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An Efficient Method for Solving a Discrete Orthogonal Approximation to Fractional Boundary Value Problems

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This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Professor Muhammed Ibrahem Syam

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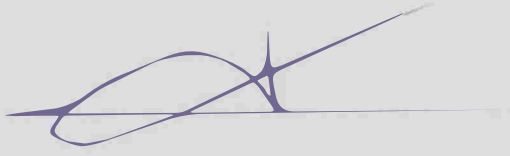
Declaration of Original Work

I, Mwaffag Husein Nahar Sharadga, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this master dissertation entitled "*An Efficient Method for Solving a Discrete Orthogonal Approximation to Fractional Boundary Value Problems*". hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Professor Muhammed Ibrahem Syam, in the College of Science at the UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

Student's Signature  Date 28/4/2015

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Copy 1 of 16

Abstract

In this thesis we developed a numerical method for solving a class of nonlinear fractional boundary value problems using the fractional order Legendre Tau-path following method. Theoretical and numerical analyses are presented. The numerical results showed that this method works properly and efficiently.

Keywords: Nonlinear Fractional Boundary Value problems, Caputo derivative, Tau method, path-following method, fractional order Legendre functions.

Title and Abstract (in Arabic)

طريقة فعالة لحل التقريب المتعامد المنفصل للمسائل المحيطية

في هذه الأطروحة قمنا بتطوير طريقة دقيقة لحل مجموعة من المعادلات المحيطية الكسرية غير الخطية بالأعتماد على طريقة" تاو والتتابع" باستخدام دوال ليجيندرا ذات الأسس الكسرية. وقد عرضنا تحليلا نظريا وعدديا لهذه الطريقة حيث اثبتت النتائج العددية بأن هذه الطريقة تعمل بشكل جيد و فعال.

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Dedication

To the souls of my parents, to yours I write these words , not that I can return even a fraction of what you have given , but to remind myself and simply extend my deepest gratitude, prayer of forgiveness. May Allah's mercy be upon you and may Allah reward you with paradise in the best way parents can be rewarded.

To human role models, to the owner of the high-creation, to the person who line illuminated imprints in my scientific career, to the great Prof. Muhammed Syam, I lift my hat to him, offering

for him.

Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

In this thesis, we discussed a numerical technique for solving a class of fractional boundary value problems of the form:

$$D^{\alpha} y = f(x, y, y'), x \in]-1,1[, 1 < \alpha \leq 2 \quad (1)$$

$$y(-1) = y_{-} \text{ and } y(1) = y_{+} \quad (2)$$

where y_{-} and y_{+} are constants and $f \in C^2([-1,1] \times \Re \times \Re)$. The fractional derivative is described in the Caputo sense. [62].

Fractional calculus is a branch of Mathematics that deals with generalization of the well-known operations of differentiations and integrations to arbitrary non-integer orders. Fractional derivative provides an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with the classical integer-order models, in which such effects are in fact neglected. The idea of modelling dynamic systems by fractional deferential equations can be used in many fields of Science and engineering including electrochemical process [27] and [32], dielectric polarization [26], earthquakes [30], fluid-dynamic traffic model [31], solid mechanics [51], bioengineering [41-43], and economics [7]. Fractional derivative and Fractional integrals also appear in theory of control of dynamical systems when control system and the controller are described by fractional differential equations.

In the recent years, a number of methods have been proposed and applied successfully to approximate various types of fractional differential equations. The most used methods are Adomian Decomposition Method [20] and [45-46], Variational Iteration Method [14,56,60,63,64], Homotopy Perturbation Method [44]

and [6] , Homotopy Analysis Method [29], Fractional Differential Transform Method[4-5], [13], and [21-22], Power Series Method [47] and other methods [15-17], [35] and [40].

Recently, wavelet based operational matrix has been also applied for the solutions of the fractional differential equations. In 2010, Li et al. [39] constructed Haar wavelet operational matrix of fractional integration with the use at Block plus functions and successfully applied for getting solution of special type of fractional differential equation. In the same year Li [38] used another operational matrix based on chebyshev wavelet for the same problem. In 2011, Saadatmandi and Dehghan [52] used the concept of orthogonal polynomial and constructed for Legendre operational matrix of differentiation for solving such problems. Bernstein polynomials have been used for solving numerically partial differential equations [8]. More recently Bernstein approximation have used for stable solution of problems of Abel inversion [54-55] and generalized Abel integral equations arising in classic theory of elasticity [48].

To identify the difficulties of this problem, we first apply our approach on the ordinary derivative when $\alpha = 2$. This issue is presented in chapter two. In addition, it starts from the linear problem. Then, there is a discussion on the solution of non-linear problem. Examples and conclusions are presented is this chapter.

In Chapter Three, we present our approach for solving problem (1)-(2). As we did in chapter two, we start from linear fractional boundary value problem. We present some of our experimental examples to explain how our approach works efficiently. Furthermore, some conclusions will be drown.

When we apply Tau method to solve non-linear fractional boundary value problems, the result is a non-linear system of equations. The classical methods for solving such problems are Newton and Secant methods. However, the necessary condition for these methods to converge to the exact solution is to start with a "good" initial guess. But such initial guess is not always available. For this reason, most of researches do not use the Tau method for this purpose.

Siyyam and Syam, [62], used the path- following method to trace implicitly defined curves. This thesis tries to relate the Tau-method and the path-following method. In this approach, the initial guess is avoided for solving the non-linear system.

The rest of this chapter will be organised as follows; in Section1, there is some historical information about the spectral methods. The idea of the Tau method will be presented in section 2. In section 3 we talk about the fractional derivatives. Section 4, has the main properties of the Legendre polynomials that are used hereafter. Finally, section 5 presents the idea of the path-following method.

1.1 Spectral Methods

Spectral methods may be viewed of an extreme development the class of discretization schemes for differential equations, of the form $L(V) = F$, known generically as the Method of Weighted Residuals (MWR).

The key elements of the MWR are the trial functions and the test functions. The first set of functions used as the basis functions for a truncation series expansion for the function. The second set of functions is used to ensure that the differential equation $L(V) = F$ is satisfied with minimum error as possible as we can by the

approximate solution v_N . It can be got that by minimizing the residual $L(V_N)-F$, i.e. the remainder when substituting the approximate solution in the differential equation. This is equivalent to force the residual to be orthogonal with respect to the test functions.

The choice of the trial functions is one of the main points that distinguish spectral methods from the finite difference methods and the finite elements methods. These functions have usually infinitely many derivatives on the real line such as the Legendre, Laguerre and Chebyshev polynomials. On the other hand in the finite element methods the domain is divided into subdomains which means that the trial functions is specified in each element. So, the trial functions are local in character. In the finite difference methods the trial functions are likewise local.

The choice of the test functions distinguishes between the three most famous Spectral methods which are Galerkin, Tau and Collocation methods. In the first method, the test functions are the same as the trial functions which implies that each function must satisfy the boundary conditions. Also, in this method we force the residual to be orthogonal with every trial function. In the Tau method, none of the test functions should satisfy the boundary conditions. Hence, supplementary equations are used to apply the boundary conditions. In the Collocation approach, the test functions are chosen to be the translated Dirac delta functions centered at special points. Usually, these points are chosen to be the extreme of some orthogonal polynomials such as the Legendre, Laguerre and Chebyshev polynomials. In this approach the differential equation is forced to be satisfied exactly at the collocation points.

In 1934, Slater, [58], found and used the collocation method. This method was the first to appear because it was the simplest Spectral methods. In 1937 Frazer

[23], developed it and used in the ordinary differential equations. In 1938, Duncan [18], established for the first time that the proper choice of trial functions and test functions and he found that it is crucial to the accuracy of the Solution. This method was reviewed in 1959 by Shuleshko [53] and in 1966 by Pomraning [49]. These studies discussed some solutions for initial value problems. In 1966, Kaplan [33], developed this method for the boundary value problems. In 1970, Hall [28], used Spectral Collocation methods to find the Fourier and pseudospectral methods.

The first series application of spectral method to partial differential equations was the Galerkin method in 1938 by Duncan [19]. It became practical for high resolution calculations of such nonlinear problems after transformation methods; it was developed in 1970 by Fuller [24].

The Tau method is a modification of the Galerkin method which is applicable to problems with non-periodic boundary conditions. In 1938, Lanczos [36], developed the Spectral Tau method although it is too difficult to apply to non- linear problems.

The first unifying mathematical assessment of the theory of Spectral methods was contained in the monograph by Gattlieb and Orszay [25]. Since then, the theory has been extended to cover some of the problems. For more historical details, see [26].

1.2 Tau method for solving Boundary Value Problem

This section discusses the idea of Tau method for solving boundary value problem (BVP). Let us consider the following BVP

$$y'' = f(x), \quad x \in [a,b] \tag{1.2.1}$$

$$y(a)=y_{-} \quad , \quad y(b)=y_{+}.$$

where y_{-} , y_{+}, a and b are constants.

Let us assumes that $\{\varphi_i\}_{i=0}^{N}$ be a complete set of linearly independent orthogonal function with respect to weight function $w(x)$ on $[a,b]$ such that $\varphi_0,\varphi_1,.....\varphi_{N+2}\in C^2(\Re)$. These functions are called the trail functions. Approximate $y(x)$ and $f(x)$ as

$$f_N(x)=\sum_{k=0}^{N} a_k\varphi_k(x) \quad , \quad \bar{y}_N(x)=\sum_{k=0}^{N+2} y_k\varphi_k(x).$$

Then the residual is

$$R(\bar{y}_N)=\bar{y}_N''-f_N(x).$$

The second derivative of y_N is given by

$$\bar{y}_N''(x)=\sum_{k=0}^{N} y_k\varphi_k''(x).$$

The idea of Tau method is to select the undetermined coefficients y_0 , y_1 ,... y_N, such that

$$\langle R(\bar{y}_N),\varphi_J\rangle=\int_a^b \varphi_J(x)R(\bar{y}_N)w(x)dx=0$$

for $J=0,1,...N-2$. Thus

$$\sum_{k=o}^{N} y_k\int_a^b \varphi_J(x)\varphi_k''(x)w(x)dx=f_J\int_a^b \varphi_J^2(x)w(x)dx \qquad (1.2.2)$$

for $J=0,1,...N-2$. From the boundary conditions, the following equations are generated:

$$y_- = \sum_{k=0}^{N} y_k \varphi_k(a) \quad , \quad y_+ = \sum_{k=0}^{N} y_k \varphi_k(b) \tag{1.2.3}$$

Equations (1.2.2) and (1.2.3) imply the following $(N+1)\times(N+1)$ linear system

$$AY = X$$

where

$$A = \begin{bmatrix} \int_a^b \varphi_0\varphi_0'' w dx & \int_a^b \varphi_0\varphi_1'' w dx & \cdots & \int_a^b \varphi_0\varphi_N'' w dx \\ \int_a^b \varphi_1\varphi_0'' w dx & \int_a^b \varphi_1\varphi_1'' w dx & \cdots & \int_a^b \varphi_1\varphi_N'' w dx \\ \vdots & \vdots & \vdots & \vdots \\ \int_a^b \varphi_{N-1}\varphi_0'' w dx & \int_a^b \varphi_{N-1}\varphi_1'' w dx & \cdots & \int_a^b \varphi_{N-1}\varphi_N'' w dx \\ \varphi_0(a) & \varphi_1(a) & \cdots & \varphi_N(a) \\ \varphi_0(b) & \varphi_1(b) & \cdots & \varphi_N(b) \end{bmatrix}$$

$$Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ \vdots \\ \vdots \\ y_{N-2} \\ y_{N-1} \\ y_N \end{bmatrix} , X = \begin{bmatrix} f_0 \int_a^b \varphi_0^2 w(x) dx \\ f_1 \int_a^b \varphi_1^2 w(x) dx \\ \vdots \\ f_{N-2} \int_a^b \varphi_{N-2}^2 w(x) dx \\ y_- \\ y_+ \end{bmatrix}.$$

For more details, see [10] - [12] - [25] and [57].

1.3 Fractional Derivatives

Recently, many papers on fractional boundary value problems have been studied extensively. Several forms of them have been proposed in standard models, and there has been significant interest in developing numerical schemes for their solutions. Several numerical techniques are used to solve such problems such as Laplace and Fourier transforms Adomain decomposition and variational iteration methods, eigenvector expansion, differential transform, finite differences methods, power series method, collocation method, and wavelet method. Many applications of fractional calculus on various branches of science such as engineering, physics, and economics are investigated. Considerable attention has been given to the theory of fractional ordinary differential equations and integral equations. Additionally, the existence of solutions of ordinary and fractional boundary value problems using monotone iterative sequences has been investigated.

Definition (1.3.1): The Riemann-Liouville fractional integral operator I^{α} of order $\alpha > 0$ on the usual Lebesgue space $L_1[0,1]$ is given by $I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt$,

$I^0 f(x) = f(x)$, where $\Gamma(\xi) = \int_o^{\infty} t^{\xi-1} e^{-t} dt$ is the Euler Gamma function.

For any $f \in L_1[0,1], \alpha, \beta \geq 0$, and $\gamma > -1$, the following properties hold:

(1) I^{α} exists for any $x \in [0,1]$,

(2) $I^{\alpha} I^{\beta} = I^{\alpha+\beta}$,

(3) $I^{\alpha} I^{\beta} = I^{\beta} I^{\alpha}$,

(4) $I^{\alpha} x^{\gamma} = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)) x^{\alpha+\gamma}$.

Definition (1.3.2): The Caputo fractional derivative of order α is defined by

$$D^{\alpha} f(x) = I^{n-\alpha} D^{n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt,$$

provided that the integral exists, where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of the positive real number α, $x > 0$.

The Caputo fractional derivative satisfies the following properties for $f \in L_1[0,1]$ and $\alpha, \beta \geq 0$.

(1) $D^{\alpha} I^{\alpha} f(x) = f(x)$,

(2) $I^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)(x^k / k!)$,

(3) $D^{\alpha} D^{\beta} f(x) = D^{\beta} D^{\alpha} f(x) = D^{\alpha+\beta} f(x), x > 0$,

(4) $D^{\alpha} c = 0$, where c is constant,

(5) $D^{\alpha} x^{\gamma} = \begin{cases} 0, & \gamma < \alpha, \gamma \in \{0,1,2,...\} \\ (\Gamma(\gamma+1)/\Gamma(\gamma-\alpha+1))\, x^{\gamma-\alpha}, & otherwise \end{cases}$,

(6) $D^{\alpha}(\sum_{i=0}^{m} c_i f_i(x)) = \sum_{i=0}^{m} c_i D^{\alpha} f_i(x)$, where $c_0, c_1, c_2, ..., c_m$ are constants.

For example, let us compute $D^{3/2} x^2$. Let $f(x) = x^2$. Then

$$D^{3/2} f(x) = \frac{1}{\Gamma(n-3/2)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{3/2-n+1}} dt,$$

where $n = \left[\frac{3}{2}\right] + 1 = 2$. Then,

$$D^{3/2} f(x) = \frac{1}{\Gamma(1/2)} \int_0^x \frac{2}{(x-t)^{1/2}}\, dt = \frac{-1}{\Gamma(1/2)} 2^2 \left[(x-t)^{1/2}\right]_0^x = \frac{4\sqrt{x}}{\Gamma(1/2)} = \frac{4\sqrt{x}}{\sqrt{\pi}}.$$

1.4 Properties of Legendre Polynomials

This section mentions the definition of one type of the orthogonal polynomial which is the Legendre polynomials. It also states some properties of these polynomials which are used hereafter.

Definition (1.4.1): The Legendre polynomials $\{P_k(x), k = 0,1,.....\}$ are the Eigen-functions of the Singular Sturm-Liouville Problem

$$\left(\left(1-x^2\right)P'_k(x)\right)' + k(k+1)P_k(x) = 0$$

Among the properties of the Legendre Polynomials the following three properties are listed:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} (n & \\ 0 & \text{if } n \neq m \end{cases} \qquad (1.4.1)$$

for $n, m \geq 0$

$$P_{n+1}(x) = \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x), n \geq 1, \qquad (1.4.2)$$

where $P_0(x) = 1$ and $P_1(x) = x$, and the endpoint relation

$$P_n(\pm 1) = (\pm 1)^n, \ n \geq 0. \qquad (1.4.3)$$

This thesis studies the relation between the coefficients of the Legendre series expansion of a given function and the coefficients of the Legendre series expansion of its first and second derivatives.

Suppose that $f(x) \in c^2[-1,1]$ and $f''(x)$ is a piecewise continuous function on $[-1,1]$. To approximate the first and second derivative of f, let

$$\Pi f(x) = \frac{d^q}{dx^q} f(x),$$

for $q = 1, 2$, then

$$\Pi f(x) = \sum_{n=0}^{r} f_n^{(q)} P_n(x)$$

which converges uniformly on $[-1,1]$, where

$$f_n^{(1)} = (2n+1) \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{\infty} f_p , \tag{1.4.4}$$

$$f_n^{(2)} = (n+0.5) \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{\infty} [p(p+1) - n(n+1)] f_p , \tag{1.4.5}$$

and

$$f_n = (n+0.5) \int_{-1}^{1} f(x) P_n(x) dx. \tag{1.4.6}$$

For more details, see [61].

If the domain is $[0,1]$, the shifted Legendre polynomials are used and will be defined by:

$$S_i(z) = P_i(2z-1).$$

Using the change of variable $x = 2z-1$, $S_i(z)$ has the following properties:

$$(1)\int_0^1 S \qquad 1 \qquad 2i \qquad \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$(2)\, S_{i+1}(z) = \frac{2i+1}{i+1}(2z-1)S_i(z) - \frac{i}{i+1}S_{i-1}(z), \quad \text{for } i \geq$$

$$(3)\, S_i(0) \qquad S_i(1) = 1.$$

The closed form of $S_i(z)$ is given by

$$S_i(z) = \sum_{j=0}^{i} (-1)^{i+j} \frac{(i+j)!}{(i-j)!} \frac{z^j}{(j!)^2}.$$

One of the efficient methods for solving fractional differential equations of order $\alpha > 0$ is using the series expansion of the form $\sum_{j=0}^{i} c_j t^{\alpha j}$. For this reason; this study defines the fractional-order Legendre function by F_i^{α}

of variable $z = t^{\alpha}$, one can show that

$$(1)\left((t-t^{i+}\right. \qquad ,$$

$$(2)\, F_i \qquad \frac{}{i+1} \qquad (t) - \frac{i}{i+1} F_{i-1}^{\alpha}(t), i$$

$$(3)\, F_0^{\alpha}(t) = 1 \text{ and } F_i^{\alpha}$$

$$(4)\, F_i$$

$$(5)\int_0^1 F_i^{\alpha}(t) F_J^{\alpha}(t) w(t)\, dt = \frac{1}{(2i+1)\alpha}\delta_{i,} \text{ where } w(t) = t^{\alpha-1}, t \in (0,1),$$

$$(6)\, F_i^{\alpha}(t) = \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{t^{\alpha}}{(k!)^2}.$$

Using properties of Caputo fractional derivative one can get

$$D^{\alpha} F_i^{\alpha}(t) = \sum_{k=m}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)} \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} t^{(k-1)k},$$

where $m = [\alpha]+1$. In the next theorem, we state one of the results which we will use in this thesis.

Theorem (1.4.1): for any nonnegative integers p *and* q,

$$F_p^{\alpha}(t)F_q^{\alpha}(t)=\sum_{k=0}^{[(p+q/2)]}A_{2k}F_{P+q-2k}^{\alpha}(t)$$

where

$$A_{2k}=\left((2p+2q-4k+1)(2k)!(2p-2k)!\times(2q-2k)!2^{p+q-k}((p+q-k)!)^2\right)$$

$$\times(2^{p+q}$$

Proof: For any nonnegative integers p *and* q,

$$P_p(x)P_q(x)=\sum_{k=0}^{[(p+q/2)]}A_{2k}^{\alpha}P_{p+q-2k}(x),$$

where

$$A_{2k}=\left((2p+2q-4k+1)(2k)!(2p-2k)!\times(2q-2k)!2^{p+q-k}((p+q-k)!)^2\right)$$

$$\times(2^{p+q-k}(2p+2q-2k+1)(k!)^2\times((p-q)!)^2((q-k)!)^2(2p+2q-2k)!)^{-1}.$$

For the proof of this case, see [14]. Using the change of variables $x=2z-1$ *and* $z=t^{\alpha}$, we obtain the result of the theorem.

Theorem (1.4.2): Let $u\in C[0,1]$ *and* $u'(t)$ be a piecewise continuous function on $[0,1]$. Then, $u(t)$ can be written in the infinite expansion as $u(t)$

$$u(t)=\sum_{k=0}^{\infty}u_kF_k^{\alpha}(t),$$

where

$$u_k=(2i+1)\alpha\int_0^1 u(t)F_k^{\alpha}(t)w(t)\,dt,\ and\ w(t)=t^{\alpha}.$$

Proof: If $u \in C[-1,1]$ *and* $u(x)$ is a piecewise continuous function on $[-1,1]$. $\sum_{k=0}^{\infty} v_k p_k(x)$ converges uniformly to $u(x)$ on $[-1,1]$, see [12] and [21-22]. Let $h:[0,1] \to [-1,1]$ by $h(t) = 2t^{\alpha} - 1$. Since $h(t)$ is a bijective continuous function, $\sum_{k=0}^{\infty} u_k^{\alpha} F_k^{\alpha}(t)$ converges uniformly to $u(t)$ on $[0,1]$. The values of u_k follows from the orthogonality relation of $\{F_i^{\alpha}(t) : i = 0,1,2,...\}$ with respect to the weight function $w(t) = t^{\alpha-1}$ on $[0,1]$. Theorem (1.4.3) gives the relation between the coefficient of the series solution of $D^{\alpha}u(t)$ and the coefficient of the series expansion of $u(t)$.

Theorem (1.4.3): let $u \in C^2[0,1]$ *and* $u'''(t)$ be a piecewise continuous function on $[0,1]$. Then $\sum_{k=0}^{\infty} u_k^{(\alpha)} F_k^{\alpha}(t)$ converges uniformly *on* $[0,1]$ *to* $D^{\alpha}u(t)$ *where*

$$u_k^{(\alpha)} = \sum_{j=k+1}^{\infty} a_{jk} u_j,$$

$$a_{jk} = (2k+1)\alpha \int_0^1 D^{\alpha} F_j^{\alpha}(t) w(t)\, dt, \qquad k = 0,1,2,..., j = k+1, k+2,....$$

Proof: let $S_n(t) = \sum_{k=0}^{n} u_k F_k^{\alpha}(t)$ for $n = 0,1,2,....$ From Theorem (1.4.2). $S_n(t)$ Converges uniformly to $u(t)$ *on* $[0,1]$. Since $u \in C^2[0,1]$ and $u'''(t)$ is a piecewise continuous function on [1,0], $\frac{d^2}{dt^2}\left(\lim_{n\to\infty} s_n(t)\right) = \lim_{n\to\infty}\left(\frac{d^2}{dt^2} s_n(t)\right)$ and $(d^2/dt^2)s_n(t)$ converges uniformly to $(d^2/dt^2)u(t)$ *on* $[0,1]$. Thus, $\int_0^x (s_n''(t)/(x-t)^{\alpha})dt$ converges uniformly to $\int_0^x (u''(t)/(x-t)^{\alpha})dt$ *on* $[0,1]$ which gives the result of the theorem. The value of a_{jk} follows from the orthogonality

relation of $\{F_i^{\alpha}(t): i = 0,1,2,...\}$ with respect to the weight function $w(t) = t^{\alpha}$ *on* $[0,1]$.

As a result of the last two theorems, one can show that for

$$\alpha = 1, \quad u_k^{(1)} = \sum_{j=k+1}^{\infty} a_{jk} u_j$$

where $$a_{jk} = (2k+1)\int_0^1 F_j^{\alpha'}(t) F_k^{\alpha}(t) w(t)\, dt$$

for $k = 0,1,2,..., \quad j = k+1, k+2,...$

1.5 Path- following methods

In this section, we give an idea about the path- following methods, for solving linear and nonlinear systems, that will be very helpful in this thesis.

Path-following methods have long served as useful tools in modern mathematics. Their use can be traced back at least to 19th Century. The use of deformations to solve nonlinear systems of equations may be traced back at least to 80 years ago. The classical embedding methods may be regarded as a forerunner of the Predictor-Corrector methods.

Suppose one wish to obtain a solution to a system of n nonlinear equations in n variables, say $F(x) = 0$, where $F: \Re^n \to \Re^n$. For purposes of beginning a discussion, we will assume that F is smooth. When saying a map is smooth, it shall mean that it has as many continuous derivatives as the subsequent discussion requires. This is done to make the statements less cumbersome. Let us consider the situation in which very little a priori knowledge concerning zero point of F is available. Certainly, if on the contrary, a good approximation x_0 of a zero point $\bar{x}$ of

F is available, it is advisable to calculate $\bar{x}$ via a Newton-type algorithm defined by an iterative formula such as

$$x_{i+1} = x_i - A_i F(x_i), i = 0,1,.... \qquad (1.5.1)$$

where A_i is some reasonable approximation of the inverse Jacobian of the $F'(x_i)$. Since we assume that such a priori knowledge is not available, the iteration (1.5.1) will often fail, because poor starting values are likely to be chosen. As a possible remedy, one defines a homotopy or deformation:

$$H : \Re^n \times [0,1] \to \Re^n$$

such that:

$$H(x,0) = G(x), H(x,1) = F(x),$$

where $G : \Re^n \to \Re^n$ is a trivial smooth map having known zero point, say x_0. Then H is also smooth. Typically, one may choose a convex homotopy such as

$$H(x,\lambda) = \lambda F(x) + (1-\lambda) G(x)$$

and attempt to trace an implicitly defined curve $c(S) \in H^{-1}(0)$ from a starting point $(x_0,0)$ to a solution point $(\bar{x},1)$. If this succeeds, then a zero point $\bar{x}$ of F is obtained. Several questions will immediately arise.

1) When is it assured that a curve $c(S) \in H^{-1}(0)$ with $(x_0,0) \in$ range (C) exist and is smooth?
2) How can we numerically trace such a curve?

To answer the first question we need the following

Definition (1.5.1): Let $f : \Re^p \to \Re^q$ be a smooth map. A point $x \in \Re^p$ is called a regular point of f if the Jacobian $f'(x)$ has maximal rank, i.e.; min $\{p,q\}$. A

value $y \in \Re^q$ is called a regular value of f if x is a regular point of f for all $x \in f^{-1}(y)$. Points and values are called singular if they are not regular.

Also, we need the following theorem.

Theorem (1.5.1) (Implicit Function Theorem): Let $H : \Re^{N+k} \to \Re^N$ be a smooth map such that $0 \in range(H)$. Then

$$M = \left\{ x \in \Re^{N+K} : H(x) = 0, x \text{ is a regular point of } H \right\}$$

is a smooth K-dimensional manifold .

Here, the Implicit Function Theorem is applied with $k = 1$. Now, the first question is answered by the Implicit Function Theorem, namely if $(x_0, 0)$ is a regular zero point of H, i.e. the Jacobian $H'(x_0, 0)$ has full rank N, then a curve $c(S) \in H^{-1}(0)$ with initial value $C(S) = (x_0, 0)$, and tangent $C'(0) \neq 0$ will exist at least locally, i.e. on some open interval around zero. Furthermore, if zero is a regular value of H, i.e. all zero points of H are regular points, and then the curve is diffeomorphic to a circle or the real line. This can be seen by a more sophisticated application of the Implicit Function Theorem as given by Milnor (1969).

The curve C, now parameterized with respect to arc length S, may be regarded as the solution of an initial value problem which is obtained by differentiating the equation

$$H(c(s)) = 0 \tag{1.5.2}$$

with respect to s. Thus,

$$H'(C)C' = 0, \ \|C'\| = 1, \ C(0) = (x_0, 0). \tag{1.5.3}$$

Now it is clear that methods for numerically solving initial value problems may be applied to (1.5.3). However, one may suspect that this is an unnatural approach, since (1.5.3) seems to be a more complicated problem than to solve (1.5.2). In fact, we should not lose sight of the fact that the solution curve c consists of zero points of H, and as such it enjoys powerful local contractive properties with respect to iterative methods such as those of Newton-type. Hence, one is led to numerically integrate (1.5.3) very coarsely and then locally use an iterative method for solving (1.5.2) as a stabilizer. For more details, see [27].

Next, we give an idea about the basic principles of path-following methods and the predictor-corrector (PC) methods.

Definition (1.5.2): Let A be an $n\times(n+1)$ matrix with $\text{rank}(A) = n$. The unique vector $t(A) \in \Re^{n+1}$ satisfying the three conditions:

$$(1)\ At = 0;$$
$$(2)\ \|t\| = 1;$$
$$(3)\ det\begin{pmatrix} A \\ t^T \end{pmatrix} > 0;$$

is called the tangent vector induced by A.

Definition (1.5.3): Let A be a $n\times(n+1)$ matrix with maximal rank. Then the Moore-Penrose Inverse of A is defined by

$$A^+ = A^T\left(AA^T\right)^{-1}.$$

Now assume that A be an $n\times(n+1)$ matrix with $rank(A)=n$, and that a decomposition

$$A^T = Q\begin{pmatrix} R \\ 0^T \end{pmatrix}$$

is given, where Q is an $(n+1)\times(n+1)$ orthogonal matrix, and R is a non-singular $n\times n$ upper triangular matrix. If Z denotes the last column of Q, then $AZ=0$ and $\|Z\|=1$, the remaining task is to choose the sign of Z so that

$$det\begin{pmatrix} A \\ Z^T \end{pmatrix} > 0.$$

Now

$$\left(A^T Z\right) = Q\begin{pmatrix} R & 0 \\ 0^T & 1 \end{pmatrix}$$

implies

$$det\begin{pmatrix} A \\ Z^T \end{pmatrix} = det\left(A^T, Z\right) = det(Q)det(R).$$

Hence, $t(A)=\pm Z$ according as the determinant is positive or negative.

Since

$$A^T = Q\begin{pmatrix} R \\ 0^T \end{pmatrix} \quad and \quad A=\left(R^T,0\right)Q^T.$$

One can easily show that

$$A^{+} = Q\begin{pmatrix} (R^{T})^{-1} \\ 0^{T} \end{pmatrix}.$$

Next assume that:

1) $H : \Re^{n+1} \to \Re^{n}$ is a smooth map ;

2) There is a point $u \in \Re^{n+1}$ such that:

   i) $H(u) = 0$;

   ii) The Jacobian matrix $H'(u)$ has maximum rank i.e. $rank(H'(u)) = n$

Then it follows from the Implicit Function Theorem that: there exists a smooth curve $C : J \to \mathbb{R}^{n+1}$ for some open interval J containing zero such 'that for all $a \in J$:

1) $c(a) = u$;

2) $H(c(a)) = 0$

3) Rank $(H'(c(a))) = n$;

4) $c'(a) \neq 0$

By differentiating equation (2) it follows that the tangent $c'(a)$ satisfies the equation

$$H'(c(a))c'(a) = 0$$

and hence, $c'(a)$ is orthogonal to all rows of $H'(c(a))$. This study use the Predictor- Corrector path-following method to numerically trace the solution curve C . The Predictor step will be used is called the Euler -predictor which is given by:

$$V = u + ht(H'(u)) \qquad (1.5.4)$$

where u is a point lying along the solution curve C, $t\left(H'(u)\right)$ is the tangent at u, and $h > 0$ represents a step size. The corrector iteration in this study is called the Gauss-Newton Corrector which is given by solving the equation

$$H'(v)(w-v) = -H(v) \qquad (1.5.5)$$

for w. In this approach only the fixed step size is used throughout the path-following technique. This study uses a particular version of the predictor-corrector method incorporating with an Euler predictor step, and the solution of the Gauss-Newton corrector (1.5.5) as follows

$$w = v - H'\langle v\rangle^{+} H\langle v\rangle,$$

as a corrector step. Note that the traversing is stopped when reaching the level $\lambda = 1$.

For more details, see [22] and [8].

Chapter 2: Second Order Ordinary Boundary Value Problem

In this chapter, the Tau-method is used to solve problem (1)-(2) when $\alpha = 2$. This chapter is divided into two sections. In section one, the case is studied when the differential equation is linear while in section two, the nonlinear case is discussed.

2.1 Numerical Technique for solving linear boundary value problems

This section presents a numerical technique for solving the following problem

$$y'' = a(x)y' + b(x)y + c(x) \tag{2.1.1}$$

$$y(-1) = y_-, y(1) = y_+ \tag{2.1.2}$$

where y_- and y_+ are constants and $a(x), b(x), c(x) \in C[-1,1]$.

Approximate $y(x)$ in terms of the Legendre polynomials as follows

$$y_N(x) = \sum_{i=0}^{N+2} y_i p_i(x).$$

Then, y'_N and y''_N can be written as

$$y'_N(x) = \sum_{i=0}^{N+1} y_i^{(1)} p_i(x),$$

and

$$y''_N(x) = \sum_{i=0}^{N} y_i^{(2)} p_i(x).$$

For y_N, the residual is given by

$$R(y_N) = y''_N - a(x)y'_N - b(x)y_N - c_N(x)$$

where

$$c_N(x) = \sum_{i=0}^{N} c_i p_i(x)$$

is an approximate function to $c(x)$. Thus

$$R(y_N) = \sum_{i=0}^{N} y_i^{(2)} p_i(x) - \sum_{i=0}^{N+1} y_i^{(1)} a(x) p_i(x) - \sum_{i=0}^{N+2} y_i b(x) p_i(x) - \sum_{i=0}^{N} c_i p_i(x).$$

Orthogonalize the residual with respect to j^{th} Legendre polynomials,

as

$$\langle R(y_N), p_j(x) \rangle = \int_{-1}^{1} R(y_N) p_j(x) dx = 0 \qquad (2.1.3)$$

for $j = 0,1,2,...,N$. This implies that

$$(j+0.5)^{-1} y_j^{(2)} - \sum_{i=0}^{N+1} a_{ij} y_i^{(1)} - \sum_{i=0}^{N+2} b_{ij} y_i = c_j (j+0.5)^{-1} \qquad (2.1.4)$$

where

$$a_{ij} = \int_{-1}^{1} a(x) p_i(x) p_j(x) dx,\ b_{ij} = \int_{-1}^{1} b(x) p_i(x) p_j(x) dx, \text{ and } j = 0,1,2,\ldots,N.$$

Let

$$Y = [y_0\ y_1\ .....y_{N+2}]^t \quad , \quad Y^{(1)} = [y_0^{(1)}\ y_1^{(1)}\ .....y_{N+1}^{(1)}]^t$$

$$Y^{(2)} = [y_0^{(2)}\ y_1^{(2)}\ .....y_N^{(2)}]^t \quad \text{and} \quad F = \left[2c_0\ \ \frac{2}{3}c_1\ \ \frac{2}{5}c_2\ .....(N+0.5)^{-1}c_N\right]^t$$

where t means the transpose of the given vector, we can rewrite Equation (2.1.4) in the matrix form as

$$A_2 Y^{(2)} - A_1 Y^{(1)} - A_0 Y = F \tag{2.1.5}$$

where A_2, A_1, *and* A_0 are $(N+1)\times(N+1), (N+1)\times(N+2), (N+1)\times(N+3)$ matrices, respectively, such that

$$A_{2ij} = \begin{cases} (J+0.5)^{-1}, \text{if } i = j \\ 0 \qquad\quad , \text{if } i \end{cases}$$
$$A_{1ij} = a_{ij} \quad \text{and} \quad A_{0ij} = b_{ij}.$$

From Equations (1.4.4) and (1.4.5) there exist two matrices B_1 and B_2 of orders $(N+2)\times(N+3)$ and $(N+1)\times(N+3)$, respectively, such that

$$Y^{(2)} = B_2 Y$$
$$Y^{(1)} = B_1 Y.$$

Thus, system (2.1.5) becomes

$$\left(A_2 B_2 - A_1 B_1 - A_0\right) Y = F. \tag{2.1.6}$$

By the endpoint relations we see that

$$y_- = y(-1) = \sum_{i=0}^{N+2} y_i p_i(-1) = \sum_{i=0}^{N+2} (-1)^i y_i$$

and

$$y_+ = y(1) = \sum_{i=0}^{N+2} y_i p_i(1) = \sum_{i=0}^{N+2} y_i.$$

Let

$$B_3 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1 & \cdots & (-1)^{N+2} \end{bmatrix}$$

be $2\times(N+3)$ matrix. Then,

$$B_3 Y = \begin{bmatrix} y_+ \\ y_- \end{bmatrix}. \tag{2.1.7}$$

Combine systems (2.1.6) and (2.1.7) together to get

$$\Omega Y = R \tag{2.1.8}$$

where

$$\Omega = \begin{bmatrix} A_2 B_2 - A_1 B_1 - A_0 \\ B_3 \end{bmatrix}, R = \begin{bmatrix} F \\ y_+ \\ y_- \end{bmatrix}.$$

Since the original ordinary boundary value problem has a unique solution, Ω is nonsingular. The major problem of the system (2.1.8) is that the matrix Ω is ill conditioned matrix. The sensitivity of the linear system will be measured by the condition number $k(\Omega) = \|\Omega\|\|\Omega^{-1}\|$. This thesis uses the $\infty-$norm which is given by $\|\Omega\| = \max\left\{\sum_{j=1}^{N+3} |c_{i,j}| : i = 1 : N+3\right\}$. The sensitivity of the system (2.1.8) will not make Gauss-elimination is a good method to solve such problems. Iterative methods such as Jacobi, Gauss Seidal and SOR method need a "good " initial guess which is not available. For this reason, this thesis uses the path- following method to solve the system (2.1.8). The function $G : \Re^{N+3} \times [0,1] \to \Re^{N+3}$ is defined by $G(Y) = \Omega Y - R$. Then, define the convex homotopy $H : \Re^{N+3} \times [0,1] \to \Re^{N+3}$ by $H(Y, \lambda) = \Omega Y - \lambda R$. Then, $Z_0 = (0,0)$ is a zero for H with $\partial H / \partial Y = \Omega$ is a nonsingular matrix. Thus, using the technique described in section (1.5), it will generate a sequence of points $Z_0, Z_1, \ldots$. We stop our iteration method when the last component of Z_{k+1} is greater than 1 while the last component of Z_k is less than1. In this case, Y will be the first

$(N+3)$ components of Z_k. The following two examples explain how our approach works efficiently.

Example (2.1.1): Consider the following linear boundary value problem

$$y''+(\cos \pi x)y'+(\pi \sin \pi x)y=-\pi^2 \cos \pi x, -1<x<1$$
$$y(-1)=-1, \quad y(1)=-1.$$

The exact solution is $y_{\text{exact}}(x)=\cos(\pi x)$. Table (2.1.1) gives the relation between the size of the matrix Ω and its condition number.

N	Cond (Ω)
4	53.1
6	404.2
8	2982.1
10	$2.21*10^5$
12	$1.62*10^5$
14	$1.22*10^6$
16	$8.84*10^6$
18	$6.63*10^7$

Table 2.1.1.Relation between the size of the matrix Ω and its condition number

Graphs of $(N,\text{cond}(\Omega))$ and $(N,\log(\text{cond}(\Omega)))$ are given in figures 2.1.1 and 2.1.2 respectively. From these figures, it is clear that the condition number of Ω grows exponentially. Thus, system (2.1.8) is ill conditioned when N is large. Next, the path–following method will be used for solving the same system.

Table 2.1.2 gives the relation between the size of the matrix Ω and the max point-wise error e_N using different step sizes $h=0.01, 0.001, 0.0001$, respectively, where

$$e_N=\max\{|y_{exact}-y_N|: x\in\{-1,-0.9,....,1\}\}.$$

N	$h = 0.01$	$h = 0.001$	$h = 0.0001$
4	$1.3 * 10^{-4}$	$1.2 * 10^{-4}$	$1.3 * 10^{-4}$
6	$2.1 * 10^{-6}$	$2.0 * 10^{-6}$	$1.7 * 10^{-6}$
8	$3.3 * 10^{-7}$	$3.2 * 10^{-7}$	$3.1 * 10^{-7}$
10	$4.1 * 10^{-9}$	$2.7 * 10^{-9}$	$2.1 * 10^{-9}$
12	$6.6 * 10^{-11}$	$4.4 * 10^{-11}$	$4.2 * 10^{-11}$
14	$1.4 * 10^{-13}$	$1.1 * 10^{-13}$	$1.1 * 10^{-13}$
16	$2.1 * 10^{-15}$	$1.7 * 10^{-15}$	$1.5 * 10^{-15}$

Table 2.1.2:The error in the approximate solution for different values of h

One can see that the step size h dose not effects too much in the linear case since the tangent vector is fixed during all iterations. For this reason it is enough to choose reasonable small step size such as $h = 0.01$. It is easy to see that the path–following method is working efficiently.

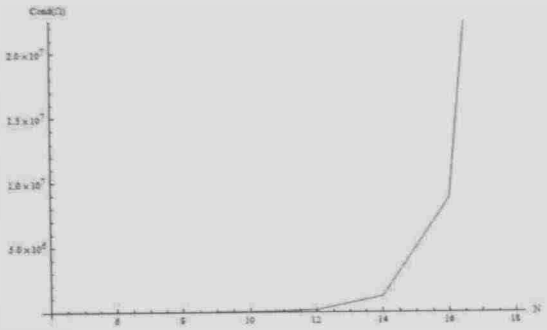

Figure 2.1.1: Graph of (N, Cond (Ω))

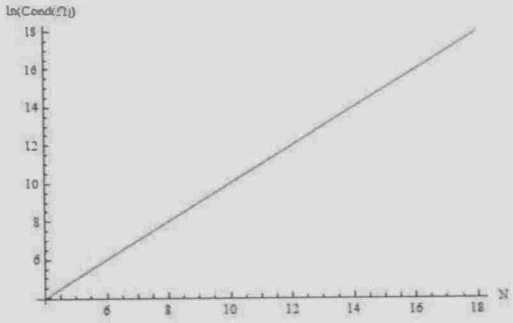

Figure 2.1.2: Graph of (N, ln (Cond (Ω))

2.2 Numerical technique for solving nonlinear boundary value problems

This section, presents a numerical technique for solving the following boundary value problem (BVPs) of the form

$$y'' = f(x, y, y') \ , \ x \in (-1,1) \tag{2.2.1}$$

$$y(-1) = y_- \ , \ y(1) = y_+ \tag{2.2.2}$$

where y_- and y_+ are constants and $f \in c^2\left([-1,1]\times\Re\times\Re\right)$. We approximate $y(x)$ in terms of the Legendre polynomials as we did in the linear case. Hence, the residual is given by

$$R(y_N) = y_N'' - f(x, y_N, y_N').$$

Orthogonalize the residual with respect to the Legendre polynomials as

$$\left\langle R(y_N), P_j(x)\right\rangle = \int_{-1}^{1} R(y_N)\, P_j(x)dx = 0 \tag{2.2.3}$$

For $j = 0,1,\ldots..,N$. Hence, Equation $(2.2.3)$ leads to

$$(j+0.5)^{-1} y_j^{(2)} = \int_{-1}^{1} P_j(x) f\left(x, \sum_{i=0}^{N+2} y_i P_i(x), \sum_{i=0}^{N+1} y_i^{(1)}{}_i P_i\right) dx. \tag{2.2.4}$$

Simple calculations imply that $\sum_{i=0}^{N+2} y_i p_i = D_1 Y$ and $\sum_{i=0}^{N+1} y_i^{(1)} p_i = D_2 Y^{(1)}$

where

$$D_1 = [p_0 \ p_1 \ \cdots \ p_{N+2}] \quad \text{and} \quad D_2 = [p_0 \ p_1 \ \cdots \ p_{N+1}].$$

From Equation (1.4.4) there exists a matrix B_1 of order $(N+2)\times(N+3)$ such that

$$Y^{(1)} = B_1 Y.$$

Thus

$$\sum_{l=0}^{N+1} y_l^{(1)} p_l = D_2 B_1 Y$$

Equation (2.2.4) becomes $y_j^{(2)} = (j+0.5)\int_{-1}^{1} p_j(x) f(x, D_1 Y, D_2 B_1 Y)\,dx.$

Let

$$g_j(Y) = (j+0.5)\int_{-1}^{1} p_j(x) f(x, D_1 Y, D_2 B_1 Y)\,dx.$$

Then

$$y_j^{(2)} = g_j(Y) \qquad for\ j = 0, 1, \ldots, N.$$

We can rewrite last equation in the matrix form as

$$Y^{(2)} = G(Y)$$

where

$$G(Y) = \begin{bmatrix} g_0(Y) \\ g_1(Y) \\ \vdots \\ g_N(Y) \end{bmatrix}.$$

From Equation (1.4.5), there exists a matrix B_2 of order $(N+1)\times(N+3)$ such that

$$Y^{(2)} = B_2 Y.$$

Thus,

$$B_2 Y = G(Y). \tag{2.2.5}$$

By the endpoint relations we see that

$$B_3 Y = \begin{bmatrix} y_+ \\ y_- \end{bmatrix} \tag{2.2.6}$$

where

$$B_3 = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & -1 & . & . & . & (-1)^{N+2} \end{bmatrix}$$

Combine systems (2.2.5) and (2.2.6) to get

$$\Omega Y = R(Y) \tag{2.2.7}$$

where $\Omega = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$ *and* $R(Y) = \begin{bmatrix} G(Y) \\ y_+ \\ y_- \end{bmatrix}$.

The matrix Ω is non-singular. Example for $N = 5$,

$$\Omega = \begin{bmatrix} 0 & 0 & 3 & 0 & 10 & 0 & 21 & 0 \\ 0 & 0 & 0 & 15 & 0 & 42 & 0 & 81 \\ 0 & 0 & 0 & 0 & 35 & 0 & 90 & 0 \\ 0 & 0 & 0 & 0 & 0 & 63 & 0 & 154 \\ 0 & 0 & 0 & 0 & 0 & 0 & 99 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 143 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

the condition number of Ω is $k(\Omega)$= 186.7133. To solve system (2.2.7) we apply the path following method as described in section 2.1. Next, we present two examples.

Example (2.2.1): Consider the following nonlinear boundary value problem

$$y'' + xe^{-x} y(x) y'(x) + (1 + x^2) y = (2 + x + x^2) e^x$$
$$y(-1) = e^{-1} \quad , \quad y(1) = e.$$

Where the exact solution is $y(x) = e^x$. In Table 2.2.1 we present the relation between the number of terms N in the approximate solution $y_N(x)$ and the max point-wise error in the approximate solution e_N using the step size $h = 0.001$.

N	e_N
6	$1.321 * 10^{-7}$
8	$3.529 * 10^{-9}$
10	$4.123 * 10^{-11}$
12	$5.681 * 10^{-12}$
14	$1.321 * 10^{-14}$
16	$9.725 * 10^{-15}$
18	$2.486 * 10^{-15}$

Table 2.2.1: The error in the approximate solution in Example (2.2.1).

Example (2.2.2): Consider the nonlinear boundary value problem

$$y''(x)+xy'(x)+\left(y(x)\right)^2 = x^2+x.$$

Then, the exact solution is x. We make an entirely analogous analysis to that of Example (2.2.1), the result is presented in Table 2.2.2.

N	e_N
4	$3.218 * 10^{-9}$
6	$6.923 * 10^{-10}$
8	$2.451 * 10^{-12}$
10	$1.976 * 10^{-15}$
12	$1.721 * 10^{-15}$
14	$1.123 * 10^{-15}$

Table 2.2.2: The error in the approximate solution in Example (2.2.2).

From Tables 2.2.1 and 2.2.2 it is clear that the proposed method works properly and efficiently.

Chapter 3: Fractional Boundary Value Problems

In this chapter, the Tau-method is used for solving problem $(1)-(2)$ when $1<\alpha<2$. This chapter is divided into two sections. The first one, it studies the linear case while the second one, it studies the nonlinear case.

3.1 Linear fractional boundary value problems

In this section, a numerical approach is presented for solving the following class of fractional linear BVP's.

$$D^{\alpha}y=a(x)y'+b(x)y+c(x),x\in(0,1),1<\alpha<2 \qquad (3.1.1)$$

$$y(0)=y_{-}\quad,\quad y(1)=y_{+} \qquad (3.1.2)$$

where y_{-} , y_{+} are constants and $a,b,c\in C[-1,1]$.

Approximate $y(x),a(x),b(x)$ and $c(x)$ in terms of the fractional-order Legendre functions as:

$$Y_N(x)=\sum_{k=0}^{N+2}y_kF_k^{\alpha}(x),\ a_N(x)=\sum_{k=0}^{N}a_kF_k^{\alpha}(x),\ b_N(x)=\sum_{k=0}^{N}b_kF_k^{\alpha}(x)$$

,and

$$C_N(x)=\sum_{k=0}^{N}C_kF_k^{\alpha}(x).$$

Thus, $D^{\alpha}y(x)$ and $y'(x)$ can be approximated by $D^{\alpha}Y_N(x)=\sum_{k=0}^{N+1}y_k^{(\alpha)}F_k^{\alpha}(x)$ and

$$Y_N'(x)=\sum_{k=0}^{N+1}y_k^{(1)}F_k^{\alpha}(x).$$

For Y_N, the residual is given by

$$R(Y_N) = D^{\alpha}Y_N - a_N(x)Y_N'(x) - b_N(x)Y_N(x) - c_N(x)$$
$$= \sum_{k=0}^{N} y_k^{(\alpha)} F_k^{\alpha}(x) - \left(\sum_{k=0}^{N} a_k F_k^{\alpha}(x)\right)\left(\sum_{k=0}^{N+1} y_k^{(1)} F_k^{\alpha}(x)\right) -$$
$$\left(\sum_{k=0}^{N} b_k F_k^{\alpha}(x)\right)\left(\sum_{k=0}^{N+2} y_k F_k^{\alpha}(x)\right) - \sum_{k=0}^{N} c_k F_k^{\alpha}(x)$$

or

$$R(Y_N) = \sum_{k=0}^{N} y_k^{(\alpha)} F_k^{\alpha}(x) - \sum_{k=0}^{2N+1}\sum_{j=0}^{k} a_j y_{k-j}^{(1)} F_j^{\alpha}(x) F_{k-j}^{\alpha}(x)$$
$$- \sum_{k=0}^{2N+2}\sum_{j=0}^{k} b_j y_{k-j} F_j^{\alpha}(x) F_{k-j}^{\alpha}(x) - \sum_{k=0}^{N} c_k F_k^{\alpha}(x).$$

Using Theorem (1.4.1), the residual can be written as

$$R(Y_N) = \sum_{k=0}^{N} y_k^{(\alpha)} F_k^{\alpha}(x) - \sum_{k=0}^{2N+1}\sum_{j=0}^{k}\sum_{l=0}^{[k/2]} A_{2l} a_j y_{k-j}^{(1)} F_j^{\alpha}(x) F_{k-2l}^{\alpha}(x)$$
$$- \sum_{k=0}^{2N+2}\sum_{j=0}^{k}\sum_{l=0}^{[k/2]} A_{2l} b_j y_{k-j} F_{k-2l}^{\alpha}(x) - \sum_{k=0}^{N} c_k F_k^{\alpha}(x).$$

where

$$a_j = 0, b_j = 0, y_j = 0, c_j = 0 \quad for\ j \geq N+2 \quad \text{and}$$
$$y_j^{(\alpha)} = 0 \quad for\ j \geq N.$$

Since the focus is on the first n terms only, the higher order terms are ignored and the residual will be written as $R(Y_N) = \sum_{k=0}^{N} \beta_k F_k^{\alpha}(x) - \sum_{k=0}^{N} c_k F_k^{\alpha}(x)$.

Orthogonalize the residual with respect to the fractional order Legendre function to get $\int_0^1 R(Y_N) F_j^{\alpha}(x) w(x)\,dx = 0, \quad j = 0:N$. Thus $\beta_k = c_k$ for $k = 0:N$.

Let

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} \quad \text{and} \quad R_1 = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix}.$$

Then,

$$\beta = R_1 \tag{3.1.3}$$

Following the procedure described in section (2.1) and the properties of the Fractional-order Legendre functions mentioned in section (1.4), we can rewrite equation (3.1.3) as

$$AY = R_1$$

where A *is* $(N+1)(N+3)$ matrix and

$$Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N+2} \end{bmatrix}.$$

From the boundary conditions, one can see that

$$y_- = Y_N(0) = \sum_{k=0}^{N+2} y_k F_k^{\alpha}(0) = \sum_{k=0}^{N+2} (-1)^k y_k,$$
$$y_+ = Y_N(1) = \sum_{k=0}^{N+2} y_k F_k^{\alpha}(1) = \sum_{k=0}^{N+2} y_k.$$

Thus,

$$\begin{bmatrix} A \\ \Lambda \end{bmatrix} Y = \begin{bmatrix} R_1 \\ y_- \\ y_+ \end{bmatrix} \tag{3. 1.4}$$

where

$$\Lambda = \begin{bmatrix} 1 & -1 & \cdots & (-1)^{N+2} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Thus, system $(3.1.4)$ can be written as

$$\Omega Y = B. \qquad (3.1.5)$$

where Ω and B *are* $(N+3)(N+3)$ and $(N+3)\times 1$ matrices. The major problem in system (3.1.5) is that the matrix Ω is ill conditioned matrix. The path-following method will be used as we did in section (2.1) to overcome this problem. The following are two examples to show the efficiency of the proposed method.

Example (3.1.1): Consider the following linear fractional boundary value problem of the form

$$D^{3/2}y + 2y' + 3y = 3x^2 + 4x + \frac{2}{\pi}\sqrt{x} + 3, \quad 0 < x < 1$$
$$y(0) = 1, \quad y(1) = 2.$$

The exact solution is $y_{exact}(x) = x^2 + 1$. Approximate the solution by

$$y_N(x) = \sum_{k=0}^{N+2} y_k F_k^{3/2}(x).$$

Let $e_N = \max\{|y_{exact}(x) - Y_N(x)| : x = 0, 0.1, ..., 1\}$. Then the errors for different values of N are given in Table 3.1.1.

N	e_N
4	$3.1*10^{-8}$
6	$7.8*10^{-10}$
8	$5.2*10^{-10}$
10	$1.3*10^{-10}$

Table 3.1.1: The error in the approximate solution in Example (3.1.1).

From Table 3.1.1, we see that Y_N is an accurate approximation to $y_{exact}(x)$. It is worth to mention that $h = 10^{-2}$ is used in the path following method.

Example (3.1.2): Consider the following linear fractional boundary value problem of the form

$$D^{1.2}y + x^2 y' = f(x), \quad 0 < x < 1$$
$$y(0) = 1 \ , \quad y(1) = e.$$

where $f(x) = x^2 E_{1,-0.2} + x^2 e^x$ *and* $E_{\alpha,\beta}(z)$ is the two parameter function of Mettag-leffler type. The exact solution is $y_{exact}(x) = e^x$. Approximate the solution by

$$y_N(x) = \sum_{k=0}^{N+2} y_k F_k^{1.2}(x).$$

Then the errors for different values of N are given in Table 3.1.2.

N	e_N
6	$4.3*10^{-6}$
8	$7.6*10^{-8}$
10	$3.5*10^{-9}$
12	$2.9*10^{-10}$
14	$1.8*10^{-12}$
16	$2.7*10^{-13}$

Table 3.1.2: The error in the approximate solution in Example (3.1.2).

From Table 3.1.2, we get an accurate approximation to $y_{exact}(x)$. In this example, $h = 10^{-2}$ is used for the path following method.

Example (3.1.3): Consider the following fractional linear boundary value problem [50] and [37],

$$D^{\alpha} y(x) + \tfrac{3}{57} y(x) = x + \frac{3x^{\alpha+1}}{57\Gamma(\alpha+2)}, \quad 0 < x < 1$$

$$y(0)=0,\ y(1)=\frac{1}{57\Gamma(\alpha+2)}.$$

The exact solution is $y_{exact}(x)=\frac{x^{\alpha+1}}{\Gamma(\alpha+1)}$. In this example; we will compare our results with Rahman and khan results [50], Wu and Li results [37], where the first researchers were used wavelet operational matrices for the solutions of the fractional differential equations, while the others were used reproducing kernel method. Approximate the solution by

$$y_N(x)=\sum_{k=0}^{N+2} y_k F_k^{\alpha}(x).$$

x	$\alpha = 1.2$ In[50], n=32	$\alpha = 1.2$ In[37], n=10	$\alpha = 1.2$ our method (n=10)
0.1	1.53×10^{-6}	5.41×10^{-6}	3.13×10^{-9}
0.2	1.52×10^{-7}	2.87×10^{-6}	2.14×10^{-9}
0.3	8.08×10^{-7}	7.77×10^{-7}	9.21×10^{-10}
0.4	6.31×10^{-7}	4.76×10^{-7}	8.75×10^{-10}
0.5	5.19×10^{-7}	1.76×10^{-7}	6.32×10^{-10}
0.6	1.83×10^{-6}	7.03×10^{-9}	5.98×10^{-10}
0.7	2.43×10^{-6}	1.67×10^{-7}	3.67×10^{-10}
0.8	3.12×10^{-6}	1.30×10^{-7}	1.71×10^{-10}
0.9	4.00×10^{-6}	2.75×10^{-7}	2.39×10^{-10}

Table 3.1.3: Comparison between the errors in the proposed method and the errors in [37], [50].

Example (3.1.4): Consider the following fractional linear BVP [59],

$$D^{1.1}y(x) = x^2 y'(x) + (1+x)y(x) - f(x), \qquad 0 < x < 1$$

$$y(0) = 1 \quad , \quad y(1) = 5$$

where the function f is chosen such that the exact solution is

$$y(x) = x^{1.1} + x^{0.2} + 1 + 3x - 7x^2 + 4x^3 + x^4.$$

In this example, we will compare our results with Stynes [59] results. Who was used finite difference method. Approximate the solution by

$$y_N(x) = \sum_{k=0}^{N+2} y_k F_k^{\alpha}(x).$$

N	e_N in[59]
64	1.464×10^{-1}
128	7.547×10^{-2}
256	3.843×10^{-2}
512	1.941×10^{-2}
1024	9.761×10^{-3}
2048	4.895×10^{-3}

Table 3.1.4: The errors in results of [59]

However, for $N = 10$, the error in our results is 3.2×10^{-7}.

3.2 Nonlinear fractional boundary value problems

In this section, we discuss the solution of problem (1) and (2) for $1 < \alpha < 2$, $x \in (0,1)$.

We approximate $y(x)$ in terms of the fractional-order Legendre functions as

$$Y_N(x) = \sum_{K=0}^{N+2} y_k F_k^{\alpha}(x).$$

Then,

$$D^{\alpha}Y_N(x)=\sum_{k=0}^{N+1}y_k^{(\alpha)}F_k^{\alpha}(x)$$

and

$$Y'(x)=\sum_{k=0}^{N+1}y_k^{(1)}F_k^{\alpha}(x).$$

Let

$$Y=\begin{bmatrix}y_0\\y_1\\\vdots\\y_{N+2}\end{bmatrix},\quad Y^{(\alpha)}=\begin{bmatrix}y_0^{(\alpha)}\\y_1^{(\alpha)}\\\vdots\\y_N^{(\alpha)}\end{bmatrix}\quad\text{and}\quad Y^{(1)}=\begin{bmatrix}y_0^{(1)}\\y_1^{(1)}\\\vdots\\y_{N+1}^{(1)}\end{bmatrix}.$$

Now, the residual is given by

$$R(y_N)=y_N^{(\alpha)}-f(x,Y_N,Y_N').$$

Orthogonalize the residual to get

$$\int_0^1 R(Y_N)F_j^{\alpha}(x)w(x)dx=0\quad\text{for } j=0:N.$$

Thus,

$$(j+0.5)^{-1}y_j^{(2)}=\int_0^1 F_j^{\alpha}(x)f(x,\sum_{i=0}^{N+2}y_iF_i^{\alpha}(x),\sum_{i=0}^{N+1}y_i^{(1)}F_i^{\alpha}(x))w(x)dx.$$

One can see that

$$\sum_{i=0}^{N+2}y_iF_i^{\alpha}(x)=D_1Y\qquad\text{and}\qquad\sum_{i=0}^{N+1}y_i^{(1)}F_i^{\alpha}(x)=D_2Y^{(1)}$$

where

$$D_1=\begin{bmatrix}F_0^{\alpha}\cdots F_{N+2}^{\alpha}\end{bmatrix}\text{ and }\ D_2=\begin{bmatrix}F_0^{\alpha}\cdots F_{N+1}^{\alpha}\end{bmatrix}.$$

From section (1.4), we can rewrite $Y^{(1)}$ and $Y^{(\alpha)}$ as

$$Y^{(1)} = B_1 Y \text{ and } Y^{(\alpha)} = B_2 Y$$

where B_1 *and* B_2 *are* $(N+2)\times(N+3)$ *and* $(N+1)\times(N+3)$ matrices.

Thus,

$$B_2 Y = G(Y) \tag{3.2.1}$$

where

$$g_j(Y) = (j+0.5)\int_0^1 F_j^{\alpha}(x) f(x, D_1 Y, D_2 B_1 Y) w(x) dx$$

for $j = 0 : N$ and

$$G(Y) = \begin{bmatrix} g_0(Y) \\ g_1(Y) \\ \vdots \\ g_N(Y) \end{bmatrix}.$$

Using the boundary conditions, we get

$$B_3(Y) = \begin{bmatrix} y_- \\ y_+ \end{bmatrix} \tag{3.2.2}$$

where

$$B_3 = \begin{bmatrix} 1 & -1 & \cdots & (-1)^{N+2} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

From equations (3.2.1) and (3.2.2), we get

$$\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} Y = \begin{bmatrix} G(Y) \\ y_- \\ y_+ \end{bmatrix}$$

or

$$\Omega Y = R(Y) \tag{3.2.3}$$

where

$$\Omega = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} \quad \text{and} \quad R(Y) = \begin{bmatrix} G(Y) \\ y_- \\ y_+ \end{bmatrix}.$$

To solve system (3.2.3), the path-following method is applied as described in section (2.1). Now, here are two of our examples.

Example (3.2.1): Consider the following fractional nonlinear equation

$$D^{1.5}y(x) = 2y^2 + \frac{8}{\sqrt{\pi}}x^{1.5} - 2y^6 \qquad 0 < x < 1$$

$$y(0) = 1,\ y(1) = 2.$$

The exact solution is $y(x) = x^3 + 1$. Approximate the solution by

$$y_N(x) = \sum_{k=0}^{N+2} y_k F_k^{1.5}(x).$$

Using the proposed technique with $h = 0.01$. For the path-following and $N = 4$,

for the Tau method, we get

$$y_0 = 1.66666667$$
$$y_1 = 0.5$$
$$y_2 = 1.33333334$$
$$y_3 = y_4 = y_5 = y_6 = 0$$

Thus,

$y_4(x) = \frac{1}{6}F_2^{1.5}(x) + \frac{1}{2}F_1^{1.5}(x) + \frac{8}{6} = x^3 + 1$. Then, we get the exact solution.

Example (3.2.2): Consider the following fractional Bratu-type equation

$$D^{1.8}y(x) = 2e^{y(x)}, \qquad 0 < x < 1$$

$$y(0) = 0\,,\ y(1) = 1.5.$$

Approximate the solution by $y_N(x) = \sum_{k=0}^{N+2} y_k F_k^{1.8}(x)$. Using $h = 0.01$ and $N = 6$, the first few values of y_k are given below

k	y_k
0	0.107243
1	0.165249
2	0.0621845
3	0.00528223
4	0.00124226
5	0.000142879
6	$3.45414 * 10^{-6}$
7	$+1.23296 * 10^{-6}$
8	$8.21974 * 10^{-8}$

Table 3.2.1: The values of the coefficients y_k

Thus,

$$\begin{aligned} y_6(x) &= 0.107243F_0^{1.8}(x) + 0.165249F_1^{1.8}(x) + 0.0621845F_2^{1.8}(x) \\ &+0.00528223F_3^{1.8}(x) + 0.00124226F_4^{1.8}(x) + 0.000142879F_5^{1.8}(x) \\ &+3.45414\times10^{-6}F_6^{1.8}(x) + 1.23296\times10^{-6}F_7^{1.8}(x) + 8.21974\times10^{-8}F_8^{1.8}(x). \end{aligned}$$

Figure 3.2.1 represents the graph of the approximation solution.

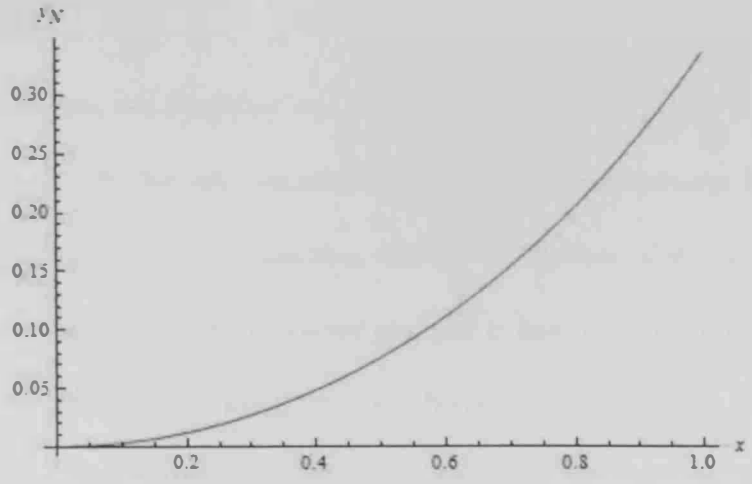

Figure 3.2.1: Graph of the approximate solution

3.3 Conclusions

In this thesis, we modified the Tau-method for the solution a class of non-linear fractional BVPs. We also succeed to find the relation between the coefficient of the series solution of a given function and the coefficient of the series expansion of its fractional derivative. The proposed method is based on the Tau Legendre and path -following methods. Numerical and theoretical results are presented. The numerical results give an evidence of the accuracy of the approximate solution of fractional boundary value problems. We compare our results with others researchers and the comparison shows that our technique works more faster and more efficiently.

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