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Scrollar Invariants of Tropical Chains of Loops

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Scrollar Invariants of Tropical Chains of Loops

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
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ABSTRACT OF DISSERTATION

Scrollar Invariants of Tropical Chains of Loops

We define scrollar invariants of tropical curves with a fixed divisor of rank 1. We examine the behavior of scrollar invariants under specialization, and compute these invariants for a much-studied family of tropical curves. Our examples highlight many parallels between the classical and tropical theories, but also point to some substantive distinctions.

KEYWORDS: tropical geometry, divisor theory, scrollar invariants, Maroni invariant, young tableaux

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April 8, 2020

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For my family: biological, spiritual, and mathematical.
Your love means the most.

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Chapter 1 Introduction

1.1 History

This is the story of scrollar invariants on tropical chains of loops, what they are, and how one might hope to calculate them. We begin with some mathematical and historical motivation and context.

Our question belongs to the field of algebraic geometry, where the basic objects are varieties, or sets whose elements are solutions to a given system of polynomial equations. Points are the only zero-dimensional varieties, so the first interesting natural questions concern the geometry of curves. One of the goals of this field is to categorize curves into families with similar behaviors. We aim to use characteristics that are independent of embedding the curve in an ambient space, though certain embeddings may provide useful information.

In most cases, the abstract nature of algebraic curves makes them difficult to analyze. The goal of Brill-Noether theory is to infer information about the geometry of a curve by studying the maps it admits into projective space. That is, we examine the existence and behavior of its linear series. This approach has been fruitful from the beginning, yielding notable results like Max Noether's theorem and the Enriques-Babbage theorem by the close of the nineteenth century.

The early twentieth century saw the evolution of two separate ideas that would become central to our study. The first important development was the theory of scrollar invariants, which we define carefully in 2.2. We study curves equipped with a rank 1 linear series, whose behavior is described by these invariants. The first well-known exploration of this idea came in the 1940s from Maroni, who studied the case where there is only a single scrollar invariant (later called the Maroni invariant). This situation has since been well studied, and it is known that there exist genus g curves of Maroni invariant m for all $0 \leq m \leq \frac{g+2}{3}$ [16].

In the general case where there are several scrollar invariants, the situation is more complicated and many natural questions remain unanswered. It is usually quite difficult to calculate the scrollar invariants of a given curve, and it is unknown which sequences of integers a_i arise as scrollar invariants of k-gonal curves. Even in cases where a curve with given scrollar invariants is known to exist, it is unknown whether the space of such curves is irreducible or what its dimension is.

While some developed these ideas in the classical setting, others transitioned to studying general curves rather than rigid fixed curves. As it became clear that considering general points of the moduli space of curves M_g was a valuable perspective, it became standard to use techniques based on degeneration of curves. In the late 1980s, many geometers began to consider a special type of nodal curve with only rational components. Here the combinatorics of the way the components meet each other encodes the interesting geometry of the curve. Early work in this direction used certain curves built out of projective lines called graph curves to establish properties of general curves, as in [4].

Over the next decades, this idea of creating a "combinatorial shadow" that preserves some properties of the original curve grew into the field of tropical geometry. In this field, we consider geometry over the tropical semiring $\mathbb{R} \cup \infty$, with operations of addition and minimum rather than the usual multiplication and addition. The use of this semiring was pioneered by Imre Simon, whose Brazilian heritage is the origin of the adjective "tropical." The development of this theory enabled the expansion of degeneration techniques in a rigorous way.

These two stories have coexisted peacefully for some time, but this paper is part of a recent reunion of ideas. The systematic approach to degeneration arguments is one of the key features of tropical Brill-Noether theory and a valuable tool in the study of scrollar invariants. In 2012, Cools, Draisma, Payne, and Robeva reproved the Brill-Noether theorem in [6], using the tropical theory of divisors on metric graphs originally developed by Baker and Norine [3]. This theory provides insight analogous to the Eisenbud and Harris theory of limit linear series, expanding the idea of formal degeneration. In [18], Pflueger generalized the techniques of Cools, et.al., providing powerful tools that were ready to be applied to other problems. Papers such as [7] consider scrollar invariants through the lens of degenerations, but ours is the first approach to this problem using the formalism of tropical geometry. The remainder of this section gives a summary of our results. For the research article version of this work, we refer the interested reader to [14].

1.2 Summary

In a family of curves of gonality k, the scrollar invariants are not lower semicontinuous. It is therefore often easier to consider the composite scrollar invariants, which we now define. If we order the scrollar invariants

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1},$$

we define the *composite scrollar invariant* σ_j to be the sum of the first j scrollar invariants:

$$\sigma_j = a_1 + a_2 + \cdots + a_j.$$

Of course, the scrollar invariants themselves can be recovered from the set of composite scrollar invariants. The composite scrollar invariants are known to be lower semicontinuous.

In this article, we define tropical analogues of composite scrollar invariants. Key to our study is the observation that the scrollar invariants are determined by the ranks of the line bundles $\pi^*\mathcal{O}_{\mathbb{P}^1}(c)$. Combining this observation with the Baker-Norine theory of divisors on tropical curves, we obtain definitions of tropical composite scrollar invariants. We refer the reader to Section 2.4 for precise definitions.

We prove that composite scrollar invariants cannot increase under specialization.

Theorem 1.2.1. *Let X be a curve over a nonarchimedean field with skeleton isometric to Γ, and let D be a divisor of degree k and rank 1 on X. Then*

$$\sigma_j(X, D) \geq \sigma_j(\Gamma, \operatorname{Trop} D) \text{ for all } j.$$

Having established this relationship between the composite scrollar invariants of a curve and those of its tropicalization, we then compute composite scrollar invariants of certain metric graphs. Of primary interest to us are the *chains of loops*, a much-studied family of metric graphs that has played a central role in tropical proofs of the Brill-Noether Theorem [6] and the Gieseker-Petri Theorem [10], as well as establishing new results such as the Maximal Rank Conjecture for quadrics [11,12] and an analogue of the Brill-Noether Theorem for curves of fixed gonality [5, 13, 17].

By varying the edge lengths, we obtain chains of loops of various gonalities. More precisely, the divisor theory of a chain of loops is determined by its *torsion profile*. We refer the reader to Definition 2.5.1 for a definition. In order for a chain of loops to be hyperelliptic, it must have a specific torsion profile. The torsion profiles corresponding to trigonal chains of loops of genus g are determined by a pair of integers a and b between 1 and g, as described in Corollary 3.0.4. Given such a pair of integers, let

$$\ell = \left\lceil \frac{b-a+4}{2} \right\rceil,$$

and let n be the smallest integer such that

$$g \leq \left\lfloor \frac{3}{2}n + \frac{1}{2}(\ell - 1) \right\rfloor.$$

If $a \neq b$, then the corresponding chain of loops possesses a unique divisor of degree 3 and rank 1, which we denote $D_{a,b}$.

Theorem 1.2.2. *Let Γ be the trigonal chain of loops corresponding to the integers a and b, and let $D_{a,b}$ be the divisor of degree 3 and rank 1 on Γ. Then*

$$\sigma_1(\Gamma, D_{a,b}) = \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

Combining Theorems 1.2.1 and 1.2.2, we see that if X is a curve over a nonarchimedean field with skeleton isometric to Γ, and D is a divisor of rank 1 on X that specializes to $D_{a,b}$, then

$$\sigma_1(X, D) \geq \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

Indeed, we will see that ℓ is the smallest positive integer such that $\mathrm{rk}(\ell D_{a,b}) > \ell$. It follows from Baker's Specialization Lemma that ℓ is a lower bound for $\sigma_1(X, D)$. In general, however, this lower bound is not tight. The integer n has a similar interpretation – it is the smallest positive integer such that $K_\Gamma - nD_{a,b}$ is not effective. It follows from Baker's Specialization Lemma that n is a lower bound for $a_2(X, D)$. Again, this lower bounds is typically not tight. On the curve X, the invariants σ_1 and a_2 satisfy the relationship $a_2 = g + 2 - \sigma_1$, but on the metric graph Γ, the invariants ℓ and n do not satisfy this relationship. Theorem 1.2.2 shows that we can obtain a stronger bound on $\sigma_1(X, D)$ by *averaging* the two invariants ℓ and n.

As the gonality increases, so too does the number of torsion profiles for which the corresponding chain of loops has the given gonality. In these cases, we do not have a closed formula for composite scrollar invariants analogous to Theorem 1.2.2.

Nevertheless, given a torsion profile, we can algorithmically compute the composite scrollar invariants, and we have implemented this algorithm in a Sage program, which can be found in Appendix A or online at

`https://github.com/kalilajo/numberboxes`.

If X is an algebraic curve and D is a divisor of degree k and rank 1 on X, then the datum of the scrollar invariants is equivalent to that of the sequence of ranks $\mathrm{rk}(cD)$. More precisely, the sequence of ranks $\mathrm{rk}(cD)$ is a convex, piecewise linear function in c, and the scrollar invariants correspond to the "bends" between domains of linearity (see Eq. (2.1)). For a tropical curve, however, the sequence of ranks is not necessarily convex. This is perhaps most striking in the trigonal case – that is, when $k = 3$. In this case, the sequence of ranks $\mathrm{rk}(cD)$ exhibits substantively different behavior.

Proposition 1.2.3. *Let Γ be the trigonal chain of loops corresponding to the integers a and b, and let $D_{a,b}$ be the divisor of degree 3 and rank 1 on Γ. Then for $0 \le i < n$, we have*

$$rk((\ell+i)D_{a,b}) = \begin{cases} rk((\ell+i-1)D_{a,b}) + 1 & \text{if } i \text{ is odd} \\ rk((\ell+i-1)D_{a,b}) + 2 & \text{if } i \text{ is even.} \end{cases}$$

It is our hope that the study initiated here could be used to resolve outstanding questions concerning scrollar invariants of classical curves. In order to do this, we would need a lifting result for scrollar invariants. We pose this as an open question.

Question 1.2.1. *Let Γ be a chain of loops, and let D be a divisor of degree k and rank 1 on Γ. Under what circumstances does there exist a curve X, over a nonarchimedean field, with skeleton Γ and a rank 1 divisor D_X on X specializing to D, such that $\sigma_j(X, D_X) = \sigma_j(\Gamma, D)$?*

Chapter 2 Preliminaries

2.1 Classical Setting

We begin by introducing the objects involved in our study. A *variety* is the set of solutions to a system of polynomial equations. In particular, an *algebraic curve* is a variety of dimension one. Our goal is to arrive at a statement about the geometry of a curve by examining collections of its divisors known as *linear series*.

Definition 2.1.1. A *divisor* D on a smooth curve X is a formal sum of points of X with integer coefficients. That is, $D = \sum_{p \in X} D(p) \cdot p$, where only finitely many $D(p) \in \mathbb{Z}$ are nonzero. We say that D is *effective* if $D(p) \geq 0$ for all points p on X.

To gain intuition, it is common to think of a divisor as a collection of signed poker chips placed on the curve. A basic invariant of a divisor is the "net" number of chips involved in this analogy, or technically, its degree.

Definition 2.1.2. The *degree* of a divisor is the sum of coefficients $\sum_{p \in X} D(p)$, denoted $\deg(D)$.

We define addition and scalar multiplication of divisors pointwise, as one would hope. In fact, the set of all divisors on X forms an abelian group, denoted $\mathrm{Div}(X)$. To fully understand the behaviour of divisors, we relate them to rational functions on X. For algebraic background on these objects, we refer the reader to [8].

Given a curve X, we denote its *function field* by $K(X)$. For any point p on X, there is a valuation on $K(X)$ given by the order of vanishing at p, denoted ord_p. The order of vanishing of a rational function is nonzero at only finitely many points on X, so any nonzero function $f \in K(X)^*$ defines a divisor $\mathrm{div}(f) := \sum_{p \in X} \mathrm{ord}_p(f) \cdot p$.

This construction gives an equivalence between divisors.

Definition 2.1.3. We say two divisors D_1 and D_2 are *linearly equivalent* if their difference is the divisor associated to some rational function on X. That is, there is some function $f \in K(X)^*$ such that $\mathrm{div}(f) = D_1 - D_2$. In this case, we write $D_1 \sim D_2$.

It is straightforward to check that this is indeed an equivalence relation. We note that the degree of a divisor is invariant under linear equivalence, but the property of being effective is not. For our purposes, it suffices to consider effective divisors up to linear equivalence, that is, complete linear series.

Definition 2.1.4. The *complete linear series* of D is $|D| := \{E \sim D | E \text{ is effective}\}$.

The idea of linear equivalence allows us to define an important invariant of a divisor which will form the basis for much of our study.

Definition 2.1.5. A divisor D on X has *rank* at least r if $D-E$ is equivalent to an effective divisor for every effective divisor E of degree r. If D is not equivalent to an effective divisor, we say that $\mathrm{rk}(D) = -1$.

Note that the rank is the same for any divisor representative of a linear series. In fact, when studying the linear series of a curve, it is common to stratify them by rank. In particular, there is an object of interest called the *Brill-Noether Locus* of X,

$$W_d^r(X) := \{D \in \mathrm{Div}(X) | \deg(D) = d \text{ and } \mathrm{rk}(D) \geq r\}.$$

Another basic invariant of X is its *genus*, which we define in the standard topological sense and denote throughout by g. In addition to providing a useful stratification of $\mathrm{Div}(X)$, the Brill-Noether loci assist in the definition of our last invariant of X, which measures some relation between the degrees and ranks of the divisors on X.

Definition 2.1.6. The *gonality* k of a curve X is the minimal integer such that $W_k^1(X)$ is non-empty, that is, there is a divisor on X of degree k and rank at least 1.

Work such as [5] and [15] explores the geometry of components of $W_d^r(X)$; we take a different direction. Instead, we fix the gonality k and a divisor $D \in W_k^1(X)$ and examine its scalar multiples $cD \in W_{ck}^r(X)$. Our goal is to find the values of r for which this locus contains a multiple of D, with the aim of calculating the *scrollar invariants* of X, described below.

We note that the theory of divisors can also be developed in terms of line and vector bundles. However, since this approach is more complicated and less intuitive, we limit ourselves to the above discussion and refer the interested reader to [8] or [19].

2.2 The Maroni Invariant and Scrollar Invariants

Given the situation in section 2.1, we ask whether we may stratify the divisors of degree k and rank 1 on X by any other invariants to gain further insight. suppose X has been canonically embedded in $\mathbb{P}^{g-1}$, and that we have fixed a degree k, rank 1 divisor. By geometric Riemann-Roch, a divisor of degree k and rank 1 spans a linear space of dimension $k-2$. Such divisors are parameterized by $\mathbb{P}^1$, and X lies on a rational normal scroll, which admits the following description.

Given that X, with a fixed divisor, is canonically embedded in $\mathbb{P}^{g-1}$, there are integers $a_1, a_2, \ldots, a_{k-1}$ so we may fix $k-1$ linear subspaces of dimension $a_1, a_2, \ldots, a_{k-1}$ (in nondecreasing order). In each subspace, we have a parameterized rational normal curve of degree equal to the subspace dimension. Then drawing a $k-2$-plane through the corresponding points for any choice of parameter t gives a rational normal scroll. The integers $a_1, a_2, \ldots, a_{k-1}$ are called the *scrollar invariants* of the pair (X, D).

Scrollar invariants also have a convenient definition in terms of vector bundles, as follows. Let X be a curve of genus g and $\pi : X \to \mathbb{P}^1$ a dominant map of degree $k \geq 3$. The map π induces a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \pi_* \mathcal{O}_X \to \mathcal{E}^\vee \to 0.$$

The sheaf $\mathcal{E}$ is a vector bundle of rank $k-1$ on $\mathbb{P}^1$, called the *Tschirnhausen bundle* of the map π. Since every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, we may write

$$\mathcal{E} = \bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

The integers a_i are known as the *scrollar invariants* of the map π. We order them so that

$$a_1 \leq a_2 \leq \cdots \leq a_{k-1}.$$

We define the jth *composite scrollar invariant* to be the sum of the first j scrollar invariants:

$$\sigma_j = a_1 + a_2 + \cdots + a_j.$$

The scrollar invariants determine, and are determined by, the sequence of integers $h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c))$. Setting $a_0 = 0$, this can be seen by the following calculation:

$$\begin{aligned} h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c)) &= h^0(\mathbb{P}^1, \pi_*\mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^1}(c)) \\ &= \sum_{i=0}^{k-1} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(c-a_i)) \\ &= \sum_{i=0}^{k-1} \max\{0, c+1-a_i\} \\ &= \max\{(c+1)(j+1) - \sigma_j\}. \end{aligned} \tag{2.1}$$

Note in particular that $h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c))$ is convex as a function in c.

Because $h^0(X, \mathcal{O}_X) = 1$, we see that each of the scrollar invariants a_i is strictly positive. Moreover, for c sufficiently large, we have $h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^1}(c)) = ck - g + 1$, so we see that $\sigma_{k-1} = g + k - 1$.

When $k = 3$, the scrollar invariants are determined by the single value $|a_2 - a_1|$, which is known as the *Maroni invariant* of the trigonal curve. The simplicity of this case is advantageous, and the Maroni invariant has been well studied. The parity of the Maroni invariant agrees with that of g. The space of trigonal curves with given Maroni invariant m is known to be irreducible and, except in the case $m = 0$, it has codimension $m - 1$ in the space of all trigonal curves.

When the gonality of X is at least 4, the situation is more mysterious. One defines the *Maroni locus* $M(\mathcal{E})$ to be the space of k-gonal curves with Tschirnhausen bundle isomorphic to $\mathcal{E}$. In general, given a vector bundle $\mathcal{E}$, it is not even known whether $M(\mathcal{E})$ is empty. That is, there is no known answer to the question of whether, given a gonality k and a sequence of scrollar invariants, there is some divisor D on a k-gonal curve X that has the given scrollar invariants.

Leaving the specifics to our sources, "If we summarize what we have here very sketchily pointed out, disregarding a thousand detailed proofs and objections, we are led to conclude" [9] that the calculation of scrollar invariants is extremely difficult. In particular, we have no method for computing scrollar invariants or finding a k-gonal curve X with a divisor D that has prescribed scrollar invariants.

2.3 Divisor Theory on Metric Graphs

Given the difficulty of investigating scrollar invariants in the classical case, we are led to search for an alternate strategy. We instead use specialization to translate our problem to a question about metric graphs. In this section we give a brief review of divisor theory in this setting, and refer the interested reader to [1] for further details.

Recall that a *metric graph* is a compact, connected metric space Γ obtained by identifying the edges of a graph G with line segments of fixed positive real length. With minor technical differences, the definitions and machinery closely mirror those in the classical case.

Definition 2.3.1. A *divisor* D on a metric graph Γ is a finite formal $\mathbb{Z}$-linear combination of points of Γ. That is, $D = \sum_{v \in \Gamma} D(v) \cdot v$, where $D(v) \in \mathbb{Z}$ is zero for all but finitely many v.

The group of all divisors on a metric graph Γ is simply the free abelian group on points of the metric space Γ, called the *divisor group* $\mathrm{Div}(\Gamma)$ of Γ. Divisors on metric graphs should be thought of as the tropical analogues of divisors on algebraic curves. The analogy of placing signed poker chips on the graph is again helpful for intuition.

As before, we relate our divisors to a tropical analogue of rational functions.

Definition 2.3.2. A *rational function* on a metric graph Γ is a continuous piecewise-linear function $\varphi : \Gamma \to \mathbb{R}$ with integer slopes. The rational functions on Γ form a group under pointwise addition, denoted $\mathrm{PL}(\Gamma)$. Given $\varphi \in \mathrm{PL}(\Gamma)$ and $v \in \Gamma$, we define the *order of vanishing* of φ at v, $\mathrm{ord}_v(\varphi)$, to be the sum of the incoming slopes of φ at v.

Note that $\mathrm{ord}_v(\varphi)$ is nonzero for only finitely many points $v \in \Gamma$. We define the divisor associated to φ as

$$\mathrm{div}(\varphi) = \sum_{v \in \Gamma} \mathrm{ord}_v(\varphi) \cdot v.$$

Definition 2.3.3. We say that two divisors D and D' on a metric graph Γ are *linearly equivalent* if their difference $D - D'$ is equal to $\mathrm{div}(\varphi)$ for some rational function $\varphi \in \mathrm{PL}(\Gamma)$.

As before, it is straightforward to show that linear equivalence is in fact an equivalence relation. For our purposes, it suffices to consider linear equivalence classes of divisors.

A basic invariant of a divisor D is its *degree*, defined to be the integer

$$\deg(D) = \sum_{v \in \Gamma} D(v).$$

In analogy with divisors on algebraic curves, we say that a divisor D is *effective* if $D(v) \geq 0$ for all $v \in \Gamma$. Similarly, we say that a divisor D is *special* if both D and $K_\Gamma - D$ are equivalent to effective divisors, where K_Γ is the canonical divisor

$$K_\Gamma = \sum_{v \in \Gamma} (\mathrm{val}(v) - 2)v.$$

Perhaps the most important invariant of a divisor on a metric graph is its Baker-Norine rank.

Definition 2.3.4. A divisor D has *rank* at least r if $D - E$ is equivalent to an effective divisor for all effective divisors E of degree r. If D is not equivalent to an effective divisor, we say $\mathrm{rk}(D) = -1$.

2.4 Composite Scrollar Invariants and Specialization

To translate our problem from algebraic curves to metric graphs, we make use of specialization. We recall here the basic properties of specialization, and refer the reader to [1] for details. Let K be an algebraically closed field that is complete with respect to a nontrivial valuation

$$\mathrm{val} : X \to \mathbb{R}^*.$$

Let X be an algebraic curve over K. A *skeleton* of X is a certain type of subset of the set of valuations on the function field $K(X)$ that extend the given valuation on K. A skeleton of X is endowed with a topology, giving it the structure of a metric graph. There is a natural map from X to its skeleton Γ. Extending linearly yields the tropicalization map on divisors

$$\mathrm{Trop} : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma).$$

The tropicalization map satisfies an important property, known as Baker's Specialization Lemma.

Lemma 2.4.1. *[1] Let D_X be a divisor on X. Then*

$$rk(D_X) \leq rk(\mathrm{Trop}\, D_X).$$

We now define composite scrollar invariants of divisors on metric graphs.

Definition 2.4.2. Let Γ be a metric graph and D a divisor of degree k and rank 1 on Γ. We define the jth *composite scrollar invariant* of the pair (Γ, D) to be

$$\sigma_j(\Gamma, D) := \min\{m | \mathrm{rk}(cD) \geq (c+1)(j+1) - (m+1) \text{ for all } c\}.$$

Note that $\mathrm{rk}(cD) \geq c$ for all c, with equality if $c = 0$, so $\sigma_0 = 0$. By Riemann-Roch, we have $\mathrm{rk}(cD) \geq ck - g$ with equality if c is sufficiently large, so $\sigma_{k-1} = g + k - 1$.

We note that there are several other ways we could define tropical analogues of these invariants. For example, we could define σ_1 to be the minimum value of c such that $\mathrm{rk}(cD) > c$. For algebraic curves, these two definitions of σ_1 agree because the rank sequence $\mathrm{rk}(cD)$ is convex as a function in c. For metric graphs, however, the rank sequence is not necessarily convex, so these two definitions do not agree.

We now prove a specialization lemma for composite scrollar invariants.

Theorem 2.4.3. *Let X be a curve over a nonarchimedean field with skeleton isometric to Γ, and let D be a divisor of degree k and rank 1 on X. Then*

$$\sigma_j(X, D) \geq \sigma_j(\Gamma, \operatorname{Trop} D) \text{ for all } j.$$

Proof. By Eq. (2.1), for any value of j we have

$$\operatorname{rk}(cD) \geq (c+1)(j+1) - (\sigma_j(X, D) + 1).$$

Simultaneously, by Baker's Specialization Lemma, we have

$$\operatorname{rk}(cD) \leq \operatorname{rk}(c \operatorname{Trop} D) \text{ for all } c.$$

It follows that

$$\operatorname{rk}(c \operatorname{Trop} D) \geq (c+1)(j+1) - (\sigma_j(X, D) + 1) \text{ for all } c.$$

Since $\sigma_j(\Gamma, \operatorname{Trop} D)$ is defined to be the minimum value of m such that

$$\operatorname{rk}(c \operatorname{Trop} D) \geq (c+1)(j+1) - (m+1) \text{ for all } c,$$

we see that

$$\sigma_j(\Gamma, \operatorname{Trop} D) \leq \sigma_j(X, D).$$

□

2.5 Divisors on Chains of Loops

In the remainder of our work, we will consider equivalence classes of special divisors on the metric graph pictured in Figure 2.1. This graph, known as the *chain of loops*, has appeared in several articles that use tropical techniques to develop results in algebraic geometry [5, 6, 10–13, 17, 18]. This graph is particularly nice because of its combinatorial properties and the fact that it can be constructed recursively.

We denote by v_k the point where the k^{th} loop meets a bridge on the left and by w_k the point where the k^{th} loop meets a bridge on the right. We label edges by their initial and terminal vertices when traversing the loop counter-clockwise. For example, w_2v_2 denotes the top edge of the second loop.

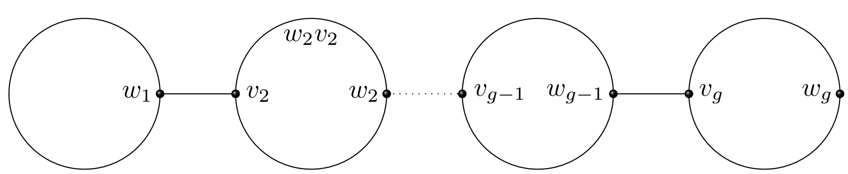


Figure 2.1: A chain of loops Γ

In this section we summarize the main result of [18] and draw a few corollaries.

Definition 2.5.1. Let ℓ_i denote the length of the i^{th} cycle, and let $\ell(w_i v_i)$ denote the length of the counterclockwise edge from w_i to v_i. If $\ell(w_i v_i)$ is an irrational multiple of ℓ_i, then the i^{th} *torsion order* m_i is 0. Otherwise, m_i is the minimum positive integer such that $m_i \cdot \ell(w_i v_i)$ is an integer multiple of ℓ_i. We record the torsion order of each loop as the vector $\underline{m} = (m_1, m_2, \ldots, m_g)$, called the *torsion profile* of Γ.

To represent divisors on chains of loops, we use the fact that the Picard group $\mathrm{Pic}(\Gamma)$ has a natural coordinate system. Denote by $\langle x \rangle_i$ the point on the i^{th} loop of Γ located $x \cdot \ell(w_i v_i)$ units clockwise from w_i. Note that $\langle x \rangle_i = \langle y \rangle_i$ if and only if $x \equiv y \pmod{m_i}$.

By the Tropical Abel-Jacobi theorem [2], every divisor class D of degree d on Γ has a unique *break divisor* representative

$$D \sim (d-g)w_g + \sum_{i=1}^{g} \langle \xi_i(D) \rangle_i$$

for some $\xi_i(D) \in \mathbb{R}/m_i\mathbb{Z}$. These divisors are our primary object of study. We also define a helpful combinatorial object.

Definition 2.5.2. An $\underline{m}$*-displacement tableau* on a partition λ is a function $t : \lambda \to \{1, \ldots, g\}$ such that:

1. t increases across each row and column of λ, and
2. if $t(x,y) = t(x',y') = i$, then $y - x \equiv y' - x' \pmod{m_i}$.

Each such tableau t defines a locus $\mathbb{T}(t) \subseteq \mathrm{Pic}^d(\Gamma)$ homeomorphic to a torus of dimension equal to g minus the number of symbols appearing in t. Specifically,

$$\mathbb{T}(t) = \{D \in \mathrm{Pic}^d(\Gamma) | \xi_{t(x,y)}(D) \equiv y - x \pmod{m_{t(x,y)}} \text{ for all } (x,y) \in \lambda\}.$$

Note that if the function t is not surjective, then there is a symbol i not appearing in the tableau, and a corresponding value ξ_i upon which no restrictions are placed.

Recall that $W^r_d(\Gamma)$ is the set of all divisor classes of degree d and rank at least r on Γ. Pflueger's main result in [18] is the following.

Theorem 2.5.3. *[18] Let Γ be a chain of loops of genus g and torsion profile $\underline{m}$, and let r and d be positive integers with $r > d - g$. Let λ be the rectangular partition of dimensions $(r+1) \times (g-d+r)$. Then*

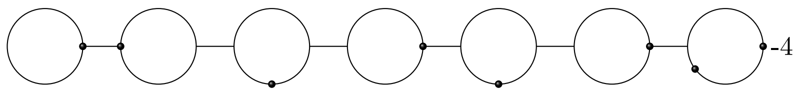
$$W^r_d(\Gamma) = \bigcup_t \mathbb{T}(t),$$

where t ranges over all $\underline{m}$-displacement tableaux on λ.

Corollary 2.5.4. *A chain of loops with torsion profile $\underline{m}$ has gonality k if and only if there is an $\underline{m}$-displacement tableau on a rectangle λ of dimensions $(g-k+1) \times 2$ and no such tableau on a rectangle of dimensions $(g-k+2) \times 2$.*

The following example illustrates the correspondence of Theorem 2.5.3.

Example 2.5.5. Suppose Γ has torsion profile $\underline{m} = (0, 2, 0, 3, 3, 0, 0)$. Figure 2.2 shows a divisor of degree 3 and rank 1 on a chain of 7 loops, along with its corresponding tableau.



1	2
2	4
3	5
4	6
5	7

Figure 2.2: A divisor and its corresponding tableau

The following lemma will prove to be a crucial step in our analysis of trigonal chains of loops in Chapter 3.

Lemma 2.5.6. *Given a divisor D on Γ, denote by $\xi_i^c := \xi_i^c(D)$ the coordinate on the i^{th} loop of Γ in the break divisor representative of cD. Then $\xi_i^{c+1} = \xi_i^c + \xi_i^1 - (i-1)$. It follows by induction on c that $\xi_i^c = c\,\xi_i^1 - (c-1)(i-1)$.*

Proof. By [18, Remark 3.4], the function

$$\widetilde{\xi_i} := \xi_i - (i-1)$$

is linear. This gives

$$\begin{aligned}
\xi_i^{c+1} &= i - 1 + \widetilde{\xi}_i^{c+1} \\
&= i - 1 + \widetilde{\xi}_i^c + \widetilde{\xi}_i^1 \\
&= i - 1 + \xi_i^c - (i-1) + \xi_i^1 - (i-1) \\
&= \xi_i^c + \xi_i^1 - (i-1).
\end{aligned}$$

□

Chapter 3 Results in Gonality Three

For the remainder of the paper, we compute composite scrollar invariants for a specific family of tropical curves, the chains of loops. In this section, we classify chains of loops of gonality three. Given a chain of loops Γ and a divisor D on Γ of degree 3 and rank 1, we compute $\mathrm{rk}(cD)$ for all values of c. We begin with the following observation.

Lemma 3.0.1. *The following is the unique tableau Λ on the rectangular partition $(g-1) \times 2$.*

1	*2*
2	*3*
3	*4*
$\vdots$	$\vdots$
g-2	*g-1*
g-1	*g*

Proof. The boxes of Λ must contain integers between 1 and g so that the entries strictly increase in each row and column. There cannot be a g in the zeroth column, since the box to the right of it must contain a larger number. Similarly, there cannot be a 1 in the first column. This leaves exactly $g-1$ distinct symbols that may appear in each column, which must appear in increasing order. This yields the above tableau. $\square$

By Lemma 3.0.1, we see that there is a unique hyperelliptic chain of loops.

Corollary 3.0.2. *A chain of loops Γ is hyperelliptic if and only if its torsion profile (termwise) divides $\underline{m} = (0, 2, 2, \ldots, 2, 0)$. In this case, there is a divisor D on Γ of degree 2 and rank 1 whose corresponding tableau is Λ.*

Proof. By Corollary 2.5.4, Γ is hyperelliptic if and only if there is an $\underline{m}$-displacement tableau on a rectangle of dimensions $(g-1) \times 2$.

By Lemma 3.0.1, we see that Λ is the unique tableau on a $(g-1) \times 2$ rectangle. Since the symbols 1 and g appear only once, Λ imposes no conditions on the torsion of the first or last loops of Γ. Each symbol i in the range $1 < i < g$ appears twice in Λ, in boxes $(0, i-1)$ and $(1, i-2)$, which are lattice distance 2 from each other. Thus we must have $m_i = 2$ and the torsion profile of Γ is as above. $\square$

We will denote by $\lambda_{a,b}$ the tableau on the rectangular partition $(g-2) \times 2$ obtained by deleting boxes $(1, a-2)$ and $(0, b-1)$ from Λ. Note that the symbols appearing in these boxes are a and b, respectively. This defines a tableau if and only if $b \geq a-1$. Tableaux of the form $\lambda_{a,b}$ are of interest for the following reason.

Proposition 3.0.3. *All tableaux on a rectangle λ of dimensions $(g-2) \times 2$ are of the form $\lambda_{a,b}$ for some $b \geq a-1$.*

Proof. Let t be a displacement tableau on λ. We must show that $t = \lambda_{a,b}$ for some $b \geq a-1$. Note that t has $g-2$ distinct entries in each column, which must be between 1 and g. As in Corollary 3.0.2, there may not be a g in the zeroth column or a 1 in the first column, so there is exactly one integer "missing" from each column. Let b be the integer that is missing from the zeroth column, and let a be the integer that is missing from the first column. Moreover, note that the missing box in the first column may not be strictly below the missing box in the zeroth column. From this, we achieve the desired result. □

Proposition 3.0.3 allows us to classify trigonal chains of loops.

Corollary 3.0.4. *A chain of loops Γ is trigonal if and only if it is not hyperelliptic, and has torsion profile that (termwise) divides*

$$\underline{m} = (0, 2, \ldots, 2, 0, 3, \ldots, 3, 0, 2, \ldots, 2, 0).$$

Proof. By Corollary 2.5.4, Γ is trigonal if and only if there is an $\underline{m}$-displacement tableau on a rectangle λ of dimensions $(g-2)\times 2$ and none on a rectangle of dimensions $(g-1)\times 2$. By Proposition 3.0.3, every tableau on λ is of the form $\lambda_{a,b}$ for some a and b. The tableau $\lambda_{a,b}$ imposes no conditions on the torsion of loops 1, a, b, and g, but the torsion of each other loop is determined by the tableau. In particular, if $i < a$, the symbol i appears twice in $\lambda_{a,b}$, both in boxes $(0, i-1)$ and $(1, i-2)$. These boxes are lattice distance 2 from each other, so we must have $m_i = 2$. In the same way, $m_i = 2$ for symbols i in the range $b < i < g$. Similarly, if $a < i < b$, the symbol i appears in boxes $(0, i-1)$ and $(1, i-3)$, which are lattice distance 3 apart, so $m_i = 3$. □

Having classified trigonal chains of loops, we now turn to the problem of computing their scrollar invariants. Given a chain of loops Γ and a divisor D on Γ of degree 3 and rank 1, our goal is to compute the rank of cD for all c. Note that if $a \neq b$, then there is a unique divisor class $D_{a,b} \in \mathbb{T}(\lambda_{a,b})$. For the remainder of this section, we fix integers a and b, and assume both that $D_{a,b} \in \mathbb{T}(\lambda_{a,b})$ and that Γ has the corresponding torsion profile.

Given this setup, we define the integer

$$\ell := \left\lceil \frac{b-a+4}{2} \right\rceil.$$

Note that ℓ depends only on the number of torsion 3 loops, $b-a-1$. Let n be the smallest integer such that

$$g \leq \left\lfloor \frac{3}{2}n + \frac{1}{2}(\ell - 1) \right\rfloor.$$

Remark 3.0.5. We will see in Corollary 3.0.8 below that ℓ is the smallest positive integer such that $\mathrm{rk}(\ell D_{a,b}) > \ell$. Similarly, we will see in Corollary 3.0.8 that n is the smallest positive integer such that $K_\Gamma - nD_{a,b}$ is not effective. On an algebraic curve, the integers ℓ and n defined in this way satisfy a natural relationship. Specifically, in the classical case, we would have $\ell = \sigma_1$ and $n = a_2 = g + 2 - \sigma_1$. These tropical invariants, however, do not satisfy this relationship.

By Theorem 2.5.3, the divisor $cD_{a,b}$ has rank at least r if and only if there exists a tableau $\lambda^c_{a,b}$ on a rectangle with $r+1$ columns and $g-3c+r$ rows such that $cD_{a,b} \in \mathbb{T}(\lambda^c_{a,b})$. By Lemma 2.5.6, $cD_{a,b} \in \mathbb{T}(\lambda^c_{a,b})$ if and only if, whenever $\lambda^c_{a,b}(x,y) = i$, we have

$$y - x = \begin{cases} i-1 \pmod{m_i} & \text{if } i \leq b, \\ i-1-3c \pmod{m_i} & \text{if } i > b. \end{cases} \tag{3.1}$$

Our goal is therefore to construct the largest possible $\underline{m}$-displacement tableau satisfying the above congruence conditions. Note in particular that if $i \leq b$, then the congruence conditions above are independent of c.

We will proceed in two steps. First, we will construct a tableau $\lambda^c_{a,b}$ satisfying the congruence conditions above. After constructing this tableau, we will then prove that there does not exist a larger tableau satisfying the congruence conditions.

Definition 3.0.6. Let $\alpha(y) \in \{-1, 0, 1\}$ be congruent to $y - a \pmod 3$. Let $\gamma(c) \in \{0, 1\}$ be congruent to $c - \lfloor \frac{a-b}{2} \rfloor \pmod 2$. We define the tableau $\lambda^c_{a,b}$ as follows.

$$\lambda^c_{a,b}(x,y) = \begin{cases} x+y+1 & \text{if } x+y+1 < a \\ 2x+y+1 & \text{if } y \geq \max\{a-4, a-x-1\} \text{ and} \\ & \quad 2x+y+1 < b \\ 2x+2y-(a-4)-\alpha(y) & \text{if } y < a-4 \text{ and} \\ & \quad a < 2x+2y-(a-4)-\alpha(y) < b \\ x+y+c+1 & \text{if } 2x+y+1 \geq b \text{ and} \\ & \quad x \leq c < \ell \\ x+y+\ell+1+\gamma(c) & \text{if } c \geq \ell \text{ and} \\ & \quad b \leq \min\{2x+y+1, 2x+2y-(a-4)-\alpha(y)\} \\ g & \text{if } c < \ell,\ x = c, \text{ and } y = g-2c-1, \\ & \text{or if } \ell \leq c < n, \\ & \quad x = \left\lceil \frac{3}{2}c - \frac{1}{2}(\ell-1) \right\rceil, \text{ and} \\ & \quad y = g-1-\left\lfloor \frac{3}{2}c + \frac{1}{2}(\ell-1) \right\rfloor. \end{cases}$$

In order to help the reader understand the formula above, we also describe it algorithmically. To assist the reader in navigating this algorithm, we note that the cases in the statement correspond (in order) to the six regions pictured in Fig. 3.1.

To produce the tableau $\lambda^c_{a,b}$ as described, we first fill in the triangle above the $(a-1)$st diagonal by placing the symbols 1 through $a-1$ on successive diagonals. More precisely, we place the symbol s in every box (x,y) along the diagonal $x+y+1 = s$.

We then place the symbols a through $b-1$ in regions 2 and 3, as shown in Fig. 3.2. Each of these symbols appears in every column of region 2. Specifically, we place the symbol s in the box $(0, s-1)$, and then make "knight moves" to the right 1 box and up 2 boxes, placing the symbol s until we exit region 2. Region 3 is filled similarly, except that we alternate between knight moves to the right 2 boxes and up 1 box, and knight moves to the right 1 box and up 2 boxes.

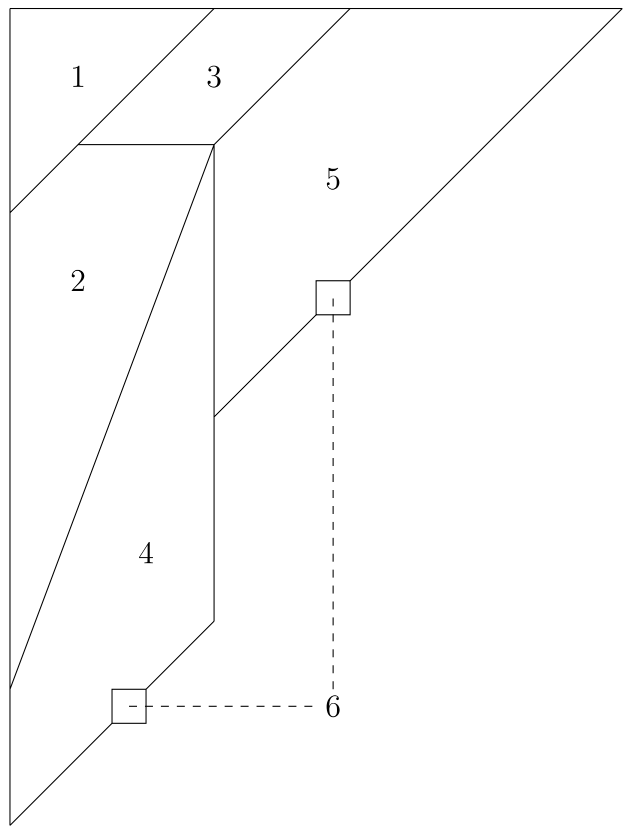


Figure 3.1: The characteristic regions of $\lambda^c_{a,b}$

Next, we place the symbols b through $g-1$ in regions 4 and 5. As in region 1, we place these symbols along an entire diagonal, starting with the first diagonal that contains an empty box.

Finally, we place the symbol g in a single box. Like the symbols a through $b-1$, the symbol g first makes knight moves to the right 1 box and up 2 boxes, until it crosses the line $y = a-4$. At this point, we alternate between knight moves to the right 2 boxes and up 1 box, and knight moves to the right 1 box and up 2 boxes.

We now show that this is the most efficient way to construct a tableau satisfying Eq. (3.1).

Theorem 3.0.7. *Suppose that $a \neq b$ and $m_a = m_b = m_g = 0$. Let t be a tableau such that $cD_{a,b} \in \mathbb{T}(t)$. Then $t(x,y) \geq \lambda^c_{a,b}(x,y)$ for all x, y.*

Proof. We prove this by induction. The base case, that $t(0,0) \geq 1$, is immediate. We assume that $t(x',y') \geq \lambda^c_{a,b}(x',y')$ for all x', y' satisfying either $x' < x$, $y' \leq y$ or $x' \leq x$, $y' < y$, and we show that $t(x,y) \geq \lambda^c_{a,b}(x,y)$.

We prove this for each region separately. To begin, if (x,y) is in region 1, then $t(x,y) \geq x+y+1$ because the rows and columns of a tableau are increasing.

Similarly, in regions 2 and 3 we fill column 0 with consecutive integers, which is clearly optimal. If $x > 0$ and $x+y+1 = a$, then since $m_a = 0$, we see that $t(x,y) > a$. If $t(x,y) < \lambda^c_{a,b}(x,y)$, then the symbols in this region correspond to torsion 3 loops,

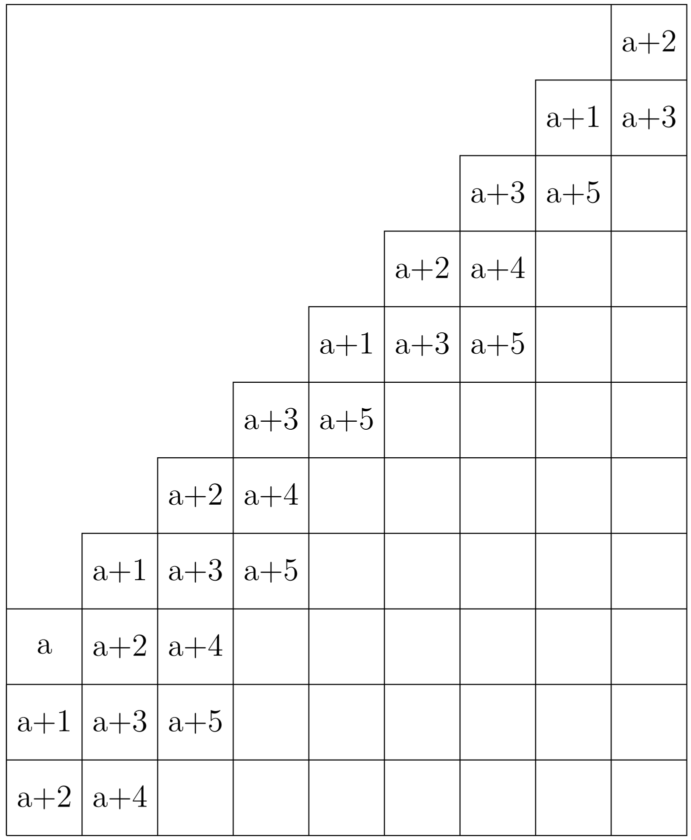


Figure 3.2: Filling regions 2 and 3 with torsion 3 symbols

so we must have $t(x,y) \equiv y - x + 1 \pmod 3$. It follows that $t(x,y) \geq a + 2 - \alpha(y)$. Otherwise, if $x > 0$ and $x + y + 1 > a$, we must have

$$t(x,y) \geq t(x-1,y) + 1.$$

But if $t(x,y) < \lambda^c_{a,b}(x,y)$, then again, the symbols in this region correspond to torsion 3 loops, and $y - x \not\equiv y - (x-1) + 1 \pmod 3$. It follows that we may not have equality in the displayed equation above. In other words, $t(x,y) \geq t(x-1,y) + 2$. Since equality holds for $\lambda^c_{a,b}$, we see that $\lambda^c_{a,b}$ is optimal in these regions.

After filling regions 1,2, and 3, we find the empty box (x,y) that minimizes $x+y$. Because $b+1$ and $b+2$ correspond to torsion 2 loops, one of $\{b, b+1, b+2\}$ can be placed in this box, and we make the minimal choice. This is clearly optimal. If (x,y) is in region 4 or 5 and does not minimize $x+y$, then since

$$t(x,y) > t(x,y-1) \geq \lambda^c_{a,b}(x,y-1) = \lambda^c_{a,b}(x,y) - 1,$$

we see that $t(x,y) \geq \lambda^c_{a,b}(x,y)$.

Finally, if $\lambda^c_{a,b}(x,y) = g$, then $\lambda^c_{a,b}(x-1,y) = g-1$. Since $t(x,y) > t(x-1,y) \geq \lambda^c_{a,b}(x-1,y)$, we see that $t(x,y) \geq g$ as well. $\square$

Corollary 3.0.8. *We have*

$$rk(cD_{a,b}) = r(c) := \begin{cases} c & \text{if } c < \ell \\ \left\lceil \frac{3}{2}c - \frac{1}{2}(\ell - 1) \right\rceil & \text{if } \ell \leq c < n \\ 3c - g & \text{if } c \geq n. \end{cases}$$

Proof. By Theorem 2.5.3, the divisor $cD_{a,b}$ has rank at least r if and only if there exists a tableau t on a rectangle with $r+1$ columns and $g - 3c + r$ rows such that $cD_{a,b} \in \mathbb{T}(t)$. The tableau $\lambda^c_{a,b}$ has $r(c)+1$ columns and $g - 3c + r(c)$ rows, where $r(c)$ is as defined above. It follows that $\mathrm{rk}(cD_{a,b}) \geq r(c)$.

Now, if $\mathrm{rk}(cD_{a,b}) > r(c)$, then there exists a tableau t with $r(c)+2$ columns and $g - 3c + r(c) + 1$ rows such that $cD_{a,b} \in \mathbb{T}(t)$. By Theorem 3.0.7, we have $t(x, y) \geq \lambda^c_{a,b}(x, y)$ for all (x, y). In particular, $t(r(c), g - 3c + r(c) - 1) \geq g$. This is impossible, because this implies that $t(r(c), g-3c+r(c)) > g$, but there is no symbol larger than g to place in this box. Thus $\mathrm{rk}(cD_{a,b}) \leq r(c)$, and the result follows. □

We note the following consequence of Corollary 3.0.8, which shows that the sequence of integers $\mathrm{rk}(cD_{a,b})$ is not convex, as it is in the classical case.

Corollary 3.0.9. *For* $0 \leq i \leq n - \ell$,

$$rk((\ell + i)D_{a,b}) = \begin{cases} rk((\ell + i - 1)D_{a,b}) + 1 & \text{if } i \text{ is odd} \\ rk((\ell + i - 1)D_{a,b}) + 2 & \text{if } i \text{ is even.} \end{cases}$$

Proof. This is a direct consequence of Corollary 3.0.8. □

We now compute the composite scrollar invariant σ_1.

Theorem 3.0.10. *We have*

$$\sigma_1(\Gamma, D_{a,b}) = \left\lfloor \frac{n + \ell}{2} \right\rfloor.$$

Proof. By Corollary 3.0.8, we have

$$2(n - 1) + 1 - \mathrm{rk}((n - 1)D_{a,b}) = 2(n - 1) + 1 - \left\lceil \frac{3}{2}(n - 1) - \frac{1}{2}(\ell - 1) \right\rceil$$

$$= 1 + \left\lfloor \frac{1}{2}(n - 1) + \frac{1}{2}(\ell - 1) \right\rfloor = \left\lfloor \frac{n + \ell}{2} \right\rfloor.$$

Thus, by the definition of σ_1, we have

$$\sigma_1(\Gamma, D_{a,b}) \geq \left\lfloor \frac{n + \ell}{2} \right\rfloor.$$

It therefore suffices to show that

$$\mathrm{rk}(cD_{a,b}) \geq 2c + 1 - \left\lfloor \frac{n + \ell}{2} \right\rfloor \text{ for all } c.$$

By Corollary 3.0.8, if $i > 0$, then

$$\mathrm{rk}((n-i)D_{a,b}) \geq \mathrm{rk}((n-1)D_{a,b}) - 2(i-1) = 2(n-i) + 1 - \left\lfloor \frac{n+\ell}{2} \right\rfloor,$$

and if $i \geq 0$, then

$$\mathrm{rk}((n+i)D_{a,b}) \geq \mathrm{rk}((n-1)D_{a,b}) + 2(i+1) = 2(n+i) + 1 - \left\lfloor \frac{n+\ell}{2} \right\rfloor.$$

$\square$

Chapter 4 Higher Gonality Generalizations

4.1 k-gonal Results

As in the classical case, we find that the situation becomes more complex in the case where Γ has gonality ≥ 4. In this section, we imitate our approach in the trigonal case in order to provide an algorithm for computing the scrollar invariants of a divisor on a k-gonal chain of loops. We begin with a natural generalization of of Proposition 3.0.3.

Proposition 4.1.1. *Every tableau on $(g-k+1) \times 2$ may be obtained by removing $k-2$ boxes from each column of Λ (as defined in Lemma 3.0.1) in such a way that, above any row, the number of boxes deleted from the left column of Λ does not exceed the number of boxes deleted from the right column.*

Proof. Consider the result λ of removing $k-2$ boxes from each column of Λ as described and sliding the remaining boxes together vertically. This forms a rectangle of dimensions $(g-k+1) \times 2$, and the condition on removed boxes guarantees that the entries in each row are increasing.

It remains to show that every displacement tableau t on λ can be obtained in this way. For any such tableau, note that each column of t must have $g-k+1 = g-1-(k-2)$ distinct entries, which must be between 1 and g. By the definition of tableau, there may not be a 1 in the first column of t or a g in the zeroth column, so each column contains all but $k-2$ of the symbols that appear in the corresponding column of Λ. In other words, the entries in each column may be obtained by deleting $k-2$ of the entries in the corresponding column of Λ. Requiring the entries in each row to increase exactly recovers our condition on the boxes removed, and the result follows. $\square$

Example 4.1.2. Fig. 4.1 illustrates this process for the tableau in Example 4.3.4.

This construction provides a natural classification of the tableaux corresponding to divisors of degree k and rank 1 on chains of loops. We use similar notation to the trigonal case, denoting by λ_D the tableau obtained in this manner corresponding to a divisor D on Γ. We associate a Dyck word (which we represent with matched sets of parentheses) to each tableau λ_D as follows: delete boxes from Λ to form D, from top to bottom. As each box is deleted, add a $\big($ or a $\big)$ to the end of the word if the box is deleted from the zeroth or first column, respectively.

We say two tableaux are of the same *combinatorial type* if they have the same associated Dyck word. Since it is known that Dyck words are enumerated by the Catalan numbers, the following is immediate.

Corollary 4.1.3. *The number of combinatorial types of tableaux corresponding to divisors of degree k and rank 1 on chains of loops is equal to the $(k-2)^{nd}$ Catalan number, C_{k-2}.*

1	2		1	2		1	2
2	3		2	3		2	3
3	4		3			3	5
4	5		4	5		4	7
5	6					6	9
6	7		6	7		8	10
7	8	$\longrightarrow$			$\longrightarrow$	10	11
8	9		8	9		11	12
9	10			10		12	13
10	11		10	11		13	14
11	12		11	12		14	15
12	13		12	13			
13	14		13	14			
14	15		14	15			

Figure 4.1: Making λ_D from Λ

This result has significant computational implications. In the trigonal case, all tableaux have the same combinatorial type, which allows us to define the tableau $\lambda^c_{a,b}$ representing $cD_{a,b}$ in Definition 3.0.6 with a (relatively) small number of cases. In higher gonality cases, the tableau λ^c_D depends on the *i-blocks* of $\underline{m}$, which we now define.

Definition 4.1.4. Let $i > 1$ be an integer. A collection $\{a+1, \ldots, b-1\}$ of consecutive integers in $\{1, \ldots, g\}$ is called an *i-block* if

1. i is a multiple of m_j for $a < j < b$, and
2. i is not a multiple of m_a or m_b.

Each combinatorial type of λ_D corresponds to a different distribution of i-blocks. In particular, if the symbol i appears only once in the tableau λ_D, then the ith torsion torsion order m_i is arbitrary. Otherwise, the ith torsion order m_i must divide

$$\begin{aligned} &2{+}\#\,(\text{ symbols } < i \text{ missing from column } 0) \\ &-\#\,(\text{ symbols } < i \text{ missing from column } 1)\,. \end{aligned}$$

Definition 4.1.5. Let λ_D be a rectangular tableau of dimensions $(g-k+1) \times 2$ containing each of the symbols in $\{1, \ldots, g\}$. We say that the torsion profile $\underline{m}$ is *nondegenerate* if it satisfies the following conditions:

1. if i appears only once in the tableau λ_D, then $m_i = 0$, and
2. otherwise,

$$\begin{aligned} m_i = {} &2{+}\#\,(\text{ symbols } < i \text{ missing from column } 0) \\ &-\#\,(\text{ symbols } < i \text{ missing from column } 1)\,. \end{aligned}$$

Corollary 4.1.3 implies that the number of combinatorial types grows exponentially with respect to k. It is therefore unfeasible to describe λ_D^c for every combinatorial type. Instead, we use the tools developed in Section 3 to construct λ_D^c recursively for each value of c. Recording the widths of the tableaux λ_D^c is equivalent to recording the rank sequence of our tropical divisor, and is therefore sufficient to calculate the sequence of composite scrollar invariants.

As in the trigonal case, Theorem 2.5.3 gives that the divisor cD has rank at least r if and only if there exists a tableau λ_D^c on a rectangle with $r+1$ columns and $g-kc+r$ rows such that $cD \in \mathbb{T}(\lambda_D^c)$. Again, by Lemma 2.5.6, $cD \in \mathbb{T}(\lambda_D^c)$ if and only if, whenever $\lambda_D^c(x,y)=i$, we have

$$y - x \equiv \xi_i^c \pmod{m_i}. \tag{4.1}$$

To produce the largest possible $\underline{m}$-displacement tableau satisfying this congruence condition, we make use of some original SAGE code available at

`https://github.com/kalilajo/numberboxes`.

In the remainder of this section, we describe the algorithm implemented by this code, prove that the resulting tableaux are optimal, and provide a few corollaries.

4.2 Algorithm for constructing λ_D^c from λ_D^{c-1}

Definition 4.2.1. For $c \geq 2$, let

$$j := k - (\mathrm{rk}(cD) - \mathrm{rk}((c-1)D)).$$

In other words, λ_D^c has j fewer rows and $k-j$ more columns than λ_D^{c-1}.

Note that identifying $j \in \{1, \dots, k-1\}$ is the overall goal of our calculation. Given λ_D^{c-1}, we construct λ_D^c recursively as follows.

Step 1: Set $j = 1$. We begin by setting $j = 1$, and we attempt to construct λ_D^c so that it has j fewer rows and $k-j$ more columns than λ_D^{c-1}.

Step 2: Start with the diagonal $x + y = 0$. To construct λ_D^c, we "traverse" each diagonal defined by fixing the sum of the coordinates, beginning with $x + y = 0$.

Step 3: Traverse the diagonal. When traversing a diagonal, we start with its leftmost box. Each time we arrive at a new box (x, y), we fill it with the smallest $s \in \{1, \dots, g\}$ that is larger than both the entry $\lambda_D^c(x, y-1)$ above it and the entry $\lambda_D^c(x-1, y)$ to the left of it, and such that Eq. (4.1) is satisfied. If there is no value of s such that these conditions hold, we increase the value of j by 1 and return to Step 2.

If we fill the box (x, y), we proceed to the box $(x+1, y-1)$ above and to the right of the current box, along the same diagonal. If the box (x, y) is the rightmost box on this diagonal, we increase the sum $x + y$ by 1 and repeat Step 3. If (x, y) is the bottom right corner of the rectangle, terminate the algorithm and output the rectangular tableau λ_D^c.

4.3 Verifying the algorithm

We apply this algorithm recursively to find the largest tableau λ_D^c such that $cD \in \mathbb{T}(\lambda_D^c)$ for each value of c. It remains to show the tableaux generated by this algorithm are optimal.

Proposition 4.3.1. *Suppose that the symbols removed to form λ_D as in Proposition 4.1.1 are distinct. Let t be a tableau such that $cD \in \mathbb{T}(t)$. Then $t(x,y) \geq \lambda_D^c(x,y)$ for all x, y.*

Proof. As in the proof of Theorem 3.0.7, we proceed by induction. The base case, $t(0,0) \geq 1$ is again immediate. We assume that $t(x',y') \geq \lambda_D^c(x',y')$ for all x', y' such that either $x' < x$, $y' \leq y$ or $x' \leq x$, $y' < y$ and show that $t(x,y) \geq \lambda_D^c(x,y)$. By construction, $\lambda_D^c(x,y)$ is the smallest symbol greater than both $\lambda_D^c(x-1,y)$ and $\lambda_D^c(x,y-1)$ that satisfies Eq. (4.1). Our inductive hypothesis implies that $t(x,y)$ must satisfy these conditions as well. We must therefore have $t(x,y) \geq \lambda_D^c(x,y)$. $\square$

We make a simple observation on the output of our algorithm. We show that a row of λ_D^c contains only every $(i-1)^{st}$ symbol in an i-block.

Lemma 4.3.2. *Suppose that the torsion profile $\underline{m}$ is nondegenerate. If $\lambda_D^c(x,y)$ and $\lambda_D^c(x,y)+i-1$ are in the same i-block, then*

$$\lambda_D^c(x+1,y) \geq \lambda_D^c(x,y)+i-1.$$

Proof. By definition, we have

$$y - x \equiv \xi_{\lambda_D^c(x,y)} \equiv c\xi^1_{\lambda_D(x,y)} - (c-1)(i-1) \pmod{i}.$$

Since $\underline{m}$ is nondegenerate and $\lambda_D(x,y)$ and $\lambda_D(x,y)+i-1$ are in the same i-block, we see that

$$\xi^1_{\lambda_D(x,y)+j} = \xi^1_{\lambda_D(x,y)} + j \text{ for all } 0 \leq j \leq i-1,$$

so

$$\xi^c_{\lambda_D(x,y)+j} \equiv \xi^c_{\lambda_D(x,y)} + j \pmod{i}$$

for all j in the same range. It follows that $i-1$ is the smallest value of j such that

$$\xi_{\lambda_D^c(x,y)+j} \equiv \xi_{\lambda_D^c(x,y)} - 1 \pmod{i}.$$

We therefore see that $\lambda_D^c(x+1,y) \geq \lambda_D^c(x,y)+i-1$. $\square$

As a consequence, we see that there is a torsion profile that maximizes the composite scrollar invariants. The torsion profile below corresponds to the tableau where the symbols $g-k+2, \ldots, g$ are missing from column zero, and the symbols $1, \ldots, k-1$ are missing from column one. We note that this torsion profile has been used in several papers to examine the behavior of general curves of gonality k [5, 13, 17]. Corollary 4.3.3 provides further evidence that this chain of loops behaves like a general curve of gonality k, as it has the scrollar invariants of a general curve.

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ξ_s^2	0	1	0	3	-2	3	-2	3	-2	1	0	1	0	1	4
m_s	0	2	2	0	0	0	0	0	0	2	2	2	2	2	0

Figure 4.2: Relevant data for placing s in λ_D^2

Corollary 4.3.3. *Suppose*

$$\underline{m} = (0, \ldots, 0, k, \ldots, k, 0, \ldots, 0).$$

Then $rk(cD) = c$ *for all* c *such that* $g > c(k-1)$. *In other words, we have*

$$\sigma_j(\Gamma, D) = \left\lceil \frac{j(g+k-1)}{k-1} \right\rceil \text{ for all } j.$$

Proof. Suppose that λ_D^c has more than $c+1$ columns. By Lemma 4.3.2, $\lambda_D^c(c+1, 0) \geq (c+1)(k-1)$. It follows that

$$\lambda_D^c(c+1, g - c(k-1)) \geq g - c(k-1) + (c+1)(k-1) = g + k - 1 > g,$$

which is impossible. It follows that λ_D^c has at most $c+1$ columns, and $\mathrm{rk}(cD) = c$. $\square$

On the other hand, if the torsion profile is more exotic, then the composite scrollar invariants can vary in interesting ways. We illustrate this phenomenon using an example.

Example 4.3.4. Let $g = 15$, $k = 5$, and let λ_D be the tableau constructed in Fig. 4.1 by removing the symbols 5, 7, and 9 from the zeroth column and 4, 6, and 8 from the first column. The output of the SAGE code can be seen in Fig. 4.6. We reproduce these results manually by using the algorithm in Definition 4.2.1 as follows.

First, we build λ_D^2 (labeled 2D in the figure) from λ_D. We naively assume λ_D^2 has $j = 1$ fewer rows and more columns than λ_D. We traverse and fill the diagonals as in steps 2 and 3 of the algorithm.

While doing this, we may only place symbol s in box (x, y) if Eq. (4.1) is satisfied; we list the relevant values in Fig. 4.2. Using this data and Eq. (4.1), we traverse and fill the diagonals of a 10×6 tableau as we are able. The result is shown in Fig. 4.3.

We see that this attempt was unsuccessful, as there were not enough symbols to fill the whole tableau. We therefore repeat this process with a tableau of dimensions 9×5, that is, assuming $j = 2$. This is similarly unsuccessful, as is letting $j = 3$. Both tableaux are shown in Fig. 4.4. Note that each tableau is the restriction of the previous one to a rectangle of smaller dimensions. Specifically, each rectangle has one fewer row and one fewer column than the previous one. Our procedure restricts to smaller and smaller rectangles until every box is filled.

1	2	3	10	11	12
2	3	10	11	12	13
3	10	11	12	13	14
4	11	12	13	14	
11	12	13	14		
12	13	14			
13	14	15			
14					

Figure 4.3: Attempting to build λ_D^2 with $j = 1$

1	2	3	10	11
2	3	10	11	12
3	10	11	12	13
4	11	12	13	14
11	12	13	14	
12	13	14		
13	14	15		
14				

1	2	3	10
2	3	10	11
3	10	11	12
4	11	12	13
11	12	13	14
12	13	14	
13	14	15	
14			

Figure 4.4: Attempting to build λ_D^2 with $j = 2$ and $j = 3$

Finally, when $j = 4$, we succeed in building the rectangular tableau shown in Fig. 4.5. We label this tableau by $2D$ in Fig. 4.6. We then repeat this process from the beginning to obtain the tableaux λ_D^3 and λ_D^4.

1	2	3
2	3	10
3	10	11
4	11	12
11	12	13
12	13	14
13	14	15

Figure 4.5: λ_D^2, attained when $j = 4$

For the benefit of the reader, we have chosen an example where the genus is relatively small in comparison to the gonality. Because of this, the tropical rank sequence happens to be convex. In examples of larger genus, this is typically not the case.

```
The genus is:15
Enter a_1 through a_k-2 as a list of numbers separated by spaces:4 6 8
Enter b_1 through b_k-2 as a list of numbers separated by spaces:5 7 9
_m_= [2, 2, 0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 0]
k= 5
D
  1  2
  2  3
  3  5
  4  7
  6  9
  8 10
 10 11
 11 12
 12 13
 13 14
 14 15
2 D
  1  2  3
  2  3 10
  3 10 11
  4 11 12
 11 12 13
 12 13 14
 13 14 15
3 D
  1  2  3 11 12
  2  3 11 12 13
  3  8 12 13 14
  4 10 13 14 15
4 D
  1  2  3 10 11 12 13 14
  2  3  8 11 12 13 14 15

The rank sequence is: [0, 1, 2, 4, 7]
The scrollar invariants are: {0: 0, 1: 3, 2: 7, 3: 13, 4: 19}
```

Figure 4.6: A sample calculation in SAGE

Appendix: SAGE Code

The following is the SAGE code that can be used to perform the calculations described in the last section. This code can also be found at

https://github.com/kalilajo/numberboxes.

```
#Define things, collect info to build tableau, validate input
while True:
    try:
        g = int(raw_input("The genus is:"))
        _a_ = [int(n) for n in raw_input("Enter a_1 through a_k-2 as a list
            of numbers separated by spaces:").split()]
        _b_ = [int(n) for n in raw_input("Enter b_1 through b_k-2 as a list
            of numbers separated by spaces:").split()]
        k = len(_a_)+2
    except ValueError:
        print("Oops! That doesn't look right... Try again!")
        continue

    #Remove boxes from each column (as a list)
    a = range(2,g+1) #Symbols in column 1 of Lambda
    for n in _a_:
        a.remove(n) #Symbols in column 1 of lambda_D
    b = range(1,g) #Symbols in column 0 of Lambda
    for n in _b_:
        b.remove(n) #Symbols in column 0 of lambda_D

    #Verify that the given values form a tableau
    if all(a[n]>b[n] for n in range(g-k+1)) and len(a)==len(b):
        break
    else:
        print("Oops! This isn't a valid displacement tableau. Be sure the
            number of b-values smaller than each n is no more than the
            number of a values.")
        continue

#Construct torsion profile
def torsion(n):
    if a.count(n)==0 or b.count(n)==0:
        return 0
    else:
        #(y in column 0)-(x in column 1) +1 horizontal unit gives lattice
            distance
        return b.index(n)-a.index(n)+1
```

```
_m_= [torsion(n) for n in range(2,g+1)]

#To use in places where we do modular calculations since mod 0 breaks
   things
def calc_torsion(n):
    if a.count(n)==0 or b.count(n)==0:
        return 104729
    else:
        return torsion(n)

print "_m_=", _m_

#Build lambda_D
lambda1 = Tableau([])
for n in range(g-k+1):
    lambda1 = lambda1.add_entry((n,0),b[n])
    lambda1 = lambda1.add_entry((n,1),a[n])

print "k=",k
print "D"
lambda1.pp()
#Make a width function
def width(t):
return t.shape()[0] #takes the width of the top row

#Calculate initial placement of each symbol
def xi1(s): #for s in range(1,g+1)
    if s<g+1 and b.count(s)>0:
        return b.index(s)
    elif s<g+1 and a.count(s)>0:
        return a.index(s)-1
    else:
        return -1

#Calculate placement in subsequent tableaux
def xi(c,s):
    if torsion(s)>0:
         return ((c*xi1(s))-((c-1)*(s-1))) % _m_[s-2]
    else:
         return ((c*xi1(s))-((c-1)*(s-1)))

#Build a tableau for the next value of c corresponding to a rank jump of j
def tryj(i,j):
    lambdac = Tableau([])
    #Tracks which diagonal we're on
    coordinatesum = 0
```

```
for coordinatesum in range(0,g):
    #Returns 0 if coordinatesum is less than max y value, 0 else
        x = max(0, (coordinatesum - (heights[i-1]-j-1)))
        y = coordinatesum - x

        while x < ranks[i-1]+k-j+1 and y>-1:
            #Find entries above and left of (x,y)
            if x == 0 and y == 0:
                p = 0
                q = 0
            elif x == 0:
                p = 0
                q = lambdac.entry((y-1,0))
            elif y == 0:
                p = lambdac.entry((0,x-1))
                q = 0
            else:
                if ((y-1,x) in lambdac.cells()) and ((y,x-1) in lambdac.
                    cells()):
                    p = lambdac.entry((y,x-1))
                    q = lambdac.entry((y-1,x))

                else:
                    #Break if we're missing a box, so the tableau can't
                        be rectangular
                    return lambdac

            s = max(p,q)+1 #Smallest potential symbol to put in (x,y)

            #Check xi^c_s
            while ((y-x) - xi(i,s)) % calc_torsion(s) != 0:
                if s>g:

                    if lambdac.height() == 1:
                        return lambdac
                    break
                else:
                    s+=1

            if s < g+1:
            #Add smallest possible symbol that has correct xi^c, if it
                exists
                lambdac=lambdac.add_entry((y,x), s)
                if s == g:
                    return lambdac

            if x==ranks[i-1]+k-j and y==heights[i-1]-j-1: #if we've
```

```
                built a rectangular tableau corresponding to a rank jump
                of j
                return lambdac
                #break
            else:
                x+=1
                y-=1

        #coordinatesum += 1
    return lambdac


ranks = [0,1] #We know the ranks of 0D and D; will add to this list as we
    build tableaux (but same as widths-1)
heights = [0,g-k+1] #Record the size of lambda^c



def tabc(i): #Build the tableau corresponding to iD; must be run for all i
    >1, in order, since it's recursive
    #Try to make it j rows shorter and wider than lambda(i-1) for j
        starting at 1 and less than k, increase j until the result is
        rectangular
    j = 1 #=k-rank jump

    if tryj(i,j) == Tableau([]):
        return Tableau([])

    while tryj(i,j).is_rectangular() == False:
        if j < k-1:
            j+=1

        else:
            return Tableau([])
    return tryj(i,j)

#Calculate rank sequence and print tableau for each cD
for c in range(2,(2*g-2)):
    thistableau = tabc(c)
    if thistableau == Tableau([]) or width(thistableau)-ranks[c-1]-1+
        heights[c-1]-thistableau.height()!=k:
        break
    else:
        heights.append(thistableau.height())
        ranks.append(width(thistableau)-1)
        print c, "D"
        thistableau.pp()
```

```
#add the rank of the first non-effective K-cD so we have a complete picture
ranks.append(len(ranks)*k-g)

#Calculate Scrollar Invariants from rank sequence
sis = {i : [] for i in range(0,k)}
m = 0
i = 0
while i < k:
    if sis[i] == []:
        if all(ranks[c] >= i*c + i + c - m for c in range(0,len(ranks))):
            sis[i] = m
            i += 1
        else:
            m += 1


#print "The heights of cD are:", heights
print "The rank sequence is:", ranks
print "The scrollar invariants are:",sis
```

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