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An Accurate Method for Solving Higher Order Differential Equations Based On the Implicit Hybrid Method

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This thesis is submitted in partial fulfilment of the requirements for the degree of Master of Science in Mathematics

Under the Supervision of Professor Muhammed Ibrahim Syam

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Declaration of Original Work

I, Ahlam Mohammed S Almokhmari, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*An Accurate Method for Solving Higher Order Differential Equations Based on the Implicit Hybrid Method*", hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Muhammed Ibrahim Syam, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In this thesis, we present numerical techniques to solve higher order differential equations based on the implicit Hybrid method. In these methods, we use the collocation and interpolating methods. Then, we derive the main schemes and their block methods. We investigate some theoretical results such as order of the method, consistency, convergence, and region of absolute stability. Some numerical results and simulations are provided to show the efficiency of the proposed methods using Mathematica.

Keywords: Differential equations, Order of the method, Collocation, Interpolating, Taylor Expansion, Region of absolute stability, Error, consistency, Zero stability, Convergence.

Title and Abstract (in Arabic)

طريقة فعالة لحل المعادلات التفاضلية العليا اعتماداً على طريقة التهجين الضمنية

الملخص

في هذه الأطروحة، تم عرض طرق عددية لحل المعادلات التفاضلية من الرتب العليا اعتماداً على طريقة التهجين الضمنية، في هذه الطرق، استخدمنا طريقتي التجميع والاستكمال الداخلي، ثم اشتقينا المعادلات الرئيسية والأنظمة التراكمية التابعة له، وبحثنا في بعض نتائج النظرية مثل رتبة الطريقة، التناسق، التقارب، ومنطقة الاستقرار المطلق، تم عرض بعض النتائج العددية لتوضيح مدى فعالية الطرق المقترحة باستخدام برنامج ماثيماتيكا.

مفاهيم البحث الرئيسية: المعادلات التفاضلية، رتبة الطريقة، التجميع، الاستكمال الداخلي، توسعة تايلور، منطقة الاستقرار المطلق، الخطأ، التناسق، الاستقرار عند الصفر، التقارب.

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Finally, I express my gratitude to my parents and family for providing me with unfailing support and continuous encouragement.

Dedication

To my beloved parents and family

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Chapter 1: Introduction

Differential equations have several applications in physics, engineering, chemistry, medicine, biology, economic, and others such as Blasius equation [1], which describes the boundary layer flow over a moving plate with velocity zero, Lane-Emden type singular equation [2], and cubic free undamped Duffing oscillator equation [3]. Several numerical methods to solve such problems can be found in [4]-[16]. In this thesis, we derive numerical methods based on the implicit hybrid method. We use collocation and interpolation techniques on the power series approximate solution to derive them. The proposed methods have high order which are very accurate comparing with other methods such as the one-offstep methods. In one-offstep methods such as Euler method or Runge-Kutta methods, we have to convert the higher order differential equations into system of first order differential equations. This makes the method is costly in terms of function evaluations. Other methods which can be used to solve such problems are the Adam Bashforth method and Adam Multon method. They need more than one initial condition which forces us to generate them using the one-offstep method. This will effect on the accuracy of the method.

Implicit hybrid methods comparing with other method is cheaper and more efficient methods. We derive these methods for first, second, and third order differential equations. However, we can use the same technique to derive methods for higher derivatives. We investigate the consistency, zero stable, convergence, the order, error constant, and region of absolute stability of the proposed method. In addition, we study the zero stability, the order, and the error constant of the block methods which are generated from the proposed methods. Numerical results are presented to show the efficiency of the proposed method. Now, we review the preliminaries of the implicit hybrid methods as well as some definitions related to these methods. Let $\{t_0, t_1, \dots, t_M\}$ be a uniform partition of [0,1] with $t_i = i\,\epsilon$, $i = 0,1,\dots,M$, and $\epsilon = \frac{1}{M}$.

Definition 1.1: A k-step hybrid formula is defined by

$$\sum_{i=0}^{k} a_i u_{n+i} + \sum_{i=0}^{l} a_{n+v_i} u_{n+v_i} = \epsilon^p \sum_{i=0}^{k} b_i h_{n+i} + \epsilon^p \sum_{i=0}^{l} b_{n+v_i} h_{n+v_i}$$

where $a_k = 1$, a_0 and b_0 are nonzeros, $v \notin \{0,1,\dots,k\}$, $u_{n+i} = u(t_n + i\,\epsilon)$ and $h_{n+v_i} = h\left(t_{n+v_i}, t_{n+v_i}\right)$. For more details, see [17].

Definition 1.2: Let

$$\mathcal{L}[u[t_n];\epsilon] = \sum_{i=0}^{k} a_i u_{n+i} + \sum_{i=0}^{l} a_{n+v_i} u_{n+v_i} - \epsilon^p \sum_{i=0}^{k} b_i h_{n+i} - \epsilon^p \sum_{i=0}^{l} b_{n+v_i} h_{n+v_i}$$

$$= c_0 u_n + c_1 u_n' + \cdots.$$

If $c_0 = 0, c_1 = 0, \dots, c_{p+1} = 0, c_{p+1} \neq 0$, then the order of the method is r and the error constant is c_{p+1}, see [17].

Definition 1.3 [17]: A linear multistep method is said to be consistent if it has order at least one.

Definition 1.4 [17]: If no zeros of the first characteristic polynomial have modulus greater than one and every root of modulus one has multiplicity not greater than one, then it is called zero stable.

Definition 1.5 [17]: If the method is consistent and zero stable, it is convergent.

Chapter 2: First Order Initial Value Problems

In this chapter, we derive the one-offstep and two-offstep implicit hybrid methods to solve the first order initial value problems. We investigate some theoretical results that are related to these methods. Numerical results are presented to show the efficiency of the proposed methods.

2.1 One-offstep method

Consider the following first order initial value problem of the form

$$y'(t) = f(t, y), \tag{2.1.1}$$

subject to

$$y(0) = y_0 \tag{2.1.2}$$

where y_0 is constant.

2.1.1 Method of solution

In this section, we derive the proposed method using one-offstep hybrid method. The solution is approximated by

$$y(t) = \sum_{i=0}^{3} a_i t^i \tag{2.1.1.1}$$

and its derivative by

$$y'(t) = \sum_{i=1}^{3} i\, a_i t^{i-1}. \tag{2.1.1.2}$$

Let $\{t_0 = 0, t_1 = \Delta, \ldots, t_M = M\Delta = T\}$ be a uniform partition of $[0,T]$ where $t_i = i\Delta, i = 0{:}M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (2.1.1.1) at $t_{n+\frac{1}{2}}$ and collocate Eqn. (2.1.1.2) at $t_{n+\frac{j}{2}}, j = 0,1,2$, to get the following linear system

$$Aa = R \tag{2.1.1.3}$$

where

$$A = \begin{pmatrix} 1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} \\ 0 & 1 & 2t_n & 3t_n^2 \\ 0 & 1 & 2t_{n+\frac{1}{2}} & 3t^2_{n+\frac{1}{2}} \\ 0 & 1 & 2t_{n+1} & 3t^2_{n+1} \end{pmatrix}, a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, R = \begin{pmatrix} y_{n+\frac{1}{2}} \\ f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix}.$$

Let

$$t_{n+1} = t - \Delta\, s + \frac{\Delta}{2},$$

$$t_{n+\frac{1}{2}} = t - \Delta\, s\, , t_n = t - \Delta\, s - \frac{\Delta}{2}.$$

Then, using the above change of variables and solving System (2.1.1.3), we get

$$a_0(s) = 1\, ,$$

$$a_1(s) = -\frac{\Delta s^2}{2} + \frac{2\Delta s^3}{3},$$

$$a_2(s) = \Delta s - \frac{4\Delta s^3}{3},$$

$$a_3(s) = \frac{\Delta s^2}{2} + \frac{2\Delta s^3}{3}.$$

When $t = t_{n+1}$, $t_{n+\frac{1}{2}} = t_{n+1} - \Delta\, s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{1}{2}}}{\Delta} = \frac{\Delta/2}{\Delta} = \frac{1}{2}.$$

Similarly, when $t = t_n, t_{n+\frac{1}{2}}, t_{n+1}$, $s = \frac{-1}{2}, 0, \frac{1}{2}$ respectively. Thus, at $s = \frac{1}{2}, \frac{-1}{2}$, we get

$$y_{n+1} = y_{n+\frac{1}{2}} + \Delta\left(\frac{-f_n}{24} + \frac{f_{n+\frac{1}{2}}}{3} + \frac{5f_{n+1}}{24}\right), \tag{2.1.1.4}$$

$$y_n = y_{n+\frac{1}{2}} + \Delta\left(\frac{-5f_n}{24} + \frac{f_{n+1}}{24} - \frac{f_{n+\frac{1}{2}}}{3}\right). \tag{2.1.1.5}$$

Let

$$Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \end{pmatrix}, Y_{2,n} = (y_n), F_{1,n} = (f_n), F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, A_3 = \begin{pmatrix} \frac{-\Delta}{24} \\ \frac{-5\Delta}{24} \end{pmatrix}, A_4 = \begin{pmatrix} \frac{\Delta}{3} & \frac{5\Delta}{24} \\ \frac{-\Delta}{3} & \frac{\Delta}{24} \end{pmatrix}.$$

Then, System (2.1.1.4) and (2.1.1.5) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}. \tag{2.1.1.6}$$

Multiply both sides of Eqn. (2.1.1.6) by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n} \tag{2.1.1.7}$$

where $B_1 = I_2$,

$$B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, B_3 = \begin{pmatrix} \frac{5\Delta}{24} \\ \frac{\Delta}{6} \end{pmatrix}, B_4 = \begin{pmatrix} \frac{\Delta}{3} & \frac{-\Delta}{24} \\ \frac{2\Delta}{3} & \frac{\Delta}{6} \end{pmatrix}.$$

Then, we solve System (2.1.1.7) iteratively to find the unknowns.

2.1.2 Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of main equation

$$y_{n+1} = y_{n+\frac{1}{2}} + \Delta\left(\frac{-f_n}{24} + \frac{f_{n+\frac{1}{2}}}{3} + \frac{5f_{n+1}}{24}\right). \quad (2.1.2.1)$$

In addition, we study the zero stability, the order, and the error constant of the block method (2.1.1.7). The first and second characteristic functions are given by

$$\tau_1(z) = z - z^{\frac{1}{2}}$$

and

$$\tau_2(z) = \frac{-1}{24} + \frac{5}{24}z + \frac{1}{3}z^{\frac{1}{2}}.$$

Then,

1. $\tau_1(1) = 0,$
2. $\tau_1'(1) = \frac{1}{2},$
3. $1!\,\tau_2(1) = \tau_1'(1) = \frac{1}{2},$
4. The roots of $\tau_1(z)$ for which $|z| = 1$ are simple.

Thus, Eqn. (2.1.2.1) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z) = \frac{\tau_1(z)}{\tau_2(z)}, z = e^{i\emptyset}, \emptyset \in [0,2\pi].$$

Then, the interval of absolute stability is $(0.370153, 2.73029)$ and the region of absolute stability is given in Figure 2.1.2.1.

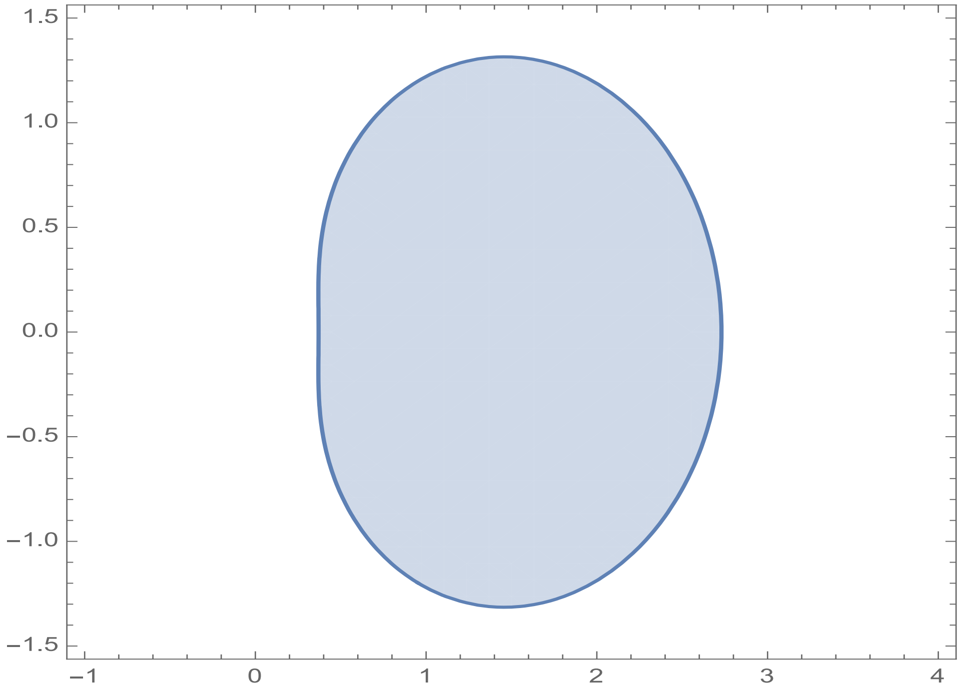

Figure 2.1.2.1: Region of absolute stability, first order IVP, one offstep-point

Normalize B_2 in Eqn. (2.1.1.7) to get

$$\hat{B}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = s(s-1).$$

Since the roots of the above equation have modulus less than1 is, the block method is zero stable as $\Delta \rightarrow 0$. Using the Taylor series, Eqn. (2.1.2.1) becomes

$$y_{n+1} = \mathrm{y}_{\mathrm{n}+\frac{1}{2}} + \Delta\left(\frac{-\mathrm{f}_{\mathrm{n}}}{24} + \frac{\mathrm{f}_{n+\frac{1}{2}}}{3} + \frac{5\mathrm{f}_{n+1}}{24}\right) = -\frac{\Delta^4\ \mathrm{y}_{\mathrm{n}}^{(4)}}{384} + \cdots.$$

Thus, the order of Eqn. (2.1.2.1) is 3 and the error constant is $-\frac{\Delta^4}{384}$. Similarly, the Taylor expansion of System (2.1.1.7) is give as

$$B_1 Y_{1,n} - B_2 Y_{2,n} - B_3 F_{1,n} - B_4 F_{2,n}$$

$$= \begin{pmatrix} -\frac{\Delta^4\ \mathrm{y}_{\mathrm{n}}^{(4)}}{384} + \cdots \\ -\frac{\Delta^4\ \mathrm{y}_{\mathrm{n}}^{(4)}}{384} + \cdots \end{pmatrix}.$$

Thus, the block method (2.1.1.7) has the following order

$$(3,3)^T$$

with error constant $\begin{pmatrix} -\frac{\Delta^4}{384} \\ -\frac{\Delta^4}{384} \end{pmatrix}$.

2.2 Two-offstep hybrid method

2.2.1 Method of solution

In this section, we derive the proposed method. Approximate the solution of Eqn. (2.1.1) by

$$y(t) = \sum_{i=0}^{4} a_i t^i. \qquad (2.2.1.1)$$

Then, the first derivative of the solution of Eqn. (2.2.1.1) is given by

$$y'(t) = \sum_{i=1}^{4} i a_i t^{i-1}. \tag{2.2.1.2}$$

Let $\{t_0 = 0, t_1 = \Delta, \dots, t_M = M\Delta = T\}$ be a uniform partition of $[0, T]$ where $t_i = i\Delta, i = 0{:}M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (2.2.1.1) at $t_{n+\frac{1}{3}}$ and collocate Eqn. (2.2.1.2) at $t_{n+\frac{j}{3}}, j = 0, \dots, 3$ to get the following linear system

$$Aa = R \tag{2.2.1.3}$$

where

$$A = \begin{pmatrix} 1 & t_{n+\frac{1}{3}} & t^2_{n+\frac{1}{3}} & t^3_{n+\frac{1}{3}} & t^4_{n+\frac{1}{3}} \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 \\ 0 & 1 & 2t_{n+\frac{1}{3}} & 3t^2_{n+\frac{1}{3}} & 4t^3_{n+\frac{1}{3}} \\ 0 & 1 & 2t_{n+\frac{2}{3}} & 3t^2_{n+\frac{2}{3}} & 4t^3_{n+\frac{2}{3}} \\ 0 & 1 & 2t_{n+1} & 3t^2_{n+1} & 4t^3_{n+1} \end{pmatrix}, a = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, R = \begin{pmatrix} y_{n+\frac{1}{3}} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix}.$$

Let

$$t_{n+\frac{1}{3}} = t - \Delta\, s, t_{n+1} = t - \Delta\, s + \frac{2\Delta}{3},$$

$$t_{n+\frac{2}{3}} = t - \Delta\, s + \frac{\Delta}{3}, t_n = t - \Delta\, s - \frac{\Delta}{3}.$$

Then, using the above change of variables, we get

$$a_0(s) = 1\,,$$

$$a_1(s) = \frac{\Delta}{4}(-s^2 + 6s^3 - 9s^4),$$

$$a_2(s) = \frac{\Delta}{8}(8s - 6s^2 - 24s^3 + 27s^4),$$

$$a_3(s) = \frac{\Delta}{8}(12s^2 + 12s^3 - 27s^4),$$

$$a_4(s) = \frac{\Delta}{4}(-s^2 + 9s^4).$$

When $t = t_{n+1}$, $t_{n+\frac{1}{3}} = t_{n+1} - \Delta\, s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{1}{3}}}{\Delta} = \frac{2\Delta/3}{\Delta} = \frac{2}{3}.$$

Similarly, when $t = t_n, t_{n+\frac{1}{3}}, t_{n+\frac{2}{3}}, t_{n+1}$, $s = \frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}$ respectively. Thus, at $s = \frac{-1}{3}, \frac{1}{3}, \frac{2}{3}$, Eqn. (2.1.2.1) becomes

$$y_{n+1} = y_{n+\frac{1}{3}} + \frac{\Delta}{9}\left(f_{n+\frac{1}{3}} + 4f_{n+\frac{2}{3}} + f_{n+1}\right),$$

$$y_{n+\frac{2}{3}} = y_{n+\frac{1}{3}} + \frac{\Delta}{72}\left(-f_n + 13\, f_{n+\frac{1}{3}} + 13\, f_{n+\frac{2}{3}} - f_{n+1}\right),$$

$$y_n = y_{n+\frac{1}{3}} + \frac{\Delta}{72}\left(-9f_n - 19\, f_{n+\frac{1}{3}} + 5\, f_{n+\frac{2}{3}} - f_{n+1}\right).$$

Let

$$A_1 = \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \end{pmatrix}, A_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, Y_{2,n} = (y_n),$$

$$F_{1,n} = (f_n), F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix}, A_3 = \begin{pmatrix} 0 \\ \frac{-\Delta}{72} \\ \frac{-9\Delta}{72} \end{pmatrix}$$

$$A_4=\begin{pmatrix} \frac{\Delta}{9} & \frac{4\Delta}{9} & \frac{\Delta}{9} \\ \frac{13\Delta}{72} & \frac{13\Delta}{72} & \frac{-\Delta}{72} \\ \frac{-19\Delta}{72} & \frac{5\Delta}{72} & \frac{-\Delta}{72} \end{pmatrix}.$$

Then, it can be written in the matrix form as

$$A_1Y_{1,n} = A_2Y_{2,n} + A_3F_{1,n} + A_4F_{2,n}. \tag{2.1.2.4}$$

Multiply both sides of Eqn. (2.1.2.4) by A_1^{-1} to get

$$B_1Y_{1,n} = B_2Y_{2,n} + B_3F_{1,n} + B_4F_{2,n} \tag{2.1.2.5}$$

where $B_1 = I_3$, $B_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $B_3 = \begin{pmatrix} \frac{\Delta}{8} \\ \frac{\Delta}{9} \\ \frac{\Delta}{8} \end{pmatrix}$, $B_4 = \begin{pmatrix} \frac{19\Delta}{72} & \frac{-5\Delta}{72} & \frac{\Delta}{72} \\ \frac{4\Delta}{9} & \frac{\Delta}{9} & 0 \\ \frac{3\Delta}{8} & \frac{3\Delta}{8} & \frac{\Delta}{8} \end{pmatrix}$.

Then, we solve System (2.1.2.5) iteratively.

2.2.2 Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of main equation

$$y_{n+1} = y_{n+\frac{1}{3}} + \frac{\Delta}{9}\left(f_{n+\frac{1}{3}} + 4f_{n+\frac{2}{3}} + f_{n+1}\right), \tag{2.2.2.1}$$

In addition, we study the zero stability, the order, and the error constant of the block method (2.1.2.5). The first and second characteristic functions are given by

$$\tau_1(z) = z - z^{\frac{1}{3}}$$

and

$$\tau_2(z) = \frac{1}{9}z^{\frac{1}{3}} + \frac{4}{9}z^{\frac{2}{3}} + \frac{1}{9}z.$$

Then,

1. $\tau_1(1) = 0,$
2. $\tau_1'(1) = \frac{2}{3},$
3. $1!\,\tau_2(1) = \frac{2}{3} = \tau_1'(1),$
4. The roots of $\tau_1(z)$ for which $|z| = 1$ are simple.

Thus, Eqn. (2.2.2.1) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z) = \frac{\tau_1(z)}{\tau_2(z)}, z = e^{i\emptyset}, \emptyset \in [0,2\pi].$$

Then, the region of absolute stability is $(-3.62406, 0.317404)$ and the region of absolute stability is given in Figure 2.2.2.1.

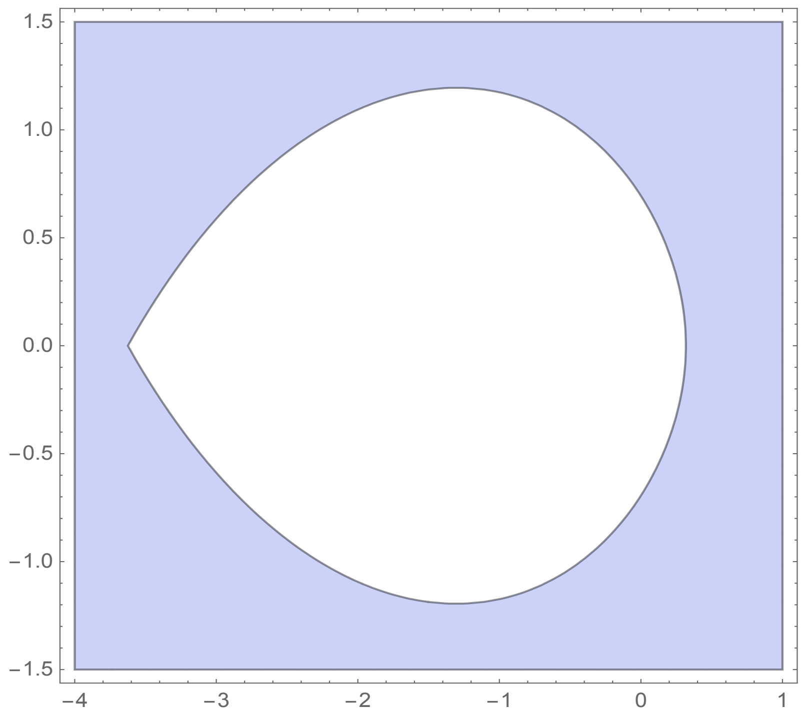

Figure 2.2.2.1: Region of absolute stability, first order IVP, two offstep-point

Normalize B_2 in Eqn. (2.2.1.5) to get

$$\hat{B}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s-1)s^2.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\Delta \rightarrow 0$. Using the Taylor series, Eqn. (2.2.1.5) becomes

$$y_{n+1} = \mathrm{y}_{\mathrm{n}+\frac{1}{3}} + \Delta\left(\frac{1}{9}\mathrm{f}_{\mathrm{n}+\frac{1}{3}} + \frac{4}{9}\mathrm{f}_{\mathrm{n}+\frac{2}{3}} + \frac{1}{9}\mathrm{f}_{\mathrm{n}+1}\right), = -\frac{19\Delta^5\ \mathrm{y}_\mathrm{n}^{(5)}}{174960} + \cdots.$$

Thus, the order of Eqn. (2.2.2.1) is 4 and the error constant is $-\frac{19\Delta^5}{174960}$. Similarly, the Taylor expansion of System (2.2.1.5) is give as

$$B_1Y_{1,n} - B_2Y_{2,n} - B_3F_{1,n} - B_4F_{2,n}$$

$$= \begin{pmatrix} -\frac{19\Delta^5\ \mathrm{y}_\mathrm{n}^{(5)}}{174960} + \cdots \\ -\frac{\Delta^5\ \mathrm{y}_\mathrm{n}^{(5)}}{21870} + \cdots \\ -\frac{\Delta^5\ \mathrm{y}_\mathrm{n}^{(5)}}{6480} + \cdots \end{pmatrix}.$$

Thus, the block method (2.2.1.5) has the following order

$(4,4,4)^T$ with error constant $\begin{pmatrix} -\frac{19\Delta^5}{174960} \\ -\frac{\Delta^5}{21870} \\ -\frac{\Delta^5}{6480} \end{pmatrix}$.

2.3 Numerical results

In this section, we present two of our examples to show the efficiency of the proposed methods in the previous sections.

Example 2.3.1 Consider the following first order initial value problem

$$y'(t) = y^2, 1 \geq t \geq 0,$$

$$y(0) = -1.$$

Then, the exact solution is $y(t) = \frac{-1}{t+1}$. Let $\Delta = 0.1$. Then, the absolute error at $t = 0, 0.1, \dots, 1$ are given in Table 2.3.2.

Table 2.3.1: The absoluter errors for Example 2.3.1

t	One-offstep method	Two-offstep method
0	0	0
0.1	1.1×10^{-6}	2.2×10^{-8}
0.2	1.5×10^{-6}	2.6×10^{-8}
0.3	1.9×10^{-6}	2.9×10^{-8}
0.4	2.2×10^{-6}	3.3×10^{-8}
0.5	2.6×10^{-6}	3.4×10^{-8}
0.6	3.1×10^{-6}	3.8×10^{-8}
0.7	3.7×10^{-6}	4.4×10^{-8}
0.8	4.2×10^{-6}	4.7×10^{-8}
0.9	5.0×10^{-6}	5.1×10^{-8}
1	5.8×10^{-6}	5.7×10^{-8}

Example 2.3.2 Consider the following first order initial value problem

$$y'(t) = \frac{1}{5}(y + y^2), 1 \geq t \geq 0,$$

$$y(0) = 1.$$

Then, the exact solution is $y(t) = \frac{1}{2}\frac{e^{\frac{t}{5}}}{1-\frac{e^{\frac{t}{5}}}{2}}$. Let $\Delta = 0.1$. Then, the absolute error at $t =$ 0,0.1, … ,1 are given in Table 2.3.2.

Table 2.3.2: The absoluter errors for Example 2.3

t	One-offstep method	Two-offstep method
0	0	0
0.1	2.1×10^{-6}	1.7×10^{-8}
0.2	2.3×10^{-6}	2.1×10^{-8}
0.3	2.5×10^{-6}	2.3×10^{-8}
0.4	2.6×10^{-6}	2.6×10^{-8}
0.5	2.8×10^{-6}	2.7×10^{-8}
0.6	3.0×10^{-6}	2.9×10^{-8}
0.7	3.1×10^{-6}	3.2×10^{-8}
0.8	3.3×10^{-6}	3.5×10^{-8}
0.9	3.4×10^{-6}	3.7×10^{-8}
1	3.6×10^{-6}	3.9×10^{-8}

Chapter 3: Second Order Initial Value Problem

In this chapter, we derive the two-offstep and three-offstep implicit hybrid methods to solve the second order initial value problems. We investigate some theoretical results that are related to these methods. Numerical results are presented to show the efficiency of the proposed methods.

3.1 Two-offstep hybrid method

Consider

$$y''(t) = g(t, y, y'), 0 \leq t \leq T \tag{3.1.1}$$

subject to

$$y(0) = \alpha_0, y'(0) = \alpha_1. \tag{3.1.2}$$

3.1.1. Method of solution

In this section, we derive the proposed method. Approximate the solution of Eqn. (3.1.1) by

$$y(t) = \sum_{i=0}^{5} a_i t^i. \tag{3.1.1.1}$$

Then, the first and second derivatives of the solution of Eqn. (3.1.1.1) are given by

$$y'(t) = \sum_{i=1}^{5} i a_i t^{i-1}. \tag{3.1.1.2}$$

and

$$y''(t) = \sum_{i=2}^{5} i(i-1)a_i t^{i-2}. \tag{3.1.1.3}$$

Let $\{t_0 = 0, t_1 = \Delta, \dots, t_M = M\Delta = T\}$ be a uniform partition of $[0,T]$ where $t_i = i\Delta, i = 0{:}M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (3.1.1.1) at $t_{n+\frac{1}{3}}, t_{n+\frac{2}{3}}$ and collocate Eqn. (3.1.1.3) at $t_{n+\frac{j}{3}}, j = 0,1,2,3,$ to get the following linear system

$$\begin{pmatrix} 1 & t_{n+\frac{1}{3}} & t^2_{n+\frac{1}{3}} & t^3_{n+\frac{1}{3}} & t^4_{n+\frac{1}{3}} & t^5_{n+\frac{1}{3}} \\ 1 & t_{n+\frac{2}{3}} & t^2_{n+\frac{2}{3}} & t^3_{n+\frac{2}{3}} & t^4_{n+\frac{2}{3}} & t^5_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 \\ 0 & 0 & 2 & 6t_{n+\frac{1}{3}} & 12t^2_{n+\frac{1}{3}} & 20t^3_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6t_{n+\frac{2}{3}} & 12t^2_{n+\frac{2}{3}} & 20t^3_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ f_n \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix} \tag{3.1.1.4}$$

Let

$$t_{n+\frac{2}{3}} = t - \Delta\, s, t_{n+1} = t - \Delta\, s + \frac{\Delta}{3},$$

$$t_{n+\frac{1}{3}} = t - \Delta\, s - \frac{\Delta}{3}, t_n = t - \Delta\, s - \frac{2\Delta}{3}.$$

Then, using the above change of variables and solving System (3.1.1.4), we get

$$a_0(s) = -3s,$$

$$a_1(s) = 1 + 3s,$$

$$a_2(s) = -\frac{\Delta^2 s(7 - 90s^2 + 243s^4)}{1080},$$

$$a_3(s) = \frac{1}{360}\Delta^2 s(22 - 180s^2 + 135s^3 + 243s^4),$$

$$a_4(s) = \frac{1}{360}\Delta^2 s(43 + 180s + 90s^2 - 270s^3 - 243s^4),$$

$$a_5(s) = \frac{\Delta^2 s(-8 + 180s^2 + 405s^3 + 243s^4)}{1080}.$$

When $t = t_{n+1}$, $t_{n+\frac{2}{3}} = t_{n+1} - \Delta\, s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{2}{3}}}{\Delta} = \frac{\Delta}{3\Delta} = \frac{1}{3}.$$

Similarly, when $t = t_{n+\frac{2}{3}}, t_{n+\frac{1}{3}}, t_n$, $s = 0, -\frac{1}{3}, -\frac{2}{3}$, respectively. Thus, at $s = \frac{1}{3}, -\frac{2}{3}$, Eqn. (3.1.1.1) becomes

$$y_{n+1} = -y_{n+\frac{1}{3}} + 2y_{n+\frac{2}{3}} + \Delta^2\left(\frac{f_{n+\frac{1}{3}}}{108} + \frac{5f_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108}\right), \tag{3.1.1.5}$$

$$y_n = 2y_{n+\frac{1}{3}} - y_{n+\frac{2}{3}} + \Delta^2\left(\frac{f_n}{108} + \frac{5f_{n+\frac{1}{3}}}{54} + \frac{f_{n+\frac{2}{3}}}{108}\right).$$

Using the change of variable $t_{n+\frac{2}{3}} = t - \Delta\, s$, we have

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{\Delta}\frac{dy}{ds}.$$

Hence,

$$\frac{dy}{dt} = \frac{1}{\Delta}\left(\sum_{i=1}^{2} a'_{i-1}(s) y_{n+\frac{i}{3}} + \sum_{i=3}^{6} a'_{i-1}(s) f_{n+\frac{i-3}{3}}\right). \tag{3.1.1.6}$$

At $s = \frac{1}{3}, 0, -\frac{1}{3}, -\frac{2}{3}$, Eqn. (3.1.1.2) implies that

$$y'_{n+1} = \frac{-3y_{n+\frac{1}{3}} + 3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{f_{n+\frac{1}{3}}}{120} + \frac{23f_{n+\frac{2}{3}}}{60} + \frac{127f_{n+1}}{1080}\right),$$

$$y'_{n+\frac{2}{3}} = \frac{-3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{f_{n+\frac{1}{3}}}{120} + \frac{23f_{n+\frac{2}{3}}}{60} - \frac{127f_{n+1}}{1080}\right), \qquad (3.1.1.7)$$

$$y'_{n+\frac{1}{3}} = \frac{-3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(-7\frac{f_n}{1080} + \frac{11f_{n+\frac{1}{3}}}{180} + \frac{43f_{n+\frac{2}{3}}}{360} - \frac{f_{n+1}}{135}\right),$$

$$y'_n = \frac{-3y_{n+\frac{1}{3}}+3y_{n+\frac{2}{3}}}{\Delta} + \Delta\left(\frac{f_n}{135} - \frac{43f_{n+\frac{1}{3}}}{360} - \frac{11f_{n+\frac{2}{3}}}{180} + \frac{7f_{n+1}}{1080}\right).$$

Let

$$A_1 = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & \Delta \\ 3 & -3 & 0 & 0 & \Delta & 0 \\ 3 & -3 & 0 & \Delta & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}, Y_{1,n} = \begin{pmatrix} y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{2}{3}} \\ y'_{n+1} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\Delta \end{pmatrix}, Y_{2,n} = \begin{pmatrix} y_n \\ y'_n \end{pmatrix}, F_{1,n} = (f_n), F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 \\ \frac{\Delta^2}{108} \\ \frac{\Delta^2}{135} \\ -\frac{7\Delta^2}{1080} \\ \frac{\Delta^2}{135} \\ -\frac{127\Delta^2}{1080} \end{pmatrix}, A_4 = \begin{pmatrix} \frac{\Delta^2}{108} & \frac{5\Delta^2}{54} & \frac{\Delta^2}{108} \\ \frac{5\Delta^2}{54} & \frac{\Delta^2}{108} & 0 \\ -\frac{\Delta^2}{120} & \frac{23\Delta^2}{60} & \frac{127\Delta^2}{1080} \\ \frac{11\Delta^2}{180} & \frac{43\Delta^2}{360} & -\frac{\Delta^2}{135} \\ -\frac{43\Delta^2}{360} & -\frac{11\Delta^2}{180} & \frac{7\Delta^2}{1080} \\ -\frac{23\Delta^2}{60} & \frac{\Delta^2}{120} & -\frac{\Delta^2}{135} \end{pmatrix}.$$

Then, Systems (3.1.1.5) and (3.1.1.7) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}. \tag{3.1.1.8}$$

Multiply both sides of Eqn. (3.1.1.8) by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n} \tag{3.1.1.9}$$

where $B_1 = I_6$,

$$B_2 = \begin{pmatrix} 1 & \frac{\Delta}{3} \\ 1 & \frac{2\Delta}{3} \\ 1 & \Delta \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, B_3 = \begin{pmatrix} \frac{97\Delta^2}{3240} \\ \frac{28\Delta^2}{405} \\ \frac{13\Delta^2}{120} \\ \frac{\Delta}{8} \\ \frac{\Delta}{9} \\ \frac{\Delta}{8} \end{pmatrix}, B_4 = \begin{pmatrix} \frac{19\Delta^2}{540} & -\frac{13\Delta^2}{1080} & \frac{\Delta^2}{405} \\ \frac{22\Delta^2}{135} & -\frac{2\Delta^2}{135} & \frac{2\Delta^2}{405} \\ \frac{3\Delta^2}{10} & \frac{3\Delta^2}{40} & \frac{\Delta^2}{60} \\ \frac{19\Delta}{72} & -\frac{5\Delta}{72} & \frac{\Delta}{72} \\ \frac{4\Delta}{9} & \frac{\Delta}{9} & 0 \\ \frac{3\Delta}{8} & \frac{3\Delta}{8} & \frac{\Delta}{8} \end{pmatrix}.$$

Then, we solve System (3.1.1.9) iteratively.

3.1.2 Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of main equation

$$y_{n+1} = -y_{n+\frac{1}{3}} + 2y_{n+\frac{2}{3}} + \Delta^2\left(-\frac{f_{n+\frac{1}{3}}}{108} + \frac{5f_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108}\right). \tag{3.1.2.1}$$

In addition, we study the zero stability, the order, and the error constant of the block method. The first and second characteristic functions are given by

$$\tau_1(z) = z^{1/3} - 2z^{2/3} + z$$

and

$$\tau_2(z) = \frac{z^{1/3}}{108} + \frac{5z^{2/3}}{54} + \frac{z}{108}.$$

Then,

1. $\tau_1(1) = 0,$
2. $\tau_1'(1) = 0,$
3. $\tau_1''(1) - 2!\,\tau_2(1) = 0,$
4. The roots of $\tau_1(z)$ for which $|z| = 1$ are simple.

Thus, Eqn. (3.1.2.1) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z) = \frac{\tau_1(z)}{\tau_2(z)}, z = e^{i\emptyset}, \emptyset \in [0,2\pi].$$

Then, the interval of absolute stability is $(0.728797, 1.42445)$ and the region of absolute stability is given in Figure 3.1.2.1.

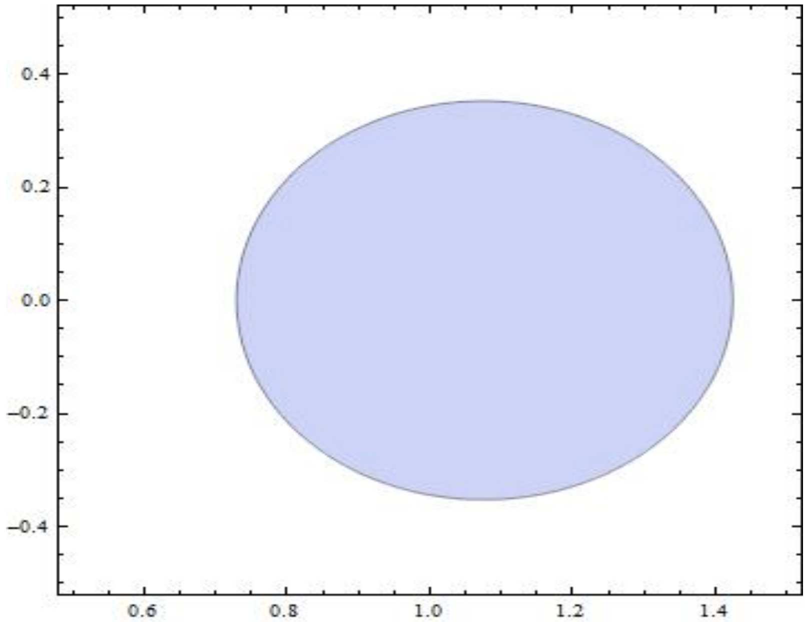

Figure 3.1.2.1: Region of absolute stability, second order IVP, two offstep-point

Normalize B_2 in Eqn. (3.1.1.9) to get

$$\hat{B}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s-1)s^5.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\Delta \to 0$. Using the Taylor series, Eqn. (3.1.2.1) becomes

$$y_{n+1} + y_{n+\frac{1}{3}} - 2y_{n+\frac{2}{3}} - \Delta^2\left(-\frac{f_{n+\frac{1}{3}}}{108} + \frac{5f_{n+\frac{2}{3}}}{54} + \frac{f_{n+1}}{108}\right) = -\frac{\Delta^6 y_n^{(6)}}{174960} + \cdots.$$

Thus, the order of Eqn. (3.1.2.1) is 4 and the error constant is $-0.000005715\Delta^6$.

Similarly, the Taylor expansion of System (3.1.1.9) is give as

$$B_1 Y_{1,n} - B_2 Y_{2,n} - B_3 F_{1,n} - B_4 F_{2,n}$$

$$= \begin{pmatrix} -0.0000200047\,\Delta^6 y_n^{(6)} + .. \\ -0.00004572473\,\Delta^6 y_n^{(6)} + \cdots \\ -0.000077160493\,\Delta^6 y_n^{(6)} + \cdots \\ -0.0001085962505\,\Delta^6 y_n^{(6)} + \cdots \\ -0.0000457247370\,\Delta^6 y_n^{(6)} + \cdots \\ -0.0001543209876\,\Delta^6 y_n^{(6)} + \cdots \end{pmatrix}.$$

Thus, the block method (3.1.1.9) has the following order

$$(4,4,4,4,4,4)^T$$

with error constant

$$\begin{pmatrix} -0.0000200047 \\ -0.00004572473 \\ -0.000077160493 \\ -0.0001085962505 \\ -0.0000457247370 \\ -0.0001543209876 \end{pmatrix} \Delta^6.$$

3.2 Three-offstep hybrid method

3.2.1 Method of solution

In this section, we derive the proposed method. Approximate the solution of Eqn. (3.1.1) by

$$y(t) = \sum_{i=0}^{6} a_i t^i. \qquad (3.2.1.1)$$

Then, the first derivative of the solution of Eqn. (3.2.1.1) is given by

$$y'(t) = \sum_{i=1}^{6} i a_i t^{i-1}. \qquad (3.2.1.2)$$

and

$$y''(t) = \sum_{i=2}^{6} i(i-1) a_i t^{i-2}. \qquad (3.2.1.3)$$

Let $\{t_0 = 0, t_1 = \Delta, \dots, t_M = M\Delta = T\}$ be a uniform partition of $[0, T]$ where $t_i = i\Delta, i = 0{:}M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (3.2.1.1) at $t_{n+\frac{1}{4}}, t_{n+\frac{1}{2}}$ and collocate Eqn. (3.2.1.3) at $t_{n+\frac{j}{4}}, j = 0,1,2,3$, to get the following linear system

$$\begin{pmatrix} 1 & t_{n+\frac{1}{4}} & t^2_{n+\frac{1}{4}} & t^3_{n+\frac{1}{4}} & t^4_{n+\frac{1}{4}} & t^5_{n+\frac{1}{4}} & t^6_{n+\frac{1}{4}} \\ 1 & t_{n+\frac{1}{2}} & t^2_{n+\frac{1}{2}} & t^3_{n+\frac{1}{2}} & t^4_{n+\frac{1}{2}} & t^5_{n+\frac{1}{2}} & t^6_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 \\ 0 & 0 & 2 & 6t_{n+\frac{1}{4}} & 12t^2_{n+\frac{1}{4}} & 20t^3_{n+\frac{1}{4}} & 30t^4_{n+\frac{1}{4}} \\ 0 & 0 & 2 & 6t_{n+\frac{1}{2}} & 12t^2_{n+\frac{1}{2}} & 20t^3_{n+\frac{1}{2}} & 30t^4_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6t_{n+\frac{3}{4}} & 12t^2_{n+\frac{3}{4}} & 20t^3_{n+\frac{3}{4}} & 30t^4_{n+\frac{3}{4}} \\ 0 & 0 & 2 & 6t_{n+1} & 12t^2_{n+1} & 20t^3_{n+1} & 30t^4_{n+1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix}$$

$$= \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ f_n \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix}. \qquad (3.2.1.4)$$

Let

$$t_{n+\frac{1}{4}} = t - \Delta\, s, t_{n+\frac{1}{2}} = t - \Delta\, s + \frac{\Delta}{4},$$

$$t_{n+\frac{3}{4}} = t - \Delta\, s + \frac{\Delta}{2}, t_{n+1} = t - \Delta\, s + \frac{3\Delta}{4}, t_n = t - \Delta\, s - \frac{\Delta}{4}.$$

Then, using the above change of variables and solving System (3.2.1.4), we get

$$a_0(s) = 1 - 4s,$$

$$a_1(s) = 4s,$$

$$a_2(s) = \frac{\Delta^2 s\left(21 - 960s^2 + 3520s^3 - 4608s^4 + 2048s^5\right)}{5760},$$

$$a_3(s) = -\frac{1}{720}\Delta^2 s(59 - 360s + 400s^2 + 800s^3 - 1920s^4 + 1024s^5),$$

$$a_4(s) = \frac{1}{960}\Delta^2 s(-55 + 960s^2 + 320s^3 - 3072s^4 + 2048s^5),$$

$$a_5(s) = -\frac{1}{720}\Delta^2 s(-9 + 240s^2 - 160s^3 - 1152s^4 + 1024s^5),$$

$$a_6(s) = \frac{\Delta^2 s(-11 + 320s^2 - 320s^3 - 1536s^4 + 2048s^5)}{5760}.$$

When $t = t_{n+1}$, $t_{n+\frac{1}{4}} = t_{n+1} - \Delta\, s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{1}{4}}}{\Delta} = \frac{3\Delta}{4\Delta} = \frac{3}{4}.$$

Similarly, when $t = t_n, t_{n+\frac{1}{4}}, t_{n+\frac{1}{2}}, t_{n+\frac{3}{4}}, s = -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$, respectively. Thus, at $s = \frac{3}{4}, \frac{1}{2}, -\frac{1}{4}$, Eqn. (3.2.1.1) becomes

$$y_{n+1} = -2y_{n+\frac{1}{4}} + 3y_{n+\frac{1}{2}} + \Delta^2\left(-\frac{f_n}{1280} + \frac{13f_{n+\frac{1}{4}}}{960} + \frac{67f_{n+\frac{1}{2}}}{640} + \frac{21f_{n+\frac{3}{4}}}{320} + \frac{17f_{n+1}}{3840}\right), \quad (3.2.1.5)$$

$$y_{n+\frac{3}{4}} = -y_{n+\frac{1}{4}} + 2y_{n+\frac{1}{2}} + \Delta^2\left(-\frac{f_n}{3840} + \frac{f_{n+\frac{1}{4}}}{160} + \frac{97f_{n+\frac{1}{2}}}{1920} + \frac{f_{n+\frac{3}{4}}}{160} - \frac{f_{n+1}}{3840}\right),$$

$$y_n = 2y_{n+\frac{1}{4}} - y_{n+\frac{1}{2}} + \Delta^2\left(\frac{19f_n}{3840} + \frac{17f_{n+\frac{1}{4}}}{320} + \frac{7f_{n+\frac{1}{2}}}{1920} + \frac{f_{n+\frac{3}{4}}}{960} - \frac{f_{n+1}}{3840}\right).$$

Using the change of variable $t_{n+\frac{1}{4}} = t - \Delta s$, we have

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{\Delta}\frac{dy}{ds}.$$

Hence,

$$\frac{dy}{dt} = \frac{1}{\Delta}\left(\sum_{i=1}^{2} a'_{i-1}(s)y_{n+\frac{i}{4}} + \sum_{i=3}^{7} a'_{i-1}(s)f_{n+\frac{i-3}{4}}\right). \quad (3.2.1.6)$$

At $s = \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, 0, -1/4$, Eqn. (3.2.1.2) implies that

$$y'_{n+1} = \frac{-4y_{n+\frac{1}{4}} + 4y_{n+\frac{1}{2}}}{\Delta} + \Delta\left(-\frac{11f_n}{1920} + \frac{71f_{n+\frac{1}{4}}}{1440} + \frac{161f_{n+\frac{1}{2}}}{960} + \frac{53f_{n+\frac{3}{4}}}{160} - \frac{95f_{n+1}}{1152}\right),$$

$$y'_{n+\frac{3}{4}} = \frac{-4y_{n+\frac{1}{4}} + 4y_{n+\frac{1}{2}}}{\Delta} + \Delta\left(\frac{f_n}{1152} + \frac{f_{n+\frac{1}{4}}}{80} + \frac{83f_{n+\frac{1}{2}}}{320} + \frac{77f_{n+\frac{3}{4}}}{720} - \frac{3f_{n+1}}{640}\right), \quad (3.2.1.7)$$

$$y'_{n+\frac{1}{2}} = \frac{-4y_{n+\frac{1}{4}} + 4y_{n+\frac{1}{2}}}{\Delta} + \Delta\left(-\frac{17f_n}{5760} + \frac{11f_{n+\frac{1}{4}}}{288} + \frac{97f_{n+\frac{1}{2}}}{960} - \frac{19f_{n+\frac{3}{4}}}{1440} + \frac{11f_{n+1}}{5760}\right),$$

$$y'_{n+\frac{1}{4}}=\frac{-4y_{n+\frac{1}{4}}+4y_{n+\frac{1}{2}}}{\Delta}+\Delta\left(\frac{7f_n}{1920}-\frac{59f_{n+\frac{1}{4}}}{720}-\frac{11f_{n+\frac{1}{2}}}{192}+\frac{f_{n+\frac{3}{4}}}{80}-\frac{11f_{n+1}}{5760}\right),$$

$$y'_n=\frac{-4y_{n+\frac{1}{4}}+4y_{n+\frac{1}{2}}}{\Delta}+\Delta\left(-\frac{481f_n}{5760}-\frac{49f_{n+\frac{1}{4}}}{160}+\frac{11\mathrm{f}_{\mathrm{n}+\frac{1}{2}}}{320}-\frac{7\mathrm{f}_{\mathrm{n}+\frac{3}{4}}}{288}+\frac{3\mathrm{f}_{\mathrm{n}+1}}{640}\right).$$

Let

$$A_1=\begin{pmatrix} 2 & -3 & 0 & 1 & 0 & 0 & 0 & 0\\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0\\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{4}{\Delta} & -\frac{4}{\Delta} & 0 & 0 & 0 & 0 & 0 & 1\\ \frac{4}{\Delta} & -\frac{4}{\Delta} & 0 & 0 & 0 & 0 & 1 & 0\\ \frac{4}{\Delta} & -\frac{4}{\Delta} & 0 & 0 & 0 & 1 & 0 & 0\\ \frac{4}{\Delta} & -\frac{4}{\Delta} & 0 & 0 & 1 & 0 & 0 & 0\\ \frac{4}{\Delta} & -\frac{4}{\Delta} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},Y_{1,n}=\begin{pmatrix} y_{n+\frac{1}{4}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{3}{4}}\\ y_{n+1}\\ y'_{n+\frac{1}{4}}\\ y'_{n+\frac{1}{2}}\\ y'_{n+\frac{3}{4}}\\ y'_{n+1} \end{pmatrix}$$

$$A_2=\begin{pmatrix} 0 & 0\\ 0 & 0\\ -1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & -1 \end{pmatrix},Y_{2,n}=\begin{pmatrix} y_n\\ y'_n \end{pmatrix},F_{1,n}=(f_n),F_{2,n}=\begin{pmatrix} f_{n+\frac{1}{4}}\\ f_{n+\frac{1}{2}}\\ f_{n+\frac{3}{4}}\\ f_{n+1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -\frac{\Delta^2}{1280} \\ -\frac{\Delta^2}{3840} \\ \frac{19\Delta^2}{3840} \\ -\frac{11\Delta}{1920} \\ \frac{\Delta}{1152} \\ -\frac{17\Delta}{5760} \\ \frac{7\Delta}{1920} \\ -\frac{481\Delta}{5760} \end{pmatrix}, A_4 = \begin{pmatrix} \frac{13\Delta^2}{960} & \frac{67\Delta^2}{640} & \frac{21\Delta^2}{320} & \frac{17\Delta^2}{3840} \\ \frac{\Delta^2}{160} & \frac{97\Delta^2}{1920} & \frac{\Delta^2}{160} & -\frac{\Delta^2}{3840} \\ \frac{17\Delta^2}{320} & \frac{7\Delta^2}{1920} & \frac{\Delta^2}{960} & -\frac{\Delta^2}{3840} \\ \frac{71\Delta}{1440} & \frac{161\Delta}{960} & \frac{53\Delta}{160} & \frac{95\Delta}{1152} \\ \frac{\Delta}{80} & \frac{83\Delta}{320} & \frac{77\Delta}{720} & -\frac{3\Delta}{640} \\ \frac{11\Delta}{288} & \frac{97\Delta}{960} & -\frac{19\Delta}{1440} & \frac{11\Delta}{5760} \\ -\frac{59\Delta}{720} & -\frac{11\Delta}{192} & \frac{\Delta}{80} & -\frac{11\Delta}{5760} \\ -\frac{49\Delta}{160} & \frac{11\Delta}{320} & -\frac{7\Delta}{288} & \frac{3\Delta}{640} \end{pmatrix}.$$

Then, Systems (3.2.1.5) and (3.2.1.7) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}. \quad (3.2.1.8)$$

Multiply both sides of Eqn. (3.2.1.8) by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n} \quad (3.2.1.9)$$

where $B_1 = I_8$,

$$B_2=\begin{pmatrix}1 & \frac{\Delta}{4}\\ 1 & \frac{\Delta}{2}\\ 1 & \frac{3\Delta}{4}\\ 1 & \Delta\\ 0 & 1\\ 0 & 1\\ 0 & 1\\ 0 & 1\end{pmatrix}, B_3=\begin{pmatrix}\frac{367\Delta^2}{23040}\\ \frac{53\Delta^2}{1440}\\ \frac{147\Delta^2}{2560}\\ \frac{7\Delta^2}{90}\\ \frac{251\Delta}{2880}\\ \frac{29\Delta}{360}\\ \frac{27\Delta}{320}\\ \frac{7\Delta}{90}\end{pmatrix}, B_4=\begin{pmatrix}\frac{3\Delta^2}{128} & -\frac{47\Delta^2}{3840} & \frac{29\Delta^2}{5760} & -\frac{7\Delta^2}{7680}\\ \frac{\Delta^2}{10} & -\frac{\Delta^2}{48} & \frac{\Delta^2}{90} & -\frac{\Delta^2}{480}\\ \frac{117\Delta^2}{640} & \frac{27\Delta^2}{1280} & \frac{3\Delta^2}{128} & -\frac{9\Delta^2}{2560}\\ \frac{4\Delta^2}{15} & \frac{\Delta^2}{15} & \frac{4\Delta^2}{45} & 0\\ \frac{323\Delta}{1440} & -\frac{11\Delta}{120} & \frac{53\Delta}{1440} & -\frac{19\Delta}{2880}\\ \frac{31\Delta}{90} & \frac{\Delta}{15} & \frac{\Delta}{90} & -\frac{\Delta}{360}\\ \frac{51\Delta}{160} & \frac{9\Delta}{40} & \frac{21\Delta}{160} & -\frac{3\Delta}{320}\\ \frac{16\Delta}{45} & \frac{2\Delta}{15} & \frac{16\Delta}{45} & \frac{7\Delta}{90}\end{pmatrix}.$$

Then, we solve System (3.2.1.9) iteratively.

3.2.2 Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, the order, error constant, and region of absolute stability of main equation

$$y_{n+1}=-2y_{n+\frac{1}{4}}+3y_{n+\frac{1}{2}}+\Delta^2\left(-\frac{f_n}{1280}+\frac{13f_{n+\frac{1}{4}}}{960}+\frac{67f_{n+\frac{1}{2}}}{640}+\frac{21f_{n+\frac{3}{4}}}{320}+\frac{17f_{n+1}}{3840}\right). \quad (3.2.2.1)$$

In addition, we study the zero stability, the order, and the error constant of the block method (3.2.1.9). The first and second characteristic functions are given by

$$\tau_1(z)=2z^{1/4}-3\sqrt{z}+z$$

and

$$\tau_2(z)=-\frac{1}{1280}+\frac{13z^{1/4}}{960}+\frac{67\sqrt{z}}{640}+\frac{21z^{3/4}}{320}+\frac{17z}{3840}.$$

Then,

1. $\tau_1(1) = 0$,
2. $\tau_1'(1) = 0$,
3. $\tau_1''(1) - 2!\,\tau_2(1) = 0$,
4. The roots of $\tau_1(z)$ for which $|z| = 1$ are simple.

Thus, Eqn. (3.2.2.1) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z) = \frac{\tau_1(z)}{\tau_2(z)}, z = e^{i\emptyset}, \emptyset \in [0,2\pi].$$

Then, the interval of absolute stability is $(0.666405, 2.27469)$ and the region of absolute stability is given in Figure 3.2.2.1.

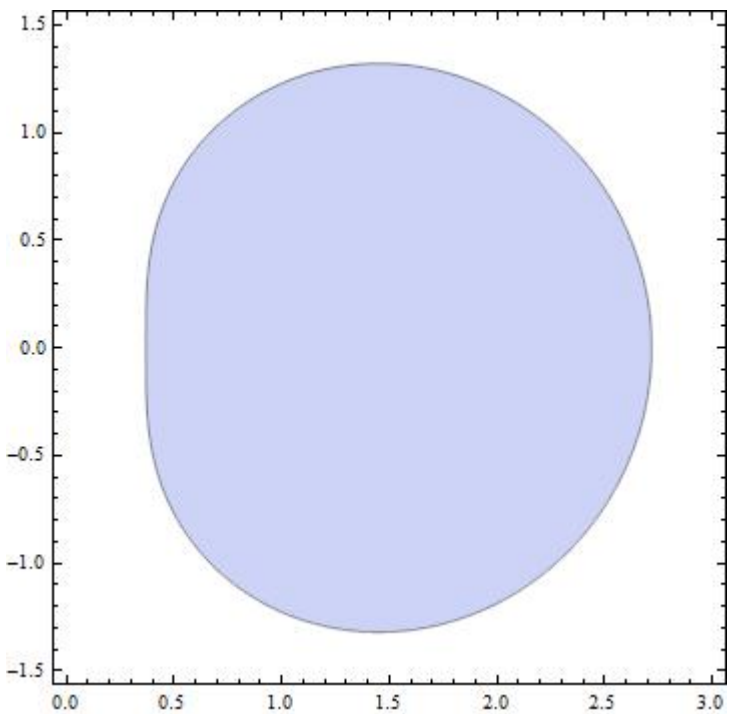

Figure 3.2.2.1: Region of absolute stability , second order IVP, three offstep-point

Normalize B_2 in Eqn. (3.2.1.9) to get

$$\hat{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s-1)s^7.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\Delta \to 0$. Using the Taylor series, Eqn. (3.2.2.1) becomes

$$y_{n+1} + \mathrm{y}_{\mathrm{n}+\frac{1}{3}} - 2\mathrm{y}_{n+\frac{2}{3}} - \Delta^2\left(-\frac{\mathrm{f}_{n+\frac{1}{3}}}{108} + \frac{5\mathrm{f}_{n+\frac{2}{3}}}{54} + \frac{\mathrm{f}_{n+1}}{108}\right)$$

$$= 6.478930\times10^{-7}\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} + \cdots.$$

Thus, the order of Eqn. (3.2.2.1) is 5 and the error constant is 6.478930×10^{-7}. Similarly, the Taylor expansion of System (3.2.1.9) is give as

$$B_1Y_{1,n} - B_2Y_{2,n} - B_3F_{1,n} - B_4F_{2,n}$$

$$= \begin{pmatrix} 6.478930276537699\times10^{-7}\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ 0.000001550099206349206\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ 0.0000024523053850446443\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ 0.0000031001984126984412\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ 0.00000457763671875\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ 0.0000027126736111111111\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ 0.00000457763671875\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \\ -5.166997354497355\times10^{-7}\Delta^7\mathrm{y}_{\mathrm{n}}^{(7)} \end{pmatrix} + \cdots.$$

Thus, the block method (3.2.1.9) has the following order

$$(5,5,5,5,5,5,5,5)^T$$

with error constant

$$\begin{pmatrix} 6.478930276537699\times10^{-7} \\ 0.0000015500992063492 06 \\ 0.0000024523053850446 43 \\ 0.0000031001984126984 12 \\ 0.00000457763671875 \\ 0.0000027126736111111 11 \\ 0.00000457763671875 \\ -5.166997354497355\times10^{-7} \end{pmatrix}.$$

3.3 Numerical results

In this section, we present some of our numerical results to show the efficiency of the proposed method, which is described in the previous sections.

Example 3.3.1. Consider the following linear Lane-Emden equation

$$y''(t) + \frac{2}{t}y'(t) + y(t) = t^3 + t^2 + 12t + 6, 0 \leq t \leq 1,$$

subject to

$$y(0) = y'(0) = 0.$$

The exact solution is $y(t) = t^3 + t^2$. Let $\Delta = \frac{1}{64}$. Then, the absolute error using the two-offstep method and reproducing Kernel method (RKM) [18] are given in Table 3.3.1.

Table 3.3.1: The absolute Error of Example 3.3.1

t_i	Abs. Error (Proposed method)	Abs. Error (RKM)
0.16	1.23124×10^{-9}	5.31280×10^{-7}
0.32	2.22167×10^{-9}	6.28587×10^{-7}
0.48	3.00125×10^{-9}	9.55642×10^{-7}
0.64	3.98747×10^{-9}	1.72184×10^{-6}
0.80	4.41836×10^{-9}	3.18416×10^{-6}
0.96	5.46519×10^{-9}	5.62569×10^{-6}

Example 3.3.2. Consider the following nonlinear Lane-Emden equation

$$y''(t) + \frac{2}{t} y'(t) + 4(2e^{y(t)} + e^{\frac{1}{2}y(t)}) = 0, 0 \le t \le 1,$$

subject to

$$y(0) = y'(0) = 0.$$

The exact solution is $y(t) = -2\ln(t^2 + 1)$. Let $\Delta = \frac{1}{64}$. Then, the absolute error using the three-offstep method and reproducing Kernel method (RKM) [18] are given in Table 3.3.2.

Table 3.3.2: The Abs. Error of Example 3.3.2

t_i	Abs. Error (Proposed method)	Abs. Error (RKM)
0.16	2.22367×10^{-11}	6.58372×10^{-7}
0.32	3.98120×10^{-11}	1.49700×10^{-7}
0.48	4.23188×10^{-11}	2.80293×10^{-7}
0.64	4.98021×10^{-11}	3.84449×10^{-6}
0.80	5.52472×10^{-11}	4.22148×10^{-6}
0.96	6.33758×10^{-11}	3.98444×10^{-6}

Chapter 4: Third Order Initial Value Problems

In this chapter, we derive four-offstep implicit hybrid methods to solve the third order initial value problems. We investigate some theoretical results that are related to these methods. Numerical results are presented to show the efficiency of the proposed methods.

4.1 Method of solution

Consider

$$y'''(t) = g(t, y, y', y''), 0 \leq t \leq T \tag{4.1.1}$$

subject to

$$y(0) = \alpha_0, y'(0) = \alpha_1, y''(0) = \alpha_2. \tag{4.1.2}$$

In this section, we derive the proposed method. Approximate the solution of Eqn. (4.1.1) by

$$y(t) = \sum_{i=0}^{8} a_i t^i. \tag{4.1.3}$$

Then, the first, the second, and the third derivative of the solution of Eqn. (4.1.3) are given by

$$y'(t) = \sum_{i=1}^{8} i a_i t^{i-1}, \tag{4.1.4}$$

$$y''(t) = \sum_{i=2}^{8} i(i-1) a_i t^{i-2}, \tag{4.1.5}$$

$$y'''(t) = \sum_{i=2}^{8} i(i-1)(i-2)a_i t^{i-3}. \quad (4.1.6)$$

Let $\{t_0 = 0, t_1 = \Delta, \dots, t_M = M\Delta = T\}$ be a uniform partition of $[0, T]$ where $t_i = i\Delta, i = 0{:}M$ and $\Delta = \frac{T}{M}$. Interpolate Eqn. (4.1.3) at $t_{n+\frac{1}{5}}, t_{n+\frac{2}{5}}, t_{n+\frac{3}{5}}$ and collocate Eqn. (4.1.1.6) at $t_{n+\frac{j}{5}}, j = 0,1,\dots,5$, to get the following linear system

$$Aa = R \quad (4.1.7)$$

where

$$A_{i,j} = \begin{cases} t_{n+\frac{i}{5}}^{j-1}, & i = 1{:}3, j = 1{:}9, \\ (j-1)(j-2)(j-3)t_{n+\frac{i-4}{5}}^{j-4}, & i = 4{:}9, j = 1{:}9, \end{cases}$$

$$R_i = \begin{cases} y(t_{n+\frac{i}{5}}), & i = 1{:}3, \\ f(t_{n+\frac{i-4}{5}}, y\left(t_{n+\frac{i-4}{5}}\right), y'\left(t_{n+\frac{i-4}{5}}\right), y''(t_{n+\frac{i-4}{5}})), & i = 4{:}9. \end{cases}$$

Let

$$t_{n+\frac{4}{5}} = t - \Delta\, s, t_{n+1} = t - \Delta\, s + \frac{\Delta}{5},$$

$$t_{n+\frac{3}{5}} = t - \Delta\, s - \frac{\Delta}{5}, t_{n+\frac{2}{5}} = t - \Delta\, s \frac{2\Delta}{5},$$

$$t_{n+\frac{1}{5}} = t - \Delta\, s - \frac{3\Delta}{5}, t_n = t - \Delta\, s \frac{4\Delta}{5}.$$

Then, using the above change of variables and solving System (4.1.7), we get

$$a_0(s) = 1 + \frac{15\,s}{2} + \frac{25\,s^2}{2},$$

$$a_1(s) = -3 - 20s - 25s^2,$$

$$a_2(s) = 3 + \frac{25s}{2} + \frac{25s^2}{2},$$

$$a_3(s) = \frac{\Delta^3(84 + 320s - 1725s^2 + 52500s^4 + 87500s^5 - 218750s^6 - 625000s^7 - 390625s^8)}{5040000},$$

$$a_4(s) = \frac{1}{5040000}\Delta^3(-252 + 180s + 15625s^2 - 350000s^4 - 525000s^5 + 1531250s^6 + 3750000s^7 + 1953125s^8),$$

$$a_5(s) = \frac{1}{2520000}\Delta^3(10164 + 73640s + 100575s^2 + 525000s^4 + 612500s^5 - 2406250s^6 - 4375000s^7 - 1953125s^8),$$

$$a_6(s) = \frac{1}{2520000}\Delta^3(10164 + 103700s + 300525s^2 - 1050000s^4 - 175000s^5 + 3718750s^6 + 5000000s^7 + 1953125s^8),$$

$$a_7(s) = -\frac{1}{5040000}\Delta^3(252 - 14640s - 198575s^2 - 840000s^3 - 1137500s^4 + 1312500s^5 + 5468750s^6 + 5625000s^7 + 1953125s^8),$$

$$a_8(s) = \frac{\Delta^3(2 + 5s)^2(21 - 160s - 1000s^2 + 6000s^3 + 28750s^4 + 37500s^5 + 15625s^6)}{5040000}.$$

When $t = t_{n+1}$, $t_{n+\frac{4}{5}} = t_{n+1} - \Delta\, s$. Thus,

$$s = \frac{t_{n+1} - t_{n+\frac{4}{5}}}{\Delta} = \frac{\Delta}{5\Delta} = \frac{1}{5}.$$

Similarly, when $t = t_{n+\frac{4}{5}}, t_{n+\frac{3}{5}}, t_{n+\frac{2}{5}}, t_{n+\frac{1}{5}}, t_n$, $s = 0, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}$ respectively.

Thus, at $s = \frac{1}{5}, 0, -\frac{4}{5}$, Eqn. (4.1.3) becomes

$$y_{n+1} = 3y_{n+\frac{1}{5}} - 8y_{n+\frac{2}{5}} + 6y_{n+\frac{3}{5}} + \Delta^3\left(\frac{f_n}{30000} - \frac{f_{n+\frac{1}{5}}}{30000} + \frac{59f_{n+\frac{2}{5}}}{5000} + \frac{247f_{n+\frac{3}{5}}}{15000} + \frac{109f_{n+\frac{4}{5}}}{30000} + \frac{f_{n+1}}{10000}\right),$$

$$y_{n+\frac{4}{5}} = y_{n+\frac{1}{5}} - 3y_{n+\frac{2}{5}} + 3y_{n+\frac{3}{5}} + \Delta^3\left(\frac{f_n}{60000} - \frac{f_{n+\frac{1}{5}}}{20000} + \frac{121f_{n+\frac{2}{5}}}{30000} + \frac{121f_{n+\frac{3}{5}}}{30000} - \frac{f_{n+\frac{4}{5}}}{20000} + \frac{f_{n+1}}{60000}\right), \tag{4.1.8}$$

$$y_n = 3y_{n+\frac{1}{5}} - 3y_{n+\frac{2}{5}} + y_{n+\frac{3}{5}} + \Delta^3\left(-\frac{f_n}{20000} - \frac{227f_{n+\frac{1}{5}}}{60000} - \frac{131f_{n+\frac{2}{5}}}{30000} + \frac{3f_{n+\frac{3}{5}}}{10000} - \frac{7f_{n+\frac{4}{5}}}{60000} + \frac{f_{n+1}}{60000}\right).$$

Using the change of variable $t_{n+\frac{4}{5}} = t - \Delta\, s$, we have

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = \frac{1}{\Delta}\frac{dy}{ds}, \frac{d^2y}{dt^2} = \frac{1}{\Delta^2}\frac{d^2y}{ds^2}.$$

Hence,

$$\frac{dy}{dt} = \frac{1}{\Delta}\left(\sum_{i=1}^{3} a'_{i-1}(s)y_{n+\frac{i}{5}} + \sum_{i=4}^{9} a'_{i-1}(s)f_{n+\frac{i-4}{5}}\right) \tag{4.1.9}$$

and

$$\frac{d^2y}{dt^2} = \frac{1}{\Delta^2}\left(\sum_{i=1}^{3} a''_{i-1}(s)y_{n+\frac{i}{5}} + \sum_{i=4}^{9} a''_{i-1}(s)f_{n+\frac{i-4}{5}}\right). \tag{4.1.10}$$

At $s = \frac{1}{5}, 0, -\frac{1}{5}, \dots, -\frac{4}{5}$, Eqn. (4.1.5) and Eqn. (4.1.6) imply that

$$y'_{n+1} = \frac{25y_{n+\frac{1}{5}} - 60y_{n+\frac{2}{5}} + 35y_{n+\frac{3}{5}}}{2\Delta} + \Delta^2\left(\frac{127f_n}{504000} - \frac{83f_{n+\frac{1}{5}}}{100800} + \frac{1431f_{n+\frac{2}{5}}}{28000} + \frac{19849f_{n+\frac{3}{5}}}{252000} + \frac{593f_{n+\frac{4}{5}}}{14400} + \frac{479f_{n+1}}{168000}\right),$$

$$y'_{n+\frac{4}{5}} = \frac{15y_{n+\frac{1}{5}}}{2\Delta} - \frac{20}{\Delta}y_{n+\frac{2}{5}} + \frac{25y_{n+\frac{3}{5}}}{2\Delta} + \Delta^2\left(\frac{f_n}{15750} + \frac{f_{n+\frac{1}{5}}}{28000} + \frac{263f_{n+\frac{2}{5}}}{9000} + \frac{1037f_{n+\frac{3}{5}}}{25200} + \frac{61f_{n+\frac{4}{5}}}{21000} - \frac{11f_{n+1}}{252000}\right),$$

$$y'_{n+\frac{3}{5}} = \frac{5y_{n+\frac{1}{5}} - 20y_{n+\frac{2}{5}} + 15y_{n+\frac{3}{5}}}{2\Delta} + \Delta^2\left(\frac{f_n}{24000} - \frac{53f_{n+\frac{1}{5}}}{504000} + \frac{2437f_{n+\frac{2}{5}}}{252000} + \frac{353f_{n+\frac{3}{5}}}{84000} - \frac{271f_{n+\frac{4}{5}}}{504000} + \frac{31f_{n+1}}{504000}\right),$$

$$y'_{n+\frac{2}{5}} = \frac{-5y_{n+\frac{1}{5}} + 5y_{n+\frac{3}{5}}}{2\Delta} + \Delta^2\left(\frac{f_n}{50400} - \frac{13f_{n+\frac{1}{5}}}{31500} - \frac{247f_{n+\frac{2}{5}}}{42000} - \frac{13f_{n+\frac{3}{5}}}{31500} + \frac{f_{n+\frac{4}{5}}}{50400}\right),$$

$$y'_{n+\frac{1}{5}} = \frac{-25y_{n+\frac{1}{5}} + 40y_{n+\frac{2}{5}} - 15y_{n+\frac{3}{5}}}{2\Delta} + \Delta^2\left(-\frac{17f_n}{100800} + \frac{551f_{n+\frac{1}{5}}}{168000} + \frac{2747f_{n+\frac{2}{5}}}{252000} - \frac{37f_{n+\frac{3}{5}}}{36000} + \frac{23f_{n+\frac{4}{5}}}{56000} - \frac{31f_{n+1}}{504000}\right),$$

$$y'_{n} = \frac{-15y_{n+\frac{1}{5}} + 20y_{n+\frac{2}{5}} - 5y_{n+\frac{3}{5}}}{2\Delta} + \Delta^2\left(\frac{37f_n}{14000} + \frac{301f_{n+\frac{1}{5}}}{7200} + \frac{893f_{n+\frac{2}{5}}}{31500} + \frac{29f_{n+\frac{3}{5}}}{42000} - \frac{f_{n+\frac{4}{5}}}{5040} + \frac{11f_{n+1}}{252000}\right), \quad (4.1.11)$$

$$y''_{n+1}=\frac{25y_{n+\frac{1}{5}}-50y_{n+\frac{2}{5}}+25y_{n+\frac{3}{5}}}{\Delta^2}+\Delta\left(\frac{103f_n}{33600}-\frac{599f_{n+\frac{1}{5}}}{33600}+\frac{7397f_{n+\frac{2}{5}}}{50400}+\frac{143f_{n+\frac{3}{5}}}{1120}+\frac{9307f_{n+\frac{4}{5}}}{33600}+\frac{6383f_{n+1}}{100800}\right),$$

$$y''_{n+\frac{4}{5}}=\frac{25y_{n+\frac{1}{5}}-50y_{n+\frac{2}{5}}+25y_{n+\frac{3}{5}}}{\Delta^2}+\Delta\left(-\frac{23f_n}{33600}+\frac{25f_{n+\frac{1}{5}}}{4032}+\frac{447f_{n+\frac{2}{5}}}{5600}+\frac{4007f_{n+\frac{3}{5}}}{16800}+\frac{7943f_{n+\frac{4}{5}}}{100800}-\frac{89f_{n+1}}{33600}\right),$$

$$y''_{n+\frac{3}{5}}=\frac{25y_{n+\frac{1}{5}}-50y_{n+\frac{2}{5}}+25y_{n+\frac{3}{5}}}{\Delta^2}+\Delta\left(\frac{17f_n}{20160}-\frac{151f_{n+\frac{1}{5}}}{33600}+\frac{1943f_{n+\frac{2}{5}}}{16800}+\frac{4867f_{n+\frac{3}{5}}}{50400}-\frac{13f_{n+\frac{4}{5}}}{1344}+\frac{37f_{n+1}}{33600}\right),$$

$$y''_{n+\frac{2}{5}}=\frac{25y_{n+\frac{1}{5}}-50y_{n+\frac{2}{5}}+25y_{n+\frac{3}{5}}}{\Delta^2}+\Delta\left(-\frac{23f_n}{33600}+\frac{283f_{n+\frac{1}{5}}}{33600}+\frac{43f_{n+\frac{2}{5}}}{10080}-\frac{83f_{n+\frac{3}{5}}}{5600}+\frac{109f_{n+\frac{4}{5}}}{33600}-\frac{43f_{n+1}}{100800}\right),$$

$$y''_{n+\frac{1}{5}}=\frac{25y_{n+\frac{1}{5}}-50y_{n+\frac{2}{5}}+25y_{n+\frac{3}{5}}}{\Delta^2}+\Delta\left(\frac{103f_n}{33600}-\frac{8069f_{n+\frac{1}{5}}}{100800}-\frac{771f_{n+\frac{2}{5}}}{5600}+\frac{353f_{n+\frac{3}{5}}}{16800}-\frac{751f_{n+\frac{4}{5}}}{100800}+\frac{37f_{n+1}}{33600}\right),$$

$$y''_{n}=\frac{25y_{n+\frac{1}{5}}-50y_{n+\frac{2}{5}}+25y_{n+\frac{3}{5}}}{\Delta^2}+\Delta\left(-\frac{6341f_n}{100800}-\frac{9349f_{n+\frac{1}{5}}}{33600}-\frac{451f_{n+\frac{2}{5}}}{16800}-\frac{463f_{n+\frac{3}{5}}}{10080}+\frac{557f_{n+\frac{4}{5}}}{33600}-\frac{89f_{n+1}}{33600}\right).$$

Let

$$A_1 = \begin{pmatrix}
-3 & 8 & -6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{25}{2} & 30 & -\frac{35}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \Delta & 0 & 0 & 0 & 0 & 0 \\
-\frac{15}{2} & 20 & -\frac{25}{2} & 0 & 0 & 0 & 0 & 0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{5}{2} & 10 & -\frac{15}{2} & 0 & 0 & 0 & 0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{2} & 0 & -\frac{5}{2} & 0 & 0 & 0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{25}{2} & -20 & \frac{15}{2} & 0 & 0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{15}{2} & -10 & \frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-25 & 50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta^2 \\
-25 & 50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta^2 & 0 \\
-25 & 50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta^2 & 0 & 0 \\
-25 & 50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta^2 & 0 & 0 & 0 \\
-25 & 50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta^2 & 0 & 0 & 0 & 0 \\
-25 & 50 & -25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\Delta & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\Delta^2
\end{pmatrix}, Y_{2,n} = \begin{pmatrix} y_n \\ y_n' \\ y_n'' \end{pmatrix}, F_{1,n} = (f_n), F_{2,n} = \begin{pmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{pmatrix},$$

$$Y_{1,n}=\begin{pmatrix} y_{n+\frac{1}{5}}\\ y_{n+\frac{2}{5}}\\ y_{n+\frac{3}{5}}\\ y_{n+\frac{4}{5}}\\ y_{n+1}\\ y'_{n+\frac{1}{5}}\\ y'_{n+\frac{2}{5}}\\ y'_{n+\frac{3}{5}}\\ y'_{n+\frac{4}{5}}\\ y'_{n+1}\\ y''_{n+\frac{1}{5}}\\ y''_{n+\frac{2}{5}}\\ y''_{n+\frac{3}{5}}\\ y''_{n+\frac{4}{5}}\\ y''_{n+1} \end{pmatrix}, A_3=\begin{pmatrix} \frac{\Delta^3}{30000}\\ \frac{\Delta^3}{60000}\\ -\frac{\Delta^3}{20000}\\ \frac{127\Delta^3}{504000}\\ \frac{\Delta^3}{15750}\\ \frac{\Delta^3}{24000}\\ \frac{\Delta^3}{50400}\\ -\frac{17\Delta^3}{100800}\\ \frac{37\Delta^3}{14000}\\ \frac{103\Delta^3}{33600}\\ -\frac{23\Delta^3}{33600}\\ \frac{17\Delta^3}{20160}\\ -\frac{23\Delta^3}{33600}\\ \frac{103\Delta^3}{33600}\\ -\frac{6341\Delta^3}{100800} \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -\frac{\Delta^3}{30000} & \frac{59\Delta^3}{5000} & \frac{247\Delta^3}{15000} & \frac{109\Delta^3}{30000} & \frac{\Delta^3}{10000} \\ -\frac{\Delta^3}{20000} & \frac{121\Delta^3}{30000} & \frac{121\Delta^3}{30000} & -\frac{\Delta^3}{20000} & \frac{\Delta^3}{60000} \\ -\frac{227\Delta^3}{60000} & -\frac{131\Delta^3}{30000} & \frac{3\Delta^3}{10000} & -\frac{7\Delta^3}{60000} & \frac{\Delta^3}{60000} \\ -\frac{83\Delta^3}{100800} & \frac{1431\Delta^3}{28000} & \frac{19849\Delta^3}{252000} & \frac{593\Delta^3}{14400} & \frac{479\Delta^3}{168000} \\ \frac{\Delta^3}{28000} & \frac{263\Delta^3}{9000} & \frac{1037\Delta^3}{25200} & \frac{61\Delta^3}{21000} & -\frac{11\Delta^3}{252000} \\ -\frac{53\Delta^3}{504000} & \frac{2437\Delta^3}{252000} & \frac{353\Delta^3}{84000} & -\frac{271\Delta^3}{504000} & \frac{31\Delta^3}{504000} \\ -\frac{13\Delta h^3}{31500} & -\frac{247\Delta^3}{42000} & -\frac{13\Delta^3}{31500} & \frac{\Delta^3}{50400} & 0 \\ \frac{551\Delta^3}{168000} & \frac{2747\Delta^3}{252000} & -\frac{37\Delta^3}{36000} & \frac{23\Delta^3}{56000} & -\frac{31\Delta^3}{504000} \\ \frac{301\Delta^3}{7200} & \frac{893\Delta^3}{31500} & \frac{29\Delta^3}{42000} & -\frac{\Delta^3}{5040} & \frac{11\Delta^3}{252000} \\ -\frac{599\Delta^3}{33600} & \frac{7397\Delta^3}{50400} & \frac{143\Delta^3}{1120} & \frac{9307\Delta^3}{33600} & \frac{6383\Delta^3}{100800} \\ \frac{25\Delta^3}{4032} & \frac{447\Delta^3}{5600} & \frac{4007\Delta^3}{16800} & \frac{7943\Delta^3}{100800} & -\frac{89\Delta^3}{33600} \\ -\frac{151\Delta^3}{33600} & \frac{1943\Delta^3}{16800} & \frac{4867\Delta^3}{50400} & -\frac{13\Delta^3}{1344} & \frac{37\Delta^3}{33600} \\ \frac{283\Delta^3}{33600} & \frac{43\Delta^3}{10080} & -\frac{83\Delta^3}{5600} & \frac{109\Delta^3}{33600} & -\frac{43\Delta^3}{100800} \\ -\frac{8069\Delta^3}{100800} & -\frac{771\Delta^3}{5600} & \frac{353\Delta^3}{16800} & -\frac{751\Delta^3}{100800} & \frac{37\Delta^3}{33600} \\ -\frac{9349\Delta^3}{33600} & -\frac{451\Delta^3}{16800} & -\frac{463\Delta^3}{10080} & \frac{557\Delta^3}{33600} & -\frac{89\Delta^3}{33600} \end{pmatrix}.$$

Then, Systems (4.1.8) and (4.1.11) can be written in the matrix form as

$$A_1 Y_{1,n} = A_2 Y_{2,n} + A_3 F_{1,n} + A_4 F_{2,n}. \tag{4.1.12}$$

Multiply both sides of Eqn. (4.1.1.12) by A_1^{-1} to get

$$B_1 Y_{1,n} = B_2 Y_{2,n} + B_3 F_{1,n} + B_4 F_{2,n} \tag{4.1.13}$$

where $B_1 = I_{15}$,

$$B_2 = \begin{pmatrix} 1 & \frac{\Delta}{5} & -\frac{\Delta^2}{50} \\ 1 & \frac{2\Delta}{5} & 0 \\ 1 & \frac{3\Delta}{5} & \frac{3\Delta^2}{50} \\ 1 & \frac{4\Delta}{5} & \frac{4\Delta^2}{25} \\ 1 & \Delta & \frac{3\Delta^2}{10} \\ 0 & 1 & -\frac{\Delta}{5} \\ 0 & 1 & \frac{\Delta}{5} \\ 0 & 1 & \frac{2\Delta}{5} \\ 0 & 1 & \frac{3\Delta}{5} \\ 0 & 1 & \frac{4\Delta}{5} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, B_3 = \begin{pmatrix} -\frac{8753\Delta^3}{5040000} \\ -\frac{141\Delta^3}{140000} \\ \frac{3761\Delta^3}{1680000} \\ \frac{5051\Delta^3}{630000} \\ \frac{261\Delta^3}{16000} \\ -\frac{431\Delta^2}{28000} \\ \frac{239\Delta^2}{24000} \\ \frac{11371\Delta^2}{504000} \\ \frac{17723\Delta^2}{504000} \\ \frac{8053\Delta^2}{168000} \\ \frac{19\Delta}{288} \\ \frac{14\Delta}{225} \\ \frac{51\Delta}{800} \\ \frac{14\Delta}{225} \\ \frac{19\Delta}{288} \end{pmatrix},$$

$$B_4 = \begin{pmatrix}
-\frac{51119\Delta^3}{5040000} & -\frac{4637\Delta^3}{2520000} & -\frac{3419\Delta^3}{2520000} & \frac{2459\Delta^3}{5040000} & -\frac{79\Delta^3}{1008000} \\
-\frac{2329\Delta^3}{180000} & -\frac{4393\Delta^3}{630000} & -\frac{121\Delta^3}{210000} & \frac{247\Delta^3}{1260000} & -\frac{43\Delta^3}{1260000} \\
-\frac{2579\Delta^3}{560000} & -\frac{3089\Delta^3}{280000} & \frac{49\Delta^3}{24000} & -\frac{17\Delta^3}{22400} & \frac{13\Delta^3}{112000} \\
\frac{9329\Delta^3}{630000} & -\frac{629\Delta^3}{63000} & \frac{3317\Delta^3}{315000} & -\frac{1529\Delta^3}{630000} & \frac{7\Delta^3}{18000} \\
\frac{45781\Delta^3}{1008000} & -\frac{2081\Delta^3}{504000} & \frac{983\Delta^3}{33600} & -\frac{1033\Delta^3}{1008000} & \frac{841\Delta^3}{1008000} \\
-\frac{5933\Delta^2}{63000} & -\frac{23\Delta^2}{1008} & -\frac{229\Delta^2}{21000} & \frac{989\Delta^2}{252000} & -\frac{\Delta^2}{1575} \\
\frac{967\Delta^2}{72000} & -\frac{1039\Delta^2}{36000} & \frac{97\Delta^2}{12000} & -\frac{223\Delta^2}{72000} & \frac{7\Delta^2}{14400} \\
\frac{11657\Delta^2}{168000} & -\frac{667\Delta^2}{84000} & \frac{1103\Delta^2}{50400} & -\frac{1171\Delta^2}{168000} & \frac{181\Delta^2}{168000} \\
\frac{63089\Delta^2}{504000} & \frac{4279\Delta^2}{252000} & \frac{17141\Delta^2}{252000} & -\frac{3449\Delta^2}{504000} & \frac{757\Delta^2}{504000} \\
\frac{90703\Delta^2}{504000} & \frac{11147\Delta^2}{252000} & \frac{643\Delta^2}{5600} & \frac{14171\Delta^2}{504000} & \frac{2483\Delta^2}{504000} \\
\frac{1427\Delta}{7200} & -\frac{133\Delta}{1200} & \frac{241\Delta}{3600} & -\frac{173\Delta}{7200} & \frac{3\Delta}{800} \\
\frac{43\Delta}{150} & \frac{7\Delta}{225} & \frac{7\Delta}{225} & -\frac{\Delta}{75} & \frac{\Delta}{450} \\
\frac{219\Delta}{800} & \frac{57\Delta}{400} & \frac{57\Delta}{400} & -\frac{21\Delta}{800} & \frac{3\Delta}{800} \\
\frac{64\Delta}{225} & \frac{8\Delta}{75} & \frac{64\Delta}{225} & \frac{14\Delta}{225} & 0 \\
\frac{25\Delta}{96} & \frac{25\Delta}{144} & \frac{25\Delta}{144} & \frac{25\Delta}{96} & \frac{19\Delta}{288}
\end{pmatrix}.$$

Then, we solve System (4.1.13) iteratively.

4.2 Analysis of the proposed method

In this section, we investigate the consistency, zero stable, convergence, order, error constant, and region of absolute stability of main equation

$$y_{n+1}=3y_{n+\frac{1}{5}}-8y_{n+\frac{2}{5}}+6y_{n+\frac{3}{5}}+\Delta^{3}\left(\frac{f_n}{30000}-\frac{f_{n+\frac{1}{5}}}{30000}+\frac{59f_{n+\frac{2}{5}}}{5000}+\frac{247f_{n+\frac{3}{5}}}{15000}+\frac{109f_{n+\frac{4}{5}}}{30000}+\frac{f_{n+1}}{10000}\right). \quad (4.2.1)$$

In addition, we study the zero stability, the order, and the error constant of the block method (4.1.13).The first and second characteristic functions are given by

$$\tau_1(z)=z-3z^{\frac{1}{5}}+8z^{\frac{2}{5}}-6z^{\frac{3}{5}}$$

and

$$\tau_2(z)=\frac{1}{30000}-\frac{1}{30000}z^{\frac{1}{5}}+\frac{59}{5000}z^{\frac{2}{5}}+\frac{247}{15000}z^{\frac{3}{5}}+\frac{109}{30000}z^{\frac{4}{5}}+\frac{1}{10000}z.$$

Then,

1. $\tau_1(1)=0,$
2. $\tau_1'(1)=0,$
3. $\tau_1''(1)=0,$
4. $\tau_1'''(1)-3!\,\tau_2(1)=0,$
5. The roots of $\tau_1(z)$ for which $|z|=1$ are simple.

Thus, Eqn. (4.2.1) is consistent and zero stable. Therefore, it is convergent. To find the region of absolute stability, let

$$\mu(z)=\frac{\tau_1(z)}{\tau_2(z)}, z=e^{i\emptyset}, \emptyset\in[0,2\pi].$$

Then, the region of absolute stability is $(0.6665, 2.27471)$ and the region of absolute stability is given in Figure 4.2.1.

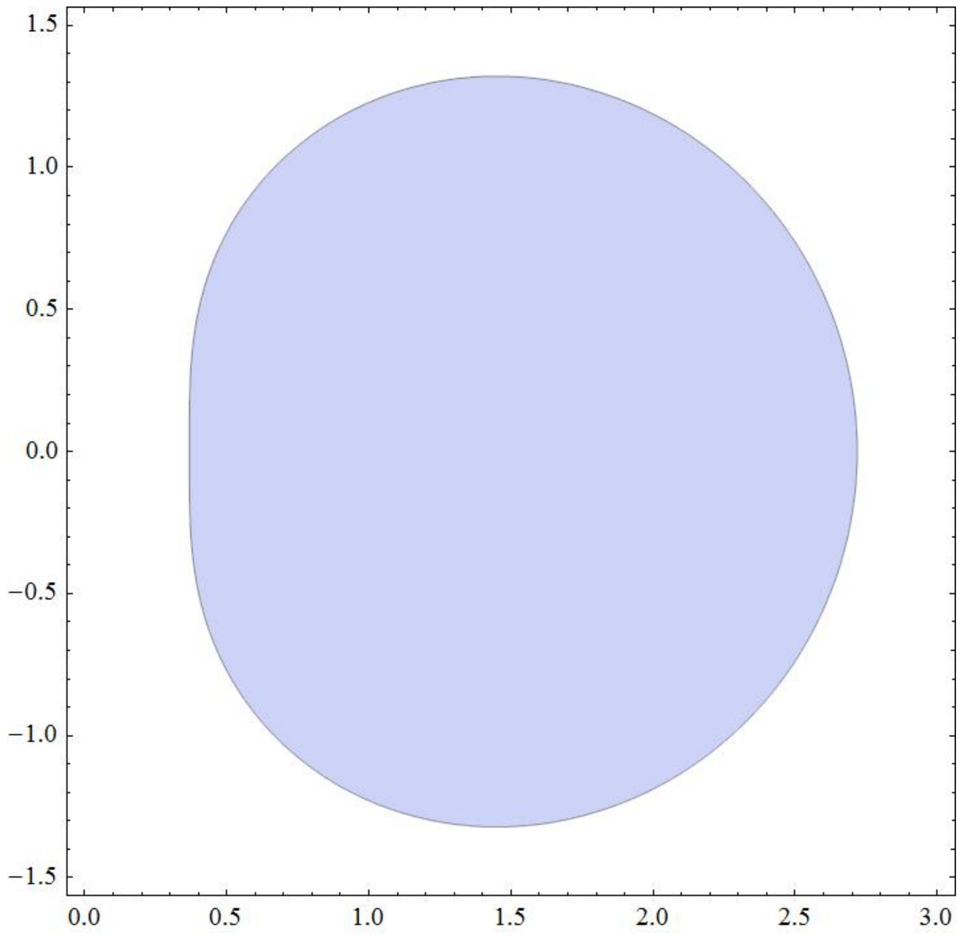

Figure 4.2.1: Region of absolute stability, four order IVP, three offstep-point

Normalize B_2 in Eqn. (4.1.13) to get

$$\hat{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$\det\left(sB_1 - \hat{B}_2\right) = (s-1)s^{14}.$$

Since the roots of the above equation which has modulus 1 is simple, the block method is zero stable as $\Delta \rightarrow 0$. Using the Taylor series, Eqn. (4.2.1) becomes

$$y_{n+1} - 3y_{n+\frac{1}{5}} + 8y_{n+\frac{2}{5}} - 6y_{n+\frac{3}{5}}$$

$$-\Delta^3\left(\frac{f_n}{30000} - \frac{f_{n+\frac{1}{5}}}{30000} + \frac{59f_{n+\frac{2}{5}}}{5000} + \frac{247f_{n+\frac{3}{5}}}{15000} + \frac{109f_{n+\frac{4}{5}}}{30000} + \frac{f_{n+1}}{10000}\right)$$

$$= -\frac{10299450000\Delta^9 y_n^{(9)}}{7441875000000000000} + \cdots.$$

Thus, the order of Eqn. (4.2.1) is 6 and the error constant is 1.38399×10^{-9}.

Similarly, the Taylor expansion of System (4.1.13) is give as

$$B_1Y_{1,n} - B_2Y_{2,n} - B_3F_{1,n} - B_4F_{2,n}$$

$$
= \begin{pmatrix}
-1.38399\times10^{-9}\Delta^9 y_n^{(9)} + \cdots \\
-8.86744\times10^{-9}\Delta^9 y_n^{(9)} + \cdots \\
-2.19086\times10^{-8}\Delta^9 y_n^{(9)} + \cdots \\
-4.10322\times10^{-8}\Delta^9 y_n^{(9)} + \cdots \\
-6.56966\times10^{-8}\Delta^9 y_n^{(9)} + \cdots \\
-2.10582\times10^{-8}\Delta^9 y_n^{(9)} + \cdots \\
-5.14709\times10^{-8}\Delta^9 y_n^{(9)} + \cdots \\
-8.05714\times10^{-8}\Delta^9 y_n^{(9)} + \cdots \\
-1.0836\times10^{-7}\Delta^9 y_n^{(9)} + \cdots \\
-1.45503\times10^{-7}\Delta^9 y_n^{(9)} + \cdots \\
-1.82646\times10^{-7}\Delta^9 y_n^{(9)} + \cdots \\
-1.25291\times10^{-7}\Delta^9 y_n^{(9)} + \cdots \\
-1.65714\times10^{-7}\Delta^9 y_n^{(9)} + \cdots \\
-1.0836\times10^{-7}\Delta^9 y_n^{(9)} + \cdots \\
-2.91005\times10^{-7}\Delta^9 y_n^{(9)} + \cdots
\end{pmatrix}.
$$

Thus, the block method (4.1.13) has the following order

$$(6,6,6,6,6,6,6,6,6,6,6,6,6,6,6)^T$$

with error constant

$$
\begin{pmatrix}
-1.38399\times10^{-9}\Delta^9 \\
-8.86744\times10^{-9}\Delta^9 \\
-2.19086\times10^{-8}\Delta^9 \\
-4.10322\times10^{-8}\Delta^9 \\
-6.56966\times10^{-8}\Delta^9 \\
-2.10582\times10^{-8}\Delta^9 \\
-5.14709\times10^{-8}\Delta^9 \\
-8.05714\times10^{-8}\Delta^9 \\
-1.0836\times10^{-7}\Delta^9 \\
-1.45503\times10^{-7}\Delta^9 \\
-1.82646\times10^{-7}\Delta^9 \\
-1.25291\times10^{-7}\Delta^9 \\
-1.65714\times10^{-7}\Delta^9 \\
-1.0836\times10^{-7}\Delta^9 \\
-2.91005\times10^{-7}\Delta^9
\end{pmatrix}.
$$

4.3 Numerical results

In this section, we present our numerical results to show the efficiency of the proposed method which is described in the previous sections. Consider the following Blasius equation of the form

$$\frac{d^3y}{d\mu^3}+\frac{1}{2}y\frac{d^2y}{d\mu^2}=0$$

subject to

$$y(0)=y'(0)=0, y'(\infty)=1.$$

The condition at infinity is replaced by $y''(0)=\theta$. Then, we find the value of θ by satisfying the condition $y'(\infty)=1$. We study the Blasius equation on the interval $[0,8]$. Using the procedure described in the previous Sections with $h=0.25$, we get $\theta=0.33206$. Then, the graphs of y, y', y'' are given in Figures 4.3.1, 4.3.2, and 4.3.3.

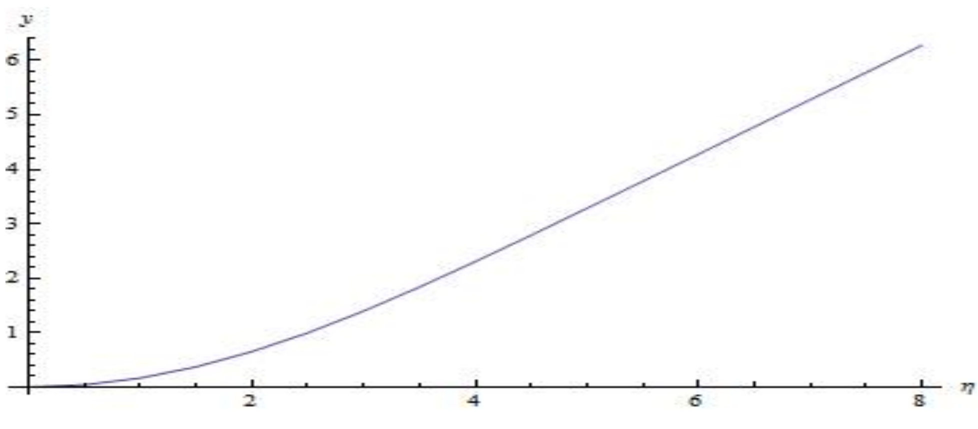

Figure 4.3.1: The graph of y

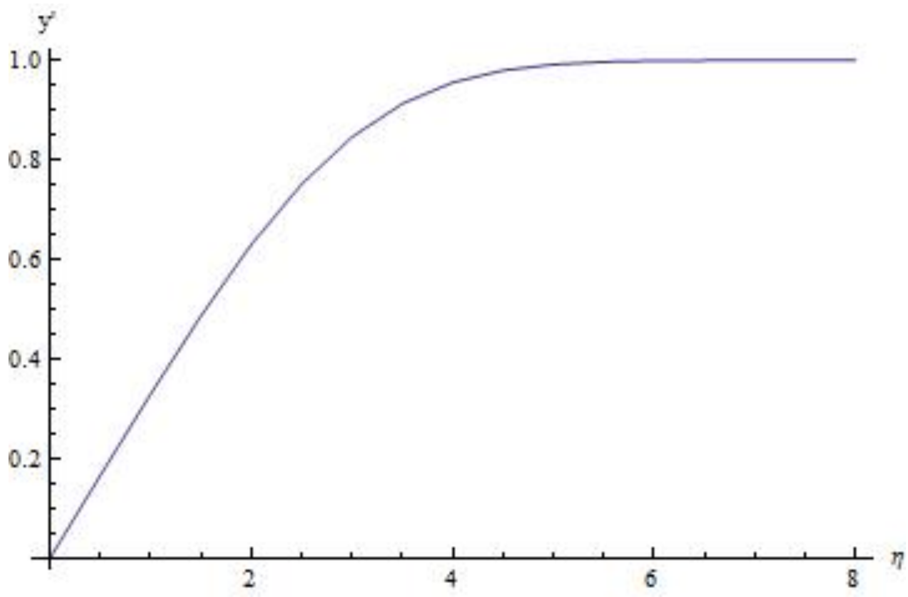

Figure 4.3.2: The graph of y'

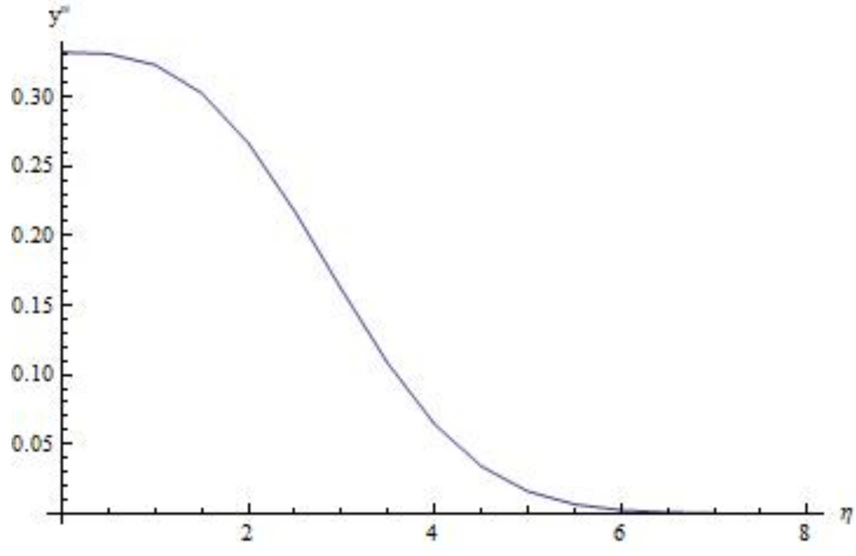

Figure 4.3.3: The graph of y''

It is worth to mention that we get the same value of θ as in [19].

Chapter 5: Conclusions

In this thesis, we investigate the solution of first, second, and third order initial value problems based on the hybrid block method. Several applications are investigated such as Blasius and the nonlinear Lane-Emden equations. In addition, we study the consistency, zero stable, convergence, order, error constant, and region of absolute stability of the proposed methods. We notice the following:

- The proposed methods are consistent.
- They are zero stable.
- They are convergent.
- They have high orders.
- We found the region of absolute stability and the interval of stability for them.
- Our numerical results show that the proposed methods are very accurate and compete other methods such as RKM and homotopy analysis methods.
- We generalize this technique to solve higher order differential initial value problems.
- We can combine this approach with the simple shooting method to solve boundary value problems as we did in Section 4.3.
- We can use this approach to solve more applications in Physics and Engineering.

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List of Publications

- M.M. Khashshan, Muhammed Syam, Ahlam Al Mokhmari, A reliable method for solving fractional Sturm-Liouville problems, Mathematics, accepted, 2018.

- Muhammed I. Syam, A.K. Alomari, Ahlam Al Mokhmari, Numerical solution of cubic free undamped Duffing oscillator equation using continuous implicit hybrid method, Italian Journal of pure and applied Mathematics, accepted, 2018.