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Series Solutions of Multi-Term Fractional Differential Equations

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SERIES SOLUTIONS OF MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS

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This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Prof. Mohammed Al-Refai

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Declaration of Original Work

I, (Moh'd Khier) Yousef Ibrahim Al Srihin, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Series Solutions of Multi-Term Fractional Differential Equations"*, hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Prof. Mohammed Al-Refai, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In this thesis, we introduce a new series solutions for multi-term fractional differential equations of Caputo's type. The idea is similar to the well-known Taylor Series method, but we overcome the difficulty of computing iterated fractional derivatives, which do not commuted in general. To illustrate the efficiency of the new algorithm, we apply it for several types of multi-term fractional differential equations and compare the results with the ones obtained by the well-known Adomian decomposition method (ADM).

Keywords: Fractional differential equations, Caputo fractional derivative, Series Solutions, Adomian decomposition method.

Title and Abstract (in Arabic)

حل المعادلات التفاضلية الكسرية متعددة الحدود من خلال المتسلسلات

الملخص

في هذه الرسالة، قمنا بتطوير طريقة حل جديدة للمعادلات التفاضلية الكسرية متعددة الحدود من النوع كبوتو باستخدام المتسلسلات. الفكرة مشابهة لطريقة متسلسلة تايلور المعروفة، ولكن قمنا بتجاوز الصعوبة الناتجة من حساب المشتقات الكسرية المتوالية والتي لا يمكن حسابها بشكل عام. لتوضيح كفاءة الطريقة الجديدة قمنا بتطبيقها على انواع مختلفة من المعادلات التفاضلية الكسرية متعددة الحدود وقارنا النتائج مع ما قمنا بتحصيله باستخدام طريقة ادومين التحليلية.

المصطلحات: المعادلات التفاضلية الكسرية، المشتقة الكسرية من النوع كبوتو، الحلول المتسلسلة، طريقة ادومين التحليلية.

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Lastly, I would like to thank my parents, and my family for their encouragement and support.

Dedication

To my wife

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Chapter 1: Fractional Calculus

Fractional calculus is a generalization of integrals and derivatives to non integer orders, see [1]. The idea of fractional calculus goes back to 1695, when L'Hopital (1661-1704) wrote a letter to Leibniz (1 646-1716) discussing the half order derivative. After that a lot of contributions have been achieved by many mathematicians in the field of fractional calculus. The well-known mathematicians L. Euler (1707-1783), J. Fourier (1768-1830), G. Riemann (1826-1866), J. Liouville (1809-1882) among others, have contributed to this field, and the reader is referred to [2] for the development and literature of fractional calculus.

1.1 Basic Functions

In this section, we introduce several functions that will appear in the thesis. These functions play an important role in the definition of fractional derivatives.

1.1.1 The Gamma and Beta Functions

The Euler Gamma function $\Gamma(z)$ is one of the basic functions in fractional calculus, and it is a generalization of the factorial function to non integer values.

Definition 1.1.1. *The Gamma function* $\Gamma(z)$ *is defined*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad \forall z \in \mathbb{R}. \tag{1.1}$$

The improper integral defined in (1.1) converges for all $z \in \mathbb{R}$, except for non positive integers. For positive integer z, we have $\Gamma(z) = (z-1)!$.

The main property of the Gamma function is the recursion relation

$$\Gamma(z+1) = z\Gamma(z),$$

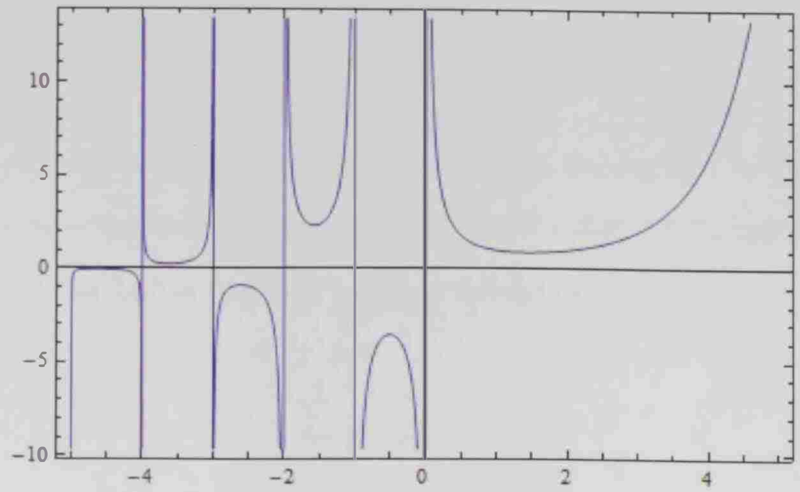

Figure 1.1: The graph of the Gamma function

which follows by integrating Eq. (1.1) by parts. Figure 1.1 depicts the Gamma function on its real domain.

Proposition 1.1.1. *The following properties of the Gamma function hold true.*

(1) $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt = \sqrt{\pi}.$

(2) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$ *for non integer z.*

(3) $\Gamma(n+\frac{1}{2}) = \frac{1\times 3\times 5\times \ldots \times (2n-1)}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi},$ *for any positive integer n.*

(4) $\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n) = (-1)^n \pi,$ *for any positive integer n.*

For the proof the reader is referred to [1, 3].

A very related function to the Gamma function is the Beta function. The Beta function is used to prove some properties of the Gamma function and it has been used in the computations of the fractional integrals and derivatives [1, 2].

Definition 1.1.2. *The Beta function* $\beta(z,w)$ *is defined*

$$\beta(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1} d\tau, \qquad \text{where } z, w > 0. \tag{1.2}$$

The Gamma function is connected with the Beta function via the relation

$$\beta(z,w) = \beta(w,z) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{1.3}$$

1.1.2 The Mittag-Leffler Function

The Mittag-Leffler function is a generalization of the Exponential function [1]. The Swedish mathematician Magnus Gustaf Mittag-Leffler (1846-1927), has introduced the one parameter Mittag-Leffler function, which is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \qquad \alpha > 0.$$

Later on, Agarwal in 1953, has defined the two parameters Mittag-Leffler function $E_{\alpha,\beta}(z)$, see [4, 5], as a generalization to the one parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha, \beta > 0. \tag{1.4}$$

The following properties of the two parameters Mittag-Leffler function hold true and the proof can be obtained directly from the definition of the Mittag-Leffler function.

(1) $E_{\alpha,\beta}(0) = \frac{1}{\Gamma(\beta)}$, for all $\alpha, \beta > 0$.

(2) $E_{\alpha,1}(z) = E_{\alpha}(z)$.

(3) $E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = e^z$.

(4) $E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z - 1)$.

(5) $E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{1}{z^2}(e^z - 1 - z)$.

(6) $E_{1,m} = \frac{1}{z^{m-1}}\left(e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right), \quad m \in \mathbb{N}$.

(7) $E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z)$.

(8) $E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}$.

Figures 1.2 and 1.3 depict the Mittag-Leffler functions for different parameters on their real domain. We have the following important recursions of the Mittag-Leffler function.

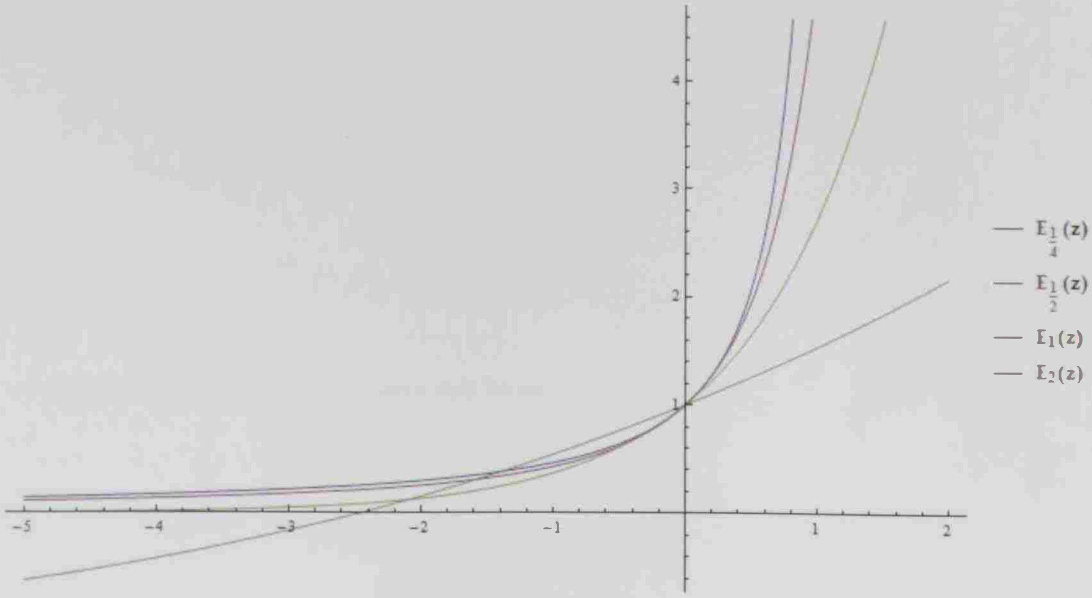

Figure 1.2: One parameter Mittag-Leffler function for different values of α.

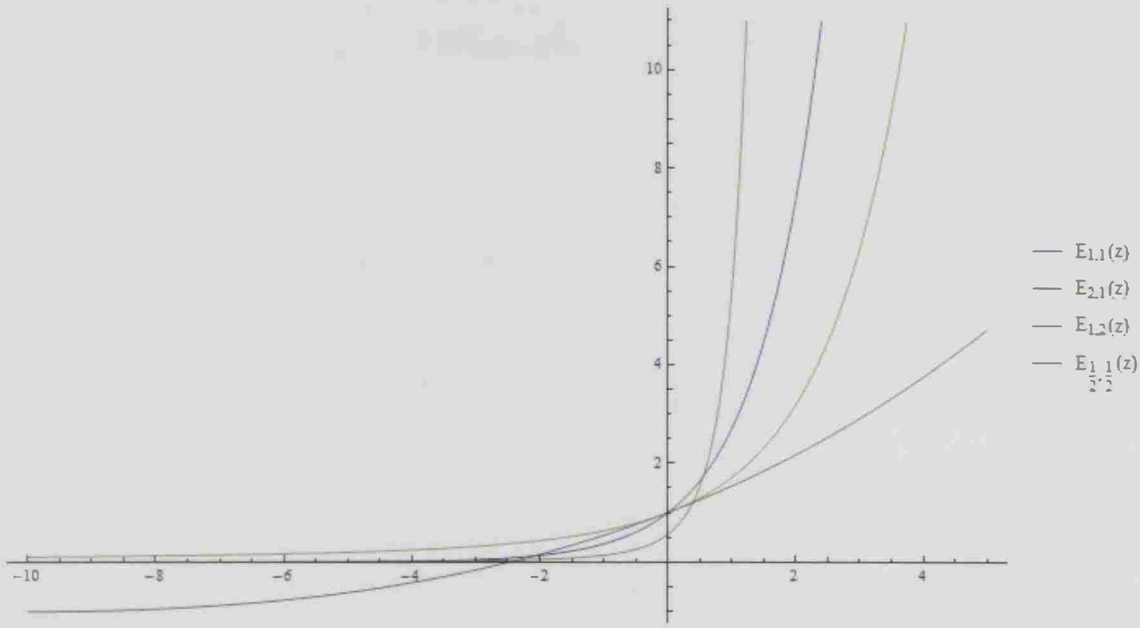

Figure 1.3: Two parameters Mittag-Leffler function for different values of α and β.

Proposition 1.1.2. *For $\alpha, \beta > 0$ and for $z \in \mathbb{R}$, the following hold true.*

(1) $\quad E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z).$

(2) $\quad E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z)$

Proof. (1) We have

$$\begin{aligned}
E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \\
&= \frac{1}{\Gamma(\beta)} + \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}.
\end{aligned}$$

By shifting the index to zero, we have

$$\begin{aligned}
E_{\alpha,\beta}(z) &= \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{z^{k+1}}{\Gamma(\alpha(k+1)+\beta)} \\
&= \frac{1}{\Gamma(\beta)} + z\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+(\alpha+\beta))} \\
&= \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z),
\end{aligned}$$

which proves the result.

(2) We start with the right hand side,

$$\begin{aligned}
\beta E_{\alpha,\beta+1}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}(z) &= \sum_{k=0}^{\infty} \frac{\beta z^k}{\Gamma(\alpha k+\beta+1)} + \sum_{k=1}^{\infty} \frac{\alpha k z^k}{\Gamma(\alpha k+\beta+1)} \\
&= \frac{\beta}{\Gamma(\beta+1)} + \sum_{k=1}^{\infty} \frac{(\alpha k+\beta)z^k}{\Gamma(\alpha k+\beta+1)} \\
&= \sum_{k=0}^{\infty} \frac{(\alpha k+\beta)z^k}{\Gamma(\alpha k+\beta+1)} \\
&= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \\
&= E_{\alpha,\beta}(z),
\end{aligned}$$

and the proof is completed.

□

Remark 1.1.1. It is worth mentioning that the power series expansion of the Mittag-Leffler function converges uniformly on $\mathbb{R}$. The proof can be obtained directly using the

Ratio test and the Weierstrass M-Test.

1.2 The Riemann-Liouville Fractional Integral

The Riemann-Liouville fractional integral is a natural extension of the following well-known Cauchy Formula for n-folds integral, see [1, 6].

$$I_a^n f(t) = \int_a^t \int_a^{\tau_{n-1}} \dots \int_a^{\tau_1} f(\tau) d\tau d\tau_1 \dots d\tau_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau. \quad (1.5)$$

The above Cauchy formula valid for $n \in \mathbb{N}$, and $a, t \in \mathbb{R}$. The new approach was introduced by replacing n by a non integer number α and $\Gamma(\alpha)$ instead of $(\alpha - 1)!$.

Definition 1.2.1. *For $\alpha > 0, t > a$, and $a, t \in \mathbb{R}$, the Riemann-Liouville fractional integral of order α is defined by*

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad t \in [a, b] \quad (1.6)$$

provided that the integral exists.

If $f(t) \in L^1[a,b]$, then the product $(t-\tau)^{\alpha-1} f(\tau) \in L^1[a,b]$, and the above integral exists.

Proposition 1.2.1. *Let f, g be two continuous functions,*

hold true.

(1) $\quad I_{a^+}^{0} f(t) = f(t)$.

(2) $\quad I_{a^+}^{\alpha} \left(c_1 f(t) + c_2 g(t) \right) = c_1 I_{a^+}^{\alpha} f(t) + c_2 I_{a^+}^{\alpha} g(t)$, *where* $c_1, c_2 \in \mathbb{R}$.

(3) $\quad I_{a^+}^{\alpha} \left(I_{a^+}^{\beta} f(t) \right) = I_{a^+}^{\beta} \left(I_{a^+}^{\alpha} f(t) \right) = I_{a^+}^{\alpha+\beta} f(t)$.

For the proof the reader is referred to ([1], p.65-67).

Proposition 1.2.2. *For $\mu > -1$ and $t > 0$, it holds that*

$$I_{0^+}^{\alpha}\left(t^{\mu}\right) = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\alpha+\mu}. \tag{1.7}$$

Proof. We have

$$\begin{aligned} I_{0^+}^{\alpha}\left(t^{\mu}\right) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{\mu} d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (1-\frac{\tau}{t})^{\alpha-1} t^{\alpha-1} \tau^{\mu} d\tau. \end{aligned}$$

By substituting $u = \frac{\tau}{t}$, we have

$$\begin{aligned} I_{0^+}^{\alpha}\left(t^{\mu}\right) &= \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{\mu} du \\ &= \beta(\alpha,\mu+1) \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \\ &= \frac{\Gamma(\alpha)\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} \frac{t^{\alpha+\mu}}{\Gamma(\alpha)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \end{aligned}$$

and the result is proved. □

Before discussing the properties of the Riemann-Liouville fractional integral, we need the following theorems.

Theorem 1.2.3 ([7], Theorem 8.2.4, pp.237). *Let (f_n) be a sequence of Riemann integrable functions on $[a,b]$, and suppose (f_n) converges uniformly on $[a,b]$ to f. Then f is Riemann integrable and*

$$\lim_{n\to\infty} \int_a^b f_n df = \int_a^b f df.$$

Theorem 1.2.4 ([7], Theorem 9.4.3, pp.267). *Suppose that the sequence of real-valued functions $f_n, n \in \mathbb{N}$, are Riemann integrable on $[a,b]$. If the series $\sum f_n$ converges to f*

uniformly on $[a,b]$, then f is Riemann integrable and

$$\sum \int_a^b f_n df = \int_a^b f df.$$

Theorem 1.2.5 ([7], Theorem 9.4.12, pp.270). *A power series can be differ* [illegible] *by-term within the interval of convergence. In fact, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, converges for $|x| < \rho$, then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ converges for $|x| < \rho$. That is, both series have the same radius of convergence ρ.*

Theorem 1.2.6. *Let $\sum_{k=0}^{\infty} f_k$ be a series of integrable functions that converges uniformly. Let $\alpha \in \mathbb{R}^+$, $n-1 < \alpha < n$, and $t > a$. Then*

$$I_{a^+}^{\alpha}\left(\sum_{k=0}^{\infty} f_k\right) = \sum_{k=0}^{\infty} I_{a^+}^{\alpha} f_k. \tag{1.8}$$

Proof. For $\alpha > 0$, we have

$$\begin{aligned} I_{a^+}^{\alpha}\left(\sum_{k=0}^{\infty} f_k\right) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \sum_{k=0}^{\infty} f_k(\tau) d\tau, \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \sum_{k=0}^{\infty} (t-\tau)^{\alpha-1} f_k(\tau) d\tau. \end{aligned}$$

Because of the uniform convergence of f_k, then $\sum_{k=0}^{\infty} (t-\tau)^{\alpha-1} f_k(\tau)$ is also uniformly convergent, and using Theorems 1.2.3 and 1.2.4, we have

$$\begin{aligned} I_{a^+}^{\alpha}\left(\sum_{k=0}^{\infty} f_k\right) &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \int_a^t (t-\tau)^{\alpha-1} f_k(\tau) d\tau \\ &= \sum_{k=0}^{\infty} I_{a^+}^{\alpha} f_k, \end{aligned}$$

which completes the proof. □

Under certain conditions, the above result valid for fractional differential operators, as we will see later on the Riemann-Liouville and Caputo fractional derivatives.

Proposition 1.2.7. *Let* $\alpha, a, b \in \mathbb{R}^+, n \in \mathbb{N}, n-1 < \alpha < n,$ *and* $t > 0$. *Then*

$$I_{0+}^{\alpha}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = t^{b-1+\alpha}E_{a,b+\alpha}(\lambda t^a). \tag{1.9}$$

Proof. Since $f(t) = t^{b-1}E_{a,b}(\lambda t^a)$ can be represented by the convergent power series

$$I_{0+}^{\alpha}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = I_{0+}^{\alpha}\left(\sum_{k=0}^{\infty}\frac{\lambda^k t^{ak+b-1}}{\Gamma(ak+b)}\right),$$

then using Theorem 1.2.6, we have

$$I_{0+}^{\alpha}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = \sum_{k=0}^{\infty}\frac{I_{0+}^{\alpha}\left(\lambda^k t^{ak+b-1}\right)}{\Gamma(ak+b)}.$$

Applying Proposition 1.2.2, we have

$$\begin{aligned} I_{0+}^{\alpha}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) &= \sum_{k=0}^{\infty}\frac{\lambda^k t^{ak+b-1+\alpha}}{\Gamma(ak+b+\alpha)} \\ &= t^{b-1+\alpha}E_{a,b+\alpha}(\lambda t^a), \end{aligned}$$

and the proof is completed. □

The recursion in Eq. (1.9) is of interests, as we can apply it to compute the fractional integrals and derivatives of many functions. Below are some examples.

(1) Consider $f(t) = e^t = E_{1,1}(t)$. Applying Eq. (1.9) with $a = 1, b = 1$, we have

$$I_{0+}^{\alpha}\left(e^t\right) = I_{0+}^{\alpha}E_{1,1}(t) = t^{\alpha}E_{1,1+\alpha}(t). \tag{1.10}$$

(2) Consider $f(t) = \cosh(t) = E_{2,1}(t^2)$. Applying Eq. (1.9) with $a = 2, b = 1$, we have

$$I^{\alpha}_{0+}\Big(\cosh(t)\Big) = I^{\alpha}_{0+}E_{2,1}(t^2) = t^{\alpha}E_{2,1+\alpha}(t^2). \tag{1.11}$$

(3) Consider $f(t) = \sinh(t) = tE_{2,2}(t^2)$. Applying Eq. (1.9) with $a = 2, b = 2$, we have

$$I^{\alpha}_{0+}\Big(\sinh(t)\Big) = I^{\alpha}_{0+}tE_{2,2}(t^2) = t^{\alpha+1}E_{2,2+\alpha}(t^2). \tag{1.12}$$

(4) Consider $f(t) = \sin(t)$. The Mittag-Leffler representation of $\sin(t)$ is,

$$\sin(t) = \frac{E_{1,1}(it) - E_{1,1}(-it)}{2i}.$$

Applying Eq. (1.9), we have

$$\begin{aligned} I^{\alpha}_{0+}\Big(\sin(t)\Big) &= \frac{I^{\alpha}_{0+}E_{1,1}(it) - I^{\alpha}_{0+}E_{1,1}(-it)}{2i} \\ &= \frac{t^{\alpha}E_{1,1+\alpha}(it) - t^{\alpha}E_{1,1+\alpha}(-it)}{2i}. \end{aligned} \tag{1.13}$$

(5) Consider $f(t) = \cos(t)$. Following analogous steps in (4), we have

$$I^{\alpha}_{0+}\Big(\cos(t)\Big) = \frac{t^{\alpha}E_{1,1+\alpha}(it) + t^{\alpha}E_{1,1+\alpha}(-it)}{2}. \tag{1.14}$$

1.3 The Riemann-Liouville Fractional Derivative

The Riemann-Liouville fractional derivative is defined as the inverse operator to the Riemann-Liouville fractional integral I^{α}_{a+}.

Definition 1.3.1. *Suppose that $\sigma = n - \alpha$, where $0 < \sigma < 1$ and n is the smallest positive integer greater than α. Let $f(t) \in C^n[0,T]$,* *derivative of $f(t)$ of order α is defined by*

$$
{}_R D^{\alpha}_{0+} f(t) = D^n (I^{\sigma}_{0+} f(t)) = \frac{1}{\Gamma(n-\alpha)} D^n \left(\int_0^t \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau \right), \tag{1.15}
$$

where D^n is the normal n^{th} derivative and I^{σ}_{0+} is Riemann-Liouville fractional integral of order σ.

It is clear that for $\alpha = k \in \mathbb{Z}$, then $n = k+1$, and

$$
{}_R D^{k}_{0+} f(t) = \frac{1}{\Gamma(1)} D^{k+1} \left(\int_0^t f(\tau) d\tau \right) = D^k f(t),
$$

which is the normal derivative of $f(t)$, see [1, 2, 6].

In the following, we present some basic results of the Riemann-Liouville fractional derivatives.

Proposition 1.3.1. *For $n-1 < \alpha < n, \sigma = n - \alpha$, and $t > 0$, we have*

$$
{}_R D^{\alpha}_{0+} \left(t^{\mu} \right) = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.
$$

Proof. Since

$$
I^{\sigma}_{0+} \left(t^{\mu} \right) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)} t^{\mu+\sigma},
$$

then

$$
\begin{aligned}
{}_R D^{\alpha}_{0+} \left(t^{\mu} \right) &= D^n \left(I^{\sigma}_{0+} t^{\mu} \right) = D^n \left(\frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)} t^{\mu+\sigma} \right) \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\sigma+1)} \frac{\Gamma(\mu+\sigma+1)}{\Gamma(\mu+\sigma+1-n)} t^{\mu+\sigma-n} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha},
\end{aligned}
$$

which completes the proof. □

Remark 1.3.1. For $\mu = 0$, we have

$$ {}_{R}D^{\alpha}_{0^+}\left(c\right) = \frac{c}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}d\tau = \frac{ct^{-\alpha}}{\Gamma(-\alpha+1)}, $$

that is the Riemann-Liouville fractional derivative of a constant function is not zero.

Proposition 1.3.2. *Suppose that $\alpha, a, b > 0$, and $n-1 < \alpha < n$, then for $t > 0$*

$$ {}_{R}D^{\alpha}_{0^+}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = t^{b-\alpha-1}E_{a,b-\alpha}(\lambda t^a) \tag{1.16} $$

Proof. For $n-1 < \alpha < n, \sigma = n-\alpha$, we have

$$ {}_{R}D^{\alpha}_{0^+}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = D^n\left(I^{\sigma}t^{b-1}E_{a,b}(\lambda t^a)\right). $$

Using Eq. (1.9), we have

$$ {}_{R}D^{\alpha}_{0^+}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = D^n\left(t^{b-1+\sigma}E_{a,b+\sigma}(\lambda t^a)\right) $$

Since the Mittag-Leffler function is written in the form of power series, using Theorem 1.2.5, we have

$$ \begin{aligned} D^n\left(t^{b-1+\sigma}E_{a,b+\sigma}(\lambda t^a)\right) &= D^n\left(\sum_{k=0}^{\infty}\frac{\lambda^k t^{ak+b-1+\sigma}}{\Gamma(ak+b+\sigma)}\right) \\ &= \sum_{k=0}^{\infty}\frac{D^n(\lambda^k t^{ak+b-1+\sigma})}{\Gamma(ak+b+\sigma)} \\ &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(ak+b+\sigma)}\frac{\Gamma(ak+b+\sigma)}{\Gamma(ak+b+\sigma-n)}t^{ak+b+\sigma-1-n}. \end{aligned} $$

Since $\alpha = n - \sigma$, then

$$ {}_{R}D_{0^+}^{\alpha}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{ak+b-\alpha-1}}{\Gamma(ak+b-\alpha)} = t^{b-\alpha-1}E_{a,b-\alpha}(\lambda t^a), $$

and the proof is completed. □

As a direct implementation of the above result, the following hold true.

(1) For $\alpha, t > 0, a = 1, b = 1, \lambda = 1$, we have

$$ {}_{R}D_{0^+}^{\alpha}\left(e^t\right) = {}_{R}D_{0^+}^{\alpha}E_{1,1}(t) = t^{-\alpha}E_{1,1-\alpha}(t). $$

(2) For $\alpha, t > 0, a = 2, b = 1, \lambda = 1$, we have

$$ {}_{R}D_{0^+}^{\alpha}\left(\cosh(t)\right) = {}_{R}D_{0^+}^{\alpha}E_{2,1}(t^2) = t^{-\alpha}E_{2,1-\alpha}(t^2). $$

(3) For $\alpha, t > 0, a = 2, b = 2, \lambda = 1$, we have

$$ {}_{R}D_{0^+}^{\alpha}\left(\sinh(t)\right) = {}_{R}D_{0^+}^{\alpha}(tE_{2,2}(t^2)) = t^{1-\alpha}E_{2,2-\alpha}(t^2). $$

(4) For $\alpha, t > 0, a = 1, b = 1$, we have

$$ \begin{aligned} {}_{R}D_{0^+}^{\alpha}\left(\sin(t)\right) &= \frac{{}_{R}D_{0^+}^{\alpha}E_{1,}}{2i} \\ &= \frac{t^{-\alpha}E_{1,}}{2i}, \end{aligned} $$

and

$$ {}_{R}D_{0^+}^{\alpha}\left(\cos(t)\right) = \frac{t^{-\alpha}E_{1,1-\alpha}(it) + t^{-\alpha}E_{1,1-\alpha}(-it)}{2}. $$

1.4 The Caputo Fractional Derivative

Some physical problems, especially in the theory of viscoelasticity and in hereditary solid mechanics, are modelled by fractional differential equations. Modelling by the Riemann-Liouville fractional derivative requires boundary and initial conditions of fractional order, which doesn't fit with the physical meaning of the problem. Therefore, M. Caputo has introduced a new definition of fractional derivative that requires natural boundary and initial conditions [1, 2, 8]. In this thesis we are interested in the Caputo fractional operator, and we denote the Caputo fractional derivative of order α by $D^{\alpha}_{0+}f(t)$.

Definition 1.4.1. *The Caputo fractional derivative of order* $\alpha > 0$, $n-1 < \alpha < n$ *of a function* $f(t) \in C^n[0,T]$ *is defined by*

$$\begin{aligned} D^{\alpha}_{0+}f(t) &= I^{n-\alpha}_{0+}\left(\frac{d^n}{dt^n}f(t)\right) \\ &= \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, & \text{if } n-1<\alpha<n \\ f^{(n)}, & \text{if } \alpha = n \in \mathbb{N}. \end{cases} \end{aligned} \tag{1.17}$$

The Caputo fractional derivative is connected with the Riemann-Liouville fractional derivative through the following relation.

Theorem 1.4.1. *Suppose that* $f \in C^n[0,T], t > 0, \alpha \in \mathbb{R}^+$, *and* $n-1 < \alpha < n \in \mathbb{N}$, *then*

$${}_RD^{\alpha}_{0+}f(t) = D^{\alpha}_{0+}f(t) + \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0). \tag{1.18}$$

Proof. We have

$${}_RD^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(n-\alpha)}D^n\left(\int_0^t \frac{f(\tau)}{(t-\tau)^{1+\alpha-n}}d\tau\right).$$

Integrating by parts $(n-1)$ times, one can verify that

$$ {}_RD^{\alpha}_{0+}f(t) = D^n\left(\sum_{k=0}^{n-1}\frac{t^{n-\alpha+k}f^{(k)}(0)}{\Gamma(n-\alpha+k+1)} + \frac{1}{\Gamma(2n-\alpha)}\int_0^t (t-\tau)^{2n-\alpha-1}f^{(n)}(\tau)d\tau\right). $$

Thus,

$$ \begin{aligned} {}_RD^{\alpha}_{0+}f(t) &= \sum_{k=0}^{n-1}\frac{\Gamma(n-\alpha+k+1)}{\Gamma(n-\alpha+k-n+1)}\frac{t^{-\alpha+k}f^{(k)}(0)}{\Gamma(n-\alpha+k+1)} \\ &+ \frac{\Gamma(2n-\alpha)}{\Gamma(n-\alpha)}\frac{1}{\Gamma(2n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \\ &= \sum_{k=0}^{n-1}\frac{t^{-\alpha+k}f^{(k)}(0)}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \\ &= \sum_{k=0}^{n-1}\frac{t^{-\alpha+k}f^{(k)}(0)}{\Gamma(k+1-\alpha)} + D^{\alpha}_{0+}f(t) \end{aligned} $$

which completes the proof. □

Remark 1.4.1. If $f^{(k)}(0) = 0$, for $k = 0, \ldots, n-1$, then the Riemann-Liouville fractional derivative is equivalent to the Caputo fractional derivative.

1.4.1 Properties of the Caputo Fractional Derivative

In the following, we present some properties of the Caputo fractional derivative. The first property is the linearity property which can be verified through the definition. For $c_1, c_2 \in \mathbb{R}$ and $\alpha \in \mathbb{R}^+$, we have

$$ D^{\alpha}_{0+}\Big(c_1 f(t) + c_2 g(t)\Big) = c_1 D^{\alpha}_{0+}f(t) + c_2 D^{\alpha}_{0+}g(t). $$

Proposition 1.4.2. *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, $f(t) \in C^{n+1}[0,T]$, then the following properties hold true*

$$ \lim_{\alpha \to n} D^{\alpha}_{0+}f(t) = f^{(n)}(t), \tag{1.19} $$

$$\lim_{\alpha \to n-1} D^{\alpha}_{0+} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0). \tag{1.20}$$

Proof. For $n-1 < \alpha < n$, we have

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau.$$

Integrating by parts with $u = f^{(n)}(\tau)$ and $dv = (t-\tau)^{n-\alpha-1} d\tau$, yields

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n)}(0)}{t^{\alpha-n}} + \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^{\alpha-n}} d\tau.$$

For $\alpha \to n$, we have

$$\begin{aligned} D^{\alpha}_{0+} f(t) &= f^{(n)}(0) + \int_0^t f^{(n+1)}(\tau) d\tau \\ &= f^{(n)}(t) \end{aligned}$$

For $\alpha \to n-1$, we have

$$\begin{aligned} D^{\alpha}_{0+} f(t) &= t f^{(n)}(0) + \int_0^t (t-\tau) f^{(n+1)}(\tau) d\tau \\ &= \int_0^t f^{(n)}(\tau \\ &= f^{(n-1)}(t) - f^{(n-1)}(0), \end{aligned}$$

and the result is followed. □

Proposition 1.4.3. *For $m \in \mathbb{N}, \alpha > 0$, we have*

$$D^{\alpha}_{0+} D^m f(t) = D^{\alpha+m}_{0+} f(t).$$

Proof. For $n-1<\alpha<n, \alpha\in\mathbb{R}^+, m,n\in\mathbb{N}$, we have

$$\begin{aligned} D^{\alpha}_{0^+}D^m f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{(D^m f(}{(t-\tau)^{\alpha+1-n}}d\tau \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n+m)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau. \end{aligned}$$

Since $n-1<\alpha<n$, then $m+n-1<\alpha+m<n+m$. Let $\beta=\alpha+m$, $N=n+m$, then

$$D^{\beta}_{0^+}f(t) = D^{\alpha+m}_{0^+}f(t) = \frac{1}{\Gamma(N-\beta)}\int_0^t \frac{f^{(N)}(\tau)}{(t-\tau)^{\beta+1-N}}d\tau.$$

Now, $N-\beta=m+n-\alpha-m=n-\alpha$ and $\beta+1-N=\alpha+m+1-m-n=\alpha+1-n$,

thus

$$D^{\alpha+m}_{0^+}f(t) \quad \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n+m)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau = D^{\alpha}_{0^+}D^m f(t),$$

which complete the proof. □

Remark 1.4.2. In general the Caputo fractional derivatives is not commutative, i.e,

$$D^{\alpha+m}_{0^+}f(t) \neq D^m D^{\alpha}_{0^+}f(t).$$

As a counter example, we consider $f(t)=t$,

We have

$$D^{\alpha+m}_{0^+}\Big(t\Big) = D^{\frac{3}{2}}_{0^+}\Big(t\Big) = 0,$$

while

$$D\Big(D^{\frac{1}{2}}_{0^+}\big(t\big)\Big) = D\left(\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}\right) = \frac{1}{\Gamma(\frac{1}{2})}t^{\frac{-1}{2}} \neq 0.$$

The following result concerning the composition between the Riemann-Lioville fractional integral and the Caputo fractional derivative, will be used in solving fractional differential equations.

Proposition 1.4.4. *For $\alpha, \beta > 0, n-1 < \beta < n \in \mathbb{N}$ and $f \in C^n[0,T]$,*

$$I_{0+}^{\alpha}\left(D_{0+}^{\beta} f(t)\right) = I_{0+}^{\alpha-\beta} f(t) - \sum_{k=0}^{n-1} \frac{\Gamma(k+1)}{\Gamma(\alpha-\beta+k+1)} \frac{t^{k+\alpha-\beta} f^{(k)}(0)}{k!}. \tag{1.21}$$

Proof. From the definition of the Caputo fractional derivative, we have

$$D_{0+}^{\beta} f(t) = I_{0+}^{n-\beta} D^n f(t).$$

Applying I_{0+}^{α} for both sides, and using the fact that the Riemann-Liouville integral is commutative, we have

$$I_{0+}^{\alpha} D_{0+}^{\beta} f(t) = I_{0+}^{\alpha} I_{0+}^{n-\beta} D^n f(t) = I_{0+}^{\alpha-\beta}(I_{0+}^{n} D^n f(t)).$$

Since

$$I_{0+}^{n} D^n f(t) = f(t) - \sum_{k=0}^{n-1} \frac{t^k f^{(k)}(0)}{k!},$$

we have

$$\begin{aligned} I_{0+}^{\alpha} D_{0+}^{\beta} f(t) &= I_{0+}^{\alpha-\beta}\Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k f^{(k)}(0)}{k!}\Big) \\ &= I_{0+}^{\alpha-\beta} f(t) - I_{0+}^{\alpha-\beta}\left(\sum_{k=0}^{n-1} \frac{t^k f^{(k)}(0)}{k!}\right) \\ &= I_{0+}^{\alpha-\beta} f(t) - \sum_{k=0}^{n-1} \frac{\Gamma(k+1)}{\Gamma(\alpha-\beta+k+1)} \frac{t^{k+\alpha-\beta} f^{(k)}(0)}{k!}, \end{aligned}$$

and the proof is completed. □

1.4.2 Computations of the Caputo Fractional Derivatives

In the following, we present the Caputo fractional derivatives of some well-known functions.

Proposition 1.4.5. *For $n-1<\alpha<n$,*

$$D_{0+}^{\alpha}\left(t^{\mu}\right)=\begin{cases}0 & \text{if } \mu\in\{0,1,...,n-1\}\\ \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha} & \text{if } \mu>n-1.\end{cases} \tag{1.22}$$

Proof. For $\mu>n-1$, we have $\frac{d^k}{dt^k}\left(t^{\mu}\right)\Big|_{t=0}=0$ for $k=0,...n-1$, and Theorem 1.4.1 yields

$$D_{0+}^{\alpha}\left(t^{\mu}\right)={}_RD_{0+}^{\alpha}t^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, \text{ for } \mu>n-1.$$

It is clear that for the case $\mu=0,1,...,n-1$,

$$D^{n}\left(t^{\mu}\right)=0, \text{ and thus } D_{0+}^{\alpha}\left(t^{\mu}\right)=0$$

which completes the proof. □

Remark 1.4.3. For $\alpha>0,\mu<n-1$, where μ is non integer number, the Caputo fractional derivative of $f(t)=t^{\mu}$ does not exist. Since of any order $\alpha>0$, we have

$$D_{0+}^{\alpha}\left(t^{\mu}\right)=\frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}I_{0+}^{n-\alpha}t^{\mu-n},$$

and the function $t^{\mu-n}$ is not integrable.

Proposition 1.4.6. *Let $f(t)=E_{1,1}(\lambda t)$. For $\alpha,t>0$, we have*

$$D_{0+}^{\alpha}f(t)=\lambda^{n}t^{-\alpha+n}E_{1,1-\alpha+n}(\lambda t). \tag{1.23}$$

Proof. Using Proposition 1.3.2, we have

$${}_RD_{0+}^{\alpha}f(t)=t^{-\alpha}E_{1,1-\alpha}(\lambda t).$$

Applying Theorem 1.4.1, we have

$$D_{0+}^{\alpha} f(t) = t^{-\alpha} E_{1,1-\alpha}(\lambda t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0).$$

Since $f^{(k)}($ [illegible] ,

$$D_{0+}^{\alpha} f(t) = \sum_{k=n}^{\infty} \frac{\lambda^k t^{k-\alpha}}{\Gamma(k+1-\alpha)}.$$

Shifting the index to zero, yields

$$\begin{aligned} D_{0+}^{\alpha} f(t) &= \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k-\alpha+n}}{\Gamma(k+1-\alpha+n)} \\ &= \lambda^n t^{-\alpha+n} E_{1,1-\alpha+n}(\lambda t) \end{aligned}$$

the proof is completed. □

Remark 1.4.4. Unlike the Riemann-Liouville fractional derivative, the Caputo fractional derivative of the Mittag-Leffler function does not always exist.

Proposition 1.4.7. *For $n-1<\alpha<n$, $a=2$, $b=1$ and $t>0$, we have*

$$D_{0+}^{\alpha} E_{2,1}(t^2) = D_{0+}^{\alpha}\Big(\cosh(t)\Big) = \begin{cases} t^{n-\alpha} E_{2,n+1-\alpha}(t^2) & \text{if } n \text{ is even number} \\ \\ t^{n+1-\alpha} E_{2,n+2-\alpha}(t^2) & \text{if } n \text{ is odd number.} \end{cases} \tag{1.24}$$

Proof. Using Theorem 1.4.1, and for $n-1<\alpha<n$, we have

$$D_{0+}^{\alpha} E_{2,1}(t^2) = {}_R D_{0+}^{\alpha} E_{2,1}(t^2) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0).$$

Since

$$f^{(k)}(0) = \frac{1-(-1)^{k+1}}{2},$$

and

$${}_RD_{0^+}^{\alpha}E_{2,1}(t^2) = t^{-\alpha}E_{2,1-\alpha}(t^2),$$

we have

$$D_{0^+}^{\alpha}E_{2,1}(t^2) = \sum_{k=0}^{\infty}\frac{t^{2k-\alpha}}{\Gamma(2k+1-\alpha)} - \sum_{\substack{k=0\\ k\text{ even}}}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}.$$

Case 1: If n is even, then $n-1$ is odd, and thus

$$D_{0^+}^{\alpha}E_{2,1}(t^2) = \sum_{k=\frac{n}{2}}^{\infty}\frac{t^{2k-\alpha}}{\Gamma(2k+1-\alpha)}.$$

Shifting the index to zero, we have

$$\begin{aligned} D_{0^+}^{\alpha}E_{2,1}(t^2) &= \sum_{k=0}^{\infty}\frac{t^{2k+n-\alpha}}{\Gamma(2k+n+1-\alpha)} \\ &= t^{n-\alpha}E_{2,n+1-\alpha}(t^2). \end{aligned}$$

Case 2: If n is odd, then $n-1$ is even, and thus

$$D_{0^+}^{\alpha}E_{2,1}(t^2) = \sum_{k=\frac{n+1}{2}}^{\infty}\frac{t^{2k-\alpha}}{\Gamma(2k+1-\alpha)}.$$

Shifting the index to zero, we have

$$\begin{aligned} D_{0^+}^{\alpha}E_{2,1}(t^2) &= \sum_{k=0}^{\infty}\frac{t^{2k+n+1-\alpha}}{\Gamma(2k+n+2-\alpha)} \\ &= t^{n+1-\alpha}E_{2,n+2-\alpha}(t^2), \end{aligned}$$

and the proof is completed. □

Proposition 1.4.8. *For $n-1<\alpha<n$, $a=2$, $b=2$ and $t>0$, we have*

$$D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right)=D^{\alpha}_{0+}\left(\sinh(t)\right)=\begin{cases} t^{n+1-\alpha}E_{2,n+2-\alpha}(t^2) & \text{if } n \text{ is even number} \\ \\ t^{n-\alpha}E_{2,n+1-\alpha}(t^2) & \text{if } n \text{ is odd number.}\end{cases} \tag{1.25}$$

Proof. Using Theorem 1.4.1, and for $n-1<\alpha<n$, we have

$$D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right)={}_R D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right)-\sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0).$$

Since

$$f^{(k)}(0)=\frac{1-(-1)^k}{2},$$

and

$${}_R D^{\alpha}_{0+}\left(t\,E_{2,1}(t^2)\right)=t^{1-\alpha}E_{2,2-\alpha}(t^2),$$

we have

$$D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right)=\sum_{k=0}^{\infty}\frac{t^{2k+1-\alpha}}{\Gamma(2k+2-\alpha)}-\sum_{\substack{k=0\\ k\text{ odd}}}^{n-1}\frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)}.$$

Case 1: If n is even, then $n-1$ is odd, and thus

$$D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right)=\sum_{k=\frac{n}{2}}^{\infty}\frac{t^{2k+1-\alpha}}{\Gamma(2k+2-\alpha)}.$$

Shifting the index to zero, we have

$$
\begin{aligned}
D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right) &= \sum_{k=0}^{\infty} \frac{t^{2k+n+1-\alpha}}{\Gamma(2k+n+2-\alpha)} \\
&= t^{n+1-\alpha} \quad {}_{2,n+2-\alpha}(t^2).
\end{aligned}
$$

Case 2: If n is odd, then $n-1$ is even, and thus

$$
D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right) = \sum_{k=\frac{n-1}{2}}^{\infty} \frac{t^{2k+1-\alpha}}{\Gamma(2k+2-\alpha)}.
$$

Shifting the index to zero, we have

$$
\begin{aligned}
D^{\alpha}_{0+}\left(t\,E_{2,2}(t^2)\right) &= \sum_{k=0}^{\infty} \frac{t^{2k+n-\alpha}}{\Gamma(2k+n+1-\alpha)} \\
&= t^{n-\alpha} E_{2,n+1-\alpha}(t^2),
\end{aligned}
$$

which completes the proof. $\square$

Proposition 1.4.9. *For $n-1<\alpha<n$ and $t>0$, we have*

$$
D^{\alpha}_{0+}\Big(\sin(t) \quad = \frac{}{2}i \qquad t \qquad E_{1,1-\alpha+n}(it) + (-1)^{n+1} t^{-\alpha+n} E_{1,1-\alpha+n}(-it) \quad ,
$$

and

$$
D^{\alpha}_{0+}\Big(\cos(t)\Big) = \frac{1}{2} i^n \Big(t^{-\alpha+n} E_{1,1-\alpha+n}(it) + (-1)^n t^{-\alpha+n} E_{1,1-\alpha+n}(-it)\Big).
$$

Proof. We represent $f(t) = \sin(t)$ by the Mittag-leffler function

$$
\sin(t) = \frac{E_{1,1}(it) - E_{1,1}(-it)}{2i}.
$$

Thus, for $n-1<\alpha<n$, we have

$$
\begin{aligned}
D_{0+}^{\alpha} f(t) &= \frac{1}{2i} D^{\alpha}\Big(E_{1,1}(it) - E_{1,1}(-it)\Big) \\
&= \frac{1}{2i}\Big(D^{\alpha} E_{1,} \text{[illegible]}
\end{aligned}
$$

Applying the result in Proposition 1.4.6, yields

$$
\begin{aligned}
D_{0+}^{\alpha}\Big(\sin(t)\Big) &= \frac{1}{2i}\Big(i^{n} t^{-\alpha+n} E_{1,} \text{[illegible]} \; {}^{n}\, {}^{-\alpha+n} \text{[illegible]} \Big) \\
&= \frac{1}{2} i^{n-1}\Big(t^{-\alpha+n} E_{1,\text{[illegible]}}(it) + (-1)^{n+1} t^{-\alpha+n} E_{1,1-\alpha+n}(-it)\Big).
\end{aligned}
$$

Applying analogous steps for $f(t)=\cos(t)$ and using the relation

$$
\cos(t) = \frac{E_{1,1}(it) + E_{1,1}(-it)}{2},
$$

yields

$$
D_{0+}^{\alpha}\Big(\cos(t)\Big) = \frac{1}{2} i^{n}\Big(t^{-\alpha+n} E_{1,1-\alpha+n}(it) + (-1)^{n} t^{-\alpha+n} E_{1,1-\alpha+n}(-it)\Big),
$$

which completes the proof. □

1.5 Fractional Differential Equations (FDE's)

Fractional differential equations are a generalization of differential equations to non integer orders. In this thesis, we are concerned with the fractional differential equations in the Caputo's sense.

The following initial value problems of the first and the second order,

$$
\begin{aligned}
y'(t) - \lambda y(t) &= 0, \qquad t>0, \\
y(0) &= a, \qquad a \in \mathbb{R},
\end{aligned}
\tag{1.26}
$$

and

$$\begin{aligned} y''(t)-\lambda y(t) &= 0, \qquad t>0,\lambda>0, \\ y^{(k)}(0) &= a_k; \qquad a\in\mathbb{R}, k=0,1, \end{aligned} \tag{1.27}$$

possesses the exact solutions

$$y(t)=ae^{\lambda t}$$

and

$$y(t)=\frac{a_1}{\sqrt{\lambda}}\sinh(\sqrt{\lambda}t)+a_0\cosh(\sqrt{\lambda}t),$$

respectively.

A natural question is, what about if we replace the integer derivatives by fractional ones? Then we have new fractional (I.V.P's) of the following forms

$$\begin{aligned} D^{\alpha}_{0+}y(t)-\lambda y(t) &= 0, \qquad 0<\alpha<1, t>0,\lambda>0, \\ y(0) &= a, \qquad a\in\mathbb{R}, \end{aligned} \tag{1.28}$$

and

$$\begin{aligned} D^{\alpha}_{0+}y(t)-\lambda y(t) &= 0, \qquad 1<\alpha<2, t>0,\lambda>0, \\ y^{(k)}(0) &= a_k, \qquad a_k\in\mathbb{R}, k=0,1. \end{aligned} \tag{1.29}$$

To find the solutions of the above (I.V.P's), we use the Laplace transform method. We have the following facts about the Laplace transformation method.

Definition 1.5.1. *Let $f(t)$ be an integrable function on $[0,\infty)$. The Laplace transform of f is the function F defined by the integral*

$$\mathscr{L}(f(t))=F(s)=\int_0^{\infty}e^{-st}f(t)dt.$$

Proposition 1.5.1. *The Laplace transformation of the convolution of $f(t)$ and $g(t)$ is*

$$\mathscr{L}(f(t)*g(t)) = \mathscr{L}(f(t))\mathscr{L}(g(t)),$$

where

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Proposition 1.5.2. *Let $f(t) \in C^n[0,1]$, then*

1. *The Laplace transformation of the n^{th} derivative of $f(t)$ is*

$$\mathscr{L}(f^{(n)}(t)) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).$$

2. *The Laplace transformation of the Caputo fractional derivative of order α of $f(t)$ is*

$$\mathscr{L}(D_{0+}^{\alpha} f(t)) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

where $n-1 < \alpha < n$.

The reader is referred to [9, 10] for more details and proofs.

The problems (1.28) and (1.29) can be solved using the Laplace transformation method.

Applying the Laplace transform for the I.V.P (1.28), we have

$$\mathscr{L}\left(D_{0+}^{\alpha} y(t)) - \lambda \mathscr{L}(y(t)\right) = 0,$$

which implies that

$$s^{\alpha} Y(s) - s^{\alpha-1} f(0) - \lambda Y(s) = 0,$$

where $Y(s) = \mathscr{L}(y(t))$. Thus

$$Y(s) = \frac{a s^{\alpha-1}}{s^{\alpha} - \lambda},$$

which is the Laplace transform of

$$y(t) = a E_{\alpha,1}(\lambda t^{\alpha}),$$

and the exact solution is obtained.

Applying analogous steps for solving the fractional I.V.P (1.29), yield

$$y(t) = a_0 E_{\alpha,1}(\lambda t^{\alpha}) + \frac{a_1}{\sqrt{\lambda}} t E_{\alpha,2}(\lambda t^{\alpha}),$$

which is the exact solution.

Since

$$\lim_{\alpha \to 1} a_0 E_{\alpha,1}(\lambda t^{\alpha}) = a_0 E_{1,1}(\lambda t) = a_0 e^{\lambda t}$$

and

$$\begin{aligned}\lim_{\alpha \to 2} a_0 E_{\alpha,1}(\lambda t^{\alpha}) + \frac{a_1}{\sqrt{\lambda}} t E_{\alpha,2}(\lambda t^{\alpha}) &= a_0 E_{2,1}(\lambda t^2) + \frac{a_1}{\sqrt{\lambda}} t E_{2,2}(\lambda t^2) \\ &= a_0 \cosh(\sqrt{\lambda} t) + \frac{a_1}{\sqrt{\lambda}} \sinh(\sqrt{\lambda} t).\end{aligned}$$

The solutions of the fractional I.V.P's (1.28) and (1.29) converge to the solutions of the I.V.P's (1.26) and (1.27) with integer orders as α approaches to 1 and 2, respectively.

1.6 The Adomian Decomposition Method (ADM)

In this thesis, we are interested in the Series solution of (FDE's). The (ADM) is one of the common methods in solving (FDE's). In the following, we present the idea of the (ADM).

Consider the following differential equation $F_y = f$, where

$$F_y = Ly + Ry + Ny = f. \tag{1.30}$$

Here L is the linear part of F, which is invertible, R is the linear operator for the remainder of the linear part, and N is the nonlinear operator for the nonlinear term of F. From Eq.(1.30), we have

$$Ly = f - Ry - Ny. \tag{1.31}$$

Applying the inverse operator L^{-1} for Eq.(1.31), we have

$$y(t) = g(t) - L^{-1}Ry - L^{-1}Ny, \tag{1.32}$$

where $g(t) = L^{-1}f(t)$.

The Adomian decomposition method represents the solution by an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

and the nonlinear term Ny is decomposed by

$$Ny = \sum_{n=0}^{\infty} A_n(t),$$

where A_n's are the well-known Adomian polynomials that can be obtained by the formula, see [11],

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k y_k(t)\right)\right)\Bigg|_{\lambda=0}. \tag{1.33}$$

Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} y_n(t) &= g(t) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n(t)\right) - L^{-1}\sum_{n=0}^{\infty} A_n(t)\\ &= g(t) - \sum_{n=0}^{\infty} L^{-1}R(y_n(t)) - L^{-1}\sum_{n=0}^{\infty} A_n(t)\end{aligned}$$

Then,

y_n is determined sequentially as follows

$$\begin{aligned} y_0(t) &= g(t) \\ y_{n+1}(t) &= -L^{-1}R(y_n(t)) - L^{-1}(A_n(t)), n \geq 0. \end{aligned} \tag{1.34}$$

Thus,

$$\sum_{n=0}^{\infty} y_n(t) = g(t) - \sum_{n=0}^{\infty} L^{-1}R(y_n(t)) - \sum_{n=0}^{\infty} L^{-1}(A_n(t)).$$

It is worth to mention that, in most cases, we choose L to be the linear part. In this case, $R = 0$ and y_n can be determined by

$$\begin{aligned} y_0(t) &= g(t) \\ y_{n+1}(t) &= -L^{-1}(A_n(t)), \quad n \geq 0. \end{aligned}$$

Also, we use to choose y_0 so that it satisfies the initial conditions given in the problem. The remaining y_i, $i \geq 1$, satisfy the homogeneous initial conditions. For more details and several applications of the (ADM) the reader is referred to [11, 12, 13, 14, 15, 16].

To explore the idea, we apply the (ADM) for a first order nonlinear (IVP).

Example 1.6.1. *Consider the I.V.P*

$$\begin{aligned} & y'(t) &= y^2(t), \qquad t > 0, \\ \textit{subiect to} \quad & y(0) &= 1. \end{aligned} \tag{1.35}$$

In this case, $L(y) = \frac{d}{dt}y, N(y) = -y^2(t)$ and $f(t) = 0$.

The inverse operator of $L = \frac{d}{dt}$, is $L^{-1} = \int_0^t (.)d\tau$.

Applying the inverse operator to the above (I.V.P), we have

$$y(t) = 1 + L^{-1}y^2.$$

Let $y(t) = \sum_{n=0}^{\infty} y_n(t)$.

The Adomian polynomials obtained by the formula (1.33) are

$$
\begin{aligned}
A_0 &= \left(\sum_{k=0}^{\infty} \lambda^k y_k(t) \right)^2 \Bigg|_{\lambda=0} = y_0^2, \\
A_1 &= \frac{d}{d\lambda} \left(\left(\sum_{k=0}^{\infty} \lambda^k y_k(t) \right)^2 \right) \Bigg|_{\lambda=0} = 2y_0 y_1, \\
A_2 &= 2y_0 y_2 + y_1^2, \\
A_3 &= 2y_0 y_3 + 2y_1 y_2, \\
A_4 &= 2y_0 y_4 + 2y_1 y_3 + y_2^2; \\
&\vdots
\end{aligned}
$$

Now, let $y_0 = 1$ and $y_{n+1} = L^{-1} A_n$, then

$$
\begin{aligned}
y_1 &= L^{-1} A_0 = L^{-1} y_0^2 = L^{-1} 1 = t, \\
y_2 &= L^{-1} A_1 = L^{-1} 2y_0 y_1 = t^2, \\
y_3 &= t^3, \\
y_4 &= t^4; \\
&\vdots
\end{aligned}
$$

Continuing with this process, we have

$$
y(t) = \sum_{n=0}^{\infty} t^n \tag{1.36}
$$

which is the Taylor series expansion of the exact solution

$$
y(t) = \frac{1}{1-t}.
$$

Chapter 2: Multi-Term Fractional Differential Equations

There are several analytical and numerical techniques for solving fractional differential equations. One of theses techniques is the series solution, where the terms of the series are determined sequentially. Such solutions can be obtained by the Adomain decomposition method and the differential transformation method, for more details about these methods the reader is referred to [12, 13, 14, 17, 18, 19]. Recently Dr. Al-Refai *et all* [20] have introduced a simple algorithm to obtain the series solutions of one-term fractional differential equation without the need of computing iterated fractional derivative, which do not commute in general. In this thesis, we generalize this algorithm to solve multi-term fractional differential equations of several types. We start with two-term fractional differential equations.

2.1 Two-Term Fractional Differential Equations

We consider the *two-term fractional initial value problems* of the form

$$c_1 D_{0^+}^{\alpha_1} u(t) + c_2 D_{0^+}^{\alpha_2} u(t) = f(t, u(t)), \quad t > 0 \tag{2.1}$$

with

$$u(0) = b, \tag{2.2}$$

where $0 < \alpha_2 < \alpha_1 < 1$, c_1 and c_2 are nonzero constants. We assume that $f(t, u(t))$ is continuous with respect to t and smooth with respect to $u(t)$. We also assume that α_1 and α_2 are rational numbers with $\alpha_1 = \frac{p_1}{q_1}$ and $\alpha_2 = \frac{p_2}{q_2}$, $p_1, p_2, q_1, q_2 \in \mathbb{N}$. Let $q = lcm(q_1, q_2)$, we have $q = sq_1 = rq_2$ for some s, $r \in \mathbb{N}$.

We expand the solution of the problem (2.1)-(2.2) in an infinite series of the form

$$u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{q}},$$

where the coefficients a_n, $n \geqslant 0$, have to be determined sequentially in the following manner. From the initial condition (2.2) we have $u(0) = b = a_0$.

For $n \geq 1$, we have

$$D_{0^+}^{\alpha_1} u(t) = \sum_{n=1}^{\infty} a_n s_n \, t^{\frac{n}{q} - \frac{p_1}{q_1}} = \sum_{n=1}^{\infty} a_n s_n \, t^{\frac{n - sp_1}{q}}, \tag{2.3}$$

and

$$D_{0^+}^{\alpha_2} u(t) = \sum_{n=1}^{\infty} a_n r_n \, t^{\frac{n}{q} - \frac{p_2}{q_2}} = \sum_{n=1}^{\infty} a_n r_n \, t^{\frac{n - rp_2}{q}}, \tag{2.4}$$

where $s_n = \frac{\Gamma(\frac{n}{q}+1)}{\Gamma(\frac{n}{q}-\alpha_1+1)}$ and $r_n = \frac{\Gamma(\frac{n}{q}+1)}{\Gamma(\frac{n}{q}-\alpha_2+1)}$.

By substituting Eq.'s (2.3) and (2.4) in Eq.(2.1) we have

$$c_1 \sum_{n=1}^{\infty} a_n s_n t^{\frac{n-sp_1}{q}} + c_2 \sum_{n=1}^{\infty} a_n r_n t^{\frac{n-rp_2}{q}} = f(t, \sum_{n=0}^{\infty} a_n t^{\frac{n}{q}}). \tag{2.5}$$

Applying the well-known Taylor series method to compute the coefficients $\{a_n; n \geq 1\}$, will lead to computing iterated fractional derivatives, which do not commute in general. To avoid this difficulty, let $t = w^q$, we have

$$c_1 \sum_{n=1}^{\infty} a_n s_n w^{n-sp_1+1} + c_2 \sum_{n=1}^{\infty} a_n r_n w^{n-rp_2+1} = f(t, \sum_{n=0}^{\infty} a_n w^n). \tag{2.6}$$

Shifting the index to zero, yields

$$c_1 \sum_{n=0}^{\infty} a_{n+1} s_{n+1} w^{n-sp_1+1} + c_2 \sum_{n=0}^{\infty} a_{n+1} r_{n+1} w^{n-rp_2+1} = f(w^q, \sum_{n=0}^{\infty} a_n w^n). \tag{2.7}$$

To avoid the singularity at $w=0$, we multiply Eq. (2.7) by w^{sp_1-1}, we have

$$c_1 \sum_{n=0}^{\infty} a_{n+1} s_{n+1} w^n + c_2 \sum_{n=0}^{\infty} a_{n+1} r_{n+1} w^{n-rp_2+sp_1} = w^{sp_1-1} f(w^q, \sum_{n=0}^{\infty} a_n w^n). \tag{2.8}$$

Now, since $\alpha_1 = \frac{p_1}{q_1} = \frac{sp_1}{q} > \frac{rp_2}{q} = \frac{p_2}{q_2} = \alpha_2$, thus $sp_1 - rp_2 > 0$, and the Eq.(2.8) has no singularity at $w=0$.

Let $k = sp_1 - rp_2 - 1 \geq 0$, and $g(w) = f(w^q, \sum_{n=0}^{\infty} a_n w^n)$, then Eq.(2.8) can be written as

$$c_1 \sum_{n=0}^{k} a_{n+1} s_{n+1} w^n + \sum_{n=k+1}^{\infty} (c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k}) w^n = w^{k+rp_2} g(w). \tag{2.9}$$

We first determine the coefficients a_n for $n \leq k$. By performing the nth derivative of Eq.(2.9) with respect to w and substituting $w=0$, we have

$$c_1\, n!\, a_{n+1}\, s_{n+1} = \frac{d^n}{dw^n}(w^{k+rp_2} g(w))|_{w=0}$$

which yields

$$a_{n+1} = \frac{1}{c_1\, n!\, s_{n+1}} \frac{d^n}{dw^n} \left(w^{k+rp_2} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n \right) \right) \Bigg|_{w=0}. \tag{2.10}$$

Since $k + rp_2 \geq n+1$ for $n \leq k$ and f is smooth then

$$\frac{d^n}{dw^n} \left(w^{k+rp_2} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n \right) \right) \Bigg|_{w=0} = 0,$$

for $n \leq k$, and hence $a_{n+1} = 0$ for $n \leq k$.

For $n \geq k+1$, by performing the nth derivative of Eq.(2.9) with respect to w and substituting $w=0$, we have

$$n!(c_1\, a_{n+1}\, s_{n+1} + c_2\, a_{n-k}\, r_{n-k}) = \frac{d^n}{dw^n}(w^{k+rp_2} g(w))|_{w=0} = \frac{d^n}{dw^n} \left(w^{k+rp_2} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n \right) \right) \Bigg|_{w=0}.$$

Using the well-known Leibniz rule for differentiating the products, we have

$$\frac{d^n}{dw^n}(w^{k+rp_2}g(w)) = \sum_{j=0}^{n}\binom{n}{j}\frac{\mathrm{d}^j}{\mathrm{d}w^j}\left(w^{k+rp_2}\right)\frac{\mathrm{d}^{n-j}}{\mathrm{d}w^{n-j}}(g(w)).$$

Since

$$\frac{\mathrm{d}^j}{\mathrm{d}w^j}\left(w^{k+rp_2}\right)|_{w=0} = \begin{cases} 0 & \text{if } j \neq k+rp_2 \\ j! & \text{if } j = k+rp_2, \end{cases}$$

we have

$$c_1 a_{n+1} s_{n+1} + c_2 a_{n-k} r_{n-k} = \frac{1}{(n-j)!}\left(\frac{\mathrm{d}^{n-j}}{\mathrm{d}w^{n-j}}g(w)\right)\Bigg|_{w=0}, \text{ where } j = k+rp_2. \quad (2.11)$$

From the last equation we can determine $a_n : n \geq k+1$ and thus the solution

$$u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{q}} = a_0 + \sum_{n=k+1}^{\infty} a_n t^{\frac{n}{q}}.$$

Remark 2.1.1. The coefficients $a_{n+1}, n \leq k$, are not necessary zeros if f is not smooth. This case will be discussed later in Example 2.3.2.

To illustrate the idea we consider the following problems.

Example 2.1.1. *Consider the two-term fractional initial value problem*

$$D_{0^+}^{\frac{1}{2}}u(t) + D_{0^+}^{\frac{1}{8}}u(t) = \frac{3\Gamma(3/4)t^{\frac{1}{4}}}{\Gamma(1/4)} + \frac{6\Gamma(3/4)t^{\frac{5}{8}}}{5\Gamma(5/8)}, \quad (2.12)$$

with

$$u(0) = 0. \quad (2.13)$$

The exact solution is $u(t) = t^{\frac{3}{4}}$.

We have $\alpha_1 = \frac{1}{2} = \frac{p_1}{q_1}$, $\alpha_2 = \frac{1}{8} = \frac{p_2}{q_2}$, $q = l.c.m(q_1, q_2) = 8$, $s = 4$ and $r = 1$. We expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{8}}$. The initial condition in (2.13) yields $a_0 = 0$.

We have $t = w^q = w^8$ and $f(w^q, u(w)) = \frac{3\Gamma(3/4)w^2}{\Gamma(1/4)} + \frac{6\Gamma(3/4)w^5}{5\Gamma(5/8)}$.

Since $f(w^q, u(w))$ is continuous with respect to w and smooth with respect to u, we have

$$a_{n+1} = 0 \text{ , for } n \leq k = sp_1 - rp_2 - 1 = 2.$$

Thus $a_1 = a_2 = a_3 = 0$. The function $f(w^q, u(w))$ satisfies the assumption of the proposed algorithm, and it holds that

$$\frac{\mathrm{d}^m}{\mathrm{d}w^m}(f(w^q, u(w))\,|_{w=0} = \begin{cases} \frac{2!\times 3\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} & \text{if} \quad m = 2, \\ \frac{5!\times\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{8})} & \text{if} \quad m = 5, \\ 0 & \textit{otherwise.} \end{cases}$$

For $n \geq 3$, Eq.(2.12) together with the last equation yield

$$a_{n+1}\, s_{n+1} + a_{n-2}\, r_{n-2} = \frac{1}{(n-3)!}\left(\frac{d^{n-3}}{dw^{n-3}} f(w^q, u(w))\right)\Bigg|_{w=0} = \begin{cases} \frac{3\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} & \text{if} \quad n = 5, \\ \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{8})} & \text{if} \quad n = 8, \\ 0 & \textit{otherwise,} \end{cases}$$

where $s_{n+1} = \frac{\Gamma(\frac{n+1}{8}+1)}{\Gamma(\frac{n+1}{8}+\frac{1}{2})}$ and $r_{n-2} = \frac{\Gamma(\frac{n-2}{8}+1)}{\Gamma(\frac{n-2}{8}+\frac{7}{8})}$.

We now apply the last recursion together with $a_0 = a_1 = a_2 = a_3 = 0$, to compute a_{n+1} for $n \geq 3$. For $n = 3$, we have $a_4 s_4 + a_1 r_1 = 0$, and thus $a_4 = 0$.

For $n = 4$, we have $a_5 s_5 + a_2 r_2 = 0$, and thus $a_5 = 0$.

For $n = 5$, we have $a_6 s_6 + a_3 r_3 = \frac{3\Gamma(3/4)}{\Gamma(1/4)}$, which yield

$$a_6 = \frac{1}{s_6}\frac{3\Gamma(3/4)}{\Gamma(1/4)} = \frac{\Gamma(5/4)}{\Gamma(7/4)} \times \frac{3\Gamma(3/4)}{\Gamma(1/4)} = 1.$$

Applying analogous argument yields to, $a_7 = a_8 = 0$.

For $n = 8$, we have $a_9 s_9 + a_6 r_6 = \frac{6\Gamma(3/4)}{5\Gamma(5/8)}$. Since $a_6 = 1$, we have

$$a_9 = \left(\frac{6\Gamma(3/4)}{5\Gamma(5/8)} - r_6\right) \times \frac{1}{s_9} = \left(\frac{6\Gamma(3/4)}{5\Gamma(5/8)} - \frac{\Gamma(14/8)}{\Gamma(13/8)}\right) \times \frac{1}{s_9} = \left(\frac{6\Gamma(3/4)}{5\Gamma(5/8)} - \frac{6\Gamma(3/4)}{5\Gamma(5/8)}\right) \times \frac{1}{s_9} = 0.$$

Since $a_8 = a_9 = 0$, we have $a_{n+1} = 0$ for $n \geq 7$.

Thus $u(t) = a_6 t^{\frac{6}{8}} = t^{\frac{3}{4}}$ and the exact solution of the problem (2.12)-(2.13) is obtained.

We compare our result with the solution obtained by the (ADM). Applying the Riemann-Liouville fractional integral operator $I_{0+}^{\frac{1}{2}}$ for both sides of Eq.(2.12) and using the fact that

$$I_{0+}^{\frac{1}{2}} D_{0+}^{\frac{1}{8}} u(t) = I_{0+}^{\frac{3}{8}} u(t) - \frac{t^{\alpha-\beta} f(0)}{\Gamma(\alpha-\beta+1)} = I_{0+}^{\frac{3}{8}} u(t),$$

we have

$$\begin{aligned} u(t) &= u(0) + I_{0+}^{\frac{1}{2}}\left(\frac{3\Gamma(3/4)t^{\frac{1}{4}}}{\Gamma(1/4)} + \frac{6\Gamma(3/4)t^{\frac{5}{8}}}{5\Gamma(5/8)}\right) - I_{0+}^{\frac{3}{8}} u(t) \\ &= t^{\frac{3}{4}} + 0.867482\, t^{\frac{9}{8}} \quad 3 \end{aligned} \tag{2.14}$$

We expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} u_n(t)$. We set

$$u_0(t) = t^{\frac{3}{4}} + 0.867482\, t^{\frac{9}{8}},$$

then $u_n, n \geq 1$, are determined sequentially from

$$u_{n+1} = -I_{0^+}^{\frac{3}{8}} u_n(t), \quad n \geq 0.$$

We have

$$u_0(t) = t^{\frac{3}{4}} + 0.867482\, t^{\frac{9}{8}}$$
$$u_1(t) = -0.867482\, t^{\frac{9}{8}} - 0.691367 t^{\frac{3}{2}},$$
$$u_2(t) = 0.691367\, t^{\frac{3}{2}} + 0.5141\, t^{\frac{15}{8}},$$
$$u_3(t) = -0.5141\, t^{\frac{15}{8}} - 0.360522\, t^{\frac{9}{4}},$$
$$u_4(t) = 0.360522\, t^{\frac{9}{4}} + 0.240312\, t^{\frac{21}{8}},$$
$$u_5(t) = -0.240312\, t^{\frac{21}{8}} - 0.153177\, t^{3}$$
$$\vdots$$
$$u_{10}(t) = 0.0175585\, t^{\frac{9}{2}} + 0.00946594\, t^{\frac{39}{8}}.$$

It is noted that we end with a telescoping sum and

$$u_{10}(t) = \sum_{n=0}^{10} u_n = t^{\frac{3}{4}} + 0.00946594\, t^{\frac{39}{8}}. \tag{2.15}$$

Figure 2.1 depicts the approximate solution obtained in Eq.(2.15) by the Adomian decomposition method together with the exact solution obtained by the proposed algorithm.

Example 2.1.2. *Consider the two-term nonlinear fractional initial value problem*

$$2D_{0^+}^{\frac{1}{2}} u(t) + 2\Gamma(\frac{13}{10}) D_{0^+}^{\frac{1}{5}} u(t) = \Gamma(\frac{1}{2}) \left(u^2 + t^{\frac{3}{10}} - t + 1\right), \tag{2.16}$$

with

$$u(0) = 0. \tag{2.17}$$

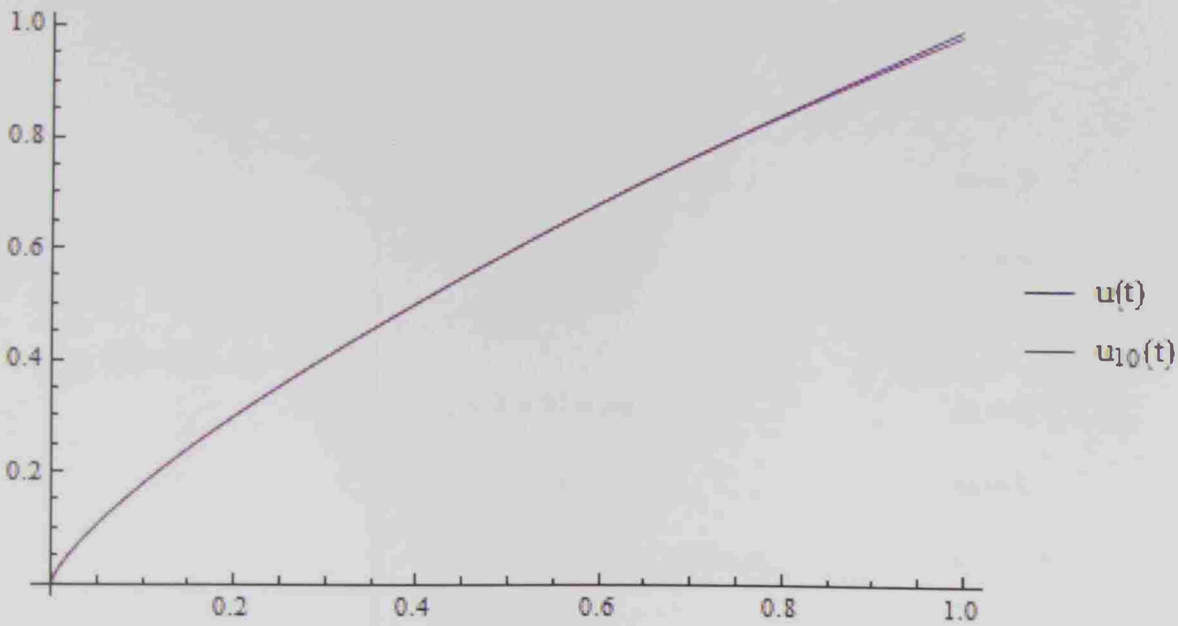

Figure 2.1: Comparison between the exact and approximate solutions obtained by the (ADM) for Example 2.1.1.

The exact solution is $u(t) = t^{\frac{1}{2}}$.

We have

$$\alpha_1 = \frac{1}{2} = \frac{p_1}{q_1}, \quad \alpha_2 = \frac{1}{5} = \frac{p_2}{q_2}, \quad q = l.c.m(q_1, q_2) = 10, \quad s = 5 \text{ and } r = 2.$$

We expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{10}}$. The initial condition in (2.17) yields $a_0 = 0$. We have $t = w^q = w^{10}$ and

$$g(w) = f(w^{10}, u(w)) = \Gamma(\frac{1}{2}) \left(\left(\sum_{n=0}^{\infty} a_n w^n \right)^2 + w^3 - w^{10} + 1 \right). \tag{2.18}$$

Since $g(w)$ is continuous with respect to w and smooth with respect to u, we have

$$a_{n+1} = 0, \text{ for } n \le k = sp_1 - rp_2 - 1 = 2.$$

Thus $a_1 = a_2 = a_3 = 0$. The function $g(w)$ satisfies the assumption of the proposed algorithm, applying Mathematica 9, we have

$$\frac{\mathrm{d}^m}{\mathrm{d}w^m}(g(w))|_{w=0} = \Gamma(\frac{1}{2})\begin{cases} 1 & m=0 \\ 3! & m=3 \\ 6!\ a_3^2 & m=6 \\ 2\times 7!\ a_3a_4 & m=7 \\ 8!\ a_4^2+2\times 8!\ a_3a_5 & m=8 \\ 2!(a_4a_5+a_3a_3) & m=9 \\ -10!+10!\ a_5^2+2\times 10!(a_4a_6+a_3a_7) & m=10 \\ \vdots & \vdots \end{cases} \tag{2.19}$$

For $n \geq 3$, substituting Eq.(2.19) in Eq.(2.11) yields

$$2\,a_{n+1}\,s_{n+1} + 2\,\Gamma(\frac{13}{10})\,a_{n-2}\,r_{n-2} = \frac{1}{(n-4)!}\frac{d^{n-4}}{dw^{n-4}}(g(w))|_{w=0}, \tag{2.20}$$

where $s_{n+1} = \dfrac{\Gamma(\frac{n+1}{10}+1)}{\Gamma(\frac{n+1}{10}+\frac{1}{2})}$ and $r_{n-2} = \frac{\Gamma(\frac{n-2}{10}+1)}{\Gamma(\frac{n-2}{10}+\frac{4}{5})}$.

We now apply the last recursion together with $a_1 = a_2 = a_3 = 0$, to compute a_{n+1} for $n \geq 3$. For $n = 3$, we have $2\,a_4\,s_4 + 2\,\Gamma(\frac{13}{10})\,a_1\,r_1 = 0$, thus $a_4 = 0$.

For $n = 4$, we have $2\,a_5\,s_5 + 2\,\Gamma(\frac{13}{10})\,a_2\,r_2 = \Gamma(\frac{1}{2})$. Since $a_2 = 0$, then

$$a_5 = \frac{\Gamma(\frac{1}{2})}{2s_5} = \frac{\Gamma(\frac{1}{2})}{2} \times \frac{1}{\Gamma(\frac{3}{2})} = 1.$$

Applying analogous arguments yield $a_6 = a_7 = 0$.

For $n = 7$, we have $2\,a_8\,s_8 + 2\,\Gamma(\frac{13}{10})\,a_5\,r_5 = \Gamma(\frac{1}{2})$. Since $a_5 = 1$, then

$$a_8 = \left(\Gamma(\frac{1}{2}) - 2\Gamma(\frac{13}{10}) \times \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{13}{10})}\right) \times \frac{1}{2s_8} = 0.$$

Applying analogous arguments yield $a_9 = a_{10} = 0$.

For $n = 10$, we have

$$2\,a_{11}\,s_{11} + 2\,\Gamma(\frac{13}{10})\,a_8\,r_8 = \Gamma(\frac{1}{2}) \times a_3^2.$$

Since $a_8 = a_3 = 0$, then $a_{11} = 0$.

Applying the same arguments for $n = 11, 12, 13$, yield $a_{12} = a_{13} = a_{14} = 0$.

For $n = 14$, we have

$$2a_{15}s_{15} + 2\Gamma(\frac{13}{10})a_{12}r_{12} = \Gamma(\frac{1}{2})(-10! + 10!a_5^2 + 2 \times 10!(a_4 a_6 + a_3 a_7)).$$

Since $a_5 = 1$ and $a_3 = a_4 = a_6 = a_7 = a_{12} = 0$, thus $a_{15} = 0$.

Following similar process we have $a_{n+1} = 0$ for $n \geq 6$.

Thus

$$u(t) = a_5 t^{\frac{5}{10}} = t^{\frac{1}{2}}$$

and the exact solution of the problem (2.16)-(2.17) is obtained.

We now apply the (ADM) to solve the proplem (2.16)-(2.17). Since $f(t,u(t))$ is nonlinear function with respect to $u(t)$, we need to compute the Adomian polynomials for

$$f(t,u) = u^2(t).$$

Applying the well-known formula for Adomian polynomials, we have

$$u^2(t) = \sum_{n=0}^{\infty} A_n$$

where

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \\ A_4 &= 2u_0u_4 + 2u_1u_3 + u_2^2, \\ &\vdots \end{aligned} \tag{2.21}$$

Applying the Riemann-Liouville fractional integral operator $I_{0+}^{\frac{1}{2}}$ to Eq.(2.16) and substituting

$$I_{0+}^{\frac{1}{2}} D_{0+}^{\frac{1}{5}} u(t) = I_{0+}^{\frac{3}{10}} u(t),$$

we have

$$\begin{aligned} u(t) &= \frac{\Gamma(\frac{1}{2})}{2} I_{0+}^{\frac{1}{2}} \left(1 + t^{\frac{3}{10}} - t\right) - \Gamma(\frac{13}{10}) I_{0+}^{\frac{3}{10}} u(t) + \frac{\Gamma(\frac{1}{2})}{2} I_{0+}^{\frac{1}{2}} u^2(t) \\ &= t^{\frac{1}{2}} + 0.853958 t^{\frac{4}{5}} - 0.666667 t^{\frac{3}{2}} - \Gamma(\frac{13}{10}) I_{0+}^{\frac{3}{10}} u(t) + \frac{\Gamma(\frac{1}{2})}{2} I_{0+}^{\frac{1}{2}} u^2(t). \end{aligned} \tag{2.22}$$

Let $u(t) = \sum_{n=0}^{\infty} u_n(t)$ and $u^2(t) = \sum_{n=0}^{\infty} A_n$, and set

$$u_0 = t^{\frac{1}{2}} + 0.853958 t^{\frac{4}{5}} - 0.666667 t^{\frac{3}{2}}.$$

Then

$$u_{n+1} = -I^{\frac{3}{10}} u_n + I^{\frac{1}{2}} A_n$$

Evaluating $u_1(t), u_2(t), u_3(t), u_4(t)$, we have

$$\begin{aligned} u_4(t) &= t^{0.5} + 0.257998 t^2 + -2.798359 t^{2.7} - 1.454522 t^3 + 8.855605 t^{3.4} + 9.799202 t^{3.7} \\ &+ 2.317936 t^4 - 11.599906 t^{4.1} - 22.429449 t^{4.4} - 11.273123 t^{4.7} + 6.685081 t^{4.8} \\ &- 1.388180 t^5 + 23.109057 t^{5.1} + 20.278401 t^{5.4} - 1.404129 t^{5.5} + 5.305238 t^{5.7} \\ &- 11.070905 t^{5.8} + 0.278605 t^6 - 17.334122 t^{6.1} - 7.918245 t^{6.4} + 2.012699 t^{6.5} \\ &- 0.885674 t^{6.7} + 7.156236 t^{6.8} + 5.827855 t^{7.1} + 1.140180 t^{7.4} - 1.153180 t^{7.5} \\ &- 2.127835 t^{7.8} - 0.742611 t^{8.1} + 0.309488 t^{8.5} + 0.244589 t^{8.8} - 0.032578 t^{9.5} \end{aligned} \tag{2.23}$$

Figure 2.2 depicts the approximate solution obtained in Eq.(2.23) by the Adomian decomposition method together with the exact solution obtained by the proposed algorithm.

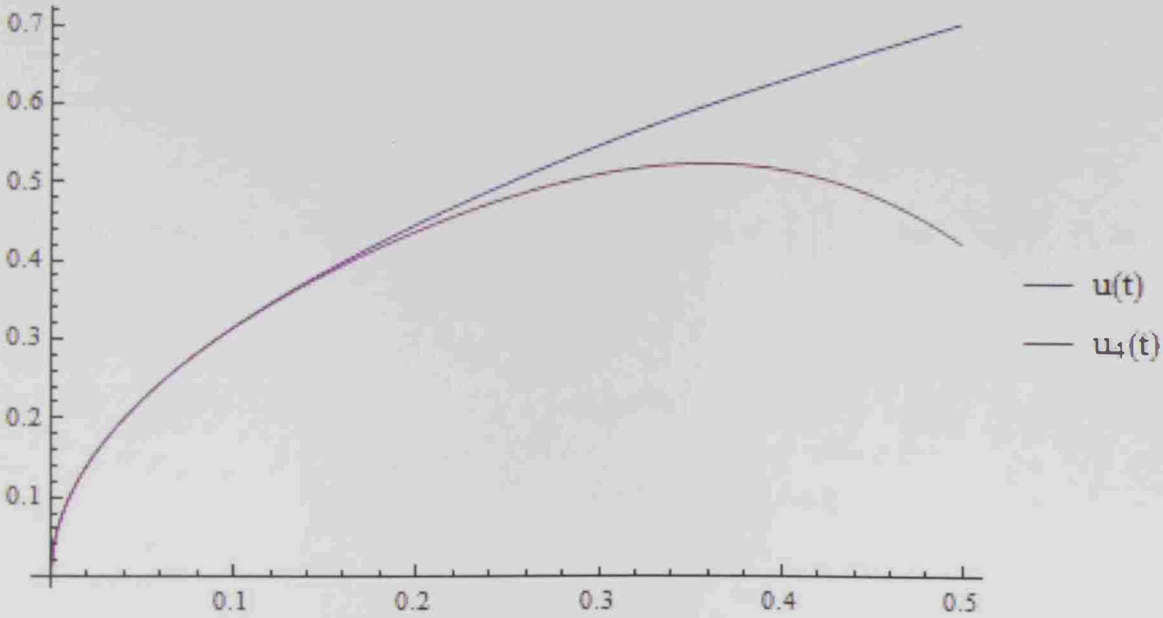

Figure 2.2: Comparison between the exact and approximate solutions obtained by the (ADM) for Example 2.1.2.

Example 2.1.3. *Consider the two-term nonlinear fractional initial value problem*

$$D_{0^+}^{\frac{4}{5}}u(t)+D_{0^+}^{\frac{1}{2}}u(t)=\frac{u(t)\sin(t)}{u^2(t)+1}, \tag{2.24}$$

$$u(0)=1. \tag{2.25}$$

Because of the non linearity of the problem, the exact solution is not available in closed form. We apply the proposed algorithm to obtain a numerical solution and then analyse the obtained solution. This example has been discussed in [21], where the problem is transformed to a fractional integral equation, then the Adams-Bashforth-Moulton method is used with step size $h=\frac{1}{50}$. At the end, the obtained numerical solution has been presented graphically.

Applying the proposed algorithm, we have

$$p_1=4, q_1=5, p_2=1, q_2=2, q=l.c.m(5,2)=10, s=2, r=5, k=2.$$

We expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{10}}$. Applying the proposed series algorithm, we have $a_0 = a_1 = ... = a_7$, and

$$a_{n+1}s_{n+1} + a_{n-2}r_{n-2} = \frac{1}{(n-7)!}\left(\frac{d^{n-7}}{dw^{n-7}} f\left(w^{10}, \sum_{n=0}^{\infty} a_n w^n\right)\right)\Bigg|_{w=0} \quad \text{for} \quad n \geq 7, \quad (2.26)$$

where

$$f\left(w^{10}, \sum_{n=0}^{\infty} a_n w^n\right) = \frac{\sin(w^{10})\left(\sum_{n=0}^{\infty} a_n w^n\right)}{\left(\sum_{n=0}^{\infty} a_n w^n\right)^2 + 1}.$$

From the initial condition (2.25), we have $a_0 = 1$. We apply Eq. (2.26) to compute the first 6 nonzero coefficients a_n for $n \geq 7$. These coefficients are $a_{18}, a_{21}, a_{24}, a_{27}, a_{30}, a_{33}$. The obtained truncated series solution is

$$\begin{aligned} u_{33}(t) &= 1 + 0.298242t^{1.8} - 0.227519t^{2.1} + 0.167717t^{2.4} \\ &- 0.119885t^{2.7} + 0.0833333t^3 - 0.0564631t^{3.3} \end{aligned} \quad (2.27)$$

To test the accuracy of the obtained solution we consider

$$\mathscr{P}u(t) = D^{\frac{4}{5}}u(t) + D^{\frac{1}{2}}u(t) - \frac{u(t)\sin(t)}{u^2(t)+1} = 0,$$

and define the error

$$E_N(t_i) = \int_0^{t_i} (\mathscr{P}(u_N(t))^2 dt, \quad t \in [0, t_i]. \quad (2.28)$$

Table 2.1 presents the error $E_{33}(t_i)$ for $0 \leq t_i \leq 1$, and Table 2.2 presents the error $E_N(1)$ for different values of N. The presented results show the efficiency of the proposed algorithm and more accuracy can be achieved by computing more terms in the power series solution.

Table 2.1: The error to Example 2.1.3.

t_i	$E_{33}(t_i)$
0.1	1.1671×10^{-10}
0.2	8.3567×10^{-9}
0.3	9.63673×10^{-8}
0.4	5.28145×10^{-7}
0.5	1.93041×10^{-6}
0.6	5.47587×10^{-6}
0.7	0.0000130817
0.8	0.0000276595
0.9	0.0000534847
1	0.0000967815

Table 2.2: The errors to Example 2.1.3 at $t = 1$.

N	$E_N(1)$
5	0.0681689
10	0.0681689
15	0.0681689
20	0.067155
25	0.0248829
30	0.00882027
33	0.0000967893

Remark 2.1.2. In order to solve the problem (2.24)-(2.25) with the (ADM), we face a problem with Mathematica in computing the Adomian polynomials for the analytic function

$$f(t,u(t)) = \frac{u(t)\sin(t)}{u^2(t)+1}.$$

2.2 Three-Term Fractional Differential Equations

We consider the *three-term fractional initial value problem* of the form

$$c_1 D_{0+}^{\alpha_1} u(t) + c_2 D_{0+}^{\alpha_2} u(t) + c_3 D_{0+}^{\alpha_3} u(t) = f(t,y(t)), \tag{2.29}$$

$$u^i(0) = u_i, i = 0,1,\dots,n-1, \tag{2.30}$$

where $0 < \alpha_3 < \alpha_2 < \alpha_1 < n$, and c_1, c_2, c_3 and b are constants. Assume that $f(t,y(t))$ is continuous with respect to t and smooth with respect to $u(t)$. We also assume that α_1, α_2

and α_3 are rational numbers with $\alpha_1 = \frac{p_1}{q_1}$, $\alpha_2 = \frac{p_2}{q_2}$ and $\alpha_3 = \frac{p_3}{q_3}$ where $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{N}$. Let $q = lcm(q_1, q_2, q_3)$, we have $q = sq_1 = rq_2 = vq_3$ for s, r and $v \in \mathbb{N}$.

Let

$$k_1 = sp_1 - rp_2 - 1 \text{ and } k_2 = sp_1 - vp_3 - 1.$$

We expand the solution $u(t)$ in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{q}}$, we have

$$\sum_{n=0}^{k_1} c_1 a_{n+1} s_{n+1} w^n + \sum_{n=k_1+1}^{k_2} (c_1 a_{n+1} s_{n+1} + c_2 a_{n-k_1} r_{n-k_1}) w^n$$
$$+ \sum_{n=k_2+1}^{\infty} (c_1 a_{n+1} s_{n+1} + c_2 a_{n-k_1} r_{n-k_1} + c_3 a_{n-k_2} v_{n-k_2}) w^n = w^{sp_1-1} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right). \tag{2.31}$$

Following analogous steps to the case of the two-term fractional differential equation, we have

- case 1: For $n \le k_1$,

$$c_1 s_{n+1} a_{n+1} = \frac{d^n}{dw^n} w^{sp_1-1} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right)_{w=0}.$$

- case 2: For $k_1 + 1 \le n \le k_2$,

$$c_1 a_{n+1} s_{n+1} + c_2 a_{n-k_1} r_{n-k_1} = \frac{d^n}{dw^n} w^{sp_1-1} f\left(w^q, \sum_{n=0}^{\infty} a_n w^n\right)_{w=0}.$$

- case 3: For $n > k_2$ and $sp_1 - 1 = j$, we have

$$c_1 a_{n+1} s_{n+1} + c_2 a_{n-k_1} r_{n-k_1} + c_3 a_{n-k_2} v_{n-k_2} = \frac{1}{(n-j)!}\left(\frac{\mathrm{d}^{n-j}}{\mathrm{d}w^{n-j}} f(w^q, \sum_{n=0}^{\infty} w^n)\right)\Big|_{w=0}. \tag{2.32}$$

Remark 2.2.1. The identify (2.32) holds true under the continuity and smoothness assumption of the function $f(t, u(t))$.

The general case happen under the assumptions continuity and smoothness for $n > k_2$, which is represent in Eq.(2.32).

Example 2.2.1. *Consider the Bagely-Torvik initial value problem*

$$D_{0+}^{\frac{5}{2}}u(t)+D_{0+}^{2}u(t)-2\sqrt{\pi}D_{0+}^{\frac{1}{2}}u(t)+4u(t)=f(t),\ t\in[0,1], \tag{2.33}$$

$$u(0)=u'(0)=u''(0)=0, \tag{2.34}$$

where

$$f(t)=4t^9-\frac{131072}{12155}t^{\frac{17}{2}}+72t^7+\frac{49152}{143\sqrt{\pi}}t^{\frac{13}{2}}.$$

The exact solution is $u(t)=t^9$.

This example has been discussed in [22] using a Chebyshev spectral method, where the solution has been approximated by the shifted Chebyshev polynomials with different degrees. The exact solution was obtained by considering the shifted Chebyshev polynomial of degree 9.

Applying the proposed series method we have,

$$\alpha_1=\frac{5}{2}=\frac{p_1}{q_1},\alpha_2=2=\frac{p_2}{q_3},\alpha_3=\frac{1}{3}=\frac{p_3}{q_3},q=lcm(q_1,q_2,q_3)=2,$$
$$s=1,r=2,v=1,k_1=sp_1-rp_2-1=0 \text{ and } k_2=sp_1-vp_3-1=3.$$

We expand the solution in infinite series of the form $u(t)=\sum_{n=0}^{\infty}a_nt^{\frac{n}{2}}$. The initial condition in (2.34) yields $a_0=0$. Let $t=w^2$ then

$$f(w^2,u(w))=4w^{18}-\frac{131072}{12155}w^{17}+72w^{14}+\frac{49152}{143\sqrt{\pi}}w^{13}-4\left(\sum_{n=0}^{\infty}a_n w^n\right). \tag{2.35}$$

Since $f(w^2,u(w))$ is continuous with respect to w, and smooth with respect to u, we have

$$a_{n+1}=0, \text{ for } n\le k_2=3.$$

Thus $a_1 = a_2 = a_3 = a_4 = 0$. The function $f(w^2, u(w))$ satisfies the assumptions of algorithm for three-terms (FDE), and it holds that

$$\frac{d^m}{dw^m} f(w^2, u(w))|_{w=0} = \begin{cases} -4\times 5!\,a_5 & \text{if } m=5 \\ -4\times 6!\,a_6 & \text{if } m=6 \\ \vdots & \vdots \\ 13!(\frac{49152}{143\sqrt{\pi}} - 4a_{13}) & \text{if } m=13 \\ 14!(72-4a_{14}) & \text{if } m=14 \\ -4\times 15!\,a_{15} & \text{if } m=15 \\ -4\times 16!\,a_{16} & \text{if } m=16 \\ 17!(-\frac{131072}{12155} - 4a_{17}) & \text{if } m=17 \\ 18!(4-4a_{18}) & \text{if } m=18 \\ -4!\,a_{19} & \text{if } m=19 \\ \vdots & \vdots \end{cases} \tag{2.36}$$

For $n \geq 5$, using Equations (2.11), (2.33) and (2.36) yields

$$a_{n+1}\, s_{n+1} + a_n\, r_n - 2\sqrt{\pi}\, a_{n-3}\, v_{n-3} = \frac{1}{(n-4)!}\left(\frac{d^{n-4}}{dw^{n-4}} f(w^2, \sum_{n=0}^{\infty} w^n)\right)\Bigg|_{w=0} \tag{2.37}$$

where $s_{n+1} = \frac{\Gamma(\frac{n+1}{2}+1)}{\Gamma(\frac{n+1}{2}-\frac{3}{2})}$, $r_n = \frac{n(n-2)}{4}$, and $v_{n-3} = \frac{\Gamma(\frac{n-3}{2}+1)}{\Gamma(\frac{n-3}{2}+\frac{1}{2})}$.

We now apply the last recursion together with $a_1 = a_2 = a_3 = a_4 = 0$, to compute a_{n+1} for $n \geq 4$. For $n = 4$, we have

$$a_5\, s_5 + a_4\, r_4 - 2a_1\, v_1 = f(0, u(0)) = 0,$$

thus $a_5 = 0$.

Applying analogous arguments yield $a_6 = a_7 = a_8 = a_9 = 0$.

For $n = 9$,

$$a_{10} s_{10} + a_9 r_9 - 2\sqrt{\pi} a_6 v_6 = -4a_5.$$

Since $a_5 = a_6 = a_9 = 0$, then $a_{10} = 0$.

Applying analogous arguments yield $a_{11} = a_{12} = a_{13} = a_{14} = a_{15} = a_{16} = a_{17} = 0$.

For $n = 17$, we have

$$a_{18} s_{18} + a_{17} r_{17} - 2\sqrt{\pi} a_{14} v_{14} = \frac{49152}{143\sqrt{\pi}} - 4a_{13}.$$

Since $a_{17} = a_{14} = a_{13} = 0$, then $a_{18} = 1$.

For $n = 18$, we have $a_{19} = 0$.

Proceeding in the same manner, we have $a_{n+1} = 0$ for $n \geq 18$. Thus

$$u(t) = a_{18} t^{\frac{18}{2}} = t^9,$$

and the exact solution of the problem (2.33)-(2.34) is obtained.

2.3 Fractional Differential Equations with Non Constant Coefficients

In the previous sections, we discussed only multi-term fractional differential equations with constant coefficients. In this section, we discuss a more general case with non constant coefficients. We follow the same algorithm in the problem (2.29)-(2.30), with minor changes in computing a_n's.

Example 2.3.1. *Consider the initial fractional value problem (Cauchy problem)*

$$D_{0+}^{1.5} u(t) + 2D_{0+} u(t) + 3\sqrt{t}\, D_{0+}^{0.5} u(t) + (1-t)u(t) = g(t), \tag{2.38}$$

$$u(0) = u'(0) = 0. \tag{2.39}$$

where $g(t) = \frac{2t^{0.5}}{\Gamma(1.5)} + 4t + \frac{4t^2}{\Gamma(1.5)} + (1-t)t^2$, *and the exact solution of this problem is* $u(t) = t^2$.

This example has been discussed in [23] using the spline collocation method.

For $q = 2$, we expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{2}}$. Let $w = t^{0.5}$, then Eq.(2.38) reduces to

$$a_1 s_1 + \sum_{1}^{2} (a_{n+1} s_{n+1} + n a_n) w^n + \sum_{n=3}^{\infty} (a_{n+1} s_{n+1} + n a_n + 3 a_{n-2} v_{n-2}) w^n = w^2 f(w^2, u(w)),$$

where

$$f(w^2, u(w)) = g(w^2) - (1 - w^2) \left(\sum_{n=0}^{\infty} a_n w^n \right),$$

$$s_{n+1} = \frac{\Gamma(0.5n + 1.5)}{\Gamma(0.5n)} \text{ and } v_{n-2} = \frac{\Gamma(0.5n)}{\Gamma(0.5n - 0.5)}.$$

From the initial condition (2.3.1), we have $a_0 = 0$. Since $f(w^2, u(w))$ is continuous with respect to w and smooth with respect $u(w)$, we have

$$a_{n+1} = 0, \text{ for } n < 2.$$

For $n = 2$, we have

$$a_{n+1} s_{n+1} + n a_n = f(0, u(0)).$$

Since $a_2 = 0$, yields $a_3 = 0$.

For $n \geq 3$, we have

$$a_{n+1} s_{n+1} + n a_n + 3 a_{n-2} v_{n-2} = \frac{1}{(n-2)!} \frac{\mathrm{d}^{n-2}}{\mathrm{d}w^{n-2}} (f(w^2, u(w))|_{w=0}, \tag{2.40}$$

where

$$\frac{d^m}{dw^m}(f(w^2,u(w))|_{w=0} = \begin{cases} \frac{2}{\Gamma(1.5)} - a_1 & \text{if } m=1 \\ 8-2a_2 & \text{if } m=2 \\ 2!(a_1-a_3) & \text{if } m=3 \\ 4!(\frac{4}{\Gamma(1.5)}+1)+4!(a_2-a_4) & \text{if } m=4 \\ 5!(a_3-a_5) & \text{if } m=5 \\ -6!+6!(a_4-a_6) & \text{if } m=6 \\ 7!(a_5-a_7) & \text{if } m=7 \\ \vdots & \vdots \end{cases} \tag{2.41}$$

we apply recursion (2.40) together with $a_0=a_1=a_2=a_3=0$, to compute a_{n+1} for $n \geq 3$.

For $n=3$, we have

$$a_4 s_4 + 3a_3 + 3a_1 v_1 = \frac{2}{\Gamma(1.5)}.$$

Since $a_1=a_3=0$, then $a_4=1$.

For $n=4$, we have

$$a_5 s_5 + 4a_4 + 3a_2 v_2 = 4 - a_2.$$

Since $a_2=0$ and $a_4=1$, yield $a_5=0$.

For $n=5$, we have $a_6=0$.

For $n=6$, we have

$$a_7 s_7 + 6a_6 + 3a_4 v_4 = (\frac{4}{\Gamma(1.5)}+1)+(a_2-a_4).$$

Since $a_2=a_6=0$ and $a_4=1$, we have $a_7=0$.

For $n=7$, we have $a_8=0$.

For $n=8$, we have

$$a_9 s_9 + 8a_8 + 3a_6 v_6 = -1 + a_4.$$

Since $a_6 = a_8 = 0$ and $a_4 = 1$, we have $a_9 = 0$.

Following this process, we have $a_{n+1} = 0$ for $n \geq 5$.

Then the solution of the problem (2.38)-(2.3.1) is $u(t) = t^2$, which is the exact solution of Cauchy problem.

Example 2.3.2. *Consider the non linear initial fractional value problem*

$$t D_{0^+}^{\frac{4}{3}} u(t) + D_{0^+}^{\frac{1}{3}} u(t) + t^{\frac{19}{6}} D_{0^+}^{\frac{1}{6}} u(t) = \frac{33 t^{\frac{8}{3}}}{4\Gamma(\frac{8}{3})} + \frac{1296 u^2(t)}{935\Gamma(\frac{5}{6})}, \tag{2.42}$$

with

$$u(0) = u'(0) = 0. \tag{2.43}$$

The exact solution for this problem is $u(t) = t^3$.

As the same criterion in the previous example, we apply the proposed series method with $q = 6$. We expand the solution in an infinite series of the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\frac{n}{6}}$. Let $w = t^{\frac{1}{6}}$, then Eq.(2.42) produce the following summation

$$\begin{aligned} &\sum_{n=0}^{19} (a_{n+1} s_{n+1} + a_{n+1} r_{n+1}) w^n + \sum_{n=20}^{\infty} (a_{n+1} s_{n+1} + a_{n+1} r_{n+1} + a_{n-19} v_{n-19}) w^n \\ &= \frac{33 w^{17}}{4\Gamma(\frac{8}{3})} + w \left(\frac{1296 (\sum_{n=0}^{\infty} a_n w^n)^2}{935\Gamma(\frac{5}{6})} \right) \end{aligned} \tag{2.44}$$

where $s_n = \frac{\Gamma(\frac{n}{6}+1)}{\Gamma(\frac{n}{6}-\frac{1}{3})}$, $r_n = \frac{\Gamma(\frac{n}{6}+1)}{\Gamma(\frac{n}{6}+\frac{2}{3})}$ and $v_n = \frac{\Gamma(\frac{n}{6}+1)}{\Gamma(\frac{n}{6}+\frac{5}{3})}$.

It is clear that for $n < 17$, $a_n = 0$.

For $n = 17$, we have

$$a_{18} s_{18} + a_{18} r_{18} = \frac{33}{4\Gamma(\frac{8}{3})},$$

which yields $a_{18} = 1$.

For $n = 37$, we have

$$a_{38} s_{38} + a_{38} r_{38} + a_{18} v_{18} = \frac{324}{\Gamma(\frac{8}{3})},$$

then $a_{38} = 0$.

Following analogous steps, we have $a_n = 0$ for $n > 37$. Thus

$$u(t) = a_{18} t^{\frac{18}{6}} = t^3,$$

which is the exact solution.

2.4 Conclusion

In this thesis, we present a new algorithm for obtaining a series solution for multi-term fractional differential equations of Caputo's type. The terms of the series are obtained sequentially, and the idea is analogous to the Taylor series method, but we overcome the difficulty of computing iterated fractional derivatives, which do not commute in general. We applied the new algorithm to several types of multi-term fractional differential equations, where accurate solutions as well as exact solutions in closed forms have been obtained. We also compared our results with the ones obtained by the Adomian decomposition method (ADM) for the two-term fractional differential equations. It is noted from the solved examples that, the new algorithm is more efficient than the (ADM), as it gives the exact solution in closed form while the (ADM) does not, it produces more accurate solutions, and it can be applied for some problems where we can not do them with the (ADM).

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