of H_q^l which contains all the eigenvalues of the system (1.8), (1.9). Similarly let $H_q = -\Delta + q$ and $\sigma(H_q)$ to be the spectrum of H_q which is the set of all the eigenvalues of the problem (1.6), (1.7). By converting the problem with operator H_q to a problem containing infinitely many Sturm-Liouville operators H_q^l, we get $\sigma(H_q) = \cup_{l=o}^{\infty} \sigma(H_q^l)$. For each angular momentum l, $\sigma(H_q^l)$ is a countable sequence of eigenvalues $\{\lambda_{l,n}(q)\}_{n>0}$.

For more details about the asymptotic of the eigenvalues, see [1] for $l = 0$, [4] for $l = 1$, and [5] for $l > 1$. By considering the Dirichlet eigenvalues as functions defined on $L^2_{\mathbb{R}}[0,1]$, they showed the following result,

Theorem 1.3.1. *For any angular momentum l and each $n \geq 1$, $\lambda_{l,n}$ is a real analytic function on $L^2_{\mathbb{R}}[0,1]$. its gradient is*

$$\frac{\partial \lambda_{l,n}}{\partial q(t)} = g^2_{l,n}(t,q).$$

Theorem 1.3.2. *For $l \geq 0$ and q in $L^2_{\mathbb{R}}[0,1]$,*

$$\sqrt{\lambda_{l,n}(q)} = \left(n + \frac{l}{2}\right)\pi + \frac{\int_0^1 q(x)\ dx - l(l+1)}{(2n+l)\pi} + \beta_{l,n}(q), \qquad \sum_{n=1}^{\infty} n\beta^2_{l,n}(q) < \infty, \tag{1.13}$$

equivalently

$$\lambda_{l,n}(q) = \left(n + \frac{l}{2}\right)^2 \pi^2 + \int_0^1 q(x)\ dx - l(l+1) + \widetilde{\lambda}_{l,n}(q), \qquad \sum_{n=1}^{\infty} \widetilde{\lambda}^2_{l,n}(q) < \infty. \tag{1.14}$$

Also,

$$\lambda_{l,n}(q) = \lambda_{l,n}(0) + O(1).$$

Corollary 1.3.1. *For $l \geq 0$ and $p, q \in L^2_{\mathbb{R}}[0,1]$, if $\lambda_{l,n}(p) = \lambda_{l,n}(q)$ for each n, then*

$$\int_0^1 (p(x) - q(x))\ dx = 0$$

Proof. From (1.14) in Theorem 1.3.2, we have

$$\int_0^1 (p(x) - q(x))\ dx = \widetilde{\lambda}_{l,n}(q) - \widetilde{\lambda}_{l,n}(p) \qquad \forall n.$$

Since $\sum_{n=1}^{\infty} \widetilde{\lambda}^2_{l,n}(p) < \infty$ and $\sum_{n=1}^{\infty} \widetilde{\lambda}^2_{l,n}(q) < \infty$ then the sequences $\{\lambda_{l,n}(p)\}_{n\geq1}$ and $\{\lambda_{l,n}(q)\}_{n\geq1}$ converge to zero as n goes to infinity. Hence by taking the limit for both sides as n goes to infinity we get,

$$\int_0^1 (p(x) - q(x))\ dx = 0.$$

□

Theorem 1.3.3. *For any $l \geq 0$ and q in $L^2_{\mathbb{R}}[0,1]$, if $\lambda(q)$ is a Dirichlet eigenvalue of* (1.8) *and* (1.9), *then*

$$\begin{aligned}\psi'(1,\lambda(q),q)\dot{\psi}(1,\lambda(q),q) &= \int_0^1 \psi^2(t,\lambda(q),q)dt \\ &= \|\psi(.,\lambda(q),q)\|^2 > 0.\end{aligned}$$

In particular, $\dot{\psi}(1,\lambda(q),q) \neq 0$. *Thus, all roots of* $\psi(1,\lambda(q),q)$ *are simple.*

We provide in this chapter the most important results of the direct eigenvalue problem of (1.8) and (1.9) which will be used later to discover some results of the inverse eigenvalue problem of the operator (1.6).

Remark: From the next chapter, for any angular momentum l and $n \in \mathbb{N}$ we will use $\psi^q_{l,n}$ to denote the n^{th} eigenfunction of q for the eigenvalue $\lambda_{l,n}(q)$.

Chapter 2 Uniqueness of the Potential of the Extended Domain

2.1 Introduction

The result in this chapter is restricted to the zero angular momentum. As we saw in the previous chapter that the eigenvalues of the problem (1.6), (1.7) is the union of the eigenvalues of infinitely many Sturm-Liouville operators on $[0,1]$ with the Dirichlet boundary conditions. This spectral data is a large set to start with. We may work on the problem and get the uniqueness by following one of these ways. Either by focusing on the eigenvalues corresponding to a single angular momenta and restricting the work on some subspaces of $L^2_{\mathbb{R}}[0,1]$ such as the subspace of the even functions, or by focusing on the eigenvalues corresponding to finitely-many angular momenta. In this chapter we will focus on the angular momentum $l = 0$. In the $l = 0$ case, one early result of Pöschel and Trubowitz, in [1], is that the uniqueness of the potential holds when the potentials are restricted to the subspace of the even functions, Theorem 1.1.3. They formulate the inverse problem of this system

$$\Big(-\frac{d^2}{dx^2} + \ q(x)\Big)\psi(x) = \lambda\psi(x) \qquad x \in (0,1)$$

and

$$\psi(1) = 0 \qquad \text{and} \quad \psi(0) \ = 0,$$

by defining a map λ from the Hilbert space $L^2_{\mathbb{R}}[0,1]$ to $\mathbb{R} \times l^2$ as following

$$q \to ([q], \widetilde{\lambda}(q))$$

where $[q] = \displaystyle\int_0^1 q(x)dx$ and $\widetilde{\lambda}(q) = \big\{\widetilde{\lambda}_{0,n}(q)\big\}_{n\geq 1}$ is the sequence of the last terms in the asymptotic of the eigenvalues in the equation (1.14). They have shown that this map is one to one if its domain is restricted on the subspace of the even functions $\mathcal{E}$ of $L^2_{\mathbb{R}}[0,1]$. Their proof of this result is based on the fact that the n^{th} eigenfunction for an even potential is odd if n is even and even if n is odd, Theorem 1.2.7 which was proved in chapter 2, [1].

This property does not hold if the potential is not even. From this point, our work in this chapter will focus potentials that are not even potentials. But we will use their method by looking to the potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ that have even extended potentials $\widetilde{p}, \widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ sharing the same eigenvalues. By proving the equality between $\widetilde{p}$ and $\widetilde{q}$, we will get the $p = q$. Let us first give a clear definition of the subspace of the even functions of $L^2_{\mathbb{R}}[-1,1]$.

Definition 2.1.1. $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ is called even if $\widetilde{q}(x) = \widetilde{q}(-x)$ for any $x \in [-1,1]$, and called odd if $\widetilde{q}(x) = -\widetilde{q}(-x)$ for any $x \in [-1,1]$.

For any $q \in L^2_{\mathbb{R}}[0,1]$, we will consider here the following eigenvalue problem for the extended domain $[-1,1]$

$$\Big(-\frac{d^2}{dx^2} + \ \widetilde{q}(x)\Big)\psi(x) = \lambda\psi(x), \qquad x \in (-1,1), \ \lambda \in \mathbb{C} \tag{2.1}$$

with the Dirichlet boundary conditions

$$\psi(1, \lambda(\widetilde{q}), \widetilde{q}) = 0 \qquad \text{and} \quad \psi(-1, \lambda(\widetilde{q}), \widetilde{q}) \ = 0. \tag{2.2}$$

where $\widetilde{q}$ is the even extension of q. The purpose of this extension is trying to prove the uniqueness of $\widetilde{q}$ which leads to the uniqueness of q. Thus the spectral data that we will focus on in this chapter is related to the extended potential $\widetilde{q}$. The result in the next lemma explains that we are still working in the same main problem of Chapter 1 but for specific type of potentials lying in $L^2_{\mathbb{R}}[0, 1]$. That is, if the two extended potentials $\widetilde{q}, \widetilde{p} \in L^2_{\mathbb{R}}[-1, 1]$ share the same eigenvalues then also the main potentials $q, p \in L^2_{\mathbb{R}}[0, 1]$, that we are trying to prove are equal, share the same eigenvalues.

Lemma 2.1.1. *Let $\widetilde{q} \in L^2_{\mathbb{R}}[-1, 1]$ be the even extension of $q \in L^2_{\mathbb{R}}[0, 1]$. Then $\left\{\lambda_{0,n}(q)\right\}_{n\geq 1} \subset \left\{\lambda_{0,n}(\widetilde{q})\right\}_{n\geq 1}$, where the first set forms the eigenvalues of the problem* (1.11)-(1.12) *and the second set forms the eigenvalues of the problem* (2.1)-(2.2)*.*

Proof. Let $\psi^q_{0,n}$ be the eigenfunction of the problem (1.11)-(1.12) on $[0, 1]$ corresponding to the n^{th} eigenvalue $\lambda_{0,n}(q)$. Consider $\widetilde{\psi}_{0,n}$ to be the odd extension of $\psi^q_{0,n}$ on $[-1, 1]$, i.e,

$$\widetilde{\psi}_{0,n}(x) = \begin{cases} \psi^q_{0,n}(x) & x \in [0, 1] \\ -\psi^q_{0,n}(-x) & x \in [-1, 0) \end{cases}$$

We choose $\widetilde{\psi}_{0,n}$ to be the odd extension of $\psi^q_{0,n}$ to guarantee the continuity of $\widetilde{\psi}_{0,n}$ at $x = 0$. Thus for any $x \in [0, 1]$ we have

$$\begin{aligned} -\frac{d^2}{dx^2}\widetilde{\psi}_{0,n}(x) + \widetilde{q}(x)\widetilde{\psi}_{0,n}(x) &= -\frac{d^2}{dx^2}\psi^q_{0,n}(x) + q(x)\psi^q_{0,n}(x) \\ &= \lambda_{0,n}(q)\psi^q_{0,n}(x) \\ &= \lambda_{0,n}(q)\widetilde{\psi}_{0,n}(x) \end{aligned}$$

and if $x \in [-1, 0)$

$$\begin{aligned} -\frac{d^2}{dx^2}\widetilde{\psi}_{0,n}(x) + \widetilde{q}(x)\widetilde{\psi}_{0,n}(x) &= \frac{d^2}{dx^2}\psi^q_{0,n}(-x) - q(x)\psi^q_{0,n}(-x) \\ &= -\lambda_{0,n}(q)\psi^q_{0,n}(-x) \\ &= \lambda_{0,n}(q)\widetilde{\psi}_{0,n}(x) \end{aligned}$$

Hence $\widetilde{\psi}_{0,n}$ is a solution of (2.1). From the definition of $\widetilde{\psi}_{0,n}$ we have $\widetilde{\psi}_{0,n}(1) = \widetilde{\psi}_{0,n}(-1) = 0$. Thus $\widetilde{\psi}_{0,n}$ is an eigenfunction of (2.1)-(2.2) corresponding to the eigenvalue $\lambda_{0,n}(q)$. Hence $\left\{\lambda_{0,n}(q)\right\}_{n\geq 1} \subset \left\{\lambda_{0,n}(\widetilde{q})\right\}_{n\geq 1}$. ☐

Before we start proving the uniqueness of the extended potential we need to know more about the eigenfunctions and the eigenvalues of the extended problem. In the next sections we present the most important properties and asymptotic estimates of them.

2.2 Eigenfunctions of the Extended Problem

When $\widetilde{q} = 0$, the equation (2.1) has the two linearly independent solutions

$$u(x, \lambda) = \frac{\sin(\omega x)}{\omega}, \qquad v(x, \lambda) = -\cos(\omega x)$$

where $\omega = \sqrt{\lambda}$. Since we are looking for two solutions satisfying the following initial conditions

$$\begin{array}{ll} u(-1) = 0 & v(-1) = -1 \\ u^{'}(-1) = 1 & v^{'}(-1) = 0 \end{array}$$

we will shift the argument of sin and cos from x to $x + 1$ to have the following two linearly independent solutions

$$\widetilde{u}(x, \lambda) = \frac{\sin(\omega(x+1))}{\omega}, \qquad \widetilde{v}(x, \lambda) = -\cos(\omega(x+1))$$

with Wronskian $W(u, v) = 1$. The eigenvalues are determined by Dirichlet boundary conditions (2.2). Since $\widetilde{u}(-1, \lambda) = 0$ for all λ, then the eigenvalues will be solutions of

$$\widetilde{u}(1, \lambda) = \frac{\sin(\omega(2))}{\omega} = 0.$$

Hence the Dirichlet spectrum is the set of zeros of $\sin(2\omega)$ which are the following infinite sequence

$$\frac{\pi^2}{4}, \frac{(2\pi)^2}{4}, \ldots., \frac{(n\pi)^2}{4}, \ldots.$$

When $\widetilde{q} \neq 0$ we consider the following solution,

$$\psi(x, \lambda, \widetilde{q}) = \frac{\sin(\omega(1+x))}{\omega} + \int_{-1}^{x} G(x, t, \lambda)\ \widetilde{q}(t)\ \psi(t, \lambda, \widetilde{q})\ dt, \qquad x \in [-1, 1]$$

where,

$$G(x, t, \lambda) = \widetilde{u}(x, \lambda)\widetilde{v}(t, \lambda) - \widetilde{u}(t, \lambda)\widetilde{v}(x, \lambda).$$

The construction of this solution and some estimates on it are justified in the next theorem.

Theorem 2.2.1. *For any* $\widetilde{q} \in L^2_{\mathbb{R}}[-1, 1]$, ψ *is a solution of* (2.1) *and satisfies the following basic estimates uniformly on bounded subsets of* $[-1, 1] \times \mathbb{R} \times L^2_{\mathbb{R}}[-1, 1]$,

$$\left|\psi(x, \lambda, \widetilde{q}) - \frac{\sin(\sqrt{\lambda}(x+1))}{\sqrt{\lambda}}\right| \leq \frac{1}{|\lambda|} exp(|Im\sqrt{\lambda}|(x+1) + \|\widetilde{q}\|\ \sqrt{x+1})$$

$$\left|\psi(x, \lambda, \widetilde{q})\right| \leq \frac{1}{|\sqrt{\lambda}|} exp\big(|Im\sqrt{\lambda}|(x+1) + \|\widetilde{q}\|\ \sqrt{x+1}\big).$$

Proof.

By Picard iteration method, we can show the definition of ψ is well defined. Let ψ be defined by

$$\sum_{k=0}^{\infty} \psi_k(x, \lambda, \widetilde{q})$$

with,
$\psi_0(x, \lambda, \widetilde{q}) = \widetilde{u}(x, \lambda)$ and
$\psi_{k+1}(x, \lambda, \widetilde{q}) = \int_{-1}^{x} G(x, t, \lambda)\ \widetilde{q}(t)\ \psi_k(t, \lambda, \widetilde{q})\ dt$ for each $k \in \mathbb{N}$.

By induction we will prove that this series converges uniformly on bounded subsets of $[-1, 1] \times \mathbb{R} \times L^2_{\mathbb{R}}[-1, 1]$, to the solution of equation (2.1).

Note that

$$\begin{aligned} \psi_1(x, \lambda, \widetilde{q}) &= \int_{-1}^{x} G(x, t_1, \lambda)\ \widetilde{q}(t_1)\ \psi_0(t_1, \lambda, \widetilde{q})\ dt_1 \\ &= \int_{-1}^{x} \frac{\sin(\sqrt{\lambda}(x - t_1))}{\sqrt{\lambda}}\ \widetilde{q}(t_1)\ \frac{\sin(\sqrt{\lambda}(t_1 + 1))}{\sqrt{\lambda}}\ dt_1 \end{aligned}$$

and ,

$$\psi_2(x, \lambda, \widetilde{q}) = \int_{-1}^{x} \frac{\sin(\sqrt{\lambda}(x - t_2))}{\sqrt{\lambda}}\ \widetilde{q}(t_2)\ \psi_1(t_2, \lambda, \widetilde{q})\ dt_2$$

$$= \int_{-1}^{x} \frac{\sin(\sqrt{\lambda}(x - t_2))}{\sqrt{\lambda}}\ \widetilde{q}(t_2)\ \int_{-1}^{t_2} \frac{\sin(\sqrt{\lambda}(t_2 - t_1))}{\sqrt{\lambda}}\ \widetilde{q}(t_1)\ \frac{\sin(\sqrt{\lambda}(t_1 + 1))}{\sqrt{\lambda}}\ dt_1\ dt_2$$

$$= \int_{-1 \le t_1 \le t_2 \le t_3 = x}^{x} \frac{\sin(\sqrt{\lambda}(t_1 + 1))}{\sqrt{\lambda}} \prod_{i=1}^{2} \widetilde{q}(t_i) \frac{\sin(\sqrt{\lambda}(t_{i+1} - t_i))}{\sqrt{\lambda}} dt_1 dt_2$$

Proceeding by induction

$$\psi_n(x, \lambda, \widetilde{q}) = \int_{-1 \le t_1 \le \ldots \le t_n \le t_{n+1} = x}^{x} \frac{\sin(\sqrt{\lambda}(t_1 + 1))}{\sqrt{\lambda}} \prod_{i=1}^{n} \widetilde{q}(t_i) \frac{\sin(\sqrt{\lambda}(t_{i+1} - t_i))}{\sqrt{\lambda}} dt_1 dt_2 \ldots dt_n$$

By the following elementary inequality

$$\cos(\sqrt{\lambda}(x + 1)) = \frac{1}{2} \Big| e^{i\sqrt{\lambda}(x+1)} - e^{-i\sqrt{\lambda}(x+1)} \Big| \le exp(Im\sqrt{\lambda}(x + 1))$$

we have, for $-1 \le x \le 1$,

$$\Big| \frac{\sin(\sqrt{\lambda}(t + 1))}{\sqrt{\lambda}} \Big| = \Big| \int_{-1}^{x} \cos(\sqrt{\lambda}(t + 1)) dt \Big| \le \frac{1}{\sqrt{\lambda}} exp(|Im\sqrt{\lambda}|(t + 1)).$$

Hence,

$$
\begin{aligned}
|\psi_n(x,\lambda,q)| &\le \frac{e^{|Im\sqrt{\lambda}(x+1)|}}{|\lambda|}\int_{-1\le t_1\le \ldots \le t_n\le x}^{x}\prod_{i=1}^{n}|\widetilde{q}(t_i)|dt_1dt_2...dt_n \\
&= \frac{e^{|Im\sqrt{\lambda}(x+1)|}}{|\lambda|\ n!}\int_{[-1,x]^n}\prod_{i=1}^{n}|\widetilde{q}(t_i)|dt_1dt_2...dt_n \\
&= \frac{e^{|Im\sqrt{\lambda}(x+1)|}}{|\lambda|\ n!}\Big(\int_{-1}^{x}|\widetilde{q}(t)|dt\Big)^n \\
&\le \frac{e^{|Im\sqrt{\lambda}(x+1)|}}{|\lambda|\ n!}(\|\widetilde{q}\|\,(\sqrt{x+1}))^n.
\end{aligned} \tag{2.3}
$$

This shows the convergence of the series to a continuous function. Since we have uniform convergence of this series then the integration and summation may be interchanged to get,

$$
\begin{aligned}
\psi(x,\lambda,\widetilde{q}) &= \psi_0(x,\lambda,\widetilde{q}) + \sum_{k=1}^{\infty}\psi_k(x,\lambda,\widetilde{q}) \\
&= u(x,\lambda) + \sum_{k=1}^{\infty}\int_{-1}^{x}G(x,t,\lambda)\ \widetilde{q}(t)\ \psi_{k-1}(t,\lambda,\widetilde{q})\ dt \\
&= u(x,\lambda) + \int_{-1}^{x}G(x,t,\lambda)\ \widetilde{q}(t)\ \sum_{k=1}^{\infty}\psi_{k-1}(t,\lambda,\widetilde{q})\ dt \\
&= u(x,\lambda) + \int_{-1}^{x}G(x,t,\lambda)\ \widetilde{q}(t)\ \psi(t,\lambda,\widetilde{q})\ dt
\end{aligned}
$$

That verifies that the integral equation of ψ is well defined, and substituting in (2.1) by ψ proves that ψ is a solution of the ODE (2.1). By taking a sum from $n = 1$ to ∞ on the estimates of $|\psi_n(x,\lambda,q)|$ in (2.3), we get the first inequality, and by adding the first term $n = 0$ we will get the second inequality.

□

2.3 Eigenvalues of the Extended Problem

In this section we present some of the properties of the eigenvalues. The first property follows from the fact that this problem contains a self-adjoint operator with boundary conditions. Thus the eigenvalues are real numbers.

Theorem 2.3.1. *The Dirichlet spectrum of $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ is a set of real numbers.*

Proof. Suppose λ is a Dirichlet eigenvalue of $\widetilde{q}$ with eigenfunction ψ then,

$$-\psi'' + \widetilde{q}(x)\psi = \lambda\psi$$

Conjugating the equation,

$$-\bar{\psi}'' + \widetilde{q}(x)\bar{\psi} = \bar{\lambda}\bar{\psi}$$

since $\widetilde{q}$ is real. Multiplying the first equation by $\bar{\psi}$, the second by ψ and taking the difference, we obtain

$$[\psi, \bar{\psi}]' = \psi\bar{\psi}'' - \psi''\bar{\psi} = (\lambda - \bar{\lambda})|\psi|^2.$$

where $[.,.]$ is the Wronskin of any two differentiable functions f and g which is given by $[f, g] = f'g - fg'$. Hence by integration,

$$[\psi, \bar{\psi}](-1) - [\psi, \bar{\psi}](1) = (\lambda - \bar{\lambda}) \int_{-1}^{1} |\psi(t)|^2 dt.$$

Since λ is an eigenvalue, then the left hand side is zero by the values of the eigenfunction at the boundary. Since the eigenfunction is non trivial solution then the integral does not equal zero. Therefore $\lambda - \bar{\lambda} = 0$, which means that λ is real.

□

As we saw that for $q = 0$, the Dirichlet spectrum is the set of zeros of $\widetilde{u}(1, \lambda)$ which are $\{\frac{(n\pi)^2}{4}\}_{n\geq 1}$. The compactness of the resolvent and Fredholm alternative shows that the Dirichlet spectrum of any potential $\widetilde{q}$ is an infinite sequence of real numbers, which is bounded below and tends to $+\infty$. the main point of the following Counting Lemma is to give an estimate of their location.

Lemma 2.3.1. *(The Counting Lemma)*

Let $\widetilde{q} \in L^2_{\mathbb{C}}$ and $N > 2e^{2\|\widetilde{q}\|}$ be an integer. Then $\psi(1, \lambda, \widetilde{q})$ has exactly N roots, counted with multiplicity, in the open half plane

$$Re\lambda < (N + \frac{1}{4})^2\pi^2,$$

and for each $n > N$, exactly one simple root in the egg shaped region

$$|\sqrt{\lambda} - \frac{n\pi}{2}| < \frac{\pi}{4}.$$

There is no other roots.

Proof. Fix $N > 2e^{2\|q\|}$, and let $K > N$ be another integer. Consider the contours

$$|\sqrt{\lambda}| = (K + \frac{1}{4})\pi$$

$$Re\sqrt{\lambda} = (N + \frac{1}{4})\pi$$

and

$$|\sqrt{\lambda} - \frac{n\pi}{2}| = \frac{\pi}{4}, \qquad n > N.$$

By Lemma A2.4 from the appendix, the estimate

$$e^{|Im 2\sqrt{\lambda}|} < 4|\sin 2\sqrt{\lambda}|$$

holds on all of them. Therefore, by the basic estimate for ψ,

$$\begin{aligned}\left|\psi(1,\lambda,\widetilde{q})-\frac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}\right| &\leq \frac{e^{2\|\widetilde{q}\|}}{|\sqrt{\lambda}|}\frac{e^{|Im 2\sqrt{\lambda}|}}{|\sqrt{\lambda}|} \\ &< \frac{2N}{|\sqrt{\lambda}|}\left|\frac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}\right| \\ &< \left|\frac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}\right|\end{aligned} \tag{2.4}$$

also holds on them. It follows that $\psi(1,\lambda,\widetilde{q})$ does not vanish on these contours. Hence, by Rouche's theorem, $\psi(1,\lambda,\widetilde{q})$ has as many roots, counted with multiplicity, as $\dfrac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}$ in each of the bounded regions and the remaining unbounded region. Since $\dfrac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}$ has only the simple roots $(\dfrac{n\pi}{2})^2$, $n \geq 1$, and since $K > N$ can be chosen arbitrary large, the lemma follows.

□

Theorem 2.3.2. *If λ is an eigenvalue of $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$, then*

$$\psi'(1,\lambda,\widetilde{q})\dot{\psi}(1,\lambda,\widetilde{q}) = \int_{-1}^{1}\psi^2(t,\lambda)dt = \|\psi(.,\lambda,\widetilde{q})\|^2 > 0.$$

In particular, $\dot{\psi}(1,\lambda,\widetilde{q}) \neq 0$. Thus, all roots of $\dot{\psi}(1,\lambda,\widetilde{q})$ are simple.

Proof. Let $\psi = \psi(x,\lambda,\widetilde{q})$. Differentiating equation (2.1) with respect to λ yields

$$-\frac{d^2}{dx^2}\dot{\psi} + \widetilde{q}(x)\dot{\psi} = \psi + \lambda\dot{\psi}$$

Multiplying this equation by ψ, the equation (2.1) by $\dot{\psi}$ and taking the difference we obtain

$$\psi^2 = \psi''\dot{\psi} - \dot{\psi}''\psi = [\dot{\psi},\psi]'$$

Integration both sides over $[-1,1]$,we will get

$$\begin{aligned}\int_{-1}^{1}\psi^2(t,\lambda,\widetilde{q})dt &= [\dot{\psi},\psi](1) - [\dot{\psi},\psi](-1) \\ &= \dot{\psi}(1,\lambda,\widetilde{q})\psi'(1,\lambda,\widetilde{q}),\end{aligned}$$

since $\psi(-1,\lambda,\widetilde{q})$ and $\dot{\psi}(-1,\lambda,\widetilde{q})$ vanish for all λ, and $\psi(1,\lambda,\widetilde{q})$ vanishes for an eigenvalue λ. Since ψ is real for real λ. then the integral is equal to $\|\psi(.,\lambda,\widetilde{q})\|^2$.

□

Now if we consider λ_n as a function on $L^2_{\mathbb{R}}[-1,1]$, then as in Theorem 3 in [1], λ_n for any n, is real analytic function on $L^2_{\mathbb{R}}[-1,1]$.

Theorem 2.3.3. *For any $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ the gradient of λ_n is given by*

$$\frac{\partial \lambda_n}{\partial \widetilde{q}(t)} = g_n^2(t, \widetilde{q}),$$

where g_n is the normalized eigenfunction.

Proof. To get an idea how to get this gradient, recall that the derivative of a map $f : E \to F$ between two Banach spaces E and F at a point $x \in E$ is a bounded linear map from $T_x E$ into $T_{f(x)} F$ which we denote by $d_x f$. Moreover, if E and F are Hilbert spaces then $T_x E = E$, $T_{f(x)} F = F$. Also if E is a Hilbert space and F is the real or complex line, then by the Riesz representation theorem, there is a unique element $\partial f / \partial x$ in E, such that for all ν in E,

$$d_x f(\nu) = \Big\langle \nu, \overline{\frac{\partial f}{\partial x}} \Big\rangle.$$

This element is the gradient of f at x. Hence, by differentiating both sides of the differential equation (2.1) of g_n in the direction ν we obtain

$$-d_{\widetilde{q}} g_n''(\nu) + \widetilde{q} d_{\widetilde{q}} g_n(\nu) + \nu g_n = \lambda_n d_{\widetilde{q}} g_n(\nu) + d_{\widetilde{q}} \lambda_n(\nu) g_n$$

If $\widetilde{q}$ is continuous, then g_n is twice continuously differentiable, and we may interchange differentiation with respect to x and q to obtain

$$-(d_{\widetilde{q}} g_n(\nu))'' + q d_{\widetilde{q}} g_n(\nu) + \nu g_n = \lambda_n d_{\widetilde{q}} g_n(\nu) + d_{\widetilde{q}} \lambda_n(\nu) g_n$$

Multiplying both sides by g_n and integrating we get

$$\Big\langle (-\frac{d^2}{dx^2} + \widetilde{q}) g_n(\nu),\ g_n \Big\rangle + \big\langle g_n^2,\ \nu \big\rangle\ = \lambda_n \big\langle d g_n(\nu),\ g_n \big\rangle + d_{\widetilde{q}} \lambda_n(\nu)$$

The first term equals

$$\Big\langle (-\frac{d^2}{dx^2} + \widetilde{q}) g_n(\nu),\ g_n \Big\rangle = \lambda_n \big\langle d_{\widetilde{q}} g_n(\nu),\ g_n \big\rangle.$$

Hence

$$d_{\widetilde{q}} \lambda_n(\nu) = \big\langle g_n^2,\ \nu \big\rangle$$

and

$$\frac{\partial \lambda_n}{\partial \widetilde{q}(t)} = g_n^2(t, \widetilde{q}).$$

∎

Theorem 2.3.4. *For $\widetilde{q}$ in $L^2_{\mathbb{R}}[-1,1]$,*

$$\begin{aligned} \lambda_n(\widetilde{q}) &= (\frac{n\pi}{2})^2 + \int_{-1}^{1} \widetilde{q}(t) dt - \frac{1}{2} \langle \cos(n\pi(x+1)), \widetilde{q} \rangle + O(\frac{1}{n}) \\ &= (\frac{n\pi}{2})^2 + \int_{-1}^{1} \widetilde{q}(t) dt + l^2(n) \end{aligned}$$

and

$$g_n(x,\widetilde{q}) = \sin(\frac{n\pi}{2}(x+1)) + O(\frac{1}{n})$$

$$g_n^{'}(x,\widetilde{q}) = \frac{\sqrt{2}n\pi}{2}\cos(\frac{n\pi}{2}(x+1)) + O(1)$$

Proof. Let $\lambda_n = \lambda_n(\widetilde{q})$. By the Counting lemma,

$$\sqrt{\lambda_m} = \frac{n\pi}{2} + O(1)$$

By the estimate of ψ,

$$\begin{aligned}\psi(x,\lambda_n,\widetilde{q}) &= \frac{\sin(\sqrt{\lambda_n}(x+1))}{\sqrt{\lambda_n}} + O(\frac{1}{|\lambda_n|})\\ &= \frac{\sin(\sqrt{\lambda_n}(x+1))}{\sqrt{\lambda_n}} + O(\frac{1}{n^2})\end{aligned}$$

Using the identity $2\sin^2(2ax) = 1 - \cos(2ax)$, we get

$$\begin{aligned}\int_{-1}^{1}\psi^2(t,\lambda_n,\widetilde{q})dt &= \int_{-1}^{1}\frac{\sin^2(\sqrt{\lambda_n}(t+1))}{\lambda_n}dt + O(\frac{1}{n^3})\\ &= \frac{1}{2\lambda_n}\Big(2 - \frac{\sin 4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\Big) + O(\frac{1}{n^3})\\ &= \frac{1}{\lambda_n}(1 + O(\frac{1}{n}))\end{aligned}$$

It follows that

$$||\psi(.,\lambda_n,\widetilde{q})||^{-1} = \sqrt{\lambda_n}(1 + O(\frac{1}{n}))$$

Hence,

$$g_n(x,\widetilde{q}) = \frac{\psi(x,\lambda_n,\widetilde{q})}{||\psi(.,\lambda_n,\widetilde{q})||} = \sin(\sqrt{\lambda_n}(x+1)) + O(\frac{1}{n}) \tag{2.5}$$

Note that,

$$\begin{aligned}\lambda_n(\widetilde{q}) - (\frac{n\pi}{2})^2 =& \lambda_n(\widetilde{q}) - \lambda_n(0)\\ =& \int_{-1}^{1}\frac{d}{dt}\lambda_n((\frac{t+1}{2})\widetilde{q})dt\\ =& \int_{-1}^{1}\langle g_n^2(x,(\frac{t+1}{2})\widetilde{q}),\widetilde{q}\rangle dt\end{aligned} \tag{2.6}$$

Since

$$\lambda_n(\widetilde{q}) = (\frac{n\pi}{2})^2 + O(1)$$

or equivalently

$$\sqrt{\lambda_n(\widetilde{q})} = \frac{n\pi}{2} + O(\frac{1}{n})$$

then,

$$g_n(x,\widetilde{q}) = \sin(\frac{n\pi}{2}(x+1)) + O(\frac{1}{n})$$

Since (2.5) holds for $\frac{t+1}{2}\widetilde{q}$, $-1 \leq t \leq 1$, then by inserting the last estimate of g_n into (2.6) and using the identity

$$2\sin^2 ax = 1 - \cos 2ax$$

we obtain

$$\begin{aligned}
\lambda_n(\widetilde{q}) - (\frac{n\pi}{2})^2 &= \int_{-1}^{1} \langle \frac{1}{2}(1 - \cos(n\pi(x+1)) + O(\frac{1}{n}), \widetilde{q}\rangle dt \\
&= \int_{-1}^{1} \widetilde{q}(t)dt - \langle \cos(n\pi(x+1)), \widetilde{q}\rangle + O(\frac{1}{n}) \\
&= \int_{-1}^{1} \widetilde{q}(t)dt + \langle \cos(n\pi x), \widetilde{q}\rangle + O(\frac{1}{n}) \\
&= \int_{-1}^{1} \widetilde{q}(t)dt + l^2(n)
\end{aligned}$$

since $\langle \cos(n\pi(x+1)), \widetilde{q}\rangle$ are the square summable Fourier coefficients of $\widetilde{q}$.

Finally, we estimate $g_n^{'}$, by using the basic estimate of $\psi^{'}$,

$$\begin{aligned}
\psi^{'}(x,\lambda_n,\widetilde{q}) &= \cos(\sqrt{\lambda_n}(x+1)) + O(\frac{1}{\sqrt{\lambda_n}}) \\
&= \cos(\frac{n\pi}{2}(x+1)) + O(\frac{1}{n})
\end{aligned}$$

Dividing by $||\psi(.,\lambda_n,\widetilde{q})||$,

$$\begin{aligned}
g_n^{'}(x,\widetilde{q}) &= \sqrt{\mu_n}\cos(\frac{n\pi}{2}(x+1)) + O(1) \\
&= \frac{\pi n}{2}\cos(\frac{n\pi}{2}(x+1)) + O(1).
\end{aligned}$$

□

2.4 Uniqueness of the Potential

The method that will be used in this section to prove uniqueness depends in the fact that if $\psi_n(x,\lambda_n(\widetilde{q}),\widetilde{q}) = \psi_n(x,\lambda_n(\widetilde{p}),\widetilde{p})$ then $\widetilde{q} = \widetilde{p}$ almost every where. This fact follow from equation (2.1)

$$\begin{aligned}
\psi_n(x,\lambda_n(\widetilde{q}),\widetilde{q})\widetilde{q} &= \lambda_n(\widetilde{q})\psi_n(x,\lambda_n(\widetilde{q}),\widetilde{q}) + \psi_n^{''}(x,\lambda_n(\widetilde{q}),\widetilde{q}) \\
&= \lambda_n(\widetilde{p})\psi_n(x,\lambda_n(\widetilde{p}),\widetilde{p}) + \psi_n^{''}(x,\lambda_n(\widetilde{p}),\widetilde{p}) \\
&= \psi_n(x,\lambda_n(\widetilde{p}),\widetilde{p})\widetilde{p}.
\end{aligned}$$

Since $\psi_n(x, \lambda_n(\widetilde{q}), \widetilde{q}) = \psi_n(x, \lambda_n(\widetilde{p}), \widetilde{p})$ then $\widetilde{q} = \widetilde{p}$. The equality between the eigenfunctions for $\widetilde{q}$ and $\widetilde{p}$ requires some specific properties. Thus, before we give the proof of the uniqueness, we will prove some important properties of the eigenfunctions that will help to prove the uniqueness. Recall that we write $\psi(x, \lambda_n(\widetilde{q}))$ for $\psi(x, \lambda_n(\widetilde{q}), \widetilde{q})$.

Theorem 2.4.1. *For $\widetilde{q}$ in $L^2_{\mathbb{R}}[-1,1]$, $\psi(1, \lambda, \widetilde{q}) = \prod_{n\geq 1} \frac{\lambda_n(\widetilde{q}) - \lambda}{(\frac{n\pi}{2})^2}$ where $\lambda_n(\widetilde{q})$ is the n^{th} eigenvalue of* (2.1)-(2.2).

Proof.

Since $\lambda_n(\widetilde{q}) = (\frac{n\pi}{2})^2 + O(1)$, then by Lemma A2.2, the infinite product $p(\lambda) = \prod_{n\geq 1}^{\infty} \frac{\lambda_n(\widetilde{q}) - \lambda}{(\frac{n\pi}{2})^2}$ is an entire function of λ and

$$p(\lambda) = \frac{\sin(2\sqrt{\lambda})}{\sqrt{\lambda}} \Big(1 + O\big(\frac{\log n}{n}\big)\Big)$$

uniformly on the circle $|\lambda| = r_n = (\frac{n}{2} + \frac{1}{2})^2\pi^2$ for n large enough.
Also the roots of $p(\lambda)$ are $\lambda_n(\widetilde{q})$, hence the quotient $\frac{p(\lambda)}{\psi(1,\lambda,\widetilde{q})}$ is an entire function. By the basic estimate for ψ we have

$$\psi(1, \lambda) = \frac{\sin(2\sqrt{\lambda})}{\sqrt{\lambda}} \Big(1 + O\big(\frac{1}{n}\big)\Big)$$

uniformly for $\lambda = r_n$. Hence,

$$\frac{p(\lambda)}{\psi(1, \lambda, \widetilde{q})} = 1 + O\Big(\frac{\log n}{n}\Big)$$

for $|\lambda_n| = r_n$, that is,

$$\sup_{|\lambda|=r_n} \Big|\frac{p(\lambda)}{\psi(1, \lambda, \widetilde{q})} - 1\Big| \to 0$$

as $n \to \infty$. It follows from the maximum principle that the difference vanishes identically. Hence $p(\lambda) = \psi$.

□

This product formula of $\psi(1, \lambda, q)$ gives us important information about the type of the eigenfunction either even or odd depending in its sign. In the next corollary we have useful consequences of the previous theorem.

Corollary 2.4.1.

- $\dot{\psi}(1, \lambda_n(\widetilde{q})) = \frac{-1}{(\frac{n\pi}{2})^2} \prod_{m\neq n} \frac{\lambda_m(\widetilde{q}) - \lambda_n(\widetilde{q})}{(\frac{m\pi}{2})^2} = \frac{(-1)^n}{\frac{n^2\pi^2}{2}} \Big(1 + O\big(\frac{\log n}{n}\big)\Big)$
- *sgn* $\dot{\psi}(1, \lambda_n(\widetilde{q})) = (-1)^n =$ *sgn* $\psi'(1, \lambda_n(\widetilde{q}))$

Proof. Part one follows from Lemma A2.3, The first identity in the second part follows from the first part, and the second identity is a consequence of the first identity and Theorem 2.3.2.

□

Theorem 2.4.2. *let $g_n(x, \widetilde{q})$ be the normalized eigenfunction of $\lambda_n(\widetilde{q})$. If $\widetilde{q}$ is even, then $g_n(x, \widetilde{q})$ is even when n is odd and odd when n is even.*

Proof. Note that $g_n(x, \widetilde{q})$ and $g_n(-x, \widetilde{q})$ are eigenfunctions of $\widetilde{q}$ for $\lambda_n(\widetilde{q})$ with norm 1. By Theorem 2.3.2, $\lambda_n(\widetilde{q})$ is simple then,

$$g_n(x, \widetilde{q}) = C g_n(-x, \widetilde{q})$$

where $C = +1$ or -1. Since $g_n^{'}(-1, \widetilde{q}) = 1$, then $g_n^{'}(1) = -C$, thus sgn C = - sgn $g_n^{'}(1) = (-1)^{n+1}$. Hence $g_n(x, \widetilde{q}) = (-1)^{n+1} g_n(-x, \widetilde{q})$.

□

Lemma 2.4.1. *Let f be a meromorphic function in the plane. If*

$$\sup_{\lambda = r_n} |f(\lambda)| = o(\frac{1}{r_n})$$

for an unbounded sequence of positive real numbers r_n, then the sum of the residues of f is zero. In particular, If the residues of f are real nonnegative, then they are all zero.

Proof. The sum of the residues of f is, by definition,

$$\lim_{n \to \infty} \frac{1}{2\pi i} \int_{|\lambda| = r_n} f(\lambda) d\lambda$$

But

$$\begin{aligned} \Big| \int_{|\lambda| = r_n} f(\lambda) d\lambda \Big| &\leq \int_{|\lambda| = r_n} |f(\lambda)| d\lambda \\ &= o(\frac{1}{r_n}) \int_{|\lambda| = r_n} d\lambda \\ &= o(1), \end{aligned} \tag{2.7}$$

so the sum is zero.

□

Theorem 2.4.3. *For $q, p \in L^2_{\mathbb{R}}[0, 1]$, let $\widetilde{q}, \widetilde{p} \in L^2_{\mathbb{R}}[-1, 1]$ be their even extensions respectively with $\lambda_n(\widetilde{q}) = \lambda_n(\widetilde{p}) = \lambda_n$ for each n, then $q = p$.*

Proof. Since $\widetilde{q}, \widetilde{p}$ are the even extensions of q and p respectively then $\widetilde{q} = \widetilde{p}$ implies $q = p$. So the goal is to prove that $\widetilde{q} = \widetilde{p}$. Notice that if $\psi(x, \lambda, \widetilde{q}) = \psi(x, \lambda, \widetilde{p})$ for some λ and each x then $\widetilde{p} = \widetilde{q}$. Hence we will assume that there is some x such that $\psi(x, \lambda, \widetilde{q}) \neq \psi(x, \lambda, \widetilde{p})$ and define,

$$f(\lambda) = -\frac{\big(\psi(x,\lambda,\widetilde{q}) - \psi(x,\lambda,\widetilde{p})\big)\big(\psi(-x,\lambda,\widetilde{q}) - \psi(-x,\lambda,\widetilde{p})\big)}{\psi(1,\lambda,\widetilde{q})}.$$

f is a meromorphic function which has simple poles at λ_n, for each $n \geq 1$ otherwise it is regular. Since $\psi(x,\lambda_n) = (-1)^{n+1}\psi(-x,\lambda_n)$ for $\widetilde{p}$ and $\widetilde{q}$ and by Corollary 2.4.1 sgn $\dot{\psi}(1,\lambda_n(\widetilde{q})) = (-1)^n$.Thus the residue of f at each λ_n is

$$R_f(\lambda_n) = \frac{\big(\psi(x,\lambda_n,\widetilde{q}) - \psi(x,\lambda_n,\widetilde{p})\big)^2}{(-1)^n\dot{\psi}(1,\lambda_n,\widetilde{q})} \geq 0.$$

Now we show that the function f satisfies the hypothesis of Lemma 2.4.1 for $r_n = (\frac{n}{2}+\frac{1}{4})^2\pi^2$. The numerator of f is bounded from above by,

$$\frac{e^{|Im\sqrt{\lambda}|(x+1)}\ e^{|Im\sqrt{\lambda}|(-x+1)}}{|\lambda|^2} = \frac{e^{|Im\lambda|}}{|\lambda|^2}$$

To bound the denominator from below we have,

$$\Big|\psi(1,\lambda,\widetilde{q}) - \frac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}\Big| \leq \frac{e^{||\widetilde{q}||}e^{2|Im\sqrt{\lambda}|}}{|\sqrt{\lambda}||\sqrt{\lambda}|} \leq \frac{e^{2|Im\sqrt{\lambda}|}}{8|\sqrt{\lambda}|}$$

for $|\lambda| \geq 8e^{||\widetilde{q}||}$. Therefore,

$$\begin{aligned} |\psi(1,\lambda,\widetilde{q})| &\geq \Big|\frac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}\Big| - \Big|\psi(1,\lambda,\widetilde{q}) - \frac{\sin 2\sqrt{\lambda}}{\sqrt{\lambda}}\Big| \\ &\geq \frac{e^{2|Im\sqrt{\lambda}|}}{4|\sqrt{\lambda}|} - \frac{e^{2|Im\sqrt{\lambda}|}}{8|\sqrt{\lambda}|} \\ &\geq \frac{e^{2|Im\sqrt{\lambda}|}}{8|\sqrt{\lambda}|} \end{aligned}$$

for $|\lambda| = r_n$, and n sufficiently large. Hence the quotient of the two bounds is $O(1/r_n^{3/4}) = O(1/r_n)$ as required, so $R_f(\lambda_n) = 0$ for each n which implies,

$$\psi(x,\lambda_n,\widetilde{q}) = \psi(x,\lambda_n,\widetilde{p}), \quad n \geq 1$$

Hence $\widetilde{q}$ and $\widetilde{p}$ have the same eigenfunctions, then by (2.1)

$$\widetilde{q}\psi = \lambda_n(\widetilde{q})\psi + \psi^{''} = \lambda_n(\widetilde{p})\psi + \psi^{''} = \widetilde{p}\psi$$

and therefore $\widetilde{q} = \widetilde{p}$ almost everywhere which implies that $q = p$.

$\square$

In this section we prove that $p = q$ by looking to their even extended potentials and studying the eigenfunctions of the extended potentials. In the next section we will present a second method to solve this problem. This method also related to the assumption that the extended potentials share the same eigenvalue to get extra spectral data. Before we start the next section we want to thank Herschenfeld, Samuel who suggested this method.

2.5 Second Method to Prove the Uniqueness of the potential

Firs, we will consider the following diffemorphism $\phi : [0,1] \to [-1,1]$ which is defined by $\phi(x) = 2x - 1$. Let $\phi^{-1} : [-1,1] \to [0,1]$ be the inverse of ϕ, so $\phi^{-1}(x) = \frac{1}{2}(x+1)$. Let $q \in L^2_{\mathbb{R}}[0,1]$, and $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ be the even extended of q. We consider the new potential $Q = 4\widetilde{q} \circ \phi$ in $L^2_{\mathbb{R}}[0,1]$. To prove the uniqueness of q we need to prove the uniqueness of Q. Before we give the proof, we will show some properties of the potential Q.

Theorem 2.5.1. *For $q \in L^2_{\mathbb{R}}[0,1]$, let $Q \in L^2_{\mathbb{R}}[0,1]$ such that $Q = 4\widetilde{q} \circ \phi$ where $\widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ is the even extended of q. Then the potential Q is even and $\sigma(H_Q) = 4\sigma(H_{\widetilde{q}})$.*

Proof. Q is even that follows from the definition of $\widetilde{q}$ and ϕ. To show that $\sigma(H_{\widetilde{q}}) = \sigma(H_Q)$, we consider the following function $\psi = \widetilde{\psi} \circ \phi$, where $\widetilde{\psi}$ is the solution of (2.1)-(2.2). For $x \in (0,1)$ let $x_1 \in (-1,1)$ such that $\phi^{-1}(x_1) = x$, so for each $\lambda_n(\widetilde{q}) \in \sigma(H_{\widetilde{q}})$ we have

$$\begin{aligned}
-\psi''(x, \lambda_n(\widetilde{q})) + Q(x)\psi(x, \lambda_n(\widetilde{q})) &= -\psi''(\phi^{-1}(x_1), \lambda_n(\widetilde{q})) + Q(\phi^{-1}(x_1))\psi(\phi^{-1}(x_1), \lambda_n(\widetilde{q})) \\
&= -4\widetilde{\psi}''(x_1, \lambda_n(\widetilde{q})) + 4\widetilde{q}(x_1)\widetilde{\psi}(x_1, \lambda_n(\widetilde{q})) \\
&= 4\lambda_n(\widetilde{q})\widetilde{\psi}(x_1, \lambda_n(\widetilde{q})) \\
&= 4\lambda_n(\widetilde{q})\psi(x, \lambda_n(\widetilde{q}))
\end{aligned}$$

and for the boundary conditions $\psi(0, \lambda_n(\widetilde{q}))) = \widetilde{\psi}(-1, \lambda_n(\widetilde{q}))) = 0$ and $\psi(1, \lambda_n(\widetilde{q}))) = \widetilde{\psi}(1, \lambda_n(\widetilde{q}))) = 0$. Hence $\psi(., \lambda_n(\widetilde{q}))$ is the n^{th} eigenfunction of $4\lambda_n(\widetilde{q})$, that show that $\sigma(H_Q) = 4\sigma(H_{\widetilde{q}})$.

□

Theorem 2.5.2. *For $q, p \in L^2_{\mathbb{R}}[0,1]$, let $\widetilde{q}, \widetilde{p} \in L^2_{\mathbb{R}}[-1,1]$ are the even extended of q and p respectively. If $\sigma(H_{\widetilde{q}}) = \sigma(H_{\widetilde{p}})$ then $q = p$.*

Proof. Define $Q = \widetilde{q} \circ \phi$ and $P = \widetilde{p} \circ \phi$. From Theorem 2.5.1, Q and P are two even potentials in $L^2_{\mathbb{R}}[0,1]$ such that $\sigma(H_Q) = \sigma(H_P)$. Hence from Pöschel and Trubowitz's Theorem 1.1.3, $Q = P$. Hence $\widetilde{q} = \frac{1}{4}Q \circ \phi^{-1} = \frac{1}{4}P \circ \phi^{-1} = \widetilde{p}$ which implies $q = p$.

□

In this chapter we focused on the zero angular momentum and the set of potentials in $L^2_{\mathbb{R}}[0,1]$ that have even extended potentials in $L^2_{\mathbb{R}}[-1,1]$ sharing the same eigenvalues of the extended problem. In the next chapter we will study the problem from geometric side. For any $p \in L^2_{\mathbb{R}}[0,1]$ and two angular momenta l_1, l_2, we will work on the dimension of two tangent spaces at any point $q \in M^{l_1}(p) \cap M^{l_2}(p)$.

Chapter 3 Dimension of the Intersection of Tangent Spaces

3.1 An Overview of the Isospectral Set of a Potential $p \in L^2_{\mathbb{R}}[0,1]$

In this chapter we focus on the dimension of the intersection of two tangent spaces corresponding to two different angular momentum l_1, l_2. The first angular momenta will be fixed $l_1 = 0$ and we will vary l_2.

Definition 3.1.1. For each angular momentum $l = 0, 1, 2, ...$ denote the isospectral set of p by $M^l(p)$ and its definition is given by

$$M^l(p) = \{q \in L^2_{\mathbb{R}}[0,1] : \lambda_{l,n}(q) = \lambda_{l,n}(p),\ n = 1, 2, 3, ...\}$$

and for each $q \in M^l(p)$ denote the tangent space of $M^l(p)$ at q by $T_qM^l(p)$ which is the set consisting all the velocity vectors at q of all smooth curves on $M^l(p)$ which pass through q.

For any angular momentum l, by considering the map

$$\lambda^l : L^2_{\mathbb{R}}[0,1] \to \mathbb{R} \times l^2$$

$$\lambda^l(q) = \big([q], \{\widetilde{\lambda}_{l,n}(q)\}_{n\geq 1}\big)$$

where $[q] = \displaystyle\int_0^1 q(x)dx$ and

$$\lambda_{l,n}(q) = \big(n + \frac{l}{2}\big)^2\ \pi^2 + \int_0^1 q(x)\ dx - l(l+1) + \widetilde{\lambda}_{l,n}(q), \qquad \sum_{n=1}^{\infty} |\widetilde{\lambda}_{l,n}(q)|^2 < \infty$$

we have the following properties of λ^l and $M^l(p)$.

Theorem 3.1.1. *λ^l is a real analytic map on $L^2_{\mathbb{R}}[0,1]$. It is derivative at q is the linear map from $L^2_{\mathbb{R}}[0,1]$ into $\mathbb{R} \times l^2$ given by*

$$\begin{aligned} d_q\lambda^l(\nu) &= \big(\langle 1, \nu\rangle, \big\{\langle \frac{\partial \widetilde{\lambda}_{l,n}}{\partial q}, \nu\rangle\big\}_{n\geq 1}\big) \\ &= \big(\langle 1, \nu\rangle, \big\{\langle g^2_{l,n} - 1, \nu\rangle\big\}_{n\geq 1}\big). \end{aligned}$$

Theorem 3.1.2. *For any angular momentum l and $p \in L^2_{\mathbb{R}}[0,1]$,*

1. *$M^l(p)$ is a real analytic submanifold of $L^2_{\mathbb{R}}[0,1]$ of infinite dimension and co-dimension lying in the hyperplane of all function with mean value $\displaystyle\int_0^1 p(t)dt$.*

2. At every point $q \in M^l(p)$ the tangent space is

$$T_qM^l(p) = span\{2\frac{d}{dx}g_{l,n}^2 : \ n \geq 1\}$$

and the normal space is

$$N_qM^l(p) = span\{g_{l,n}^2 - 1 \quad n \geq 1\}.$$

For any $q \in M^l(p)$, $\lambda^l(q)$ is a regular value of λ^l with the following splitting of $L^2_{\mathbb{R}}[0,1]$ in the sense of Definition 3.1.1

$$L^2_{\mathbb{R}}[0,1] = ker(d_q\lambda^l) \bigoplus (ker(d_q\lambda^l))^{\perp}.$$

Also the restriction of $d_q\lambda^l$ to $(ker(d_q\lambda^l))^{\perp}$ is boundedly invertible. Hence by Regular Value Theorem A2.1, $M^l(p)$ is a real analytic sub-manifold of $L^2_{\mathbb{R}}[0,1]$ with infinite dimension and co-dimension (see [1] and [8]) and

$$T_qM^l(p) = ker(d_q\lambda^l) \tag{3.1}$$

The vectors of the tangent space follow from Theorem 1.2.3 and definition of $d_q\lambda^l$, for more details see [1] for $l = 0$ and [3] for $l \geq 1$.

According to Definition A3.1 of the Fredholm operators, Shubin, in [6], proved that the intersection of $T_pM^0(p)$ and $T_pM^1(p)$ is finite dimensional for any $p \in L^2_{\mathbb{R}}[0,1]$ by defining a Fredholm operator that has $T_pM^0(p) \cap T_pM^1(p)$ as a kernel. Then Shubin and Carlson [7] have proven that these result is hold for any two angular momunta l_1 and l_2 such that $l_1 + l_2 = 1 \bmod 2$ which means that l_1 and l_2 have different parity see [7]. In this work we try to use the same method used by Shubin in [6] to prove similar results. The tangent spaces that we will consider here are at any point $q \in M^{l_1}(p) \cap M^{l_2}(p)$. In Chapter 4, we will try to connect the uniqueness of the potential with the dimension of the intersection of its tangent spaces. So the main theorem of this chapter is

Theorem 3.1.3. *For any $p \in L^2_{\mathbb{R}}[0,1]$ and $l \geq 1$, the intersection of $T_qM^0(p)$ and $T_qM^l(p)$ is finite dimensional at each fixed $q \in M^0(p) \cap M^l(p)$.*

We will discusses this problem in two different cases, one of them when one angular momentum is even and the second case when it is odd. Since that method depends on showing an operator is Fredholm, we will begin by introducing some important operators that will be used in the proof.

3.2 Some Important Operators and Their Properties

The first operator has a special property that converts squared eigenfunctions for angular momentum $l = 0$ to the eigenfunctions for $l > 0$. This property will help to analyze some operators easily. In the next definition and lemma, we will give a simple review of this operator and its properties. For more details about the proof of its properties, see [5] and [3].

Definition 3.2.1. For each positive integer l, define the operator $S_l : L^2_{\mathbb{C}}[0,1] \to L^2_{\mathbb{C}}[0,1]$ by

$$S_l[f](x) = f(x) - 4lx^{2l-1} \int_x^1 \frac{f(s)}{s^{2l}} \, ds$$

and the operator $T_l : L^2_{\mathbb{C}}[0,1] \to L^2_{\mathbb{C}}[0,1]$ by

$$T_l = (-1)^{l-1} S_l S_{l-1} ... \, S_1.$$

Lemma 3.2.1. *For each positive integer l*
1. The operator S_l has the following properties :

(i) S_l is bounded on $L^2_{\mathbb{C}}[0,1]$.

(ii) S_l is one to one and

$$S_l^{-1}[f](x) = f(x) - 4lx^{-2l} \int_x^1 f(s) s^{2l} \, ds.$$

(iii) The family S_l pairwise commutes: $S_{l_1} S_{l_2} = S_{l_2} S_{l_1}$ for any l_1, l_2.

(iv) The adjoint of S_l is

$$S_l^*[f](x) = f(x) - 4lx^{-2l} \int_0^x s^{2l-1} f(s) \, ds$$

2. The operator T_l is a bounded, one to one linear operator on $L^2_{\mathbb{C}}[0,1]$ such that for any $\xi \in L^2_{\mathbb{R}}[0,1]$ and $\lambda \geq 0$

$$\int_0^1 (2\Phi_l(\sqrt{\lambda x}) - 1) \xi(x) \, dx = \int_0^1 \cos(2\sqrt{\lambda x}) T_l[\xi](x) \, dx$$

where $\Phi_l(x) = (x j_l(x))^2$ and j_l is the spherical Bessel function of order l.

It is straightforward to compute explicit expressions for the operator T_l using the definition of S_l, hence

$$T_2[f](x) = -f(x) - 12x \int_x^1 \frac{f(s)}{s^2} \, ds + 24x^3 \int_x^1 \frac{f(s)}{s^4} \, ds$$

$$\begin{aligned} T_3[f](x) = f(x) &- 24x \int_x^1 \frac{f(s)}{s^2} \, ds + 120x^3 \int_x^1 \frac{f(s)}{s^4} \, ds \\ &- 120x^5 \int_x^1 \frac{f(s)}{s^6} \, ds \end{aligned}$$

$$\begin{aligned} T_4[f](x) = -f(x) &- 40x \int_x^1 \frac{f(s)}{s^2} \, ds + 360x^3 \int_x^1 \frac{f(s)}{s^4} \, ds \\ &- 840x^5 \int_x^1 \frac{f(s)}{s^6} \, ds + 560x^7 \int_x^1 \frac{f(s)}{s^8} \, ds \end{aligned}$$

Hence in general

$$T_l[f](x) = a_1 f(x) + a_2 x \int_x^1 \frac{f(s)}{s^2}\, ds + a_3 x^3 \int_x^1 \frac{f(s)}{s^4}\, ds$$
$$+ .... + a_{l+1} x^{2l-1} \int_x^1 \frac{f(s)}{s^{2l}}\, ds$$

where $a_1 = (-1)^{l+1}$ and $a_2, a_3, ...a_{l+1} \in \mathbb{R}$.

Following [6], the second operator we will consider here depends on l, and its composition with the previous operator T_l will be the interested operator that we will focus on.

Definition 3.2.2. For each $l \geq 1$, define the operator

$$P_l : L^2_{\mathbb{R}}[0,1] \to T_q M^0(p)$$

by

$$P_l(w) = \sum_{n\geq 1} \langle w, f_{l,n-1} \rangle h_n(x,q)$$

where $\{h_n\}$ is a basis of $T_q M^0(p)$ given by

$$\begin{aligned} h_n(x,q) &= \frac{\sqrt{2}}{2n\pi} \frac{d}{dx} \frac{\partial \widetilde{\lambda}_n^0}{\partial q} \\ &= \frac{\sqrt{2}}{2n\pi} \frac{d}{dx} (g_{0,n}(x,q)^2) \\ &= \sqrt{2} \sin(2n\pi x) + O(\frac{1}{n}) \end{aligned}$$

where $g_{0,n}$ is the normalized eigenfunction of problem (1.8)-(1.7) and

$$f_{l,n}(x,q) = \sqrt{2} \cos 2\sqrt{\lambda_{l,n}(q)} x + R_{n,l}(x,q), \qquad n \geq 1.$$

where $|R_{l,n}(x,q)| \leq Cn^{-1}$ (see [3] for more details for $f_{l,n}$).

From Theorem 1.3.2 the leading term of the asymptotic of $\lambda_{l,n}$ is $(n + \frac{l}{2})^2 \pi^2$. Thus when l is even, $f_{l,n}$ has the following asymptotic form

$$f_{l,n}(x,q) = \sqrt{2} \cos(2n\pi x) + O(1/n)$$

and when l is odd, $f_{l,n}$ has the following asymptotic form

$$f_{l,n}(x,q) = \sqrt{2} \cos((2n+1)\pi x) + O(1/n)$$

By recalling the definition of the eigenfunctions which have the spherical Bessel function as a leading term and using the properties of the operator T_l we have

$$T_l^*(1) = -1$$

$$T_l^*(f_{l,n}) = \frac{\partial \widetilde{\lambda}_{l,n}}{\partial q} = g_{l,n}^2(x,q) - 1, \qquad n \geq 1.$$

Now we return to our goal which is to define an operator with kernel as an intersection of two tangent spaces, we will consider the composition of P_l in definition 3.2.2 and T_l in definition 3.2.1 as an operator from $T_qM^0(p)$ to itself. So for any $\omega \in T_qM^0(p)$ we have,

$$\begin{aligned} P_lT_l(\omega) &= \sum_{n\geq 1} \langle T_l\omega, f_{l,n-1}\rangle h_n(x,q) \\ &= \sum_{n\geq 1} \langle \omega, T_l^* f_{l,n-1}\rangle h_n(x,q) \\ &= -\langle \omega, 1\rangle h_1(x,q) + \sum_{n\geq 2} \langle \omega, \frac{\partial \widetilde{\lambda}_{l,n-1}}{\partial q}\rangle h_n(x,q). \end{aligned}$$

Lemma 3.2.2. *For any $l > 0$, the operator*

$$P_lT_l : T_qM^0(p) \to T_qM^0(p)$$

has the following kernel

$$ker(P_lT_l) = T_qM^0(p) \cap T_qM^l(p).$$

Proof. Recall from (3.1) the tangent space of $M^l(p)$ at any $q \in M^l(p)$ is given by

$$T_qM^l(p) = ker(d_q\lambda^l)$$

where

$$d_q\lambda^l\omega = (\langle 1, \omega\rangle, \{\langle \frac{\partial \widetilde{\lambda}_{l,n}}{\partial q}, \omega\rangle\}_{n=1}^{\infty}).$$

Thus $\omega \in T_qM^l(p)$ if and only if $\langle 1, \omega\rangle = 0$ and $\langle \frac{\partial \widetilde{\lambda}_{l,n}}{\partial q}, \omega\rangle = 0$ for all n. Hence from the definition of P_lT_l and being $\{h_n\}$ is a basis of $T_qM^0(p)$ we have $\omega \in ker(P_lT_l)$ if and only if $\omega \in T_qM^0(p) \cap T_qM^l(p)$.

□

3.3 A Proof of the Main Theorem (Dimension of the Intersection of Two Tangent Spaces)

In this section, we will prove Theorem 3.1.3 which says, for any $p \in L^2_{\mathbb{R}}[0,1]$ and $l \geq 1$, the intersection of $T_qM^0(p)$ and $T_qM^l(p)$ is finite dimensional at each fixed $q \in M^0(p) \cap M^l(p)$.

From Lemma 3.2.2, the intersection of $T_qM^0(p)$ and $T_qM^l(p)$ is the kernel of P_lT_l, hence all what we need is to show that the kernel of P_lT_l has a finite dimension. Depending on Corollary A3.1, we can show that P_lT_l is a Fredholm operator by factoring P_lT_l to a sum of two operators, one of them is Fredholm and the second one is compact from $T_qM^0(p)$ to itself. Recall that

$$P_lT_l(\omega) = \sum_{n\geq 1} \langle T_l\omega, f_{l,n-1}\rangle h_n(x,q).$$

So to factor P_lT_l, we will start by some notes that will help to understand that factoring

1. Since $h_n(x,q) = \sqrt{2}\sin(2n\pi x) + O(\frac{1}{n})$, so we will consider the set $\{e_n\} = \{\sqrt{2}\sin(2n\pi x)\}$ which is an orthonormal basis for $\mathcal{O}$, the subspace of odd functions in $L^2_{\mathbb{R}}[0,1]$, because it is a set of the eigenfunctions associated to $-\dfrac{d^2}{dx^2}$ with Dirichlet boundary conditions. we will use $\|h_n - e_n\| = O(1/n)$.

2. Recall that, in case l is even we have $f_{l,n}(x,q) = \sqrt{2}\cos(2n\pi x) + O(1/n)$. Thus we will consider the set $\{F_n\} = \{\sqrt{2}\cos(2n\pi x)\}$ which is the orthonormal basis for $\mathcal{E}$, the even space of $L^2_{\mathbb{R}}[0,1]$, because it is the set of the eigenfunctions associated to $-\dfrac{d^2}{dx^2}$ with anti-periodic boundary conditions. Also we have $\|F_n - f_n\| = O(1/n)$.

3. In case l is odd, $f_{l,n}(x,q) = \sqrt{2}\cos((2n+1)\pi x) + O(1/n)$. Thus we will consider $\{F_n(x)\} = \{\cos((2n+1)\pi x)\}$ which is the orthonormal basis of the odd supspace $\mathcal{O}$ of $L^2_{\mathbb{R}}[0,1]$, because they are the eigenfunctions associated to $-\dfrac{d^2}{dx^2}$ with anti-periodic boundary conditions also we have $\|F_n - f_{l,n}\| = O(1/n)$.

4. For any $\omega \in T_qM^0(p)$, the basis of $T_qM^0(p)$ gives us a nice expression for w explained in the next lemma,

Lemma 3.3.1. *For any $\omega \in T_qM^0(p)$ there exists a sequence $\{b_n\} \in l^2$ such that $\omega = \sum_{n\geq 1} b_nh_n$ and $\|\omega\|_{L^2} \sim \|\{b_n\}\|_{l^2}$, which means there are two constants $C_1, C_2 > 0$ such that*

$$C_2\|\omega\|_{L^2} < \|\{b_n\}\|_{l^2} < C_1\|\omega\|_{L^2}.$$

Proof. Assume $\omega = \sum_{n\geq 1} b_nh_n$. From Theorem 1.2.3, for any $(n,m) \in \mathbb{N}^2$ we have $\langle a_n, \dfrac{d}{dx}g_m^2\rangle = \dfrac{1}{2}\delta_{m,n}$, where $a_n = \psi_n\phi_n$ is the product of the two linearly independent solutions (1.8) evaluated at the n^{th} eigenvalue. Taking the inner product of ω with

a_n gives $b_n = 2\langle \omega, a_n \rangle$. Thus

$$\begin{aligned} \|\{b_n\}\|_{l^2}^2 &= \sum_{n=1}^{\infty} |b_n|^2 \\ &= 4 \sum_{n=1}^{\infty} |\langle \omega, a_n \rangle|^2 \\ &\leq 4 \|\omega\|_{L^2}^2 \sum_{n=1}^{\infty} \|a_n\|^2 \\ &< \infty \end{aligned}$$

Because from the basic estimates of ψ and ϕ, there is some constant $C_1 > 0$ such that $\|a_n\| < \dfrac{C_1}{n}$ for each n, and that implies $\{b_n\} \in l^2$.

For the second claim, from the definition h_n, there is a constant $C_2 > 0$ such that $\|h_n\| < C_2$ for all n and that implies

$$\begin{aligned} \|\omega\|_{L^2}^2 &= \int_0^1 \Big| \sum_{n=1}^{\infty} b_n h_n(x) \Big|^2 dx \\ &\leq \sum_{n=1}^{\infty} |b_n|^2 \int_0^1 |h_n(x)|^2 dx \\ &\leq C_2 \|b_n\|_{l^2}^2 . \end{aligned}$$

Hence $\|\omega\|_{L^2} \sim \|\{b_n\}\|_{l^2}$.

$\square$

Now by using all these three notes we can factor $P_l T_l$ as following, take $\omega \in T_q M^0(p)$, then $\omega = \sum_{n \geq 1} a_n h_n$ for some $\{a_n\} \in l^2$, so we have

$$\begin{aligned} P_l T_l \omega &= \sum_{n,m=1}^{\infty} a_m \langle T_l h_m, f_{l,n-1} \rangle h_n \\ &= \sum_{n,m=1}^{\infty} a_m \Bigg(\langle T_l e_m, F_{n-1} \rangle e_n + \langle T_l (h_m - e_m), f_{l,n-1} \rangle h_n \\ &\quad + \langle T_l e_m, f_{l,n-1} - F_{n-1} \rangle h_n + \langle T_l e_m, F_{n-1} \rangle (h_n - e_n) \Bigg) \\ &= C + \sum_{i=1}^{3} I^i \end{aligned}$$

where

$$C = \sum_{n,m=1}^{\infty} a_m \langle T_l e_m, F_{n-1} \rangle e_n$$

$$I^1 = \sum_{n,m=1}^{\infty} a_m \langle T_l(h_m - e_m), f_{l,n-1}\rangle h_n$$

$$I^2 = \sum_{n,m=1}^{\infty} a_m \langle T_l e_m, f_{l,n-1} - F_{n-1}\rangle h_n$$

and

$$I^3 = \sum_{n,m=1}^{\infty} a_m \langle T_l e_m, F_{n-1}\rangle (h_n - e_n)$$

For the second claim above, we need to show that C is Fredholm and I^1, I^2 and I^3 are compact. At some point the proving that C is a Fredholm operator will depend on being l is odd or even, so we will start first proving the compactness of I^1, I^2 and I^3 and then we will prove that C is Fredholm.

Lemma 3.3.2. *I^1, I^2 and I^3 are uniform limits of compact operators, so by Lemma A3.5 they are compact.*

Proof. Before we prove the compactness of theses operator, recall that $\|h_m - e_m\|^2 = O(m^{-2})$, $\{F_n\}$ is an orthonormal basis for E, $\sum \|F_n - f_{l,n}\|^2 = O(n^{-2})$ and by Lemma 3.3.1 for each $\omega \in T_q M^0(p)$ there is $\{a_m\} \in l^2$ such that $\omega = \sum a_m h_m$ and $\|\omega\| \sim \|\{a_n\}\|_{l^2}$.

Now to prove I^1 is compact, consider the following sequence of operators

$$I^1_N(\omega) = \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_m \langle T_l(h_m - e_m), f_{l,n-1}\rangle h_n, \qquad N \geq 1$$

Each I^1_N is a bounded linear operator and has finite rank, so is compact. By using the Schwarz inequality we have,

$$\begin{aligned} \left\|I^1 - I^1_N(\omega)\right\|^2 &= \left\| \sum_{n>N} \sum_{m} a_m \langle T_l(h_m - e_m), f_{l,n-1}\rangle h_n \right\|^2 \\ &\leq C \sum_{n>N} \left| \sum_{m} a_m \langle T_l(h_m - e_m), f_{l,n-1}\rangle \right|^2 \\ &\leq C \|\omega\|^2 \sum_{n>N} \sum_{m} \langle T_l(h_m - e_m), f_{l,n-1}\rangle^2 \end{aligned}$$

Note that

$$
\begin{aligned}
\sum_{n>N}\sum_{m} \langle T_l(h_m-e_m), f_{l,n-1}\rangle^2 &\le 2\sum_{n}\sum_{m} \langle T_l(h_m-e_m), F_{n-1}\rangle^2 \\
&+ 2\sum_{n}\sum_{m} \langle T_l(h_m-e_m), f_{l,n-1}-F_{n-1}\rangle^2 \\
&\le 2\sum_{m} \|T_l(h_m-e_m)\|^2 \\
&+ 2\sum_{m} \|T_l(h_m-e_m)\|^2 \sum_{n} \|f_{l,n-1}-F_{n-1}\|^2 \\
&\le 2\,\|T_l\|^2 \big(\sum_{m} \|(h_m-e_m)\|^2 \\
&+ \sum_{m} \|(h_m-e_m)\|^2 \sum_{n} \|f_{l,n-1}-F_{n-1}\|^2 \big) \\
&< \infty.
\end{aligned}
$$

Hence $\|I^1 - I^1_N(\omega)\|^2 \to 0$ as N tends to infinity. Thus I^1 is the uniform limit of compact operators hence it is compact.

Now for I^2, for each $N \ge 1$ let

$$
I^2_N = \sum_{n=1}^{N}\sum_{m=1}^{\infty} a_m \langle T_l e_m, f_{l,n-1}-F_{n-1}\rangle h_n .
$$

Each I^2_N is bounded and has finite rank, so it is compact. Using the Schwarz inequality,

$$
\begin{aligned}
\left\|I^2 - I^2_N(\omega)\right\|^2 &= \left\|\sum_{n>N}\sum_{m=1}^{\infty} a_m \langle T_l e_m,\ f_{l,n-1}-F_{n-1}\rangle\ h_n\right\|^2 \\
&\le C\,\|\omega\|^2 \sum_{n>N} \Big|\langle T_l e_m,\ f_{l,n-1}-F_{n-1}\rangle\Big|^2 \\
&\le C\,\|\omega\|^2\,\|T_l\|^2 \sum_{n>N-1} \|f_{l,n}-F_n\|^2 \\
&\le C\,\|\omega\|^2\,\|T_l\|^2 \sum_{n>N-1} n^{-2}.
\end{aligned}
$$

So $\|I^2 - I^2_N(\omega)\|^2 \to 0$ as $N \to \infty$, hence I^2 is compact.
Finally, for I^3 we will consider the following sequence

$$
I^3_N = \sum_{n=1}^{N}\sum_{m=1}^{\infty} a_m \langle T_l e_m, F_{n-1}\rangle (h_n - e_n), \qquad N > 1.
$$

Each I_N^3 is bounded with finite rank, so it is compact. Using the Schwarz inequality and $\{F_n\}$ is orthonormal basis to get,

$$\begin{aligned}\left\|I^3 - I_N^3(\omega)\right\|^2 &= \left\|\sum_{n=1}^{N}\sum_{m=1}^{\infty} a_m \langle T_l e_m, F_{n-1}\rangle (h_n - e_n)\right\|^2 \\ &\leq C\,\|\omega\| \sum_{n>N} \left|\langle T_l e_m, F_{n-1}\rangle\right|^2 \sum_{n>N} \|h_n - e_n\|^2 \\ &\leq C\,\|\omega\|\,\|T_l\|^2 \sum_{n>N} n^{-2}.\end{aligned}$$

So $\|I^3 - I_N^3(\omega)\|^2 \to 0$ as $N \to \infty$, hence I^3 is compact.

□

Now to get more understanding of C we will write it as a composition of specific operators, some of these operator will be different in case l is even or odd. So the first case where l is even we consider the following operators,

1. The projection operator onto the even space $\mathcal{E}$,

$$\Pi : L^2_{\mathbb{R}}[0,1] \to \mathcal{E}, \qquad \Pi w = \sum_{n\geq 1} \langle w,\ F_n\rangle F_n$$

   where $F_n(x) = \sqrt{2}\cos(2n\pi x)$.

2. The operator $A : \mathcal{O} \to \mathcal{O}$,

$$Aw = \sum_{n\geq 1} \langle T_2 w,\ F_{n-1}\rangle e_n$$

3. The operator $S : T_q M^0(q) \to \mathcal{O}$

$$S\Big(\sum_{n\geq 1} a_n h_n\Big) = \sum_{n\geq 1} a_n e_n$$

$$S^{-1}\Big(\sum_{n\geq 1} a_n e_n\Big) = \sum_{n\geq 1} a_n h_n$$

4. The operator $H : \mathcal{E} \to \mathcal{O}$

$$H(\sum_{n\geq 1} a_n F_{n-1}) = \sum_{n\geq 1} a_n e_n$$

So A can be expressed as the composition: $A = H\Pi T_l$, and $C = AS$. In the next lemma we prove that C is Fredholm operator.

Lemma 3.3.3. *C is Fredholm operator from $T_q M^0(p)$ into $\mathcal{O}$.*

Proof. Since S is invertible, then by Lemma A3.4 we just need to show that A is Fredholm from $\mathcal{O}$ into $\mathcal{O}$. Identify $\mathcal{O}$ with $L^2_{\mathbb{R}}[0,1/2]$ by restriction. Taking $f \in L^2_{\mathbb{R}}[0,1/2]$,

$$\begin{aligned}
\Pi T_l f(x) &= 1/2(T_l f(x) + T_l f(1-x)) \\
&= \frac{1}{2}(T_l f(x) - f(1-x)) + \frac{1}{2}(T_l f(1-x) + f(1-x)) \\
&= T(f)(x) + K(f)(x)
\end{aligned}$$

where $Tf(x) = \frac{1}{2}(T_l f(x) - f(1-x))$ and $K(f)(x) = \frac{1}{2}(T_l f(1-x) + f(1-x))$. Note that for $f \in \mathcal{O}$ and $n \geq 1$ we have,

$$\begin{aligned}
\int_x^1 \frac{f(s)}{s^{2n}}\, ds &= \int_x^{1/2} \frac{f(s)}{s^{2n}}\, ds + \int_{1/2}^1 \frac{f(s)}{s^{2n}}\, ds \\
&= \int_x^{1/2} \frac{f(s)}{s^{2n}}\, ds - \int_{1/2}^1 \frac{f(1-s)}{s^{2n}}\, ds \\
&= \int_x^{1/2} \frac{f(s)}{s^{2n}}\, ds - \int_0^{1/2} \frac{f(s)}{(1-s)^{2n}}\, ds.
\end{aligned} \tag{3.2}$$

Also

$$\begin{aligned}
\int_{1-x}^1 \frac{f(s)}{s^{2n}}\, ds &= -\int_{1-x}^1 \frac{f(1-s)}{s^{2n}}\, ds \\
&= -\int_0^x \frac{f(s)}{(1-s)^{2n}}\, ds.
\end{aligned} \tag{3.3}$$

Therefore, for $0 \leq x \leq 1/2$ we have,

$$\begin{aligned}
Tf(x) &= \frac{1}{2}(T_l f(x) - f(1-x)) \\
&= 1/2\Big(a_2 x \int_x^{1/2} \frac{f(s)}{s^2}\, ds + a_3 x^3 \int_x^{1/2} \frac{f(s)}{s^4}\, ds + ... + a_{l+1} x^{2l-1} \int_x^{1/2} \frac{f(s)}{s^{2l}}\, ds \\
&- \big(a_2 x \int_0^{1/2} \frac{f(s)}{(1-s)^2}\, ds + a_3 x^3 \int_0^{1/2} \frac{f(s)}{(1-s)^4}\, ds + ... + a_{l+1} x^{2l-1} \int_0^{1/2} \frac{f(s)}{(1-s)^{2l}}\, ds\big)\Big) \\
&= \widetilde{T}(f)(x) + K_1(f)(x)
\end{aligned}$$

where

$$\widetilde{T}(f)(x) = 1/2\Big(a_2 x \int_x^{1/2} \frac{f(s)}{s^2}\, ds + a_3 x^3 \int_x^{1/2} \frac{f(s)}{s^4}\, ds + ... + a_{l+1} x^{2l-1} \int_x^{1/2} \frac{f(s)}{s^{2l}}\, ds\Big)$$

and

$$\begin{aligned}
K_1(f)(x) = -1/2\Big(a_2 x \int_0^{1/2} \frac{f(s)}{(1-s)^2}\, ds + a_3 x^3 \int_0^{1/2} \frac{f(s)}{(1-s)^4}\, ds \\
+ .... + a_{l+1} x^{2l-1} \int_0^{1/2} \frac{f(s)}{(1-s)^{2l}}\, ds\Big).
\end{aligned}$$

Also we have

$$
\begin{aligned}
K(f)(x) &= \frac{1}{2}(T_l f(1-x) + f(1-x)) \\
&= 1/2\big(a_2(1-x)\int_{1-x}^{1} \frac{f(s)}{s^2}\,ds + a_3(1-x)^3\int_{1-x}^{1} \frac{f(s)}{s^4}\,ds \\
&+ .... + a_{l+1}(1-x)^{2l-1}\int_{1-x}^{1} \frac{f(s)}{s^{2l}}\,ds\big) \\
&= -1/2\big(a_2(1-x)\int_{0}^{x} \frac{f(s)}{(1-s)^2}\,ds + a_3(1-x)^3\int_{0}^{x} \frac{f(s)}{(1-s)^4}\,ds \\
&+ .... + a_{l+1}(1-x)^{2l-1}\int_{0}^{x} \frac{f(s)}{(1-s)^{2l}}\,ds\big).
\end{aligned}
$$

Hence

$$\Pi T_2 f(x) = \widetilde{T}(f)(x) + K_1(f)(x) + K(f)(x)$$

Note that the integrals in the definition of K_1 and K have Hilbert Schmidt Kernels $\frac{xf(s)}{(1-s)^{2n}}$, and $\frac{(1-x)f(s)}{(1-s)^{2n}}$, for each $n = 1, 2, 3, ...$ on $[0, 1/2] \times [0, 1/2]$ respectively. Thus by Theorem A3.2, K_1 and K are compact operators from $L^2_{\mathbb{R}}[0, 1/2]$ to $L^2_{\mathbb{R}}[0, 1/2]$.
If $\widetilde{T}$ is Fredholm, then ΠT_l is Fredholm. So we need to show that $dim(ker\widetilde{T}) < \infty$, $dim(coker\widetilde{T}) < \infty$, and $Ran\widetilde{T}$ is closed.

1) $dim(ker\widetilde{T}) < \infty$:If $f \in ker\widetilde{T}$ then

$$0 = \Big(a_2 x\int_{x}^{1/2} \frac{f(s)}{s^2}\,ds + a_3 x^3\int_{x}^{1/2} \frac{f(s)}{s^4}\,ds + ... + a_{l+1}x^{2l-1}\int_{x}^{1/2} \frac{f(s)}{s^{2l}}\,ds\Big).$$

By differentiating this equation l times we find that f is a solution of the following system of Euler equation of degree $l-1$ with initial conditions,

$$
\begin{cases}
f^{(l-1)}(x) & + \frac{b_1}{x} f^{(l-2)}(x) + \frac{b_2}{x^2} f^{(l-3)}(x) + ...\frac{b_{l-1}}{x^{l-1}} f(x) = 0 \\
f^{(r)}(1/2) & = 0, \qquad r = 1, 2, ..., l-1 \\
f(1/2) & = 0
\end{cases}
$$

where $b_1, b_2, ..., b_{l-1} \in \mathbb{R}$. The kernel of $\widetilde{T}$ is generated by the solutions of this system. Since this system has a finite number of solutions then $dim(ker\widetilde{T}) < \infty$.

2) $dim(coker\widetilde{T}) < \infty$: Note the $coker\widetilde{T}$ is the quotient space $L^2_{\mathbb{R}}[0, 1/2]/\widetilde{T}(L^2_{\mathbb{R}}[0, 1/2]) = L^2_{\mathbb{R}}[0, 1/2]/(ker\widetilde{T}^*)^{\perp}$, hence to know the dimension of the $coker\widetilde{T}$ we need to know the kernel of $\widetilde{T}^*$.

$$\widetilde{T}^*(f)(x) = 1/2\Big(\frac{a_2}{x^2}\int_{0}^{x} f(s)s\,ds + \frac{a_3}{x^4}\int_{0}^{x} f(s)s^3\,ds + ... + \frac{a_{l+1}}{x^{2l}}\int_{x}^{1/2} f(s)s^{2l-1}\,ds\Big).$$

If $f \in ker\widetilde{T}^*$ then

$$0 = \frac{a_2}{x^2}\int_0^x f(s)s\ ds + \frac{a_3}{x^4}\int_0^x f(s)s^3\ ds + ... + \frac{a_{l+1}}{x^{2l}}\int_x^{1/2} f(s)s^{2l-1}\ ds$$

Differentiating this equation l times we get that f is a solution of the following system

$$\begin{cases} f^{(l-1)}(x) & +\frac{c_1}{x}f^{(l-2)}(x) + \frac{c_2}{x^2}f^{(l-3)}(x) + ...\frac{c_{l-1}}{x^{l-1}}f(x) = 0 \\ f^{(r)}(1/2) & = 0, \qquad r = 1, 2, ..., l-1 \\ f(1/2) & = 0 \end{cases}$$

where $c_1, c_2, ..., c_{l-1} \in \mathbb{R}$. Similarly this system has a finite number of solutions which generates the kernel of $\widetilde{T}^*$. Hence $dim(ker\widetilde{T}^*) < \infty$.

3) $ran\widetilde{T} = (ker\widetilde{T}^*)^{\perp}$ which is a closed subspace of $L^2_{\mathbb{R}}[0, 1/2]$.

Hence $\widetilde{T}$ is Fredholm operator on $L^2_{\mathbb{R}}[0, 1/2]$, therefore ΠT is the sum of a Fredholm operator $\widetilde{T}$ and two compact operators K_1 and K, so it is Fredholm operator from $\mathcal{O}$ to $\mathcal{O}$. Since $A = H\Pi T$ and $C = AS$ this prove the lemma.

□

Note that C is a Fredholm operator from $T_qM^0(p)$ into $\mathcal{O}$, but what we want to get is P_lT_l as a sum of Fredholm and compact operators from T_qM^0 into itself. For that reason we will factor C as following, for $\omega \in T_qM^0(p)$ write

$$\begin{aligned} C\omega &= \sum_{m,n} a_m\langle T_le_m, F_{n-1}\rangle h_n - \sum_{m,n} a_m\langle T_le_m, F_{n-1}\rangle(h_n - e_n) \\ &= (S^{-1}C + K_2)\omega. \end{aligned}$$

Since S^{-1} is invertible from $\mathcal{O}$ into T_qM^0, then by Lemma A3.3, $S^{-1}C$ is Fredholm operator from T_qM^0 into itself. Also note that $K_2 = P_lT_l - S^{-1}C$ is a sum of two operator from $T_qM^0(p)$ into itself and it is compact. Therefore P_lT_l is Fredholm operator from $T_qM^0(p)$ into $T_qM^0(p)$.

Now for the case where l is odd, the proof is almost similar to the even case. The difference will be in the operators that are used to factor C. For showing the operator C is Fredholm, we will consider the operators A and S are the same as in the even case, and we will consider the following two operators

1. The projection operator onto the odd space $\mathcal{O}$

$$\Pi : L^2_{\mathbb{R}}[0, 1] \to \mathcal{O}, \qquad \Pi w = \sum_{n\geq 1}\langle w,\ F_n\rangle F_n$$

   where $F_n = \cos((2n + 1)\pi x)$.

2. The operator

$$H : \mathcal{O} \to \mathcal{O}$$
$$H\Big(\sum_{n\geq 1} a_n F_{n-1}\Big) = \sum_{n\geq 1} a_n e_n.$$

So we will get $C = AS = H\Pi T_l S$. Since S and H are invertible. By Lemma A3.3 and Lemma A3.4 it is enough to show that ΠT_l is Fredholm to get C Fredholm from $\mathcal{O}$ to $\mathcal{O}$. Taking $f \in L^2_{\mathbb{R}}[0, 1/2]$,

$$\begin{aligned}
\Pi T_l f(x) &= 1/2(T_l f(x) - T_l f(1-x)) \\
&= \frac{1}{2}(T_l f(x) + f(1-x)) - \frac{1}{2}(T_l f(1-x) - f(1-x)) \\
&= T(f)(x) + K(f)(x)
\end{aligned}$$

where $Tf(x) = \frac{1}{2}(T_l f(x) - f(1-x))$ and $K(f)(x) = \frac{1}{2}(T_l f(1-x) + f(1-x))$. By using equations (3.2) and (3.3) we have

$$\Pi T_l f(x) = \widetilde{T}(f)(x) + K_1(f)(x) + K(f)(x)$$

where

$$\begin{aligned}
K_1(f)(x) = -1/2\Big(&a_2 x \int_0^{1/2} \frac{f(s)}{(1-s)^2}\, ds + a_3 x^3 \int_0^{1/2} \frac{f(s)}{(1-s)^4}\, ds \\
&+ \ldots\ldots + a_{l+1} x^{2l-1} \int_0^{1/2} \frac{f(s)}{(1-s)^{2l}}\, ds\Big)
\end{aligned}$$

and

$$\begin{aligned}
K(f)(x) = -\frac{1}{2}\Big(&a_2(1-x) \int_0^{x} \frac{f(s)}{(1-s)^2}\, ds + a_3(1-x)^3 \int_0^{x} \frac{f(s)}{(1-s)^4}\, ds \\
&+ \ldots\ldots + a_{l+1}(1-x)^{2l-1} \int_0^{x} \frac{f(s)}{(1-s)^{2l}}\, ds\Big).
\end{aligned}$$

K_1 and K are compact since they are integral operators with Hilbert-Schmidt kernels. Also, the operator

$$\widetilde{T}(f)(x) = 1/2\Big(2f(x) + a_2 x \int_x^{1/2} \frac{f(s)}{s^2}\, ds + a_3 x^3 \int_x^{1/2} \frac{f(s)}{s^4}\, ds + \ldots\ldots + a_{l+1} x^{2l-1} \int_x^{1/2} \frac{f(s)}{s^{2l}}\, ds\Big)$$

is Fredholm because

1) $dim(ker\widetilde{T}) < \infty$: If $f \in ker\widetilde{T}$ then

$$f(x) = \Big(a_2 x \int_x^{1/2} \frac{f(s)}{s^2}\, ds + a_3 x^3 \int_x^{1/2} \frac{f(s)}{s^4}\, ds + \ldots\ldots + a_{l+1} x^{2l-1} \int_x^{1/2} \frac{f(s)}{s^{2l}}\, ds\Big)$$

By differentiating this equation l times we find that f is a solution of the following system of Euler equation of degree $l-1$ with initial conditions,

$$\begin{cases} f^{(l)}(x) & +\frac{b_1}{x} f^{(l-1)}(x) + \frac{b_2}{x^2} f^{(l-2)}(x) + ...\frac{b_{l-1}}{x^{l-1}} f(x) = 0 \\ f^{(r)}(1/2) & = 0, \qquad r = 1, 2, ..., l \\ f(1/2) & = 0 \end{cases}$$

where $b_1, b_2, ..., b_{l-1} \in \mathbb{R}$. Since this system has a finite number of solutions that generate the kernel of $\widetilde{T}$, then $dim(ker\widetilde{T}) < \infty$.

2) $dim(coker\widetilde{T}) < \infty :$ Note the $coker\widetilde{T}$ is the quotient space $L^2_{\mathbb{R}}[0, 1/2]/\widetilde{T}(L^2_{\mathbb{R}}[0, 1/2]) = L^2_{\mathbb{R}}[0, 1/2]/(ker\widetilde{T}^*)^{\perp}$, hence to know the dimension of the $coker\widetilde{T}$ we need to know the kernel of $\widetilde{T}^*$.

$$\widetilde{T}^*(f)(x) = 1/2\Big(\frac{a_2}{x^2}\int_0^x f(s)s\ ds + \frac{a_3}{x^4}\int_0^x f(s)s^3\ ds + ... + \frac{a_{l+1}}{x^{2l}}\int_x^{1/2} f(s)s^{2l-1}\ ds\Big)$$

If $f \in ker\widetilde{T}^*$ then

$$f(x) = \frac{a_2}{x^2}\int_0^x f(s)s\ ds + \frac{a_3}{x^4}\int_0^x f(s)s^3\ ds + ... + \frac{a_{l+1}}{x^{2l}}\int_x^{1/2} f(s)s^{2l-1}\ ds.$$

Differentiating this equation l times we get that f is a solution of the following system

$$\begin{cases} f^{(l-1)}(x) & +\frac{c_1}{x} f^{(l-2)}(x) + \frac{c_2}{x^2} f^{(l-3)}(x) + ...\frac{c_{l-1}}{x^{l-1}} f(x) = 0 \\ f^{(r)}(1/2) & = 0, \qquad r = 1, 2, ..., l-1 \\ f(1/2) & = 0 \end{cases}$$

where $c_1, c_2, ..., c_{l-1} \in \mathbb{R}$. Similarly this system has a finite number of solutions, then $dim(ker\widetilde{T}^*) < \infty$.

3) $ran\widetilde{T} = (ker\widetilde{T}^*)^{\perp}$ which is a closed subspace of $L^2[0, 1/2]$.

Thus ΠT is a Fredholm operator and as a consequence C is Fredholm. Ending of the proof is exactly as the even case.

□

In the next Chapter we will try to explain how the result of Theorem 3.1.3 could lead to the uniqueness of the radial potential by considering a finite number of angular momenta.

Chapter 4 Some Geometric Properties of the Isospectral Set

4.1 Introduction

By Theorem 3.1.2, we know that for $p \in L^2_{\mathbb{R}}[0,1]$ and each $l \geq 1$, $dim(M^l(p))$ is infinite. Also at any point $q \in M^l(p)$ we have that $dim(T_qM^l(p))$ is infinite. These fact imply that the set of eigenvalues corresponding to a single angular momentum is not enough date to show the uniqueness. In Theorem 1.1.6, Carlson and Shubin showed that $dim(T_pM^{l_1}(p) \cap T_pM^{l_2}(p)) < \infty$ for any angular momunta l_1, l_2 such that $l_1 + l_2 = 1 \mod 2$. Also in the previous chapter, we proved Theorem 3.1.3 which says that for $p \in L^2_{\mathbb{R}}[0,1]$ and each $l \geq 1$, the dimension of $T_qM^0(p) \cap T_qM^l(p)$ is finite at each $q \in M^0(p) \cap M^l(p)$. In this chapter we will try to figure out how the finite dimension of the intersection of tangent spaces leads to the uniqueness of the potential. We will define the sequence of angular momenta that q shares their eigenvalues with p by

$$l_1(q) = 0 < l_2(q) < ...... < l_i(q) < .....$$

The reason for being $l_1(q)$ is the zero angular momentum to guarantee that

$$dim\Big(\bigcap_{i=1}^{k} T_qM^{l_i}(p)\Big) < \infty \qquad \text{for any } k > 1$$

where $l_i(q)$ is the i^{th} angular momentum. Also we will define a sequence of the dimensions of the finite intersection of the tangent spaces as following

$$d_2(q), d_3(q), d_4(q), ..., d_k(q), ...$$

where

$$d_k(q) = dim\Big(\bigcap_{i=1}^{k} T_qM^{l_i}(p)\Big).$$

Since $\bigcap_{i=1}^{k+1} T_qM^{l_i}(p) \subset \bigcap_{i=1}^{k} T_qM^{l_i}(p)$, then $\{d_k(q)\}_{k\geq 1}$ is a decreasing sequence of natural numbers that is bounded below by zero. Hence this sequence converges to some constant $c \geq 0$. The interesting question here, if we assume that $c = 0$, that is there is some k_0 such that $d_k(q) = 0$ for all $k > k_0$, then can we prove the uniqueness by using the eigenvalues that corresponds to $l_1(q), l_2(q), ..., l_{k_0}(q)$?

Before we start discuss this problem, we will give a brief overview for some properties of the isospectral set. Some of these properties were proven for each $l \geq 0$ and some of them were proven for $l = 0$. In the next section we will prove them for $l > 0$.

4.2 Overview of Some Important Properties of the Isospectral Set of Potential $p \in L^2_{\mathbb{R}}[0,1]$

The first property is illustrated in Theorem 3.1.2. It was proven by Pöschel, and Trubowitz in [1] for $l = 0$ and by Serier in [3] for $l \geq 1$. It depends on analyzing the following map

$$\lambda^l \times \kappa^l : L^2_{\mathbb{R}}[0,1] \to \mathbb{R} \times l^2 \times l^2_1$$

$$\lambda^l \times \kappa^l(q) = \Big([q], \{\widetilde{\lambda}_{l,n}(q)\}_{n\geq 1}, \{\kappa_{l,n}(q)\}_{n\geq 1}\Big)$$

where $\kappa_{l,n}$ is defined by the terminal velocity of each eigenfunction corresponding to angular momentum l

$$\kappa_{l,n}(q) = ln\left|\frac{\psi'_{l,n}(1,q)}{u'(0,q)}\right|$$

and $\widetilde{\lambda}_{l,n}$ is the last term in the asymptotic of the n^{th} eigenvalue (1.14).

Theorem 4.2.1. *For any angular momentum l, κ^l is a real analytic map on $L^2_{\mathbb{R}}[0,1]$. It is derivative is given by*

$$d_q\kappa^l(\nu) = \big(\{\langle\frac{\partial \kappa_{l,n}}{\partial q}, \nu\rangle\}_{n\geq 1}\big)$$

where

$$\frac{\partial \kappa_{l,n}}{\partial q} = -\psi_{l,n}(t,q)\phi_{l,n}(t,q) + \frac{\partial \lambda_{l,n}}{\partial q}\int_0^1 \psi_{l,n}(s,q)\phi_{l,n}(s,q)$$

and has the following estimate

$$\frac{\partial \kappa_{l,n}}{\partial q}(t) = \frac{1}{\omega_{l,n}} j_l(\omega_{l,n}t)\eta_{l,n}(\omega_{l,n}t) + O(\frac{1}{n^2}).$$

From Theorem 3.1.1 and 4.2.1 we have,

Theorem 4.2.2. *$\lambda^l \times \kappa^l$ is a real analytic map on $L^2_{\mathbb{R}}(0,1)$. Its derivative is given by*

$$d_q(\lambda^l \times \kappa^l)(v) = \Big(\langle 1, v\rangle, \big\{\langle\frac{\partial \widetilde{\lambda}_{l,n}}{\partial q}, v\rangle\big\}_{n\geq 1}, \big\{\langle\frac{\partial \kappa_{l,n}}{\partial q}, v\rangle\big\}_{n\geq 1}\Big).$$

Before describing $M^l(p)$ in details, we will set some notations that will be used later. For any angular momentum l, set

$$U_{l,0} = 1$$

$$U_{l,n} = g^2_{l,n} - 1 \qquad n \geq 1,$$

and

$$V_{l,n} = 2\frac{d}{dx}g^2_{l,n} \qquad n \geq 1,$$

where $g_{l,n}$ is the n^{th} normalized eigenfunction. These vectors depend on x and q. Also for any $\eta \in \mathbb{R} \times l^2$ and $\zeta \in l^2_2$ we will write,

$$U_{l,\eta} = \sum_{n\geq 0} \eta_n U_{l,n}$$

and

$$V_{l,\zeta} = \sum_{n \geq 0} \zeta_n V_{l,n}$$

By Theorem 1.2.3, $\left\{U_{l,\eta} : \eta \in \mathbb{R} \times l^2\right\}$ and $\left\{V_{l,\zeta} : \zeta \in l_2^2\right\}$ are perpendicular. By analyzing λ^l, Theorem 3.1.2 illustrates that $M^l(p)$ is a real analytic submanifold of $L^2_{\mathbb{R}}[0,1]$ of infinite dimension and co-dimension lying in the hyperplane of all functions with mean value $\displaystyle\int_0^1 p(t)dt$ with the tangent space given by

$$T_q M^l(p) = span\{V_{l,\zeta} : \zeta \in l_2^2\}$$

and the normal space given by

$$N_q M^l(p) = span\{U_{l,\eta} : \eta \in \mathbb{R} \times l^2\}.$$

Theorem 4.2.3. *For any angular momentum l and $p \in L^2_{\mathbb{R}}[0,1]$, κ^l is a global coordinate system on $M(p)$. Its derivative $d_q\kappa^l$ is an isomorphism between $T_q M^l(p)$ and $T_{\kappa(q)} l_1^2 \simeq l_1^2$, which is given by*

$$d_q\kappa^l(V_{l,\zeta}) = \zeta.$$

The value of $V_{l,\zeta}$ by $d_q\kappa^l$ followes from Theorem 1.2.3. Notice that every tangent vector at $q \in M^l(p)$ is of the form

$$V_{l,\zeta} = \sum_{n \geq 0} \zeta_n V_{l,n}$$

with uniquely determined coefficients $\zeta = (\zeta_1, \zeta_2, ...) \in l_1^2$. Since $V_{l,n}$ depend on q then these coefficients uniquely determine a tangent vector at every point in $M^l(p)$ by the same expression. Hence every tangent vector $V_{l,\zeta}$ at a given point determines a vector field on $M(p)$, which we denote by the same symbol. By these vector fields we will define the exponential map exp_q for any $q \in M(p)$. But first we will introduce the solutions curve.

Definition 4.2.1. For any $q \in M^l(p)$ and vector field $V_{l,\zeta}$, a curve

$$\Phi_l^t(q) = \Phi_l^t(q, V_{l,\zeta}), \qquad a < t < b$$

on $M^l(p)$ is called a solution curve of the vector field $V_{l,\zeta}$ with initial value q, if

$$\frac{d}{dt}\Phi_l^t(q) = V_{l,\zeta}(\Phi_l^t(q)) \qquad a < t < b$$

and $\Phi_l^0(q) = q$.

From Theorem 4.2.3, the vector field $V_{l,\zeta}$ is the constant vector field ζ in the κ^l coordinate system on $M^l(p)$. Hence, any solution curve in this coordinate system is a straight line, so we have

$$\kappa^l(\Phi_l^t(q, V_{l,\zeta})) = C + t\zeta.$$

Since $\Phi_l^0(q) = q$, then

$$\kappa^l(\Phi_l^t(q, V_{l,\zeta})) = \kappa^l(q) + t\zeta.$$

From this equation we can get an expression for $\Phi_l^t(q, V_{l,\zeta})$ by taking $(k^l)^{-1}$ for both sides but that can be done if we know that $\kappa^l(q) + t\zeta$ is inside the open set $\kappa^l(M^l(p))$. Thus in the next steps we will see that $\Phi_l^t(q, V_{l,\zeta})$ is defined for all time t and that implies unbounded property of $M^l(p)$.

Lemma 4.2.1. *For $n \geq 1$ and angular momentum l,*

$$\langle q, V_{l,n} \rangle = 4\delta_{l,n}(q) sinh(\kappa_{l,n}(q)).$$

where

$$\delta_{l,n}(q) = \frac{(-1)^n}{\dot{\psi}_{l,n}(1, \lambda_{l,n})}$$

Proof. Using the differential equation (1.8) and the boundary conditions we get,

$$\begin{aligned}
\langle q, \frac{d}{dx} g_{l,n}^2 \rangle &= \int_0^1 2q g_{l,n} g_{l,n}' dx \\
&= \int_0^1 2(g_{l,n}'' + \lambda_{l,n} g_{l,n}) g_{l,n}' dx \\
&= \int_0^1 \frac{d}{dx}((g_{l,n}')^2 + \lambda_{l,n} g_{l,n}^2) dx \\
&= (g_{l,n}')^2 \Big|_0^1 \\
&= \frac{(\psi'(x, \lambda_{l,n}, q))^2}{(\dot{\psi}(1, \lambda_{l,n}, q))(\psi'(1, \lambda_{l,n}, q))} \Big|_0^1 \\
&= \frac{1}{\dot{\psi}(1, \lambda_{l,n}, q)} \Big(\psi'(1, \lambda_{l,n}, q) - \frac{1}{\psi'(1, \lambda_{l,n}, q)} \Big)
\end{aligned}$$

Hence, by using the definition of $\kappa_{l,n}$ and the identity $2sinh(x) = e^x - e^{-x}$ we get,

$$\begin{aligned}
\langle q, V_{l,n} \rangle &= 2\langle q, \frac{d}{dx} g_{l,n}^2 \rangle \\
&= \frac{2(-1)^n}{\dot{\psi}(1, \lambda_{l,n}, q)} (e^{\kappa_{l,n}} - e^{-\kappa_{l,n}}) \\
&= 4\delta_{l,n}(q) \sinh(\kappa_{l,n}(q)).
\end{aligned}$$

□

Lemma 4.2.2.

$$\left\| \Phi_l^t(q, V_{l,\zeta}) \right\|^2 = \|q\|^2 + 8 \sum_{n \geq 1} \int_0^t \zeta_n \delta_{l,n}(\Phi_l^t(q)) sinh(\kappa_{l,n}(q) + t\zeta_n).$$

Proof. Set $\Phi_l^t(q, V_{l,\zeta}) = \Phi_l^t(q)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left\|\Phi_l^s(q)\right\|^2 &= \frac{1}{2}\frac{d}{ds}\int_0^1 (\Phi_l^s(q))^2(x)dx \\ &= \int_0^1 \Phi_l^s(q)(x)\frac{d}{ds}\Phi_l^s(q)(x)dx\end{aligned}$$

At $s = 0$ we have $\Phi_l^0(q) = q$ and $\frac{d}{ds}\Phi_l^s(q)(x)\big|_{s=0} = V_{l,\zeta}(q) = \sum_{n\geq 1}\zeta_n V_{l,n}(q)$. Hence by Lemma 4.2.1, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left\|\Phi_l^s(q)\right\|^2 &= \sum_{n\geq 1}\zeta_n\int_0^1 q(x)V_{l,n}(q,x)dx \\ &= \sum_{n\geq 1}4\zeta_n\delta_{l,n}(q)\sinh(\kappa_{l,n}(q))\end{aligned}$$

For the derivative at time $t \neq 0$, replace q in the last expression by $\Phi^t(q)$ and use the fact that $\Phi_l^{s+t}(q) = \Phi_l^s(\Phi_l^t(q))$.Thus we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left\|\Phi_l^t(q)\right\|^2 &= \frac{1}{2}\frac{d}{ds}\left\|\Phi_l^s(\Phi_l^t(q))\right\|^2\Bigg|_{s=0} \\ &= \sum_{n\geq 1}4\zeta_n\delta_{l,n}(\Phi_l^t(q))\sinh(\kappa_{l,n}(\Phi_l^t(q))) \\ &= \sum_{n\geq 1}4\zeta_n\delta_{l,n}(\Phi_l^t(q))\sinh(\kappa_{l,n}(q)+t\zeta_n)\end{aligned}$$

The right hand side converges uniformly on bounded intervals of time because

$$\begin{aligned}\zeta_n\delta_{l,n}(\Phi_l^t(q))\sinh(\kappa_{l,n}(q)+t\zeta_n) &= O(\zeta_n\delta_{l,n}(\Phi_l^t(q))(\kappa_{l,n}(q)+t\zeta_n)) \\ &= O(\delta_{l,n}(\Phi_l^t(q))(\zeta_n\kappa_{l,n}(q)+t\zeta_n^2))\end{aligned}$$

for bounded t and $\zeta \in l_1^2$. Thus we can integrate under the summation sign to get,

$$\begin{aligned}\left\|\Phi_l^t(q, V_{l,\zeta})\right\|^2 - \|q\|^2 &= \int_0^t \frac{d}{ds}\left\|\Phi_l^s(q)\right\|^2 ds \\ &= 8\sum_{n\geq 1}\int_0^t \zeta_n\delta_{l,n}(\Phi_l^s(q))\sinh(\kappa_{l,n}(q)+s\zeta_n)ds.\end{aligned}$$

Recall that $\delta_{l,n}(q) = \dfrac{(-1)^n}{\dot{\psi}(1,\lambda_{l,n},q)}$. In case $l = 0$, by Theorem 1.2.6 we have

$$\dot{\psi}(1,\lambda_{0,n},q) = \frac{-1}{(n\pi)^2}\prod_{m\neq n}\frac{\lambda_{0,m}-\lambda_{0,n}}{(m\pi)^2}.$$

Notice that $\dot{\psi}$ is the same function for all q in the same isospectral set. Hence $\delta_{0,n}(q) = \delta_{0,n}(\Phi^t(q))$ since q and $\Phi_l^t(q)$ belongs to the same isospectral set. Also $\delta_{0,n}(q) = \delta_{0,n}$ are constant for each n. Thus

$$\begin{aligned}
\left\|\Phi_0^t(q, V_{0,\zeta})\right\|^2 - \|q\|^2 &= \int_0^t \frac{d}{ds} \|\Phi_0^s(q)\|^2 \, ds \\
&= 8 \sum_{n \geq 1} \delta_{0,n} \int_0^t \zeta_n \sinh(\kappa_{0,n}(q) + s\zeta_n) ds \\
&= 8 \sum_{n \geq 1} \delta_{0,n} (cosh(\kappa_{0,n}(q) + t\zeta_n) - cosh(\kappa_{0,n}(q))).
\end{aligned}$$

$\square$

Theorem 4.2.4. *For every $q \in M^l(p)$ and every $\zeta \in l_1^2$, the solution curve $\Phi_l^t(q, V_{l,\zeta})$ exists for all time.*

Proof. Fix q in $M^l(p)$ and $\zeta \in l_1^2$.To prove the this theorem need to show that the stright line $\kappa^l(q) + t\zeta$ is in the open set $\kappa^l(M^l(p))$ for all t. Assume that there is some t so that $\kappa^l(q) + t\zeta$ is not in $\kappa^l(M^l(p))$, hence there is a t^* such that $k^l(q) + t^*\zeta$ is on the boundary of $\kappa^l(M^l(p))$ while the segment $\kappa^l(q) + t\zeta$, $0, t, t^*$ is in its interior. By Lemma 4.2.2

$$\sup_{0<t<t^*} \left\|\Phi_l^t(q)\right\| < \infty.$$

Thus we can choose a sequence t_n converging to t^* from below such that Φ^{t_n} converges weakly to some point q_* in $M^l(p)$ with

$$\kappa(q_*) = \kappa^l(q) + t^*\zeta.$$

Since $\kappa_{l,n}$ is compact functions on $L^2_{\mathbb{R}}[0,1]$. That implies $\kappa^l(q) + t^*\zeta$ lies in the interior of $\kappa^l(M)$. That is a contradiction. $\square$

Now we can define the exponential map on $T_qM^l(p)$ by the solution curve with initial value q as following,

Definition 4.2.2. For any $q \in M^l(p)$, the exponential map at q

$$exp_q^l : T_qM^l(p) \to M(p)$$

is defined by

$$exp_q^l(V_\zeta) = \Phi_l^1(q, V_\zeta)$$

Theorem 4.2.5. *For all $q \in M^l(p)$, the exponential map exp_q^l is a real analytic isomorphism between $T_qM^l(p) \simeq l_1^2$ and $M^l(p)$. It satisfies*

$$\kappa^l(exp_q(V_\zeta)) = \kappa(q) + \zeta.$$

Hence $M^l(p)$ is connected and simply connected.

Proof. To prove that the exponential map is an isomorphism map, we will define the translated map

$$(exp_q^l)^{-1} : M^l(p) \to T_q M^l(p)$$
$$(exp_q^l)^{-1}(\tilde{q}) = \kappa^l(\tilde{q}) - \kappa^l(q)$$

which is a real analytic isomorphism between $M^l(p)$ and l_1^2 by Theorem 4.2.3. For one direction we have,

$$\begin{aligned}(exp_q^l)^{-1}(exp_q^l)(V_\zeta) &= (exp_q^l)^{-1}(\Phi^1(q, V_\zeta)) \\ &= \kappa^l(\Phi^1(q, V_\zeta) - \kappa^l(q) \\ &= \kappa^l(q) + \zeta - \kappa^l(q) = \zeta.\end{aligned}$$

For the second direction,

$$\begin{aligned}exp_q^l((exp_q^l)^{-1})(\tilde{q}) &= exp_q^l(\kappa^l(\tilde{q}) - \kappa^l(q)) \\ &= exp_q^l\big(\sum_{n\geq 1} \tilde{\eta}_n V_{l,n}(q) - \sum_{n\geq 1} \eta_n V_{l,n}(q)\big) \\ &= exp_q^l(V_{\tilde{\eta}-\eta}) \\ &= \Phi^1(q, V_{\tilde{\eta}-\eta}).\end{aligned}$$

Notice that,

$$\begin{aligned}\kappa^l(\Phi^1(q, V_{\tilde{\eta}-\eta})) &= \kappa^l(q) + \tilde{\eta} - \eta \\ &= \kappa^l(q) + \kappa^l(r) - \kappa^l(q) \\ &= \kappa^l(r)\end{aligned}$$

which implies $\Phi^1(q, V_{\tilde{\eta}-\eta}) = r$ by Theorem 4.2.3. The identity

$$\kappa^l(exp_q(V_\zeta)) = \kappa(q) + \zeta$$

follows from the definition of $\exp_q^l$ and the value of $\kappa^l(\Phi_l^1(q, V_\zeta))$. □

4.3 Uniqueness of the Potential by Using a Finite Number of Angular Momentum

In this section we will try to prove the uniqueness of the potential by using the geometric properties of the isospectral sets and the finite dimension of the intersection of the tangent spaces. From Theorem 4.2.5 we have that $M^l(p)$ is simply connected for each l which implies that $M^l(p)$ is path connected. In case we have that a finite intersection of isospectral sets $\cap_{i=1}^k M^{n_i}(p)$ are path connected, then with assuming some conditions on the dimension of the intersection of the tangent spaces at each point $q \in \cap_{i=1}^k M^{n_i}(p)$ we get the following result.

Theorem 4.3.1. *let* $p \in L^2_{\mathbb{R}}[0,1]$ *and a finite sequence of angular momentum*

$$l_1 = 0 < l_2 < l_3 < \dots . . < l_k$$

for some $k \in \mathbf{N}$ *such that*

- $\cap_{i=1}^{k} M^{l_i}(p)$ *is simply connected*
- *dim* $\cap_{i=1}^{k} T_q M^{l_i}(p) = 0$ *for all* $q \in \cap_{i=1}^{k} M^{l_i}(p)$

Then $\cap_{i=1}^{k} M^{l_i}(p) = \{p\}$.

Proof. Let $q \in \cap_{i=1}^{k} M^{l_i}(p)$. Since $\cap_{i=1}^{k} M^{l_i}(p)$ is simply connected then there is a path

$$\delta : [a, b] \to \cap_{i=1}^{k} M^{l_i}(p)$$

such that $\delta(a) = q$ and $\delta(b) = p$. Note that $\delta([a, b]) \subset M^{l_i}(p)$ for all $i = 1, 2, .., k$. Thus $\dfrac{d\delta}{dt}(t_o) = v_{t_0} \in \cap_{i=1}^{k} T_{\delta(t_0)} M^{l_i}(p)$ for all $t_0 \in [a, b]$. Since $dim(\cap_{i=1}^{k} T_{\delta(t_0)} M^{l_i}) = 0$ for all $\delta(t_0) \in \cap_{i=1}^{k} M^{l_i}(p)$ then $v_{t_0} = 0$ for all t_0. Hence δ is a constant path which means $q = \delta(a) = \delta(b) = p$.

□

In Theorem 4.3.1 we set some assumptions on the intersections of the isospectral sets and the dimension of the intersection of the tangent spaces to the isospectral sets at each points to prove the uniqueness. If we can prove these assumption then the theorem will be more stronger and include all applications. Hence in the next chapter we will write these assumptions as open problems. Also we will list all others conjectures for the inverse eigenvalue problem for the Schrödinger equation on the unit ball of $\mathbb{R}^3$ that were raised when we worked in this research. Some of these conjectures we worked on but we could not get results.

Chapter 5 Open Problems

Starting with Pöschel and Trubowitz result, in case the angular momentum $l = 0$, they proved the uniqueness of two even potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ that share the same eigenvalues. The proof of this result depends on the fact that $\psi^q_{0,n}(x) = (-1)^{n+1}\psi^q_{0,n}(1-x)$ for any even potential q. That is, $\psi^q_{0,n}(1-x)$ is also an eigenfunction for q, and satisfies

$$-\frac{d^2}{dx^2}\psi^q_{0,n}(1-x) + q(x)\psi^q_{0,n}(1-x) = \lambda\psi^q_{0,n}(1-x), \qquad x \in (0,1).$$

In case $l > 0$, this fact can not be proved because the last part of the Sturm-Lioville operator

$$H^l_q = -\frac{d^2}{dx^2} + q + \frac{l(l+1)}{x^2}$$

which is $\frac{l(l+1)}{x^2}$ does not have the symmetric property. In the second chapter of this dissertation, we use the Pöschel and Trubowitz method to prove Theorem 2.4.3 which says that in case $l = 0$, if two potentials $p, q \in L^2_{\mathbb{R}}[0,1]$ have even extended potentials $\widetilde{p}, \widetilde{q} \in L^2_{\mathbb{R}}[-1,1]$ sharing the same eigenvalues then $p = q$. In the extended problem, the even potentials are with respect to 0, i.e. $\widetilde{q}(x) = \widetilde{q}(-x)$. One of our attempt is trying to generalize the result of Theorem 2.4.3 for any angular momentum $l > 0$. Because by the extended domain, the last part of H^l_q satisfies $\frac{l(l+1)}{x^2} = \frac{l(l+1)}{(-x)^2}$. From this point we conjectured that for $l > 0$, the function $\widetilde{\psi}^q_{l,n}(-x)$ may be a solution of the extended problem. But the main problem we faced that the extended eigenfunctions are not defined at $x = 0$. Because the singular part $\frac{l(l+1)}{x^2}$. It may there is another way to define the extended eigenfunctions to get the desired result, so the next open problem is

Problem: For $l > 0$ and $p, q \in L^2_{\mathbb{R}}[0,1]$, let $\widetilde{p}$ and $\widetilde{q}$ in $L^2_{\mathbb{R}}[-1,1]$ be the even extended potentials of p and q respectively. Can we prove that $p = q$ by assuming that $\widetilde{p}$ and $\widetilde{q}$ share the same eigenvalues of the extended problem?

In chapter four, the main result, Theorem 3.1.3. It says that for any $l > 0$ and $p \in L^2_{\mathbb{R}}[0,1]$, the $dim(T_qM^0(p) \cap T_qM^l(p)) < \infty$ at each fixed $q \in M^0(p) \cap M^l(p)$. In this theorem we fixed the first angular momentum to be the zero angular momentum and we vary the second one $l > 0$. The proof depends on showing that the operator

$$P_lT_l : T_qM^0(p) \to T_qM^0(p)$$

is a Fredholm operator with kernel $ker(P_lT_l) = T_qM^0(p) \cap T_qM^l(p)$. Our work is similar to Shubin's work in [6]. She proved this result for $l_1 = 0$ and $l_2 = 1$. Carlson and Shubin [7], by using a different method, proved this result in case the isospectral sets corresponding to two different angular momentum l_1 and l_2 such that $l_1 + l_2 = 1 \bmod 2$. Thus, the open problem we thought about is

Problem: For l_1 and l_2 such that $l_1 + l_2 = 0 \bmod 2$, can one prove the same result. Can we construct a Fredholm operator with kernel $T_qM^{l_1}(p) \cap T_qM^{l_2}(p)$ to get that $dim(T_qM^{l_1}(p) \cap T_qM^{l_2}(p)) < \infty$? Can the maps $P_{l_1}T_{l_1}$ and $P_{l_2}T_{l_2}$ be helpful to create that map?

In chapter five, we focused in the geometric properties of the isospectral sets and tried to see how these properties lead to the uniqueness. We tried to show how the finite dimension of the intersection of tangent spaces could give a finite dimention of the intersection of the isospectral sets for finitely many angular momentum. We build Theorem 4.3.1 based on some assumptions. We considered $l = l_1, l_2, ..., l_k$ to be all angular momentum where q and p share their eigenvalues. We assumed that $\cap_{i=1}^{k} M^{l_i}(p)$ is simply connected to get the path connected property. By Theorem 4.2.5, $M^l(p)$ is simply connected for any l. That also implies that $M^l(p)$ is path connected. So the next open problem is

Problem: Can we show that $\cap_{i=1}^{k} M^{l_i}(p)$, for $k > 1$, is simply connected or path connected ?

In Theorem 4.2.5 we proved that the exponential map exp_q^l is an isomorphism between $T_qM^l(p)$ and $M^l(p)$ for each $q \in M^l(p)$ and each l. Hence it may useful to involve the exponential maps to create a path connected between any two distinct points in $\cap_{i=1}^{k} M^{l_i}(p)$.

The second assumption in Theorem 4.3.1 is $dim \cap_{i=1}^{k} T_qM^{l_i}(p) = 0$ for all point $q \in \cap_{i=1}^{k} M^{l_i}(p)$. By Theorem 4.2.3, we know that $d_q\kappa^l$ is an isomorphism between $T_qM^l(p)$ and $T_{\kappa(q)}l_1^2 \simeq l_1^2$ for each $l \geq 0$. Thus $T_{q_1}M^l(p)$ and $T_{q_2}M^l(p)$ are isomorphic for each $q_1, q_2 \in \cap_{i=1}^{k} M^{l_i}(p)$. If this isomorphic relation could give any idea about the dimension of the intersection of tangent spaces at two distinct points q_1 and q_2. Thus the next open problems related to these assumptions is,

Problem: For two different potentials $q_1, q_2 \in \cap_{i=1}^{k} M^{l_i}(p)$, can we show that $dim \cap_{i=1}^{k} T_{q_1}M^{l_i}(p) = dim \cap_{i=1}^{k} T_{q_2}M^{l_i}(p)$?

The last open problem we would like to mention here is related to the main question of this dissertation: Does $\sigma(H_q) = \sigma(H_p)$ lead to $q = p$? Recall that $\sigma(H_q) = \cup_{l=0}^{\infty} \sigma(H_q^l)$, hence proving the uniqueness by assuming the potential q and p share eigenvalues associated to a finite number of angular momentum leads to answer this question. By working in this idea and focusing in each angular momenta we notice some results could be helpful to answer the main question.Thus, first we will discuss this result in the next section, then we will form the related open problem for these result.

5.1 Results Related to Each Angular Momentum

We will start by assuming that two potentials p and q share the same sequence of eigenvalues $\left\{\lambda_{l,n}\right\}_{n=1}^{\infty}$ corresponding to a fixed angular momentum l. From the equation (1.8), we get,

$$\int_0^1 (p(x) - q(x)) \ \psi_{l,n}^p(x) \ \psi_{l,n}^q(x) \ dx = 0, \quad \text{for each } n,$$

where $\psi^p_{l,n}$ and $\psi^q_{l,n}$ are the eigenfunctions of $\lambda_{l,n}$ for p and q respectively. Since our goal is to show the equality between p and q and we have the inner product between $p-q$ and $\psi^p_{l,n}\ \psi^q_{l,n}$ is equal zero for each n, then Showing this set $\left\{\psi^p_{l,n}\ \psi^q_{l,n}\right\}_{n=0}^{\infty}$ is a basis for $L^2_{\mathbb{R}}[0,1]$ leads to the desired result. Notice that if for a single angular momentum l this set $\left\{\psi^p_{l,n}\ \psi^q_{l,n}\right\}_{n=0}^{\infty}$ is a basis then the isospectral set $M^l(q)$ will have a zero dimension and that is contradiction to the result proven by Carlson about infinite dimension of $M^l(q)$ for any $q \in L^2[0,1]$. Hence our conjecture for this problem is if we consider a finite number of angular momenta $l_1, l_2, \ldots l_m$, such that q and p share their eigenvalues, then we may get that $\cup_{i=1}^{m}\left\{\psi^p_{l_i,n}\ \psi^q_{l_i,n}\right\}_{n=0}^{\infty}$ is a basis.

Since the zero potential has a nice eigenfunction with well known eigenvalues, we will begin by fixing p to be the zero potential. Recall that the eigenfunctions of the zero potential are given by Spherical Bessel functions

$$\psi^0_{l,n}(x) = u(x, \lambda_{l,n}) = \frac{x}{\omega_{l,n}^{l+1}} j_l(x\omega_{l,n})$$

where $\omega = \sqrt{\lambda}$. Thus by the boundary conditions, if $\lambda_{l,n}$ is an eigenvalues for 0 then it is a solution of $j_l(\omega_{l,n}) = 0$. Hence $\omega_{l,n} \simeq (n + \frac{l}{2})\pi$. Let q be a potential sharing the eigenvalues with zero potential, then the eigenfunction of $\lambda_{l,n}$ for q is

$$\psi^q_{l,n}(x) = \psi^0_{l,n}(x) + K_{l,n}(x,q)$$

where $K_{l,n}(x,q) = \displaystyle\int_0^x G(x,t,\lambda_{l,n})\ q(t)\ \psi^q_{l,n}(t)\ dt$. Hence

$$\psi^0_{l,n}(x)\psi^q_{l,n}(x) = (\psi^0_{l,n})^2(x) + \psi^0_{l,n}(x)K_{l,n}(x,q).$$

Since q shares the eigenvalues with the zero potential then

$$\int_0^1 q(x)dx = 0$$

by Corollary 1.3.1. Combining this fact with

$$\int_0^1 q(x)\ \psi^0_{l,n}(x)\ \psi^q_{l,n}(x)\ dx = 0, \qquad \text{for each } n,$$

we get,

$$\int_0^1 q(x)\ \left\{1 - 2(n\pi)^2(\psi^0_{l,n})^2(x)\ + -2(n\pi)^2\psi^0_{l,n}(x)K_{l,n}(x,q)\right\}\ dx = 0, \qquad \text{for each } n, \tag{5.1}$$

In case $l = 0$, $1 - 2(n\pi)^2(\psi^0_{0,n})^2(x) = \cos(2n\pi x)$, so the integral (5.1) becomes,

$$\int_0^1 q(x)\ \left\{\cos(2n\pi x)\ - 2(n\pi)^2\psi^0_{0,n}(x)K_{0,n}(x,q)\right\}\ dx = 0, \qquad \text{for each } n.$$

For $l \geq 1$, we will use the following property of the operator T_l which is given in the definition 3.2.1

$$\int_0^1 (2\Phi_l(\sqrt{\lambda}x) - 1)\xi(x)\ dx = \int_0^1 \cos(2\sqrt{\lambda}x)T_l[\xi](x)\ dx$$

where $\Phi_l(\sqrt{\lambda x}) = x^2 j_l^2(\sqrt{\lambda x})$ for any $\xi \in L^2_{\mathbb{R}}[0,1]$ to get

$$\int_0^1 q(x)\ \left\{2\omega_{l,n}^{2l+2}(\psi_{l,n}^0)^2(x) - 1\ + 2\omega_{l,n}^{2l+2}\psi_{l,n}^0(x)K_{l,n}(x,q)\right\}\ dx = \int_0^1 T_l[q(x)]\{\cos(2n+l)\pi x) + \widetilde{K}_{l,n}(q,x)\}dx$$

where $T_l^*[\widetilde{K}_{l,n}(x,q)] = 2\omega_{l,n}^{2l+2}\psi_{l,n}^0(x)K_{l,n}(x,q)$. Hence (5.1) becomes

$$\int_0^1 T_l[q(x)]\{\cos(2n+l)\pi x) + \widetilde{K}_{l,n}(q,x)\}dx = 0$$

5.1.1 Conjecture for the Case $l_1 = 0$ and $l_2 = 1$

If we consider the angular momenta $l_1 = 0$ and $l_2 = 1$ then we have the following system,

$$\begin{cases} \displaystyle\int_0^1 q(x)\ \left\{\ \cos(2n\pi x)\ + 2(n\pi)^2\psi_{0,n}^0(x)K_{0,n}(x,q)\right\}\ dx = 0, & \text{for each } n. \\ \displaystyle\int_0^1 T_1[q(x)]\{\cos(2n+1)\pi x) + \widetilde{K}_{1,n}(q,x)\}dx = 0 & \text{for each } n \end{cases}$$

The second part of each integral has the following estimate of

$$|2(n\pi)^2\psi_{0,n}^0(x)K_{0,n}(x,q)| = O(\frac{1}{n})$$

$$|\widetilde{K}_{1,n}(q,x)| = O(\frac{1}{n})$$

For this monument, we will ignore the error parts and prove the following theorem

Theorem 5.1.1. *For $q \in L^2_{\mathbb{R}}[0,1]$, if we have*

$$\begin{cases} \displaystyle\int_0^1 q(x)\cos(2n\pi x)\ dx = 0, & \textit{for each } n \\ \displaystyle\int_0^1 T_1[q](x)\cos((2n+1)\pi x)\ dx = 0 & \textit{for each } n \end{cases}$$

then $q = 0$.

Proof. Since $\{cos(n\pi x)\}_{n\geq 1}$ is an orthonormal basis of $L^2[0,1]$, then from the first integral we can say that q is an odd function which we will denote by $q = q_O$. So now, we have

$$\int_0^1 T_1[q_O](x)\cos((2n+1)\pi x)\ dx = 0 \quad \text{for each } n$$

which implies that the odd part of $T_1[q_O]$ is zero. By using the definition of T_1

$$T_1[q_O] = q_O + 4x \int_x^1 \frac{q_O(s)}{s^2} ds$$

we get

$$\begin{aligned} 0 &= \frac{1}{2}\Big(T_1[q_O](x) - T_1[q_O](1-x)\Big) \\ &= \frac{1}{2}\Big(2q_O(x) - 4x \int_x^1 \frac{q_O(s)}{s^2} ds + 4(1-x) \int_{1-x}^1 \frac{q_O(s)}{s^2} ds\Big) \end{aligned}$$

Since q_O is odd then

$$\int_{1-x}^1 \frac{q_O(s)}{s^2} ds = -\int_0^x \frac{q_O(s)}{(1-s)^2} ds$$

Hence,

$$q_O(x) = 2x \int_x^1 \frac{q_O(s)}{s^2} ds + 2(1-x) \int_0^x \frac{q_O(s)}{(1-s)^2} ds$$

Because these integrals in the form of q_O are absolute continuous, then we can take the derivative of q_O to get

$$q_O^{'}(x) = 2 \int_x^1 \frac{q_O(s)}{s^2} ds - 2 \int_0^x \frac{q_O(s)}{(1-s)^2} ds - 2\frac{q_O(x)}{x} + 2\frac{q_O(x)}{(1-x)} \quad \text{a.e}$$

Since also these integrals in the form of $q_o^{'}$ are absolute continuous then we can take the derivative to get

$$(1-x)xq_O^{''}(x) - (4x-2)q_O^{'} = 0 \quad \text{a.e}$$

Hence, $\Big((1-x)^2x^2q_O^{'}\Big)^{'} = 0$. Integration from 0 to x with using $\lim_{x\to 0}(1-x)^2x^2q_O(x) = 0$ to get

$$(1-x)^2x^2q_O^{'}(x) = 0$$

Thus $q_O^{'}(x) = 0$ almost everywhere, hence q_O is constant. Since q_O is a constant and odd function, then q_O is zero.

□

Now return back to the main problem which contains the error parts. we have this sysytem

$$\begin{cases} \displaystyle\int_0^1 q(x)\ \left\{\cos(2n\pi x) + 2(n\pi)^2\psi_{0,n}^0(x)K_{0,n}(x,q)\right\}\ dx = 0, & \text{for each } n. \\ \displaystyle\int_0^1 T_1[q(x)]\{\cos(2n+1)\pi x) + \widetilde{K}_{1,n}(q,x)\}dx = 0 & \text{for each } n \end{cases}$$

If we can show that $E = \{\cos(2n\pi x) + 2(n\pi)^2\psi^0_{0,n}(x)K_{0,n}(x,q)\}_{n\geq 1}$ is a basis of the subspace of the even functions of $L^2_{\mathbb{R}}[0,1]$ and $O = \{\cos(2n+1)\pi x) + \widetilde{K}_{1,n}(q,x)\}_{n\geq 1}$ is a basis of the subspace of the odd functions of $L^2_{\mathbb{R}}[0,1]$ then by the process as above we will get $q = 0$. One of our attempt for this case is trying to use the following results related to complete sets of Hilbert spaces,

Definition 5.1.1. A sequence of vectors $\{\psi_n\}_{n\geq 1}$ in an infinite dimensional Hilbert space H is said to be a basis for H if to each vector $u \in H$ there is a unique sequence $\{\mu_n\}_{n\geq 1}$ of complex numbers such that

$$u = \sum_{n\geq 1} \mu_n \psi_n$$

Corollary 5.1.1. *Let $\{\phi_n\}_{n\geq 1}$ be a complete orthonormal family in H and $\{\psi_n\}_{n\geq 1}$ be a sequence of nonzero vectors of H. Then*

$$\sum_{n\geq 1} \|\phi_n - \psi_n\| < 1$$

is a sufficient condition for $\{\psi_n\}_{n\geq 1}$ to be a basis.

Corollary 5.1.2. *Let $\{\phi_n\}_{n\geq 1}$ be a complete orthonormal family in H. Suppose $\{\psi_n\}_{n\geq 1}$ is another sequence of vectors in H that either spans or is linearly independent. If, in addition*

$$\sum_{n\geq 1} \|\phi_n - \psi_n\|^2 < \infty \tag{5.2}$$

then $\{\psi_n\}_{n\geq 1}$ is a basis of H. Moreover, the map

$$u \to (\langle u, \psi_n\rangle)_{n\geq 1}$$

is a linear isomerism between H and l^2.

First we tried to apply Corollary 1.1, by considering $\{\phi_n\}_{n\geq 1} = \{\cos(n\pi x)\}_{n\geq 1}$ and $\{\psi_n\}_{n\geq 1} = E \cup O$. From Theorems 1.2.1 and 1.2.4 we have,

$$\|\phi_n - \psi_n\| = \left\|2(n\pi)^2\psi^0_{0,n}(x)K_{0,n}(x,q)\right\| \leq \frac{1}{n}e^{\|q\|}$$

in case $\psi_n \in E$ and

$$\|\phi_n - \psi_n\| = \left\|\widetilde{K}_{1,n}(q,x)\right\| \leq \frac{1}{n}e^{C\|q\|}$$

in case $\psi_n \in O$. Since $\|\phi_n - \psi_n\|$ depends on the norm of q, then applying Corollary 5.1.1 will give a condition on the norm of q to guarantee that $\{\psi_n\}_{n\geq 1}$ is a basis. But the problem here is the asymptotic of these terms which is $\|\phi_n - \psi_n\| = O(\frac{1}{n})$. Hence the sum over n of theses terms gives diverges series. Another way to handle this problem we consider is to apply Corollary 5.1.2, where the sum here will be for these terms

$$\|\phi_n - \psi_n\|^2 = O(\frac{1}{n^2})$$

That is good since these squared summable, but the basic problem in applying this corollary is to show that $\{\psi_n\}_{n\geq 1} = E \cup O$ is linearly independent sets or spans of $L^2_{\mathbb{R}}[0,1]$, which is difficult since these two sets depends on different two angular moment. In 1946, Borg [13] proved that for $l = 0$ and $p, q \in \mathcal{E}$, this set $\{\psi^q_{0,n}\psi^p_{0,n} - 1\}_{n\geq 1}$ is linearly independent. Then by using Corollary 5.1.2 and

$$\int_0^1 (p(x) - q(x))\ (\psi^p_{0,n}(x)\ \psi^q_{0,n}(x) - 1)\ dx = 0, \qquad \text{for each } n,$$

he proved $p = q$. So the open problem for this part is

Problem: For $l = 0, 1$ can we show that E and O are a basis of the subspace of the even functions and odd functions respectively?

5.1.2 Conjecture for the Case $l_1 > 0$ and $l_2 = l_1 + 1$

Now let's consider two different angular momentum l_1, l_2 such that $l_2 = l_1 + 1$. From Definition 3.2.1 we have $T_{l_2} = S_{l_2} T_{l_1}$. Assuming q shares the eigenvalues associated with l_1 and l_2 with the zero potential, and using the properties of T_{l_1} and T_{l_2} (3.2.1) leads to the following system,

$$(*) \quad \begin{cases} \displaystyle\int_0^1 T_{l_1}[q](x)\{\cos(2(n + \frac{l_1}{2})\pi x) + \widetilde{K}_{l_1,n}(q,x)\}dx = 0 & \text{for each } n \\ \\ \displaystyle\int_0^1 S_{l_2}[T_{l_1}[q]](x)\{\cos(2(n + \frac{l_2}{2})\pi x) + \widetilde{K}_{l_2,n}(q,x)\}dx = 0 & \text{for each } n \end{cases}$$

Consider the two following sets $A_{l_1} = \{\cos(2(n + \frac{l_1}{2})\pi x) + \widetilde{K}_{l_1,n}(q,x)\}_{n\geq 1}$ and $A_{l_2} = \{\cos(2(n + \frac{l_2}{2})\pi x) + \widetilde{K}_{l_2,n}(q,x)\}_{n\geq 1}$. If l_1 is even, A_{l_1} and A_{l_2} are perturbation sets of $\{\cos(2n\pi x)\}_{n\geq 1}$ and $\{\cos(2(n+1)\pi x)\}_{n\geq 1}$ respectively. So by ignoring the error parts $\widetilde{K}_{l_1,n}$ and $\widetilde{K}_{l_2,n}$, the first integral in $(*)$ tells that $T_{l_1}[q]$ is odd. Then from the second integral we will get that the odd part of $S_{l_2}[T_{l_1}[q]]$ is zero. So we have the following equation,

$$\frac{1}{2}\Big\{S_{l_2}[T_{l_1}[q]](x) - S_{l_2}[T_{l_1}[q]](1-x)\Big\} = 0 \tag{5.3}$$

So our conjecture is solving (5.3) for $T_{l_1}[q]$ giving $T_{l_1}[q] = 0$. Since T_{l_1} is a one to one map then $q = 0$. Hence for the main problem without ignoring the error parts, showing that A_{l_1} and A_{l_2} are bases for the subspace of the even and odd functions could lead to $q = 0$. Similarly if l_1 is odd, showing that A_{l_1} and A_{l_2} are base for the subspace of the odd and even functions could lead to the same result. So the last open problem is,

Problem: For l_1 and $l_2 = l_1 + 1$, in case l_1 is even can we show that A_{l_1} and A_{l_2} are bases for the subspace of the even and odd functions of $L^2_{\mathbb{R}}[0,1]$? In case l_1 is odd can we show that A_{l_1} and A_{l_2} are bases for the subspace of the odd and even functions of $L^2_{\mathbb{R}}[0,1]$?

Appendix

A1 Spherical Bessel Functions

A1.1 Series Forms of Spherical Bessel functions

1- The fist spherical Bessel function of order l is j_l:

$$\begin{aligned} xj_l(x) &= \sqrt{\frac{x\pi}{2}} J_{l+1/2}(x) \\ &= \sqrt{\frac{x\pi}{2}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+l+\frac{3}{2})} \Big(\frac{x}{2}\Big)^{2s+l+\frac{1}{2}} \\ &= \Big(\frac{x}{2}\Big)^{l+1} \sqrt{\pi} \frac{1}{\Gamma(l+\frac{3}{2})} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(l+\frac{3}{2})_s} \Big(\frac{x}{2}\Big)^{2s} \end{aligned}$$

Since $\Gamma(l+3/2) = \sqrt{\pi}\dfrac{(2l+1)!!}{2^l}$ then we have

$$xj_l(x) = \Big(\frac{x}{2}\Big)^{l+1} \frac{2^l}{(2l+1)!!} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(l+\frac{3}{2})_s} \Big(\frac{x}{2}\Big)^{2s} \tag{4}$$

2- The second spherical Bessel function of order l:

$$\eta_l(x) = (-1)^{l+1} j_{-l-1}(x)$$

Hence

$$\begin{aligned} x\eta_l(x) &= \sqrt{\frac{x\pi}{2}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s-l+\frac{1}{2})} \Big(\frac{x}{2}\Big)^{2s-l-\frac{1}{2}} \\ &= \Big(\frac{2}{x}\Big)^l \sqrt{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s-l+\frac{1}{2})} \Big(\frac{x}{2}\Big)^{2s-l-\frac{1}{2}} \end{aligned}$$

Since $\Gamma(s-l+(\frac{1}{2})) = \Gamma(\frac{1}{2}-l)(\frac{1}{2}-l)_s = \frac{\sqrt{\pi}(-2)^l}{(2l-1)!!}(\frac{1}{2}-l)_s$ we have

$$x\eta_l(x) = \Big(\frac{2}{x}\Big)^l \frac{(2l-1)!!}{(-2)^l} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(\frac{1}{2}-l)_s} \Big(\frac{x}{2}\Big)^{2s} \tag{5}$$

A1.2 Trigonometric Polynomial Forms of Spherical Bessel functions

1- The trigonometric polynomial form of $xj_l(x)$ and $x\eta_l(x)$ are given by

$$xj_l(x) = \sin\Big(x-\frac{l\pi}{2}\Big)P_l\Big(\frac{1}{x}\Big) + \cos\Big(x-\frac{l\pi}{2}\Big)I_l\Big(\frac{1}{x}\Big) \tag{6}$$

and

$$x\eta_l(x) = \cos(x - \frac{l\pi}{2})P_l(\frac{1}{x}) - \sin(x - \frac{l\pi}{2})I_l(\frac{1}{x}) \tag{7}$$

where

$$P_l(x) = \sum_{s=0}^{[l/2]} \frac{(-1)^s(l+2s)!}{(2s)!(l-2s)!}(\frac{1}{2x})^{2s} \quad \text{and} \quad I_l(x) = \sum_{s=0}^{[l/2]} \frac{(-1)^s(l+2s+1)!}{(2s+1)!(l-2s-1)!}(\frac{1}{2x})^{2s}$$

For any $x, t \in \mathbb{C}$, we define Now we will define $\widetilde{G}(x,t,\lambda) = xj_l(\omega x)t\eta(\omega t) - tj_l(\omega t)x\eta(\omega x)$ where $\omega = \sqrt{\lambda}$. In the next

A1.3 Upper Bound of Spherical Bessel Functions

For any $x \in \mathbb{C}$ there is a constant C_l depending on l such that

$$|xj_l(x)| \le C_l e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1} \tag{8}$$

$$|x\eta_l(x)| \le C_l e^{|Imx|}\Big(\frac{1+|x|}{|x|}\Big)^{l} \tag{9}$$

Also for any x, t such that $0 \le t \le x \le 1$, there is C such that

$$|\widetilde{G}(x,t,\lambda)| \le C\Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1}\Big(\frac{1+|\omega|t}{|\omega|t}\Big)^{l} \tag{10}$$

and for each $x \le t \le 1$

$$|\widetilde{G}(x,t,\lambda)| \le C\Big(\frac{1+|\omega|x}{|\omega|x}\Big)^{l}\Big(\frac{|\omega|t}{1+|\omega|t}\Big)^{l+1} \tag{11}$$

To verify (8) and (9), we will use the trigonometric forms in case $|x| > 1$, and the series forms in case $|x| \le 1$.

In case $|x| > 1$, we have

$$1 \le e^{|Imx|} = e^{|Imx|}\Big(\frac{1+|x|}{1+|x|}\Big)^{l+1} \le e^{|Imx|}2^{l+1}\Big(\frac{|x|}{1+|x|}\Big)^{l+1}$$

Hence from (6)

$$\begin{aligned}
|xj_l(x)| &\le e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1}2^{l+1}\left(|P_l(\frac{1}{x})| + |I_l(\frac{1}{x})|\right) \\
&\le e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1}2^{l+1}\left(\sum_{s=0}^{[l/2]} \frac{(l+2s)!}{(2s)!(l-2s)!}(\frac{1}{2})^{2s} + \sum_{s=0}^{[l/2]} \frac{(l+2s+1)!}{(2s+1)!(l-2s-1)!}(\frac{1}{2})^{2s}\right)
\end{aligned}$$

Thus

$$|xj_l(x)| \leq C_{j_l} e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1}.$$

where

$$C_{j_l} = 2^{l+1}\left(\sum_{s=0}^{[l/2]} \frac{(l+2s)!}{(2s)!(l-2s)!}(\frac{1}{2})^{2s} + \sum_{s=0}^{[l/2]} \frac{(l+2s+1)!}{(2s+1)!(l-2s-1)!}(\frac{1}{2})^{2s}\right)$$

In case $|x| \leq 1$, notice that $2^{l+1} > (1+|x|)^{l+1}$. Hence from (4) we have

$$\begin{aligned} |xj_l(x)| &\leq \Big(\frac{|x|}{1+|x|}\Big)^{l+1} \frac{2^l}{(2l+1)!!} \sum_{s=0}^{\infty} \frac{1}{s!(l+\frac{3}{2})_s}(\frac{1}{2})^{2s} \\ &\leq e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1} \frac{2^l}{(2l+1)!!} \sum_{s=0}^{\infty} \frac{1}{2^s} \end{aligned}$$

Since $(2l+1)!! > 2^l$ for each $l > 0$ then $\frac{2^l}{(2l+1)!!} < 1$ also $\sum_{s=0}^{\infty} \dfrac{1}{2^s} = 1$ then we have

$$|xj_l(x)| \leq e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1}$$

Hence For any $x \in \mathbb{C}$ we have

$$|xj_l(x)| \leq C_{j_l} e^{|Imx|}\Big(\frac{|x|}{1+|x|}\Big)^{l+1} \tag{12}$$

Now for the estimate in (9)

In case $|x| > 1$, we have $|x|^l < (1+|x|)^l$ hence $1 < \dfrac{(1+|x|)^l}{|x|^l}$, then from (7)

$$\begin{aligned} |x\eta_l(x)| &\leq e^{|Imx|}\Big(\frac{1+|x|}{|x|}\Big)^{l}\left(|P_l(\frac{1}{x})| + |I_l(\frac{1}{x})|\right) \\ &\leq e^{|Imx|}\Big(\frac{1+|x|}{|x|}\Big)^{l+1}\left(\sum_{s=0}^{[l/2]} \frac{(l+2s)!}{(2s)!(l-2s)!}(\frac{1}{2})^{2s} + \sum_{s=0}^{[l/2]} \frac{(l+2s+1)!}{(2s+1)!(l-2s-1)!}(\frac{1}{2})^{2s}\right) \end{aligned}$$

Hence,

$$|x\eta_l(x)| \leq C_{\eta_l} e^{|Imx|}\Big(\frac{1+|x|}{|x|}\Big)^{l} \tag{13}$$

where

$$C_{\eta_l} = \left(\sum_{s=0}^{[l/2]} \frac{(l+2s)!}{(2s)!(l-2s)!}(\frac{1}{2})^{2s} + \sum_{s=0}^{[l/2]} \frac{(l+2s+1)!}{(2s+1)!(l-2s-1)!}(\frac{1}{2})^{2s}\right)$$

In case $|x| \leq 1$, Since $1^l < (1+|x|)^l$, then from (5) we have

$$
\begin{aligned}
|x\eta_l(x)| &\leq \Big(\frac{1+|x|}{|x|}\Big)^l 2^l \frac{(2l-1)!!}{(-2)^l} \sum_{s=0}^{\infty} \frac{1}{s!(\frac{1}{2}-l)_s} \big(\frac{1}{2}\big)^{2s} \\
&\leq \Big(\frac{1+|x|}{|x|}\Big)^l (2l-1)!! \sum_{s=0}^{\infty} \frac{1}{2^s} \\
&\leq e^{|Imx|} \Big(\frac{1+|x|}{|x|}\Big)^l (2l-1)!!
\end{aligned}
$$

Hence,

$$
|x\eta_l(x)| \leq \tilde{C}_{\eta_l} e^{|Imx|} \Big(\frac{1+|x|}{|x|}\Big)^l \tag{14}
$$

where $\tilde{C}_{\eta_l} = (2l-1)!!$. By taking $C_l = max\{C_{j_l}, C_{\eta_l}, \tilde{C}_{\eta_l}\}$, we prove (8) and (9).

Now for the estimate of $\widetilde{G}$, from (8) and (9) we have

$$
\begin{aligned}
|\widetilde{G}(x,t,\lambda)| &\leq C_l^2 \Bigg\{ \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l + \Big(\frac{|\omega|t}{1+|\omega|t}\Big)^{l+1} \Big(\frac{1+|\omega|x}{|\omega|x}\Big)^l \Bigg\} \\
&\leq C_l^2 \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l \Bigg\{ 1 + \Big(\frac{|\omega|t}{1+|\omega|t}\Big)^{2l+1} \Big(\frac{1+|\omega|x}{|\omega|x}\Big)^{2l+1} \Bigg\}
\end{aligned}
$$

In case $|\omega|x \leq 1$, the we will use $(1+|\omega|x)^{2l+1} \leq 2^{2l+1}$ to get

$$
\begin{aligned}
|\widetilde{G}(x,t,\lambda)| &\leq C_l^2 \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l \Bigg\{ 1 + 2^{2l+1} \Bigg\} \\
&\leq 2^{2l+2} C_l \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l
\end{aligned}
$$

In case $|\omega|x \geq 1$, we will use (6) and (7) to find an estimate for $\widetilde{G}$:

$$
\begin{aligned}
\widetilde{G}(x,t,\lambda) =& \big(P_l(\omega x)P_l(\omega t) + I_l(\omega x)I_l(\omega t)\big)\sin(\omega(x-t)) \\
&- \big(P_l(\omega x)I_l(\omega t) + I_l(\omega x)(P_l(\omega t)\big)\cos(\omega(x-t))
\end{aligned}
$$

If $|\omega|t \geq 1$ and we have $|\omega|x \geq 1$ then $|P_l(\omega x)| \leq C_1$,$|P_l(\omega t)| \leq C_1$,$I_l(\omega x) \leq C_2$ and $I_l(\omega t) \leq C_2$

$$
C_1 = \sum_{s=0}^{[l/2]} \frac{(l+2s)!}{(2s)!(l-2s)!} \big(\frac{1}{2}\big)^{2s} \quad \text{and} \quad C_2 = \sum_{s=0}^{[l/2]} \frac{(l+2s+1)!}{(2s+1)!(l-2s-1)!} \big(\frac{1}{2}\big)^{2s}
$$

.

By considering $C_{1,2} = \max\{C_1^2, C_2^2, C_1C_2\}$, $1 = \Big(\frac{1+|\omega|x}{1+|\omega|x}\Big)^{l+1} \leq 2^{l+1}\Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1}$ and $1 \leq \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l$, we will get

$$
|\widetilde{G}(x,t,\lambda)| \leq 2^{l+1} C_{1,2} \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l
$$

Last case when $|\omega|x \geq 1$ and $|\omega|t \leq 1$ we have

$$|P_l(\omega t)| \leq \frac{C_1}{|\omega t|^l} \leq C_1 \big(\frac{1+|\omega|t}{|\omega|t}\big)^l$$

$$|I_l(\omega t)| \leq \frac{C_2}{|\omega t|^l} \leq C_2 \big(\frac{1+|\omega|t}{|\omega|t}\big)^l$$

and $1 \leq 2^{l+1}\Big(\dfrac{|\omega|x}{1+|\omega|x}\Big)^{l+1}$. Thus

$$|\widetilde{G}(x,t,\lambda)| \leq 2^{l+1} C_{1,2} \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l$$

Hence for any $\lambda \in \mathbb{C}$ and $0 \leq t \leq x \leq 1$ we have,

$$|\widetilde{G}(x,t,\lambda)| \leq C \Big(\frac{|\omega|x}{1+|\omega|x}\Big)^{l+1} \Big(\frac{1+|\omega|t}{|\omega|t}\Big)^l$$

where

$$C = \max\{2^{l+1} C_{1,2}, 2^{2l+2} C_l\}. \tag{15}$$

For more interested information of Spherical Bessel functions and their estimate, see [15], [17], and [18].

A2 Infinite Product

Lemma A2.1. *Suppose $a_{mn}, m, n > 1$, are complex numbers satisfying*

$$|a_{mn}| = O\Big(\frac{1}{|m^2 - n^2|}\Big), \quad m \neq n$$

then

$$\prod_{\substack{m \neq n \\ m \geq 1}} (1 + a_{mn}) = 1 + O\Big(\frac{\log n}{n}\Big), \qquad n \geq 1,$$

Lemma A2.2. *Suppose $z_m = (\frac{m\pi}{2})^2 + O(1)$ then,*

- $\displaystyle\prod_{m \geq 1}^{\infty} \frac{z_m - \lambda}{\big(\frac{m\pi}{2}\big)^2}$ *is an entire function of λ with roots $z_m,\ m \geq 1$*
- $\displaystyle\prod_{m \geq 1}^{\infty} \frac{z_m - \lambda}{\big(\frac{m\pi}{2}\big)^2} = \frac{\sin(2\sqrt{\lambda})}{2\sqrt{\lambda}} \Big(1 + O\big(\frac{\log n}{n}\big)\Big)$

Proof.

- Note that $z_m - (\frac{m\pi}{2})^2$ is uniform bounded, so we have here

$$\sum_{m\geq 1} \frac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2} - 1 = \sum_{m\geq 1} \frac{z_m - \big(\dfrac{m\pi}{2}\big)^2 - \lambda}{\big(\dfrac{m\pi}{2}\big)^2}$$

converges uniformly on bounded subsets of $\mathbb{C}$, therefore $\prod_{m\geq 1}^{\infty} \frac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2}$ converges to an entire function of λ and its roots $z_m,\ m \geq 1$.

- Note that $\dfrac{\sin(2\sqrt{\lambda})}{2\sqrt{\lambda}} = \prod_{m\geq 1}^{\infty} \dfrac{(m\pi)^2 - 4\lambda}{(m\pi)^2} = \prod_{m\geq 1}^{\infty} \dfrac{\big(\dfrac{m\pi}{2}\big)^2 - \lambda}{\big(\dfrac{m\pi}{2}\big)^2}$

Hence
$\prod_{m\geq 1}^{\infty} \dfrac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2} \div \dfrac{\sin(2\sqrt{\lambda})}{2\sqrt{\lambda}} = \prod_{m\geq 1}^{\infty} \dfrac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2 - \lambda}.$

On the circle $|\lambda| = (\frac{n}{2} + \frac{1}{2})^2\pi^2$ the uniform estimates given by

$$\frac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2 - \lambda} = \begin{cases} 1 + O(\frac{1}{n}) & m = n \\ 1 + O\big(\frac{1}{|m_2 - n^2|}\big) & m \neq n \end{cases}$$

Hence

$$\begin{aligned}\prod_{m\geq 1}^{\infty} \frac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2 - \lambda} &= \frac{z_n - \lambda}{\big(\dfrac{n\pi}{2}\big)^2 - \lambda} \prod_{m\neq n}^{\infty} \frac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2 - \lambda} \\ &= \Big(1 + O(\frac{1}{n})\Big)\Big(1 + O\big(\frac{\log n}{n}\big)\Big) \\ &= 1 + O\big(\frac{\log n}{n}\big)\end{aligned}$$

The last line follow from lemma A2.1.

□

Lemma A2.3. *suppose $z_m, m \geq 1$, is a sequence of a complex numbers such that $z_m = \dfrac{m^2\pi^2}{4} + O(1)$ then for each $n \geq 1$*

$$\prod_{\substack{m\neq n\\ m\geq 1}} \frac{z_m - \lambda}{m^2\pi^2}$$

is an entire function of λ such that

$$\prod_{\substack{m\neq n\\ m\geq 1}} \frac{z_m - \lambda}{\big(\dfrac{m\pi}{2}\big)^2} = \frac{1}{2}(-1)^{n+1}\Big(1 + \big(\frac{\log n}{n}\big)\Big)$$

uniformly for $\lambda = n^2\pi^2 + O(1)$.

Lemma A2.4. *If* $|z - n\pi| \geq \frac{\pi}{4}$ *for all integers* n*, then*

$$e^{|Imz|} < 4|\sin z|.$$

Proof. Write $z = x + iy$ with real x and y. since $|\sin z|$ is even and periodic with period π, it is suffices to prove the lemma for $0 \leq x \leq \pi/2$ and $|z| \geq \pi/4$.
we have

$$|\sin z|^2 = cosh^2 y - \cos^2 x$$

For $\pi/6 \leq x \leq \pi/2$,

$$\cos^2 x \leq \frac{3}{4} \leq \frac{3}{4}\, cosh^2 y$$

for all real y. For $0 \leq x \leq \pi/6$, the assumption $|z| \geq \pi/4$ implies $y^2 \geq (\pi/4)^2 - x^2 \geq \frac{5}{144} \geq \frac{1}{3}$ and hence

$$cosh^2 y \geq 1 + y^2 \geq \frac{4}{3} \geq \frac{4}{3}\cos^2 x$$

as before. Thus, in both cases we have

$$|\sin z|^2 \geq \frac{1}{4} cosh^2 y > \frac{1}{16} e^{2|y|}$$

and the result follows. □

Definition A2.1. (Regular Value)
Let $f : A \to F$ be a continuously differentiable map from an open subset A of a Banach space E into another Banach space F. A point $c \in F$ is a regular value of f, if for every point x in the level set

$$M_c = \{x \in A : f(x) = c\}$$

there exists a splitting $E = E_h \bigoplus E_v$, such that $d_x f|E_v$, the restriction of $d_x f$ to E_v is a linear isomorphism between E_v and F.

Theorem A2.1. *(Regular Value Theorem)*
Suppose $f : A \to F$ *is a real analytic map from an open subset* A *of a Banach space* E *into another Banach space* F*. If* $c \in F$ *is a regular value of* f*, then*

$$M_c = \{x \in A : f(x) = c\}$$

is a real analytic submanifold of E*. Moreover,*

$$T_x M_c = ker(d_x f)$$

at every point $x \in M_c$*. For the proof of this theorem see [1].*

A3 Fredholm and Compact Operators

Definition A3.1. Let X and Y be Banach spaces, and let $F : X \to Y$ be a bounded linear operator. F is said to be Fredholm operator if the following hold,

- $ker(F)$ is finite dimensional,
- $coker(F)$ is finite dimensional,
- $ran(F)$ is closed.

Lemma A3.1. *Let X, Y, and Z be Banach spaces. If $F : X \to Y$ and $S : Y \to Z$ be two linear operators such that $dim(kerF) < \infty$ and S invertible, then $dim(kerSF) < \infty$.*

Proof. Since S is ivertiable then $kerS = \{0\}$. let $f \in KerSF$ then

$$SF(f) = 0 \iff F(f) = 0 \iff f \in kerF$$

hence $kerSF = kerF$ which proves the claim.

□

Lemma A3.2. *Let X, Y, and Z be Banach spaces. If $S : X \to Y$ and $F : Y \to Z$ be two linear operators such that $dim(kerF) < \infty$ and S invertible, then $dim(kerFS) < \infty$.*

Proof. Since S is invertible then there is a linear operator S^{-1} such that $SS^{-1}(Y) = Y$ and $S^{-1}S(X) = X$. Assume $kerF = span\{f_1, f_2, ..., f_n\}$. Let $f \in kerFS$ then $FS(f) = 0$. Since $S^{-1}(Y) = X$ and $f \in X$ then there is $g \in Y$ such that $S^{-1}(g) = f$. Thus $0 = FS(f) = FSS^{-1}(g) = F(g)$ which implies that $g \in kerF$. Hence $g = \sum_{i=1}^{n} a_i f_i$, therefore $f = \sum_{i=1}^{n} a_i S^{-1}(f_i)$ so $f \in span\{S^{-1}(f_1), S^{-1}(f_2)...., S^{-1}f_n\}$. Thus $dimker(FS) \leq n$.

□

Lemma A3.3. *Let X, Y, and Z be Banach spaces. If $F : X \to Y$ and $S : Y \to Z$ be two linear operators such that F is a Fredholm operator and S is invertible, then SF is Fredholm.*

Proof. SF is Fredholm operator if $dimker(SF) < \infty$, $dimcoker(SF) < \infty$ and $Ran(SF)$ is closed. Lemma A3.1 shows the $ker(SF)$ has a finite dimension. For the second condition, note that

$$coker(SF) = Z/SF(Y) = Z/(ker(SF)^*)^{\perp}$$

Hence $dimcoker(SF) = dim(ker(SF)^*) = dim(ker(F^*S^*))$
Note that F is Fredholm then $dimker(F^*) = dimcoker(F) < \infty$, also S is invertible which implies S^* is invertible. Hence by lemma A3.2 we have $dimcoker(SF) = dim(ker(F^*S^*)) < \infty$.

For the last condition, Let $\{f_n\}_{n=1}^{\infty} \in Ran(SF)$ be a convergent sequence to $f \in Z$, we need to show that $f \in Ran(SF)$. Note that for each n there is $g_n \in X$ such that $SF(g_n) = f_n$. since S^{-1} is invertible then $S^{-1}(f_n) = F(g_n)$ is convergent to some point ω in the range of F because $Ran(F)$ is closed. $\omega = F(g)$ for some $g \in X$. Since S is invertible then $SF(g_n)$ converges to $SF(g)$ and since the limit is unique then $SF(g) = f$ hence $f \in Ran(SF)$. □

Lemma A3.4. *Let X, Y, and Z be Banach spaces. If $S : X \to Y$ and $F : Y \to Z$ be two linear operators such that F is a Fredholm operator and S is invertible, then FS is Fredholm.*

Proof. Similar to the proof of Lemma A3.3 . □

Theorem A3.1. *(Atkinson)*
For a bounded operator $F : X \to Y$ between Banach spaces the following statements are equivalent.

(1) F is Fredholm operator.

(2) There exist compact operators K_1 from X to itself and K_2 from Y to itself and a bounded operator S from X to Y such that $SF = I_X - K_1$ and $FS = I_Y - K_2$.

(3) There exist compact operators K_1 from X to itself and K_2 from Y to itself and a bounded operator S_1 and S_2 from X to Y such that such that $S_1F = I_X - K_1$ and $FS_2 = I_Y - K_2$.

Corollary A3.1. *If $F : X \to Y$ is a Fredholm operator, then for each compact operator $K : X \to Y$ the operator $F + K$ is also Fredholm.*

Proof. Since F is Fredholm then by Lemma A3.1 there exist compact operators K_1 from X to itself and K_2 from Y to itself and a bounded operator S_1 and S_2 from X to Y such that such that $S_1F = I_X - K_1$ and $FS_2 = I_Y - K_2$. It follows that
$S_1(F + K) = I_X - (K_1 - S_1K)$ and $(F + K)S_2 = I_Y - (K_2 - KS_2)$.
Since the operators $K_1 - S_1K$ and $K_2 - KS_2$ are compact and that implies by Lemma A3.1 $F + K$ is a Fredholm operator.

□

For more properties of Fredholm operators see [9].

Definition A3.2. (Compact operator)
Let H be a Hilbert space. A linear operator T on H is compact, if it maps weekly converging sequences into strongly converging sequences. Equivalently, a compact operator T maps bounded subsets into relatively compact subset of H.

Lemma A3.5. *The uniform limit of compact operators is compact.*

Proof. Let T be the uniform limit of compact operators T_n. Let x_m be a weakly converging sequence, and let x be its weak limit. Then, by the Principle of uniform boundedness,

$$\|x\| \leq \sup_m \|x_m\| \leq M < \infty$$

hence

$$\begin{aligned}\|Tx_m - Tx\| &\leq \|Tx_m - T_n x_n\| + \|T_n x_m - T_n x\| + \|T_n x - Tx\| \\ &\leq \|T - T_n\| M + \|T_n x_m - T_n x\| + \|T - T_n\| M\end{aligned}$$

The first and third terms can be made small by choosing n sufficiently large. Then the middle term can be made equally small by choosing m sufficiently large, since all T_n are compact. It fallows that

$$\|Tx_m - Tx\| \to 0$$

as m tends to infinity. That is, Tx_n converge strongly to Tx. Thus, T is compact. □

Definition A3.3. (Hilbert Schmidt Kernel)
Let Ω be an open and connected set in $n-$dimensional Euclidean space $\mathbb{R}^n$, a Hilbert–Schmidt kernel is a function $k : \Omega \times \Omega \to \mathbb{C}$ with

$$\int_\Omega \int_\Omega |k(x,y)|^2 \; dx \; dy < \infty.$$

Theorem A3.2. *Let T be an integral operator on $L^2[0,1]$ given by*

$$T(f)(x) = \int_0^1 \int_0^1 k(x,y) f(y) \; dx \; dy$$

where $K \in L^2([0,1] \times [0,1])$ is a Hilbert Schmidt kernel.Then T is a compact operator.

Proof. Let $\{e_n(x)\}$ be an orthonormal basis of $L^2[0,1]$, then $\{e_n(x)e_m(y)\}$ is an orthonormal basis of $L^2[0,1]$. Thus

$$k(x,y) = \sum_{m,n=1}^{\infty} k_{n,m} e_n(x) e_m(y)$$

where

$$k_{n,m} = \int_0^1 \int_0^1 k(x,y) e_n(x) e_m(y) \; dx \; dy.$$

Define

$$T_N(f)(x) = \int_0^1 k_N(x,y) f(y) dy.$$

where

$$k_N(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{\infty} k_{n,m} e_n(x) e_m(y)$$

Note that the range of T_N is generated by $\{e_1, e_2, ..., e_n\}$, so T_N is bounded and has a finite rank, so it is compact.

Note that

$$\begin{aligned}
\|Tf - T_N f\|^2 &= \left\| \int_0^1 (k(x,y) - k_N(x,y) f(y)\ dy \right\|^2 \\
&= \int_0^1 \Big(\int_0^1 (k(x,y) - k_N(x,y) f(y)\ dy \Big)^2 dx \\
&\leq \|f\|^2 \int_0^1 \int_0^1 |k(x,y) - k_N(x,y)|^2\ dy\ dx \\
&= \|f\|^2 \sum_{n=1+N}^{\infty} \sum_{m=1}^{\infty} |k_{n,m}|^2
\end{aligned}$$

since

$$\sum_{n=1+N}^{\infty} \sum_{m=1}^{\infty} |k_{n,m}|^2 \leq \|k\|^2 < \infty$$

Hence as $N \to \infty$, $\|Tf - T_N f\|^2 \to 0$. Thus T is the uniform limit of compact operators, hence it is compact by previous lemma A3.5.

$\square$

Notation Index

The following is a list of symbols used in this dissertation

$\mathbb{R}^3$	n-dimensional real Euclidean space
$\mathbb{R}$	1-dimensional real Euclidean
$\mathbb{C}$	field of complex numbers
$L^2_{\mathbb{C}}[0,1]$	the Hilbert space of all complex-valued square-integrable functions on $[0,1]$
$L^2_{\mathbb{R}}[0,1]$	the Hilbert space of all real-valued square-integrable functions on $[0,1]$
$\mathcal{E}$	supspace of the even functions of $L^2_{\mathbb{R}}[0,1]$ with respect to $x = 1/2$
$\mathcal{O}$	supspace of the odd functions of $L^2_{\mathbb{R}}[0,1]$ with respect to $x = 1/2$
l^2	the Hilbert space of all real real sequences $(\alpha_1, \alpha_2, ...)$ such that $\sum_{n\geq 1} \alpha_n^2$ is finite
l^2_k	the Hilbert space of all real real sequences $(\alpha_1, \alpha_2, ...)$ such that $\sum_{n\geq 1}(n^k\alpha_n)^2$ is finite for $k > 0$
$\sigma(H)$	set of the eigenvalues of the operator H
j_l	Spherical Bessel function of the first type of order l.
η_l	Spherical Bessel function of the second type of order l.
$\dot{f}$	derivative of the function f with respect to λ
f'	derivative of the function f with respect to x
$d_x f$	derivative of a map f between two Banach spaces at a point x
$[f, g]$	the Wronskin of any two differentiable functions f and g

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