2019

BOUNDING THE NUMBER OF COMPATIBLE SIMPLICES IN HIGHER DIMENSIONAL TOURNAMENTS

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Digital Object Identifier: https://doi.org/10.13023/etd.2019.203

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DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

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2019

ABSTRACT OF DISSERTATION

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A tournament graph G is a vertex set V of size n, together with a directed edge set $E \subset V \times V$ such that $(i, j) \in E$ if and only if $(j, i) \notin E$ for all distinct $i, j \in V$ and $(i, i) \notin E$ for all $i \in V$. We explore the following generalization: For a fixed k we orient every k-subset of V by assigning it an orientation. That is, every facet of the $(k-1)$-skeleton of the $(n-1)$-dimensional simplex on V is given an orientation. In this dissertation we bound the number of compatible k-simplices, that is we bound the number of k-simplices such that its $(k-1)$-faces with the already-specified orientation form an oriented boundary. We prove lower and upper bounds for all $k \geq 3$. For $k = 3$ these bounds agree when the number of vertices n is q or $q + 1$ where q is a prime power congruent to 3 modulo 4. We also prove some lower bounds for values $k > 3$ and analyze the asymptotic behavior.

KEYWORDS: Discrete Mathematics, Number Theory

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Date: May 8, 2019

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Dedicated to my wife Deepthi, my & her parents, Kre8Now Makerspace, our dear uncle & aunt in Philly, the 16 US States we toured and our Ford & our Scion.

ACKNOWLEDGMENTS

The authors thank Dr. D. Leep for the proof of Lemma 1.4.4 and Dr. P. Sarnak for directing us to the cross ratio. I also thank Dr. X. Shao for pointing us to results about "primes in short intervals". I thank committee members Dr. M. Readdy, Dr. J. Jaromczyk, Dr. M. Yip, Dr. D. Young and my advisor Dr. R. Ehrenborg. Last but not the least we thank Deepthi.

TABLE OF CONTENTS

LIST OF TABLES

Chapter 1 The number-theoretic and asymptotic views

1.1 Orientations

For a set X of cardinality n define $\Lambda^k(X)$ to be the set of all k-tuples of distinct elements from the set X, that is,

$$\Lambda^k(X) = \{(x_1, x_2, \ldots, x_k) \in X^k \ : \ x_i \neq x_j \text{ for } 1 \leq i < j \leq k\}.$$

An *orientation* s of the set $\Lambda^k(X)$ is a function $s : \Lambda^k(X) \longrightarrow \{1, -1\}$ such that for all permutations $\pi \in \mathfrak{S}_k$ we have $s(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) = (-1)^\pi \cdot s(x_1, x_2, \ldots, x_k)$, where $(-1)^\pi$ denotes the sign of the permutation π. We view $(x_1, x_2, \ldots, x_k)$ as an oriented $(k-1)$-dimensional simplex in the $(k-1)$-dimensional skeleton of the $(n-1)$-dimensional simplex with vertex set X.

The notion of an orientation of $\Lambda^2(X)$ is equivalent to the notion of a tournament T on the complete graph on the vertex set X. That is, given $s(x, y) = 1$, we orient the edge (x, y) to be the directed edge $x \to y$. Szele [23, Equation (28)] proved that for a tournament T on n vertices the number of directed 3-cycles is bounded above by

$$\begin{cases} (n+1) \cdot n \cdot (n-1)/24 & \text{if } n \text{ is odd,} \\ (n+2) \cdot n \cdot (n-2)/24 & \text{if } n \text{ is even.} \end{cases} \tag{1.1.1}$$

See also Clark in the paper by Gale [7, Theorem 4]. For more background, see [14, Section 5]. Furthermore, this bound is sharp, that is for every n there are tournaments that reach this upper bound.

The notion of directed 3-cycles in a tournament can be reformulated as an oriented triangle (x_1, x_2, x_3) such that the induced orientation of its boundary agrees with the tournament, that is, $s(x_1, x_2) = 1$, $s(x_1, x_3) = -1$ and $s(x_2, x_3) = 1$. This notion is inspired by the oriented simplices in the definition of simplicial homology; see [15, Section 5]. Hence we naturally extend it to higher dimensions. Thus given an orientation s of $\Lambda^k(X)$, a k-dimensional simplex $\{x_1, x_2, \ldots, x_{k+1}\}$ is *compatible* with the orientation s if

$$(-1)^i \cdot s(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = (-1)^j \cdot s(x_1, x_2, \ldots, \widehat{x_j}, \ldots, x_{k+1})$$

for $1 \leq i < j \leq k+1$. In other words, the signs of the sequence $s(x_1, x_2, \ldots, x_k)$, $s(x_1, x_2, \ldots, x_{k-1}, x_{k+1})$, $s(x_1, x_2, \ldots, x_{k-2}, x_k, x_{k+1})$, …, $s(x_2, x_3, \ldots, x_{k+1})$ alternate.

Given $k \geq 3$ and a set X of cardinality n, it is natural to ask for an upper bound on the number of oriented k-dimensional simplices whose oriented boundary agrees with the orientation of $\Lambda^k(X)$. In this chapter, we give such an upper bound. Furthermore, we show when $k = 3$ that this bound is sharp when n or $n-1$ is a prime power q congruent to 3 modulo 4.

1.2 The upper bound

In order to prove the upper bound, we introduce the notion of a witness. Given an orientation s of $\Lambda^k(X)$, we call a subfacet $G = \{x_1, x_2, \ldots, x_{k-1}\}$ of a simplex $F = \{x_1, x_2, \ldots, x_{k+1}\}$ a *witness* if $s(x_1, x_2, \ldots, x_{k-1}, x_k) = s(x_1, x_2, \ldots, x_{k-1}, x_{k+1})$. That is, the subfacet G is a witness if the orientation of the two facets $G \cup \{x_k\}$ and $G \cup \{x_{k+1}\}$ implies that F is not compatible with the orientation s.

Lemma 1.2.1. *Given a k-dimensional simplex $F = \{x_1, x_2, \ldots, x_{k+1}\} \subseteq X$, the number of subfacets of F that are witnesses is of the form $a \cdot (k+1-a)$ for $0 \leq i \leq k+1$.*

Proof. Let $F = \{x_1, x_2, \ldots, x_{k+1}\}$. Divide up the $k+1$ facets of F into two subsets according to

$$A^+ = \{\{x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}\} \ : \ s(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = (-1)^{k+1-i}\},$$
$$A^- = \{\{x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}\} \ : \ s(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = -(-1)^{k+1-i}\},$$

and let a be the cardinality of A^+. If $a = 0$ or $a = k+1$ the simplex F matches the orientation s. In these two cases, there are no witnesses. In the remaining cases, $1 \leq a \leq k$, note that the set A^+ forms a $(k-1)$-ball on the boundary of the simplex. Similarly, A^- forms the complementary ball, that is, the boundary of A^+ is the boundary of A^-. Note that subfacets in $\partial A^+ = \partial A^-$ are the witnesses. Since each two facets of F intersect in subfacet, we have $a \cdot (k+1-a)$ witnesses, since $|A^+| = a$ and $|A^-| = k+1-a$. □

Geometrically, the set of witnesses can be seen as the boundary of the projection of a k-simplex onto a hyperplane. For instance, when $k = 3$ this boundary is either a triangle or a quadrilateral, demonstrating the values $3 \cdot 1$ and $2 \cdot 2$, respectively.

Corollary 1.2.2. *Given a k-dimensional simplex $F = \{x_1, x_2, \ldots, x_{k+1}\} \subseteq X$, the number of subfacets of F that are witnesses is bounded above by $\lfloor (k+1)/2 \rfloor \cdot \lceil (k+1)/2 \rceil$.*

Theorem 1.2.3. *Let X be a set of cardinality n. An upper bound for the number of compatible k-dimensional simplices for any orientation of $\Lambda^k(X)$ is given by*

$$\binom{n}{k+1} - \frac{\left(\binom{n-k+1}{2} - \left\lfloor \frac{n-k+1}{2} \right\rfloor \cdot \left\lceil \frac{n-k+1}{2} \right\rceil\right)}{\lfloor (k+1)/2 \rfloor \cdot \lceil (k+1)/2 \rceil} \cdot \binom{n}{k-1}.$$

Proof. Given an orientation s of $\Lambda^k(X)$, let M be the number of k-dimensional simplices that are not compatible with the orientation s. For $0 \leq a \leq \lfloor (k+1)/2 \rfloor$ let

N_a be the number of simplices which has $a \cdot (k+1-a)$ witnesses. A lower bound for M is obtained as follows:

$$\begin{aligned}
\left\lfloor \frac{k+1}{2} \right\rfloor \cdot \left\lceil \frac{k+1}{2} \right\rceil \cdot M &= \sum_{a=1}^{\lfloor \frac{k+1}{2} \rfloor} \left\lfloor \frac{k+1}{2} \right\rfloor \cdot \left\lceil \frac{k+1}{2} \right\rceil \cdot N_a \\
&\geq \sum_{a=0}^{\lfloor \frac{k+1}{2} \rfloor} a \cdot (k+1-a) \cdot N_a \\
&= \sum_{F \in \binom{X}{k+1}} \text{number of witnesses of } F \\
&= \sum_{G \in \binom{X}{k-1}} \left| \left\{ F \in \binom{X}{k+1} \;:\; G \text{ is a witness for } F \right\} \right|. \qquad (1.2.1)
\end{aligned}$$

Given a subfacet $G = \{x_1, x_2, \ldots, x_{k-1}\} \subseteq X$, divide the $n-k+1$ elements of $X - G$ into two classes according to

$$\begin{aligned}
Y^+ &= \{y \in X - G \;:\; s(x_1, x_2, \ldots, x_{k-1}, y) = 1\}, \\
Y^- &= \{y \in X - G \;:\; s(x_1, x_2, \ldots, x_{k-1}, y) = -1\}.
\end{aligned}$$

Then a face F that has G as a witness is constructed as the union $G \cup \{y, z\}$ where the two distinct elements y and z are both from Y^+ or are both from Y^-. That is, we have the inequality

$$\begin{aligned}
&\left| \left\{ F \in \binom{X}{k+1} \;:\; G \text{ is a witness for } F \right\} \right| \\
&= \binom{|Y^+|}{2} + \binom{|Y^-|}{2} \\
&= \binom{n-k+1}{2} - |Y^+| \cdot |Y^-| \\
&\geq \binom{n-k+1}{2} - \left\lfloor \frac{n-k+1}{2} \right\rfloor \cdot \left\lceil \frac{n-k+1}{2} \right\rceil. \qquad (1.2.2)
\end{aligned}$$

Now combining the two inequalities (1.2.1) and (1.2.2) we obtain

$$M \geq \frac{\left(\binom{n-k+1}{2} - \left\lfloor \frac{n-k+1}{2} \right\rfloor \cdot \left\lceil \frac{n-k+1}{2} \right\rceil \right)}{\lfloor (k+1)/2 \rfloor \cdot \lceil (k+1)/2 \rceil} \cdot \binom{n}{k-1}$$

Since the number of simplices that are compatible with the orientation s is given by $\binom{n}{k+1} - M$, we obtain the desired lower bound. □

Corollary 1.2.4. *Let X be a set of cardinality n. An upper bound for the number of oriented 3-dimensional simplices whose boundary agrees with an orientation of $\Lambda^3(X)$ is given by*

$$\begin{cases} \lfloor n^2 \cdot (n-1) \cdot (n-2)/96 \rfloor & \textit{if } n \textit{ even}, \\ \lfloor (n+1) \cdot n \cdot (n-1) \cdot (n-3)/96 \rfloor & \textit{if } n \textit{ odd}. \end{cases}$$

The upper bound in Corollary 1.2.4 is not always sharp. When $n = 6$ it yields an upper bound of 7, whereas the next lemma shows that 6 is an upper bound. Later on, we will give a construction showing that 6 is sharp.

Lemma 1.2.5. *Let X be a set of cardinality n and assume $n \geq k+2$. Then the number of compatible k-simplices for an orientation of $\Lambda^k(X)$ is at most*

$$\frac{2 \cdot \binom{n}{k+2}}{n-k-1}.$$

Proof. We begin to prove this upper bound for the case $n = k+2$ where the upper bound is 2. Without loss of generality we assume that $\{x_1, x_2, \ldots, x_k, x_{k+1}\}$ and $\{x_1, x_2, \ldots, x_k, x_{k+2}\}$ are two compatible simplices. Hence we have

$$s(x_1, \ldots, \widehat{x_i}, \ldots, x_k, x_{k+1}) = (-1)^{k-i+1} \cdot s(x_1, x_2, \ldots, x_k) = s(x_1, \ldots, \widehat{x_i}, \ldots, x_k, x_{k+2}).$$

This implies that $G = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k\}$ is a witness that $G \cup \{x_{k+1}, x_{k+2}\}$ is not a compatible simplex, just leaving us with two compatibles simplices.

Now assume $n \geq k+3$ and let s be an orientation of $\Lambda^k(X)$. Consider p, the number of pairs (F, Z) where F is a compatible k-simplex to the orientation s and Z is a $(k+2)$-element set such that $F \subseteq Z \subseteq X$. By first choosing Z in $\binom{n}{k+2}$ ways, and then picking a compatible k-simplex $F \subseteq Z$ in at most 2 ways by the previous argument, we obtain $p \leq 2 \cdot \binom{n}{k+2}$. However, since a compatible k-simplex is contained in exactly $n-k-1$ $(k+2)$-element sets, we have that the number of compatible simplices is bounded above by $p/(n-k-1) \leq 2 \cdot \binom{n}{k+2}/(n-k-1)$. □

1.3 A polynomial lower bound

We now give a lower bound for the maximal possible number of compatible simplices. We begin by first weakening the conditions of an orientation. A *partial orientation* s of the set $\Lambda^k(X)$ is a function $s : \Lambda^k(X) \longrightarrow \{0, 1, -1\}$ such that for all permutations $\pi \in \mathfrak{S}_k$ we have $s(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(k)}) = (-1)^{\pi} \cdot s(x_1, x_2, \ldots, x_k)$. We view the $(k-1)$-dimensional faces that get assigned the value 0 as not yet being assigned an orientation. Furthermore, a k-dimensional simplex $\{x_1, x_2, \ldots, x_{k+1}\}$ is *compatible* with the partial orientation s if

$$(-1)^i \cdot s(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_{k+1}) = (-1)^j \cdot s(x_1, x_2, \ldots, \widehat{x_j}, \ldots, x_{k+1}) \neq 0$$

for $1 \leq i < j \leq k+1$.

Let X be the set $\{1, 2, \ldots, n\}$ and define two partial orientations κ_0 and κ_1 by the following, where $x = (x_1, x_2, \ldots, x_k)$:

$$\kappa_0(x) = \begin{cases} 1 & \text{if } x_1 \not\equiv x_2 \not\equiv \cdots \not\equiv x_k \bmod 2, \\ (-1)^i & \text{if } x_1 \not\equiv x_2 \not\equiv \cdots \not\equiv x_i \equiv x_{i+1} \not\equiv x_{i+2} \not\equiv \cdots \not\equiv x_k \bmod 2, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3.1}$$

$$\kappa_1(x) = \begin{cases} 1 & \text{if } 1 \equiv x_1 \not\equiv x_2 \not\equiv \cdots \not\equiv x_k \bmod 2, \\ (-1)^i & \text{if } 0 \equiv x_1 \not\equiv x_2 \not\equiv \cdots \not\equiv x_i \equiv x_{i+1} \not\equiv x_{i+2} \not\equiv \cdots \not\equiv x_k \bmod 2, \\ -1 & \text{if } 0 \equiv x_1 \not\equiv x_2 \not\equiv \cdots \not\equiv x_k \bmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.2}$$

We begin by studying the case when k is even.

Lemma 1.3.1. *For k even, the number of k-simplices that are compatible with the partial orientation κ_0 in equation* (1.3.1) *is given by*

$$\begin{cases} 2 \cdot \binom{(n+k)/2}{k+1} & \text{if } n \equiv 0 \bmod 2, \\ \binom{(n+k+1)/2}{k+1} + \binom{(n+k-1)/2}{k+1} & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

This function is a quasi-polynomial of period 2 *in the variable n where the leading term is $n^{k+1}/(2^k \cdot (k+1)!)$.*

Proof. Note that a simplex $\{y_1 < y_2 < \cdots < y_{k+1}\}$ is compatible with the partial orientation κ_0 if and only if the parities of $y_1, y_2, \ldots, y_{k+1}$ alternate. Thus the k-simplex $\{y_1 < y_2 < \cdots < y_{k+1}\}$ is compatible with κ_0 if and only if all the differences $y_2 - y_1, y_3 - y_2, \ldots, y_{k+1} - y_k$ are odd. Hence by the bijection between subsets of $\{1, 2, \ldots, n\}$ and compositions of $n+1$, we are interested in the number of compositions $(c_1, c_2, \ldots, c_{k+2})$ of $n+1$ into $k+2$ parts, so that $c_2, c_3, \ldots, c_{k+1}$ are odd. Call this number $\alpha_{n,k}$. The generating function is given by

$$\begin{aligned} \sum_{n \geq k+1} \alpha_{n,k} \cdot t^{n+1} &= \frac{t}{1-t} \cdot \left(\frac{t}{1-t^2}\right)^k \cdot \frac{t}{1-t} \\ &= t^{k+2} \cdot (1+t)^2 \cdot \frac{1}{(1-t^2)^{k+2}} \\ &= t^{k+2} \cdot (1+t)^2 \cdot \sum_{j \geq 0} \binom{k+j+1}{k+1} \cdot t^{2j}. \end{aligned}$$

The result follows now by comparing the coefficients for t^{n+1}. □

Lemma 1.3.2. *For k odd, the number of k-simplices that are compatible with the partial orientation κ_1 in equation* (1.3.2) *is given by*

$$\binom{\lfloor (n+k)/2 \rfloor}{k+1}.$$

This function is a quasi-polynomial of period 2 *in the variable* n *where the leading term is* $n^{k+1}/(2^{k+1}\cdot(k+1)!)$.

Proof. Note that a simplex $\{y_1 < y_2 < \cdots < y_{k+1}\}$ is compatible with the partial orientation κ_1 if and only if y_1 is even and the parities of $y_1, y_2, \ldots, y_{k+1}$ alternate. Hence the proof differs from that of Lemma 1.3.1 in that we also need to assume that the first part is even. The generating function is given by

$$\sum_{n\geq k+1} \alpha_{n,k}\cdot t^{n+1} = \frac{t^2}{1-t^2}\cdot\left(\frac{t}{1-t^2}\right)^k\cdot\frac{t}{1-t} = t^{k+3}\cdot(1+t)\cdot\sum_{j\geq 0}\binom{k+j+1}{k+1}\cdot t^{2j}.$$

The result follows again by comparing the coefficients for t^{n+1}. □

Let $\mathrm{Comp}(k)$ be the collection of all compositions of k, that is, all vectors of positive integer entries, such that that their sum is k. For a vector $x = (x_1, x_2, \ldots, x_k)$ of k integers define its *type modulo* a to be the composition of k consisting of lengths of runs of congruence classes modulo a. More formally, we define $\mathrm{type}_a(x) = \vec{c} = (c_1, c_2, \ldots, c_r)$ where

$$x_{c_i+1} \equiv x_{c_i+2} \equiv \cdots \equiv x_{c_{i+1}} \bmod a$$

for all $1 \leq i \leq r$ and $x_{c_i} \not\equiv x_{c_i+1} \bmod a$ for $1 \leq i \leq r-1$.

Define $\mathrm{Comp}^{\mathrm{odd},\geq 3}(k)$ to be all compositions of k such that all entries are odd and all the entries are at least 3. More formally, we have

$$\begin{aligned}\mathrm{Comp}^{\mathrm{odd},\geq 3}(k) = \{(c_1, c_2, \ldots, c_r) \in \mathrm{Comp}(k) \ : \ & c_1, c_2, \ldots, c_r \text{ are all odd} \\ & \text{and } c_1, c_2, \ldots, c_r \geq 3\}.\end{aligned}$$

Let $\vec{e}_i$ denote the i^{th} unit vector, that is, $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Also introduce the short hand $x_{[a,b]}$ to denote the vector $(x_a, x_{a+1}, \ldots, x_b)$.

We now extend the partial orientations κ_0 and κ_1 in equations (1.3.1) and (1.3.2) by equation (1.3.3), where we let $\vec{c} = (c_1, c_2, \ldots, c_r)$ be the composition $\mathrm{type}_2(x)$ and I_j be the interval $[c_1 + \cdots + c_{j-1} + 1, c_1 + \cdots + c_{j-1} + c_j]$. Furthermore, let $\circ$ denote concatenation of compositions and hence $(1)^c$ denotes the composition $(1, 1, \ldots, 1)$ of the positive integer c. Finally, let i be the unique index such that the part c_i is even. Define

$$\lambda_p(x) = \begin{cases} \kappa_p(x) & \text{if } \kappa_p(x) \neq 0, \\ (-1)^{i-1} & \text{if } \mathrm{type}_4(x_{I_\ell}) = (1)^{c_\ell} \text{ for all } 1 \leq \ell \leq r, \\ (-1)^{i-1+j} & \text{if } \mathrm{type}_4(x_{I_\ell}) = (1)^{c_\ell} \text{ for all } \ell \neq i, 1 \leq \ell \leq r \\ & \text{and } \mathrm{type}_4(x_{I_i}) = (1)^{j-1} \circ (2) \circ (1)^{c_i-j-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.3}$$

Proposition 1.3.3. *The number of compatible k-simplices with the orientation λ_p in equation* (1.3.3) *is of order*

$$\begin{cases} \left(2^{-k} + \alpha_k \cdot 2^{-2k-1}\right) \cdot n^{k+1}/(k+1)! + O(n^k) & \text{if } k \equiv 0 \bmod 2, \\ \left(2^{-k-1} + \alpha_k \cdot 2^{-2k-1}\right) \cdot n^{k+1}/(k+1)! + O(n^k) & \text{if } k \equiv 1 \bmod 2, \end{cases}$$

where the constants α_k are given by the generating function

$$\sum_{k \geq 0} \alpha_k \cdot u^{k+1} = (1-u^2)/(1-u^2-2u^3)$$
$$= 1 + 2u^3 + 2u^5 + 4u^6 + 2u^7 + 8u^8 + 10u^9 + 12u^{10} + \cdots.$$

Proof. The compatible simplices to the partial orientation λ_p that are not compatible to κ_p have the form $y = (y_1, y_2, \ldots, y_{k+1})$ where $y_1 < y_2 < \cdots < y_{k+1}$, $\text{type}_2(y) \in \text{Comp}^{\text{odd},\geq 3}(k+1)$ and $\text{type}_4(y) = (1)^{k+1}$. Thus given a composition $\vec{d} = (d_1, d_2, \ldots, d_r)$ in $\text{Comp}^{\text{odd},\geq 3}(k+1)$ the generating function for the number of compatible k-simplices y such that $\text{type}_2(y) = \vec{d}$ is given by

$$A(t) = \frac{t}{1-t} \cdot \left(\frac{t^2}{1-t^4}\right)^{d_1-1} \cdot \frac{t+t^3}{1-t^4} \cdot \left(\frac{t^2}{1-t^4}\right)^{d_2-1} \cdot \frac{t+t^3}{1-t^4}$$
$$\cdots \frac{t+t^3}{1-t^4} \cdot \left(\frac{t^2}{1-t^4}\right)^{d_r-1} \cdot \frac{t}{1-t}.$$

Collecting factors in this expression yields

$$A(t) = \frac{t^{r+1} \cdot (1+t^2)^{r-1}}{(1-t)^2 \cdot (1-t^4)^k} = \frac{1}{(1-t)^{k+2}} \cdot \frac{t^{r+1} \cdot (1+t^2)^{r-1}}{(1+t+t^2+t^3)^k}.$$

Note that

$$A(t) \sim (1-t)^{-k-2} \cdot 2^{r-1-2k} \quad \text{as} \quad t \longrightarrow 1^-.$$

Now applying the Hardy–Littlewood–Karamata Tauberian theorem (see for instance Theorem 9 in [2]), we obtain

$$[t^n]A(t) \sim \frac{2^{r-1-2k} \cdot n^{k+1}}{(k+1)!} \quad \text{as} \quad n \longrightarrow \infty.$$

It remains to sum over all compositions $\vec{d} = (d_1, d_2, \ldots, d_r)$ in $\text{Comp}^{\text{odd},\geq 3}(k+1)$. Note that the generating function is given by

$$\sum_{\vec{d} \in \text{Comp}^{\text{odd},\geq 3}(k+1)} 2^r \cdot u^{k+1} = \frac{1}{1 - 2 \cdot \frac{u^3}{1-u^2}} = \frac{1-u^2}{1-u^2-2u^3}. \qquad \square$$

We believe that the lower bound for the maximal number of compatible k-dimensional simplices in Proposition 1.3.3 is far from the truth. In the construction of this proposition the compatible k-simplices do not share any facets. In order to obtain better bounds sharing of facets has to occur. In the remainder of this chapter, we will concentrate on the $k = 3$ case.

1.4 The Legendre symbol over finite fields and Möbius transformations

For a prime power q let F_q denote the finite field of order q. We begin recalling the Legendre symbol for fields of odd order. Call a non-zero square in the field a *quadratic residue*. Similarly, a non-zero non-square is called a *quadratic non-residue*.

Definition 1.4.1. *For an odd prime power q, and a an element in the field F_q, define*

$$\left(\frac{a}{q}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue in } F_q, \\ -1 & \text{if } a \text{ is a quadratic non-residue in } F_q, \\ 0 & \text{if } a = 0. \end{cases}$$

To begin we note that the Legendre symbol is multiplicative.

Lemma 1.4.2. *For a and b in the field F_q of odd order, the following hold:*

$$\left(\frac{a \cdot b}{q}\right) = \left(\frac{a}{q}\right) \cdot \left(\frac{b}{q}\right)$$
$$\left(\frac{a \cdot b^{-1}}{q}\right) = \left(\frac{a}{q}\right) \cdot \left(\frac{b}{q}\right) \quad b \neq 0$$

Proof. The first identity is directly true if $a = 0$ or $b = 0$. Otherwise, let γ be a primitive element in the field, that is, the order of γ is $q - 1$. Now we observe that $\left(\frac{\gamma^k}{q}\right) = (-1)^k$ from which the multiplicative statement follows. The second statement follows from $\left(\frac{\gamma^k}{q}\right) = (-1)^k = (-1)^{-k} = \left(\frac{\gamma^{-k}}{q}\right)$. □

Lemma 1.4.3. *Let q be an odd prime power. The element -1 is quadratic residue in the field F_q if and only if $q \equiv 1 \bmod 4$.*

Proof. Let γ be a primitive element in the field F_q. Note that $(\gamma^{(q-1)/2})^2 = 1$. Since γ has order $q - 1$, we conclude that $\gamma^{(q-1)/2} = -1$. If $q \equiv 1 \bmod 4$ then $(q-1)/2$ is even and hence -1 is a quadratic residue. If $q \equiv 3 \bmod 4$ then $(q-1)/2$ is odd and hence -1 is not a quadratic residue. □

Another way to state this lemma is $\left(\frac{-1}{q}\right) = (-1)^{(q-1)/2}$.

Lemma 1.4.4. *Let $q = p^k$ be an odd prime power. The element 2 is a quadratic residue in the field F_q if and only if $q \equiv \pm 1 \bmod 8$.*

The following proof was communicated to us by David Leep.

Proof of Lemma 1.4.4. We will work in F_{q^2}, the unique quadratic extension of F_q. The multiplicative group of F_{q^2} is a cyclic group of order $q^2 - 1$. Since q is odd, we have $q^2 - 1$ is divisible by 8. Let β be a generator of this cyclic group and let $\alpha = \beta^{(q^2-1)/8}$. Thus the element α has order 8. We observe that $(\alpha + \alpha^{-1})^2 = \alpha^2 + \alpha^{-2} + 2 = \alpha^{-2} \cdot (\alpha^4 + 1) + 2 = 2$, because $\alpha^4 = -1$. Thus $\pm(\alpha + \alpha^{-1})$ are the two square roots of 2 in the field F_{q^2}.

The question becomes: When is $\alpha + \alpha^{-1}$ an element of F_q? We have $\alpha + \alpha^{-1} \in F_q$ if and only if $(\alpha + \alpha^{-1})^q = \alpha + \alpha^{-1}$. Since q is a power of p, the characteristic of our fields, we have that $(\alpha + \alpha^{-1})^q = \alpha^q + \alpha^{-q}$. If $q \equiv \pm 1 \bmod 8$, then $\alpha^q + \alpha^{-q} = \alpha + \alpha^{-1}$ since α has order 8. If $q \equiv 3, 5 \bmod 8$, then we have $\alpha^q + \alpha^{-q} = \alpha^3 + \alpha^5 = \alpha^4 \cdot (\alpha + \alpha^{-1}) = -(\alpha + \alpha^{-1})$. $\square$

Let F be a field. Recall that a *Möbius transformation* is a function on the set $F \cup \{\infty\}$ to itself of the form

$$f(z) = \frac{a \cdot z + b}{c \cdot z + d},$$

where $a, b, c, d \in F$ and $a \cdot d \neq b \cdot c$. Note that we define $f(\infty) = a/c$ and $f(-d/c) = \infty$. For more on Möbius transformations on finite fields, see the book [6, Section 3.1]. The set of Möbius transformations forms the group $\mathrm{PGL}_2(F)$. When the field F is finite of order q the group is denoted by $\mathrm{PGL}_2(q)$ and has order equalling $(q+1) \cdot q \cdot (q-1)$.

Let G be the set $\{x, 1-x, 1/(1-x), x/(x-1), (x-1)/x, 1/x\}$ consisting of Möbius transformations. Observe that G is closed under composition, that is, G is a subgroup of $\mathrm{PGL}_2(F)$. In fact, G is isomorphic to the symmetric group on three elements since G permutes the three elements $\{0, 1, \infty\}$.

Definition 1.4.5. *The cross-ratio of a 4-tuple $(a, b, c, d) \in \Lambda^4(F)$ is defined by*

$$\mathrm{Cr}(a, b, c, d) = \frac{(a-b) \cdot (c-d)}{(b-c) \cdot (d-a)}.$$

We extend this definition to $\Lambda^4(F \cup \{\infty\})$ by

$$\begin{aligned} \mathrm{Cr}(\infty, b, c, d) &= -(c-d)/(b-c), \\ \mathrm{Cr}(a, \infty, c, d) &= -(c-d)/(d-a), \\ \mathrm{Cr}(a, b, \infty, d) &= -(a-b)/(d-a) \end{aligned}$$

and

$$\mathrm{Cr}(a,b,c,\infty) = -(a-b)/(b-c).$$

Note that the cross-ratio is invariant under the action of two disjoint 2-cycles to its entries:

$$\mathrm{Cr}(a,b,c,d) = \mathrm{Cr}(b,a,d,c) = \mathrm{Cr}(c,d,a,b) = \mathrm{Cr}(d,c,b,a). \tag{1.4.1}$$

Lemma 1.4.6. *Assume that* $\mathrm{Cr}(a_1,a_2,a_3,a_4) = x$. *Then for any permutation* $\pi \in \mathfrak{S}_4$ *we have* $\mathrm{Cr}(a_{\pi(1)},a_{\pi(2)},a_{\pi(3)},a_{\pi(4)}) \in \{x, 1-x, 1/(1-x), x/(x-1), (x-1)/x, 1/x\}$. *In fact, this mapping provides an epimorphism from the symmetric group* $\mathfrak{S}_4$ *to* $\mathfrak{S}_3$.

Proof. Observe that $\mathrm{Cr}(a_2,a_3,a_4,a_1) = 1/x$ and $\mathrm{Cr}(a_1,a_3,a_2,a_4) = 1-x$. Since $\mathfrak{S}_4$ is generated by $(2,3,4,1)$ and $(2,3)$, the result follows. □

A classic result is the following statement connecting Möbius transformations and the cross-ratio.

Theorem 1.4.7. *The cross-ratio is invariant under Möbius transformations, that is,*

$$\mathrm{Cr}(f(a), f(b), f(c), f(d)) = \mathrm{Cr}(a,b,c,d)$$

for any Möbius transformation f *and* a,b,c,d *in* $F \cup \{\infty\}$.

Lemma 1.4.8. *Given* $k+1$ *elements* $a_1, a_2, \ldots, a_k, e$ *in* $F_q \cup \{\infty\}$, *the number of Möbius transformations* f *in* $\mathrm{PGL}_2(q)$ *such that* $f(a_i) \neq e$ *for all* $1 \leq i \leq k$ *is given by* $q \cdot (q-1) \cdot (q+1-k)$.

Proof. The number of Möbius transformations g such that $g(e) = a_i$, for a given index i, is $q \cdot (q-1)$. Hence the number of Möbius transformations g such that $g(e) \notin \{a_1, a_2, \ldots, a_k\}$ is given by $q \cdot (q-1) \cdot (q+1-k)$. By letting f to be the compositional inverse of g, the result follows. □

1.5 Construction for $k = 3$ where $n-1$ is an odd prime power

It remains now to show that the bound given in Corollary 1.2.4 is sharp when n or $n-1$ is a prime power congruent 3 modulo 4. We begin with the case $n = q+1$, where q is a prime power. We now introduce the orientation on set $\Lambda^3(X)$ where $X = F_q \cup \{\infty\}$.

Definition 1.5.1. *For an odd prime power q define the* VL *function σ_q on $\Lambda^3(F_q \cup \{\infty\})$ by the following four Legendre symbols*

$$\sigma_q(a,b,c) = \begin{cases} \left(\frac{(a-b)\cdot(b-c)\cdot(a-c)}{q}\right) & \text{if } a,b,c \in F_q, \\ \left(\frac{b-c}{q}\right) & \text{if } b,c \in F_q \text{ and } a = \infty, \\ \left(\frac{c-a}{q}\right) & \text{if } a,c \in F_q \text{ and } b = \infty, \\ \left(\frac{a-b}{q}\right) & \text{if } a,b \in F_q \text{ and } c = \infty. \end{cases}$$

Lemma 1.5.2. *Let q be an odd prime power. If $q \equiv 1 \bmod 4$ then the VL function σ_q is symmetric. If $q \equiv 3 \bmod 4$ then the VL function σ_q is anti-symmetric, that is, σ_q is an orientation.*

Proof. When $q \equiv 1 \bmod 4$ we have $\left(\frac{-1}{q}\right) = 1$. The VL function σ_q is then symmetric, that is, when transposing two variables, the function σ_q does not change sign. When $q \equiv 3 \bmod 4$ we have $\left(\frac{-1}{q}\right) = -1$. The VL function σ_q is then anti-symmetric. □

Hereafter, in the case of $q \equiv 3 \bmod 4$ we will call the VL function σ_q a VL orientation. The last three cases in Definition 1.5.1 should be understood by viewing the element ∞^2 as a quadratic residue. For instance, when $a = \infty$ we have

$$\left(\frac{(a-b)\cdot(b-c)\cdot(a-c)}{q}\right) = \left(\frac{(b-c)\cdot\infty^2}{q}\right) = \left(\frac{b-c}{q}\right).$$

Similarly, when $b = \infty$ we have

$$\left(\frac{(a-b)\cdot(b-c)\cdot(a-c)}{q}\right) = \left(\frac{-(a-c)\cdot\infty^2}{q}\right) = \left(\frac{c-a}{q}\right).$$

Lemma 1.5.3. *For an odd prime power q, for the VL function $\sigma = \sigma_q$ and for four distinct elements $a,b,c,d \in F_q \cup \{\infty\}$ the following identity holds*

$$\sigma(a,b,c)\cdot\sigma(b,c,d) = \left(\frac{\mathrm{Cr}(b,a,c,d)}{q}\right).$$

Proof. We begin with the case when all the elements a, b, c, d belong to the field F_q:

$$\begin{aligned}
\sigma(a,b,c)\cdot\sigma(b,c,d) &= \left(\frac{(a-b)\cdot(b-c)\cdot(a-c)}{q}\right)\cdot\left(\frac{(b-c)\cdot(b-d)\cdot(c-d)}{q}\right)\\
&= \left(\frac{\frac{(b-a)\cdot(c-d)}{(a-c)\cdot(d-b)}}{q}\right)\\
&= \left(\frac{\mathrm{Cr}(b,a,c,d)}{q}\right).
\end{aligned}$$

Next we have the four cases when one of elements is equal to ∞. First when $a=\infty$ we have:

$$\begin{aligned}
\sigma(\infty,b,c)\cdot\sigma(b,c,d) &= \left(\frac{(b-c)}{q}\right)\cdot\left(\frac{(b-c)\cdot(b-d)\cdot(c-d)}{q}\right)\\
&= \left(\frac{-\frac{c-d}{d-b}}{q}\right)\\
&= \left(\frac{\mathrm{Cr}(b,\infty,c,d)}{q}\right).
\end{aligned} \tag{1.5.1}$$

Second, when b is equal to ∞:

$$\sigma(a,\infty,c)\cdot\sigma(\infty,c,d) = \left(\frac{c-a}{q}\right)\cdot\left(\frac{c-d}{q}\right) = \left(\frac{-\frac{c-d}{a-c}}{q}\right) = \left(\frac{\mathrm{Cr}(\infty,a,c,d)}{q}\right). \tag{1.5.2}$$

When c is equal to ∞ we have $\sigma(a,b,\infty)\cdot\sigma(b,\infty,d) = \sigma(d,\infty,b)\cdot\sigma(\infty,b,a) = \left(\frac{\mathrm{Cr}(\infty,d,b,a)}{q}\right) = \left(\frac{\mathrm{Cr}(b,a,\infty,d)}{q}\right)$, by using (1.5.2) and (1.4.1). Similarly, when $d=\infty$ we have $\sigma(a,b,c)\cdot\sigma(b,c,\infty) = \sigma(\infty,c,b)\cdot\sigma(c,b,a) = \left(\frac{\mathrm{Cr}(c,\infty,b,a)}{q}\right) = \left(\frac{\mathrm{Cr}(b,a,c,\infty)}{q}\right)$ by using (1.5.1) and again (1.4.1). □

Proposition 1.5.4. *Let q be a prime power such that $q \equiv 3 \bmod 4$. Let S be a subset of the finite field F_q such that S consists of quadratic non-residues and that S is closed under the action of the group $G = \{x, 1-x, 1/(1-x), x/(x-1), (x-1)/x, 1/x\}$. Then every simplex $\{a,b,c,d\} \subseteq F_q \cup \{\infty\}$ such that $\mathrm{Cr}(a,b,c,d) \in S$ is compatible with the VL orientation σ_q.*

Proof. Let $\{a,b,c,d\}$ be a 3-dimensional simplex such that $\mathrm{Cr}(a,b,c,d) \in S$. Since S is closed under operations of the group G of order 6 we have $\mathrm{Cr}(b,a,c,d) \in S$. Especially, $\left(\frac{\mathrm{Cr}(b,a,c,d)}{q}\right) = -1$. Now by Lemma 1.5.3 we have $\sigma(a,b,c)\cdot\sigma(b,c,d) = -1$, that is, $\sigma(a,b,c) = -\sigma(b,c,d)$. Also $\sigma(a,b,c) = -\sigma(d,a,b) = -\sigma(a,b,d)$ and $\sigma(a,b,c) = \sigma(b,c,a) = -\sigma(c,a,d) = \sigma(a,c,d)$. The last three identities show that $\{a,b,c,d\}$ is compatible. □

Proposition 1.5.5. *Let q be a prime power such that $q \equiv 3 \bmod 4$. Then there exists a subset S of the finite field F_q such that S consists of quadratic non-residues, S is closed under the action of the group $G = \{x, 1-x, 1/(1-x), x/(x-1), (x-1)/x, 1/x\}$ and the cardinality of S is given by $(q+1)/4$.*

Proof. Consider the action of G on the set $F_q \cup \{\infty\}$ and the orbits of this action. The set S is a disjoint union of these orbits. However, we need to select orbits that consist of quadratic non-residues only. We begin to discuss all the orbits.

(i) The set $\{0, 1, \infty\}$ is an orbit. Note that this orbit has one element which is a quadratic residue and no quadratic non-residues.

(ii) The set $\{2, 1/2, -1\}$ is an orbit. Since $\left(\frac{-1}{q}\right) = -1$, the element -1 is a quadratic non-residue. By Lemma 1.4.4 we have that the elements 2 and $1/2$ are quadratic non-residues if $q \equiv 3 \bmod 8$ and quadratic residues if $q \equiv 7 \bmod 8$. Furthermore, if q is a power of 3, then this orbit consists of only one element, that is the case $q \equiv 3 \bmod 24$.

(iii) If $q \equiv 1 \bmod 6$ then there two primitive 6^{th} roots of unity: $\gamma^{(q-1)/6} = \omega$ and ω^5. The set $\{\omega, \omega^5\}$ forms an orbit since these elements are the roots of the polynomial $x^2 - x + 1$, the 6^{th} cyclotomic polynomial. Note that the condition $q \equiv 3 \bmod 4$ implies that $q \equiv 7 \bmod 12$. Hence $(q-1)/6$ is always odd and we conclude that when this orbit occurs it consists only of quadratic non-residues.

(iv) Finally, there are orbits of size 6. Note that $-1 = x \cdot 1/(1-x) \cdot (x-1)/x$. Hence
$$-1 = \left(\frac{x}{q}\right) \cdot \left(\frac{1/(1-x)}{q}\right) \cdot \left(\frac{(x-1)/x}{q}\right).$$
Thus among the three elements x, $1/(1-x)$ and $(x-1)/x$ there are one or three quadratic non-residues. Hence among the elements of the orbit x, $1/x$, $1/(1-x)$, $1-x$, $(x-1)/x$ and $x/(x-1)$ there are two or six quadratic non-residues. We will assume among these orbits that there are $m_{0,6}$ orbits $C_{0,6}$ containing six quadratic non-residues and $m_{4,2}$ orbits $C_{4,2}$ containing two quadratic non-residues.

Note that there is no prime power such that $q \equiv 15 \bmod 24$ since it would imply that 3 divides q and hence that q is 3 to an odd power, which implies that $q \equiv 3 \bmod 24$, a contradiction.

The proof now consists of the five cases $q \equiv 3, 7, 11, 19, 23 \bmod 24$. The proof is summarized in Table 1.1. First, the possible orbits are listed with how many quadratic residues versus the non-residues. Then using the fact that there are $(q-1)/2$ quadratic residues, we can solve for $m_{4,2}$, the number of orbits $C_{4,2}$ with four

Table 1.1: The possible orbits of G when acting on $F_q \cup \{\infty\}$ in the case when $q \equiv 3 \bmod 4$. The number of orbits of type $C_{0,6}$ is $m_{0,6}$ and the number of orbits of type $C_{4,2}$ is $m_{4,2}$. Finally, observe that the set S has cardinality $(q+1)/4$ in each of the five cases.

	3 mod 24		7 mod 24		11 mod 24		19 mod 24		23 mod 24	
	qr	qnr	qr	qnr	qr	qnr	qr	qnr	qr	qnr
$\{0,1,\infty\}$	1	0	1	0	1	0	1	0	1	0
$\{2,1/2,-1\}$	0	1	2	1	0	3	0	3	2	1
$\{\omega,\omega^5\}$	0	0	0	2	0	0	0	2	0	0
$C_{0,6}$	0	6	0	6	0	6	0	6	0	6
$C_{4,2}$	4	2	4	2	4	2	4	2	4	2
$m_{4,2}$	$(q-3)/8$		$(q-7)/8$		$(q-3)/8$		$(q-3)/8$		$(q-7)/8$	
$m_{0,6}$	$(q-3)/24$		$(q-7)/24$		$(q-11)/24$		$(q-19)/24$		$(q+1)/24$	
$\lvert S\rvert$	$(q+1)/4$		$(q+1)/4$		$(q+1)/4$		$(q+1)/4$		$(q+1)/4$	

quadratic residues. Using the value of $m_{4,2}$ and that there are $(q-1)/2$ quadratic non-residues we can solve for $m_{0,6}$. Finally, the cardinality of S, that is, the union of the orbits of consisting only quadratic non-residues is determined. In all five cases we obtain $|S| = (q+1)/4$. □

By combining Propositions 1.5.4 and 1.5.5, we obtain

Theorem 1.5.6. *Let q be a prime power such that $q \equiv 3 \bmod 4$. Then there are*

$$\frac{(q+1)^2 \cdot q \cdot (q-1)}{96}$$

3-dimensional simplices compatible with the VL orientation σ_q of the set $\Lambda^3(F_q \cup \{\infty\})$.

Proof. There are $\binom{q+1}{3}$ three-element subsets $\{a,b,c\}$ of $F_q \cup \{\infty\}$. There are exactly $(q+1)/4$ ways to pick an element t in the set S given in Proposition 1.5.5. Now we can solve for the unique d in the equation $\mathrm{Cr}(a,b,c,d) = t$, yielding the simplex $\{a,b,c,d\}$. However, this simplex is obtained in four different ways depending on which 2-dimensional face we started with. Hence the number of simplices is $\binom{q+1}{3} \cdot (q+1)/4 \cdot 1/4$. □

Lemma 1.5.7. *The VL orientation σ_q is uniform in the following sense:*

(i) Every vertex $a \in F_q \cup \{\infty\}$ lies in $(q+1) \cdot q \cdot (q-1)/24$ compatible 3-simplices.

(ii) Every edge $\{a,b\} \subseteq F_q \cup \{\infty\}$ lies in $(q+1) \cdot (q-1)/8$ compatible 3-simplices.

(iii) *Every triangle* $\{a, b, c\} \subseteq F_q \cup \{\infty\}$ *lies in* $(q+1)/4$ *compatible* 3*-simplices.*

Proposition 1.5.8. *Let* q *be a prime power such that* $q \equiv 3 \bmod 4$ *and let* σ_q *be the VL orientation. Assume that* $X = F_q \cup \{\infty\}$ *is the disjoint union of the two sets* Y *and* Z *where the* Y *has cardinality* k*. Let* M *be the number of compatible* 3*-simplices in the restriction* $\Lambda^3(Y)$*. Then the number of compatible* 3*-simplices of the orientation* σ_q *restricted to* $\Lambda^3(Z)$ *is given by*

$$\frac{(q+1)^2 \cdot q \cdot (q-1)}{96} - \frac{(q+1) \cdot q \cdot (q-1)}{24} \cdot k + \frac{(q+1) \cdot (q-1)}{8} \cdot \binom{k}{2} - \frac{(q+1)}{4} \cdot \binom{k}{3} + M. \tag{1.5.3}$$

Proof. The proof is by inclusion-exclusion. □

Corollary 1.5.9. *Let* q *be a prime power such that* $q \equiv 3 \bmod 4$*. Then the number of compatible* 3*-simplices of the VL orientation* σ_q *restricted to* $\Lambda^3(F_q)$ *is equal to upper bound given in Corollary 1.2.4.*

Proof. It is enough to observe

$$\frac{(q+1)^2 \cdot q \cdot (q-1)}{96} - \frac{(q+1) \cdot q \cdot (q-1)}{24} = \frac{(q+1) \cdot q \cdot (q-1) \cdot (q-3)}{96}.$$ □

For small values of k, we can a give a lower bound for the maximal number of compatible 3-simplices. Define the partial function $f_q(k)$ by

$$f_q(k) = \begin{cases} 0 & \text{if } 0 \le k \le 3, \\ 2k-6 & \text{if } k \equiv 0 \bmod 3 \text{ and } k \le (q+5)/4, \\ 2k-7 & \text{if } k \equiv 1 \bmod 3 \text{ and } k \le (q+9)/4, \\ 2k-8 & \text{if } k \equiv 2 \bmod 3 \text{ and } k \le (q+13)/4. \end{cases}$$

Proposition 1.5.10. *Let* q *be a prime power such that* $q \equiv 3 \bmod 4$*. Then for* Z *a set of cardinality* $q+1-k$ *there is an orientation on* $\Lambda^3(Z)$ *with the number of compatible* 3*-simplices given by*

$$\frac{(q+1) \cdot (q+1-2k) \cdot (q^2 - (2k+1) \cdot q + 2k^2 - 2k)}{96} + f_q(k).$$

Proof. We apply Proposition 1.5.8. For $k \leq 3$ use any set Y of cardinality k.

Let $\{x, 1/x, (x-1)/x, x/(x-1), 1/(1-x), 1-x\}$ be an orbit contained in the set S and let A be the subset $\{x, (x-1)/x, 1/(1-x)\}$. Assume that the set A has cardinality 3, that is, three elements in A are distinct. Note that we have

$$\begin{aligned} \mathrm{Cr}(x, 0, 1, \infty) &= \mathrm{Cr}(0, x, (x-1)/x, 1/(1-x)) = x, \\ \mathrm{Cr}((x-1)/x, 0, 1, \infty) &= \mathrm{Cr}(1, x, (x-1)/x, 1/(1-x)) = (x-1)/x \end{aligned}$$

and

$$\mathrm{Cr}(1/(1-x), 0, 1, \infty) = \mathrm{Cr}(\infty, x, (x-1)/x, 1/(1-x)) = 1/(1-x),$$

which yields six compatible 3-simplices supported on the set $\{0, 1, \infty\} \cup A$.

We begin in the case when $k \equiv 0 \bmod 3$. We have to find $j = k/3 - 1$ such three-element sets $A_1, A_2, \ldots, A_j$. There are $2 \cdot m_{0,6}$ such sets and possibily the orbit $\{2, 1/2, -1\}$ if the containment $\{2, 1/2, -1\} \subseteq S$ holds. Using Table 1.1 we obtain at least $(q-7)/12$ such sets. Hence the union of these sets, together with $\{0, 1, \infty\}$, yields the bound $k = 3 \cdot (j+1) \leq (q+5)/4$. Finally, each set A_i together with $\{0, 1, \infty\}$ supports 6 compatible simplices, yielding at least $6 \cdot j = 2k - 6$ compatible simplices supported on the set $Y = \{0, 1, \infty\} \cup \bigcup_{i=1}^{j} A_i$.

When $k \equiv 1 \bmod 3$ we use the above construction to obtain $k-1$ elements supporting $2 \cdot (k-1) - 6$ compatible simplices. In each case we can find another element y from the set S, either from the orbits $\{\omega, \omega^5\}$ or $\{2, 1/2, -1\}$. In the case when $q \equiv 11 \bmod 24$, we pick the element y from one of the six-element orbits that were not used. We obtain one more compatible simplex on the set $\{y, 0, 1, \infty\}$. Hence we obtain k elements supporting $2 \cdot (k-1) - 6 + 1 = 2k - 7$ compatible simplices. Lastly, the bound on k is given by $k - 1 \leq (q+5)/4$, that is, we have $k \leq (q+9)/4$.

Finally, when $k \equiv 2 \bmod 3$ we again use the above construction to obtain $k-2$ elements supporting $2 \cdot (k-2) - 6$ compatible simplices. When $q \equiv 7, 19 \bmod 24$ we pick the two extra elements ω and ω^5 which support the two simplices $\{\omega, 0, 1, \infty\}$ and $\{\omega^5, 0, 1, \infty\}$. When $q \equiv 3, 11, 23 \bmod 24$ we pick the two elements from one of the six-element orbits that were not used. Hence we obtain k elements supporting $2 \cdot (k-2) - 6 + 2 = 2k - 8$ compatible simplices. Finally, the bound on k is given by $k - 2 \leq (q+5)/4$, that is, $k \leq (q+13)/4$. □

Given an orientation s on $\Lambda^3(X)$ and a subset Y of X, we can duplicate the nodes in the set Y to create a partial orientation on a set of size $|X| + |Y|$. Let X' be the set $X \times \{1\} \cup Y \times \{2\}$ and let ϕ be the projection map $\phi : X' \longrightarrow X$ defined by $\phi(x, i) = x$. Then define the function s' on X' by $s'(x, y, z) = s(\phi(x), \phi(y), \phi(z))$. It follows directly from the definition that s' is antisymmetric, that is, that s' is a partial orientation. Furthermore, there are $\binom{k}{2}$ simplices of the form $\{(x,1),(x,2),(y,1),(y,2)\}$ that do not share any 2-dimensional faces between themselves and with the previously-defined orientation. Thus we can extend s by $s((x,2),(y,1),(y,2)) = -s((x,1),(y,1),(y,2)) = s((x,1),(x,2),(y,2)) = -s((x,1),(x,2),(y,1)) = 1$ and obtain $\binom{k}{2}$ more compatible simplices.

Proposition 1.5.11. *Let q be a prime power such that $q \equiv 3 \bmod 4$ and let σ_q be the* VL *orientation. Let Y be a subset of $X = F_q \cup \{\infty\}$ of cardinality k. Let N be the number of compatible 3-simplices in the restriction $\Lambda^3(Y)$. Then the number of compatible 3-simplices when the nodes in Y are duplicated is given by*

$$\begin{aligned} &\frac{(q+1)^2 \cdot q \cdot (q-1)}{96} + \frac{(q+1)\cdot q \cdot (q-1)}{24} \cdot k \\ &+ \frac{(q+1)\cdot(q-1)}{8} \cdot \binom{k}{2} + \frac{(q+1)}{4} \cdot \binom{k}{3} + N + \binom{k}{2}. \end{aligned} \tag{1.5.4}$$

Proof. A compatible 3-dimensional simplex F in $F_q \cup \{\infty\}$ will appear $2^{|F \cap Y|}$ times in the duplication and this number is accounted for in the $|F \cap Y| + 1$ first terms in the sum (1.5.4). □

By combining Proposition 1.5.11 and the construction in the proof of Proposition 1.5.10, we have the next result.

Proposition 1.5.12. *Let q be a prime power such that $q \equiv 3 \bmod 4$. Then there is an orientation on $\Lambda^3(X')$ where X' has cardinality $q+1+k$ and this orientation has the following number of compatible orientations:*

$$\frac{(q+1)\cdot(q^3 + 4q^2k + 6qk^2 + 4k^3 - 10qk - 18k^2 - q + 14k)}{96} + f_q(k) + \binom{k}{2}.$$

A different way to view the compatible 3-simplices is as follows. Let S be the set in Proposition 1.5.5 of cardinality $(q+1)/4$. For s in the set S, consider the quadruple $(s,0,1,\infty)$. Note that the cross-ratio is given by $\mathrm{Cr}(s,0,1,\infty) = s$. Consider all the images of this quadruple when acting upon by the Möbius transformations in $\mathrm{PGL}_2(q)$. Finally, view these quadruples as 3-simplices, that is, we remove the order between the 4 elements, obtaining the set

$$\{\{f(s), f(0), f(1), f(\infty)\} \;:\; f \in \mathrm{PGL}_2(q)\}.$$

This is the set of compatible 3-simplices and its cardinality is directly given by $|S| \cdot |\mathrm{PGL}_2(q)|/4!$.

This viewpoint yields a different proof of the second part of Corollary 1.5.9.

Second proof of Corollary 1.5.9. The number of Möbius transformations f such that $f(s), f(0), f(1)$ and $f(\infty)$ are all different from ∞ is given by $(q-1)\cdot q\cdot(q-3)$ by Lemma 1.4.8. Hence the number of compatible 3-simplices in $\Lambda^3(F_q)$ is given by $|S|\cdot(q-1)\cdot q\cdot(q-3)/4!$, which is the upper bound. $\square$

1.6 Products of two prime powers

We now will explore the componentwise action of $\mathrm{PGL}_2(q_1)\times\mathrm{PGL}_2(q_3)$ on the Cartesian product $(F_{q_1}\cup\{\infty\})\times(F_{q_3}\cup\{\infty\})$, where $q_i \equiv i \bmod 4$ for $i=1,3$. We start by investigating the orbits of the action of the group G on $F_q\cup\{\infty\}$ in the case when $q\equiv 1 \bmod 4$ to obtain results analogous to Table 1.1.

Proposition 1.6.1. *Let q be a prime power such that $q\equiv 1 \bmod 4$. Then the possible orbits of the action of the group $G=\{x, 1-x, 1/(1-x), x/(x-1), (x-1)/x, 1/x\}$ are listed in Table 1.2.*

Proof. Consider the action of G on the set $F_q\cup\{\infty\}$ and the orbits of this action. The set S is a disjoint union of these orbits. However, we need to pick out orbits that consist of quadratic non-residues only. We begin to discuss all the orbits.

(i) The set $\{0,1,\infty\}$ is an orbit. Note that this orbit has one element which is a quadratic residue and no quadratic non-residues.

(ii) The set $\{2, 1/2, -1\}$ is an orbit. Since $\left(\frac{-1}{q}\right)=1$, the element -1 is a quadratic residue. By Lemma 1.4.4 we have that the elements 2 and $1/2$ are quadratic residues if $q\equiv 1 \bmod 8$ and quadratic non-residues if $q\equiv 5 \bmod 8$. Furthermore, if q is a power of 3, then this orbit consists of only one element, that is the case $q\equiv 9 \bmod 24$.

(iii) If $q\equiv 1 \bmod 6$ then there two primitive 6^{th} roots of unity, namely, $\gamma^{(q-1)/6}=\omega$ and ω^5. The set $\{\omega,\omega^5\}$ forms an orbit since these elements are the roots of the polynomial x^2-x+1, the 6^{th} cyclotomic polynomial. Note that the condition $q\equiv 1 \bmod 4$ implies that $q\equiv 1 \bmod 12$. Hence $(q-1)/6$ is always even and we conclude that when this orbit occurs it consists only of quadratic residues.

(iv) Finally, there are orbits of size 6. Note that $-1 = x\cdot 1/(1-x)\cdot(x-1)/x$. Hence
$$1=\left(\frac{x}{q}\right)\cdot\left(\frac{1/(1-x)}{q}\right)\cdot\left(\frac{(x-1)/x}{q}\right).$$
Thus among the three elements x, $1/(1-x)$ and $(x-1)/x$ there are one or three quadratic residues. Hence among the elements of the orbit x, $1/x$, $1/(1-x)$, $1-x$, $(x-1)/x$ and $x/(x-1)$ there are two or six quadratic

Table 1.2: The possible orbits of G when acting on $F_q \cup \{\infty\}$ in the case when $q \equiv 1 \bmod 4$. The number of orbits of type $C_{6,0}$ is $m_{6,0}$ and the number of orbits of type $C_{2,4}$ is $m_{2,4}$.

	1 mod 24		5 mod 24		9 mod 24		13 mod 24		17 mod 24	
	qr	qnr	qr	qnr	qr	qnr	qr	qnr	qr	qnr
$\{0, 1, \infty\}$	1	0	1	0	1	0	1	0	1	0
$\{2, 1/2, -1\}$	3	0	1	2	1	0	1	2	3	0
$\{\omega, \omega^5\}$	2	0	0	0	0	0	2	0	0	0
$C_{6,0}$	6	0	6	0	6	0	6	0	6	0
$C_{2,4}$	2	4	2	4	2	4	2	4	2	4
$m_{2,4}$	$(q-1)/8$		$(q-5)/8$		$(q-1)/8$		$(q-5)/8$		$(q-1)/8$	
$m_{6,0}$	$(q-25)/24$		$(q-5)/24$		$(q-9)/24$		$(q-13)/24$		$(q-17)/24$	

residues. We assume among these orbits that there are $m_{6,0}$ orbits $C_{6,0}$ containing six quadratic residues and $m_{2,4}$ orbits $C_{2,4}$ containing two quadratic residues.

Note that there is no prime power such that $q \equiv 21 \bmod 24$ since it would imply that 3 divides q and hence that q is 3 to an even power, which implies that $q \equiv 9 \bmod 24$ a contradiction.

The proof now consists of the five cases $q \equiv 1, 5, 9, 13, 17 \bmod 24$. The proof is summarized in Table 1.2. First the possible orbits are listed with how many quadratic residues versus quadratic non-residues. Then using that there are exactly $(q-1)/2$ quadratic non-residues, we can solve for $m_{2,4}$, the number of orbits $C_{2,4}$ with four quadratic non-residues. Using the value of $m_{2,4}$ and that there are $q/2$ quadratic residues we can solve for $m_{6,0}$. $\square$

Let q_1 and q_3 be two prime powers such that $q_i \equiv i \bmod 4$. We will now construct a partial orientation on the set $\Lambda^3(X)$ where the set $X = X_1 \times X_3$ where $X_i = F_{q_i} \cup \{\infty\}$. Let R be the ring $F_{q_1} \times F_{q_3}$ where the operations are defined component-wise. Define the Jacobi symbol $\left(\frac{x}{q_1 \cdot q_3}\right)$ as the product of the two Legendre symbols

$$\left(\frac{x}{q_1 \cdot q_3}\right) = \left(\frac{x_1}{q_1}\right) \cdot \left(\frac{x_3}{q_3}\right),$$

where $x = (x_1, x_3)$. Observe $\left(\frac{-1}{q_1 \cdot q_3}\right) = \left(\frac{(-1,-1)}{q_1 \cdot q_3}\right) = \left(\frac{-1}{q_1}\right) \cdot \left(\frac{-1}{q_3}\right) = 1 \cdot (-1) = -1$.

Define a partial orientation $\sigma_{q_1 \cdot q_3}$ as the product of the VL functions σ_{q_1} and σ_{q_3}, that is,

$$\sigma_{q_1\cdot q_3}((a_1,a_3),(b_1,b_3),(c_1,c_3)) = \sigma_{q_1}(a_1,b_1,c_1)\cdot\sigma_{q_3}(a_3,b_3,c_3). \tag{1.6.1}$$

By Lemma 1.5.2 we know that σ_{q_1} is symmetric and σ_{q_3} is anti-symmetric. Hence the product $\sigma_{q_1}\cdot\sigma_{q_3}$ is anti-symmetric, that is, $\sigma_{q_1\cdot q_3}$ is a partial orientation.

Let the group $\mathrm{PGL}_2(q_1)\times\mathrm{PGL}_2(q_3)$ act upon the product $X = X_1\times X_3$ by componentwise action. Also let the group G of order 6, act on X by the diagonal action, that is, $g\circ(x_1,x_3) = (g(x_1),g(x_3))$.

We also need analogues of Lemma 1.5.3 and Theorem 1.5.4.

Lemma 1.6.2. *Let q_1 and q_3 be two odd prime powers. For the orientation σ in equation* (1.6.1)*, four distinct elements a_1, b_1, c_1 and d_1 in $F_{q_1}\cup\{\infty\}$ and four distinct elements a_3, b_3, c_3 and d_3 in $F_{q_3}\cup\{\infty\}$, the following identity holds:*

$$\sigma(a,b,c)\cdot\sigma(b,c,d) = \left(\frac{\mathrm{Cr}(b,a,c,d)}{q_1\cdot q_3}\right),$$

where $a=(a_1,a_3)$, $b=(b_1,b_3)$, $c=(c_1,c_3)$ and $d=(d_1,d_3)$.

Proof. This is a direct application of Lemma 1.5.3:

$$\begin{aligned}\sigma(a,b,c)\cdot\sigma(b,c,d) &= \sigma_{q_1}(a_1,b_1,c_1)\cdot\sigma_{q_3}(a_3,b_3,c_3)\cdot\sigma_{q_1}(b_1,c_1,d_1)\cdot\sigma_{q_3}(b_3,c_3,d_3)\\ &= \left(\frac{\mathrm{Cr}(b_1,a_1,c_1,d_1)}{q_1}\right)\cdot\left(\frac{\mathrm{Cr}(b_3,a_3,c_3,d_3)}{q_3}\right).\end{aligned}$$ □

Proposition 1.6.3. *Let q_1 and q_3 be two prime powers such that $q_i\equiv i \bmod 4$. Let S be a subset of the product ring $F_{q_1}\times F_{q_3}$ such that S consists of elements whose Jacobi symbol is negative and that S is closed under the action of the group $G=\{x,1-x,1/(1-x),x/(x-1),(x-1)/x,1/x\}$. Then for four distinct elements a_1, b_1, c_1 and d_1 in $F_{q_1}\cup\{\infty\}$, and four distinct elements a_3, b_3, c_3 and d_3 in $F_{q_3}\cup\{\infty\}$, such that $\mathrm{Cr}((a_1,a_3),(b_1,b_3),(c_1,c_3),(d_1,d_3))\in S$ the set $\{(a_1,a_3),(b_1,b_3),(c_1,c_3),(d_1,d_3)\}$ is a compatible simplex with the orientation σ in equation* (1.6.1).

Proof. The Jacobi symbol satisfies $\left(\frac{-1}{q_1\cdot q_3}\right) = -1$. The remainder of the argument is the same as the proof of Proposition 1.5.4. □

We now observe how the product of two orbits of the group G decomposes.

Lemma 1.6.4. *Let D_i be an orbit of G in X_i, for $i=1,3$. Then the Cartesian product $D_1\times D_3$ decomposes into orbits of G. The multiset of the cardinalities of these orbits is* *given by the table below, where the superscripts denote multiplicity:*

	$\lvert D_3\rvert = 1$	$\lvert D_3\rvert = 2$	$\lvert D_3\rvert = 3$	$\lvert D_3\rvert = 6$
$\lvert D_1\rvert = 1$	$\{1\}$	$\{2\}$	$\{3\}$	$\{6\}$
$\lvert D_1\rvert = 2$	$\{2\}$	$\{2^2\}$	$\{6\}$	$\{6^2\}$
$\lvert D_1\rvert = 3$	$\{3\}$	$\{6\}$	$\{3,6\}$	$\{6^3\}$
$\lvert D_1\rvert = 6$	$\{6\}$	$\{6^2\}$	$\{6^3\}$	$\{6^6\}$

Our goal is to obtain orbits of the set X such that all the elements of the orbits have negative Jacobi symbol. Returning to Tables 1.1 and 1.2 we note that there are seven types of orbits, when considering the quadratic residues versus non-residues. In Table 1.1 we have the orbits $\{0, 1, \infty\}$, $E_{0,1}$, $E_{2,1}$, $E_{0,3}$, $\{\omega_3, \omega_3^5\}$, $C_{0,6}$ and $C_{4,2}$, where $E_{i,j}$ denotes the orbit $\{2, -1, 1/2\}$ with i quadratic residues and j quadratic non-residues. For instance, the orbit $E_{0,1}$ only appears when q is a power of 3. Similarly, in Table 1.2 we have the orbits $\{0, 1, \infty\}$, $E_{1,0}$, $E_{1,2}$, $E_{3,0}$, $\{\omega_1, \omega_1^5\}$, $C_{6,0}$ and $C_{2,4}$.

Lemma 1.6.5. *Let D_i be an orbit of G in X_i, for $i = 1, 3$. Then the Cartesian product $D_1 \times D_3$ decomposes into orbits of G. The multiset of the cardinalities of these orbits, where all the elements have negative Jacobi symbol, is given by:*

	$E_{0,1}$	$E_{2,1}$	$E_{0,3}$	$\{\omega_3, \omega_3^5\}$	$C_{0,6}$	$C_{4,2}$
$E_{1,0}$	$\{1\}$	–	$\{3\}$	$\{2\}$	$\{6\}$	–
$E_{1,2}$	–	$\{3\}$	–	–	–	$\{6\}$
$E_{3,0}$	$\{3\}$	–	$\{3,6\}$	$\{6\}$	$\{6^3\}$	–
$\{\omega_1, \omega_1^5\}$	$\{2\}$	–	$\{6\}$	$\{2^2\}$	$\{6^2\}$	–
$C_{6,0}$	$\{6\}$	–	$\{6^3\}$	$\{6^2\}$	$\{6^6\}$	–
$C_{2,4}$	–	$\{6\}$	–	–	–	$\{6^2\}$

Proposition 1.6.6. *Let q_1 and q_3 be two prime powers such that $q_i \equiv i \bmod 4$. Then there exists a subset it of the ring $F_{q_1} \times F_{q_3}$ such that S consists of elements with negative Jacobi symbols and is closed under the action of the group $G = \{x, 1-x, 1/(1-x), 1/x, x/(x-1), (x-1)/x\}$ and cardinality of S is given by $((q_1 - 2) \cdot (q_3 - 2) - 3)/4$.*

Proof. The result follows from multiplying the three matrices

$$\begin{pmatrix} 0 & 0 & 1 & 1 & \frac{q_1-25}{24} & \frac{q_1-1}{8} \\ 0 & 1 & 0 & 0 & \frac{q_1-5}{24} & \frac{q_1-5}{8} \\ 1 & 0 & 0 & 0 & \frac{q_1-9}{24} & \frac{q_1-1}{8} \\ 0 & 1 & 0 & 1 & \frac{q_1-13}{24} & \frac{q_1-5}{8} \\ 0 & 0 & 1 & 0 & \frac{q_1-17}{24} & \frac{q_1-1}{8} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 3 & 2 & 6 & 0 \\ 0 & 3 & 0 & 0 & 0 & 6 \\ 3 & 0 & 9 & 6 & 18 & 0 \\ 2 & 0 & 6 & 4 & 12 & 0 \\ 6 & 0 & 18 & 12 & 36 & 0 \\ 0 & 6 & 0 & 0 & 0 & 12 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ \frac{q_3-3}{24} & \frac{q_3-7}{24} & \frac{q_3-11}{24} & \frac{q_3-19}{24} & \frac{q_3+1}{24} \\ \frac{q_3-3}{8} & \frac{q_3-7}{8} & \frac{q_3-3}{8} & \frac{q_3-3}{8} & \frac{q_3-7}{8} \end{pmatrix}.$$

The first matrix encodes Table 1.2. The rows of this matrix correspond to the cases $q_1 \equiv 1, 5, 9, 13, 17 \bmod 24$ and the columns to the orbits $E_{1,0}$, $E_{1,2}$, $E_{3,0}$, $\{\omega_1, \omega_1^5\}$, $C_{6,0}$ and $C_{2,4}$. The second matrix enumerates the elements in the array in Lemma 1.6.5. Finally, the third matrix encodes Table 1.1, where the rows correspond to the orbits $E_{0,1}$, $E_{2,1}$, $E_{0,3}$, $\{\omega_3, \omega_3^5\}$, $C_{0,6}$ and $C_{4,2}$ and the columns to the cases $q_3 \equiv 3, 7, 15, 19, 23 \bmod 24$. The resulting matrix product is given by $(q_1 \cdot q_3 - 2 \cdot q_1 - 2 \cdot q_3 + 1)/4$ times the matrix J of all ones. □

Combining Propositions 1.6.3 and 1.6.6 yields the next result.

Proposition 1.6.7. *Let q_1 and q_3 be two prime powers such that $q_i \equiv i \bmod 4$. Then there are*

$$\frac{((q_1-2)\cdot(q_3-2)-3)\cdot(q_1+1)\cdot q_1\cdot(q_1-1)\cdot(q_3+1)\cdot q_3\cdot(q_3-1)}{96}$$

3-simplices on the set $(F_{q_1} \cup \{\infty\}) \times (F_{q_3} \cup \{\infty\})$ compatible with the partial orientation in equation (1.6.1).

We can slightly improve the construction in Proposition 1.6.7 by changing the partial orientation as follows.

Theorem 1.6.8. *Let q_1 and q_3 be two prime powers such that $q_i \equiv i \bmod 4$. Assume there is an orientation on $\Lambda^3(Y)$, where Y has cardinality $q_1 + 1$, with M compatible 3-simplices. Then there is a partial oreintation on $\Lambda^3((F_{q_1} \cup \{\infty\}) \times (F_{q_3} \cup \{\infty\}))$ with the number of compatible 3-simplices given by*

$$\frac{((q_1-2)\cdot(q_3-2)-3)\cdot(q_1+1)\cdot q_1\cdot(q_1-1)\cdot(q_3+1)\cdot q_3\cdot(q_3-1)}{96}$$
$$+ (q_1+1)\cdot\frac{(q_3+1)^2\cdot q_3\cdot(q_3-1)}{96} + M\cdot(q_3+1).$$

Proof. View the set Y as $F_{q_1} \cup \{\infty\}$ and let τ' denote the partial orientation on $\Lambda^3(Y)$. Use the following partial orientation τ, where $a = (a_1, a_3)$, $b = (b_1, b_3)$ and $c = (c_1, c_3)$:

$$\tau(a,b,c) = \begin{cases} \sigma_{q_1,q_3}(a,b,c) & \text{if } a_1, b_1, c_1 \text{ distinct and } a_3, b_3, c_3 \text{ distinct,} \\ \sigma_{q_3}(a_3,b_3,c_3) & \text{if } a_1 = b_1 = c_1, \\ \tau'(a_1,b_1,c_1) & \text{if } a_3 = b_3 = c_3. \end{cases} \qquad \square$$

We note that the improvement from Proposition 1.6.7 to Theorem 1.6.8 is slight, since the leading term in the proposition is $(q_1 \cdot q_3)^4/96$, whereas the number of added simplices in the theorem is at most of the order $q_1 \cdot q_3^4/96 + q_1^4 \cdot q_3/96$.

Corollary 1.6.9. *Let q_1 and q_3 be two prime powers such that $q_i \equiv i \bmod 4$. Assume there is an orientation on $\Lambda^3(Y)$, where Y has cardinality q_1, with M' compatible 3-simplices. Then there is a partial orientation on $\Lambda^3(F_{q_1} \times F_{q_3})$ with the number of compatible 3-simplices given by*

$$\frac{((q_1-2)\cdot(q_3-2)-3)\cdot q_1\cdot(q_1-1)\cdot(q_1-3)\cdot q_3\cdot(q_3-1)\cdot(q_3-3)}{96} + q_1 \cdot \frac{(q_3+1)\cdot q_3\cdot(q_3-1)\cdot(q_3-3)}{96} + M' \cdot q_3.$$

1.7 Asymptotic behavior

In a recent paper Baker, Harman and Pintz [1] showed the following bound on the prime gap.

Theorem 1.7.1. *There exists a real number x_0 such that for all $x > x_0$, the interval $[x - x^{0.525}, x]$ contains prime numbers.*

In this section we are interested in primes congruent to 3 modulo 4. Hence it is reasonable to make the following conjecture.

Conjecture 1.7.2. *There exist a real number $\alpha \in (0,1)$ and a real number x_0 such that for all $x > x_0$, the interval $[x, x + x^\alpha]$ contains a prime number congruent to 3 modulo 4.*

Theorem 1.7.3. *Assume that Conjecture 1.7.2 is true. Then for $n > x_0$ there exists an orientation of $\Lambda^3(X)$, where X is an n-element set, with at least*

$$\frac{n^4}{96} - \frac{n^{3+\alpha}}{24} + O(n^3)$$

number of compatible simplices.

Proof. Given $n > x_0$ there exists a prime q such that $n \leq q \leq n + n^\alpha$ and congruent to 3 modulo 4. Hence we have an orientation of the set $F_q \cup \{\infty\}$ with $(q+1)^2 \cdot q \cdot (q-1)/96$ compatible simplices. Let k be the difference $q + 1 - n$. By part (i) Lemma 1.5.7 we remove at most $(q+1) \cdot q \cdot (q-1)/24 \cdot k$ number of compatible 3-simplices when we restrict the orientation on $F_q \cup \{\infty\}$ to a subset of size n. Thus a lower bound is given by

$$\begin{aligned}
&\frac{(q+1)^2 \cdot q \cdot (q-1)}{96} - \frac{(q+1) \cdot q \cdot (q-1)}{24} \cdot k \\
&\geq \frac{(q+1)^2 \cdot q \cdot (q-1)}{96} - \frac{(q+1) \cdot q \cdot (q-1)}{24} \cdot (n^\alpha + 1) \\
&\geq \frac{n^4}{96} + O(n^3) - \left(\frac{n^3}{24} + O(n^2)\right) \cdot (n^\alpha + 1) \\
&\geq \frac{n^4}{96} - \frac{n^{3+\alpha}}{24} + O(n^3).
\end{aligned}$$

□

1.8 Concluding remarks

It is tempting to suggest that for $k \geq 4$ and q a prime power such that $q \equiv 3 \bmod 4$ that the following orientation of $\Lambda^k(F_q)$ would maximize the number of compatible k-simplices:

$$\sigma(a_1, a_2, \ldots, a_k) = \left(\frac{\prod_{1 \leq i < j \leq k}(a_j - a_i)}{q}\right). \tag{1.8.1}$$

Note that the alternating property of the Vandermonde product and the Legendre symbol $\left(\frac{-1}{q}\right) = -1$ imply that σ is an orientation. When k is even, it is not clear how to extend this σ to $F_q \cup \{\infty\}$. However, we conjecture that the following enumeration of compatible 4-simplices holds:

Conjecture 1.8.1. *Let q be a prime power congruent to* 3 *modulo* 4*. Then for $k = 4$ the number of compatible* 4*-simplices to the orientation in* (1.8.1) *is given by*

$$\frac{q \cdot (q-1) \cdot (q-7) \cdot (q-3) \cdot (q+1)}{1920}.$$

Note this conjecture has the same leading term as Lemma 1.3.1, hence it is overtaken by Proposition 1.3.3.

Conjecture 1.8.2. *Let q be a prime power congruent to* 3 *modulo* 4*. Then for $k = 5$ there are no compatible* 5*-simplices to the orientation in* (1.8.1).

These two conjectures show that new techniques are needed to find good orientations to maximize the number of compatible simplices.

Table 1.3: For $k = 3$, the upper bound and the best constructions for $4 \leq n \leq 83$.

n	upper bound	lower bound		n	upper bound	lower bound	
4	1	1		44	36421	36421	Thm. 1.5.6
5	2	2		45	39847	39732	Prop. 1.5.10 & 1.5.12
6	6	6	Lem. 1.2.5	46	43642	43516	Prop. 1.5.10
7	14	14	Cor. 1.5.9	47	47564	47564	Cor. 1.5.9
8	28	28	Thm. 1.5.6	48	51888	51888	Thm. 1.5.6
9	45	42	Prop. 1.5.10	49	56350	56212	Prop. 1.5.12
10	75	70	Prop. 1.5.10	50	61250	60813	Prop. 1.5.12
11	110	110	Cor. 1.5.9	51	66300	65742	Prop. 1.5.10
12	165	165	Thm. 1.5.6	52	71825	71233	Prop. 1.5.10
13	227	220	Prop. 1.5.12	53	77512	77057	Prop. 1.5.10
14	318	296	Prop. 1.5.10	54	83713	83226	Prop. 1.5.10
15	420	402	Prop. 1.5.10	55	90090	89752	Prop. 1.5.10
16	560	536	Prop. 1.5.10	56	97020	96656	Prop. 1.5.10
17	714	700	Prop. 1.5.10	57	104139	103950	Prop. 1.5.10
18	918	900	Prop. 1.5.10	58	111853	111650	Prop. 1.5.10
19	1140	1140	Cor. 1.5.9	59	119770	119770	Cor. 1.5.9
20	1425	1425	Thm. 1.5.6	60	128325	128325	Thm. 1.5.6
21	1732	1710	Prop. 1.5.10 & 1.5.12	61	137097	136880	Prop. 1.5.12
22	2117	2090	Prop. 1.5.10	62	146552	145900	Prop. 1.5.10
23	2530	2530	Cor. 1.5.9	63	156240	155790	Prop. 1.5.10
24	3036	3036	Thm. 1.5.6	64	166656	166176	Prop. 1.5.10
25	3575	3542	Prop. 1.5.10 & 1.5.12	65	177320	177072	Prop. 1.5.10
26	4225	4186	Prop. 1.5.10	66	188760	188496	Prop. 1.5.10
27	4914	4914	Cor. 1.5.9	67	200464	200464	Cor. 1.5.9
28	5733	5733	Thm. 1.5.6	68	212993	212993	Thm. 1.5.6
29	6597	6552	Prop. 1.5.10 & 1.5.12	69	225802	225522	Prop. 1.5.10 & 1.5.12
30	7612	7560	Prop. 1.5.10	70	239487	239190	Prop. 1.5.10
31	8680	8680	Cor. 1.5.9	71	253470	253470	Cor. 1.5.9
32	9920	9920	Thm. 1.5.6	72	268380	268380	Thm. 1.5.6
33	11220	11160	Prop. 1.5.12	73	283605	283290	Prop. 1.5.12
34	12716	12520	Prop. 1.5.12	74	299811	298866	Prop. 1.5.10
35	14280	14026	Prop. 1.5.10	75	316350	315702	Prop. 1.5.10
36	16065	15793	Prop. 1.5.10	76	333925	333241	Prop. 1.5.10
37	17926	17717	Prop. 1.5.10	77	351851	351500	Prop. 1.5.10
38	20035	19806	Prop. 1.5.10	78	370870	370500	Prop. 1.5.10
39	22230	22068	Prop. 1.5.10	79	390260	390260	Cor. 1.5.9
40	24700	24520	Prop. 1.5.10	80	410800	410800	Thm. 1.5.6
41	27265	27170	Prop. 1.5.10	81	431730	431340	Prop. 1.5.10 & 1.5.12
42	30135	30030	Prop. 1.5.10	82	453870	453460	Prop. 1.5.10
43	33110	33110	Cor. 1.5.9	83	476420	476420	Cor. 1.5.9

It is remarkable in Propositions 1.5.5 and 1.6.6 that the end result of the cardinality of the set S does not depend on the congruence class of q, q_1 and q_3 modulo 24. This fact suggests that there is proof of these results that do not split the argument in 5, respectively 25, cases.

Given an orientation s of $\Lambda^k(X)$, instead of enumerating the boundaries of k-simplices, we can ask the same question for other k-dimensional simplicial polytopes. For tournaments several authors have given a sharp upper bound on the number of 4-cycles, that is, the number of compatible squares; see [5, 11, 13]. For results on pentagon, hexagons and heptagons, see [3, 12, 19, 20]. For general values of k, the first step in this direction would be to find an upper bound for the number of boundaries of bipyramids of $(k-1)$-simplices. More challenging would be to find upper bounds for the number of boundaries of k-dimensional cross-polytopes or cyclic polytopes.

Chapter 2 The representation-theoretic view

2.1 Introduction

The problem of maximizing the number of compatible simplices is a combinatorial optimization problem. One recent method to solve such a problem is the sum-of-squares certificate method, introduced in the papers [8], [16] and [17]. This technique has been useful in industrial fields of enquiry like dynamical systems and control theory. For more information see the papers [4] and [24]. We begin with an example to show the role of a sum of squares in accomplishing upper bounds. The idea is to express the problem in terms of a polynomial in several variables, then rewrite it as a sum of squares of other polynomials. Since squares of polynomials are non-negative the polynomial will reveal a bound for the cardinality of the combinatorial object we are trying to bound. Below is an example of a proof of a lower bound for a polynomial in x and y, where x and y are constrained.

Example 2.1.1. *(due to A. Raymond) We claim that* $1 - y \geq 0$ *on the* $x^2 + y^2 = 1$*. To prove this we rewrite*

$$1 - y = 1 - y + \frac{1}{2}(x^2 + y^2 - 1) = \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y-1}{\sqrt{2}}\right)^2 \tag{2.1.1}$$

Thus the right-hand side of equation (2.1.1) *is a sum-of-squares expression for* $1 - y$ *modulo the ideal* I *generated by the polynomial* $f(x, y) = x^2 + y^2 - 1$ *over the field* $\mathbb{R}$*. Note that on the unit circle* $f(x, y) = 0$*. Thus we have that* $1 - y$ *is non-negative on* $x^2 + y^2 = 1$*.*

In combinatorial applications the variables take one of two possible values. Usually in the literature those values are 0 and 1. Hence the polynomials are optimized over the vertices of a hypercube. However, in our application it is more convenient to optimize over the cube $\{-1, 1\}^N$, that is the two values are ± 1 and we have the identity that each variable squared is 1. Furthermore in many combinatorial optimization problems there is an underlying symmetry. For instance in the problem described in Section 1.1 if we relabel the vertex set X we do not change the problem. The papers [8], [16] and [17] describe a theory how such a symmetry is helpful in finding sum-of-squares expressions.

2.2 Sum-of-squares Certificate Proofs

Definition 2.2.1. *Let* R *be a commutative ring with unity and let* $x_1, x_2, \ldots, x_r$ *be indeterminates. A* sum-of-squares certificate modulo an ideal *I for a polynomial p is a* *sum $\sum_{i=1}^k {p_i}^2$ of squares of polynomials p_i in the polynomial ring $R[x_1, x_2, \ldots, x_r]$ such that there exist a constants c, α satisfying $cp + \alpha \equiv \sum_{i=1}^k {p_i}^2 \mod I$.*

Remark 2.2.2. *Definition 2.2.1 implies the following bounds:*

$$\begin{aligned} c < 0 &\implies p \leq -\alpha/c, \\ c > 0 &\implies p \geq -\alpha/c. \end{aligned}$$

We are now able to prove the bound in Equation (1.1.1).

Theorem 2.2.3. *The number of directed 3-cycles in a tournament graph on n vertices is bounded above by*

$$\begin{cases} (n-1) \cdot n \cdot (n+1)/24 & n \text{ is odd,} \\ (n-2) \cdot n \cdot (n+2)/24 & n \text{ is even.} \end{cases}$$

Furthermore there are tournaments that attain this bound for each n.

Proof. Let $x_{i,j}$ be 1 if the edge ij is oriented $i \to j$ and otherwise let $x_{i,j} = -1$. That is, for all variables we have $x_{i,j} = -x_{j,i}$. Then 4 times number of 3-cycles in a directed graph is given by

$$\begin{aligned} 4s &= \frac{1}{2} \sum_{i<j<k} ((1 + x_{i,j}) \cdot (1 + x_{j,k}) \cdot (1 + x_{k,i}) + (1 - x_{i,j}) \cdot (1 - x_{j,k}) \cdot (1 - x_{k,i})) \\ &= \sum_{i<j<k} (1 + x_{i,j}x_{j,k} + x_{j,k}x_{k,i} + x_{k,i}x_{i,j}) \\ &= \binom{n}{3} + \sum_{i<j<k} x_{i,j}x_{j,k} + \sum_{i<j<k} x_{j,k}x_{k,i} + \sum_{i<j<k} x_{k,i}x_{i,j} \\ &= \binom{n}{3} - \sum_{j<i<k} x_{i,j}x_{i,k} - \sum_{k<j<i} x_{i,j}x_{i,k} - \sum_{i<j<k} x_{i,k}x_{i,j} \\ &= \binom{n}{3} - 1/2 \cdot \sum_i \sum_{j,k \neq i} x_{i,j}x_{i,k} \\ &= \binom{n}{3} - 1/2 \cdot \sum_i \left(\left(\sum_{j \neq i} x_{i,j} \right)^2 - (n-1) \right) \\ &= \binom{n}{3} + \binom{n}{2} - \sum_i \left(\frac{1}{\sqrt{2}} \sum_{j \neq i} x_{i,j} \right)^2 . \end{aligned}$$

where in the fourth step we switched i and j in the first sum and switched i and k in the second sum and in the sixth step we used $\sum_{j \neq i} x_{i,j}^2 = n - 1$. The inference is that like in Example 2.1.1 we have shown a sum-of-squares certificate for $\binom{n}{3} + \binom{n}{2} - 4s$. Thus we obtain

$$s \leq \frac{1}{4} \left(\binom{n}{3} + \binom{n}{2} \right).$$

When n is even the sharper inequality $\left(\sum_{j\neq i} x_{i,j}\right)^2 \geq 1$ yields the better bound

$$s \leq \frac{1}{4}\left(\binom{n}{3} + \binom{n}{2} - \frac{n}{2}\right).$$

For n odd the inequality in the bound is an equality when $\sum_{j\neq i} x_{i,j} = 0$ for all $1 \leq i \leq n$. That is the in-degree equals the out-degree of each of the vertices i. Such a tournament is obtained by taking an Eulerian circuit of K_n and orienting the edges along the circuit.

For n even the inequality in the bound is an equality when $\sum_{j\neq i} x_{i,j} = 1$ for all $1 \leq i \leq n$. That is the in-degree and the out-degree of each of the vertices i differ by 1. To form such a tournament, first a complete matching M of K_n is picked. Note that the edges of $K_n - M$ form a regular graph with even vertex degree. We therefore orient along an Eulerian circuit in $K_n - M$ and then orient the edges of M in any way. This forms the required tournament. □

Theorem 2.2.4. *The number of 4-cycles in a tournament graph is bounded above by*

$$\begin{cases} (n-1)\cdot n\cdot(n+1)\cdot(n-3)/48 & \text{if } n \text{ is odd,} \\ (n-2)\cdot n\cdot(n+2)\cdot(n-3)/48 & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, for each n there are tournaments that attain this bound.

Proof. As in the proof of Theorem 2.2.3 we let $x_{i,j} = \pm 1$ according to how the edge ij is oriented $i \to j$. Note that a tournament on 4 vertices has either as 0 or 1 directed 4-cycles. Hence a bound on the number of 4-cycles is given by removing the number of subgraphs on 4 vertices where one vertex is dominating over the 3 other vertices from $\binom{n}{4}$.

$$s \leq \binom{n}{4} - \frac{1}{8}\sum_i \sum_{\substack{j<k<\ell \\ j,k,\ell\neq i}} (1+x_{i,j})\cdot(1+x_{i,k})\cdot(1+x_{i,\ell}).$$

We have a similar bound by removing subgraphs where one vertex is dominated by the 3 other vertices:

$$s \leq \binom{n}{4} - \frac{1}{8}\sum_i \sum_{\substack{j<k<\ell \\ j,k,\ell\neq i}} (1-x_{i,j})\cdot(1-x_{i,k})\cdot(1-x_{i,\ell}).$$

By adding these two bounds and multiplying by 4 we have

$$\begin{aligned}
8 \cdot s &\leq 8 \cdot \binom{n}{4} - \sum_i \sum_{\substack{j<k<\ell \\ j,k,\ell \neq i}} (1 + x_{i,j}x_{i,k} + x_{i,j}x_{i,\ell} + x_{i,k}x_{i,\ell}) \\
&= 8 \cdot \binom{n}{4} - n \cdot \binom{n-1}{3} - \sum_i \sum_{\substack{j<k<\ell \\ j,k,\ell \neq i}} (x_{i,j}x_{i,k} + x_{i,j}x_{i,\ell} + x_{i,k}x_{i,\ell}) \\
&= 8 \cdot \binom{n}{4} - 4 \cdot \binom{n}{4} - \frac{1}{6} \cdot \sum_i \sum_{\substack{j,k,\ell \neq i \\ j,k,\ell \text{ distinct}}} (x_{i,j}x_{i,k} + x_{i,j}x_{i,\ell} + x_{i,k}x_{i,\ell}) \\
&= 4 \cdot \binom{n}{4} - \frac{n-3}{2} \cdot \sum_i \sum_{\substack{j,k \neq i \\ j,k \text{ distinct}}} x_{i,j}x_{i,k} \\
&= 4 \cdot \binom{n}{4} - \frac{n-3}{2} \cdot \sum_i \left(\left(\sum_{j \neq i} x_{i,j} \right)^2 - (n-1) \right) \\
&\leq 4 \cdot \binom{n}{4} + (n-3) \cdot \binom{n}{2}.
\end{aligned}$$

Again, when n is even we have a sharper bound $\left(\sum_{j \neq i} x_{i,j}\right)^2 \geq 1$, giving

$$8 \cdot s \leq 4 \cdot \binom{n}{4} - \frac{(n-3) \cdot n}{2} + (n-3) \cdot \binom{n}{2}. \qquad \square$$

The examples discussed at the end of the proof of Theorem 2.2.3 show that the bound is attained for each n.

2.3 Introduction to polynomial rings

We now describe the polynomial we intend to optimize over the set $\{-1,1\}^{\binom{n}{3}}$. Recall the definition of orientation in the beginning of Section 1.1. We try to maximize the number of compatible 3-simplices over all orientations $s : \Lambda^3([n]) \to \{-1,1\}$. Set $s(i,j,k) = x_{i,j,k}$. Thus the polynomial to optimize is

$$\begin{aligned}
p = \frac{1}{8} \sum_{1 \leq i<j<k<\ell \leq n} &\big(1 - x_{i,j,k}x_{i,j,\ell} + x_{i,j,k}x_{i,k,\ell} - x_{i,j,k}x_{j,k,\ell} - x_{i,j,\ell}x_{i,k,\ell} + x_{i,j,\ell}x_{j,k,\ell} \\
&- x_{i,k,\ell}x_{j,k,\ell} + x_{i,j,k}x_{i,j,\ell}x_{i,k,\ell}x_{j,k,\ell}\big). \qquad (2.3.1)
\end{aligned}$$

It can be checked that for each set i, j, k, ℓ of four integers the corresponding summand evaluates to 8 if (i, j, k, ℓ) is a compatible 3-simplex and 0 otherwise. Thus p counts the number of oriented 3-simplices and hence maximizing p solves the problem.

The goal is to find a real number α as small as possible such that $\alpha - p$ is a sum-of-squares modulo the ideal I in the sense of Example 2.1.1.

We now determine precisely the polynomial ring p lives in. For this we need some more definitions.

Definition 2.3.1. *An* $\mathbb{R}G$-module *is a real vector space with the action of a group G by linear transformations.*

Definition 2.3.2. *For a polynomial ring $\mathfrak{R}$ define $\mathfrak{R}_{(d)}$ to be the vector subspace spanned by the degree-d monomials in $\mathfrak{R}$.*

Definition 2.3.3. *Define $\mathcal{S}_{n,m}$ to be the set of symbols $\{x_S : S \in \Lambda^m([n])\}$.*

Definition 2.3.4. *Define $\mathbb{R}^{\text{ext}}[\mathcal{S}_{n,m}]$ to be the polynomial ring $\mathbb{R}[\mathcal{S}_{n,m}]$ modulo relations $x_S = (-1)^\sigma \cdot x_{\sigma S}$ for every $S \in \mathcal{S}_{n,k}$ and every permutation $\sigma \in \mathfrak{S}_S$ of S.*

Definition 2.3.5. *For any set X, we denote by $F(X)$ the set of all formal linear combinations of elements of X over the field $\mathbb{R}$.*

Definition 2.3.6. *Let G be a group. Define the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ of $\mathbb{R}G$-modules $V_1, V_2, \ldots, V_m$ by $F(V_1 \times V_2 \times \cdots \times V_m)/\sim$ where the following relations hold for all $1 \leq i \leq k$.*

$$\begin{aligned}&(v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_m) + (v_1, v_2, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_m)\\ &\qquad \sim (v_1, v_2, \ldots, v_{i-1}, v_i + v_i', v_{i+1}, \ldots, v_m),\\ &c(v_1, v_2, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_m) \sim (v_1, v_2, \ldots, v_{i-1}, cv_i, v_{i+1}, \ldots, v_m).\end{aligned}$$

The tensor product is equipped with the induced action of G from the action of G on each V_i that is

$$g \cdot (v_1, v_2, \ldots, v_m) \sim (g \cdot v_1, g \cdot v_2, \ldots, g \cdot v_m).$$

Definition 2.3.7. *The symmetric product* $\mathrm{Sym}(V_1, V_2, \ldots, V_m)$ *of m copies of V, also denoted as* $\mathrm{Sym}^k(V)$ *is defined as*

$$\mathrm{Sym}(V_1, V_2, \ldots, V_m) = \left\{ \sum_{\sigma \in \mathfrak{S}_n} \bigotimes_{i=1}^{m} v_{\sigma i} : \bigotimes_{i=1}^{m} v_i \in V_1 \otimes V_2 \otimes \cdots \otimes V_m \right\}.$$

We denote the symmetric product of $v_i \in V_i$ for $1 \leq i \leq m$ by $v_1 v_2 \cdots v_m$.

Definition 2.3.8. *The exterior product* $\mathrm{Ext}(V_1, V_2, \ldots, V_m)$ *of m copies of V, also denoted as* $\mathrm{Ext}^m(V)$ *is defined as*

$$\mathrm{Ext}(V_1, V_2, \ldots, V_m) = \left\{ \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \bigotimes_{i=1}^{m} v_{\sigma i} : \bigotimes_{i=1}^{m} v_i \in V_1 \otimes V_2 \otimes \cdots \otimes V_m \right\}.$$

We denote the exterior product of $v_i \in V_i$ for $1 \leq i \leq m$ by $v_1 \wedge v_2 \wedge \cdots \wedge v_m$.

Remark 2.3.9. $\mathrm{Sym}^m(V)$ *and* $\mathrm{Ext}^m(V)$ *are* $\mathbb{R}G$*-modules whenever* V *is an* $\mathbb{R}G$*-module.*

Lemma 2.3.10. *For positive integers* $m \leq n$ *and a sequence* $\mathrm{x} = x_1, x_2, \ldots, x_n$ *we have the isomorphism of* $\mathbb{R}\mathfrak{S}_n$ *modules,*

$$\phi : \mathbb{R}^{\mathrm{ext}}[\mathcal{S}_{n,m}]_{(1)} \to \mathrm{Ext}^m(\mathbb{R}[\mathrm{x}]_{(1)})$$

satisfying $\phi(x_{i_1,i_2,\ldots,i_n}) = x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n}$. □

Since the variables $x_{i,j,k}$ in the polynomial in Equation 2.3.1 all are ± 1 we only need to optimize over the quotient ring $\mathbb{R}^{\mathrm{ext}}[\mathcal{S}_{n,3}]/I$ of polynomials that vanish on I, where I is the ideal generated by ${x_{i,j,k}}^2 - 1$ for all $1 \leq i < j < k \leq n$. The polynomial terms are degree either 4, 2 or 0. Note that multiplying ${x_{i,j,k}}^2$ to any element of the quotient ring leaves it unchanged, it is enough to consider the degree 4 terms. We conclude that we need to optimize over the ring

$$\mathrm{Sym}^4(\mathrm{Ext}^3(\mathbb{R}[\mathrm{x}]_{(1)})).$$

where the Sym^k-term above covers the degree-k terms.

2.4 General theory of sum-of-squares expressions

We now introduce the general theory of sum-of-squares optimization. See the papers [8], [16] and [17] for more details. Recall real symmetric positive semi-definite matrices are matrices all of whose eigenvalues are non-negative.

Definition 2.4.1. *We say* p *is a* d-sum-of-squares *(or simply* d-SOS*) if* p *can be expressed as the sum of squares of degree* d *polynomials.*

Theorem 2.4.2. *Let* p *be a polynomial* $p(x_1, x_2, \ldots, x_n)$ *of degree* d*. Let* v *be the (column) vector where the entries are the monomials of degree* d *or less (and therefore has length* $\binom{n+d}{d}$*). The polynomial* p *is a* d*-SOS if and only if there exists a* $\binom{n+d}{d} \times \binom{n+d}{d}$ *symmetric positive semi-definite matrix* M *such that*

$$v^T M v = p$$

Proof. Set $m = \binom{n+d}{d}$. We need a preliminary result. We claim that any real symmetric positive-definite matrix M has a factorization $M = A^T A$. To prove it let us diagonalize M via an orthogonal matrix U. That is

$$M = U^T D U \tag{2.4.1}$$

where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ has only non-negative eigenvalues. Define a matrix $V = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_m})$. Now we have $D = V^T V$ and thus

$$M = U^T V^T V U.$$

Setting $A = VU$ our claim is proved.

Now we assume $p = v^T M v$. By means of the factorization above we have $p = v^T A^T A v = \|Av\|^2$ which makes p a d-SOS.

Conversely, if p is a d-SOS, then p occurs as the sum of squares of polynomials each of which is a linear combination of monomials. Thus the list of polynomials whose squares add up to p is encoded in the column vector Av for some $m \times m$ matrix A. Then the sum of squares of the concerned polynomials is nothing but $p = \|Av\|^2 = v^T A^T A v$. However, $A^T A$ is a real symmetric positive semi-definite matrix and we are done. □

2.5 Classic theory of Specht modules

In general, finding SOS certificates is hard, which is why we need to use the technique of semi-definite programming to optimize polynomials. See [8] for various formulations of a semi-definite programming problem and algorithms for their solutions. The optimization algorithms of our interest involves decomposing modules of an appropriate ring into irreducible submodules. In [16] it is proven that the aforementioned algorithms can be improved so that the number of *irreducible submodules* in the decomposition depends only on the degree of the polynomial to be optimized. We will define these terms precisely. The key ingredient in this process is what is known as a Specht module. We now introduce some definitions with the aim of defining a Specht module. Recall the definition of $\mathbb{R}G$-module from 2.3.1.

Definition 2.5.1. *An* $\mathbb{R}G$-submodule *of V is a subset U of V which is also an $\mathbb{R}G$-module.*

Definition 2.5.2. *An $\mathbb{R}G$-module U is said to be* irreducible *if the only $\mathbb{R}G$-submodules of U are U and $\{0\}$.*

Throughout we focus on the case $G = \mathfrak{S}_n$ where $\mathfrak{S}_n$ is the symmetry group on n letters.

Definition 2.5.3. *A* Young tableau *of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, or a λ-*tableau *is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k$ in that order, where each cell is populated with entries from $[n]$.*

Definition 2.5.4. *A* standard Young tableau (SYT) *of shape $(\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$ is a Young tableau in which the cells are populated with entries in $[n]$ so that they increase from left to right as well as top to bottom.*

Example 2.5.5. *Below A and B are Young Tableaux of shapes $(4,2,1)$ and $(3,3,1)$ respectively, but only A is a SYT.*

$$A = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}, \qquad B = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 3 & 7 \\ \hline 5 \\ \cline{1-1}\end{array}.$$

The group $\mathfrak{S}_n$ acts on tableaux by permuting labels as usual. See example below:

Example 2.5.6. *For a permutation $\pi = (12)(345)$ and a tableau $\tau = \begin{array}{|c|c|c|c|}\hline 5 & 2 & 4 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$, we have*

$$\pi \cdot \tau = \begin{array}{|c|c|c|c|}\hline 3 & 1 & 5 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

Definition 2.5.7. *A* tabloid *of shape λ or a λ-*tabloid *is an equivalence class $\{\tau\}$ of a tableaux τ, where $\tau \sim v$ whenever for every i, the i^{th} rows of τ and v have the same elements.*

Example 2.5.8. *Let $\tau = \begin{array}{|c|c|c|c|}\hline 7 & 2 & 4 & 1 \\ \hline 6 & 3 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$ and $v = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$ be tableaux. Then we have the following equation of tabloids:*

$$\{\tau\} = \{v\} = \begin{array}{cccc}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}.$$

Definition 2.5.9. *If a tableau τ has rows $R_1, R_2, \ldots, R_k$ and columns $C_1, C_2, \ldots, C_\ell$ the* row stabilizer *R_τ is given by*

$$R_\tau = \mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \cdots \times \mathfrak{S}_{R_k}$$

and the column stabilizer *is given by*

$$C_\tau = \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \times \cdots \times \mathfrak{S}_{C_\ell}.$$

Definition 2.5.10. *With τ from Example 2.5.8 we have the following stabilizers.*

$$\begin{aligned} R_\tau &= \mathfrak{S}_{\{1,2,4,7\}} \times \mathfrak{S}_{\{3,6\}} \times \mathfrak{S}_{\{5\}}, \\ C_\tau &= \mathfrak{S}_{\{5,6,7\}} \times \mathfrak{S}_{\{2,3\}} \times \mathfrak{S}_{\{4\}} \times \mathfrak{S}_{\{1\}}. \end{aligned}$$

Remark 2.5.11. *Note that the equivalence class $\{\tau\}$ can be expressed as $\{\tau\} = R_\tau \cdot \tau$.*

Definition 2.5.12. *For any subset $H \subseteq \mathfrak{S}_n$ we define the following formal sums:*

$$\begin{aligned} H^+ &= \sum_{\pi \in H} \pi, \\ H^- &= \sum_{\pi \in H} (-1)^\pi \cdot \pi. \end{aligned}$$

Definition 2.5.13. *We define κ_τ for a partition τ by $\kappa_\tau = C_\tau^-$.*

Definition 2.5.14. *For a partition τ the* associated polytabloid *e_τ is defined by $\kappa_\tau \cdot \{\tau\}$.*

Example 2.5.15. *For $\tau = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$ and* Id *the identity permutation we have*

$$\kappa_\tau = (\mathrm{Id} - (12))(\mathrm{Id} - (34)),$$

$$e_\tau = \begin{array}{ccc}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 2 & 3 & 5 \\ \hline 1 & 4 & \\ \cline{1-2}\end{array} - \begin{array}{ccc}\hline 1 & 4 & 5 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array} + \begin{array}{ccc}\hline 2 & 4 & 5 \\ \hline 1 & 3 & \\ \cline{1-2}\end{array}.$$

Definition 2.5.16. *The space M^λ is defined as the formal linear span of all λ-tabloids. In other words M^λ is said to be the collection of all polytabloids.*

Definition 2.5.17. *The* Specht module *S^λ is the subspace of M^λ generated by e_τ for all λ-tableaux τ.*

Example 2.5.18. *Let $\tau_1 = \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}$, $\tau_2 = \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$, $\tau_3 = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ be tabloids. These tabloids are of shape $\lambda = (2, 1)$. We have the following.*

$$\begin{aligned} M^\lambda &= \{c_1\tau_1 + c_2\tau_2 + c_3\tau_3 : c_1, c_2, c_3 \in \mathbb{R}\}, \\ S^\lambda &= \{c_1\tau_1 + c_2\tau_2 + c_3\tau_3 : c_1, c_2, c_3 \in \mathbb{R}, c_1 + c_2 + c_3 = 0\}. \end{aligned}$$

The latter is true because here S^λ is generated by the polytabloids $\tau_2 - \tau_1$ and $\tau_3 - \tau_1$.

Definition 2.5.19. *The inner product $\langle \cdot, \cdot \rangle$ on M^λ where $\lambda \vdash n$ is a partition is defined as follows*

$$\langle \{\tau\}, \{v\} \rangle = \delta_{\{\tau\},\{v\}}, \text{ where } \{\tau\}, \{v\} \text{ are } \lambda\text{-tabloids}$$

and extended linearly to all of M^λ.

Theorem 2.5.20. [10, Theorem 3] *The irrecudible $\mathbb{R}\mathfrak{S}_n$-modules are precisely the Specht modules S^λ where λ is a partition of n.*

2.6 Polynomial rings in terms of Specht modules

Recall that for a tableau τ of shape $\lambda \vdash n$, we have that R_τ is the subgroup of $\mathfrak{S}_n$ consisting of all permutations that leave the rows of τ invariant.

Definition 2.6.1. *For an $\mathbb{R}\mathfrak{S}_n$-module, define*

$$V^{R_\tau} = \{\mathbf{v} \in V : \sigma \cdot \mathbf{v} = \mathbf{v} \quad \forall \sigma \in R_\tau\}.$$

Theorem 2.6.2. *[17, Theorem 4.9] For τ_λ a tableau of shape λ, the following holds*

$$V^{R_{\tau_\lambda}} \subseteq \bigoplus_{\mu \trianglerighteq \lambda} V^\mu$$

where $V^\mu = (S^\mu)^{\oplus m_\mu}$ for some positive integers m_μ.

Theorem 2.6.3. *The Specht module S^λ has dimension equalling the number of SYTs of shape λ.*

Proof. Follows directly from [18, Theorem 2.5.9 and Theorem 2.6.4] which show that the set

$$\{e_\tau : \tau \text{ is a SYT of shape } \lambda\}$$

is a basis for S^λ. □

Example 2.6.4. *For an integer $k \geq 0$, we count the number of SYTs of shape $(n-k, 1^k)$ - known as* hook *partitions. In an SYT the values ascend as you go down a column or across to the right along a row. Thus a choice of k integers (neither of which can be 1) along the tail (the 1^k-part) of the hook completely determines the SYT. Thus there are $\binom{n-1}{k}$ SYTs of shape $(n-k, 1^k)$. By Theorem 2.6.3 the dimension of $S^{(n-k,1^k)}$ is $\binom{n-1}{k}$.*

Theorem 2.6.5. *Let $\mathrm{x} = x_1, x_2, \ldots, x_n$ be a sequence. Then the following decomposition holds true:*

$$\mathbb{R}[\mathrm{x}]_{(1)} = S^{(n-1,1)} \oplus S^{(n)}.$$

Proof. The monomial x_i is fixed by the tableau τ of shape $(n-1, 1)$ where i is in the tail and the elements other than i are inserted into τ. Thus we have

$$V \subseteq \sum_{\text{shape}(\tau)=(n-1,1)} V^{R_\tau} \subseteq \bigoplus_{\mu \trianglerighteq (n-1,1)} V^\mu = (S^{(n-1,1)})^{\oplus m_1} \oplus (S^{(n)})^{\oplus m_2}$$

where the latter containment and the last equality are due to Theorem 2.6.2. We know that $V^{(n-1,1)} = (S^{(n-1,1)})^{\oplus m_1}$ and $V^{(n)} = (S^{(n)})^{\oplus m_2}$ for some $m_1, m_2 \geq 1$. By Example 2.6.4 the dimension of $(S^{(n-1,1)})^{\oplus m_1} \oplus (S^{(n)})^{\oplus m_2}$ is $m_1 \cdot (n-1) + m_2 \cdot 1$. However this cannot exceed the dimension of V, so we have $m_1(n-1) + m_2 = n$. Since n was arbitrary, we must have $m_1 = m_2 = 1$. □

One can show the following results using Definition 2.5.17.

Lemma 2.6.6. *For $1 \leq k < n$ integers, we have the following isomorphisms*

$$\mathrm{Ext}^k(S^{(n-1,1)}) \cong S^{(n-k,1^k)},$$

$$\mathrm{Ext}^k(S^{(n)}) \cong \{0\} \text{ if } k \geq 2.$$

Lemma 2.6.7. *For integers $1 \leq k < n$ we have the isomorphism*

$$\mathrm{Ext}^k(S^{(n-1,1)} \oplus S^{(n)}) \cong S^{(n-k.1^k)} \oplus S^{(n-(k-1),1^{k-1})}.$$

Proof. The result follows from Lemma 2.6.6 and the fact that

$$\mathrm{Ext}^k(A \oplus B) \cong \sum_{\substack{i+j=k \\ i,j\geq 0}} \mathrm{Ext}^i(A) \otimes \mathrm{Ext}^j(B).$$ □

When $k = 3$ in Lemma 2.6.7 above we have

$$\mathrm{Ext}^3(S^{(n-1,1)} \oplus S^{(n)}) \cong S^{(n-3,1^3)} \oplus S^{(n-2,1^2)}. \tag{2.6.1}$$

By Theorem 2.6.5 we need to work in the ring

$$\mathrm{Sym}^4(S^{(n-3,1^3)})$$

There is no known formula for the multiplicities of the Specht factors of symmetric products. However, we do know by definition that symmetric products are submodules of the corresponding tensor products. We find an upper bound on the multiplicities of the Specht modules in the tensor product. These will be obvious upper bounds for the case of the symmetric product. The following sections will elaborate on this process.

2.7 Permutations and permutation matrices

First we introduce definitions and notations. Let $\mathfrak{S}_n$ denote the group of permutations. All permutations hereafter will be defined in one-line notation. We will denote by $[n]$ the set of all positive integers not exceeding n, i.e., $\{1, 2, ..., n\}$. We also denote by $\mathfrak{P}_n$ the set of all permutation matrices of order n.

Definition 2.7.1. *For $u \in \mathfrak{S}_n$ in one-line notation, the* descent set $D(u) \subseteq [n-1]$ *is the set of all positions i such that the i^{th} letter of u exceeds the $(i+1)^{th}$ letter. That is, if $u = a_1a_2 \ldots a_n$ then $D(u) = \{i : 1 \leq i \leq n-1, a_i > a_{i+1}\}$.*

Definition 2.7.2. *The* descent composition $C(u)$ *of* $u \in \mathfrak{S}_n$ *is given by*

$$C(u) = (a_1, a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}, n - a_k)$$

where the descent set $D(u)$ *is given by* $\{a_1, a_2, \dots, a_k\}$.

Example 2.7.3. *For* $u = 7562413 \in \mathfrak{S}_7$, *we have* $D(u) = \{1, 3, 5\}$ *and* $C(u) = (1, 2, 2, 2)$.

Definition 2.7.4. *We define* $\omega : \mathfrak{P}_n \to \mathfrak{S}_n$ *by* $(\delta_{i,\sigma j})_{i,j} \mapsto \sigma$, *where* $\sigma \in \mathfrak{S}_n$ *and* δ *is the Kronecker delta.*

Lemma 2.7.5. *If* $U, V \in \mathfrak{P}_n$ *are matrices, then we have* $\omega(UV) = \omega(U)\omega(V)$.

Proof. Follows by the definition of matrix multiplication. □

Corollary 2.7.6. *For permutations* $u, v \in \mathfrak{S}_n$, *we have* $\omega^{-1}(u)\omega^{-1}(v) = \omega^{-1}(uv)$.

Definition 2.7.7. *Let* $1 \le a, b \le n + 1$ *be a two integers,* M *be an* $n \times n$ *matrix. We write the matrix* M *in block form below*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A *is an* $(a - 1) \times (b - 1)$ *matrix. We define,*

$$(M)_{a,b} := \begin{pmatrix} A & \mathbf{0} & B \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ C & \mathbf{0} & D \end{pmatrix}.$$

where $\mathbf{0}$*'s are (column) zero vectors of appropriate length.*

Whenever $u = \omega(U)$ we set $(u)_{a,b} = \omega((U)_{a,b})$.

Lemma 2.7.8. *For* $1 \le a, b, c \le n + 1$ *and* M, N *matrices of order* $n \times n$ *we have,*

$$(M)_{a,b} \cdot (N)_{b,c} = (MN)_{a,c}.$$

Proof. The proof follows by definition of matrix multiplication. □

Corollary 2.7.9. *For integers* $1 \le a, b, c \le n + 1$ *and two permutations* $u, v \in \mathfrak{S}_n$ *we have,*

$$(u)_{a,b} \cdot (v)_{b,c} = (uv)_{a,c}.$$

Proof. Follows by multiplicativity property in Lemma 2.7.5 and the definition of matrix multiplication. □

Corollary 2.7.10. *For a permutation $u \in \mathfrak{S}_n$, integers $a, b \in [n+1]$ and $b = u(a)$, we have $(u)_{b,a}{}^{-1} = (u^{-1})_{a,b}$.*

Proof. By definition of permutation matrices, we have

$$(u)_{b,a}^{-1} = \omega((U)_{b,a})^{-1} = \omega((U)_{b,a}^{T}) = \omega((U^T)_{a,b}) = (u^{-1})_{a,b}. \qquad \square$$

In what follows we denote by $\pi \circ \mu$ the concatenation of compositions (or partitions) π and μ.

Lemma 2.7.11. *Suppose $u \in \mathfrak{S}_n$ is a permutation, a, n, r integers such that $1 \leq a \leq n - r$, $b = u(a)$ and π a composition of r. Then we have the equivalence*

$$C(u) = (n-r) \circ \pi \iff C((u)_{b,a}) = (n-r+1) \circ \pi.$$

Proof. First assume $C(u) = (n-r) \circ \pi$. Let $(U)_{b,a} = \omega^{-1}((u)_{b,a})$. Since $a \leq n-r$, we use Definition 2.7.7 and conclude that $(u)_{b,a}$ is ascending for the first $n-r+1$ letters. Examining the blocks in $(U)_{b,a}$ it is also clear that the descent composition of the word formed by the last r letters of $(u)_{b,a}$ is π just as in case of u. Conversely, we delete the row and the column containing the entry at (b, a) and conclude that the descent composition of u is $(n-r) \circ \pi$. □

We now define a left-inverse of the family of operations defined in Definition 2.7.7.

Definition 2.7.12. *For M an $n \times n$ matrix and indices $1 \leq a, b \leq n$ let $(M)^{a,b}$ be the matrix M where the a^{th} row and b^{th} column removed.*

Whenever $u = \omega(U)$ and $u(b) = a$ we denote $(u)^{a,b} = \omega((U)^{a,b})$.

Lemma 2.7.13. *Suppose we have three integers $1 \leq a, b, c \leq n$ and two matrices in block form*

$$M_i = \begin{pmatrix} A_i & \mathbf{0} & B_i \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ C_i & \mathbf{0} & D_i \end{pmatrix}, \quad i = 1, 2$$

where the singleton block 1 in M_1 and M_2 are in positions (a, b) and (b, c) respectively. Then we have

$$(M_1 M_2)^{a,b} = (M_1)^{a,b} (M_2)^{b,c}. \tag{2.7.1}$$

Proof. Follows by definition of matrix multiplication. □

Corollary 2.7.14. *For integers $1 \leq a, b, c \leq n+1$ and permutations $u, v \in \mathfrak{S}_n$ such that $u(b) = a$ and $v(c) = b$ we have,*

$$(u)^{a,b} \cdot (v)^{b,c} = (uv)^{a,c}.$$

Proof. Follows by multiplicativity property in Lemma 2.7.5 and the definition of matrix multiplication. □

Corollary 2.7.15. *If $u \in \mathfrak{S}_n$ is a permutation, $a \in [n]$ and $b = u(a)$, then we have $\left((u)^{b,a}\right)^{-1} = (u^{-1})^{a,b}$.*

Proof. The proof is analogous to that of Corollary 2.7.10. □

Corollary 2.7.16. *If $M = \begin{pmatrix} A & \mathbf{0} & B \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ C & \mathbf{0} & D \end{pmatrix}$ is a matrix where the singleton block containing '1' is at position (b, a), then*

$$((M)^{b,a})_{b,a} = M. \tag{2.7.2}$$

Proof. Follows by Definitions 2.7.7 and 2.7.12. □

Corollary 2.7.17. *If $v \in \mathfrak{S}_n$ is a permutation satisfying $v(a) = b$, then*

$$((v)^{b,a})_{b,a} = v. \tag{2.7.3}$$

Proof. One proves this statement by passing to the permutation matrices and by using Corollary 2.7.16. □

Lemma 2.7.18. *For $n \geq 1$ if $v \in \mathfrak{S}_{n+1}$ satisfies $v(a) = b$ and $v(a+1) = b+1$ for some $1 \leq a, b \leq n+1$, then there exists unique $u \in \mathfrak{S}_n$ such that $u(a) = b$ and $(u)_{b,a} = v$, and $u = (v)^{b,a}$.*

Proof. We pass to permutation matrices and note by Definition 2.7.12 that $u = (v)^{b,a}$ satisfies $u(a) = b$. Now by Corollary 2.7.16 the existence follows. The map which sends u to $(u)_{b,a}$ is injective by definition so the uniqueness follows. □

2.8 Ascent behavior of permutations

We need a few preliminary definitions for this study on ascent behavior of permutations.

Definition 2.8.1. *For integers $1 \leq a \leq n$ the a^{th}* nerve *of a sequence* $\mathbf{u}$ *of permutations* $(u_1, u_2, \ldots, u_k) \in (\mathfrak{S}_n)^k$ *is a sequence* $\mathbf{a} = (a_0, a_1, \ldots, a_k) \in [n]^{k+1}$ *with* $a_0 = a$ *and* $a_j = u_j(a_{j-1})$ *for all* $1 \leq j \leq k$*. We use notation* $\mathbf{a} = \mathfrak{N}_{a_0}(\mathbf{u})$*.*

Definition 2.8.2. *The* cumulation *of a sequence* $\mathbf{u} = (u_1, u_2, \ldots, u_k) \in (\mathfrak{S}_n)^k$ *is a sequence* $\overline{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_k) \in (\mathfrak{S}_n)^{k+1}$ *such that* σ_0 *is the identity and* $\sigma_i = u_i \sigma_{i-1}$ *for all* $i \geq 1$*. We use notation* $\overline{\sigma} = \mathfrak{C}(\mathbf{u})$*.*

Remark 2.8.3. *Note that for all* $1 \leq i \leq k$*, the permutation* σ_i *above is an* i^{th} *partial product of the sequence* $(u_1, u_2, \ldots, u_k)$*. That is* $\sigma_i = u_i u_{i-1} \cdots u_1$*. Further, if* $\overline{\sigma} = \mathfrak{C}(\mathbf{u})$ *then the* a^{th} *nerve of* $\mathbf{u}$ *is given by* $\mathbf{a} = (\sigma_0(a), \sigma_1(a), \ldots, \sigma_k(a))$*.*

Lemma 2.8.4. *If* $v \in \mathfrak{S}_n$ *is a permutation (in one-line notation) and* $a, b \in [n]$ *such that* $a \leq b$ *and* v *is ascending from position* a *thru* b*, then* $v(b) - v(a) \geq b - a$*.*

Proof. The proof follows from the fact that each ascending step is at least 1. □

Theorem 2.8.5. *Suppose the sequence* $\mathbf{v} \in (\mathfrak{S}_n)^k$ *has a nerve* $\mathfrak{N}_{a_0}(\mathbf{v}) = (a_0, a_1, \ldots, a_k)$ *and cumulation* $\mathfrak{C}(\mathbf{v}) = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ *and the following holds - for each* $1 \leq i \leq k$*,* v_i *is ascending from* $\sigma_{i-1}(a_0)^{th}$ *to* $\sigma_{i-1}(a_0+1)^{th}$ *position. Then the nerve* $\mathfrak{N}_{a_0+1}(\mathbf{v})$ *satisfies:*

$$\sigma_k(a_0+1) = a_k + 1 \implies \mathfrak{N}_{a_0+1}(\mathbf{v}) = (a_0+1, a_1+1, \ldots, a_k+1).$$

Proof. By Remark 2.8.3 we have

$$\sigma_j(a_0) = a_j \quad \forall 1 \leq j \leq k. \tag{2.8.1}$$

By Lemma 2.8.4 applied in succession, the following inequalities hold:

$$\sigma_{j+1}(a_0+1) - \sigma_{j+1}(a_0) \geq \sigma_j(a_0+1) - \sigma_j(a_0) \geq 1 \quad \forall 1 \leq j \leq k-1. \tag{2.8.2}$$

Now assume $v(a_0+1) = a_k + 1$. We have $v(a_0) = \sigma_k(a_0) = a_k$. Thus we have

$$v(a_0+1) - v(a_0) = 1.$$

Thus for all $1 \leq j \leq k-1$ from the inequality (2.8.2) we have

$$\sigma_j(a_0+1) - \sigma_j(a_0) \leq 1.$$

But permutations are injective maps, therefore

$$\sigma_j(a_0+1) - \sigma_j(a_0) = 1. \tag{2.8.3}$$

From equations (2.8.1) and (2.8.3) we have that

$$\sigma_j(a_0+1) = a_j + 1 \quad \forall 1 \leq j \leq k. \tag{2.8.4}$$

Now from (2.8.4) and the definition of cumulation we have for $1 \leq j \leq k$ that

$$v_j(a_{j-1}+1) = v_j\sigma_{j-1}(a_0+1) = \sigma_j(a_0+1) = a_j + 1. \qquad \square$$

Corollary 2.8.6. *Suppose $(v_1, v_2, \ldots, v_k) \in (\mathfrak{S}_n)^k$ has a nerve $\mathfrak{N}_{a_0}(\mathbf{v}) = (a_0, a_1, \ldots, a_k)$ and cumulation $\mathfrak{C}(\mathbf{v}) = (\sigma_0, \sigma_1, \ldots, \sigma_k)$ where $v = \sigma_k$. Suppose further that $0 \leq r_1, r_2, \ldots, r_k$ are integers such that $n \geq a_0 + \sum_{i=1}^{k} r_i$ and $C(v_i) = (n - r_i) \circ \mu_i$ with μ_i composition of r_i for all $1 \leq i \leq k$. Then we have for all $1 \leq j \leq k$ the inequality*

$$a_{j-1} \leq n - r_j.$$

Proof. By Remark 2.8.3 we need to show that $\sigma_{j-1}(a_0) \leq n - r_j$. We prove this inequality for all $1 \leq j \leq k$ by proving the two inequalities below

$$\sigma_{j-1}(a_0) \leq a_0 + \sum_{i=1}^{j-1} r_i, \tag{2.8.5}$$

$$a_0 + \sum_{i=1}^{j-1} r_i \leq n - r_j. \tag{2.8.6}$$

The inequality (2.8.6) is true since $n - \sum_{i=1}^{j} r_i - a_0 \geq n - \sum_{i=1}^{k} r_i - a_0 \geq 0$ by hypothesis. We now prove the second inequality in (2.8.5) by induction. For $j = 1$ this becomes $\sigma_0(a_0) \leq a_0$ which is true since σ_0 is the identity. This establishes the basic step. Inductively assume (2.8.5) holds for a particular $1 \leq j < k$. This combined with (2.8.6) shows that $\sigma_{j-1}(a_0) \leq n - r_j$. Thus the ascending behavior of v_j ensures that v_j is increasing from $\sigma_{j-1}(a_0)^{th}$ to $(n-r_j)^{th}$ position thus Lemma 2.8.4 be comes applicable. Thus we have

$$\begin{aligned} n - v_j(\sigma_{j-1}(a_0)) &\geq v_j(n-r_j) - v_j(\sigma_{j-1}(a_0)) \\ &\geq n - r_j - \sigma_{j-1}(a_0) \\ &\geq n - r_j - \left(a_0 + \sum_{i=1}^{j-1} r_i\right) \end{aligned}$$

where the first inequality is because $v_j \in \mathfrak{S}_n$, the second by Lemma 2.8.4 and the third by induction hypothesis. Rearranging the terms yields

$$n - v_j(\sigma_{j-1}(a_0)) \geq n - \left(a_0 + \sum_{i=1}^{j} r_i\right). \tag{2.8.7}$$

Canceling the term n in the equation (2.8.7) we have $a_j = \sigma_j(a_0) = v_j\sigma_{j-1}(a_0) \leq a_0 + \sum_{i=1}^{j} r_i$ which proves the induction step. $\square$

Corollary 2.8.7. *Suppose n, r_i where $1 \le i \le k$ are integers with $n \ge 1 + \sum_{i=1}^{k} r_i$ and μ_i is a composition of r_i for each i. Let $\mathbf{v} = (v_1, v_2, \dots, v_k) \in (\mathfrak{S}_{n+1})^k$ with nerve $\mathfrak{N}_1(\mathbf{v}) = (1, a_1, \dots, a_k)$ and cumulation $\mathfrak{C}(\mathbf{v}) = (\sigma_0, \sigma_1, \dots, \sigma_k)$ satisfying the following equality*

$$C(v_i) = (n + 1 - r_i) \circ \mu_i \quad \forall 1 \le i \le k.$$

Then we have the implication

$$\sigma_k(2) = a_k + 1 \implies \mathfrak{N}_2(\mathbf{v}) = (2, a_1 + 1, a_2 + 1, \dots, a_k + 1).$$

Proof. Firstly let $\sigma_k(2) = a_k + 1$. By Remark 2.8.3 we have $\sigma_k(1) = a_k$. Thus we have as an immediate consequence,

$$\sigma_k(2) - \sigma_k(1) = 1.$$

Again by Remark 2.8.3 we get $\sigma_j(1) = a_j$ for all $0 \le j \le k$. If we show

$$\sigma_{j-1}(1) < \sigma_{j-1}(2) \le n - r_j + 1 \tag{2.8.8}$$

then we are done because we know hypothesis implies that v_j is ascending for the first $n - r_j + 1$ letters; this forces v_j to ascend from $\sigma_{j-1}(1)^{th}$ to $\sigma_{j-1}(2)^{th}$ position for every j and we are done by Theorem 2.8.5. Thus it suffices to prove (2.8.8). By hypothesis we have $n + 1 \ge 2 + \sum_{i=1}^{k} r_i$. Thus by Corollary 2.8.6 with $n + 1$ in place of n and $a_0 = 2$ the second inequality in (2.8.8) is true.

We turn to proving the first part of equation (2.8.8). For $j = 1$ the inequality holds since σ_0 is the identity. Let us assume the inequality holds for some $1 \le j < k$. By the second inequality in (2.8.8) we conclude

$$\sigma_{j-1}(1) < \sigma_{j-1}(2) \le n - r_j + 1.$$

The ascending behavior of v_j now ensures

$$\sigma_j(1) = v_j \sigma_{j-1}(1) < v_j \sigma_{j-1}(2) = \sigma_j(2)$$

and the proof follows by induction. Thus (2.8.8) holds and the theorem follows. □

2.9 Specht factors of the tensor product of Specht modules

In this section, we are interested in the multiplicities of Specht modules in the direct-sum decomposition of the tensor product of finitely many Specht modules. We begin by stating a remark which is straightforward to check.

Remark 2.9.1. *It can be checked that there is at most one SYT τ for any permutation $u \in \mathfrak{S}_n$, such that the letters of u inserted in order from left to right, bottom to top forms the SYT τ. This is because $C(u)$ determines the rows of τ uniquely. That brings us to the below definition.*

Definition 2.9.2. *For a permutation $u \in \mathfrak{S}_n$, if the letters of u inserted in order from left to right, bottom to top forms a SYT τ of shape λ then we say that the* companion tableau $\mathfrak{T}(u)$ *is τ and the* companion partition $T(u)$ *is λ. Else we write $\mathfrak{T}(u) = \emptyset$ and $T(u) = \emptyset$.*

Example 2.9.3. *For $u = 5271346$, we have $T(u) = (4, 2, 1)$ since we have the following companion tableau:*

$$\mathfrak{T}(u) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$$

On the other hand for $v = 6571423$, we have $T(v) = \emptyset$.

Definition 2.9.4. *A tableau A is said to be* truncatable *if deleting the first entry, shifting the first row one cell to the left and then subtracting 1 from each entry gives a valid tableau which is called the* truncation *of A denoted by A^{tr}.*

Example 2.9.5. *Among the tableaux below, A is not truncatable whereas B is truncatable with truncation given below.*

$$A = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad B = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad B^{tr} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

Definition 2.9.6. *Let $(Q, \tau) \in (P_n)^k \times P_n$, $Q = (\mu_1, \mu_2, \ldots, \mu_k)$. We define $\mathcal{P}_{Q,\tau}$ to be the set of all tuples $(u_1, u_2, \ldots, u_k, u) \in (\mathfrak{S}_n)^{k+1}$ satisfying the following:*

i. $C(u_i) = \mu_i$ *for* $1 \leq i \leq k$;

ii. $u = u_k u_{k-1} \cdots u_1$; *and*

iii. $T(u^{-1}) = \tau$.

For τ, μ_i as above let the multiplicity of S^τ in the tensor product $\bigotimes_{1 \leq i \leq k} S^{\mu_i}$ be denoted by $\left\langle S^\tau, \bigotimes_{1 \leq i \leq k} S^{\mu_i} \right\rangle$.

Theorem 2.9.7. *[9, Theorem 17] (Gessel) For an integer $n \geq 1$, and $Q \in (P_n)^k$ where $Q = (\mu_1, \mu_2, \ldots, \mu_k)$ and a partition $\tau \vdash n$, we have*

$$\left\langle S^\tau, \bigotimes_{1 \leq i \leq k} S^{\mu_i} \right\rangle = \left| \mathcal{P}_{Q,\tau} \right|.$$

For the rest of this article, we refer to partitions of n whose first part is $\lambda_1 = n-r$ for some $r \geq 0$. We denote such a partition by $(n-r,\pi)$ where $\pi \vdash r$.

Theorem 2.9.8. *For $n \geq 0$ and $1 \leq k \leq n-1$ integers we have*

$$\left\langle S^{(n)}, S^{(n-k,1^k)} \otimes S^{(n-m,1^m)} \right\rangle = \delta_{k,m}.$$

Proof. Set $Q = ((n-k,1^k),(n-m,1^m))$ and $\tau = (n)$ the trivial partition.

Suppose $k = m$. By Definition 2.9.6, $\mathcal{P}_{Q,\tau}$ must only have tuples (u,v,Id) which satisfy

$$D(u) = D(v) = [n-k, n-1].$$

This forces $u(n-k) = v(n-k) = n$. By definition of $\mathcal{P}_{Q,\tau}$ we have $v = u^{-1}$. Consequently we have that $v(n) = u(n) = n-k$. Thus, we deduce directly from the descent sets that $u = v$ and u is the involution that reverses the last $k+1$ letters. Thus $\mathcal{P}_{Q,\tau} = \{(u,u,\mathrm{Id})\}$ with u as above. By Theorem 2.9.7 the trivial summand has multiplicity 1 and the case $k = m$ is disposed.

Now we claim that $\mathcal{P}_{Q,\tau} \neq \emptyset$ implies $k = m$. If we prove this implication then we are done. We know

$$D(u) = [n-k, n-1] \quad \text{and} \quad D(v) = [n-m, n-1].$$

As before we have the equation $u = v^{-1}$ and consequently the equalities $u(n-k) = v(n-m) = n$, $u(n) = n-m$ and $v(n) = n-k$. We therefore have a strict decreasing sequence in $\{n,\ldots,n-m\}$ in u, of length $k+1$. The longest such sequence has length $m+1$. Thus we deduce $k \leq m$. Similarly with v instead we show that $m \leq k$. This forces $k = m$. □

Theorem 2.9.9. *For integers k, m such that $k+m \leq n-1$ and partitions $\pi \vdash k$, $\sigma \vdash m$, $\mu \vdash r$ and $\lambda = (n-r,\mu)$ we have the implication*

$$\left\langle S^{\lambda}, S^{(n-k,\pi)} \otimes S^{(n-m,\sigma)} \right\rangle \neq 0 \implies r \leq k+m.$$

Proof. Let Q be $((n-k)\circ\pi, (n-m)\circ\sigma)$. We firstly have that the tuples (u_1,u_2,v) in $\mathcal{P}_{Q,\lambda}$ must satisfy $u_2u_1 = v$ and the following descent set containments:

$$D(u_1) \subseteq [n-k, n-1] \quad \text{and} \quad D(u_2) \subseteq [n-m, n-1].$$

Every permutation u_1 with a descent set as above satisfies $u_1(1) \leq k+1 \leq n-m$. Now as a consequence of the $D(u_2)$ above, we have $v(1) = u_2(u_1(1)) \leq u_2(k+1) \leq k+m+1$. Therefore, we have

$$v(1) \leq k+m+1. \tag{2.9.1}$$

This shows that v^{-1} written in one-line notation must have 1 occuring in position $v(1) \leq 1+k+m$. Now, a valid SYT must have in its first row, exactly the letters of v^{-1} starting from $v(1)^{th}$ to n^{th} position, which is $n-r = n-v(1)+1$ letters. Thus from (2.9.1) we have,

$$r = v(1) - 1 \leq k + m. \qquad \square$$

Corollary 2.9.10. *For I a finite index set and $r, r_i > 0$ integers $\mu \vdash r$ and $\mu_i \vdash r_i$ for all $i \in I$ a finite ordered indexing set, the following implication holds*

$$\left\langle S^{(n-r,\mu)}, \bigotimes_{i\in I} S^{(n-r_i,\mu_i)} \right\rangle \neq 0 \implies r \leq \sum_{i\in I} r_i.$$

Proof. Note that the case $|I| = 2$ is Theorem 2.9.9. Thus the proof is very similar. $\square$

Theorem 2.9.11. *Suppose for $r_i \geq 0$ integers, $\mu_i \vdash r_i$, $\mu \vdash r$ partitions and $r \geq 1$ we have*

$$S = \{(n-r_i) \circ \mu_i\}_{i=1}^{k}, \quad \tau = (n-r) \circ \mu,$$
$$S' = \{(n-r_i+1) \circ \mu_i)\}_{i=1}^{k}, \quad \tau' = (n-r+1) \circ \mu.$$

Let $\phi : \mathcal{P}_{S,\tau} \to \mathcal{P}_{S',\tau'}$ be defined by

$$\phi(u_1, u_2, \ldots, u_k, u) = ((u_1)_{a_1,a_0}, (u_2)_{a_2,a_1}, \ldots, (u_k)_{a_k,a_{k-1}}, (u)_{a_k,a_0}).$$

where $(a_0, a_1, \ldots, a_k)$ is the first nerve of $(u_1, u_2, \ldots, u_k)$. Then the following holds:

(a) If $n \geq 1 + \sum_{j=1}^{k} r_j$ then ϕ is well-defined and injective.

(b) If $n \geq r + \sum_{j=1}^{k} r_j$ then ϕ is surjective.

Proof. We start by proving the first statement. The map ϕ is injective by definition if well-defined. Thus we only need to show that the image of any element in the domain lies in the range. Let σ_0 be the identity permutation and $\sigma_j = u_j\sigma_{j-1}$ for all $1 \leq j \leq k$. Also let $(a_0, a_1, \ldots, a_k)$ be the first nerve of $(v_1, v_2, \ldots, v_k)$. Thus we get $a_j = \sigma_j(1)$ for all $0 \leq j \leq k$. By Corollary 2.8.6 we have for all $1 \leq i \leq k$ that

$$a_{i-1} = \sigma_{i-1}(1) \leq n - r_i.$$

Since $C(u_i) = (n-r_i) \circ \mu_i$ it follows from Lemma 2.7.11 that for all $1 \leq i \leq k$ we have

$$C\left((u_i)_{a_i,a_{i-1}}\right) = (n-r_i+1) \circ \mu_i. \tag{2.9.2}$$

We additionally have $T(u^{-1}) = (n-r) \circ \mu$ by hypothesis. Thus we have the equalities

$$T({(u)_{a_k,1}}^{-1}) = T((u^{-1})_{1,a_k}) = (n-r+1) \circ \mu, \tag{2.9.3}$$

where the equalities follow from Corollary 2.7.10 and from Lemma 2.7.11 in that order. Now, by Corollary 2.7.9 we know that

$$(u_k)_{a_k,a_{k-1}}(u_{k-1})_{a_{k-1},a_{k-2}}\cdots(u_1)_{a_1,a_0} = (u)_{a_k,a_0}. \tag{2.9.4}$$

From equations (2.9.2), (2.9.3) and (2.9.4) and Definition 2.9.6 it follows that ϕ maps into $\mathcal{P}_{S',\tau'}$ and is therefore well-defined.

We proceed to proving the second statement. We do this by directly computing the pre-image of each element in the range under ϕ and showing that it lies in the domain. Using the fact that the composition of increasing functions is increasing one shows that $v = v_k v_{k-1}\cdots v_1$ is ascending in the first $n - \left(\sum_{i=1}^{k} r_i\right) + 1$ letters. Thus v is ascending until position $r+1 \leq n - \left(\sum_{i=1}^{k} r_i\right) + 1$. Consequently, v^{-1} must have the letters 1, 2, ..., $r+1$ appear in that order a priori not necessarily successively.

However, if v^{-1} fits in an SYT then the letter 1 cannot be followed by a descent. Thus the letters 1 through $r+1$ must be successive letters in v^{-1} and hence the first row of the tableau has the entries $1, 2, \ldots, r+1$ in succession and possibly more entries, all in ascending order. Since $r \geq 1$, the letter 2 must follow 1 in tableau B. Thus letting $v^{-1}(a_k) = 1$ we have $v^{-1}(a_k + 1) = 2$. Therefore we infer

$$v(2) = a_k + 1. \tag{2.9.5}$$

Since $\mu \vdash r$, the second row has r or fewer entries each of which exceeds $r+1$. Set $\mathfrak{T}(v^{-1}) = B$. It can be checked that the first two rows of B have the form given below where $j \leq r$ and $r+1 < m_1 < m_2 < \cdots < m_{n-r-1}$.

1	2	$\cdots$	j	$j+1$	$\cdots$	$r+1$	m_1	m_2	$\cdots$	m_{n-r-1}
n_1	n_2	$\cdots$	n_j							

Further the tableau $B^{tr} = A$ has shape $(n-r)\circ\mu$ and we have $A = \mathfrak{T}((v^{-1})^{1,a_k})$. Thus we get

$$T((v^{-1})^{1,a_k}) = (n-r)\circ\mu. \tag{2.9.6}$$

Let $(1, a_1, \ldots, a_{k-1}, b_k)$ be the first nerve of $(v_1, v_2, \ldots, v_k)$. Thus we have

$$a_k = v(a_0) = v_k v_{k-1}\cdots v_1(a_0) = b_k.$$

This implies $(v_1, v_2, \ldots, v_k)$ has first nerve $(1, a_1, \ldots, a_k)$. This fact combined with (2.9.5) implies that the tuple $(v_1, v_2, \ldots, v_k)$ and the sequence $(1, a_1, \ldots, a_k)$ now satisfy the hypotheses of Theorem 2.8.7 and we therefore have for all $1 \leq j \leq k$ that

$$v_j(a_{j-1}+1) = a_j + 1. \tag{2.9.7}$$

Since $v(1) = a_k$ and $v_j(a_{j-1}) = a_j$ for all j, by equations (2.9.5) and (2.9.7) and Lemma 2.7.18 we have

$$(u)_{a_k,a_0} = v \text{ and } (v)^{a_k,a_0} = u, \tag{2.9.8}$$
$$u(a_0) = a_k, \tag{2.9.9}$$
$$(u_j)_{a_j,a_{j-1}} = v_j \text{ and } (v_j)^{a_j,a_{j-1}} = u_j \quad \forall 1 \leq j \leq k, \tag{2.9.10}$$
$$u_j(a_{j-1}) = a_j \quad \forall 1 \leq j \leq k. \tag{2.9.11}$$

Thus $\phi(u_1, u_2, \ldots, u_k, u) = (v_1, v_2, \ldots, v_k, v)$ and by (2.9.11) we have $(a_0, a_1, \ldots, a_k)$ is the first nerve of $(u_1, u_2, \ldots, u_k)$. It remains to check that u satisfies the conditions in Definition 2.9.6. By Lemma 2.7.10 and equation (2.9.8) we have $u^{-1} = (v^{-1})^{a_0,a_k}$. Thus by (2.9.6) we have the third condition

$$T(u^{-1}) = (n-r) \circ \mu.$$

By the multiplicativity property in Corollary 2.7.14 we have the second condition

$$u_k u_{k-1} \cdots u_1 = (v_k)^{a_k,a_{k-1}} (v_{k-1})^{a_{k-1},a_{k-2}} \cdots (v_1)^{a_1,a_0} = (v)^{a_k,a_0} = u.$$

Finally it now follows from Lemma 2.7.11 that the first condition below also holds:

$$C(u_i) = (n - r_i) \circ \mu_i \quad \forall 1 \leq i \leq k. \qquad \square$$

Corollary 2.9.12. *Given $r_i > 0$ are integers and $\mu_i \vdash r_i$, $\mu \vdash r$ ($r \geq 1$) are partitions, the coefficient $\left\langle S^{(n-r)\circ\mu}, \bigotimes_{i=1}^k S^{(n-r_i)\circ\mu_i} \right\rangle$ is an increasing function of n for $n \geq 1 + \sum_{i=1}^k r_i$ and constant for $n \geq r + \sum_{i=1}^k r_i$.*

Proof. Follows directly from Theorem 2.9.7 and Theorem 2.9.11. $\square$

2.10 Concluding Remarks

Unfortunately we were unable to carry out the calculation of the decomposition of the $\mathfrak{S}_n$ module $\mathrm{Sym}^4(\mathrm{Ext}^3(V))$ into Specht modules. Thus we decomposed $(\mathrm{Ext}^3(V))^{\otimes 4}$ which contains $\mathrm{Sym}^4(\mathrm{Ext}^3(V))$ as a submodule. The decomposition of the tensor product is computed in the Appendix.

The computations were done via a computer search using the programming language of Python, for k-tuples of permutations satisyfing the conditions in the Definition 2.9.6 for $\mathcal{P}_{Q,\tau}$ where Q is a k-tuple of partitions of n. Let m_λ and n_λ be multiplicities of S^λ in $\mathrm{Sym}^4(\mathrm{Ext}^3(V))$ and $(\mathrm{Ext}^3(V))^{\otimes 4}$ respectively. The multiplicities n_λ are tabulated in the Appendix. We have chosen values of n for each table large enough that the hypotheses in Corollary 2.9.12 are satisfied. Thus n_λ is stable at its values given in the Appendix. By [16, Theorem 2.2], we need to solve a semi-definite programming problem of size $\sum_\lambda m_\lambda$. We have $\sum_\lambda m_\lambda \leq \sum_\lambda n_\lambda = 911300$

as the maximum size of the problem, where the last equality is arrived at using equations (A.1) and (A.2) on the multiplicities tabulated in the Appendix. Since the symmetric power has a total dimension of approximately $1/4!^{th}$ of that of the tensor power here, this upper bound is possibly a very crude one.

Appendix: Multiplicities of Specht modules

We tabulate the multiplicities

$$\left\langle S^{(n-k,\mu)}, (S^{(n-3,1^3)})^{\oplus i} \otimes (S^{(n-2,1^2)})^{\oplus j} \right\rangle,$$

where $i + j = 2$ and μ is a partition of k. We do the same for $i + j = 4$. Recall that by Corollary 2.9.12, the multiplicities stabilize for $n \geq k + 3i + 2j$. Also by Corollary 2.9.10 we have $k \leq 3i + 2j$. So partitions $(n-k) \circ \mu$ with $k > 3i + 2j$ have zero multiplicity. In all the tables below we set $Q_{i,j}$ to the multiset $\{(n-3, 1^3)^i, (n-2, 1^2)^j\}$ of partitions of n. Also, each cell in the table is the cardinality of $\mathcal{P}_{Q,\tau}$ for the corresponding partition τ and the multiset $Q = Q_{i,j} \subseteq P_n$. For each table we choose a value $n = k + 3(i + j) \geq k + 3i + 2j$ thus the multiplicities are stable in the sense of Corollary 2.9.12. The exception is Table **??** where we use Theorem 2.9.8. The total multiplicity of $(\mathrm{Ext}^3(V))^{\otimes 2}$ and $(\mathrm{Ext}^3(V))^{\otimes 4}$ can be computed using the below equations:

$$\begin{aligned}
(\mathrm{Ext}^3(V))^{\otimes 2} &= (S^{(n-3,1^3)} \oplus S^{(n-2,1^2)})^{\oplus 2} \\
&= \sum_{i=0}^{2} \left((S^{(n-3,1^3)})^{\otimes i} \otimes (S^{(n-2,1^2)})^{\otimes 2-i} \right)^{\oplus \binom{2}{i}} \\
&= \sum_{i=0}^{2} (Q_{2-i,i})^{\oplus \binom{2}{i}} \qquad \text{(A.1)}
\end{aligned}$$

Similarly to above equation (A.1) we have

$$(\mathrm{Ext}^3(V))^{\otimes 4} = \sum_{i=0}^{4} (Q_{4-i,i})^{\oplus \binom{4}{i}} \qquad \text{(A.2)}$$

The multiplicity of S^τ in the tensor product $\bigoplus_{\sigma \in Q} S^\sigma$ were computed using 2.9.7 that is by enumerating the set of all permutations in $\mathcal{P}_{Q,\tau}$.

Multiplicities for $i + j = 2$

$$\tau = (n-6) \circ \mu \quad \mu \vdash 6 \quad n = 12$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
$(n-6, 1^6)$	1	0	0
$(n-6, 2, 1^4)$	1	0	0
$(n-6, 2^2, 1^2)$	1	0	0
$(n-6, 2^3)$	1	0	0

$$\tau = (n-5) \circ \mu \quad \mu \vdash 5 \quad n = 11$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
$(n-5, 1^5)$	1	1	0
$(n-5, 2, 1^3)$	2	1	0
$(n-5, 2^2, 1)$	2	1	0
$(n-5, 3, 1^2)$	1	0	0
$(n-5, 3, 2)$	1	0	0

$$\tau = (n-4) \circ \mu \quad \mu \vdash 4 \quad n = 10$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
$(n-4, 1^4)$	1	1	1
$(n-4, 2, 1^2)$	2	2	1
$(n-4, 2, 2)$	2	1	1
$(n-4, 3, 1)$	2	1	0
$(n-4, 4)$	1	0	0

$$\tau = (n-3) \circ \mu \quad \mu \vdash 3 \quad n = 9$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
$(n-3, 1^3)$	1	1	1
$(n-3, 2, 1)$	2	2	2
$(n-3, 3)$	2	1	1

$$\tau = (n-2) \circ \mu \quad \mu \vdash 2 \quad n = 8$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
$(n-2, 1^2)$	1	1	1
$(n-2, 2)$	2	1	2

$$\tau = (n-1, 1) \quad n = 7$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
$(n-1, 1)$	1	1	1

$$\tau = (n) \quad n = 4$$

τ	$Q_{2,0}$	$Q_{1,1}$	$Q_{0,2}$
(n)	1	0	1

Multiplicities for $i + j = 4$

$$\tau = (n-12) \circ \mu \quad \mu \vdash 12 \quad n = 24$$

μ	$Q_{4,0}$	$Q_{3,1}$	$Q_{2,2}$	$Q_{1,3}$	$Q_{0,4}$
(1^{12})	1	0	0	0	0
$(2,1^{10})$	3	0	0	0	0
$(2^2,1^8)$	6	0	0	0	0
$(2^3,1^6)$	10	0	0	0	0
$(2^4,1^4)$	11	0	0	0	0
$(2^5,1^2)$	9	0	0	0	0
(2^6)	4	0	0	0	0
$(3,1^9)$	3	0	0	0	0
$(3,2,1^7)$	8	0	0	0	0
$(3,2^2,1^5)$	15	0	0	0	0
$(3,2^3,1^3)$	16	0	0	0	0
$(3,2^4,1)$	11	0	0	0	0
$(3^2,1^6)$	6	0	0	0	0
$(3^2,2,1^4)$	15	0	0	0	0
$(3^2,2^2,1^2)$	15	0	0	0	0
$(3^2,2^3)$	6	0	0	0	0
$(3^3,1^3)$	10	0	0	0	0
$(3^3,2,1)$	8	0	0	0	0
(3^4)	1	0	0	0	0
$(4,1^8)$	1	0	0	0	0
$(4,2,1^6)$	3	0	0	0	0
$(4,2^2,1^4)$	6	0	0	0	0
$(4,2^3,1^2)$	6	0	0	0	0
$(4,2^4)$	3	0	0	0	0
$(4,3,1^5)$	3	0	0	0	0
$(4,3,2,1^3)$	8	0	0	0	0
$(4,3,2^2,1)$	7	0	0	0	0
$(4,3^2,1^2)$	6	0	0	0	0
$(4,3^2,2)$	3	0	0	0	0
$(4^2,1^4)$	1	0	0	0	0
$(4^2,2,1^2)$	3	0	0	0	0
$(4^2,2^2)$	2	0	0	0	0
$(4^2,3,1)$	3	0	0	0	0
(4^3)	1	0	0	0	0

$\tau = (n-11) \circ \mu \quad \mu \vdash 11 \quad n = 23$

μ	$Q_{4,0}$	$Q_{3,1}$	$Q_{2,2}$	$Q_{1,3}$	$Q_{0,4}$
(1^{11})	6	1	0	0	0
$(2,1^9)$	24	3	0	0	0
$(2^2,1^7)$	54	6	0	0	0
$(2^3,1^5)$	84	9	0	0	0
$(2^4,1^3)$	90	9	0	0	0
$(2^5,1)$	60	6	0	0	0
$(3,1^8)$	36	3	0	0	0
$(3,2,1^6)$	102	8	0	0	0
$(3,2^2,1^4)$	162	13	0	0	0
$(3,2^3,1^2)$	156	12	0	0	0
$(3,2^4)$	66	5	0	0	0
$(3^2,1^5)$	84	6	0	0	0
$(3^2,2,1^3)$	156	12	0	0	0
$(3^2,2^2,1)$	126	9	0	0	0
$(3^3,1^2)$	72	6	0	0	0
$(3^3,2)$	42	3	0	0	0
$(4,1^7)$	24	1	0	0	0
$(4,2,1^5)$	72	3	0	0	0
$(4,2^2,1^3)$	108	5	0	0	0
$(4,2^3,1)$	84	4	0	0	0
$(4,3,1^4)$	72	3	0	0	0
$(4,3,2,1^2)$	120	6	0	0	0
$(4,3,2^2)$	60	3	0	0	0
$(4,3^2,1)$	48	3	0	0	0
$(4^2,1^3)$	24	1	0	0	0
$(4^2,2,1)$	36	2	0	0	0
$(4^2,3)$	12	1	0	0	0
$(5,1^6)$	6	0	0	0	0
$(5,2,1^4)$	18	0	0	0	0
$(5,2^2,1^2)$	24	0	0	0	0
$(5,2^3)$	12	0	0	0	0
$(5,3,1^3)$	18	0	0	0	0
$(5,3,2,1)$	24	0	0	0	0
$(5,3^2)$	6	0	0	0	0
$(5,4,1^2)$	6	0	0	0	0
$(5,4,2)$	6	0	0	0	0

$$\tau = (n-10) \circ \mu \quad \mu \vdash 10 \quad n = 22$$

μ	$Q_{4,0}$	$Q_{3,1}$	$Q_{2,2}$	$Q_{1,3}$	$Q_{0,4}$
(1^{10})	25	6	1	0	0
$(2,1^8)$	115	24	3	0	0
$(2^2,1^6)$	273	51	6	0	0
$(2^3,1^4)$	403	72	8	0	0
$(2^4,1^2)$	381	66	7	0	0
(2^5)	161	27	3	0	0
$(3,1^7)$	216	36	3	0	0
$(3,2,1^5)$	608	93	8	0	0
$(3,2^2,1^3)$	864	129	11	0	0
$(3,2^3,1)$	660	96	8	0	0
$(3^2,1^4)$	503	69	6	0	0
$(3^2,2,1^2)$	765	105	9	0	0
$(3^2,2^2)$	376	51	4	0	0
$(3^3,1)$	259	36	3	0	0
$(4,1^6)$	208	24	1	0	0
$(4,2,1^4)$	561	63	3	0	0
$(4,2^2,1^2)$	693	78	4	0	0
$(4,2^3)$	328	36	2	0	0
$(4,3,1^3)$	510	54	3	0	0
$(4,3,2,1)$	608	66	4	0	0
$(4,3^2)$	129	15	1	0	0
$(4^2,1^2)$	160	15	1	0	0
$(4^2,2)$	120	12	1	0	0
$(5,1^5)$	103	6	0	0	0
$(5,2,1^3)$	240	15	0	0	0
$(5,2^2,1)$	231	15	0	0	0
$(5,3,1^2)$	186	12	0	0	0
$(5,3,2)$	131	9	0	0	0
$(5,4,1)$	52	3	0	0	0
(5^2)	3	0	0	0	0
$(6,1^4)$	21	0	0	0	0
$(6,2,1^2)$	39	0	0	0	0
$(6,2^2)$	24	0	0	0	0
$(6,3,1)$	21	0	0	0	0
$(6,4)$	3	0	0	0	0

$$\tau = (n-9) \circ \mu \quad \mu \vdash 9 \quad n = 21$$

μ	$Q_{4,0}$	$Q_{3,1}$	$Q_{2,2}$	$Q_{1,3}$	$Q_{0,4}$
(1^9)	85	25	6	1	0
$(2,1^7)$	426	112	24	3	0
$(2^2,1^5)$	993	243	48	6	0
$(2^3,1^3)$	1346	315	60	7	0
$(2^4,1)$	1007	231	43	5	0
$(3,1^6)$	886	201	36	3	0
$(3,2,1^4)$	2337	500	84	8	0
$(3,2^2,1^2)$	2853	591	97	9	0
$(3,2^3)$	1328	271	44	4	0
$(3^2,1^3)$	1798	356	55	6	0
$(3^2,2,1)$	2084	403	62	6	0
(3^3)	417	78	12	1	0
$(4,1^5)$	968	181	24	1	0
$(4,2,1^3)$	2340	417	54	3	0
$(4,2^2,1)$	2214	387	50	3	0
$(4,3,1^2)$	1836	306	38	3	0
$(4,3,2)$	1246	206	26	2	0
$(4^2,1)$	464	70	8	1	0
$(5,1^4)$	579	82	6	0	0
$(5,2,1^2)$	1149	156	12	0	0
$(5,2^2)$	640	86	7	0	0
$(5,3,1)$	687	87	7	0	0
$(5,4)$	117	13	1	0	0
$(6,1^3)$	178	15	0	0	0
$(6,2,1)$	256	21	0	0	0
$(6,3)$	82	6	0	0	0
$(7,1^2)$	22	0	0	0	0
$(7,2)$	18	0	0	0	0

$$\tau = (n-k) \circ \mu \quad \mu \vdash k \quad k = 7, 8 \quad n = k + 12$$

τ	$Q_{4,0}$	$Q_{3,1}$	$Q_{2,2}$	$Q_{1,3}$	$Q_{0,4}$
$(n-8,1^8)$	251	84	25	6	1
$(n-8,2,1^6)$	1273	388	109	24	3
$(n-8,2^2,1^4)$	2761	803	213	45	6
$(n-8,2^3,1^2)$	3205	904	232	48	6
$(n-8,2^4)$	1470	406	104	21	3
$(n-8,3,1^5)$	2634	732	186	36	3
$(n-8,3,2,1^3)$	6208	1658	398	75	8
$(n-8,3,2^2,1)$	5839	1518	356	66	7
$(n-8,3^2,1^2)$	4106	1038	232	42	6
$(n-8,3^2)$	2763	684	150	27	3
$(n-8,4,1^4)$	2858	715	155	24	1
$(n-8,4,2,1^2)$	5853	1404	290	45	3
$(n-8,4,2^2)$	3272	765	157	24	2
$(n-8,4,3,1)$	3539	801	155	24	3
$(n-8,4^2)$	538	112	20	3	1
$(n-8,5,1^3)$	1735	376	63	6	0
$(n-8,5,2,1)$	2632	542	88	9	0
$(n-8,5,3)$	927	178	27	3	0
$(n-8,6,1^2)$	579	99	10	0	0
$(n-8,6,2)$	483	77	8	0	0
$(n-8,7,1)$	96	10	0	0	0
$(n-8,8)$	6	0	0	0	0
$(n-7,1^7)$	643	230	83	25	6
$(n-7,2,1^5)$	2992	1030	350	106	24
$(n-7,2^2,1^3)$	5724	1911	626	183	42
$(n-7,2^3,1)$	5092	1666	533	154	36
$(n-7,3,1^4)$	5671	1860	590	171	36
$(n-7,3,2,1^2)$	11159	3551	1088	302	66
$(n-7,3,2^2)$	6173	1929	581	159	36
$(n-7,3^2,1)$	5591	1712	501	131	30
$(n-7,4,1^3)$	5594	1724	501	130	24
$(n-7,4,2,1)$	8561	2558	718	180	36
$(n-7,4,3)$	2962	852	228	54	12
$(n-7,5,1^2)$	3031	859	220	46	6
$(n-7,5,2)$	2586	709	176	36	6
$(n-7,6,1)$	862	216	45	6	0
$(n-7,7)$	103	21	3	0	0

$$\tau = (n-k) \circ \mu \quad \mu \vdash k \quad k \leq 6 \quad n = k + 12$$

τ	$Q_{4,0}$	$Q_{3,1}$	$Q_{2,2}$	$Q_{1,3}$	$Q_{0,4}$
$(n-6,1^6)$	1291	507	209	82	25
$(n-6,2,1^4)$	5278	2044	810	312	103
$(n-6,2^2,1^2)$	8320	3162	1221	462	153
$(n-6,2^3)$	4353	1628	624	232	81
$(n-6,3,1^3)$	8624	3256	1234	460	156
$(n-6,3,2,1)$	12500	4619	1716	625	212
$(n-6,3^2)$	3523	1272	460	163	53
$(n-6,4,1^2)$	7046	2562	922	325	106
$(n-6,4,2)$	5952	2118	753	258	87
$(n-6,5,1)$	2890	997	338	108	31
$(n-6,6)$	481	154	49	13	3
$(n-5,1^5)$	1923	849	390	188	81
$(n-5,2,1^3)$	6688	2906	1312	613	274
$(n-5,2^2,1)$	7748	3318	1477	682	311
$(n-5,3,1^2)$	8773	3738	1648	747	342
$(n-5,3,2)$	6981	2932	1280	575	267
$(n-5,4,1)$	5164	2144	919	403	186
$(n-5,5)$	1156	463	193	81	37
$(n-4,1^4)$	2101	1028	528	291	167
$(n-4,2,1^2)$	5804	2796	1417	771	439
$(n-4,2,2)$	3749	1778	900	483	285
$(n-4,3,1)$	5385	2542	1274	679	390
$(n-4,4)$	1682	774	385	199	118
$(n-3,1^3)$	1644	874	496	306	209
$(n-3,2,1)$	3174	1660	935	573	396
$(n-3,3)$	1541	789	442	268	187
$(n-2,1^2)$	874	496	306	209	162
$(n-2,2)$	876	484	301	202	165
$(n-1,1)$	286	168	112	83	74
(n)	45	25	19	14	16

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