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Entanglement, uncertainty and relativity in fundamental mechanics with an application in QKD

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ENTANGLEMENT, UNCERTAINTY AND RELATIVITY IN
FUNDAMENTAL MECHANICS
WITH AN APPLICATION IN QKD

A Dissertation

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Contents

Abstract

In this dissertation I will probe the innate uncertainty of quantum mechanics. After deriving the necessary tools I will takle Popper's experiment, a long misunderstood thought experiment with recent experimental results. I will then discuss how uncertainty changes when making measurements from different relativistic reference frames and resolve some on the tension between quantum mechanics and relativity. Finally I utilize the practical aspect of quantum uncertainty and describe a practical quantum key distribuition scheme.

Chapter 1
Introduction

1.1 General Introduction

The quantum world is fraught with uncertainty. This uncertainty is fundamental to the workings of our universe. In the classical world uncertainty simply comes from a lack of knowledge and if you measure a thing precisely enough then everything can be known about it. In the quantum world there are things that just cannot be known no matter how precise your measurement is. A fundamental understanding of this innate and unavoidable uncertainty is crucial to understanding the physics that govern our universe. Quantum uncertainty is strange and frustrating but is ultimately a key that unlocks many mysteries. In this journey to understand quantum uncertainty and use it for practical purposes I will delve into three problems that are solved with uncertainty.

I will start with a review the concepts necessary to understand the three examples I am presenting in this dissertation. I will begin with quantum mechanics. I will then move on to two fundamental demonstrations of quantum mechanics that will be indispensable to further discussions; diffraction and the thought experiment by Einstein, A; B Podolsky and N Rosen. After a short review of relativity I will explain the three systems in which quantum uncertainty play a major role. The first system discussed in chapter 2 involves uncertainty and entanglement in the long misunderstood thought experiment of Karl Popper. I then, in chapter 3, discuss uncertainty, entanglement and relativity and resolve some tension between those concepts. Finally in chapter 4 the innate uncertainty in quantum mechanics is used constructively to detect an eavesdropper.

1.2 Quantum Mechanics

In the late 19th and early 20th century there were some mysteries that appeared in which physical theory up to that time could not explain. One of these mysteries was the photoelectric effect. If a electromagnetic source shines on metal surface then electrons can be freed. However this only happens above a threshold frequency. Below that threshold frequency, no matter the intensity of the source, electrons are not produced. Einstein published a paper in 1905 [1] about the photoelectric effect in which he states that light exists as discrete particles with discrete energy. He states:

> According to the assumption considered here, in the propagation of a light ray emitted from a point source, the energy is not distributed continuously over ever-increasing volumes of space, but consists of a finite number of energy quanta localized at points of space that move without dividing, and can be absorbed or generated only as complete units.

This model postulated by Einstein has a source of electromagnetic radiation emitting not a wave, but discrete photons with fixed direction and energy. Then to observe the photoelectric effect these quanta of light would need to have an energy greater than the energy needed to dislodge an electron from the metal. This explains why a beam of great intensity but low frequency generates no electrons. It also explains why when the frequency of the photons from the electromagnetic source increases then the average energy of the freed electrons also increases. The problem with these explanations is that they are at odds with the traditional Maxwellian idea of radiation and infinite divisibility. It would seem that light behaves as discrete quantized particles in the case of the photoelectric effect. We

find later that photons have a both wave like and particle like behavior. We will pick up this thread again in section 1.2.4, but for now more theoretical background on the fundamentals of quantum mechanics.

1.2.1 Shrödinger's Equation

It was shown by De Broglie in [2] that even massive particles, such as electrons, have a wavelength. Shrödinger reasoned that if particles behave as waves then they should be able to be described by a wave equation which when solved should accurately describe a quantum system. He derived this wave equation which we now call Shrödinger's equation [3]. Much of quantum mechanics comes down to solving his equation. It is shown below in it's common differential form.

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial^2 x} + V\Psi(x,t) \tag{1.1}$$

Where $\Psi(x,t)$ is called the time dependent wave function, m is mass, t is the time, x is the position and V is the potential. Eqn. (1.1) is the time dependent Shrödinger equation, but if Ψ can be separated into spatial and time components, $\Psi(x,t) = \psi(x)\phi(t)$, then we can write down the time independent Shrödinger equation as:

$$E\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial^2 x} + V\psi(x) \tag{1.2}$$

Where E is a separation constant related or equal to the energy. The goal for defining a quantum mechanical system is to solve this equation.

1.2.2 Wave Functions and Plane Waves

It might be a natural question now to ask what $\psi(x)$ is. The wave function is a probability amplitude. If the probability amplitude is squared we get a probability density which can tell you the likelihood of getting a certain value. Why do we need

to square it and what does the unsquared wave function mean? Let's ignore that question, move on and just say that for $\psi(x)$ the probability density $\psi^*(x)\psi(x)$ gives the probability of finding the particle at the position x. The probability of finding a particle at any position $-\infty < x < \infty$ should be unity so we also require the wave function to be normalized $\int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = 1$.

The rest of this dissertation only uses one kind of particle, a photon in free space. In free space all the infinite values of position and momentum are eigenvalues in the Hilbert space. This can sometimes lead to trouble, so care is taken below to fully derive all the tools necessary for the rest of the dissertation. As well as an infinite eigenvalues, in free space there is no potential which reduces Shrödinger's equation to:

$$E\psi(x) = -\frac{1}{2m}\frac{\partial^2\psi(x)}{\partial^2 x} \tag{1.3}$$

Where I have now set $\hbar = 1$. A solution to Eqn. (1.3) is:

$$\psi(x) = e^{ipx}$$

Where $p^2 = -2mE$. This is the time independent part. To get the time dependent part we solve:

$$E\phi(t) = i\frac{\partial\phi(t)}{\partial t}$$

We then find the solution to the above equation $\phi(t)$ and write the complete wave function $\Psi(x, t)$:

$$\phi(t) = e^{-iEt}$$
$$\Psi(x, t) = \psi(x)\phi(t) = e^{i(px - Et)} \tag{1.4}$$

4

This describes a plane wave and has everything we need to know about a particle in free space. However, there is a problem with this wave function. It is usually necessary for wave functions to be normalized, but the plane wave solution defies all attempts at normalization.

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = \infty \neq 1$$

The probability density $\psi^*(x)\psi(x) = 1$, meaning that there is an equal probability of finding the particle anywhere $-\infty < x < \infty$ in space. This does not mean that the plane wave solutions are not useful though. A localized superposition of plane waves as is found in most real circumstances and that is normalizable. An example of such a superposition using a Gaussian function is:

$$\psi_G(x) = \left(\frac{2a^2}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-p^2 a^2} e^{ip(x-x_0)} \, \mathrm{d}p = \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{4}} e^{-\frac{(x-x_0)^2}{4a^2}}$$

This represents a group of plane waves centered around position x_0 with an uncertainty in position of a. This state, unlike a lone plane wave, is normalizable with:

$$\int_{-\infty}^{\infty} \psi_G^*(x)\psi_G(x) \, \mathrm{d}x \;=\; \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2a^2}} \, \mathrm{d}x = 1$$

We will use such superpositions of plane waves in chapters 2 and 3.

1.2.3 Notation and Operators

A Hilbert space is simply a set of vectors. In this space an inner product is defined between two vectors that allows length and angle to be measured. The most common example of a Hilbert space is Euclidean space consisting of all the possible three dimensional vectors and an inner product between two vectors gives the

angle between them and a magnitude. Dirac notation gives a very clear way of describing and manipulating Hilbert spaces. In this notation a vector in the space is denoted by a ket, $|\Psi\rangle$. The conjugate is a bra, $\langle\Psi|$ and a bra and a ket next to each other is a bracket $\langle\Psi|\Psi'\rangle$ and denotes an inner product. States are constructed out of one or more vectors in the Hilbert space. A basis is a group of vectors that equal or can be summed to be equal to any other vector in the Hilbert space. An orthogonal basis is basis with inner products of each element with each other are zero. Operators, defined with a hat above them $\hat{O}$, are functions that take one state into another $\hat{O}|\Psi\rangle = |\Psi'\rangle$. An eigenstate is a state which is unchanged by an operator except for a constant called the eigenvalue $\hat{O}|\Psi\rangle = O|\Psi\rangle$.

We are primarily interested in the one dimensional infinite Hilbert space containing all position and momentum vectors $|x\rangle$ and $|p\rangle$. All states $|x\rangle$ with $-\infty < x < \infty$ are eigenstates of the $\hat{x}$ operator with eigenvalue x and all states $|p\rangle$ with $-\infty < p < \infty$ are eigenstates of the $\hat{p}$ operator with eigenvalue p.

$$\hat{x}|x\rangle = x|x\rangle$$

$$\hat{p}|p\rangle = p|p\rangle$$

The Hamiltonian operator is the sum of the kinetic energy and potential energy operators $\hat{H} = \hat{T} + \hat{V}$. It has eigenvalues of total energy $\hat{H}|\Psi\rangle = E|\Psi\rangle$. If we look back at the Shrödinger's equation in Eqn. (1.2) and use $\hat{T} = \frac{1}{2m}\hat{p}^2$ we can see that $\hat{T} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} = \frac{1}{2m}\hat{p}^2$, so we can define the momentum operator by $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. The converse is also true and we can define the position operator by $\hat{x} = -i\hbar\frac{\partial}{\partial p}$.

$$\hat{p}|x\rangle = -i\hbar\frac{\partial}{\partial x}|x\rangle$$

$$\hat{x}|p\rangle = -i\hbar\frac{\partial}{\partial p}|p\rangle$$

The order in how the position and momentum operators are applied matters, $\hat{x}\hat{p}|x\rangle \neq \hat{p}\hat{x}|x\rangle$. This is because their commutator defined by $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i$ is not zero.

A wave function is a representation of a quantum state in a specific basis. The wave function $\psi(x)$ is the representation of $|\Psi\rangle$ in the position basis and the wave function $\phi(p)$ is the representation of $|\Psi\rangle$ the momentum basis. In Dirac notation they are written:

$$\langle x|\Psi\rangle = \psi(x)$$

$$\langle p|\Psi\rangle = \phi(p)$$

A projector is defined by the projection operator $\hat{P}_p = \frac{1}{\sqrt{|\psi|^2}}|p\rangle\langle p|$. This will take the wave function Ψ and project it onto the state with a momentum p. The $\frac{1}{\sqrt{|\psi|^2}}$ term is there for normalization. The integral of all the possible projectors, $\int_{-\infty}^{\infty}|p\rangle\langle p|dp = 1$, is one. We can use the projector to transfer between the x and p representation by projecting onto all possible momentum or position states.

$$\langle x|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\langle p|x\rangle\langle x|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x e^{+ixp}\langle x|\Psi\rangle$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x e^{+ixp}\psi(x) \tag{1.5}$$

$$\langle p|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\langle x|p\rangle\langle p|\Psi\rangle = \int_{-\infty}^{\infty} \mathrm{d}p e^{-ixp}\langle p|\Psi\rangle$$

$$= \int_{-\infty}^{\infty} \mathrm{d}p e^{-ixp}\phi(x) \tag{1.6}$$

Where $\langle x|p\rangle$ and $\langle p|x\rangle$ are found by solving the differential equation $\langle x|\hat{p}|x\rangle = p\langle x|p\rangle = -i\frac{\partial}{\partial x}\langle x|p\rangle$.

$$\langle x|p\rangle = e^{-ixp}$$

$$\langle p|x\rangle = e^{+ixp}$$

Notice that Eqns. [1.5] and [1.5] are simply the Fourier transform of the conjugate wave function. We can now test the orthogonality of the $|x\rangle$ and $|p\rangle$ states thusly:

$$\langle x'|x\rangle = \int_{-\infty}^{\infty} \mathrm{d}p \langle x'|p\rangle \langle p|x\rangle = \int_{-\infty}^{\infty} e^{i(x-x')p} \,\mathrm{d}p = \delta(x - x')$$

$$\langle p'|p\rangle = \int_{-\infty}^{\infty} \mathrm{d}x \langle p'|x\rangle \langle x|p\rangle = \int_{-\infty}^{\infty} e^{i(p-p')x} \,\mathrm{d}x = \delta(x - x')$$

The average value of the position operator $\hat{x}$ in Dirac notation is derived by twice projecting onto all possible position states.

$$
\begin{aligned}
\langle \hat{x} \rangle &= \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}x'' \langle \psi|x'\rangle \langle x'|\hat{x}|x''\rangle \langle x''|psi\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}x'' \langle \Psi|x'\rangle x'' \langle x'|x''\rangle \langle x''|\Psi\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}x'' \langle \Psi|x'\rangle x'' \delta(x' - x'') \langle x''|\Psi\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}x'' \psi^*(x') x'' \delta(x' - x'') \psi(x'') \\
&= \int_{-\infty}^{\infty} \psi^*(x') x' \psi(x') \,\mathrm{d}x'
\end{aligned}
$$

The average value of any operator is found the same way. As an example we can find the average position $\langle \hat{x} \rangle$ and the square of the average position $\langle \hat{x}^2 \rangle$ of the gaussian superposition in Eqn. (1.5).

$$\langle \Psi_G | \hat{x} | \Psi_G \rangle = \int_{-\infty}^{\infty} \psi_G^*(x) x \psi_G(x) \, \mathrm{d}x$$

$$= \left(\frac{1}{2\pi a^2} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-x_0)^2}{2a^2}} = x_0 \tag{1.7}$$

$$\langle \Psi_G | \hat{x}^2 | \Psi_G \rangle = \int_{-\infty}^{\infty} \psi_G^*(x) x^2 \psi_G(x) \, \mathrm{d}x \tag{1.8}$$

$$= \left(\frac{1}{2\pi a^2} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-x_0)^2}{2a^2}} = a^2 + x_0^2$$

Where $\psi_G(x) = \langle x | \Psi_G \rangle$.

1.2.4   Measurement

Measurement in quantum mechanics not only returns a value, but also irreversibly affects the quantum state. When a measurement is made by a detector the quantum state is then projected onto one of the possible states of the detector's subspace. This is called the wave function collapse. Before the measurement the wave function has a probability to be in many possible states and then when a measurement is made the wave function collapses into only one of those possible states. For example, a position measurement is made on $|\Psi\rangle$, which could have many or infinite possible position values, and a value of x_0 is returned. The state is then projected into state which has a definite position x_0. As can be seen below the state $|\Psi\rangle$ after a measurement that returns a position x_0 transforms into the position eigenstate $|x_0\rangle$

$$\psi'(x) = \langle x | \Psi' \rangle = \frac{\langle x | x_0 \rangle \langle x_0 | \Psi \rangle}{\sqrt{|\psi(x_0)|^2}} = \delta(x - x_0) \frac{\langle x_0 | \Psi \rangle}{\sqrt{|\psi(x_0)|^2}}$$

$$= \delta(x - x_0) = \langle x | x_0 \rangle$$

$$|\Psi'\rangle = |x_0\rangle$$

$$\tag{1.9}$$

Returning to Einstein's quote in chapter 1.2. He wrote that fully formed photons are emitted from a beam of light with predetermined energy and direction. However if the source randomly emits photons in any direction this is not necessarily the case. Imagine a source of photons surrounded by a spherical detector that will detect the position of the photon when it passes through the sphere. When the sphere make a position measurement it is then tempting to believe that the photon was emitted from the source and traveled to that point on the sphere. After all, this is what Einstein stated. This is, however, not quite true. Before the detector makes a measurement the behavior of the photon is regulated by its wave function which has a probability to be in many difference states. The wave function in this case being a radial plane wave at all possible points in the sphere $\psi(r) = \langle r|\Psi\rangle = e^{-irp}$ where r is a completely unknown variable. When the detector makes a measurement on the photon it collapses the wave function and projects it into the state $\psi'(r)$ with an average position at $\langle \hat{r}\rangle = r_0$.

$$
\begin{aligned}
\psi'(r) &= \langle r|\hat{P}_r|\Psi\rangle = \langle r|r_0\rangle\langle r_0|\Psi\rangle \\
&= \delta(r - r_0)\langle r_0|\Psi\rangle = \delta(r - r_0)e^{-ir_0p} \\
\langle \hat{r}\rangle &= \langle \psi'(r)|\hat{r}|\psi'(r)\rangle = \langle \psi(r)|r_0\rangle\langle r_0|\hat{r}|r_0\rangle\langle r_0|\psi(r)\rangle \\
&= \int_{-\infty}^{\infty} e^{+irp}\delta(r - r_0)r_0\delta(r - r_0)e^{-irp}\,\mathrm{d}r \\
&= \int_{-\infty}^{\infty} r_0\delta^2(r - r_0)\,\mathrm{d}r = r_0
\end{aligned}
$$

Where I have now dropped the cumbersome normalization from the projector. Since we should always be able to normalize the wave function at the end of the calculations, normalization will be assumed from now on. Before the measurement the particle was at all points r within the sphere. The photon begins as a uniform

sphere of probability originating from the source until a measurement is performed on it. Only then can we say that the photon was in a particular place. So while Einstein was right about the photons in [1] he did not have the complete picture yet. Even if our spherical detector had a radius of light years, the wave function that spans that entire space collapses and coalesces into a point instantly over all that distance when a measurement is made.

Until this point in this dissertation when a measurement is made we have only projected into a single state of the detectors subspace. This is called a Von Neumann measurement, but it is not the only kind of measurement. A positive operator values measurement (POVM) is a measurement that can project the initial state into more than one possible states of the detectors subspace. The POVM $\hat{M}$ transforms the quantum state $|\Psi\rangle$ into a state $|\Psi'\rangle$ like:

$$\langle x|\Psi'\rangle = \langle x|\hat{M}|\Psi\rangle = \int_{-\infty}^{\infty} \langle x|\hat{M}|x'\rangle\langle x'|\Psi\rangle dx = \int_{-\infty}^{\infty} M(x')\psi(x')dx'$$

Where $M(x')$ is a function that describes the subspace in which the quantum state is being projected into. A POVM is less restrictive, and therefore often more useful, than a Von Neumann measurement. It will be used to explain the single slit experiment in chapter 1.3.1, popper's experiment in chapter 2 and extensively in chapter 3. When a Von Neumann measurement is made, complete information about the measured state is obtained. According to Heisenberg's uncertainty principle this means that no information about the conjugate variable can be obtained. If we perform a POVM then we can get information about both conjugate variables, but not too much information about either.

1.2.5 Heisenberg's Uncertainty

In 1952 Werner Heisenberg wrote [4]:

It can be expressed in its simplest form as follows: One can never know with perfect accuracy both of those two important factors which determine the movement of one of the smallest particlesits position and its velocity. It is impossible to determine accurately both the position and the direction and speed of a particle at the same instant.

This is not only true about position and momentum, but about any two non-commuting observables. I should like to make one point very clear. When we are talking about uncertainty we are not talking about the uncertainty of a single measurement. For if you measure one particle to be at one position then that particle is at that position with zero uncertainty. The uncertainty of which quantum mechanics and Heisenberg speak of is the uncertainty of multiple measurements. If that particle were to be measured many times, the value found for position may change and so the range of values that the position took will have an uncertainty. If we take the example of a spherical detector around a source of photons describes in chapter 1.2.4, a single measurement will give a point on that sphere. There is no uncertainty involved in that measurement, but if we keep collecting data we would find that the detections will randomly happen over the entire sphere. This is what has uncertainty and in this case the uncertainty spans the sphere. Quantum mechanics is a probabilistic theory and suggesting that single measurements have uncertainties can be dangerous.

Heisenberg's uncertainty principle can be defined in terms of the commutator of two operators, in this case $\hat{x}$ and $\hat{p}$.

$$\sigma_x^2 \sigma_p^2 \geq \langle \frac{1}{2i}[\hat{x}, \hat{p}] \rangle^2$$

Where σ_x is the standard deviation of $\hat{x}$ and is defined by $\sigma_x^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2$. The commutator was already found in chapter 1.2.3 to be $[\hat{x}, \hat{p}] = i$. For the $\hat{x}$ and $\hat{p}$ operators the uncertainty relation is:

$$\sigma_x^2 \sigma_p^2 \geq \frac{1}{4} \qquad (1.10)$$

This tells us that when measuring position and momentum at the same time we are not able to have completely certain information about either. It is a common misconception that the uncertainty principle is due to the so called "observer effect". That the measurement of one variable disturbs the conjugate variable in such a way to make it uncertain. While this is mostly a harmless assumption it is not quite true. The Heisenberg uncertainty relation is a fundamental relation that hold no matter how well an observer makes a measurement and can be considered a tenant of quantum mechanics.

As an example of using the HUP we can find the uncertainty relation of the gaussian superposition from Eqn. (1.5). Using Eqns. (1.7 - 1.8) we find the uncertainty in position σ_x to be:

$$
\begin{aligned}
\sigma_x^2 &= \langle \Psi_G | \hat{x}^2 | \Psi_G \rangle - \langle \Psi_G | \hat{x} | \Psi_G \rangle^2 \\
&= a^2 + x_0^2 - x_0^2 = a^2
\end{aligned}
$$

Using the HUP we can quickly calculate the momentum uncertainty $\sigma_p^2 = \left(\frac{1}{2a} \right)^2$. This is easily verified by deriving the momentum uncertainty σ_p from the average values of $\hat{p}$ and $\hat{p}^2$.

$$
\begin{aligned}
\langle \Psi_G | \hat{p} | \Psi_G \rangle &= \langle \Psi_G | \hat{p} | \Psi_G \rangle = \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \langle \Psi_G | x' \rangle \langle x' | \hat{p} | x \rangle \langle x | \Psi_G \rangle \, \mathrm{d}x \\
&= -i \int_{-\infty}^{\infty} \langle \Psi_G | x \rangle \frac{\partial}{\partial x} \langle x | \Psi_G \rangle \, \mathrm{d}x \\
&= -i \int_{-\infty}^{\infty} \psi_G^*(x) \frac{\partial}{\partial x} \psi_G(x) \, \mathrm{d}x \\
&= -i \left(\frac{1}{2\pi a^2} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{2a^2} (x - x_0) e^{-\frac{(x-x_0)^2}{2a^2}} \, \mathrm{d}x = 0 \\
\langle \Psi_G | \hat{p^2} | \Psi_G \rangle &= \langle \Psi_G | \hat{p} | \Psi_G \rangle = -\int_{-\infty}^{\infty} \psi_G^*(x) \frac{\partial^2}{\partial x^2} \psi_G(x) \, \mathrm{d}x \\
&= -\left(\frac{1}{2\pi a^2} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4a^2}} \frac{\partial^2}{\partial x^2} e^{-\frac{(x-x_0)^2}{24a^2}} \, \mathrm{d}x = \frac{1}{4a^2} \\
\sigma_p^2 &= \langle \Psi_G | \hat{p^2} | \Psi_G \rangle - \langle \Psi_G | \hat{p} | \Psi_G \rangle^2 = \left(\frac{1}{2a} \right)^2 \\
\sigma_x^2 \sigma_p^2 &= a^2 \left(\frac{1}{2a} \right)^2 = \frac{1}{4}
\end{aligned}
$$

1.3 Diffraction

Now that the mathematical formalism and notation of the quantum mechanics that will be used in this dissertation is over, we will use the tools above to explore some important examples of which complete understanding will be needed for the following chapters. The most instructive demonstration of the uncertainty in quantum mechanics can be performed simply be shining a laser at one or more slits. It is amazing how such a simple experiment exposes the core of quantum mechanics.

1.3.1 Single Slit Diffraction

When coherent laser light passes through a slit it diffracts. If you put a screen after the slit as shown in Fig. (1.1) you will observe the diffraction pattern in Fig. (1.2). This phenomena can be seen with ocean waves passing through a narrow opening, so the diffraction pattern from the laser passing through the slit can be explained
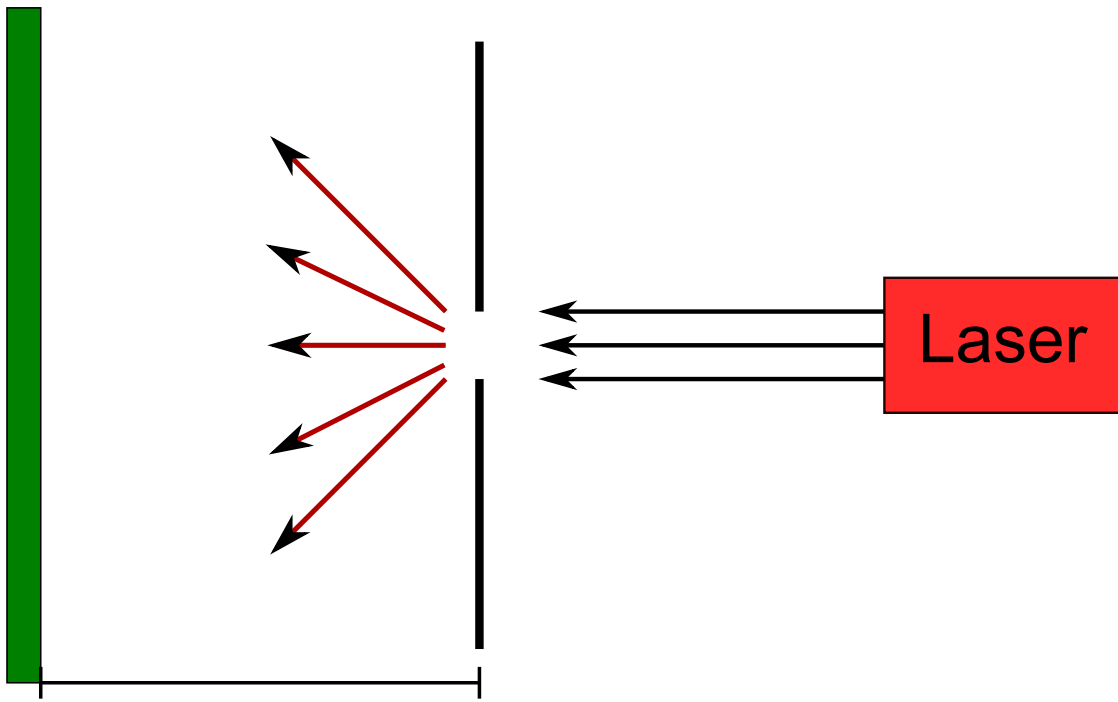
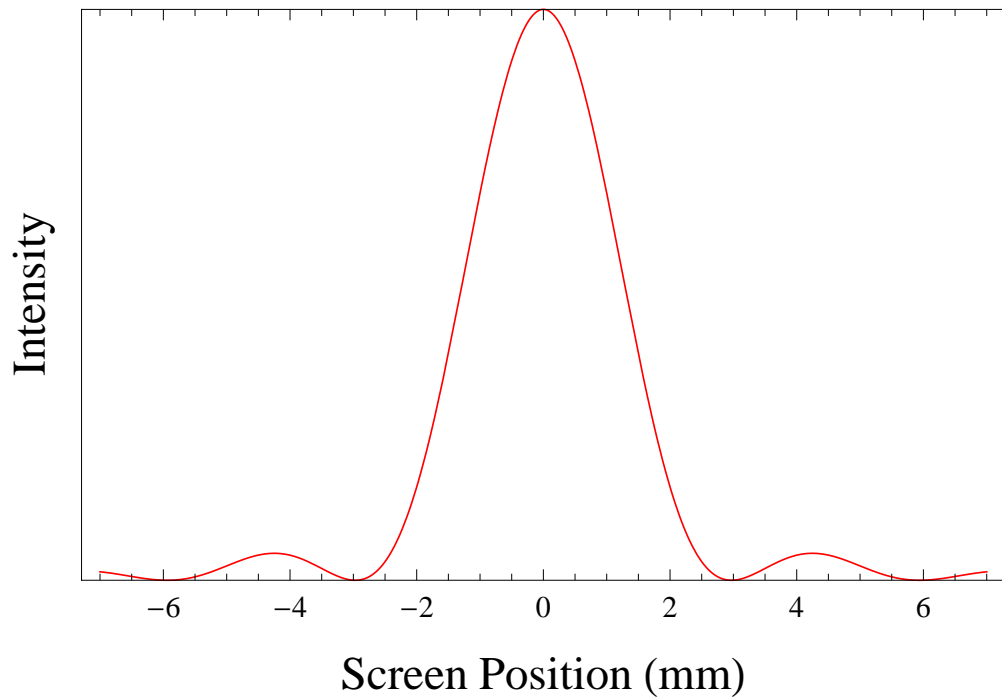FIGURE 1.1. When a laser shines through a slit the light that makes it through is diffracted.

FIGURE 1.2. The diffraction pattern produced on a screen by light that has passed through a single slit.

by assuming the light entering the slit is a coherent wave front and the interference pattern results from that wave interfering with itself after it passes through the slit. However, a very interesting thing happens when the laser is turned down so that only single photons are emitted. The diffraction pattern, after many photons have passed, remains the same. In this situation we can no longer consider a wave front passing through the slit. Single photons are passing through the slit one at a time.

The diffraction pattern can be explained by thinking of the slit as a position measurement on the incoming photon which we will model as a plane wave $\psi(y) = \langle y|\Psi\rangle = e^{-iyp}$. If a photon passes through the slit we know that it was within the width of the slit. We define the slit operator to be a unit box with $\langle y|\hat{\Pi}|y\rangle = \Pi(y) = \frac{1}{d}$ for $-\frac{d}{2} \leq y \leq \frac{d}{2}$. The wave function for the photon after the slit is derived below.

$$
\begin{aligned}
\phi(p) &= \langle y|\hat{\Pi}|\Psi\rangle = \int_{-\infty}^{\infty} \langle y|\hat{\Pi}|y\rangle\langle y|\Psi\rangle dy \\
&= \int_{-\infty}^{\infty} \Pi(y)e^{-iyp}dy = \int_{-\frac{d}{2}}^{\frac{d}{2}} e^{-iyp}dy = \frac{\sin(\frac{d}{2}p)}{p} \\
|\phi(p)|^2 &= \frac{\sin^2(\frac{d}{2}p)}{p^2}
\end{aligned}
\tag{1.11}
$$

Which is exactly the intensity distribution in Fig. (1.2). Measuring the position of the photon at the slit changed how certain we can be of the photons momentum represented on the screen as the diffraction pattern. This is Heisenberg's uncertainty principle at work. The uncertainty in momentum from Eqn. (1.11) is $\sigma_p = \frac{1}{d}$ and the uncertainty in position comes from the width of the slit $\sigma_y = \frac{d}{2}$. This gives the same uncertainty principle $\sigma_x^2\sigma_p^2 = \frac{1}{4}$ as in Eqn. (1.10).
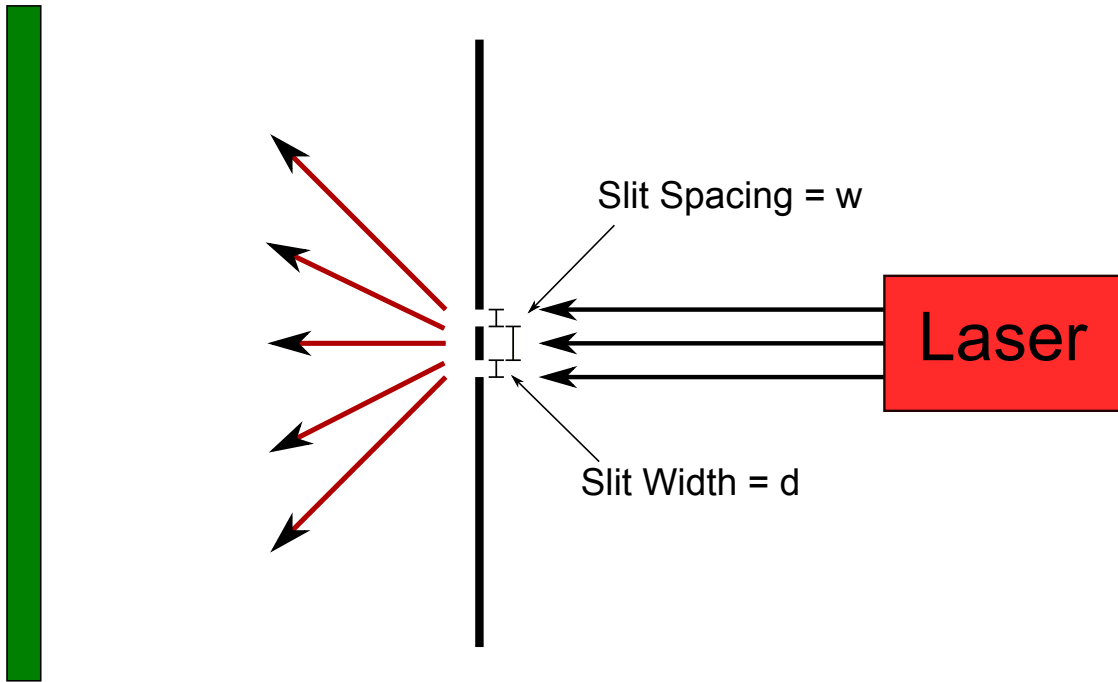
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FIGURE 1.3. When a laser shines through a double slit the light that makes it through is diffracted.
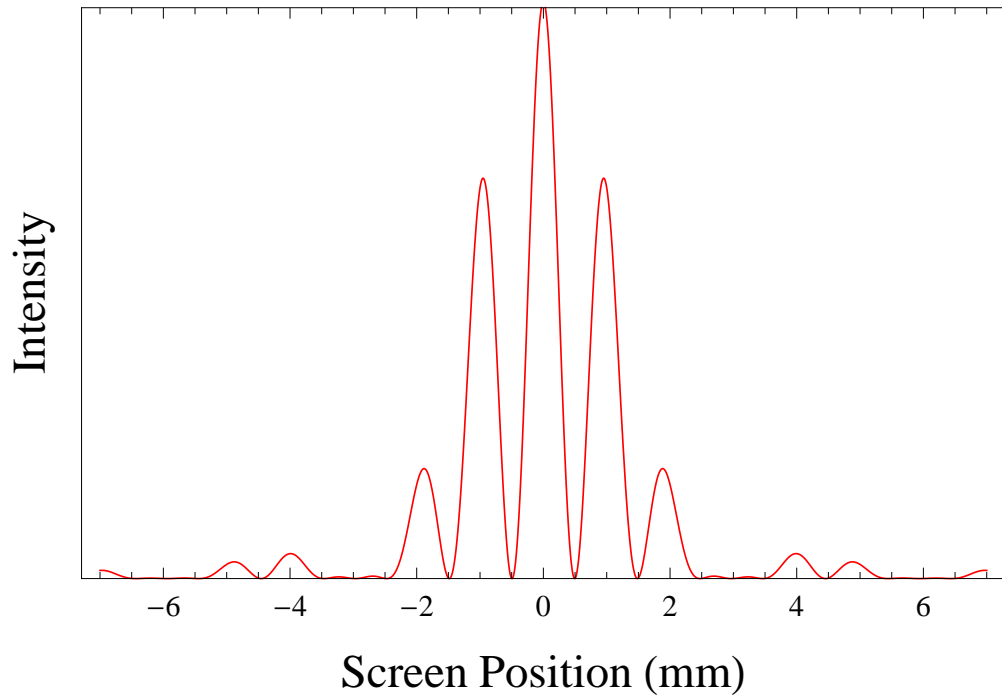

FIGURE 1.4. The diffraction pattern produced on a screen by light that has passed through a double slit.

1.3.2 Double Slit

The double slit experiment represented in Fig. (1.3) is treated much the same way. The interference pattern shown in Fig. (1.4) resulting from the spacing of the slits w and the slit width d can also be explained by assuming a wave front. Again the same diffraction pattern is observed when the source is sending only single photons and we can derive the wave function in the same way as above. $\hat{T}$ represents the double slit operator.

$$
\begin{aligned}
\phi(p) &= \langle y|\hat{T}|\Psi\rangle = \int_{-\infty}^{\infty} \langle y|\hat{T}|y\rangle \langle y|\Psi\rangle dy \\
&= \int_{-\infty}^{\infty} T(y)e^{-iyp}dy = \int_{\frac{w-d}{2}}^{\frac{w+d}{2}} e^{-iyp}dy + \int_{\frac{-w-d}{2}}^{\frac{-w-d}{2}} e^{-iyp}dy \\
&= \frac{e^{-ip\frac{w}{2}}\sin(\frac{d}{2}p) + e^{ip\frac{w}{2}}\sin(\frac{d}{2}p)}{p} \\
&= \frac{\cos(\frac{w}{2}p)\sin(\frac{d}{2}p)}{p} \\
|\phi(p)|^2 &= \frac{\cos^2(\frac{w}{2}p)\sin^2(\frac{d}{2}p)}{p^2}
\end{aligned}
$$

Which is the exact intensity distribution shown in Fig. (1.4). The same measurement principles apply as the single slit experiment. A measurement of the photons's position meant more uncertainty of the photons momentum.

There is another interpretation of the results of the double slit experiment which will be useful in chapter 4. The photon incident on the double slits does not go through one slit or the other. If that were so, all that would be produced on the screen is two diffraction patterns. If there is no knowledge of which slit the photon passed then the probability of the photon entering each slit is the same and the wave function does not collapse. The photon goes through both slits, or more accurately the probability for the photon to go through either slit is the same. If

we measure which path the photon took, through the upper or the lower slit, then the interference pattern disappears and we are again left with just the diffraction pattern from a single slit. In fact if we have only partial which path information then the interference pattern will only partially go away. There is a relationship $W + V = 1$ where W is a number from zero to one and represents how much path information we have about the photon and V is the visibility of the interference pattern. In the absence of which path information the visibility is one and then declines smoothly to zero as the the which path information goes to one.

1.4 Entanglement and EPR

Up until now we have only been describing states of one particle. It is possible for two or more particles to share a state. When this happens it is called entanglement. When two particles are entangled, a measurement on one particle will collapse the wave function and determine the state of the other particle. These particles need not be in the same place at the time of measurement for the wave function collapse to affect both particles. The collapse of the wave function is thought to occur simultaneously even if the two particles are far away from each other. In fact, a series of experiments beginning with [21] proved that the speed of wave function collapse is orders of magnitude greater than the speed of light and can be considered instantaneous.

1.4.1 The EPR Paradox and States

In 1935 Einstein, A; B Podolsky and N Rosen (EPR) proposed a thought experiment [8] that cut to the very heart of the strangeness of quantum entanglement. They posited that if the position of one of two spatially entangled and spatially separated particles were measured by Alice and simultaneously Bob measured the momentum of the other particle then, because the two particles are entangled, both

the position and momentum of the two particles will be precisely known. This is forbidden according to Heisenberg's uncertainty principle. The common solution to this paradox was given by Bell in 1964 by his famous inequality [9] which I will briefly explain in chapter 1.4.2. Bell's solution, however, speaks to the non-locality of entanglement and does not directly derive the results of a possible EPR experiment. I will do that now and at the same time go over some of the mathematical fundamentals of the EPR paradox that will be used throughout this document.

An EPR state is a state involving two particles that share a common variable. In the original EPR paper, they used position and it's conjugate variable momentum. The position and momentum wave functions can be represented by a the dirac delta function.

$$\psi(x_A, x_B) = \langle x_A, x_B | \Psi \rangle = \delta(x_A - x_B)$$

$$\phi(p_A, p_B) = \langle p_A, p_B | \Psi \rangle = \delta(p_A + p_B)$$

Where x_A, p_A, x_B, p_B are the positions and momentums of the photons that are measured at Alice and Bob's detectors. The wave functions can also be defined in terms of their conjugate variables which takes the form of a Fourier transform.

$$\psi(x_A, x_B) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(p_A + p_B) e^{i(x_A p_A + x_B p_B)} dp_A dp_B = \delta(x_A - x_B)$$

$$\phi(p_A, p_B) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x_A - x_B) e^{i(x_A p_A + x_B p_B)} dx_A dx_B = \delta(p_A + p_B)$$

When Alice measures the position of one of the particles, she projects the wave function into her position eigenstate, $|X_A\rangle$.

$$\begin{aligned}
\psi_A(x_A, x_B) &= \langle x_A, x_B | x_A = X_A, x_B \rangle \langle x_A = X_A, x_B | \Psi \rangle \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i((x_A - X_A)p_A)} \delta(p_A - p_B) e^{i(x_A p_A + x_B p_B)} dp_A dp_B \\
&= \delta(X_A - x_B)
\end{aligned}$$

Since the wave function is collapsed into a definite state position, Heisenberg's uncertainty principle tells us that the momentum state should be completely unknown, but the EPR paradox suggests that we can now measure the momentum of the second particle and therefore have both position and momentum information for both particles. We now find the average value of the momentum at Bob's detector after Alice's position measurement.

$$\begin{aligned}
\langle \Psi_A | \hat{p_B} | \Psi_A \rangle &= \langle \Psi_A | p_A, p_B \rangle \langle p_A, p_B | \hat{p_B} | p_A, p_B \rangle \langle p_A, p_B | \Psi_A \rangle \\
&= \int_{-\infty}^{\infty} \mathcal{F}[\psi_A(x_A, x_B)]^* p \mathcal{F}[\psi_A(x_A, x_B)] dp \\
&= \int_{-\infty}^{\infty} e^{-i(X_A - x_B)p} p e^{i(X_A - x_B)p} dp \\
&= \int_{-\infty}^{\infty} p \, dp = \infty
\end{aligned}$$

Which tells us that after many measurements the average momentum of the second particle is completely unknown. This agrees with Heisenberg's uncertainty principle and disagrees with the EPR thought experiment, which is expected since the solution to the EPR paradox is well known. As in the spherical detector example in chapter 1.2.4 the EPR state before measurement spans the entire space between Alice and Bob (assuming the detector has been on for a long time). Only when Alice or Bob performs a measurement does the state collapse into a position eigenstate at Alice's detector or a momentum eigenstate at Bob's detector.

1.4.2 The Solution: Bell Inequality

I will briefly discuss Bell Inequalities since a deep understanding of them is not necessary for understanding the rest of this dissertation.

It can be speculated that the photons coming from an EPR source have some as yet unknown hidden variable determining what state they will collapse into. This would avoid the notion of the wave function collapsing instantaneously over long distances or as Einstein called it, "spooky action at a distance"[10]. In the above discussion pains have been taken to state that the wave function collapses only when a measurement takes place and this delayed choice affects the Alice and Bob's statistics. If the state is predetermined by a hidden variable then it is no surprise that this would affect the measurement statistics. Bell uses this fact to construct an experiment that can tell the difference between an entangled states such as the EPR states and states predetermined by a hidden variable. If there are hidden variables then the statistics would obey an inequality and if not we say that state violates a Bell inequality.

1.4.3 Impossible Communication and Marginal Distributions

Even though the collapse of the wave function effects both particles of an EPR pair simultaneously, this does not mean that faster than light (FTL) communication is possible. If Alice measured one particle and Bob the other, they would still need to compare measurement results over a classical, non FTL, channel for any information to be sent.

Let us imagine a FTL communication scheme. There is a source of spin entangled pair of particles. One particle goes to Alice and the other to Bob who are light years apart. If Alice encodes here message by encoding the message in bits where a 0 is encoded by measuring her particle in the spin $\pm z$ basis and a 1 is encoded

by measuring in the $\pm x$ basis. She then she knows that Bob's particle will also be in those bases. Since the collapse of the wave function is instantaneous it would seem that the both share a message that was sent faster than the speed of light. This is not the case since Bob has no way of knowing which basis Alice measured in. If Bob measures in the $\pm x$ basis when Alice is measuring in the $\pm z$ basis then the message will be entirely scrambled. They must compare their basis choice over a classical channel before any message can be distilled.

We can see this by introducing the concept of marginal probabilities. The joint probability density $|\psi(x_A, x_B)|^2$ is the function describing how the variables x_A and x_B behave with respect to each other. The marginal probability density is calculated by integrating out the value of which no information is known. This gives the probability density for one variable when there is nothing known about the other variable. If we wanted to know the probability density at Alice's detector before she and Bob have compared data, the marginal distribution is:

$$|\psi(x_A)|^2 \;\; = \;\; \int_{-\infty}^{\infty} |\psi(x_A, x_B)|^2 dx_B = \int_{-\infty}^{\infty} \delta^2(x_A - x_B)dx_B = 1$$

Which is an equal probability over all space of detecting a photon. This means that Alice's position measurement is completely random and unrelated to Bob's measurement until she communicates with Bob classically and compares data.

1.5 A Relativistic Tool Box

This section is meant to give use the tools necessary to explore topics in relativity presented in chapter 3. We begin, of course, with Einstein.

1.5.1 Relativity

The same year Einstein published his paper on the photoelectric effect he also published the paper [5] on special relativity. In the paper he worked from two basic assumptions, the first of which is:

> ...the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. ... which will hereafter be called the Principle of Relativity.

More broadly this states that the physical laws of the universe are the same no matter what reference frame you are in. This may seem trivial, but at the time there were theories involving the aether which might change the mechanics of a system dependent on relative velocity. Secondly he writes:

> ...light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies...

No matter how fast or slow an observer is going relative to a light source they will always measure the speed of light to be the constant c. This is the more strange of the two postulates as many people on trains with flashlights can tell you. This is all it took for him to develop one of the most successful theories in history.

1.5.2 Lorentz Transforms

Working with the two postulates above and the work of Lorentz [6] Einstein was able in the same paper [5] to mathematically explain how space and time transform between different reference frames. Given an event at space-time coordinate (x, t)

Einstein claimed the space-time coordinate (x', t') in a reference frame which has a relative velocity of β can be found using the Lorentz transform equations below.

$$
\begin{aligned}
t' &= \gamma(t - \beta x) \\
x' &= \gamma(x - \beta t) \\
t &= \gamma(t' + \beta x') \\
x &= \gamma(x' + \beta t')
\end{aligned}
$$

Where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and I have set $c = 1$. As can be seen, as the relative velocity $\beta \to 0$, $\gamma \to 1$ and the two observers will see the event happen at the same place and the same time ($x' = x, t' = t$). As $\beta \to 1$ the position and time in the boosted frame go to infinity and the observer in the boosted frame will observe the event to have happened an infinite time ago and at an infinite distance.

The Lorentz transforms can also be used to calculate how time intervals Δt and distances Δx change in different reference frames. First we find the Lorentz transforms for Δt and Δx.

$$
\begin{aligned}
\Delta t' &= t'_A - t'_B = \gamma(t_A - \beta x_A - t_B + \beta x_B) = \gamma(\Delta t - \beta \Delta x) \quad (1.12) \\
\Delta x' &= x'_A - x'_B = \gamma(x_A - \beta t_A - x_B + \beta t_B) = \gamma(\Delta x - \beta \Delta t) \\
\Delta t &= t_A - t_B = \gamma(t'_A + \beta x'_A - t'_B - \beta x'_B) = \gamma(\Delta t' + \beta \Delta x') \\
\Delta x &= x_A - x_B = \gamma(x'_A + \beta t'_A - x'_B - \beta t'_B) = \gamma(\Delta x' + \beta \Delta t') \quad (1.13)
\end{aligned}
$$

If a clock is at rest in the lab frame then $\Delta x = 0$ and Δt is the time between clicks, then the time interval in the boosted frame is calculated from Eqn. (1.12) to be $\Delta t' = \gamma \Delta t$. As the velocity $\beta \to 1$ the time between clicks in the boosted

frame gets longer. An observer in the boosted frame would see a watch in the lab frame run slow and vice-versa. This is called time dilation.

To see how distances change when observing from a different reference frame, consider a ruler in the lab frame. When the measurement of the ruler is performed in the boosted frame, the measurement happens at both sides of the ruler simultaneously, which gives $\Delta t' = 0$. We can use this along with Eqn. (1.13) to find $\Delta x' = \frac{\Delta x}{\gamma}$. An observer in the boosted frame sees distances shrink in the lab frame and vice-versa. This is called length contraction.
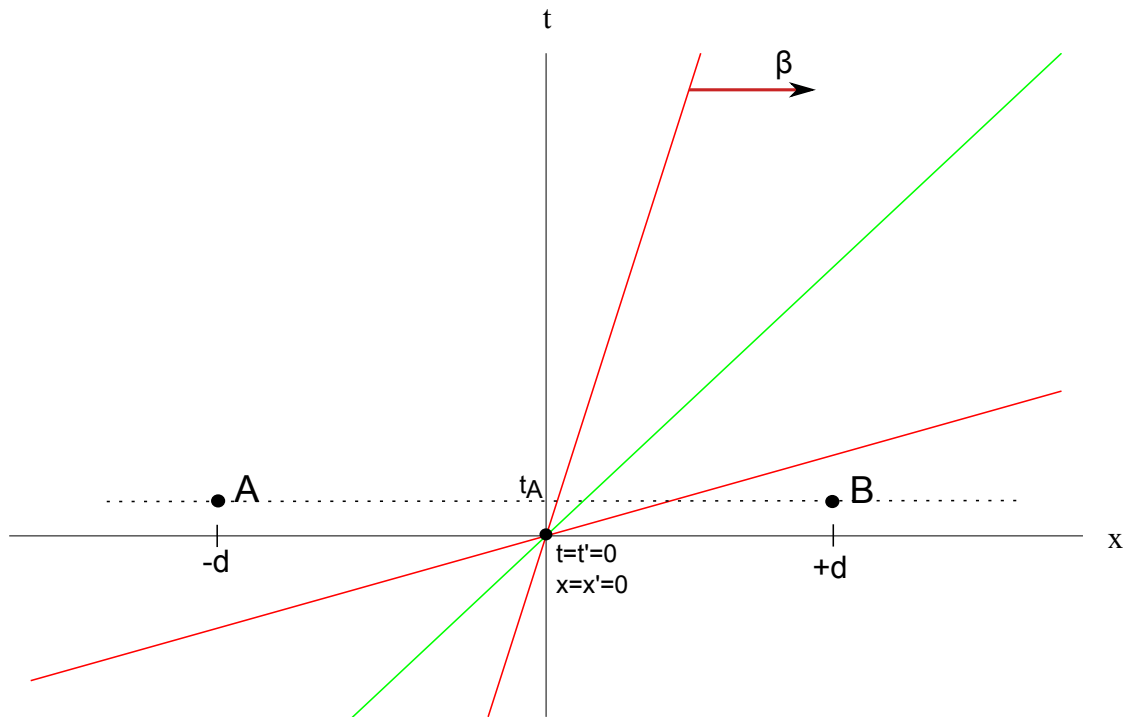
1.5.3 Space-Time Diagrams

FIGURE 1.5. A space time diagram. The red boosted frame is traveling with a velocity β relative to the lab frame. The green line with unity slope represents a trajectory of a photon going through the origin. It can be seen that the event A and event B occur simultaneously in the lab frame, but the observer in the boosted frame observes event A to happen after event B.

To visualize relativistic scenarios in one dimension there is no better tool than the space-time diagram. A space-time diagram is a graph with time on the y axis

and position on the x axis. The reference frame with the observer at rest is called the lab frame and the reference frame in motion relative to the lab frame is called the boosted frame. Because distance and time are equivalent to a photon traveling at the speed of light, a line with a slope of ± 1 represents the trajectory of a photon. If we assume that communication takes place with photons, then lines with slope ± 1 also represent how information travels.

To represent a reference frame traveling with some velocity relative to the lab frame on the same diagram we must find how the boosted frame relates to the lab frame. If we draw the trajectory of the boosted observer on the space-time diagram of the lab frame it makes a line with a slope $\frac{1}{\beta}$ and we call this the world line of the boosted observer. We assume that it passes the observer in the lab frame at the origin. Because the speed of light is a limit there can be no world line with a slope $|m| > 1$ since this would mean an observer is traveling faster than the speed of light. Of course the boosted observer is at rest in his own reference frame, so this line is the boosted observers time axis and it is called the boosted observers line of constant position. The world line for an observer at rest would have an infinite slope which then would match the axis of the lab reference frame. We can find the line of constant time in the boosted frame by by solving the Lorentz transform at the origin $x = 0 = \gamma(x' - \beta t')$. Solving this gives the line of constant time to be a line with a slope of β. As a check we can see how the boosted observer views the speed of a photon. It should be obvious that the trajectory of a photon travels an equal distance in an equal amount of time in the boosted reference frame, so the line of a photon still has a slope of ± 1 in the boosted frame. If we draw all these lines as in Fig. (1.5), the red axes belong to the boosted frame which is traveling with a velocity β relative to the lab frame. The green line with unity slope represents

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a trajectory of a photon going through the origin. Space-time diagrams easily and visually represents how events are observed in different reference frames.

1.5.4 Simultaneity

If we look again at Fig. (1.5) there are two events on the plot, event A at space time coordinate $(-d, t_A)$ and event B at $(+d, t_B = t_A)$ where $\beta d \geq t_A$. It can be seen that the event A and event B occur simultaneously at different positions to an observer in the lab frame. To an observer in the boosted frame, however, the order of the events change. The observer in the boosted frame observes event B occur at a negative time $t'_A = \gamma(t_A + \beta d)$ below their x' axis and event A occur at a positive time $t'_B = \gamma(t_A - \beta d)$ above it. The observer in the boosted frame perceives event A to happen before event B by an amount:

$$\Delta t' = t'_B - t'_A = \gamma(t_A - \beta d - t_A - \beta d) = 2\gamma\beta d \tag{1.14}$$

This result forces us to abandon the notion of simultaneity and realize that simultaneity is relative notion depending on the reference frame of the observer. Is uncertainty then also a relative quantity? Will it change when viewed from a different reference frame?

Chapter 2
Uncertainty in Entangled Photons

While the EPR thought experiment discussed in chapter 1.4 is very useful for discussing topics such as hidden variable and realism, a discussion of uncertainty in entangled pairs is better viewed in the context of Popper's thought experiment. It is very similar to the EPR thought experiment, but it involves two dimensions and the diffraction that was discussed in chapter 1.3. We are lucky that there are recent experiments that tested Popper's thought experiment.

In 1999 Kim and Shih implemented Popper's thought experiment in the lab [11]. The experiment, while well done with clear results, was not able to answer all questions and has instigated many interpretations [12][13][14] of the results. The results show some correlation between entangled photons, but not in the way that Popper thought, nor in the way a simple application of quantum mechanics might predict. A different experiment on ghost-imaging done in 1995 by Strekalov, *et al.* [15] sheds light on the physics behind Popper's thought experiment and the results found by Kim and Shih, but does not try to directly test Popper's thought experiment.

I will use these experiments and the quantum mechanical tools from chapter 1.2 to build the physics of Popper's thought experiment from the ground up and show how the results of both of these experiments reinforce each other and the theory of quantum mechanics. This chapter borrows heavily from work I did in [26]. A handy table containing all the experimental constants is provided in appendix 5.
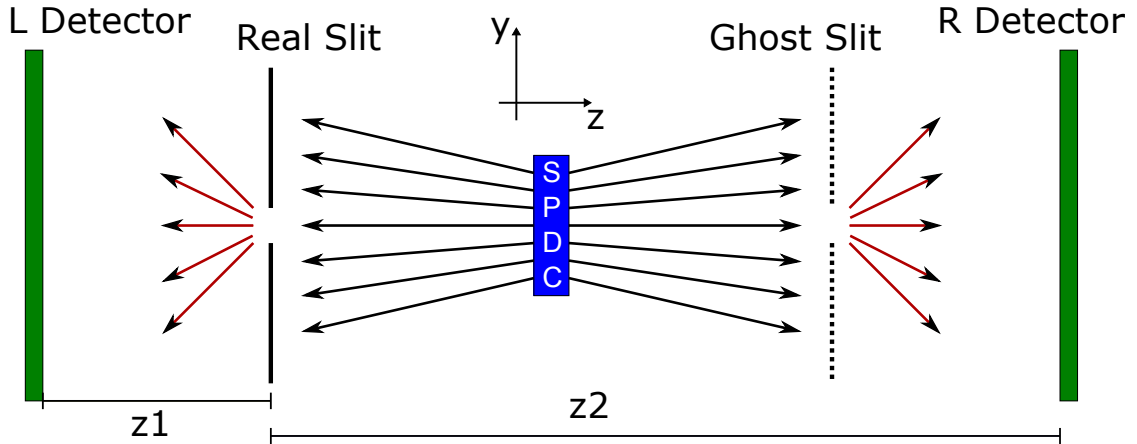
FIGURE 2.1. What Popper's original thought experiment may look like using a SPDC source of photons. The source emits a pair of momentum entangled photons in opposite directions. The photon on the left encounters a slit that causes diffraction. The "ghost" slit is the theoretical slit that Popper and others think exist due to the action of the real slit on the left.

2.1   Popper's Thought Experiment

Karl Popper posed an interesting thought experiment in 1934 [7]. With it, he meant to question the completeness of quantum mechanics. He claimed, in the same way that Einstein, Podolsky and Rosen did [8], that the notion of quantum entanglement leads to absurd scenarios that cannot be true in real life and that an implementation of his thought experiment would not give the results that quantum mechanics predicts. Unfortunately for Popper, it has taken until recently to perform an experiment that tested his claims. However, the results of the experiment do not refute quantum mechanics as Popper predicted, but neither do they confirm what Popper claimed quantum mechanics predicted.

Popper proposed an experiment Fig. (2.1) in which two photons entangled in position and momentum were sent in opposite directions [7]. The photon on the left passes through a slit. The result of many of those photons passing through the slit would produce an interference pattern on a screen behind the slit. This is the well-understood single slit diffraction experiment. The action of the slit can be
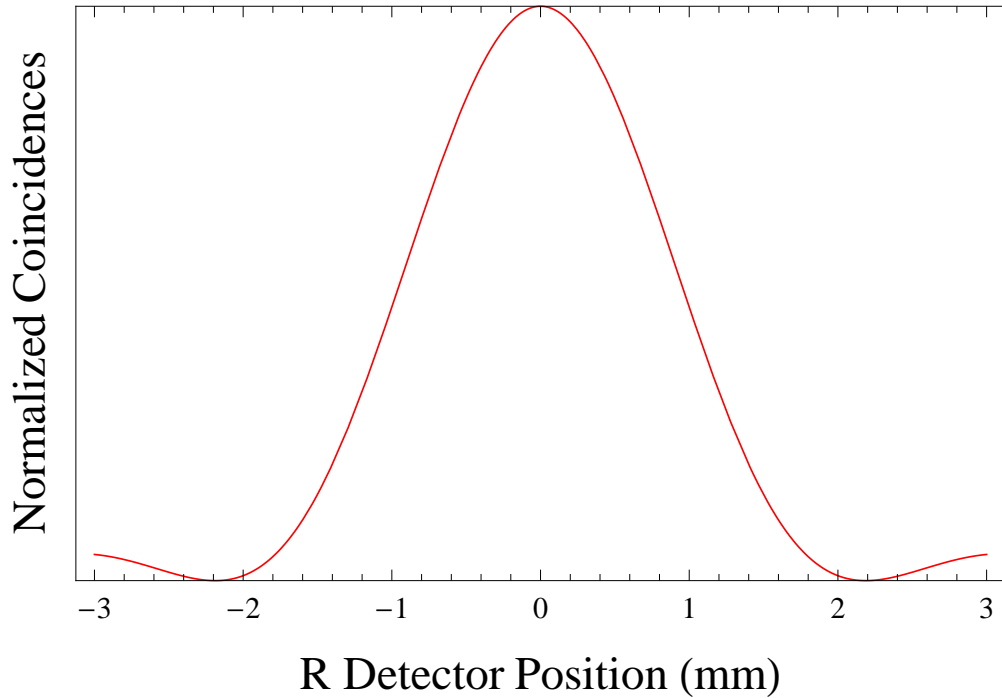
FIGURE 2.2. The theoretical distribution of photons on the right that Popper thought quantum mechanics predicts. It is a single slit interference pattern governed by the real slit width.

thought of as a measurement of the y_L position of the photon. The diffraction of the photon can be thought of as a direct consequence of Heisenberg's uncertainty principle. Since the photon's y_L position was measured, then it's momentum p_L is uncertain. Now, since we are dealing with an entangled source, Popper claimed that quantum mechanics tells us that if one photon's position is measured, then the other photon's position y_R is also known. Therefore, Popper argued the momentum p_R of the other photon must also be uncertain even though it did not pass through a slit. Then, according to Popper, this photon too, when measured over many trials, should produce an identical interference pattern Fig. (2.2) compared to the photon that passes through the real slit.

Popper did not think that this would happen in an experiment. He claimed that this sort of instantaneous action at a distance was incorrect. He argued that the diffraction of the right side photon by a ghost slit is what quantum mechanics
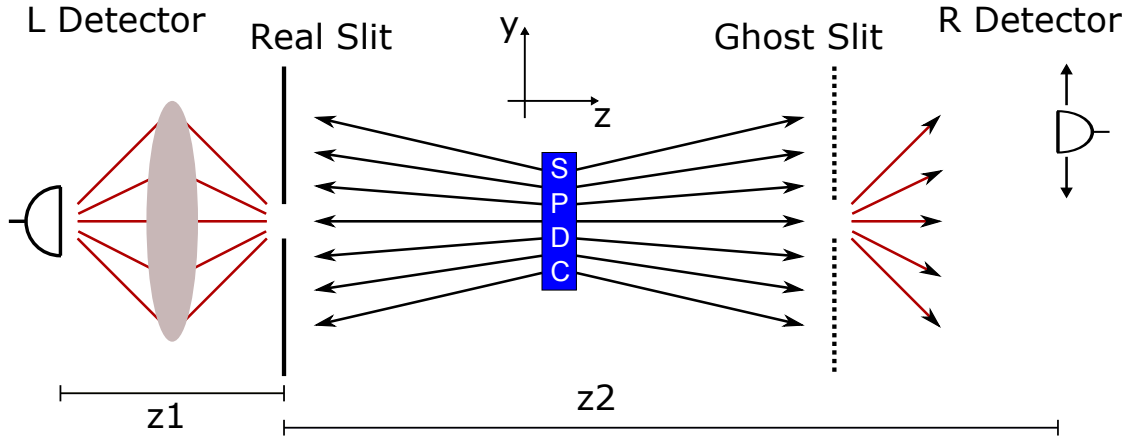
FIGURE 2.3. Kim and Shih's experimental setup with a theoretical ghost slit. The SPDC source emits a pair of momentum entangled photons in opposite directions. The photon traveling to the left travels through a slit. All the photons on the left that make it through the slit are collected into one fixed detector. The detector on the right is free to scan the y axis. Only detection events in which detectors fired in coincidence are reported.

predicts, but when the experiment was performed, no diffraction of the undisturbed photon would appear and this would therefore prove that quantum mechanics is wrong or at least incomplete.

2.2 Kim and Shih's Experiment

Kim and Shih's experiment [11] tried to directly set up and run Popper's original thought experiment. They used a source of spontaneous parametric down conversion (SPDC) photons and sent one half through a slit and then a lens to collect all the photons that made it through the slit. On the other side, they scanned the y-axis for coincidence counts Fig. (2.3). The difference between this experiment and Popper's original thought experiment is that all the photons on the left side are collected through a lens and sent to one detector and the photons on the right are scanned in the y axis instead of landing on some sort of screen. The lens should have no bearing on what we see from the detector on the right, since the photon on the left still traveled through the real slit.
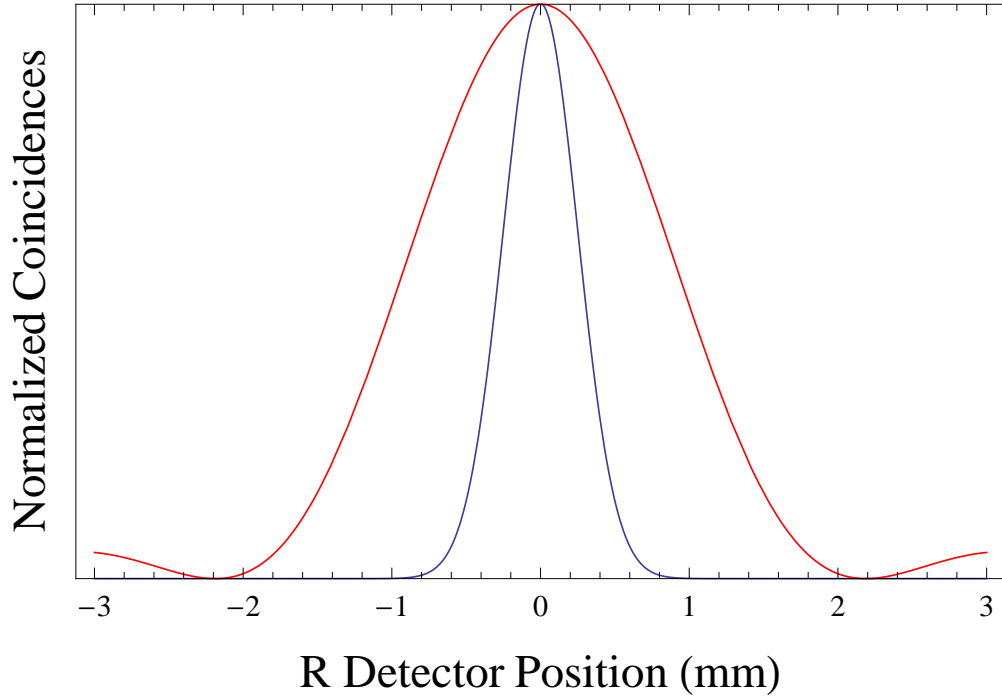
FIGURE 2.4. The inside curve is a reproduction of the results of Kim and Shih's experiment. The outside curve is the theoretical interference pattern from a single slit. The experimental results show a momentum uncertainty less than that of single slit interference pattern. If the photon on the right actually passes through a ghost slit of the same width of the real slit, then some violation of the uncertainty principle is suggested.

What Kim and Shih found is that the momentum distribution of the photon passing through the ghost slit is less than the the spread in momentum due to a real slit, and also less than the original momentum distribution from the SPDC source Fig. (2.4). This seems to be what Popper claimed would happen with an incomplete quantum theory. The diffraction at both location is different even though the photons are entangled. This could also suggest a violation the Heisenberg Uncertainty Principle (HUP). If the photon at the ghost slit was located to a width, d = 0.16 mm, then the uncertainty principle says that it's momentum uncertainty should have increased more than the experiment shows. The uncertainty in momentum from the experiment is $\Delta p_R \approx 3 \, \hbar \, \text{mm}^{-1}$, so calculating the product of the position and momentum uncertainties for the photon passing through the ghost slit gives $\Delta p_R \Delta y_R \approx \frac{\hbar}{4} < \frac{\hbar}{2}$. This would violate violate the HUP. Kim and Shih
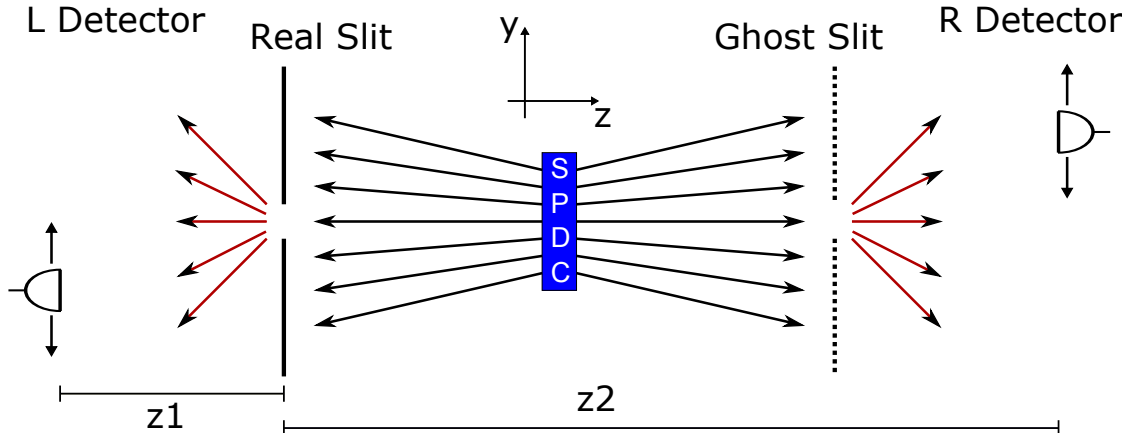
FIGURE 2.5. The experimental setup of Strekalov, *et al.* with a theoretical ghost slit. The SPDC source emits a pair of momentum entangled photons in opposite directions. The photon traveling to the left travels through a slit. The detectors on both sides are free to scan the y axis, so not all of the photons that pass through the slit on the left are collected. Only detection events in which detectors fired in coincidence are reported.

explain this by stating that the actual measurement taking place is the sum of the momentum of the bi-photon and not the two photons individual momentum. It is well known that while the position and uncertainty uncertainties of a single photon cannot violate HUP, the uncertainties of the addition of momentum and the difference of position commute with each other, $[\Delta(y_R - y_L), \Delta(p_R + p_L)] = 0$, and therefore need not obey HUP. The inequality, $\Delta(y_R - y_L)\Delta(p_R + p_L) < \frac{\hbar}{2}$, can be true. While we agree with Kim and Shih that the HUP is not violated, we have a alternate interpretation of why.

2.3 Ghost Imaging Experiment

The experiment by Strekalov, *et al.* [15], was not meant to be a direct test of Popper's experiment. It was designed to observe ghost imaging using SPDC photons. It does, however, give a great deal of insight into Popper's experiment.

The experiment is setup in much the same way as Kim and Shih's. A SPDC source sends entangled photons in opposite directions. On one side there is a slit and on the other there is not. This experiment differs from Kim and Shih's in that
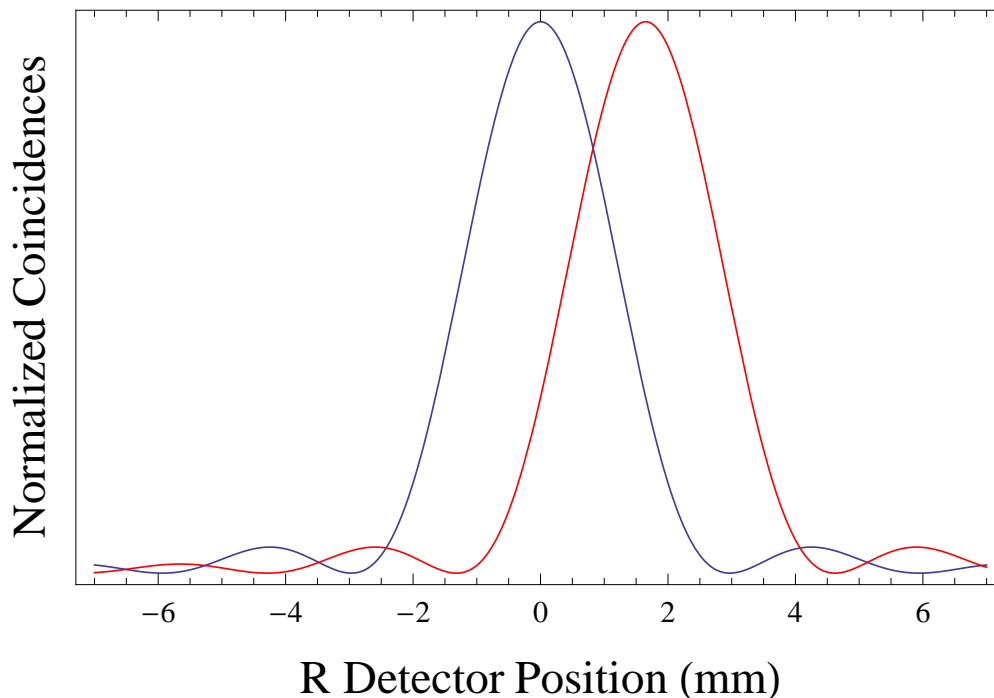
FIGURE 2.6. Reproduction of experimental results of the experiment by Strekalov, *et al.* The curve centered at the origin is the scan of the y axis of the detector on the right side when the detector on the left side is stationary at the origin. The displaced curve is the scan of the y axis of the detector on the right side when the detector on the left side is displaced from the origin. Both curves are that of a single slit interference pattern, although not of the width of what would be produced by a slit of the same width as on the left side.

both sides can be scanned on the y-axis. In Kim and Shih's experiment, all the photons that passed through the slit were collected and detected, whereas in the experiment by Strekalov, *et al.*, there is no lens to collect all the photons and the detector is free to scan the y-axis exactly like the detector behind the ghost slit.

The results of the Strekalov, *et al.* experiment are strikingly different from Kim and Shih's. When the detector behind the real slit is kept stationary and the detector behind the ghost slit is scanned, a single slit interference pattern was observed Fig. (2.6) in coincidence counts from the detectors. On the surface this seems to be exactly what Popper claimed quantum mechanics predicts and exactly what did not happen in Kim and Shih's experiment. Strekalov, *et al.* noted two other things about the results. When the detector behind the real slit was displaced

from the center of the y-axis, the single slit interference pattern behind the ghost slit was also displaced, but in the opposite direction. Also, the width between the fringes was dependent on the distance between the real slit and the detector behind the ghost slit, a distance they labeled as z_2 Fig. (2.5).

The dependence of the fringe width on the distance z_2 is interesting. As Tittel, et al. have shown [16], entanglement is valid over extremely long distances. If one entangled photon is measured to have a certain momentum then we know exactly the other photon's momentum, regardless of the distance between them. It does not matter how far away two entangled particles are, their correlations are still exact.

2.4 Analysis of Popper's Experiment

In the following analysis the z component, the axis perpendicular to the slit, of position and momentum will be ignored since it holds no surprises and does not contribute to the analysis. In the simplest case, plane waves with wave functions that are spread about the entire two dimensional space would be used. It is more instructive to use a more real world starting point for the wave function from the SPDC. We could start with a real life SPDC wave function, but the complexity either does not add to the analysis or reduces to the following wave function. I will start with a Gaussian distribution wave function describing a momentum entangled pair of photons traveling in opposite directions. To get the position representation of the wave function one starts by taking a Fourier transform of the momentum wave function.

$$
\begin{aligned}
\psi(y_R, y_L) &= \int_{-\infty}^{\infty} N e^{-p^2 a^2} e^{-ip(y_L - y_R)} dp \qquad (2.1) \\
&= \mathcal{F}(N e^{-p^2 a^2}) = N e^{\frac{-(y_L - y_R)^2}{4a^2}} \qquad (2.2)
\end{aligned}
$$

Where both photons have equal and opposite momentum p, $\mathcal{F}$ is a Fourier transform, a is the width of the Gaussian packet, y_L, y_R are the left and right components of position of the two photons and N is the normalization, which will be dropped from further equations,.

The detectors are far away enough from the slit that we can use the far-field Fraunhofer approximation to find what action the slit has on the wave function after it passed through. The slit is taken to be a box function ($\Pi(\frac{y_L}{d})$), or the sum of two step functions, of width d. The probability amplitude in momentum space, $\Phi(p_R, p_L)$, follows.

$$
\begin{aligned}
\phi(p_R, p_L) &= \mathcal{F}[\psi(y_R, y_L)\Pi(\frac{y_L}{d})] \\
&= \int_{-\frac{d}{2}}^{\frac{d}{2}} dy_L \int_{-\infty}^{\infty} \Psi(y_L, y_R)e^{i(y_L p_L + y_R p_R)} dy_R \\
&= e^{-p_R^2 a^2} \frac{\sin(\frac{d}{2}(p_L + p_R))}{(p_L + p_R)} \qquad (2.3) \\
\Phi(p_R, p_L) &= |\phi(p_R, p_L)|^2 \\
&= e^{-2p_R^2 a^2} \frac{\sin(\frac{d}{2}(p_L + p_R))^2}{(p_L + p_R)^2} \qquad (2.4)
\end{aligned}
$$

Where p_L and p_R are the inferred momentum and are functions of the slit width, wavelength, distance from the slit and position of the detector. Note that the presence of a Gaussian in equation (2.4), with the original SPDC source beam's width, shows that the spread in momentum due to a ghost slit can never be greater than the original momentum spread of the SPDC source. If this were not so, then one could signal faster than light (FTL) just by changing the real slit width and watching for a y momentum spread change in the right side detector. It should also be noted that only when an observer has the *coincidence* information from both sides of the experiment can any correlation can be seen. This can be shown by

finding the marginal probability distributions. These show the probability densities for an observer on one side when they have no information about the other side.

$$\begin{aligned}
\Phi(p_L) &\approx \int_{-\infty}^{\infty} \delta(p_R) \frac{\sin(\frac{d}{2}(p_L + p_R))^2}{(p_L + p_R)^2} dp_R \\
&= \frac{\sin(\frac{d}{2}p_L)^2}{p_L^2} \tag{2.5} \\
\Phi(p_R) &= \int_{-\infty}^{\infty} \Phi(p_R, p_L) dp_L = e^{-2p_R^2 a^2} \tag{2.6}
\end{aligned}$$

The marginal distribution on the left, Eqn. (2.5), can be found by assuming the beam width, a, is wide compared to the slit, $a > d$. In this case the Gaussian in the joint distribution Eqn. (2.4) can be approximated as a Dirac delta function. After integrating the joint distribution Eqn. (2.4) over p_R, the marginal distribution Eqn. (2.5) gives us the single slit interference pattern that we expect from a photon going through a single slit. So with no information from the right side detector we get exactly what we thought we should get, single slit diffraction. To get the marginal distribution on the right side Eqn. (2.6), we integrate the joint distribution Eqn. (2.4) over p_L. Only a Gaussian with the original width remains. So, without any information from the left side, we are left with what was originally emitted from the SPDC source. Only in coincidence counts will any interference pattern be seen, and since the coincidence counts are transmitted over a classical channel, we are again safe from any FTL signaling.

These equations have the correct form for the results of the experiment by Strekalov, *et al.*, but there is one problem. Strekalov, *et al.* noticed that the probability distribution is dependent on the aforementioned distance z_2 from the real slit to the detector behind the ghost slit. But if we stay with the notion of a ghost slit, then one would assume the probability distribution behind the ghost slit would

only depend on the distance from the ghost slit to the detector and on the width of the ghost slit. This is not the case.

2.5   Resolution of the Two Experiments
2.5.1   Bi-Photon

The problem comes from having assumed that the ghost slit is placed at a certain position; that some measurement of the z position has been made. If this were so, then we could justify the presence of a ghost slit at the position opposite that of the real slit. For, as we know, if the position of one photon is measured, then the position of the other is known. But, the z axis components of the wave function has in no way, weak or otherwise, been measured or restricted. No ghost slit can been made by only measuring the y-component of the wave function at the real slit. So, we should not use the location of a ghost slit as a reference point. This leaves the only point of reference for the partial collapse of the initial wave function to be the position of the real slit. So all momentum changes must be referenced from that point. This is where the notion of two entangled photons traveling in opposite directions breaks down and is what has lead Shih and others to refer to this entangled state as a "bi-photon". The measurement due to the slit on the left side can not be thought to measure only one photon and then through entanglement have an effect on the other photon. The action of the real slit must be understood to affect the bi-photon wave function.

Since we are abandoning the notion of a ghost slit, the distance between the ghost slit and the detector on the right is not a number that can be used. The momentum vectors of both sides need to have their origin at the real slit. Fig. (2.7) shows that the angle, A', referenced from the ghost slit will be larger than the angle, A, referenced from the real slit. The translation has the effect on the right side of extending the distance z_1 that is normally seen in the distributions of
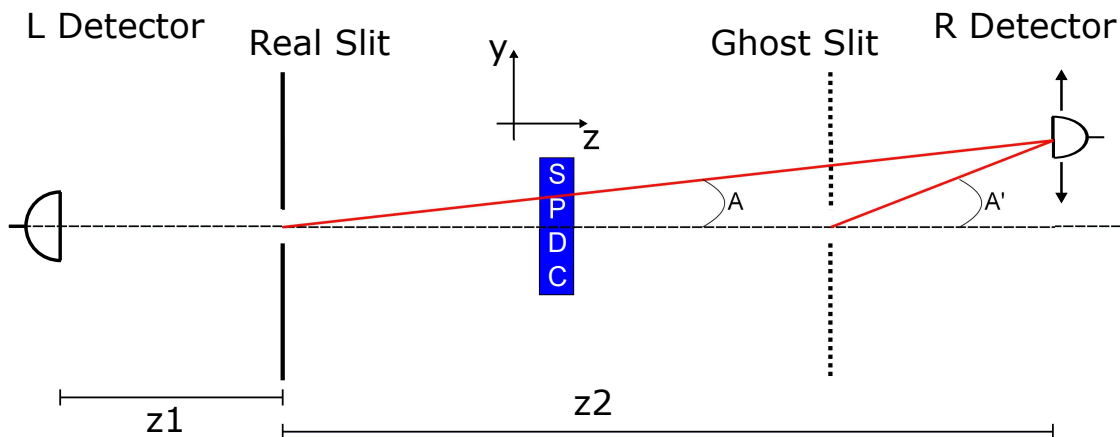
FIGURE 2.7. The angle of diffraction changes depending on which reference frame you choose. In this case, the angle from the reference frame of the real slit, A, is smaller than the angle, A', taken from the reference frame of the ghost slit.

single slit interference patterns to the much larger distance z_2 that runs from the plane of the real slit to the plane of the detector behind the ghost slit. Then the probability amplitude at the detector on the right is:

$$
\begin{aligned}
\Phi(y_R, y_L) &= e^{-\frac{8\pi^2 y_R^2 a^2}{\lambda^2 z_2^2}} \frac{\sin\left(\frac{\pi d}{\lambda}\left(\frac{y_L}{z_1} + \frac{y_R}{z_2}\right)\right)^2}{\left(\frac{y_L}{z_1} + \frac{y_R}{z_2}\right)^2} \\
&\approx \frac{\sin\left(\frac{\pi d}{\lambda}\left(\frac{y_L}{z_1} + \frac{y_R}{z_2}\right)\right)^2}{\left(\frac{y_L}{z_1} + \frac{y_R}{z_2}\right)^2}
\end{aligned}
\tag{2.7}
$$

2.5.2 The two Experiments Resolved

Equation 2.7 gives us all the results that the experiment by Strekalov, *et al.* measured Fig. (2.9). Note that this theoretical figure is normalized by probability rather than coincidence counts as is the case for the figure of experimental results Fig. (2.6). One can see that the probability for detecting photons when the left detector is displaced from the large central peak is much less.

Kim and Shih found that the width of the distribution after the ghost slit was less then the original beam width and less than the width of the single slit interference pattern of a real slit at the ghost slit's position. This does not agree with the assumption that the ghost slit will cause an increase in the momentum spread. It
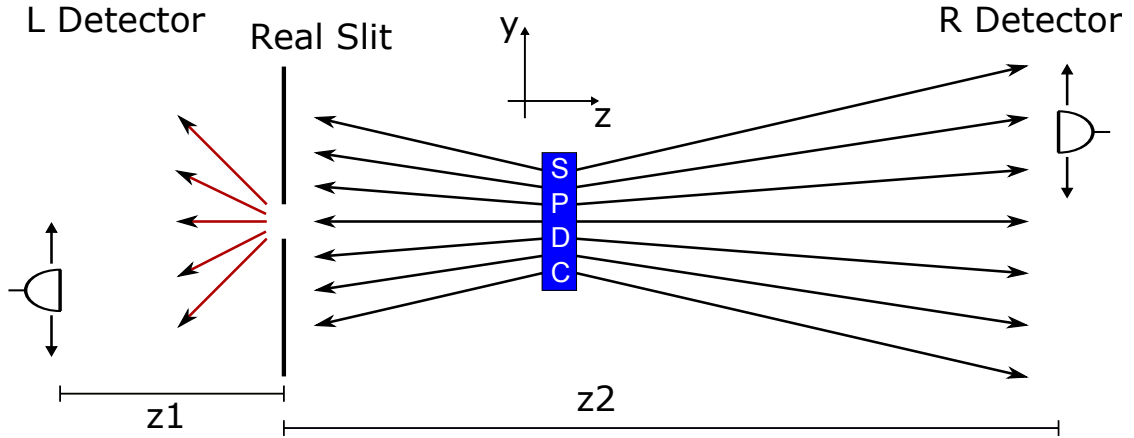
FIGURE 2.8. The experimental setup of Strekalov, *et al.* without the notion of a ghost slit. The SPDC source emits a pair of momentum entangled photons in opposite directions. The left part of the bi-photon passes through a slit. This has a measurable effect on the right side of the bi-photon, but the notion of a ghost slit on the right side is discarded. The detectors on both sides are free to scan the y axis, so only some of the photons that pass through the slit on the left are collected. Only detection events in which detectors fired in coincidence are reported.

also does not seem to agree with Strekelov's result that indicates the width will be that of single slit interference pattern, augmented by the distance z_2 between the detector behind the ghost slit and the real slit. The key to understanding Kim and Shih's results is to realize that the only substantial difference between their experiment and Strekelov's experiment is that all of the photons were collected by a lens after the real slit. They, in effect, took a weighted average of all the photons that passed through the slit. With the weighting function, $W(y_R, y_L)$, being that of a single slit interference pattern. So if we multiply the probability amplitude of joint detection by the weighting function and then integrate over all the detected photons that came through the left slit, we get the counting rate, $R(y_R)$ that Kim and Shih measured in their experiment:
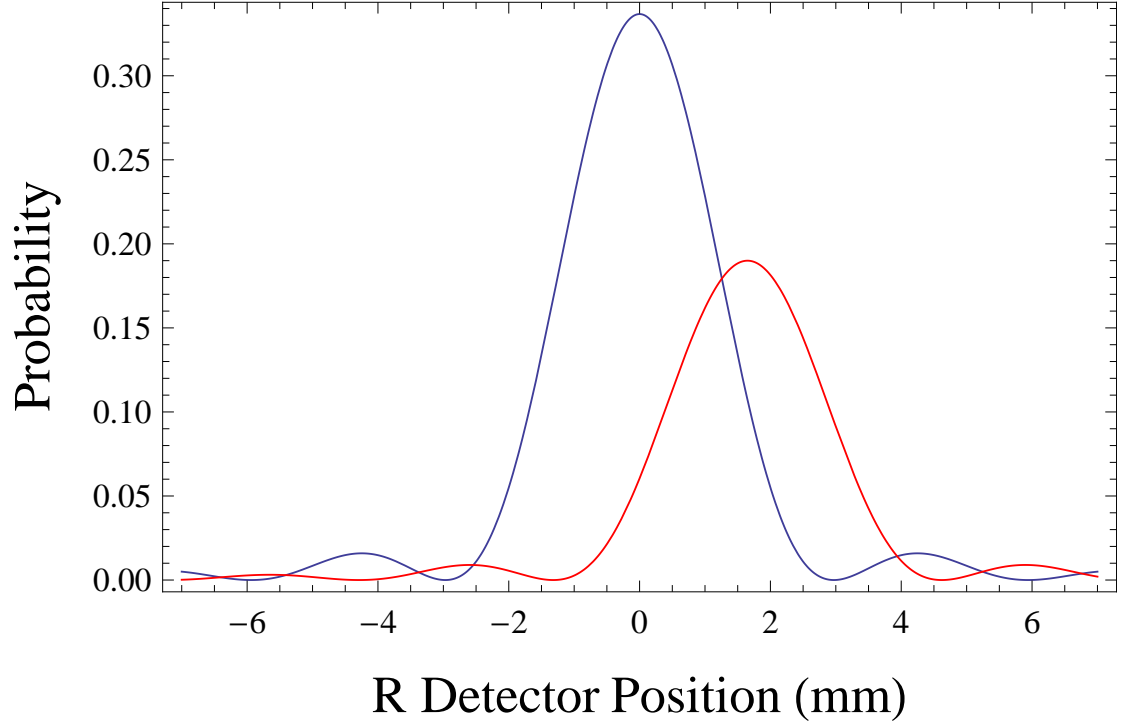
FIGURE 2.9. Theoretical probabilities for detecting photons at the detector on the right for the experiment by Strekalov, *et al.* The curve centered at the origin is the theoretical probability distribution of intensity on the y axis on the right side when the detector on the left side is stationary at the origin. The displaced curve is the theoretical probability distribution of intensity on the y axis of the detector on the right side when the detector on the left side is displaced from the origin. Both curves are that of a single slit interference pattern. The width of the pattern is governed by the distance between the real slit on the left to the detector on the right. These theoretical results match those given in experiment.

$$
\begin{aligned}
R(y_R) &= \int_{-\infty}^{\infty} W(y_R, y_L)\Phi(y_R, y_L)dy_L \\
&= \int_{-\infty}^{\infty} e^{-\frac{8\pi^2 y_R^2 a^2}{\lambda^2 z_2^2}} \frac{\sin(\frac{\pi d}{\lambda}(\frac{y_L}{z_1}))^2}{(\frac{y_L}{z_1})^2} \frac{\sin(\frac{\pi d}{\lambda}(\frac{y_L}{z_1} + \frac{y_R}{z_2}))^2}{(\frac{y_L}{z_1} + \frac{y_R}{z_2})^2} dy_L \\
&= \frac{e^{\frac{8\pi^2 y_R^2 a^2}{\lambda^2 z_2^2}}(\frac{\pi y_R d}{z_2 \lambda} - \sin[\frac{\pi y_R d}{z_2 \lambda}])}{y_R^3}
\end{aligned}
\tag{2.8}
$$

And if we insert the values that Kim and Shih used in their experiment we get

Fig. (2.11); where the red line on the outside is what one would get for single slit

diffraction at a real slit and the blue line on the inside is what Kim and Shih got on the detector on the right, behind the ghost slit.

If one were to assume a ghost slit were present, that would mean that the position of the photon on the right was known to be in the region of the slit width, but it's momentum uncertainty is less than what would be allowed by Heisenberg's uncertainty principle. But as I have said, no measurement of the z position was taken and there is no ghost slit, therefore no violation of the uncertainty principle. What we end up with in Kim and Shih's experiment is the sum of all those displaced single slit interference patterns noticed in Strekelov's experiment weighted by the probability of the interference pattern of the real single slit. The intensity distribution is something that looks like a Gaussian with a width dependent on the distance between the real slit and the detector behind the ghost slit, the beam width, the wavelength and the slit width. The edges of this curve are greatly attenuated by the width of the beam. If one were to take the limit of $R(y_R)$ as the beam width gets large, $a \to 0$, one could see from the rescaled Fig. (2.12) that the curve becomes much broader.

The results from both experiments flow from the same fundamentals and do not commit any egregious crimes against quantum mechanics such as FTL signaling and HUP violation. The notion of a ghost slit never comes out of the analysis, it is an artificial assumption forced onto the experiment. I agree that the ghost slit sounds right and I imposed it when first looking into Popper's experiment, but the data from two well run experiments and the preceding analysis forces it to be abandoned. The fact that the results of Popper's thought experiment were never correctly calculated in the seventy plus years since it was proposed illustrates just how unintuitive and strange quantum mechanics can be. Unlike Popper, we are fortunate to have decades of research showing just how correct quantum mechan-
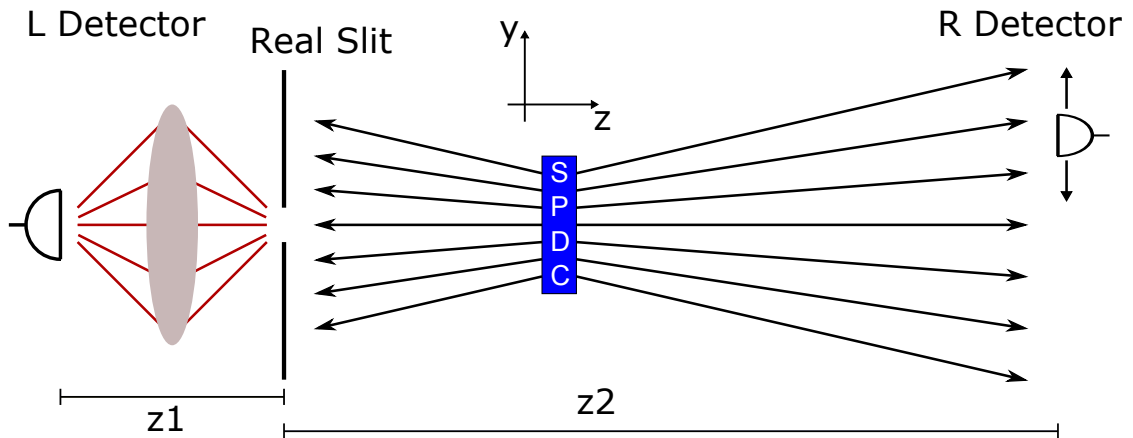
FIGURE 2.10. Kim and Shih's experimental setup without the notion of a ghost slit. The SPDC source emits a pair of momentum entangled photons in opposite directions. The left part of the bi-photon passes through a slit. This has a measurable effect on the right side of the bi-photon, but the notion of a ghost slit on the right side is discarded. All the photons on the left that make it through the slit are collected into one fixed detector. The detector on the right is free to scan the y axis. Only detection events in which detectors fired in coincidence are reported.
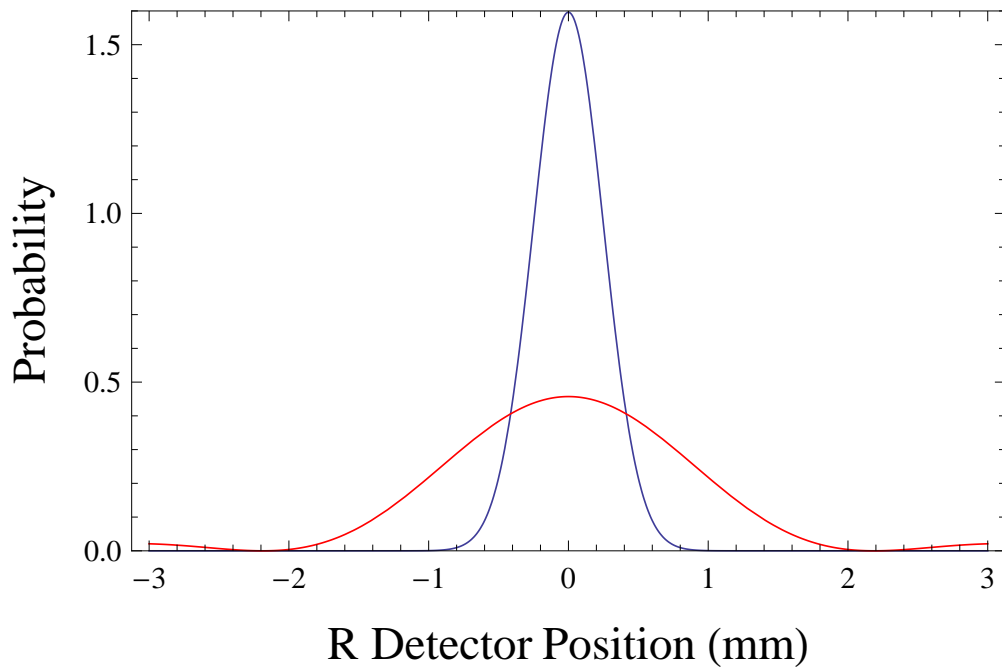

FIGURE 2.11. The inside curve is the theoretical probability distribution of intensity on the y axis on the right side for the experiment by Kim and Shih. The outside curve is the probability distribution of intensity for single slit diffraction. These theoretical results match those given in experiment.
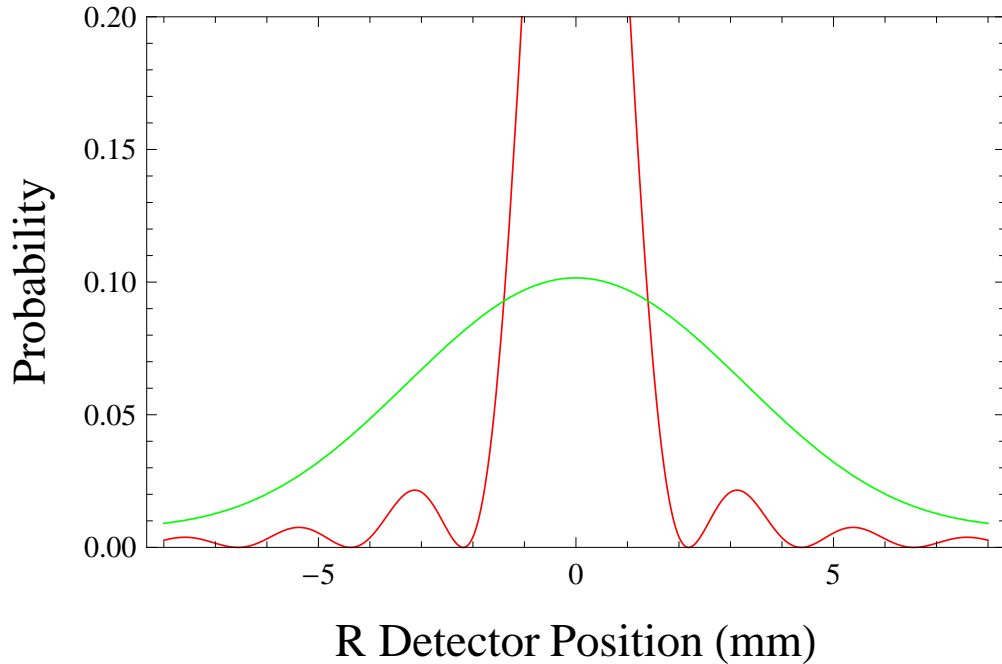
FIGURE 2.12. The outside curve is the theoretical probability distribution of intensity on the y axis on the right side for the experiment by Kim and Shih if the beam width is taken to go to infinity. The inside curve is the probability distribution of intensity for single slit diffraction.

ics is, so instead of using this experiment as a test of quantum mechanics it should be viewed as an interesting and subtle application of quantum mechanical fundamentals that gives up a deeper understanding of quantum mechanics. Popper argued that quantum mechanics was incomplete and that an experiment would not show spooky action at a distance at a ghost slit. He was right about the ghost slit, but wrong about quantum mechanics. Assuming that the photon on the right that does not pass through the slit is still measured at a ghost slit is incorrect. No measurement of the z component of the wave function was made, so placing a ghost slit at a specific place on the z axis should not be done. Being able to show that the two different experimental results come from the same quantum mechanics principles gives credence to this conclusion.

Chapter 3
Uncertainty with Entangled Pairs in Different Reference Frames

The theories of quantum mechanics and relativity are both wildly successful with a vast amount of empirical evidence to back them up. The basic postulates of both theories do not contradict each other and even work well together in some cases. For example, Dirac was able to derive a relativistic form of Shrödinger's equation [18] and there is promising work on uniting general relativity with quantum mechanics being done called loop quantum gravity by Gambini, Pullin [19] and others. However, there are situations in which the two theories seem to clash. This is most evident with the notion of wave function collapse. We have shown in chapter 1.5.4 that simultaneity is relative, but we have also states in chapter 1.2.4 that the collapse of the wave function is instantaneous across all space. The following section explores and ultimately resolves some of this tension between relativity and quantum mechanics.

3.1 The Zbinden, Brendel, Gisin and Tittel Experiment

There is a paradox, first suggested by Suarez and Scarani [20] and later tested by Zbinden, Brendel, Gisin and Tittel [21], in which two observers can both believe that they measured one half of an entangled pair of photons before the other. The collapse of the wave function is considered an irreversible and instantaneous process and so for a pair of entangled photons that are spatially separated the collapse occurs for both photons when a measurement is made on either of the pair. Given this, it is reasonable to ask which photon was measured first or when was the wave function collapsed, and expect a definite answer.
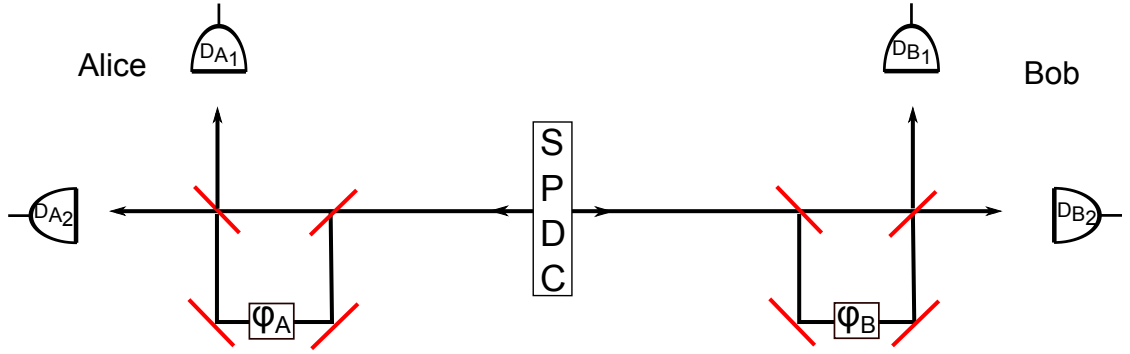
FIGURE 3.1. This Franson interferometer can be used to violate a Bell inequality. The possible paths the photons can take cause an interference at the detectors that can not be reproduced by classical light.

I will resolve this paradox by proposing an experiment to test it. I will show that the experiment constrains the system in such a way that an uncertainty is induced into the system that makes the question unanswerable and therefore the time ordering of entangled measurements does not change no matter the reference frame.

To do this we must apply a couple simple concepts. The first is that measurements in different reference frames may not yield the same answers. I are not saying that the two reference frames obey different physical laws. This would contradict the first of Einstein's postulate from chapter 1.5.1. I am simply noting that two observers in different reference frames may not always see the same thing, which is completely consistent with Einstein's relativity. The second being that in an experiment involving two observers in different reference frames, the observers must have information about both reference frames for reasonable communication to occur. When the observers communicate their results to each other, their respective reference frames must be accounted for.

It was proposed by Suarez and Scarani [20] that if two observers both think they measured one half of an entangled pair first then it in some way distinguishes them and the correlations between the two should disappear. So if one were to perform a
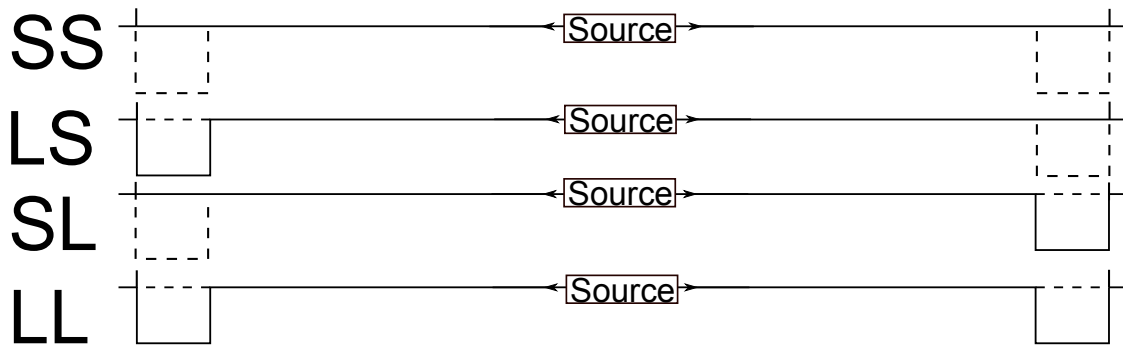
SS

LS

SL

LL

FIGURE 3.2. Possible paths a photon can take to get from the source to Alice and Bob's detectors: short-short (SS), long-long (LL), short-long (SL), and long-short (LS). The SS and LL paths, which are indistinguishable from each other are used to violate a Bell inequality.
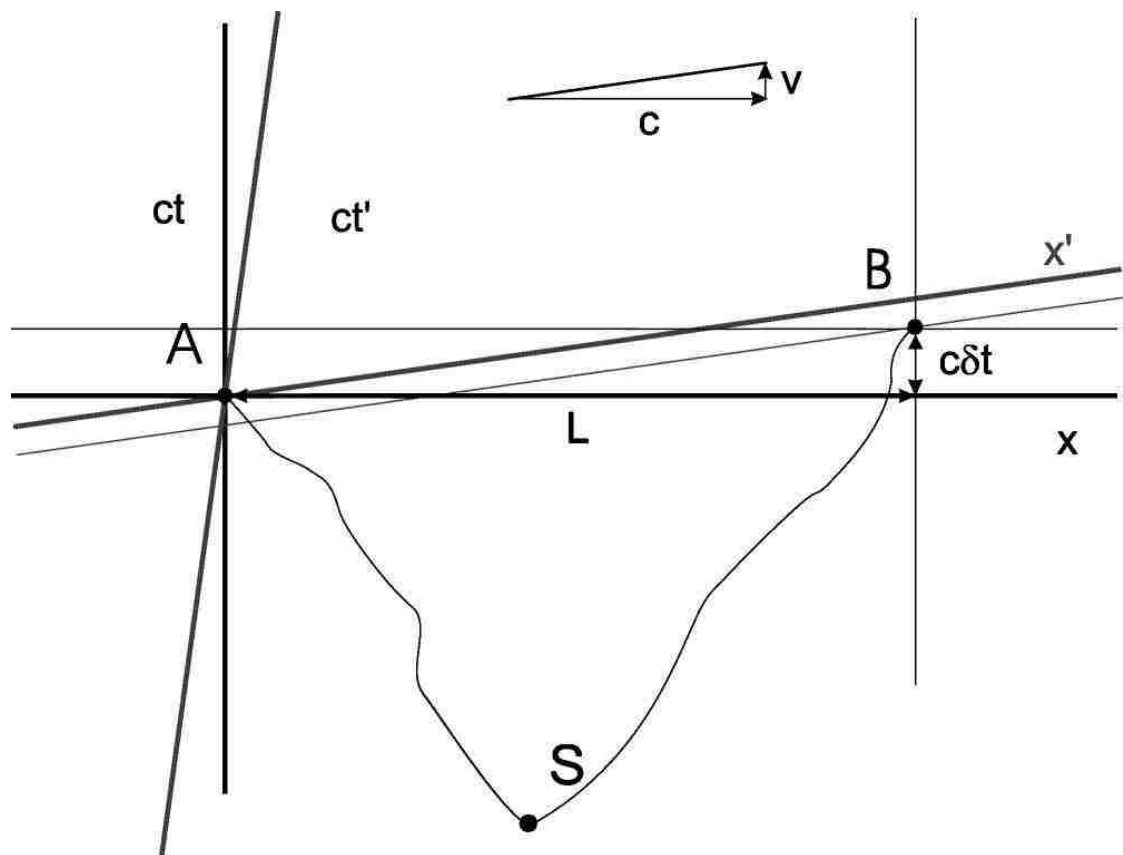
FIGURE 3.3. Diagram of the experiment by Zbinden, Brendel, Gisin and Tittel [21]. An entangled pair is released from a source(S) and travels to Alice(A) and Bob(B) who is in motion relative to Alice. In the situation pictured, both Alice and Bob measure first.

Bell inequality test, the inequality would only show only classical correlations and not quantum ones. This is exactly the kind of experiment that Zbinden, Brendel, Gisin and Tittel [21] performed. They used a Franson interferometer pictured in Fig. (3.1) than ran under Lake Geneva to perform a Bell inequality test.

A photon in a Franson interferometer can take two paths, the long path (L) and the short path (S) as seen in Fig. (3.2). Each side of the interferometer has the same path length difference. Because the two photons are entangled in time, there will be quantum interference visible when Alice and Bob compare data. To insure that the interference is from two photon entanglement, the path length difference is kept greater that the single particle coherence time and less than the two particle coherence time. The state of the system is at first:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|S\rangle|S\rangle + |L\rangle|S\rangle + |S\rangle|L\rangle + |L\rangle|L\rangle)$$

The LS and SL paths can be distinguished from their time of arrival information, so they can be ignored in post-selection and the effective state becomes:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|S\rangle|S\rangle + |L\rangle|L\rangle)$$

The detection probabilities for joint detection then become:

$$P(D_{A_1}, D_{B_1}) = P(D_{A_2}, D_{B_2}) = \frac{1}{4}(1 + cos(\phi_A + \phi_B)) \tag{3.1}$$

$$P(D_{A_1}, D_{B_2}) = P(D_{A_2}, D_{B_2}) = \frac{1}{4}(1 - cos(\phi_A + \phi_B)) \tag{3.2}$$

In which quantum correlations can clearly be seen and in which a Bell inequality can be violated. Part of what Zbinden, Brendel, Gisin and Tittel were testing in their experiment was the possibility that Eqns. [3.1] and [3.2] were not valid.

Building on the work by Suarez and Scarani they theorized that if both observers believe they measure first then the probabilities at each observer reduce to those determined only by the local states. In this case the probability for a photon to be detected in either detector at Alice's and Bob's end is equal which gives:

$$P(D_{A_1}, D_{B_1}) = P(D_{A_2}, D_{B_2}) = \frac{1}{4}$$
$$P(D_{A_1}, D_{B_2}) = P(D_{A_2}, D_{B_2}) = \frac{1}{4}$$

The stated purpose of their experiment was to test the speed of wave function collapse, but we will ignore that part and focus instead on what happened with the Bell test. Their experiment is depicted in Fig. (3.3). An entangled two photon source fed two photons under lake geneva to Alice and Bob. Alice is at rest relative to the source and Bob is on a wheel that allows him to have a velocity relative to Alice and the source. The path to Bob is slight longer than the path to Alice by an amount $c\delta t$. The time difference δt is then the time difference between Alice and Bob's measurements when Alice and Bob are at rest. When Bob gains velocity the time difference changes to:

$$\delta t' = \gamma(\delta t - \frac{\beta L}{c})$$

Where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. So if the situation occurs when,

$$\frac{\beta L}{c} \geq \delta t \geq 0$$

Then $\delta t'$ becomes negative and the time ordering in Bob's reference frame changes and Bob thinks he measured first. However, in Alice's reference frame, she believes that she measured first. So, did this experiment show that when this
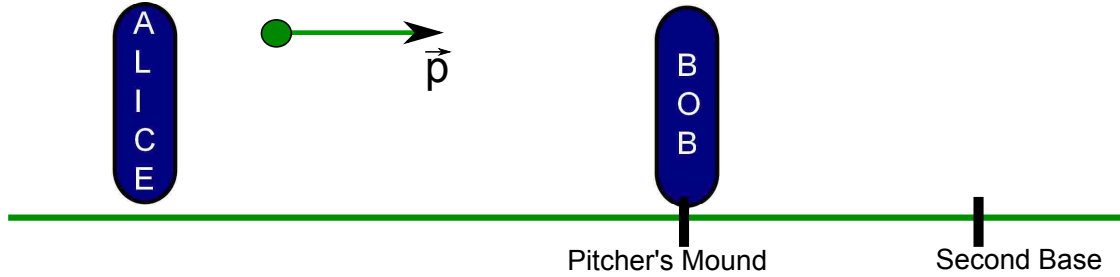
FIGURE 3.4. Alice throws a baseball with momentum p to Bob. If Bob catches the ball then Alice and Bob know that the ball was caught at the position of the pitcher's mound.

situation happens that the quantum correlations disappear? No, they did not find any change in quantum correlations when Bob was still, in which both Alice and Bob agree that Alice measured first, versus when Bob was in motion, in which both Alice and Bob believe that they measured first. And yet seemingly the wave function must have been collapsed by someone first. In the following I will resolve this paradox.

3.2 Motivation

Before diving into the proposed experiment it will be helpful to go over some examples of systems in which two observers are not in the same reference frame. The first example is a classical example with an observer in a reference frame in motion with respect to the other observer. However, reference frames need not only be based on difference in velocity. The second example has two observers that observe the spin of a particle from two different reference frames. From these two examples the notion of measurements between observers in different reference frames should become clearer.

3.2.1 Batting Practice

Imagine Alice and Bob are at batting practice as is depicted in Fig. (3.4). It's Alice's turn to bat, so she is hitting the ball and Bob is catching. Alice hits a line drive and Bob catches it at the pitchers mound. Alice asks Bob the simple
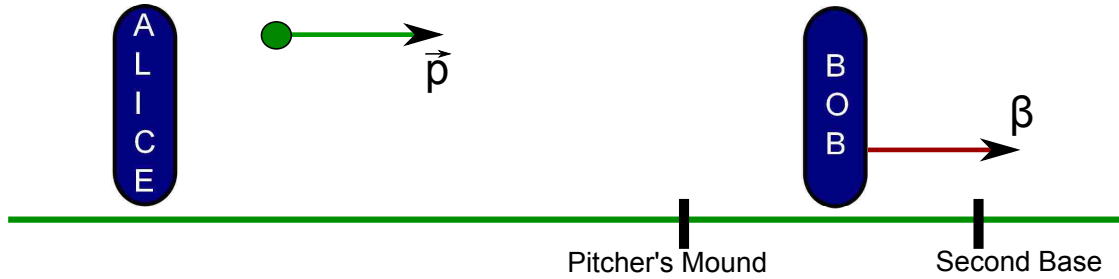
FIGURE 3.5. Alice throws a baseball with momentum p to Bob. Bob is moving with a velocity β away from Alice and catches the ball at second base. If Bob catches the ball then Alice and Bob not only know that the ball was caught as second base, but also that it was traveling with a minimum speed of β.

question "Where did you catch the ball Bob?" Bob answers the obvious "I caught the ball at the pitchers mound." Alice also asks the question "How fast was the ball going?", but Bob, not having a speed sensor, cannot answer with any certainty. Alice and Bob now change their relative reference frame by deciding that when Bob hears the crack of the ball he should start running backwards as fast as he can, which he knows is β, as is depicted in Fig. (3.5). When Alice hits the ball, Bob starts running backwards and catches the ball at second base. Bob can still tell Alice that he caught the ball at second base, but in addition Bob can now also tell Alice that the ball had a velocity of at least β. If the ball went any slower it would have never reached Bob. When Bob communicated his measurement to Alice he gave her momentum information as well as position information. It is because Bob knows something about Alice's reference frame that he can make such an inference about the baseballs momentum. This may seem trivial, but in the quantum world Bob made a measurement of the balls momentum and that measurement has consequences. A measurement of position made from a reference frame in motion relative to Alice and then communicated to Alice also becomes a measurement of momentum.
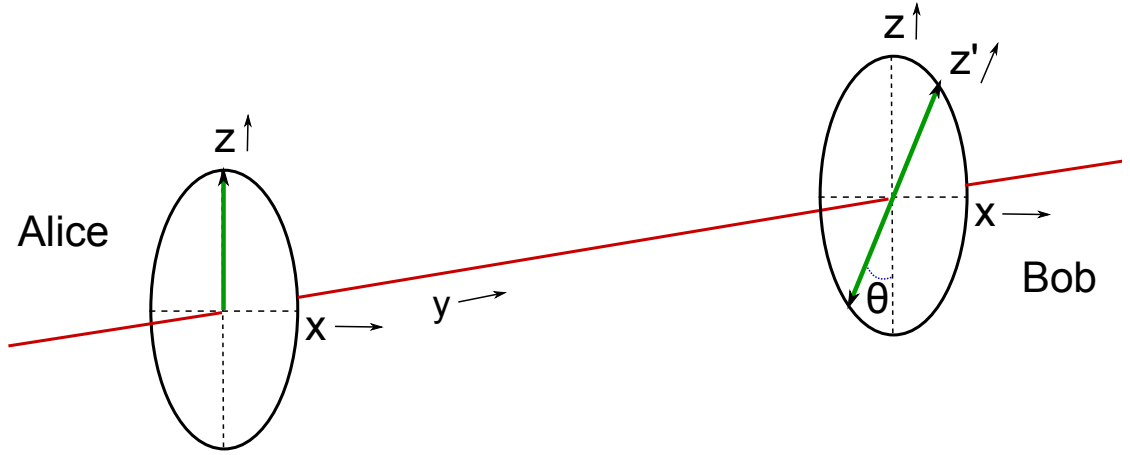
FIGURE 3.6. Alice sends Bob a particle with spin in the $+z$ direction. Bob does not share the same reference frame as Alice. Bob's frame is rotated about the by θ relative to Alice's reference frame. If Bob measures a particle with a spin of $+z'$ in his own reference frame, he cannot tell Alice with certainty that the particle she sent had a spin in the $+z$ direction.

Now let us say that there is a maximum momentum that Alice can impart to the baseball $p_{max} = mv_{max}$. Then if Bob were moving faster that $\beta > v_{max}$, he would never detect the baseball. If Bob were traveling at just under the maximum velocity of the baseball $\beta \approx v_{max}$, then we can determine that the momentum baseball is $p_{max} = mv_{max} = m\beta$. The momentum of the baseball has completely determined simply because Bob made the measurement. If this were a quantum baseball then this begs us to ask what happens to the position. The HUP would tell us that the position measurement over many trials would be completely uncertain. Baseballs aren't the best quantum particles which leads us to move to a simple example in the next section using a particle with spin.

3.2.2 Spin Motivation

Imagine Alice and Bob agree on a reference frame and Bob has a measurement device that measures spin in the z direction. If Alice sends Bob a particle of spin $+z$ and Bob measures that particle, Alice can ask Bob "Does the the particle have spin in the $+z$ direction?". Bob can answer with certainty "Yes, the particle I

detected has a spin in the $+z$ direction." This is an easy answer because when Bob measures the particle, he projects it into the state with spin in the $\pm z$ direction. However, if Alice and Bob change their relative reference frames by rotating in the z, x plane by θ as depicted in Fig. (3.6), then when Bob makes a measurement in his reference frame the state relative to Alice is projected into the rotated reference frame. Now even though Bob is completely certain when he measures $+z'$ in his own frame of reference the particle has a spin in the $+z'$ direction, if Alice asks Bob the same question, Bob cannot answer with certainty what the spin would be in Alice's reference frame. The uncertainty would be $\sigma_{+z} = \sin(\frac{\theta}{2})$.

As Bob rotates further to Alice's $+x$ axis he becomes completely uncertain if the particle has a spin in Alice's $+z$ direction. However, Bob can tell Alice that he has some certainty that the particle has a spin component in Alice's $+x$ direction. When Bob rotates completely to Alice's $+x$ axis he can tell Alice with complete certainty that the particle has a spin in Alice's $+x$ direction even though in Bob's reference frame he measured in the $+z'$ direction. A measurement of spin by Bob in the $+z'$ direction made from a rotated reference frame relative to Alice and then communicated to Alice also becomes a measurement of the spin in the $+z$ and $+x$ directions. Again, the change in reference frame gives information about a conjugate variable.

3.3 The Proposed Experiment
3.3.1 The Experimental Device

Motivated by the Zbinden, Brendel, Gisin and Tittel experiment [21], I describe a experiment that reproduces the paradox in which two observers measure first which will measure the temporal order of two measurements in different reference frames. The thought experiment is represented through space-time diagrams in Figs. (3.7 - 3.10). The fundamental constants are set to unity, $\hbar = c = 1$. Bob is
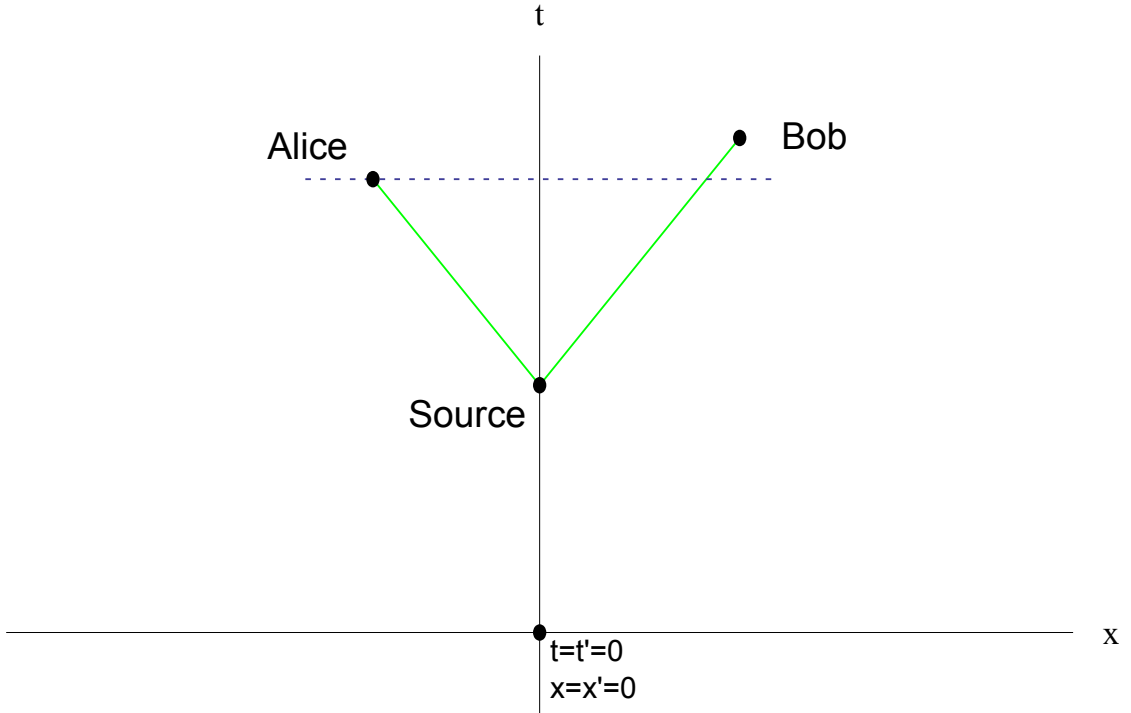
FIGURE 3.7. Bob is stationary. At some time t_e after Bob passes the bi-photon source, a bi-photon is emitted. In both Alice and Bob's frame, Bob detects a photon after Alice.
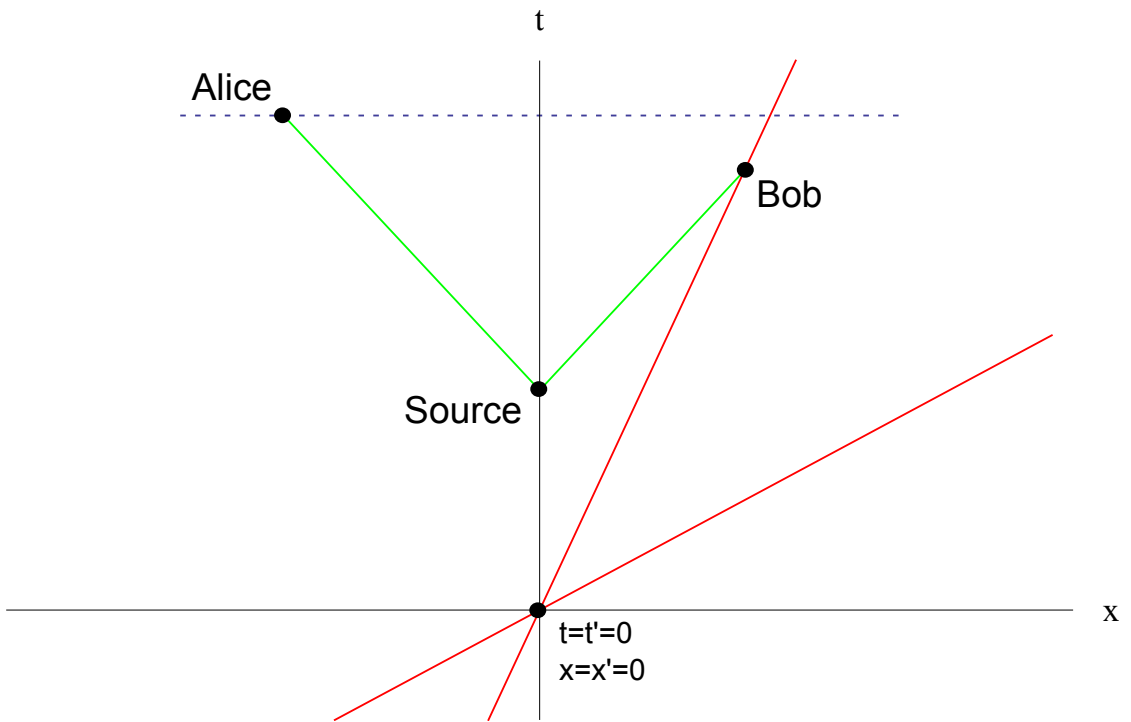

FIGURE 3.8. At some time t_e where $t_e < t_{e_{min}} = d(\frac{1}{\beta} - 1)$ after Bob passes the bi-photon source, a bi-photon is emitted. In both Alice and Bob's frame, Bob detects a photon before Alice.
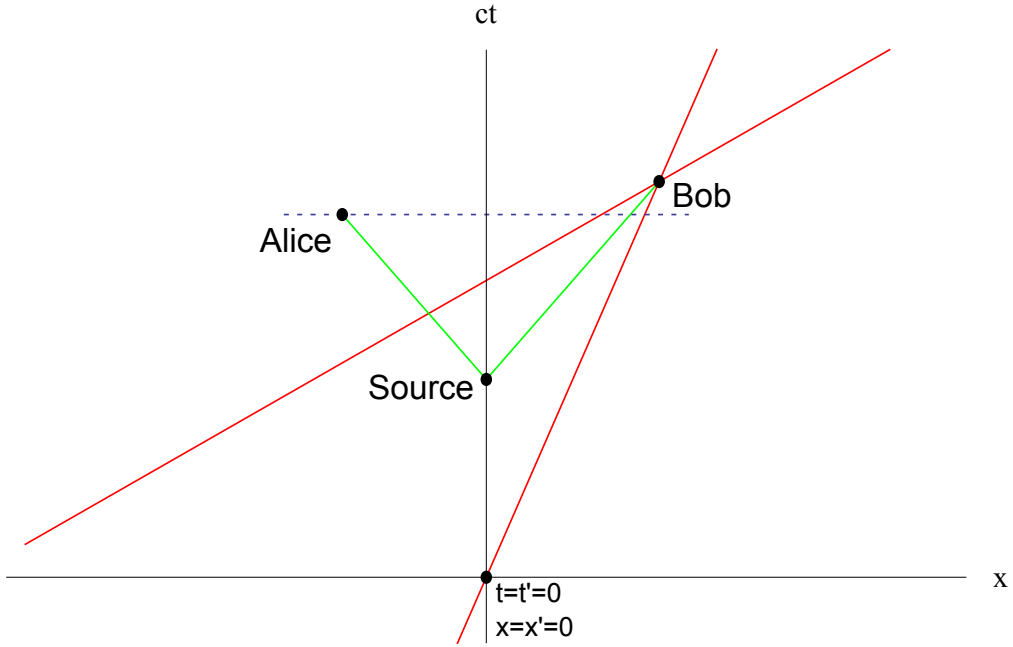
FIGURE 3.9. At some time t_e where $d(\frac{1}{\beta} - 1) = t_{e_{min}} < t_e < t_{e_{max}} = d\frac{1+\beta}{1-\beta}(\frac{1}{\beta} - 1)$ after Bob passes the bi-photon source, a bi-photon is emitted. In Alice's reference frame, the lab frame, Bob detects a photon after Alice. In Bob's reference frame the measurement order is reversed.

FIGURE 3.10. At some time t_e where $t_e > t_{e_{max}} = d\frac{1+\beta}{1-\beta}(\frac{1}{\beta} - 1)$ after Bob passes the bi-photon source, a bi-photon is emitted. In both Alice and Bob's frame, Bob detects a photon after Alice.
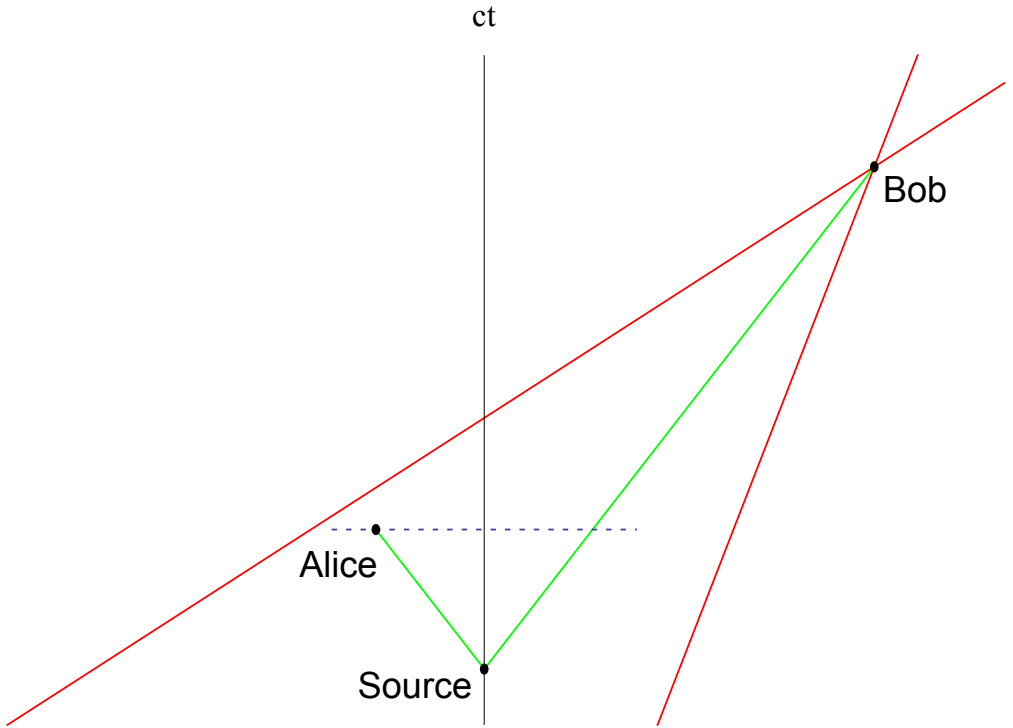
in motion relative to Alice with a velocity β. At $(x_{B_0}, t_{B_0}) = (x'_{B_0}, t'_{B_0}) = (0,0)$ Bob passes a source a source of energy-time entangled particles and Alice is at $(x_{A_0}, t_{A_0}) = (-d, 0)$. At some later time t_e the source emits a bi-photon. Alice receives her photon at $(x_A, t_A) = (-d, t_A)$ and Bob receives his photon at (x_B, t_B). It is assumed that Alice sends Bob a signal when her detector triggers and Bob can use it to determine Alice's measurement time and compare events. The Lorentz transform in this case becomes:

$$
\begin{aligned}
\Delta t' &= t'_B - t'_A \\
&= \gamma(\Delta t - \beta \Delta x) \\
&= \gamma(\Delta t - \beta(2d - \beta \Delta t)) \\
&= \gamma(\Delta t(1 - \beta^2) - 2\beta d)
\end{aligned}
$$

It is of vital importance to realize that to test the situation in which it is possible for the measurement time to swap in Bob's reference frame we must require that $t_B \geq t_A$ and $x_B \geq x_A$. In the experiment by Zbinden, Brendel, Gisin and Tittel experiment [21] the time the entangled particles will be emitted can be determined from how long the wheel is in a position to take a measurement. If it is known what time the detector on the wheel will be in a position to take a measurement then it is known when the bi-photon will be emitted. The wheel is, however, more restrictive than necessary and will exaggerate the results. In this experiment the photons must be emitted within a time Δt_e otherwise no paradox will occur. If they are emitted before or after, like in figures 3.8 and 3.10, there can be no result since both observers will agree on the time ordering.

The time constraint can be derived with some simple algebra. For Bob to be in a position to measure after Alice in Alice's frame, he must at least be equidistant from the source, which puts Bob's minimum position and time at:

$$x_{B_{min}} = d$$

$$t_{B_{min}} = \frac{d}{\beta}$$

When Bob leaves the maximum time in which Alice believes she measured first, we can make a line from Alice that has the slope of Bob's velocity $y_A = \beta x + d(1 + \beta) + t$. When we set this equal to Bob's line $y_B = \frac{x}{\beta}$, we get Bob's maximum position and time:

$$x_{B_{max}} = d\frac{1 + \beta}{1 - \beta}$$

$$t_{B_{max}} = \frac{x_{B_{max}}}{\beta}$$

$$= \frac{d}{\beta}\frac{1 + \beta}{1 - \beta}$$

Then to get the constraint on the emission time Δt_e we take Bob's measurement times and subtract the amount of time it took the photon to get to Bob.

$$t_{e_{min}} = t_{B_{min}} - d = d(\frac{1}{\beta} - 1)$$

$$t_{e_{max}} = t_{B_{max}} - x_{B_{max}}$$

$$= \frac{d}{\beta}\frac{1 + \beta}{1 - \beta} - d\frac{1 + \beta}{1 - \beta}$$

$$= d\frac{1 + \beta}{1 - \beta}(\frac{1}{\beta} - 1)$$

Therefore, the time constraint Δt_e is:

$$\Delta t_e = t_{e_{max}} - t_{e_{min}} = 2d. \tag{3.3}$$

Simply posing the thought experiment in such a way that allows Alice and Bob to compare measurements in different frames adds this time constraint to the bi-photon. This gives us information about the entangled pair or, in other words a measurement on the bi-photon. Any knowledge about the bi-photon is a measurement on the bi-photon and will reduce or collapse some of the entanglement in the bi-photon system. This will lead to an uncertainty which balances the time difference induced by the Lorentz transform of Alice's measurement time.

3.3.2 Analysis of the Experiment

To begin I start with the original EPR [8] states translated into the energy-time representation. This can be dangerous to do. After all, there is no real time operator and I would hesitate to do this with a massive particle, but with photons the two representations can be considered relatively interchangeable. For a photon, position is time and momentum is energy and I will henceforth consider the arrival time of a photon a measurement of it's time. The EPR states in energy-time representation are:

$$
\begin{aligned}
\psi(t_A, t_B) &= \langle t_A, t_B | \Psi \rangle = \delta(t_A - t_B) \\
&= \int_{-\infty}^{\infty} e^{-iE(t_A - t_B)} \, \mathrm{d}E \\
\phi(E_A, E_B) &= \langle E_A, E_B | \Psi \rangle = \delta(E_A + E_B) \\
&= \int_{-\infty}^{\infty} e^{-it(E_A - E_B)} \, \mathrm{d}t
\end{aligned}
$$

The time constraint $\Delta t_e = 2d$ when applied to $\psi(t_A, t_B)$ will yield no additional uncertainty when a time measurement is made. It will, however, in accordance

with the HUP have an effect on any possible energy measurements. I have chosen the function for the time constraint to be a Gaussian with width $(2d)$. This choice is completely arbitrary and substituting any well behaved probability distribution would work just as well. If we apply the constraint and derive $\phi(E_A, E_B)$ we get:

$$
\begin{aligned}
\phi_C(E_A, E_B) &= \langle E_A, E_B | \Psi_C \rangle = \int_{-\infty}^{\infty} e^{-\frac{t_A^2}{4(2d)^2}} e^{-it_A(E_A+E_B)} \, \mathrm{d}t_A \\
&= e^{-(E_A+E_B)^2(2d)^2}
\end{aligned}
$$

The above equation is a Gaussian with a width of $\frac{1}{(2d)}$. As $(2d) \to \infty$, then $\phi_L(E_A, E_B) \to \delta(E_A + E_B)$ and entanglement is unchanged, as would be expected. To find the wave function on Bob's side after coincidence counting, we require the wave function with full knowledge of Alice's measurement. This is done by projecting onto Alice's basis and then tracing out her side.

$$
\begin{aligned}
\psi_B(t_B) &= \langle t_B | \Psi_B \rangle = \langle t_A = t_0 | \Psi_C \rangle = \langle t_A = t_0 | t_A, t_B \rangle \langle t_A, t_B | \Psi_C \rangle \\
&= \delta(t_B - t_0) \\
\phi_B(E_B) &= \langle E_B | \Psi_B \rangle = \langle E_A = E_0 | \Psi_C \rangle \\
&= \langle E_A = E_0 | E_A, E_B \rangle \langle E_A, E_B | \Psi_C \rangle \\
&= \int_{-\infty}^{\infty} e^{-(E_A+E_B)^2(2d)^2} \delta(E_A - E_0) \, \mathrm{d}E_A \\
&= e^{-(E_B+E_0)^2(2d)^2}
\end{aligned} \tag{3.4}
$$

Equation 3.4 would not usually make a difference since if Alice and Bob were in the same reference frame, the energy component of the wave function would not be measured. However, since the measurement is being done in a different reference frame, we must project onto Bob's reference frame. I have chosen the projector in Eqn. (3.5) based on three assumptions. Firstly, a particle at rest is in an eigenstate

of position. Secondly, a particle traveling at the speed of light is in a momentum eigenstate. Thirdly, there should be a smooth transition between eigenstates as the velocity goes from zero to the speed of light. The chosen projector that takes a state from Alice's reference frame into Bob's reference frame is:

$$|t'_B\rangle\langle t'_B| = \int_{-\infty}^{\infty} dt_B (1 - \beta^2)|t_B\rangle\langle t_B| + \int_{-\infty}^{\infty} dE_B \beta^2 |E_B\rangle\langle E_B| \tag{3.5}$$

Where $(1 - \beta^2) + \beta^2 = 1$ and projecting twice returns the original projector as a well formed projector should:

$$
\begin{aligned}
|t'_B\rangle\langle t'_B|t'_B\rangle\langle t'_B| &= \iint_{-\infty}^{\infty} dt_B (1 - \beta^2)^2 |t_B\rangle\langle t_B|t_B\rangle\langle t_B| \\
|t'_B\rangle\langle t'_B|t'_B\rangle\langle t'_B| &= \int_{-\infty}^{\infty} dt_B \int_{-\infty}^{\infty} dt_B (1 - \beta^2)^2 |t_B\rangle\langle t_B|t_B\rangle\langle t_B| \\
&\quad + \int_{-\infty}^{\infty} dE_B \int_{-\infty}^{\infty} dE_B \beta^4 |E_B\rangle\langle E_B|E_B\rangle\langle E_B| \\
&\quad + \int_{-\infty}^{\infty} dt_B \int_{-\infty}^{\infty} dE_B \beta^2 (1 - \beta^2)|t_B\rangle\langle t_B|E_B\rangle\langle E_B| \\
&\quad + \int_{-\infty}^{\infty} dE_B \int_{-\infty}^{\infty} dt_B \beta^2 (1 - \beta^2)|E_B\rangle\langle E_B|t_B\rangle\langle t_B| \\
&= \int_{-\infty}^{\infty} dt_B (1 - \beta^2)^2 |t_B\rangle\langle t_B| + \int_{-\infty}^{\infty} dt_B \beta^4 |E_B\rangle\langle E_B| \\
&\quad + \int_{-\infty}^{\infty} dt_B \int_{-\infty}^{\infty} dE_B \beta^2 (1 - \beta^2)e^{+it_B E_B}|t_B\rangle\langle E_B| \\
&\quad + \int_{-\infty}^{\infty} dE_B \int_{-\infty}^{\infty} dt_B \beta^2 (1 - \beta^2)e^{-it_B E_B}|E_B\rangle\langle t_B| \\
&= \int_{-\infty}^{\infty} dt_B (1 - \beta^2)^2 |t_B\rangle\langle t_B| + \int_{-\infty}^{\infty} dE_B \beta^4 |E_B\rangle\langle E_B| \\
&\quad + \int_{-\infty}^{\infty} dt_B \beta^2 (1 - \beta^2)|t_B\rangle\langle t_B| \\
&\quad + \int_{-\infty}^{\infty} dE_B \beta^2 (1 - \beta^2)|E_B\rangle\langle E_B| \\
&= \int_{-\infty}^{\infty} dt_B (1 - \beta^2)|t_B\rangle\langle t_B| + \int_{-\infty}^{\infty} dE_B \beta^2 |E_B\rangle\langle E_B| \\
&= |t'_B\rangle\langle t'_B|
\end{aligned}
$$

The uncertainty in position, σ'_{t_B}, after a projective measurement of $|t'_B\rangle\langle t'_B|$ is:

$$
\begin{aligned}
\langle \hat{t}'_B \rangle &= \langle \Psi_B | t'_B \rangle \langle t'_B | \hat{t}_B | t'_B \rangle \langle t'_B | \Psi_B \rangle \\
&= \langle \Psi_B | [\int_{-\infty}^{\infty} dt_B \int_{-\infty}^{\infty} dt_B (1 - \beta^2)^2 |t_B\rangle\langle t_B | \hat{t}_B | t_B \rangle \langle t_B | \\
&\quad + \int_{-\infty}^{\infty} dE_B \int_{-\infty}^{\infty} dE_B \beta^4 |E_B\rangle\langle E_B | \hat{t}_B | E_B \rangle \langle E_B | \\
&\quad + \int_{-\infty}^{\infty} dt_B \int_{-\infty}^{\infty} dE_B \beta^2 (1 - \beta^2) |t_B\rangle\langle t_B | \hat{t}_B | E_B \rangle \langle E_B | \\
&\quad + \int_{-\infty}^{\infty} dE_B \int_{-\infty}^{\infty} dt_B \beta^2 (1 - \beta^2) |E_B\rangle\langle E_B | \hat{t}_B | t_B \rangle \langle t_B |] | \Psi_B \rangle \\
&= \int_{-\infty}^{\infty} dt_B (1 - \beta^2)^2 \langle \Psi_B | t_B \rangle t_B \langle t_B | \Psi_B \rangle \\
&\quad + \int_{-\infty}^{\infty} dE_B \beta^4 \langle \Psi_B | E_B \rangle \frac{d}{dE_B} \langle E_B | \Psi_B \rangle \\
&\quad + \int_{-\infty}^{\infty} dt_B \int_{-\infty}^{\infty} dE_B \beta^2 (1 - \beta^2) \langle \Psi_B | t_B \rangle \frac{d}{dE_B} e^{+it_B E_B} \langle E_B | \Psi_B \rangle \\
&\quad + \int_{-\infty}^{\infty} dE_B \int_{-\infty}^{\infty} dt_B \beta^2 (1 - \beta^2) \langle \Psi_B | E_B \rangle \frac{d}{dE_B} e^{-it_B E_B} \langle t_B | \Psi_B \rangle \\
&= \int_{-\infty}^{\infty} (1 - \beta^2) \psi_B t_B \psi_B dt_B + \int_{-\infty}^{\infty} \beta^2 \phi_B \frac{d}{dE_B} \phi_B dE_B = 0 \\
\langle \hat{t}'^2_B \rangle &= \int_{-\infty}^{\infty} (1 - \beta^2) \psi_B t_B^2 \psi_B dt_B + \int_{-\infty}^{\infty} \beta^2 \phi_B \frac{d^2}{dE_B^2} \phi_B dE_B = \beta^2 (2d)^2 \\
\sigma'^2_{t_B} &= \langle \hat{t}'^2_B \rangle - \langle \hat{t}'_B \rangle^2 = \beta^2 (2d)^2
\end{aligned}
$$

So as $\beta \to 1$, the uncertainty becomes our original time constraint. Alice now asks Bob at what time he measured the photon and Bob answers:

$$
\begin{aligned}
\Delta t' &= \gamma(\Delta t \pm \sigma_t - \beta(2d - \beta\Delta t)) \\
&= \gamma(\Delta t(1 - \beta^2) - 2\beta d \pm \sigma_t) \\
&= \gamma(\Delta t(1 - \beta^2) - 2\beta d \pm 2\beta d) \\
\Delta t'_+ &= \gamma(\Delta t(1 - \beta^2) - 2\beta d + 2\beta d) \\
&= \gamma(\Delta t(1 - \beta^2)) \tag{3.6}
\end{aligned}
$$

Equation 3.6 will always have a positive value. The time uncertainty always out paces the time difference induced by the change in reference frames. Neither Alice nor Bob will ever, with certainty, observe the two measurements swap temporal order. I would like to reiterate here what I said in chapter 1.2.5. Quantum mechanics is a probabilistic theory. In the above it may seem like I have been talking about the uncertainty of a single measurement, but that is not how quantum mechanics works. In the analysis above it is completely possible that when Alice and Bob measure one entangled pair they both get results which are consistent with them both measuring first. It will only be after many runs of the experiment that the uncertainty becomes evident. In the experiment proposed above Bob would have to start from the source for each trial and only after many trials will the measurement order, or lack thereof, become evident.

Using the above it can also be said that if a measurement on an entangled bi-photon is simultaneous in one reference frame, then it can be considered simultaneous in all reference frames. I say this because if one were to try and determine how the temporal order changes, the experiment will introduce enough uncertainty to make it impossible. Therefore, there need not be a preferred reference frame for wave function collapse. The attempt to determine what reference frame the wave function collapse takes place in would lead to an uncertainty that would make it impossible to determine. That is not to say that all measurements on entangled particles are simultaneous. There are many situations in which one can determine the order of measurement, but if it can be determined in one reference frame then it will be the same in all reference frames. Simply posing a way to measure the paradox induces an uncertainty in the system that makes the paradox unanswerable. I have turned a paradox into a catch-22.

Chapter 4
A Practical Application in QKD

4.1 Quantum Key Distribution

Uncertainty was used in the previous examples to explain complex system, but we can also treat uncertainty as a resource. From the discussion in chapter 1.2.4 we know that a measurement collapses a wave function into one of the possible states of the detector's Hilbert space. If a subsequent measurement using a detector with an alternate Hilbert space is done on that state then the result will be uncertain. We can use this uncertainty to detect an eavesdropper if we use quantum states to encode a secret message. To encode messages we first need to secretly distribute a key.

Quantum key distribution is rapidly emerging as an elegant application of quantum information theory with immense practical value. The advent of quantum computing compromises classical encryption schemes which are dependent on computational difficulty for security. Fortunately, quantum information theory solves the exact problem it creates. If a transmitter, Alice, wants to exchange a message with a receiver, Bob, then the fundamental principles of quantum mechanics allow them to generate a key that cannot be obtained by an eavesdropper, Eve [22][23][24]. The problem that we are addressing is that in real life QKD protocols there are practical compromises that lead to security vulnerabilities. In the following I discuss a decoy state scheme developed by me and others in [25] to combat one of these vulnerabilities. The protocol that we will base our decoy scheme on is the BB84 protocol.
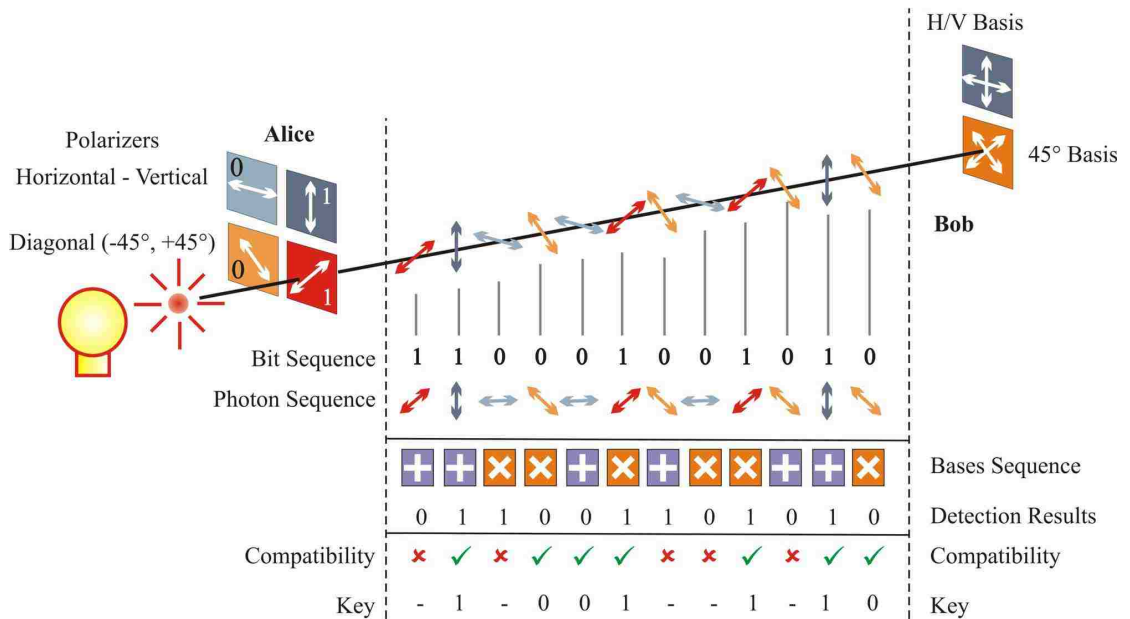
FIGURE 4.1. A representation of the BB84 protocol from [30]. Alice sends bits to Bob in either the Horizontal-Vertical or the Diagonal $(-45°, +45°)$ basis and Bob randomly chooses which basis to measure in. When they compare their basis choice they can distill a key. Ideally an Eavesdropper would upset the statistics which Alice an Bob would be able to detect when they compare their basis choices.

4.2 The BB84 Protocol

In the theoretic framework of BB84, Alice sends a sequence of single photon pulses to Bob. These photons are prepared in two randomly chosen bases. In the receiving lab, Bob has the same two bases in which to measure the photon and randomly alternates between them. So, if Alice send a key bit in the Horizontal-Vertical (HV) bases and Bob happens to be receiving in that bases, then when Alice and Bob get together to discuss which basis they measured in, they will know they shared that bit of information and therefore a bit of the key. If Alice sends a bit in the HV bases, but Bob measured in the Diagonal basis, then when they compare basis for that bit they realize they don't match up and it will not contribute to the shared key. If Eve tries measuring Alice's photon and then sending the result of her measurement to Bob, the eavesdropper will introduce errors into the key, since she
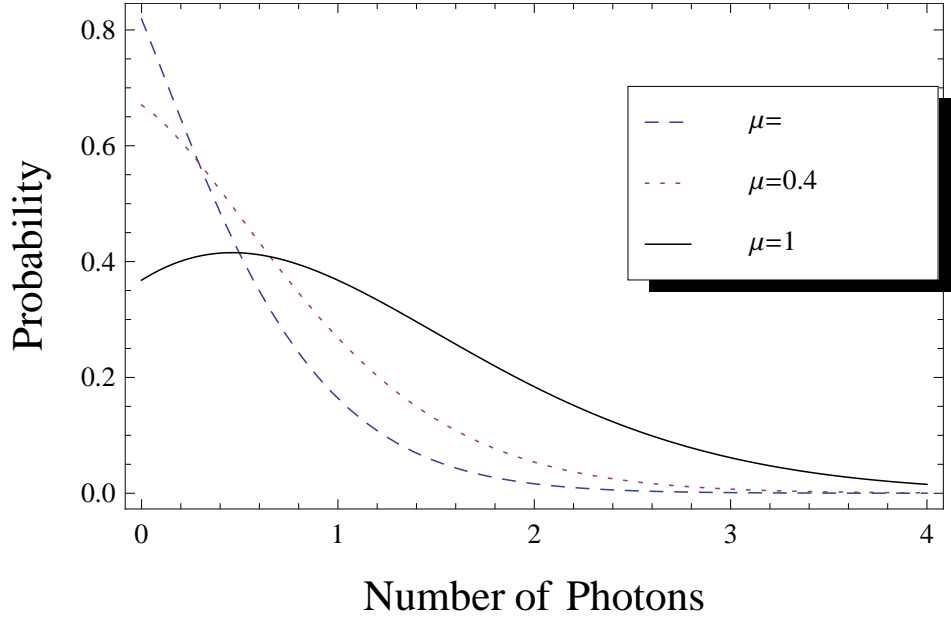
FIGURE 4.2. QKD implementations use attenuated lasers as single photon sources. The average photon number of attenuated lasers follow a Poisson distribution. This makes them vulnerable to a photon number splitting attack.

does not know in which basis the photon is being sent nor does she know in which basis Bob will measure. For example if both Alice and Bob are in the Diagonal basis and Eve is in the vertical basis then after Eve makes a measurement the photon is then diagonal and Bob's measurement may not be consistent with what Alice sent him. Alice and Bob can then use these errors to detect the eavesdropper's presence and determine the security of the key [4].

4.3   PNS Attack

However, in many experimental settings, Alice does not have a true single photon source, so she sends weak laser pulses (WLP) instead. This coherent light photon number probability follows a Poisson distribution. The probability of a pulse
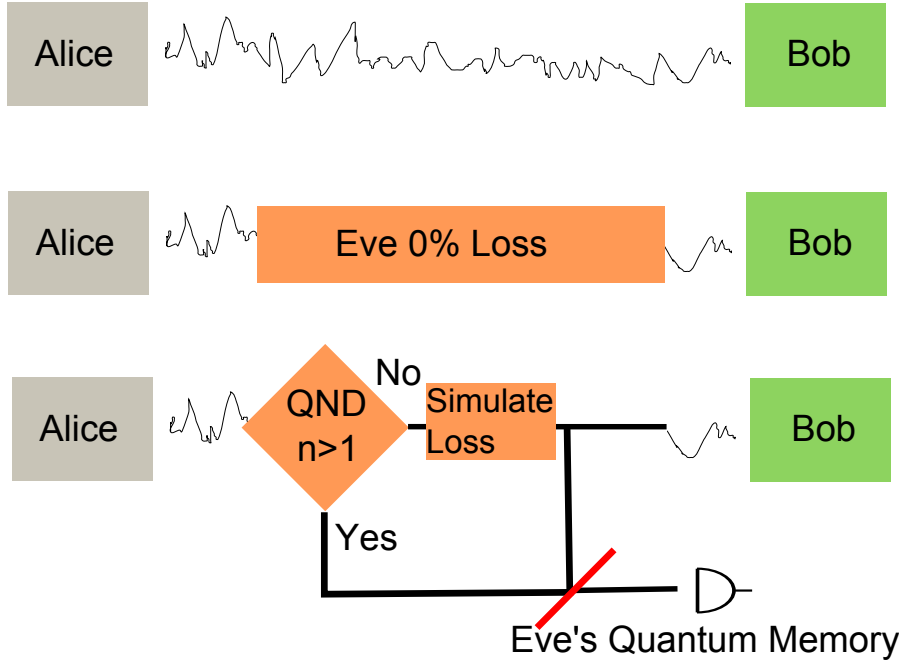
FIGURE 4.3. The eavesdropper replaces part or all of the lossy channel with a lossless channel. Eve then performs a QND measurement of number Eve blocks a fraction of single photon pulses and splits off multi-photon pulses If loss is high, she can obtain a significant fraction of the key without being detected by Alice and Bob.

containing n photons is

$$P_n = \frac{\mu^n}{n!e^\mu} \tag{4.1}$$

where μ is the mean photon number which will be taken to be a positive number less than one to avoid pulses with more than one photon. However, multiple photon pulses will still occur with probability $P_M = 1 - e^{-\mu} - \mu e^{-\mu}$. This exposes the scheme to the photon number splitting (PNS) attack.

To perform the PNS attack, Eve replaces the high loss channel that Alice and Bob are using with a lossless channel. Eve then performs a quantum non-demolition (QND) measurement on each pulse to obtain number information without perturbing the bases in which the information is encoded. When she determines a pulse with a single photon is in the line, Eve simulates the loss of the original line by blocking a fraction of these pulses. When Eve observes a pulse that has multiple photons, she splits the pulse and stores a photon in a quantum memory. Eve then

sends the rest of the pulse to Bob. After Alice and Bob perform public discussion and announce the bases used for each pulse, Eve can retrieve the photons from her quantum memory and obtain a significant fraction of the key without being detected by Alice and Bob [5-9].

In general, all losses must be attributed to eavesdropping and privacy amplification methods are used to distill a smaller secret key from the raw key generated via the BB84 protocol. In single photon BB84, the distilled secure key rate has approximately linear dependence on the transmittivity. However, for WLP BB84, the PNS attack reduces the secure key rate to approximately quadratic dependence on the channel's transmittivity [10]. In a typical high loss situation, this presents a major problem for the key rate. One solution is to use coherent decoy states, a technique which has met with multiple experimental successes [11-18]. Another alternative is to use entanglement to effectively trump Eve's use of the PNS attack. This is the impetus for the development of our entanglement enhanced scheme for BB84. For convenience and clarity, we will refer to this entanglement enhanced WLP BB84 as EE BB84.

Most entanglement based quantum key distribution schemes rely on violations of Bell's inequalities to ensure security [19]. However, this is not the strategy that our EE BB84 employs here. Instead, we detect Eve by introducing an entangled quantum state into the system that is not used to transmit key bits but only to detect Eve's QND measurements. In figure 1 we schematically illustrate how such an entanglement ancilla may be generated. This allows for a recovery of an approximately linear dependence on transmittivity for the key rate. EE BB84 shares this advantage with coherent decoy state protocols as well as schemes that utilize strong phase reference pulses to eliminate Eve's ability to send Bob vacuum signals [10].
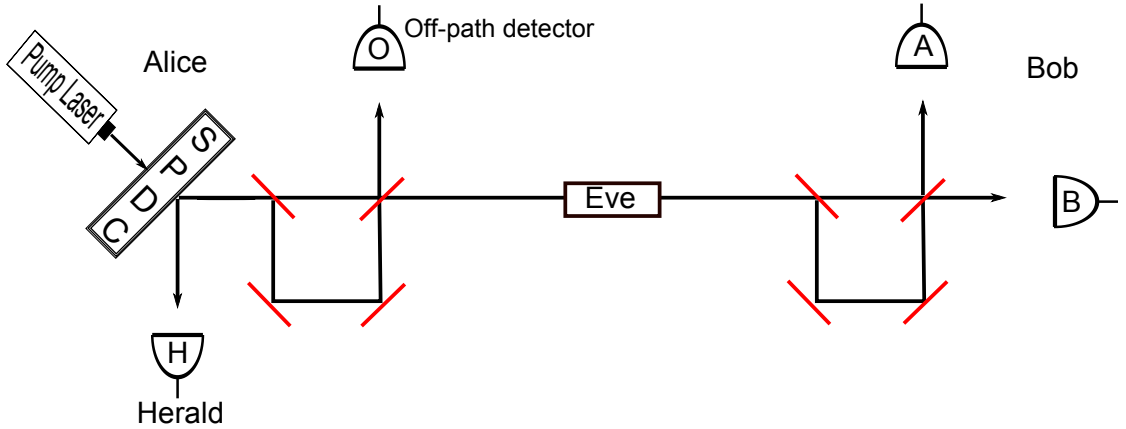
FIGURE 4.4. In the entanglement ancilla, for each photon pair generated by Alice, one is detected in her lab to obtain time information. The other is sent into a beamsplitter and then recombined at a second beamsplitter in the lab to create a pulse with halves that have a time delay that results from a length difference in the paths between the two beamsplitters. This pulse is then sent through the channel to Bob, who passes the pulse through two beamsplitters in his lab that have path differences identical to those in Alice's lab.

4.4 Using Entanglement to Detect a PNS Attack

4.4.1 EE BB84 Scheme

In our EE BB84, Alice and Bob randomly alternate between implementing WLP BB84 and an entangled decoy state ancilla. The entangled states are not primarily used to distribute key bits. Instead, Alice and Bob use the entangled states to detect the presence of an eavesdropper. Alice sends the entangled pulses randomly mixed with the weak laser pulses to guard against the use of a QND measurement device. When Eve measures photon number in the PNS attack on unaugmented WLP BB84, she avoids detection. The QND measurement collapses the coherent state into a number state, which Bob cannot distinguish from the coherent state. This is related to the fact that the number operator commutes with the prepared bases. However, phase and number do not commute, as they are conjugate variables. Therefore, Alice and Bob can use the phase information provided by phase
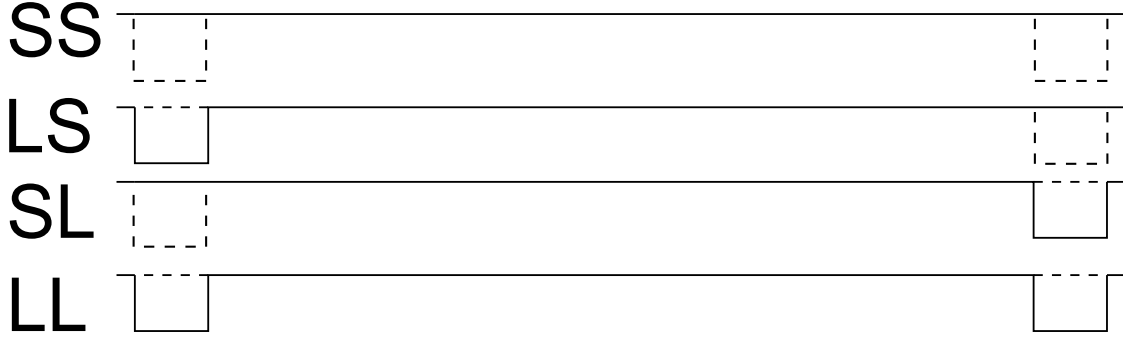
FIGURE 4.5. Possible paths a photon can take to get from Alice to Bob's detector: short-short (SS), long-long (LL), short-long (SL), and long-short (LS). Alice's time information allows the SL and LS paths, which are indistinguishable from each other to be distinguished from both SS and LL.

entangled decoy states to detect Eve whenever she chooses an attack scheme that involves measuring number.

In the entangled state mode, we generate two time-entangled photons using spontaneous parametric down conversion (SPDC). Alice measures one photon in the pair to obtain an accurate time of emission for the other photon. This combination of pump laser, SPDC, and detection of one of the pair of photons gives us a heralded single photon source. As in BB84, the heralded photon is randomly assigned either a horizontal, vertical, diagonal, or anti-diagonal polarization. Then, the heralded photon is sent to a beam splitter which leads to the state:

$$\lvert\Psi\rangle = \frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$$

Half of the state travels down the longer arm, while the other half travels down the shorter arm. The halves recombine at the second beam splitter where there is a probability for the state to leave the quantum channel (see figure 1). A detector will distinguish these possibilities and allows them to be ignored. However, when the pulse does exit into the quantum channel, it is an entangled pulse, where half is delayed in time due to extra path length of the long arm.

70

When Bob receives the test pulse from Alice in his lab, he detects the pulse by sending it through a beam splitter which puts the pulse through long and short arms identical to the setup in Alice's lab. The pulse then encounters the final beam splitter. In this process, there are three possibilities for the pulse. The strong time information from the photon initially detected by Alice allows for the differentiation between these three outcomes. One possibility is that the photon takes the short path both times, labeled SS in figure 4.5. Another outcome is that the photon takes the long path both times, labeled as LL. These two possibilities do not yield strong information about Eve's activities. However, the other possibility is that the photon travels down one long path and one short path, labeled LS or SL. This possibility can detect the use of a quantum non-demolition measurement device [44]. The photon's self-interference will result in a bright port and a dark port in Bob's detection apparatus. Yet, if Eve is measuring number for the PNS attack, then Bob's dark port will not be completely dark. Obviously, it will not be completely dark even without an eavesdropper, since a practical system will have imperfections and not identically match the ideal case. Nevertheless, Eve's actions will still introduce additional error, which can be used to detect her presence.

In our setup, Bob's detection scheme for the entangled pulses is different from his detection scheme for the signal states. This is less than ideal, because if the mode that Alice and Bob are operating in at any given time is not random, then the security of the entire protocol is compromised. If Eve can predict whether a signal state or a decoy state is being sent, then she can adjust her attack plan accordingly and render the entangled states useless. Therefore, it is critical that Eve cannot distinguish between the entangled states and the signal states. Additionally, Alice and Bob must randomly alternate between the signal and decoy modes. Felicitously, the decoy mode does not need to be run with very high frequency in
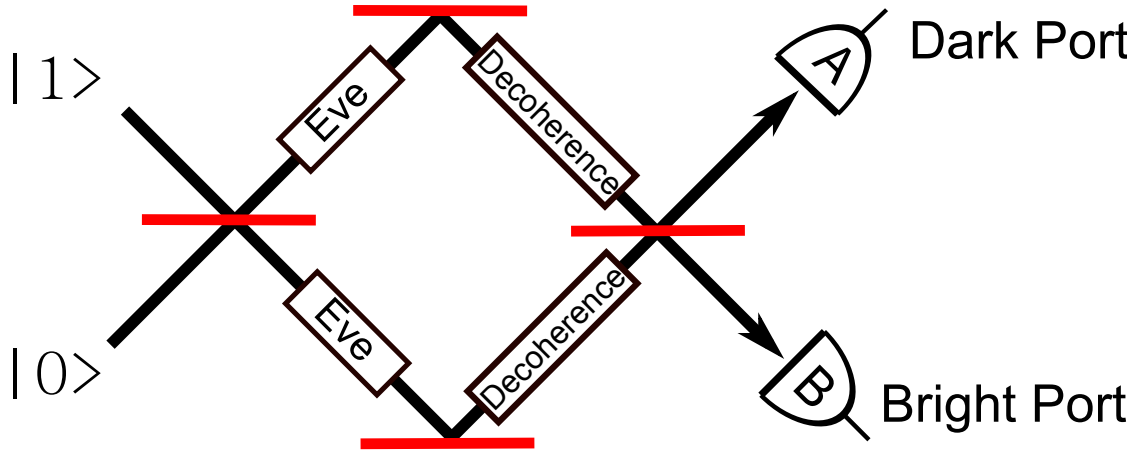
FIGURE 4.6. When Eve does a QND number measurement on the state she is effectively providing which path information, so we can model the interferometer as a Mach-Zender interferometer. The interference at Bobs end is destroyed when Eve makes a measurement. Dephasing can be caused be Eve, the environment or both.

order to detect the use of a quantum non-demolition attack. Nevertheless, since Alice and Bob must each run separate modes for the signal states and the decoy states, a fraction of the pulses they exchange will be worthless. Alice and Bob runs WLP BB84 protocol with frequencies f_{SA} and f_{SB} respectively. They implement the entangled state decoy ancilla with frequencies f_{DA} and f_{DB}. Alice and Bob exchange key information with frequency $f_{SA}f_{SB} + f_{DA}f_{SB}$, and the entangled decoy pulses yield information about the presence of a quantum non-demolition measurement device with frequency $f_{DA}f_{DB}$. With frequency $f_{SA}f_{DB}$, Alice and Bob are operating in incompatible modes, and these exchanges will provide no valuable information, because Bob does not obtain polarization information when measuring phase. Since f_{SA} and f_{SB} are much larger than f_{DA} and f_{DB}, this inefficiency is undesirable, but ultimately does not significantly diminish the practicality of the scheme. Nevertheless, it is also indicative of the trade-off in quantum cryptography between speed and security.
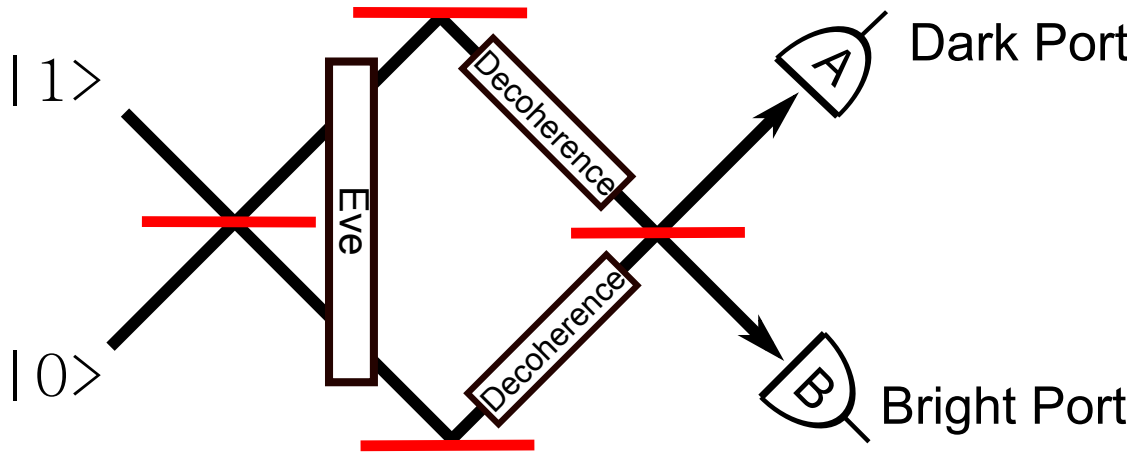
FIGURE 4.7. It must be guarantied that the duration of Eves QND measurement is shorter than the time difference of the two paths. If this is not the case, then Eves measurement will only cause an overall phase difference and no path information will be revealed. The path length difference must be greater than the time difference between the pulses of key information.

4.4.2   EE BB84 Scheme Caveats

4.4.2.1   Pulse Timing

It is instructive to model the EE BB84 scheme as a Mach-Zender interferometer. Mach-Zender interferometer like in Fig. (4.6) is a path entangling device. A photon is inserted into the first beam splitter. It then travels in a superpoisiton of the upper path and the lower path. At the second beam splitter it is recombined and there is interference which results in a light and dark port dependent on the phase change in the possible paths. If the path the photon took could be determined then the interference would disappear and there would be an equal probability for the photon to enter the light or dark port.

Our EE BB84 scheme is very similar to a Mach-Zender interferometer. There are two paths the photon can take, the LS and SL paths, and there is a light port and a dark port dependent on the action of an eavesdropper or the environment. When Eve does a number measurement on the key bit, she is determining which
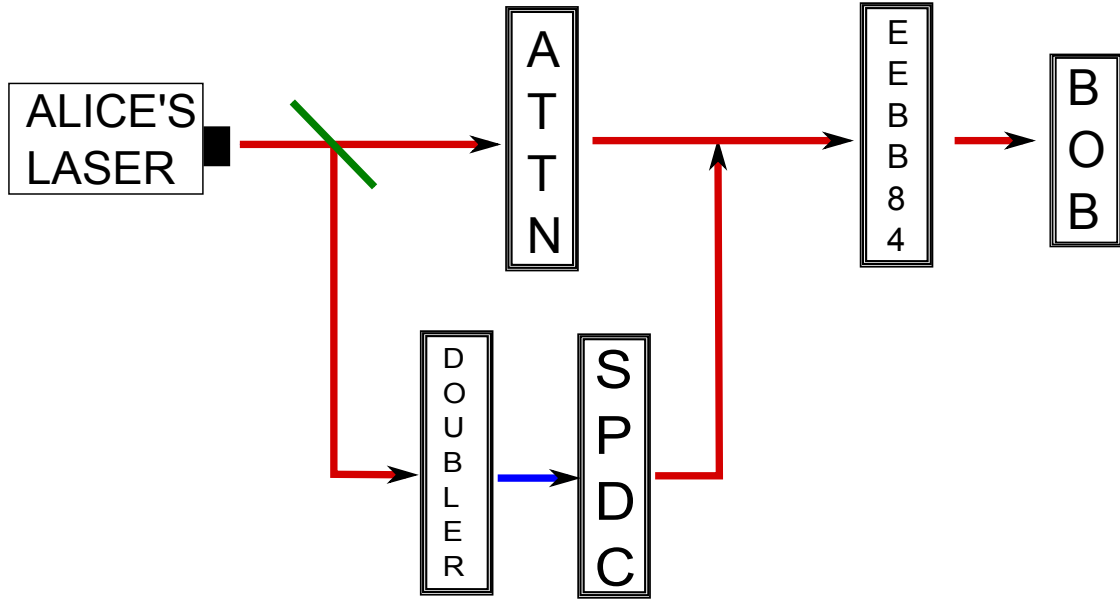
FIGURE 4.8. Part of the coherent source is fed to a frequency doubler which is in turn fed into the SPDC source. The frequency of the photons from the SPDC source will match the frequency of the photons leaving the attenuator (ATTN).

path the photon travels and thus destroys the interference resulting in an equal probability of the photon entering the bright and dark ports.

Using this representation it is easier to see that there is an important constraint we must place on the scheme. The time delay between signal photons must be shorter than the time delay inserted into the decoy state as in Fig. (4.6). If this were not so, then Eve could do a number measurement of both halves of the decoy state in a time interval that encompasses the whole decoy state as in Fig. (4.7). If Eve did this, no path information will be revealed and the decoy state would not collapse. We then would not detect the eavesdropper. The time delay in the decoy state must be large enough to let many signal photons pass, so if Eve was measuring over this long time, she would be measuring over many signal photons and hence not get any usable information.

4.4.2.2 Decoy Insertion

There is also the problem of how to insert the decoys into the attenuated beam in a way that makes it indistinguishable to an eavesdropper. For the decoy states to be indistinguishable from the key bits, they must have the same frequency. A discussion with physicists at BBN Technologies led to the possible solution pictured in Fig. (4.8). Part of the same coherent laser source used for the attenuated source is bled using a beam splitter. This is then fed into a frequency doubler which is then used to pump the SPDC source which will then halve the frequency. The two beams are then recombined. This garauntees the frequency of the photons leaving the SPDC source is the same as that of the photons from the attenuated source.

4.5 Symmetric Hypothesis testing and the Chernoff Distance

We use Chernoff distance [45] and symmetric hypothesis testing to calculate the confidence in which Eve is known to be listening or not listening [46]. For EE BB84 the null hypothesis is that Eve is not measuring number using a QND measurement device, and the alternative hypothesis is that Eve is using such a device to measure number. For the null hypothesis, the probability that the photon will enter the bright port is p, and there is $\bar{p} = 1 - p$ probability for the photon to enter the dark port. When Eve is acting on the system in the alternative hypothesis, there is a probability q for the photon to enter Bob's light port and a probability $\bar{q} = 1 - q$ for it to enter the dark port. Furthermore, the maximum probability $P_{\text{Error}}^{\text{Max}}$ of a false positive or of choosing the wrong hypothesis after n trials is:

$$P_{\text{Error}}^{\text{Max}} = \frac{1}{2}e^{-nC(p,q)} \tag{4.2}$$

where $C(p, q)$ is the Chernoff distance given by the equation:

$$C(p, q) = \xi \ln(\frac{\xi}{p}) + \bar{\xi} \ln(\frac{\bar{\xi}}{\bar{p}}) \tag{4.3}$$

where $\bar{\xi} = 1 - \xi$ and:

$$\xi = \frac{\ln(\frac{\bar{q}}{\bar{p}})}{\ln(\frac{p}{\bar{p}}) + \ln(\frac{\bar{q}}{q})} \tag{4.4}$$

We use Eqns. [4.2], [4.3] and [4.4] to calculate the number of trials needed for a given maximum uncertainty P_{Error}^{Max}:

$$n = \frac{-\ln(2P_{\text{Error}}^{\text{Max}})}{C(p, q)}. \tag{4.5}$$

An application of the above analysis will determine the number of trials necessary for a given confidence of detecting an eavesdropper for EE BB84 and coherent decoy states.

4.6 EE BB84 Statistical Analysis

In an ideal scenario, with no dephasing from the environment, we can easily construct the probabilities of the two symmetric hypotheses. For the null hypothesis, the probability that the photon will enter the bright port is $p = 1$, and there is $\bar{p} = 1 - p = 0$ probability for the photon to enter the dark port. When Eve is acting on the system in the alternative hypothesis, there is an equal probability, $q = \bar{q} = \frac{1}{2}$, for the photon to enter either of Bob's detectors. This results in a Chernoff distance of .69. Therefore, if we define a trial to be a photon sent from Alice and detected by Bob, the number of trials to detect Eve at the 99% confidence level ($P_{\text{Error}}^{\text{Max}} = 0.01$) requires an exchange of a maximum of just 6 photons between Alice and Bob.

We are only investigating the photons that reach Bob with the proper time information. Thus, unlike the coherent decoy states, loss is not the most significant

quantity to investigate quantitatively. Instead, dephasing (decoherence) is our primary concern. The environment can affect the entangled decoy state by changing the phase information in it. Since the two states are sent down the line close together, it might be assumed that any environmental factor that would affect one half of the state, would affect the other and therefore the total phase information in the state would remain unchanged. Experiments such as the one performed by Zbinden, Brendel, Gisin and Tittel [21] seem to confirm this. Phase entangled photons in that experiment traveled over 10 kilometers under lake Geneva while still retaining enough coherence to violate a Bell inequality. However, since in our framework, dephasing is what would affect the scheme the most, we still want to investigate its effect on the Chernoff distance.

When dephasing is included, the problem turns into that of determining whether a coin is fair. The question becomes: how many trials does it take to be confident that Eve is there or not? When dephasing is present, the probability for a photon to be detected in the dark port increases. It becomes more difficult to tell Eve apart from the environment. With complete dephasing the probability to find a photon in either the bright port or the dark port becomes 50-50. Fig. (4.9) shows how many trials are needed to have a 99% confidence of determining if Eve is listening or not versus the probability of finding a photon in the dark port (dephasing) regardless of Eve.

4.7 Coherent Decoy States Statistical Analysis

The alternative to EE BB84 is the popular coherent decoy state solution. In the PNS attack, Eve assumes Alice's photon source has a constant mean photon number. However, if Alice randomly alters the mean photon number of her source in a way that is known to her, but not perceivable to Eve, then she can detect the
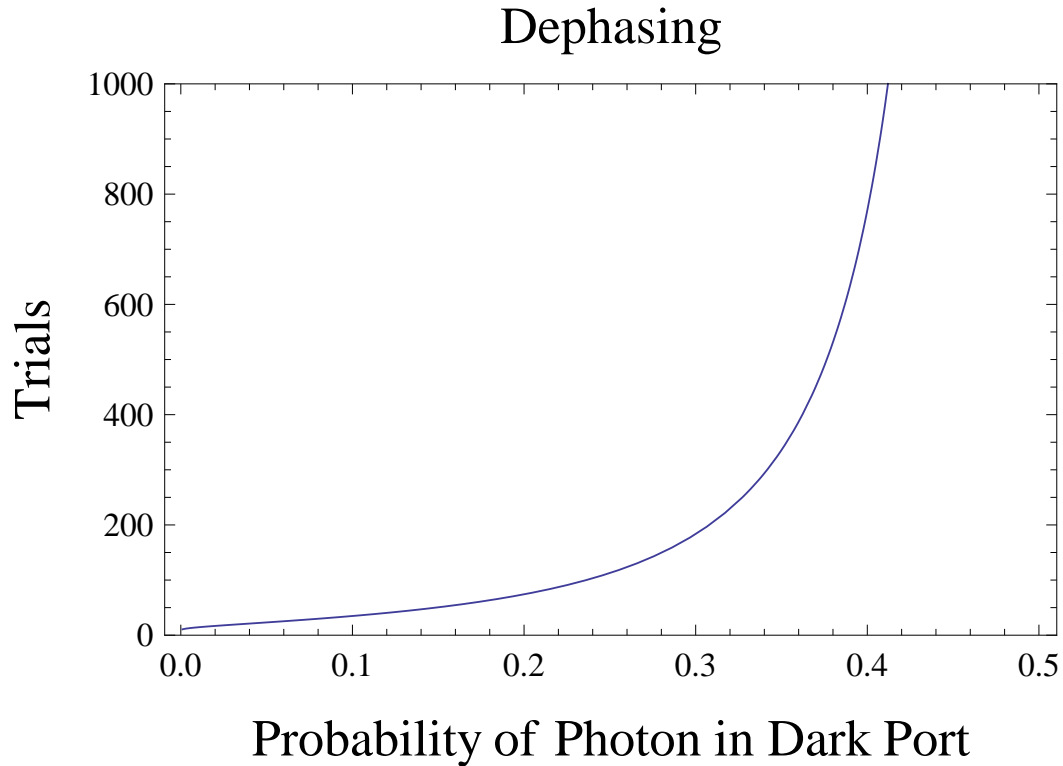
Dephasing

FIGURE 4.9. Dephasing can be caused by the environment, an eavesdropper or both. As the dephasing increases, the probability of finding a photon in the dark port increases. This causes the number of trials needed to detect an eavesdropper with a 99% confidence to increase. When the probability of detecting a photon at the light and dark port is equal, it becomes impossible to tell an eavesdropper apart from the environment.

PNS attack. This is the idea that motivates coherent decoy states. Pulses from the source with a higher mean photon number will contain a greater fraction of multi-photon pulses, which Eve will not block. Therefore, when Alice and Bob discuss the protocol, Alice can compare the loss in the line for when different mean photon numbers were used. If there is a marked difference between the loss for the decoy states and the loss for the signal states, then Alice can conclude that Eve is using the photon number splitting attack [22-26].

We treat coherent decoy states in a similar manner to EE BB84, but instead of dephasing being the key quantity of interest, loss is, because Eve hides in the loss of the system. The coherent decoy state solution uses two (or more) attenuated
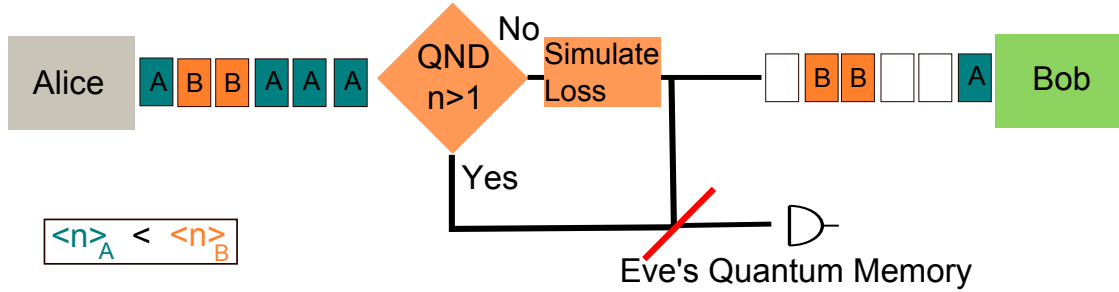
FIGURE 4.10. If an eavesdropper is not listening then the probability, p, of measuring a attenuated pulse with $\langle n \rangle_A = A$ vs. the pulse with $\langle n \rangle_B = B$ is simply equal to the original distribution of the two pulses. If there is an eavesdropper performing a PNS attack then the probability, q, of measuring a attenuated pulse with $\langle n \rangle_A = A$ vs. the pulse with $\langle n \rangle_B = B$ becomes less with increases loss.

coherent sources with different average photon numbers $\bar{n}_1$ and $\bar{n}_2$. Alice determines the percentage of each of these states that is sent down the channel. If Alice sends Bob a total of 100 pulses, of which 70 (70%) have an average photon number of $\bar{n}_1$ and 30 (30%) have $\bar{n}_2$ and we assume a loss of 50%, then Bob should receive 35 (70%) pulses with an average photon number of $\bar{n}_1$ and 15 (30%) with $\bar{n}_2$. In this scenario, we define loss as losing the whole pulse. Loss affects the total number of photons received, but not the percentage of $\bar{n}_1$ and $\bar{n}_2$. Eve performs a PNS attack by replacing all or part of the lossy transmission line with a lossless line and altering the percentage of $\bar{n}_1$ and $\bar{n}_2$ sent through to Bob. In this example we assume Eve has replaced the entire transmission line with a lossless one. Eve sits on the line and measures number until she finds a pulse containing more than one photon and then she takes one of these photons and lets the other pass. She blocks enough of the single photon pulses such that the initial loss is preserved. If $\bar{n}_1 < \bar{n}_2$, The $\bar{n}_2$ pulse will have more photons on average than the other and therefore will be allowed to pass through to Eve more than the other. So, in the presence of Eve, if Alice sends 100 pulses, of which 70 (70%) have an average photon number of $\bar{n}_1$ and 30 (30%) have $\bar{n}_2$ and we assume a loss of 50% which Eve will take over, then Bob would still receive a total of 50 pulses, but the percentages of $\bar{n}_1$ pulses will
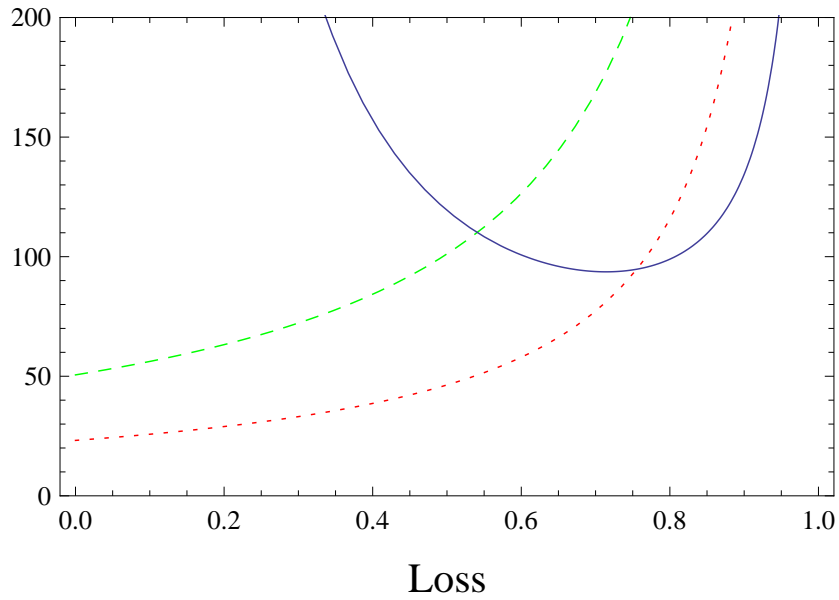
FIGURE 4.11. The solid line is the number of pulses sent by Alice (not necessarily detected by Bob) for the coherent decoy state scheme to detect an eavesdropper with a 99% confidence. The dotted and dashed lines are for the EE BB84 scheme at 10% and 30% dephasing respectively. For cases of very high loss, decoy states outperform EE BB84. However, for more moderate levels of loss, EE BB84 requires fewer pulses to confidently detect the presence of an eavesdropper compared to coherent decoy states.
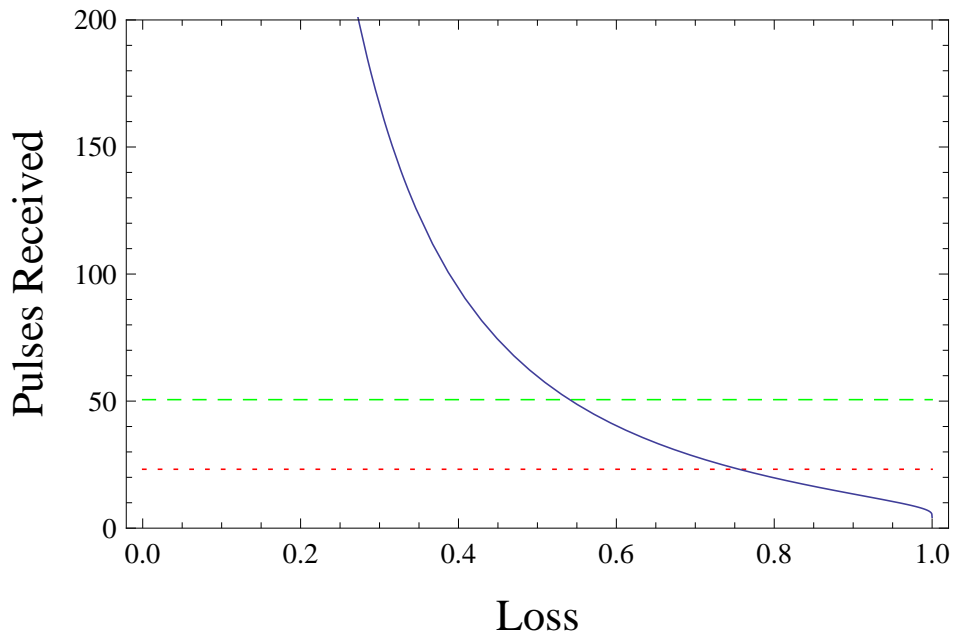

FIGURE 4.12. The solid line is the number of pulses received by Bob for the coherent decoy state scheme to detect an eavesdropper with a 99% confidence. The dotted and dashed lines are for the EE BB84 scheme at 10% and 30% dephasing respectively.

be less than 30% and the percentage of $\bar{n}_2$ pulses will be greater than 70%, which is not identical to what Alice sent. Here, we are looking at the very worst possible case of eavesdropping. We are assuming that Eve has replaced all of the noise with a noiseless channel.

Alice looks at the percentage of $\bar{n}_1$ and $\bar{n}_2$ received by Bob and compares it to the percentages she sent. If she can tell the difference between them with an acceptable confidence, then Eve is detected. This is treated in the same way we treated EE BB84 above. The Chernoff distance gives us a metric to determine the presence of Eve. The number of pulses sent from Alice to Bob that are necessary to be 99% confident of the presence of an eavesdropper can be seen in figure 4.11. The efficiency of coherent decoy states improves as loss rises because it gives Eve more space to sift the photons, but as the loss becomes too high, then obviously transmission becomes difficult for any scheme. A can be seen, coherent decoy states are poor performers at low loss, however at low loss it is more difficult for Eve to hide and the necessity for decoy states declines. The number of pulses received by Bob from Alice that are necessary to be 99% confident of the presence of an eavesdropper can be seen in figure 4.12. The lines from the EE BB84 are simply straight since decoherence is the quantity effects the necessary pulses and not loss. Again, the coherent decoy states improves as loss rises.

The crux of the coherent decoy state solution is that Eve manipulates photon number statistics in a way that Alice can detect. However, if Eve can gain information, which allows her to not alter the statistics in a detectable manner, then the coherent decoy state technique will not be a successful solution. This situation would obviously justify the implementation of EE B84, yet EE BB84 is advantageous in some other scenarios as well.

The parameters and performance of EE BB84 and coherent decoy states can vary greatly depending on environment and choice of variables. It can be seen from the figures that for loss of less than 75% and dephasing less than 10% the EE BB84 scheme outperforms the coherent decoy state scheme by requiring fewer pulses. At 50% loss the EE BB84 scheme would need to send about a third the number of pulses as the coherent decoy state to detect an eavesdropper with 99% confidence.

Coherent decoy states are a popular solution to the photon number splitting attack for a reason. They achieve linear scaling with transmittivity. Additionally, coherent decoy states can be used to distill a secret key without Bob alternating detection modes. However, in EE BB84, Bob must alternate between a polarization detection mode and a phase detection. This gives coherent decoy states an advantage over the present version of EE BB84.

At the moment, EE BB84 does not possess general superiority to coherent decoy states. Therefore, the appeal of EE BB84 is that it has some situational advantages and approaches the problem of the photon number splitting attack in a manner strategically different from that of coherent decoy states. The general strategy of coherent decoy states is to improve the secret key transmission rate by focusing on limiting the amount of information that Eve can possibly obtain while still avoiding detection. Meanwhile, the strategy behind EE BB84 is direct detection of an eavesdropper that might be performing quantum non-demolition measurements. The strategy of EE BB84 is not superior to that of coherent decoy states. It is simply different, and this difference helps generate situations where the EE BB84 scheme has specific advantages, like the case when the operation time for the key transmission is not long enough for decoy states to be a robust defense. In cases

such as this, EE BB84 has an advantage because of its ability to determine the use of quantum non-demolition measurement with a rather meager number of pulses.

Chapter 5
Conclusions

The uncertainty inherent in our world can be uncomfortable. Before Einstein accepted quantum mechanics he famously said in a letter [52] to Born in 1926:

> I, at any rate, am convinced that He does not throw dice.

This was said in relation to the indeterminacy and spooky action at a distance involved in systems like the EPR thought experiment. It is a strange concept that something like a quantum state can be completely indeterminate until it is observed. It is nonetheless true as the inequality introduced by Bell has shown [9] many times over. However, instead of viewing quantum uncertainty as a strange thing I have come to see it as a concept that makes strange things understandable and useful.

Popper's odd thought experiment from chapter 2 and the experimental results that tested it can be completely explained with quantum uncertainty. The results from both of the experiments that tested Popper's thought experiment flow from quantum fundamentals. The notion of two entangled particles behaving like two correlated particles is a concept that can lead to problems. This is evident from continued confusion about Popper's thought experiment. To understand this thought experiment it was vital realize that two entangled particles are actually a bi-photon that only present as two particles upon measurement. After we realize this it follows that the uncertainty also does not behave as the uncertainty of two particles but rather as the uncertainty of the bi-photon.

Uncertainty explains a question that can never be answered. Performing an experiment that could answer the question asked in chapter 3, "Who measured first?", makes answering the question impossible to know. Setting up an experiment that can test the temporal order of measurement on a bi-photon when the two observers are in different reference frames introduces an uncertainty that masks the measurement order. There is no preferred reference frame in which wave function collapse takes place. Trying to find such a reference frame would introduce an uncertainty that makes it impossible to find. Relativity shows simultaneity to be a relative concept and I have shown that the concept of wave function collapse, while instantaneous and across all space, does not, thanks to uncertainty, contradict relativity.

Uncertainty can be used to protect our secrets. The EE BB84 scheme in chapter 4 and quantum cryptography in general is completely dependent on uncertainty introduced to a quantum system by an eavesdropper. Using fundamental quantum mechanics it is easy to see what happens when a detector measuring in one basis measures a state prepared in a different basis. The result becomes uncertain. This is the basis for many quantum cryptography schemes. Fundamental quantum mechanics also describe what happens when a measurement determining the path of a particle in a superposition of possible paths is made. The interference between the possible paths disappears and the outcome becomes random. We use this to simple fact to detect an eavesdropper.

We live in an uncertain world, but there is opportunity in uncertainty. It is an explanation in itself and a useful tool. We can use it to dispose of paradoxes and talk to each other. The best future is an uncertain one.

References

[1] Einstein, A., *On a Heuristic Point of View Concerning the Production and Transformation of Light*, Annalen der Physik (ser. 4), 17, 132148 1905

[2] de Broglie, L. *Recherches sur la thorie des quanta [On the Theory of Quanta]*, Annales de Physique 10 (3): 22128 (1925)

[3] Schrdinger, E. *Quantisierung als Eigenwertproblem; von Erwin Schrdinger*, Annalen der Physik: 361377 (1926)

[4] Heisenberg, W., *Die Physik der Atomkerne*, Taylor and Francis, 1952, p. 30.

[5] Einstein, A. (1905-06-30), *Zur Elektrodynamik bewegter Krper*, Annalen der Physik 17 (10): 891921

[6] Lorentz, H. A. *Elektromagnetische Vorgnge*, Beibltter zu den Annalen der Physik, 29 (4): 168-170

[7] K.R.Popper, *Quantum Theory and the Schism in Physics*, Die Naturwissenshaften, 22, 807 (1934)

[8] Einstein, A.; B. Podolsky, N. Rosen, *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?*, Physical Review, 47, 77780, (1935)

[9] Bell, J.S. [1964], *On the Einstein-Podolsky-Rosen paradox*, Physics, 1: 195200.

[10] Letter from Einstein to Max Born, 3 March 1947; The Born-Einstein Letters; Correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955, Walker, New York, 1971.

[11] Y. Kim and Y.H. Shih, *Experimental realization of Popper's Experiment: Violation of the Uncertainty Principle?*, Found. Phys., 29, 1849 (1999)

[12] A.J. Short, *Popper's experiment and conditional uncertainty relations*, Found. Phys. Lett., 14, 275284, (2001)

[13] P. Sancho, *Popper's Experiment Revisited*, Found. Phys., 32,789805 (2002)

[14] T. Qureshi, *Understanding Popper's Experiment*, Am. J. Phys, 53, 541544, (2005)

[15] D.V. Strekalov, A.V. Sergienko, D.N. Klyshko, Y.H. Shih, *Observation of Two-Photon "Ghost" Interference and Diffraction*, PRL, 17, 18 (1995)

[16] W. Tittel, J. Brendel, N. Gisin, H. Zbinden, *Long-distance Bell-type tests using energy-time entangled photons*, PhysRevA, 59, 41504163, (1999)

[17] T.B. Pittman, D.V. Strekalov, A. Migdall, M.H. Rubin, A.V. Sergienko, and Y.H. Shih, *Can Two-Photon Interference be Considered the Interference of Two Photons?*, Physical Review Letters, 77, 1917, (1996)

[18] Dirac, P.A.M. (1928-02-01). *The Quantum Theory of the Electron.* Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 117 (778): 610624.

[19] Rodolfo Gambini and Jorge Pullin, *Loops, Knots, Gauge Theories and Quantum Gravity*, Cambridge University Press (1996)

[20] A. Suarez and V. Scarani, Phys. Lett. A 232, 9 (1997).

[21] H. Zbinden, J. Brendel, N. Gisin, W. Tittel, *Experimental test of non local quantum correlation in relativistic configurations*,PhysRevA, 63, 022111, (2001)

[22] Gisin N.,Ribordy G.,Tittel W. and Zbinden H. 2002 Quantum Cryptography *Rev. Mod. Phys.* **74**

[23] Scarani V., et al. 2009 The Security of Practical Quantum Key Distribution *Rev. Mod. Phys.* **81**

[24] Qi B., Qian L. and Lo H. 2010 A brief introduction of quantum cryptography for engineers

[25] Chris D. Richardson, et al. *Thwarting the Photon Number Splitting Attack with Entanglement Enhanced BB84 Quantum Key Distribution*, New J. Phys. 14 (2012) 043003

[26] Chris D. Richardson, Jonathan P. Dowling *Popper's Thought Experiment Reinvestigated*, International Journal of Quantum Information (IJQI), Volume: 10, Issue: 3(2012) 1250033

[27] Bennett C.H. and Brassard G. 1984 *Proc. IEEE Int. Conf. on Computers, Systems and Signal processing* (New York: IEEE) pp 175-179

[28] Huttner B., Imoto N., Gisin N. and Mor T. 1995 Quantum cryptography with coherent states *Phys. Rev. A* **51** 1863

[29] Brassard G., Lütkenhaus N., Mor T. and Sanders B C 2000 Limitations on Practical Quantum Cryptography *Phys. Rev. Lett.* **85** 1330

[30] *BB84 Protocol* Image ¡http://swissquantum.idquantique.com/?Key-Sifting¿ June 11, 2009

[31] Lütkenhaus N. 2000 Security against individual attacks for realistic quantum key distribution *Phys. Rev. A* **61** 052304

[32] Lütkenhaus N. and Jahma M. 2002 Quantum key distribution with realistic states: photon-number statistics in the photon-number splitting attack *New J. Phys.* **4** 44

[33] Yuen H.P. 1996 Quantum amplifiers, quantum duplicators and quantum cryptography *Quantum Semiclass. Opt.* **8** 939

[34] Lütkenhaus N. 2007 Chapter 15: Theory of Quantum Key Distribution (QKD) *Lectures on Quantum Information*

[35] Liu Y., et al. 2010 Decoy-state quantum key distribution with polarized photons over 200 km *Optics Express* **8** 008587

[36] Peng C.Z. et al. 2007 Experimental Long-Distance Decoy-State Quantum Key Distribution Based on Polarization Encoding *Phys. Rev. Lett.* **98** 010505

[37] Schmitt-Manderbach T., et al. 2007 Experimental Demonstration of Free-Space Decoy-State Quantum Key Distribution over 144 km *Phys. Rev. Lett.* **98** 010504

[38] Rosenberg D. et al. 2009 Practical long-distance quantum key distribution system using decoy levels *New J. Phys.* **11** 045009

[39] Chen T.Y. et al. 2009 Field test of a practical secure communication network with decoy-state quantum cryptography *Opt. Exp.* **17** 6450

[40] Wang Q. et al. 2008 Experimental Decoy-State Quantum Key Distribution with a Sub-Poissionian Heralded Single-Photon Source*Phys. Rev. Lett.* **100** 090501

[41] Yin Z.Q. et al. 2008 Experimental Decoy State Quantum Key Distribution Over 120 km Fibre *Chin. Phys. Lett.* **25** 3547

[42] Zhao Y., Q. Bi, Ma X., Lo H.K., and Qian L 2006 Experimental Quantum Key Distribution with Decoy States *Phys. Rev. Lett.* **96** 070502

[43] Ekert A.K. 1991 Quantum cryptography based on Bell's theorem *Phys. Rev. Lett.* **67** 661-663

[44] Kok P., Lee H., Dowling P. 2002 Single-photon quantum-nondemolition detectors constructed with linear optics and projective measurements *Phys. Rev. A* **66** 063814

[45] Chernoff H. 1952 Annals of Mathematical Statistics 23 493

[46] Anisimov P.M., Lum D.J., McCracken S.B., Hwang L. and Dowling J.P. 2010 An Invisible Quantum Tripwire *New J. Phys.* **12** 083012

[47] Wang X.B. 2005 Beating the photon-number-splitting attack in practical quantum cryptography *Phys. Rev. Lett.* **94** 230503

[48] Lo H., Ma X., Chen K. 2005 Decoy State Quantum Key Distribution *Phys. Rev. Lett.* **94** 230504

[49] Wang X.B., Hiroshima T., Tomita A., and Hayashi M. 2007 Quantum information with Gaussian states *Phys. Rep.* **448** 1

[50] Hwang W.Y. 2003 Quantum key distribution with high loss: toward global secure communication *Phys. Rev. Lett.* **91** 057901

[51] Harrington J.W., Ettinger J.M., Hughes R.J., and Nordholt J.E. Enhancing practical security of quantum key distribution with a few decoy states arxiv:quant-ph/0503002

[52] Letter to Max Born (4 December 1926); The Born-Einstein Letters (translated by Irene Born) (Walker and Company, New York, 1971)

Appendix

Experimental Constants		
	Kim and Shih	Strekalov, *et al.*
Wavelength (λ)	0.0007 mm	0.000702 mm
Slit Width (d)	0.16 mm	0.4 mm
z_1	500 mm	1000 mm
z_2	1500 mm	1650 mm
Beam Width ($\frac{1}{a}$)	3 mm	not given

TABLE 5.1. Table of experimental constants used by Strekalov, *et al.*, and Kim and Shih.

Vita

Chris Richardson (Graduate Student) earned his bachelor's degree in 2002 from Louisiana State University, where he graduated with a major in physics with a concentration in computer science. Afterwards he worked programming the database and front-end of the extension disaster education network (EDEN) website for the LSU Agcenter. During this time, in 2004 he re-entered LSU as a part-time graduate student in physics. He was awarded a teaching assistantship in 2006 and became a full time graduate student. After passing the qualifier he was awarded a research assistantship under Dr. Jonathan Dowling and funded by the Foundational Questions Institute (FQXi). He researches foundational quantum mechanics, specifically, the uncertainty principle and how it applies to entangled systems and entangled relativistic systems.