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ON THE GEOMETRY OF FUCHSIAN GROUPS

Hala Alaqad

United Arab Emirates University

College of Science

Department of Mathematics

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Hala Alaqad

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Under the Supervision of Dr. Jianhua Gong

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Declaration of Original Work

I, Hala Alaqad, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*On the Geometry of Fuchsian Groups*", hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Dr. Jianhua Gong, in the College of Sciences at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

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Approval of the Master Thesis

This Master Thesis is approved by the following Examining Committee Members:

1) Advisor (Committee Chair): Dr. Jianhua Gong

    Title: Associate Professor

    Department of Mathematical Sciences

    College of Science

    Signature_______________________________ Date _______________

2) Member : Dr. Leonard Daus

    Title: Assistant Professor

    Department of Mathematical Sciences

    College of Science

    Signature _______________________________ Date _______________

3) Member (External Examiner): Dr. Ysukasa Yashiro

    Title: Associate Professor

    Department of Mathematics and Statistics

    Sultan Qaboos University

    Signature _______________________________ Date _______________

This Master Thesis is accepted by:

Dean of the College of Science: Professor Frederick Chi-Ching Leung

Signature_______________________________ Date ______________________

Dean of the College of the Graduate Studies: Professor Nagi T. Wakim

Signature _______________________________ Date ______________________

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Abstract

In this Master thesis we consider the discrete groups with emphasis on the geometry of discrete groups, which lie at the intersection between Hyperbolic Geometry, Topology, Abstract Algebra, and Complex Analysis.

Fuchsian groups are discrete subgroups of the group PSL(2,ℝ) of linear fractional transformations of one complex variable, which is isomorphic to a quotient topological group: $PSL(2,\mathbb{R}) \cong SL(2,\mathbb{R})/\{\pm I\}$. Here SL (2,ℝ) is special linear group and I is the identity.

We study discrete groups, in particular, Fuchsian groups. We present the geometric properties of Fuchsian groups such as fundamental domains, compactness, and Dilichlet tessellations. In addition, we also present some algebraic properties of Fuchsian groups.

Keywords: Fuchsian groups, discrete groups, discontinuous groups, locally finite, fundamental domains, Dilichlet regions, and tessellations

Title and Abstract (in Arabic)

في هندسة مجموعات الفيوجين

الملخص

في هذي أطروحة الماجستير سوف نأخذ بعين الاعتبار المجموعات المنفصلة مع التركيز على هندسة المجموعات المنفصلة والتي توجد في التقاطع بين الهندسة الزائيدية (Hyperbolic Geometry) والطولوجيا (Topology) و الجبر المركب(Abstract Algebra) و التحليل العقدي(Complex Analysis).

مجموعات الفيوجين هي مجموعات فرعية منفصلة من إسقاطي خطي خاص بالمجموعة $PSL(2,\mathbb{R})$ وهي مجموعة مطابقة لمجموعة خطية خاصة$SL(2,\mathbb{R})$.

سنتناول في هذه الاطروحة مجموعات منفصلة, ولا سيما مجموعات الفيوجين وسنقدم خصائص هندسية لهذه المجموعات مثل (compactness) و (tessellation) و(fundamental domains).

بالإضافة إلى خصائص جبرية لهذه المجموعات.

الكلمات المفاتحية: مجموعات الفيوجين ، المجموعات المنفصلة ، المجموعات الغير متصلة ، التغطية الفسيفيساء ، الديمان الاساسي ، منطقة الديريتشيليت ، محدود محليا.

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Finally, I wish to thank my parents Mr. Khader and Mrs. Fayrouiz. They have always been there for me, giving endless amounts of support and encouragement.

Dedication

A special feeling of gratitude to my parents whose words of encouragement and push for tenacity ring in my ears. My siblings who have never left my side and continue to remain very dear and special.

I also dedicate this thesis to my beloved son Fares and my beloved parents who have supported me throughout the process. I will always appreciate all they have done.

Table of Contents

List of Figures

Chapter 1: Introduction

In various ways, mathematicians associate extra structure with sets and then investigate these objects. For example, by adding a binary operation to a set and validating that the set with the binary operation fulfills certain axioms, we arrive at a group. Similarly, given a set one can specify a collection of subsets fulfilling some criteria that give a topology, a sense of which points are close to which other points to that set. It is reasonable that one could associate two kinds of structure with a set. This is how a topological group is defined.

A topological group is a set, which is both a group and has a topology. Certain axioms apply to ensure the group structure and topology "play nicely". An arbitrary group can be viewed as a discrete topological group because that it can be equipped with a discrete topology. However, a subgroup of a topological group needs not to be a discrete subgroup.

A discrete group is a subgroup of a topological group G and inherits the discrete topology from G. Gehring and Martin [7], in addition, Iwaniec and Martin [13] investigated the geometry of discrete quasiconformal groups acting on the extended higher dimensional Euclidean spaces and they studied the convergence groups. Martin [14] showes that discrete quasiconformal groups are not the quasiconformal conjugates of Möbius groups for three and higher dimensional spaces.

Later many mathematicians studied various aspects of convergence groups. For example, Gong [8] for discrete groups of K-quasiconformal homeomorphisms on domains; Casson and Jungreis [4] for convergence groups and Seifert bered 3-manifolds; Tukia [18] for Convergence groups and Gromov's metric hyperbolic spaces; Hinkkanen and Martin [10] for limit functions for convergence groups and uniformly quasi regular maps.

Gabai [6] showed that convergence subgroups of Homeo(S^1) are Fuchsian groups. Katok [11] developed the theory of arithmetic Fuchsian groups. Liebecka and Shalev [12] the spaces of homomorphisms from Fuchsian groups to symmetric groups, they obtained a wide variety of applications, ranging from counting branched coverings of Riemann surfaces, to subgroup growth and random finite quotients of Fuchsian groups, as well as random walks on symmetric groups.

A generalization of Fuchsian groups are Kleinian group, that are discrete subgroups of $PSL(2,\mathbb{C})$. Beardon [3] and Maskit [15] studied the complex projective linear group $PSL(2,\mathbb{C})$ and its discrete subgroups and the two- and three-dimensional hyperbolic spaces on which they act as groups of isometries.

In this master thesis we study discrete groups, which lie at the intersection between Hyperbolic Geometry, Topology, Abstract Algebra, and Complex Analysis. We consider the topological $PSL(2,\mathbb{R})$ and its structures, also the classification of $PSL(2,\mathbb{R})$. We present algebric properties of Fuchsian groups such as cyclic and abelian.

We plan to study geometric properties of Fuchsian groups, for example a cyclic Fuchsian group can form a fundamental region and then tessellations in the upper half plan. On the other hand, properly discontiuous groups and elemantary are also some of geometric properties of Fuchsian groups.

In the first chapter There are many approachs to introduce the planar Hyperbolic Geometry. For examples, Stahl [17] takes pure geometric point of view and Anderson [1] takes the advantage from Abstract Algebra. In addition, Gong [9] established many formulae by using Analytic Geometry. We introduce the Hyperbolic Geometry in this chapter in line with the approach in [9].

In the first section we define the hyperbolic length, then we introduce hyperbolic lines as geodiscs of hyperbolic plan in the second section. In the third and fourth sections we discuss hyperbolic angles, hyperbolic trinagles and their areas.

Then, in second chapter, in this chapter we introduce topology then groups since a topological group is a set which is both a group and has a topology as in section 3. After that, in the fourth section we present the discrete groups, where we use a discrete topology to define. Also, we describe the properly discontiuous groups action and thier properties in the last section.

After, and in the third chapter, we present a Fuchsian group is a discrete subgroup of the $PSL(2, \mathbb{R})$ which is a set of special linear fractional transformations and send complex extended plane to itself. We describe three classification of $PSL(2, \mathbb{R})$ according to the trace of transformation, fixed points of transformation and conjugattion of transformation.

We then introduce the notion of Fuchsian groups and characterizations. In this chapter, we focus on the algebaic properties of Fuchsian groups such as cyclic and abelian Fuchsian groups as a result, every abelian Fuchsian groups is cyclic.

Finally, we discuss the last chapter a fundamental domains of mainly Fuchsian group as in the first section and there are three result Theorem 5.1.2, Theorem 5.1.4, Theorem 5.1.5 for infinite and finite fundamental region .

However, it is convenient to give a definition in a slightly more specify as in locally finite fundamental domains. We also introduce the Dirichlet region and its structure in the second section. Also, we desccribe a geometric properties of Fuchsian groups such as compact Fuchsian groups in the last section.

Chapter 2: Introduction to Hyperbolic Geometry

There are many approachs to introduce the planar Hyperbolic Geometry. For examples, Stahl [17] takes pure geometric point of view and Anderson [1] takes the advantage from Abstract Algebra. In addition, Gong [9] established many formulae by using Analytic Geometry. We introduce the Hyperbolic Geometry in this chapter in line with the approach in [9].

In Section 1 we define the hyperbolic length, then we introduce in Section 2 hyperbolic lines as geodiscs of hyperbolic plan. In the sections 3 and 4 we discuss hyperbolic angles, hyperbolic trinagles and their areas.

2.1 Hyperbolic Length

Given a curve γ in Euclidean plane $\mathbb{R}^2$, if its Cartesian equation given by $y = f(x), a \leq x \leq b$. Then the corresponding parametric equation is

$$x = t,\ y = f(t),\ a \leq t \leq b.$$

Similarly, if a curve γ is given by Cartesian equation $x = f(y)$, $c \leq y \leq d$. Then the corresponding parametric equation is

$$x = g(t),\ y = t,\ c \leq t \leq d.$$

In Euclidean plane $\mathbb{R}^2$, a curve γ given by a parametric equation:

$$x = x(t),\ y = y(t),\ a \leq t \leq b.$$

is called *smooth* if $x'(t)$ and $y'(t)$ exist on $[a, b]$.

We knows from Calculus that each smooth curve can be approximated locally by a tangent line segment. Thus, the element length dL associate with an infinitesimal right angled triangle. By the Pythagorean Theorem:

$$dL = \sqrt{d^2x + d^2y}.$$

Therefore, Euclidean length of the smooth curve γ is given by

$$L_E(\gamma) = \int_\gamma dL = \int_\gamma \sqrt{d^2x + d^2y} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt.$$

Definition 2.1.1 *The upper half-plane* $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ *is called the hyperbolic plane, it is also known as the Poincaré upper half-plane. All points from* $\mathbb{H}$ *are called hyperbolic points.*

Definition 2.1.2 *Given a smooth curve* γ *with parametric equation in the hyperbolic plane* $\mathbb{H}$ *:*

$$x = x(t),\ y = y(t),\ a \leq t \leq b$$

We denote the hyperbolic length of the curve γ *by* $L_H(\gamma)$*, and it's defined by*

$$L_H(\gamma) = \int_\gamma \frac{\sqrt{d^2x + d^2y}}{y} = \int_a^b \frac{\sqrt{(x'(t))^2 + (y'(t))^2}dt}{y(t)}.$$

Theorem 2.1.3 *The hyperbolic length of a vertical line segment joining the points* $P(a, y_1)$ *and* $Q(a, y_2)$*,* $0 < y_1 \leq y_2$ *is*

$$L_H(PQ) = \ln \frac{y_2}{y_1}.$$

Proof. It's clear that the parametric equation of the vertical line segment PQ is

$$x = a,\ y = t,\ 0 < y_1 \leq t \leq y_2.$$

Thus the hyperbolic length of PQ is

$$\begin{aligned} L_H(PQ) &= \int_{y_1}^{y_2} \frac{\sqrt{(x'(t))^2+(y'(t))^2}dt}{y(t)} \\ &= \int_{y_1}^{y_2} \frac{dt}{t} = [\ln|t|]_{y_1}^{y_2}, \text{ as } y_2, y_1 > 0 \\ &= \ln\frac{y_2}{y_1}. \end{aligned}$$

■

Remark 2.1.4 *P is equal to Q unless P and Q are coincide.*

$$L_E(PQ) = L_H(PQ) = 0.$$

Notice that the Euclidean length of the segment PQ is $y_2 - y_1$, that is different from the hyperbolic lenghtt $\ln\frac{y_2}{y_1}$.

Theorem 2.1.5 *Let q be a semi-circle with center $C\,(c,0)$ and radius r. If $P(x_1,y_1)$ and $Q(x_2,y_2)$ are points of q such that the radii CP and CQ make angles α and β $(\alpha < \beta)$, with the positive $x-$axis, respectively, then the hyperbolic length of the arc PQ is*

$$L_H(\text{ arc } PQ) = \ln\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha},$$

where,

$$\begin{aligned} \csc\alpha &= \frac{r}{y_1}, \ \cot\alpha = \frac{x_1 - c}{y_1}; \\ \csc\beta &= \frac{r}{y_2}, \ \cot\beta = \frac{x_2 - c}{y_2}. \end{aligned}$$

Proof. Let t be the angle from the positive $x-$axis to the radius through an arbitrary point (x,y) on q, then the parametric equation is

$$x = c + r\cos t, \ y = r\sin t, \ 0 < \alpha \leq t \leq \beta < \pi.$$

Consequently,

$$x'(t) = -r\sin t,\ y'(t) = r\cos t, \text{ and } \sqrt{x'^2(t) + y'^2(t)} = \sqrt{(-r\sin t)^2 + (r\cos t)^2} = r.$$

Then, the hyperbolic length is

$$\begin{aligned} L_H(\text{ arc } PQ) &= \int_\alpha^\beta \frac{rdt}{r\sin t} = \int_\alpha^\beta \csc t dt \\ &= [\ln|\csc t - \cot t|]_\alpha^\beta \\ &= \ln\left|\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}\right|. \end{aligned}$$

and hence, since $0 < \alpha, \beta < \pi$, $\sin\alpha > 0$, $\sin\beta > 0$ and $\cos\beta < 1, \cos\alpha < 1$

$$\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} = \frac{\sin\alpha(1-\cos\beta)}{\sin\beta(1-\cos\alpha)} > 0.$$

so,

$$L_H(\text{ arc } PQ) = \ln\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}.$$

In addition, by the Trigonometry, we have $\csc\alpha = \frac{r}{y_1}, \cot\alpha = \frac{x_1 - c}{y_1}, \csc\beta = \frac{r}{y_2}, \cot\beta = \frac{x_2 - c}{y_2}$. ■

Example 2.1.6 *Given the arc $x^2 + y^2 = 25, y > 0$ that lies between two points $P(3,4)$ and $Q(-4,3)$. Let angles between the radii OP, OQ and the positive $x-$axis be α and β $(\alpha < \beta)$, respectively. Then by the Theorem 2.1.5*

$\csc\alpha = \frac{r}{y} = \frac{5}{4}$, $\cot\alpha = \frac{x}{y} = \frac{3}{4}$; $\csc\beta = \frac{r}{y} = \frac{5}{3}$, $\cot\beta = \frac{x}{y} = \frac{-4}{3}$.

The hyperbolic length of the arc PQ is

$$L_H(\text{ arc } PQ) = \ln\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha} = \ln 6.$$

2.2 Hyperbolic Distance

As we know the Euclidean distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by $d_E(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. In particular, if two points $P(x_1, 0)$

and $Q(x_2, 0)$ on the $x-$axis then $d_E(P, Q) = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|$. If two points $P(0, y_1)$ and $Q(0, y_2)$ on the $y-$axis then $d_E(P, Q) = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|$.

Given two distinct points P and Q in Euclidean plan $\mathbb{R}^2$. There are infinitely many smooth curves joining P and Q. The line segment joining P and Q has the shortest Euclidean length among all smooth curves joining P and Q. These lines segements are called *geodesics* of Euclidean plan $\mathbb{R}^2$. As an analogue, the geodesics are called smooth curves having the shortest hyperbolic length among all smooth curves between two given points in $\mathbb{H}$.

Theorem 2.2.1 *The geodesics of hyperbolic plane $\mathbb{H}$ are the following two types:*

(a) Vertical Euclidean line segments that are perpendicular to $x-$axis.

(b) Arcs of Euclidean semi-circles centered on $x-$axis.

Proof. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two distinct points in the hyperbolic plane $\mathbb{H}$ with $0 < y_1 < y_2$ and let γ be any smooth curve joining P and Q.

We suppose that a parametric equation of γ is given by

$$x = f(t), \ y = t, \ y_1 \leq t \leq y_2.$$

Then, the hyperbolic length of γ is

$$\begin{aligned} L_H(\gamma) &= \int_{y_1}^{y_2} \frac{\sqrt{(x'(t))^2 + (y'(t))^2} dt}{y(t)} \\ &= \int_{y_1}^{y_2} \frac{\sqrt{f'^2(t) + 1}}{t} dt \\ &\geq \int_{y_1}^{y_2} \frac{dt}{t} \\ &= \ln \frac{y_2}{y_1}. \end{aligned}$$

Notice that, by Theorem 2.1.3 $\ln \frac{y_2}{y_1}$ is the hyperbolic length of vertical Euclidean line segment PQ joining P and Q, so PQ has the shortest hyperbolic length among all smooth curves between P and Q in hyperbolic plan $\mathbb{H}$.

Hence, in the case $x_1 = x_2$, the vertical Euclidean line segment joining P and Q is the geodesic of hyperbolic plan $\mathbb{H}$.

If $x_1 \neq x_2$, then the Euclidean line segment PQ is not perpendicular to $x-$axis, so there is the $x-$intercept of the perpendicular bisector of PQ, say $C\left(c,0\right).$

Relative to this polar coordinate system, suppose that the polar equation of the smooth curve γ joining P and Q is $r = f\left(\theta\right), \, \alpha \leq \theta \leq \beta.$

Then a parametric equation of γ is

$$x = c + r\cos\theta, \;\; y = r\sin\theta, \;\; \alpha \leq \theta \leq \beta.$$

It follows that

$$x'(\theta) = r'\cos\theta - r\sin\theta \text{ and } y'(\theta) = r'(\theta)\sin\theta + r\cos\theta.$$

Thus,

$$\begin{aligned}
(x'(\theta))^2 + (y'(\theta))^2 &= \left(r'\cos\theta - r\sin\theta\right)^2 + \left(r'\sin\theta + r\cos\theta\right)^2 \\
&= r'^2\left(\cos^2\theta + \sin^2\theta\right) - 2rr'\left(\cos\theta\sin\theta - \sin\theta\cos\theta\right) + r^2\left(\sin^2\theta + \cos^2\theta\right) \\
&= r'^2 + r^2.
\end{aligned}$$

Finally, the hyperbolic length of γ is

$$\begin{aligned}
L_H(\gamma) &= \int_\alpha^\beta \frac{\sqrt{(x'(\theta))^2 + (y'(\theta))^2}d\theta}{y(\theta)} = \int_\alpha^\beta \frac{\sqrt{r'^2 + r^2}}{r\sin\theta}d\theta \geq \int_\alpha^\beta \frac{\sqrt{r^2}}{r\sin\theta}d\theta = \int_\alpha^\beta \csc\theta d\theta \\
&= \ln\frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha},
\end{aligned}$$

This inequality shows that the arc between P and Q of Euclidean semi-circle centered on $x-$axis has the shortest hyperbolic length among all smooth curves between P and Q in hyperbolic plan $\mathbb{H}$.

Hence, in the case $x_1 \neq x_2$, the arc between P and Q of Euclidean semi-circle centered on $x-$axis is the geodesic of hyperbolic plan $\mathbb{H}$. ■

Definition 2.2.2 *A hyperbolic line can be defined as either:*

(a) A vertical Euclidean ray in the upper half-plane that is from and perpendicular to $x-$axis, or:

(b) An Euclidean semi-circle in the upper half-plane centered on $x-$axis.

In addition, the hyperbolic distance $d_H(P, Q)$ between P and Q in hyperbolic plane $\mathbb{H}$ can be defined as the hyperbolic length of a hyperbolic segment joining P and Q, denoted by $d_H(P, Q)$.

Now we have the following theorem by applying Theorem 2.1.3 and Theorem 2.1.5.

Theorem 2.2.3 *Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points in hyperbolic plane $\mathbb{H}$, then the hyperbolic length of a hyperbolic distance between P and Q is given by*

$$d_H(P, Q) = \begin{cases} \ln \frac{y_2}{y_1}, & \text{if } x_1 = x_2, 0 < y_1 \leq y_2; \\ \ln \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}, & \text{if } x_1 \neq x_2. \end{cases}$$

where,

$$\begin{aligned} \csc\alpha &= \frac{r}{y_1}, \quad \cot\alpha = \frac{x_1 - c}{y_1}; \\ \csc\beta &= \frac{r}{y_2}, \quad \cot\beta = \frac{x_2 - c}{y_2}. \end{aligned}$$

Example 2.2.4 *Given two points $A = (8, 4)$ and $B = (0, 8)$ in hyperbolic plane $\mathbb{H}$.*

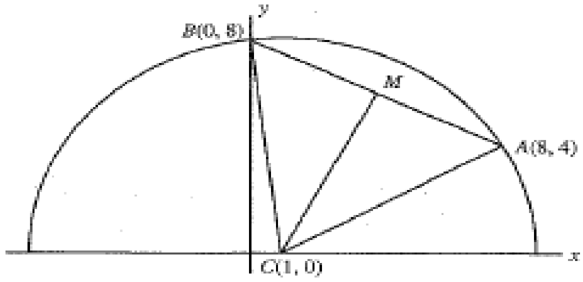

Figure 2.2.4. Hyperbolic distance $d_H(A, B)$ between A and B.

Since $x_1 = 8 \neq 0 = x_2$, *the hyperbolic segment joining points* A *and* B *is an arc of Euclidean semi-circle centered on* $x-$*axis.*

Let the center be $C = (c, 0)$, *then* $CA = CB$ *which gives*

$$\begin{aligned}
\sqrt{(8-c)^2 + (4-0)^2} &= \sqrt{(0-c)^2 + (8-0)^2} \\
(8-c)^2 + (4-0)^2 &= (0-c)^2 + (8-0)^2 \\
8^2 - 16c + c^2 + 4^2 &= c^2 + 8^2 \\
-16c + 4^2 &= 0, \;\; c = 1.
\end{aligned}$$

Thus, the center is $(c, 0) = (1, 0)$, *and the radius*

$$r = CA = CB = \sqrt{(8-1)^2 + (4-0)^2} = \sqrt{65}.$$

Therefore, the hyperbolic distance between A *and* B *is*

$$\begin{aligned}
d_H(A, B) &= \ln \frac{\csc\beta - \cot\beta}{\csc\alpha - \cot\alpha}, \\
\textit{where } \csc\alpha &= \frac{r}{y_1} = \frac{\sqrt{65}}{4}, \;\; \cot\alpha = \frac{x_1 - c}{y_1} = \frac{8-1}{4} = \frac{7}{4}; \\
\csc\beta &= \frac{r}{y_2} = \frac{\sqrt{65}}{8}, \;\; \cot\beta = \frac{0-1}{8} = \frac{-1}{8}.
\end{aligned}$$

Finally, the hyperbolic distance $d_H(A, B)$ *between* A *and* B *is* $\ln \frac{9+\sqrt{65}}{4}$.

2.3 The Angles of the Hyperbolic Triangles

The measure of hyperbolic angles are the same as Euclidean angles, the angle between two curves at an intersection is defined as the angle between two tangent lines at that point. Given two hyperbolic rays γ and δ from a point P, denote the angle at P from γ to δ (anti-clockwise) by $\angle(\gamma, \delta)$. Thus, $\angle(\delta, \gamma) = \pi - \angle(\gamma, \delta)$. We can convert hyperbolic angles into interior angles of a Euclidean triangle, so that we can use the Cosine Rule to find the interior angles.

Definition 2.3.1 *Suppose that two curved hyperbolic lines* γ_1 *and* γ_2 *intersect at* P. *The hyperbolic angle* $\angle(\gamma_1, \gamma_2)$ *is called an inner hyperbolic angle if it inside one of* *semi-circles γ_1 and γ_2 only. Otherwise, $\angle(\gamma_1, \gamma_2)$ is called outer hyperbolic angle.*

It's clear if $\angle(\gamma_1, \gamma_2)$ is an inner hyperbolic angle, then $\angle(\gamma_2, \gamma_1)$ is an outer hyperbolic angle. In addition, $\angle(\gamma_2, \gamma_1) = \pi - \angle(\gamma_1, \gamma_2)$

Theorem 2.3.2 *Let m be a vertical hyperbolic line at $D(x_0, 0)$ on the x-axis, and let q be curved hyperbolic line centered at $C(c, 0)$ with radius r. Suppose that m and q intersect at point P. Then,*

$$\angle(m, q) = \begin{cases} \angle PCD, \text{ if } x_0 > c. \\ \frac{\pi}{2}, \text{ if } x_0 = c. \\ \pi - \angle PCD, \text{ if } x_0 < c. \end{cases}$$

Proof. Case 1: If $x_0 > c$. i.e $\angle(m, q)$ and $\angle PCD$ on the same side.

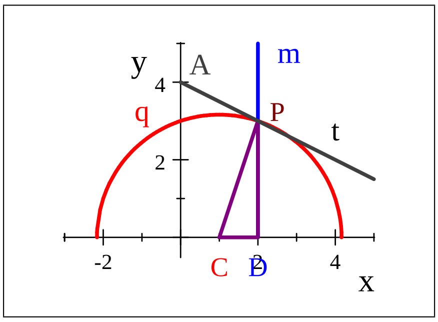

Figure 2.3.2.a The hyperbolic

angle with $x_0 > c$.

In the above figure, notice that both hyperbolic angle $\angle(m, q)$ and interior angle $\angle PCD$ are on the same side, and m is perpendicular to x-axis at D. Since the tangent line t is perpendicular to the radius CP, then

$$\angle(m, q) + \angle CP = \frac{\pi}{2}.$$

On the other hand, in the right angled triangle $\triangle PCD$,

$$\angle PCD + \angle CPD = \frac{\pi}{2}.$$

Thus, $\angle(m,q) = \angle PCD$.

Since, $CP = r$ and $CD = x_0 - c$, $\angle PCD = \cos^{-1}\frac{CD}{CP} = \cos^{-1}\frac{x_0-c}{r}$.

Case 2: If $x_0 = c$, then the raduis CP is perpendicular to the tangent line t, so

$$\angle(m,q) = \frac{\pi}{2}.$$

Case 3: If $x_0 < c$, i.e. $\angle(m,q)$ and $\angle PCD$ aren't in the same side, see Figure 2.3.2. Using the similar argument in case 1 $\angle(q,m) = \angle PCD$, and hence

$$\angle(m,q) = \pi - \angle(q,m) = \pi - \angle PCD = \pi - \cos^{-1}\frac{c - x_0}{r}.$$

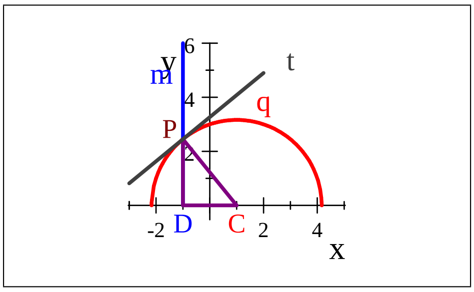

Figure 2.3.2.b The hyperbolic angle with $x_0 < c$.

In summary, we have

$$\angle(m,q) = \begin{cases} \cos^{-1}\frac{x_0-c}{r}, \text{ if } x_0 > c. \\ \frac{\pi}{2}, \text{ if } x_0 = c. \\ \pi - \cos^{-1}\frac{c-x_0}{r}, \text{ if } x_0 < c. \end{cases} \qquad \blacksquare$$

Theorem 2.3.3 *Let γ_1 and γ_2 be two curved hyperbolic lines centered at C_1 and C_2. Suppose that γ_1 and γ_2 intersect. Then the hyperbolic inner angle $\angle(\gamma_1,\gamma_2)$ is*

$$\angle(\gamma_1,\gamma_2) = \angle C_1PC_2.$$

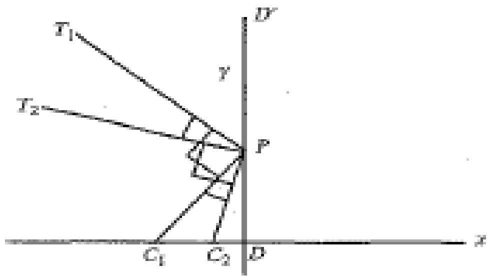

Figure 2.3.3. Hyperbolic inner angle.

Proof. Let D' be a point of γ such that P is between D and D', let PT_1 and PT_2 be the Euclidean tangent line to γ_1 and γ_1 at P (Figure 2.3.3). Since the tangent to a circle is perpendicular to the raduis through the point of contact, it follows that $\angle T_1P\ C_1 = \angle T_2P\ C_2 = \frac{\pi}{2}$. Hence, the $\angle(\gamma_1, \gamma_2) = \angle T_1P\ T_2 = \frac{\pi}{2} - \angle T_2P\ C_1 = \angle C_1PC_2$. ■

Example 2.3.4 *The angles of the hyperbolic triangle with vertices* $A(0, 1)$, $B(2, 1)$, *and* $C(4, 1)$.

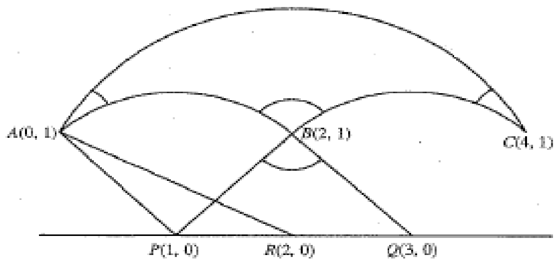

Figure 2.3.4. Angles of hyperbolic triangle.

Example 2.3.5 *It is easy to see that the Euclidean centers of the curved hyperbolic segments* AB, BC, *and* CA *are the points* $P(1, 0)$, $Q(3, 0)$, *and* $R(2, 0)$, *respectively.*

By the Cosine Rule

$$\begin{aligned}\angle PAR &= \cos^{-1}\left(\frac{AP^2 + AR^2 - PR^2}{2AP \cdot AR}\right). \\ &= \cos^{-1}\left(\frac{2+5-1}{2\sqrt{2}\sqrt{5}}\right) = \cos^{-1}\left(\frac{3\sqrt{10}}{10}\right) \approx 0.32.\end{aligned}$$

Hence, from Theorem 2.3.3

$$\angle(AB, AC) = \angle(\gamma_2, \gamma_3) = \angle PAR \approx 0.32.$$

Let γ_1, γ_2 and γ_3 be the curved hyperbolic lines including arc AB, arc BC, respectively.

By the Cosine Rule,

$$\begin{aligned}
\angle PBQ &= \cos^{-1}\left(\frac{BP^2 + BQ^2 - PQ^2}{2BP \cdot BQ}\right). \\
&= \cos^{-1}\left(\frac{2+2-4}{2\sqrt{2}\sqrt{2}}\right). \\
&= \cos^{-1}(0) = \frac{\pi}{2}.
\end{aligned}$$

By Theorem 2.3.3

$$\angle(\gamma_1, \gamma_2) = \angle PBQ \text{ since } \angle(CB, AB) = \angle(\gamma_2, \gamma_1) = \pi - \angle(\gamma_1, \gamma_2).$$

Hence,

$$\angle(CB, AB) = \pi - \angle PBQ = \frac{\pi}{2}.$$

Similarly, we have $\angle(AC, BC) = 0.32$.

Remark: *The sum of three interior angles in the hyperbolic triangle $\triangle ABC$ is less than π :*

$$\begin{aligned}
&\angle(AB, AC) + \angle(CB, AB) + \angle(AC, BC) \\
\approx\ & 0.32 + \frac{\pi}{2} + 0.32 = 2.21 < \pi.
\end{aligned}$$

2.4 The Area of the Hyperbolic Triangle

Definition 2.4.1 *If R is a region in the hyperbolic plane, then its hyperbolic area is defined by*

$$\iint_R \frac{dxdy}{y^2}.$$

Denote the hyperbolic area of R by $ha(R)$.

Lemma 2.4.2 *Let DE be a segment of a curved geodesic. If d and e are the straight geodesics above D and E, respectively, and if R is the portion of the infinite strip between d and e that lies above DE, then the hyperbolic area is $\pi - \delta - \varepsilon$, where δ and ε are the angles interior to R at D and E, respectively.*

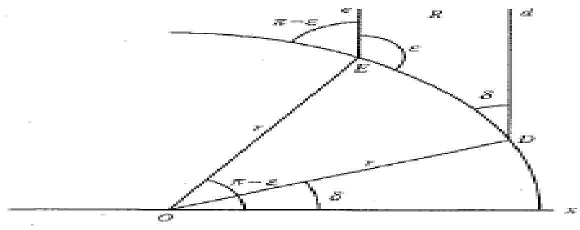

Figure 2.4.2. Hyperbolic area.

Proof. We assume without loss of generality that the curved geodesic containing DE centered at $O(0,0)$ with radius r, and hence DE satisfies the equation of the circle $x^2+y^2=r^2$. From the Theorem 2.3.3 of hyperbolic angles, $\angle xOD = \delta$ and $\angle xOE = \pi - \varepsilon$, and hence the hyperbolic rays from D and E satisfy the equations of vertical lines $x = r\cos\delta$ and $x = r\cos(\pi-\varepsilon) = -r\cos\varepsilon$, respectively.

Now the hyprbolic area of R is

$$\begin{aligned}
ha(R) &= \iint_R \frac{dxdy}{y^2} = \int\limits_{-r\cos\varepsilon}^{r\cos\delta} dx \int\limits_{\sqrt{r^2-x^2}}^{\infty} \frac{dy}{y^2} = -\int\limits_{-r\cos\varepsilon}^{r\cos\delta} \left[\frac{1}{y}\right]_{y=\sqrt{r^2-x^2}}^{y=\infty} dx \\
&= \int\limits_{-r\cos\varepsilon}^{r\cos\delta} \frac{dx}{\sqrt{r^2-x^2}} = \left[\sin^{-1}\left(\frac{x}{r}\right)\right]_{x=-r\cos\varepsilon}^{r\cos\delta} \\
&= \sin^{-1}(\cos\delta) - \sin^{-1}(-\cos\varepsilon) \\
&= \sin^{-1}\left(\sin\left(\frac{\pi}{2}-\delta\right)\right) + \sin^{-1}\left(\sin\left(\frac{\pi}{2}-\varepsilon\right)\right) \\
&= \frac{\pi}{2} - \delta + \frac{\pi}{2} - \varepsilon = \pi - \delta - \varepsilon.
\end{aligned}$$

■

Theorem 2.4.3 *Let α, β and γ be the hyperbolic angles of a hyperbolic triangle*

$\triangle ABC$, then the hyperbolic area of the hyperbolic triangle is equal to

$$ha(\triangle ABC) = \pi - \alpha - \beta - \gamma.$$

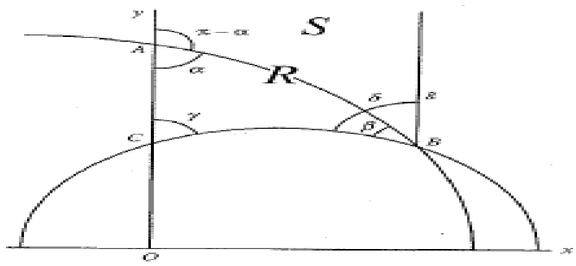

Figure 2.4.3. Area of hyperbolic triangle.

Proof. Let g be the vertical containing B, let R denote the portion of the infinite strip between g and the $y-$axis that lies above the hyperbolic segment BC, and let S denote the portion of the same strip that lies above the hyperbolic AB. Finally, let α, β and γ denote the angles of the triangle at the vertices A, B and C, respectively. Thus,

$$\text{Let } \angle(g, BC) = \delta, \text{ then } \angle(g, BA) = \delta - \beta$$

By the Lemma 2.4.2,

$$\begin{aligned} ha(\triangle ABC) &= ha(R) - ha(S) = \pi - \gamma - \delta - [\pi - (\pi - \alpha) - (\delta - \beta)] \\ &= \pi - \gamma - \delta - \alpha + \delta - \beta \\ &= \pi - \alpha - \beta - \gamma. \end{aligned}$$

■

Remark 2.4.4 *The sum of three interior hyperbolic angles of a hyperbolic triangle is less than π :*

$$\alpha + \beta + \gamma < \pi.$$

In fact, because the hyperbolic area of a hyperbolic triangle must be a positive number $\pi - (\alpha + \beta + \gamma) > 0$. If the respective angles of two hyperbolic triangles are equal, then they have the same hyperbolic area. By the contrast, if the respective angles of two Euclidean triangles are equal, then they are similar, but two similar Euclidean triangles may have different area.

Chapter 3: Discrete Groups and Discontinuous Groups

In this chapter we introduce topology then groups since a topological group is a set which is both a group and has a topology as in section 3. We present the discrete group, where we use a discrete topology to define in Section 4. Also, in Section 5 we describe the properly discontiuous groups and thier properties.

3.1 Topology

Topology is the mathematical study of shapes. It is an area of mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending. This includes such properties as connectedness, continuity and boundary. Topology developed as a field of study out of geometry and set theory, through analysis of such concepts as space, dimension, and transformation. In this section we introduce various topological spaces such as discrete space, Hausdorff space, compact space. In addition, homeomorphisms and quotient space.

Definition 3.1.1 *A topological space is a pair* $(X\ ,\tau)$, *or simple* x *if without confusion, where* X *is a set and* τ *is a collection of subsets of* X *satisfying the following conditions:*

(a) ϕ *and* $X \in \tau$.

(b) Any union of elements of τ *is in* τ.

(c) Any finite intersection of elements of τ *in* τ.

A subset U *of* X *is called open if* $U \in \tau$. *A subset* F *of* X *is called closed if its complement* $X - F$ *is open.*

Remark 3.1.2 *: The condition (c) can be replaced by the following condition :·*

$$\text{If } U_1 \in \tau \text{ and } U_2 \in \tau \text{ then } U_1 \cap U_2 \in \tau.$$

Example 3.1.3 *Given a set X, let τ be a collection of all subset of X. Obviously, $(X,\ \tau)$ here is a topological space, and τ is called a discrete topology. Thus, in a discrete topological space, every subset is both open and closed.*

Theorem 3.1.4 *A metric space is a topological space.*

Proof. Let $\tau = \{\forall x \in U, \exists \varepsilon > 0$, such that a ball $B(x, \varepsilon) \subset U\ \}$. By applying Definition 3.1.1, then

(a) $\phi \in \tau$ and $X \in \tau$

(b) Let $(U_\alpha)_{\alpha \in J}$ be a collection of open sets $\cup_{\alpha \in J} U_\alpha$ is an open set. Let $x \in \cup_{\alpha \in J} U_\alpha$, then $\exists \alpha.$, such that $x \in U_{\alpha.}$ since $U_{\alpha.}$ is open, then $\exists \varepsilon > 0$, such that $B(x, \varepsilon) \subset U_{\alpha.}$, since $U_{\alpha.} \subset \cup_{\alpha \in J} U_\alpha$, then $B(x, \varepsilon) \subset \cup_{\alpha \in J} U_\alpha$. In conclusion, $\cup_{\alpha \in J} U_\alpha$ is an open set.

(c) Let U_1 and U_2 be two open sets in τ, then $U_1 \cap U_2$ is an open set. Let $x \in U_1 \cap U_2$, since $x \in U_1, \exists \varepsilon_1 > 0$, such that $B(x, \varepsilon_1) \subset U_1$ and $x \in U_2, \exists \varepsilon_2 > 0$, such that $B(x, \varepsilon_2) \subset U_2$.

Let $\varepsilon = min(\varepsilon_1, \varepsilon_2)$, then $B(x, \varepsilon) \subset B(x, \varepsilon_1) \subset U_1$ and $B(x, \varepsilon) \subset B(x, \varepsilon_2) \subset U_2$ this implies $B(x, \varepsilon) \subset U_1 \cap U_2$, then $U_1 \cap U_2$ is an opent set. ■

Example 3.1.5 *Let τ be a topology induced by the Euclidean distance in $\mathbb{R}$, then τ is called the standard topology over $\mathbb{R}$. It is followed that a subset of $\mathbb{R}$ is open if its a union of open intervals [16]. Smilarly, $\mathbb{R}^n$ is a topological space induced by the Euclidean distance.*

The next theorem shows that topology is hereditary property.

Theorem 3.1.6 *Every subset Y of a topological space $(X,\ \tau)$ is a topological space, which is called subspace of X.*

Proof. Let a collection $\sigma = \{U_\alpha \cap Y : U_\alpha \in \tau\}$:

(a) Take $U_1 = \phi \in \tau$, then $U_1 \cap Y = \phi \in \sigma$ and take $U_2 = X \in \tau$, then $U_2 \cap Y = Y \in \sigma$.

(b) $\cup_\alpha (U_\alpha \cap Y) = (\cup_\alpha U_\alpha) \cap Y \in \sigma$, as $\cup_\alpha U_\alpha \in \tau$.

(c) $\cap_{i=1}^{n}(U_i \cap Y) = (\cap_{i=1}^{n}U_i) \cap Y \in \sigma$, as $\cap_{i=1}^{n}U_i \in \tau$.

By Definition 3.1.1 Y is a topological space. ■

Definition 3.1.7 *A topological space $(X\ ,\tau)$ is said to be a Haussdorff space if for all distinct point x and $y \in X$. There exist two open sets U and V with $x \in U$ and $y \in V$, such that $U \cap V = \phi$.*

Example 3.1.8 *$\mathbb{R}$ is Haussdorff topological space. In fact, for all $x \neq y \in \mathbb{R}$ and let, $U_x = (x - \frac{|y-x|}{2}, x + \frac{|y-x|}{2})$and $U_y = (y - \frac{|y-x|}{2}, y + \frac{|y-x|}{2})$, where $\varepsilon > 0$. Then, $U_x \cap U_y = \phi$. So, $\mathbb{R}$ is Haussdorff space.*

Definition 3.1.9 *Given a topological space$(X\ ,\tau)$. A collection $\{U_\alpha \in \tau : \alpha \in J\}$ is called an open cover of X if $\cup_\alpha U_\alpha = X$, where J is an index set, the cardiuality $|J|$ maybe uncounatable.*

A topological space X is called a compact space if every given open cover of X:$\{U_\alpha \in \tau : \cup_\alpha U_\alpha = X\}$. There exists a finite subcover $\{U_1, \ldots, U_m\}$, such that $X = \cup_{k=1}^{m} U_k$.

Example 3.1.10 *Let X be any set and $\tau = \{\phi\} \cup \{U \subseteq X : U^c = X - U$ is finite$\}$. $(X\ ,\tau)$ is a topological space and it isn't difficult to show that by Definition 3.1.1.*

Moreover, a cofinite topology is a compact space. Let $(U_\alpha)_{\alpha \in J}$ be an open cover of X, let $U_B$one of these open sets $U_B^c = X - U_B$ is finite $= \{x_1, ..., x_n\}$. Let $U_{\alpha i}$ be an open set contain x.

$X = U_B \cup U_{\alpha 1} \cup .... \cup U_{\alpha n}$ is a finite subcover of x.

Remark 3.1.11 *If X is finite, the cofinite topological space is the same of discrete space. Since every subset and complement are finite.*

Theorem 3.1.12 *Every closed subset of a compact space is compact.*

Proof. Let $(X\ ,\tau)$ be a compact. Let Y be a closed set of X. Then $Y^{\ c}$ is open in X. Let $\{U_\alpha \in \tau : \alpha \in J\}$ be an open cover of Y, then $X : \{U_\alpha \in \tau : \alpha \in J\} \cup Y^c$ is an open cover of X. Since X is compact, there is a finite subcover of X, then $Y \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ a finite subcover of Y, so Y is compact subspace. ■

Theorem 3.1.13 *[16] A subspace X of $\mathbb{R}^n$ is compact if and only if X is bounded and closed.*

Theorem 3.1.14 *Every compact subspace Y of a Hausdorff space is closed .*

Proof. Let $(Y\ ,\tau)$ be a compact. Let Y be a closed set of X. Then Y^c is open in X. Let X is Hausdorff let $y \in Y^c$ for $x \in Y, x \neq y$, then, there exists U and V open sets such that $x \in U, y \in V$ and $U \cap V = \phi, Y \subset \cup_{x \in y} U_\alpha$ (all Y is covered) is open cover of Y, since$Y$ is compact, this cover has a subcover $U_{x_1}, ..., U_{xp}$. Take $U_{x_1} \cup U_{x_2} \cup U_{xp}$ is open and $\subset Y^c, Y^c$ is open implies Y is compact. ■

Let X, Y be two topological spaces $f : X \to Y$ is continuous if each open set V in Y, such that the preimage $f^{-1}(V)$ is open in X.

If for whatever reason we prefer closed sets, we can use the following theorem

Theorem 3.1.15 *A function $f : X \to Y$ between two topological spaces X and Y is continuous if and only if each closed set F in Y, such that the preimage $f^{-1}(F)$ is closed in X.*

Definition 3.1.16 *A function $f : X \to Y$ between two topological spaces X and Y, which maps any open set U in X , then $f(U)$ is open in Y. Likewise, a closed map is a function which maps closed sets to close sets.*

Definition 3.1.17 *A function $f : X \to Y$ between two topological spaces X and Y is called a homeomorphism if it has a following properties :*

(a) f is a bijection (one to one and onto).

(b) f is continuous.

(c) The inverse function f^{-1} is continuous .

Theorem 3.1.18 *If f is one to one continuous map of a compact space X onto Hausdorff space Y, then f is homeomorphism.*

Example 3.1.19 *Any open interval of $\mathbb{R}$ is homeomorphic to any other open interval . Consider $X = (a, b)$ and $Y = (-1, 1)$.*

Lemma 3.1.20 *Every one to one and continuous mapping of an open set of the plane* $\mathbb{R}^2$ *onto a plane set* $\mathbb{R}^2$ *is homeomorphism .*

Proof. let f is an open map which is a bijection (one to one and onto) and continuous in $\mathbb{R}^2$. Let f^{-1} is continuous, $f : Y \to X$ then there exist an open set $U \in X$ such that $f(U) = (f^{-1})^{-1}(U) \in Y$, so that f is homeomorphism in $\mathbb{R}^2$. ■

Definition 3.1.21 *The class of open subsets of* Y *is indeed a topology* τ_f *on* Y *and is called the quotient topology induced by* f*. With this topology* f *is automatically continuous.*

Proposition 3.1.22 *Let* X *be a topological space and suppose that* f *maps* X *onto* Y*. Let* τ *be any topology on* Y *and let* τ_f *be a quotient topology on* Y *induced by* f*.*

(a) If $f : x \to (Y, \tau)$ is continuous, then $\tau \subset \tau_f$.

(b) If $f : x \to (Y, \tau)$ is continuous and open, then $\tau = \tau_f$.

Proof. Suppose that $f : x \to (Y, \tau)$ is continuous. If V is in τ, then $f^{-1}(V)$ is in open in X and so V is in τ_f . If in addition , $f : x \to (Y, \tau)$ is an open map then V in τ_f implies that $f^{-1}(V)$ is in open in X and so $f(f^{-1}V)$ is in τ. As f is surjective, $f(f^{-1}V) = V$, so $\tau \subset \tau_f$. ■

An alternative approach to the quotient topology is by equivalence relation. If X carries an equivalence relation R with equivalence classes $[x]$, then X/R (the space of equivalence classes) inherits the quotient topology induced by the map $x \to [x]$.

Definition 3.1.23 *Let* $PSL(2, \mathbb{R})$ *be the set of all linear fractional transformations with the real coefficient from* $\overline{\mathbb{C}}$ *to itself.*

$$PSL(2, \mathbb{R}) = \left\{ T : \overline{\mathbb{C}} \to \overline{\mathbb{C}}, T(z) = \frac{az+b}{cz+d} : z \in \mathbb{C}, a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$

Remark 3.1.24 *(1) For each* $\frac{az+b}{cz+d} \in PSL(2, \mathbb{R})$ *we can assume that* $a \geq 0$*. If* $a < 0$*, then* $-a > 0$*. In fact,* $\frac{az+b}{cz+d} = \frac{-az-b}{-cz-d} \in PSL(2, \mathbb{R})$*, and* $(-a)(-d) - (-b)(-c) = ad - bc = 1$*.*

(2) If $ad-bc>0,$ *then* $\frac{az+b}{cz+d}\in PSL\left(2,\mathbb{R}\right).$ *Because that*

$$\frac{az+b}{cz+d}=\frac{\frac{a}{\sqrt{ad-bc}}z+\frac{b}{\sqrt{ad-bc}}z}{\frac{c}{\sqrt{ad-bc}}z+\frac{d}{\sqrt{ad-bc}}z}\in PSL\left(2,\mathbb{R}\right).$$

as

$$\frac{a}{\sqrt{ad-bc}}\frac{d}{\sqrt{ad-bc}}-\frac{b}{\sqrt{ad-bc}}\frac{c}{\sqrt{ad-bc}}=\frac{ad-bc}{ad-bc}=1.$$

Theorem 3.1.25 $PSL(2,\mathbb{R})$ *is a metric space and hence it is a topological space.*

Proof. Let $X=(a,b,c,d)\in\mathbb{R}^4 : ad-bc=1$, then X is a subspace of $\mathbb{R}^4$, since $\phi : PSL(2,\mathbb{R})\to X$, by $\frac{az+b}{cz+d}\longmapsto(a,b,c,d\ )$, so the metric is given by:

$$\|T\|=\sqrt{a^2+b^2+c^2+d^2}.$$

For each $f(z)=\frac{az+b}{cz+d}\in PSL\left(2,\mathbb{R}\right).$ Thus, the distance $d(T,S)=\|T-S\|$ for T and $S\in PSL(2,\mathbb{R})$. That is $PSL\left(2,\mathbb{R}\right)$ is a metric space. By Theorem 3.1.4, $PSL(2,\mathbb{R})$ is topological space. ■

3.2 Groups

In Mathematics, a group is an algebraic structure consisting of a set of elements together with an operation that combines any two elements to form a third element. The operation satisfies three conditions called the group axioms, namely associativity, identity and invertibility.

Definition 3.2.1 *A group is an ordered pair* $(G,*)$*, where* G *is a set and* $*$ *is a binary operation on* G *satisfying the following condition :*

(1) $(a*b)*c=a*(b*c)$ *for all* $a,b,c\in G$*, i.e,* $*$ *is associative.*

(2) There exists an element e *in* G*, called an identity of* G*, such that for all* $a\in G$ *we have* $a*e=e*a$*.*

(3) For each $a\in G$*, there is an element* a^{-1} *of* G*, called an inverse of* a*, such that* $a*a^{-1}=a^{-1}*a=e$*.*

Remark 3.2.2 *The group* $(G, *)$ *is called abelian (or commutative) if*

$$a * b = b * a \text{ for all } a, b \in G.$$

Example 3.2.3 $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *and* $\mathbb{C}$ *are groups under addition. On the other hand,* $\mathbb{Q}^*, \mathbb{R}^*, \mathbb{C}^*$ *are groups under multiplication.*

Let G be a group. The subset H of G is a subgroup of G if H is nonempty and H is closed under product and inverse (i.e. $x, y \in H$ implies $x^{-1} \in H$ and $xy \in H$). If H is a subgroup of G we shall write $H \leq G$.

Definition 3.2.4 *Subgroups of* G *are just subsets of* G *which are themselves groups with respect to the operation defined in* G*. When we say that* H *is a subgroup of* G *we shall always mean that the operation for the group* H *is the operation on* G *restricted to* H*.*

Example 3.2.5 $\mathbb{Z} \leq \mathbb{Q}$ *and* $\mathbb{Q} \leq \mathbb{R}$ *with the operation of addition.*

A group G is cyclic if and only if it can be generated by a single element, i.e. there is an element $x \in G$ such that $G = \{x^n : n \in \mathbb{Z}\}$, where

$$x^n = x \ * \ x \ * \ \cdots \ * \ x$$

under the operation $*$ denoted by $\langle x \rangle$. In additive notation G is cyclic

$$G = \{nx : n \in \mathbb{Z}\}$$

is called a generator of H.

Definition 3.2.6 *Let* G *be a group and* N *be a subgroup of* G*.For each* $g \in G$*. Define*

$$gN = \{gn : n \in N\} \text{ and } Ng = \{ng : n \in N\}$$

which are called a left coset and a right coset of N *in* G*, respectively.*

Any element of a coset is called a *representative* for the coset.

Definition 3.2.7 *N is called a normal subgroup if N satisfies the condition:*

$$gN = Ng \text{ for all } g \in G$$

doneted by $N \trianglerighteq G$.

Theorem 3.2.8 *Let G be a group and N be a normal subgroup, the set of all cosets of N in G, is denoted by $G\ /N$. Then $G\ /N$ is a group under operation*

$$(uN)(vN) = (uv)N. \tag{3.1}$$

$G\ /N$ is called the quotient group or factor group of G by N.

Proof. Let $G\ /N$ is closed under the given operation 3.1.

(1) N is the identity. Notice that $eN = N$ if e is the identity of G.

$$(uN)(N) = (uN)(eN)$$

for all $uN \in G\ /N$. Similarly,

$$N(uN) = uN = (ue)N = N.$$

(2) For each $uN \in G\ /N$, the inverse of uN is $u^{-1}N$, as $(uN)(u^{-1}N) = (uu^{-1})N = eN = N$.

Obviously, $G\ /N$ satisfies the Assosiative Law, becuase G satisfies the Assosiative Law. ■

For example, each subgroup of an abelian group is normal.

Example 3.2.9 *Let $G = (\mathbb{Z}, +)$, and let $N = n\mathbb{Z} = \{..., -2n, -n, 0, n, 2n, ...\}$, then $N \trianglerighteq G$, and the quotient group $\mathbb{Z}/n\mathbb{Z} = \{n\mathbb{Z}, 1+n\mathbb{Z}, 2+n\mathbb{Z}, 3+n\mathbb{Z}, \cdots, (n-1)+n\mathbb{Z}\}$.*

Definition 3.2.10 *The map $\phi : G \to H$ is called an isomorphism and G and H are said to be isomorphism type, written $G \cong H$, if*

(1) ϕ is homomorphism (i.e., $\phi(xy) = \phi(x)\phi(y)$), and

(2) ϕ is bijection.

Let $SL(2,\mathbb{R})$ be the group of all real $2\times\ 2$ matrices with determinant one under multiplication:

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$$

Definition 3.2.11 *$PSL(2,\mathbb{R})$ is group under composition.*

Let $\phi : PSL(2,\mathbb{R}) \to SL(2,\mathbb{R})/\{\pm I\}$ defined by $\frac{az+b}{cz+d} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then ϕ is a bijection from $PSL(2,\mathbb{R})$ to $SL(2,\mathbb{R})$. It's not difficult to see that matrix multiplication in $SL(2,\mathbb{R})$ is equivalent to the composition of $PSL(2,\mathbb{R})$. Which gives that, ϕ is homomorphism. Thus, ϕ is an isomorphism from $SL(2,\mathbb{R})$ to $SL(2,\mathbb{R})$, so $PSL(2,\mathbb{R})$ is isomorphic to $SL(2,\mathbb{R})/\{\pm I\}$, as we can write

$$PSL(2,\mathbb{R}) \cong SL(2,\mathbb{R})/\{\pm I\}.$$

3.3 Topological Groups

It is reasonable that one could associate two kinds of structure with a set. This is how topological group is defined. A topological group is both a group and topological space, the two structures being related by the requirement as the following definition.

Definition 3.3.1 *Let G be a group under the operation $*$. A topological space (X, τ) on G said to be a topological group if:*

(1) $(x, y) \to xy$ (of $G \times G$ onto G) is continuous.

(2) The map $x \mapsto x^{-1}$ (of G onto G) is continuous.

Remark 3.3.2 *Two topological groups are called isomorphism if there is a bijection, homomorphism and homeomorphism. This is the natural identification of topological group.*

Theorem 3.3.3 *If* H *is a normal subgroup of a topological group* G*, then* G/H *with usual structures is a topological group.*

The following groups are an examples of topological groups.

Example 3.3.4 *(1) The real numbers under addition denoted by* $(\mathbb{R}, +)$.

(2) The positive real numbers under multiplication denoted by $(\mathbb{R}^+, \times)$.

(3) The unit circle S^1 *in* $\mathbb{C}$ *is a topological group under complex multiplication denoted by* $(S^1, \cdot_{\mathbb{C}})$.

Theorem 3.3.5 $PSL(2, \mathbb{R})$ *is a topological group.*

Proof. By Theorem 3.1.25 $PSL(2, \mathbb{R})$ is topology and from theorem 3.2.11, $PSL(2, \mathbb{R})$ is group.

(1) Let show $f : X \times X \to G$ given by $(T, S) \to T \circ S$ is continuous. Since $PSL(2, \mathbb{R})$ is group under composition. $T \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $T \circ S \in PSL(2, \mathbb{R})$ is continuous.

(2) If $x \to X$ is by $T \to T^{-1}$ is continuous. Because

$$T^{-1} \to \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in PSL(2, \mathbb{R})$$

is continuous. ■

3.4 Discrete Topological Groups

Since any group can be equipped with a discrete topology, an arbitrary group can be regarded as a discrete topological group. However, a subgroup of a non-topological group need not to be a discrete subgroup with respect to the induced topology. The group of PSL $(2, \mathbb{R})$ is a topological group under composition with respect to the metric topology. A subgroup G of the group $PSL\ (2,\mathbb{R})$ is called a *discrete group* if the induced topology from the metric coincides with the discrete topology. Otherwise G is called a *non-discrete group*. One can show the following characterizations of a discrete group immediately. Notice that a topological group is topologically homogeneous.

Proposition 3.4.1 *Let G be a subgroup of a topological group $PSL\ (2,\mathbb{R})$, then the following are equivalent:*

(1) G is a discrete group.

(2) The identity of G is isolated in G.

(3) Each element of G is isolated in G.

Theorem 3.4.2 *The subgroup G of $SL(2,\mathbb{R})$ is discrete if and only if for each positive k the set*

$$G_k = \{A \in G : \|A\| \leq k\}$$

is finite and $SL(2,\mathbb{R})$ is countable $G = \cup_{n=1}^{\infty} G_n$, where G_n is finite set A in G with $\|A\| \leq n$.

Proof. (1) let's show that $G_k = \{A \in G : \|A\| \leq k\}$, for $k \in \mathbb{N}$ is finite. Suppose that G_k is infinite, then there exists an infinite sequence from G_k:

$$A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},\ a_n, b_n, c_n, d_n \in \mathbb{R}$$

Since $\|A_n\| = \sqrt{|a_n|^2 + |b_n|^2 + |c_n|^2 + |d_n|^2} \leq k$, thus $|a_n| \leq k, |b_n| \leq k, |c_n| \leq k, |d_n| \leq k$. It follows that$\{|a_n|\}$ is a sequence of real numbers and is bounded. By Bolzano-Weierstrass Theorem, there exists a subsequence $\{|a_{n_k}|\}$ which converges to $a \leq k$. Consider the bounded sequence $|b_{n_k}|$, then there exists a subsequence $\{|b'_{n_k}|\} \to b$, consider the sequence $|c_{n_k}|$, then $\exists$ subsequence $\{|c''_{n_k}|\} \to c$, consider the sequence $|d'''_{n_k}|$, then $\exists$ subsequence $\{|d_{n_{3_k}}|\} \to d$. Therefore, there is a subsequence$\{A'''_{n_k}\}$ such that $A'''_{n_k} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} = X$, for sufficiently large $n_k^3 \geq N$. That contradicts with our assumption.

(2) Notice that, $G = \cup_{n=1}^{\infty} G_n$. Since that a countable union of finite set is countable, therefore G is countable. ■

Definition 3.4.3 *A discrete subgroup of a topological group G is a subgroup of G, which a topological space inherits discrete topology from G.*

Example 3.4.4 *(1) The identity is always a discrete subgroup.*

(2) Any finite topological group is discrete topological group.

(3) Any discrete subgroup of real numbers under addition denoted by $(\mathbb{R}, +)$ is infinite cyclic.

(4) Any discrete subgroup of the unit circle S^1 in $\mathbb{C}$ under complex multiplication denoted by $(S^1, \cdot_{\mathbb{C}})$ is fiinite cyclic .

Lemma 3.4.5 *Any non-trivial discrete subgroup of $\mathbb{R}$ is infinite cyclic, where cyclic is $\forall x > 0$, $\{nx : n \in \mathbb{Z}\}$.*

Proof. Let Γ be a discrete subgroup of $\mathbb{R}$. Of course, $0 \in \Gamma$, and there exists a smallest positive $x \in \Gamma$,otherwise Γ would not be discrete. Then $\{nx : n \in \mathbb{Z}\}$ is a subgroup of Γ. Suppose there is $y \in \Gamma, y \neq nx$. We may assume y>0, otherwise we take $-y$ which also belongs to Γ. There exists an integer $k \geq 0$ such that $kx < y<(k+1)x$, and $y - kx < x$ and $y - kx \in \Gamma$ which contradicts the choice of x. ■

Lemma 3.4.6 *Any discrete subgroup of S^1 is finite cyclic.*

Proof. Let Γ now be a discrete subgroup of $S^1 = \{z \in \mathbb{C} : z = e^{i\phi}\}$. By discreteness there exists $z = e^{i\phi 0} \in \Gamma$, with the smallest argument ϕ_0, and for some $m \in \mathbb{Z}$, $m\ \phi_0 = 2\Pi$, otherwise we get a contradiction with the choice of ϕ_0. ■

3.5 Properly Discontinuous Group Action

Definition 3.5.1 *A function $f : X \to Y$ is called a transformations if it is a bijection, i.e, it is injective (one-to-one) and surjective (onto).*

Definition 3.5.2 *A group G is said to be acting on a set X if every element of G gives a transformation on X.*

Let G be a group, and let X be a set. Suppose ϕ is a function: $\phi : G \times X \to X$ defined by $(g, x) \to \phi(g, x)$, where $\phi(g, x)$ is a point of X, which can be written as $\phi(g)(x)$, or simply as $g(x)$, The function ϕ is called a group action of G on X if it satisfies the following two conditions:

(a) Compatibility: $(gh)(x) = g(h(x))$, for $g, h \in G$ and all $x \in X$, here, gh denotes the result of applying the group operation of G to the elements g and h.

(b) Identity: $e(x) = x$, for $x \in X$, here e is the identity of G.

The set X is called a G-set and the group G is said to act on X.

Let X be a topological space. If G is a group of transformation of X then the *stabilize*r G_x(in G) of x is the subgroup of G defined by

$$G_x = \{g \in G : g(x) = x\}.$$

On the other hand, the *orbit (or G-orbit)* $G(x)$ of x is the subset of X donated by

$$G(x) = \{g(x) \in X : g \in G\}.$$

While, the fixed points of transformation of g are those x which satisfy

$$g(x) = g.$$

Definition 3.5.3 *Let X be any topological space and G a group of homeomorphism of X onto itself. We say that G acts discontinuously on X if and only if for every compact subset K of X, $g(K) \cap K = \phi$, except for a finite number g in G.*

Theorem 3.5.4 *Suppose that G acts discontinuously on X : The following state- ments are true.*

(1) Every subgroup of G acts discontinuously on X.

(2) If ϕ is a homeomorphism of X onto Y, then $\phi G\,\phi^{-1}$ acts discontinuously

on Y .

(3) If Y is a G- invariant subset of X, then G acts discontinuously on Y.

(4) If $x \in X$ and if $g_1, g_2, \cdots$ are distinct elements of G, then the sequence $g_1(x)$, $g_2(x), \cdots$ cannot converge to any y in X.

(5) If $x \in X$,the stabilizer G_x is finite.

(6) If $X \subset \overline{\mathbb{R}^3}$, then G is countable, where $\overline{\mathbb{R}^3}$ is $\mathbb{R}^3 \cup \{\infty\}$.

Proof. Clearly 1 and 2 are true. If $Y \subset X$, then any compact subset of Y is also a compact subset of X and 3 follows. To prove 4, observe that if the given sequence converges to y, then $K = \{y, x, g_1(x), g_2(x), \cdots\}$ is a compact set. As $g_n(K) \cap K \neq \phi$ $(n = 1, 2, \cdots.)$ and as the g_n are distinct, G cannot act discontinuously on X : thus 4 follows. For each x in X, $\{x\}$ is compact, thus 5 is direct consequence of above definition. Finally, there is 1-1 correspondence between G/G_x and the orbit $G(x)$ and so by 5, G is countable if and only if $G(x)$ is countable. Now any countable set set in $\overline{\mathbb{R}^3}$ contains a limit point of itself and so by 4, $G(x)$ must be countable.This proves 6. ■

A continuous group action of topological group G on the topological space X is called *proper* if the map from $G \times X \to X \times X$ taking (g, x) to (gx, x) is proper. If the group G is discrete, then the action is called *propelry discontinuous.*

Theorem 3.5.5 *G acts properly discontinuously on X if and only if each point $x \in X$ has a neighborhood V such that $T(V) \cap V = \phi$, except for finitely many $T \in G$.*

Proof. Suppose G acts properly discontinuously on X, then each orbit G_x is finite. This implies that for any point x there exists a ball $B_\varepsilon(x)$ centered at x of radius ε containing no points of G_x other than x. Let $V \subset B_{\varepsilon/2}(x)$ be a neighborhood of x, then $T(V) \cap V \neq \phi$ implies that $T \in G_x$, hence it is possible for only finitely many$T \in G$. Conversely if $T(V) \cap V \neq \phi$, for finitely many $T \in G$ holds, we have to show that each G-orbit is discrete and that the stabilizer of each point z , G_z, has finite order. If G_z is not discrete, it has a limit point, say z_0, and any neighborhood of z_0 will meet infinitely many of its images under G, a contradiction with $T(V) \cap V \neq \phi$, for finitely many $T \in G$. Similarly, if $T(z) = z$, for infinitely many $T \in G$, then any neighborhood V of z meets infinitely many of its images under G. ∎

Definition 3.5.6 *Given a metric space X, a group G which is a subgroup of the isometries of X acts properly discontinuously on X if and only if one of the following three conditions hold:*

(1) The G orbit of any point is locally finite.

(2) The G orbit of any point is discrete and the stabilizer of that point is finite.

(3) For any point, there is a neighborhood of that point,V, for which only finitely many $T \in G$ satisfy $T(V) \cap V \neq \phi$.

A collection of sets S_α is considered locally finite if and only if for any compact set K, $S_\alpha \cap K \neq \phi$ for only finitely many α.

Chapter 4: Fuchsian Groups

A Fuchsian group is a discrete subgroup of the group $PSL(2,\mathbb{R})$ of linear fractional transformations of one complex variable. We describe three classification of $PSL(2,\mathbb{R})$ according to the trace, fixed points and conjugattion in Section 1. We then introduce the notion of Fuchsian groups and characterizations in Section 2. In this chapter, we focus on the algebaic properties of Fuchsian groups such as cyclic Fuchsian groups and abelian Fuchsian groups. We discuss non-elementary and elementary Fuchsian groups in Sections 3 and 4.

4.1 Three Classification of $PSL(2,\mathbb{R})$

As Fuchsian groups are discrete subgroups of the group $PSL(2,\mathbb{R})$, it helps us to understand Fuchsian groups better, if we can understand what kind of elements can be found in the group $PSL(2,\mathbb{R})$. We describe three classification of $PSL(2,\mathbb{R})$ according to the trace, fixed points and conjugattion in this section.

A way to classify these transformations is by their sets of fixed points. Suppose that we have a non-identity transformation $T(z) = \frac{az+b}{cz+d} \in PSL(2,\mathbb{R})$, and we want to find the fixed points of T. Let $\frac{az+b}{cz+d} = z$, then it's equivalent to the equation:

$$cz^2 + (d-a)z - b = 0. \tag{4.1}$$

Notice that $T(\infty) = \lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c}$. If $T(\infty) = \infty$ then $\frac{a}{c} = \infty$, and hence $c = 0$. So $T(\infty) = \infty$ is equivalent to $c = 0$.

If $c = 0$, then $T(z) = a^2z + ba$ as $ad - bc = ad = 1$, i.e., $d = \frac{1}{a}$; and hence ∞ is a fixed point. Moreover, the quadratic equation (4.1) becomes the linear equation $(d-a)z = b$, which gives that

$$z = \begin{cases} \infty \text{ and } \frac{ba}{1-a^2} \in \overline{\mathbb{R}}, \text{ if } c = 0 \text{ and } a \neq 1; \\ \infty \in \overline{\mathbb{R}}, \text{ if } c = 0 \text{ and } a = 1. \end{cases} \tag{4.2}$$

Therefore, if $c = 0$, then T has either one fixed point $\infty \in \overline{\mathbb{R}}$ when $a = 1$, or two fixed points ∞ and $\frac{ba}{1-a^2} \in \overline{\mathbb{R}}$ when $a \neq 1$.

If $c \neq 0$, then $T(\infty) \neq \infty$. We can use the quadratic formula to solve this equation (4.1):

$$z = \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c} \tag{4.3}$$

Notice that $ad - bc = 1$, hence $bc = ad - 1$. We can rearrange the discriminant $\triangle$:

$$\begin{aligned}
\triangle &= (d-a)^2 + 4bc \\
&= d^2 + a^2 - 2ad + 4bc \\
&= d^2 + a^2 - 2ad + 4(ad - 1) \\
&= d^2 + a^2 + 2ad - 4 \\
&= (d+a)^2 - 4.
\end{aligned}$$

Now we have,

$$z = \frac{a - d \pm \sqrt{(d+a)^2 - 4}}{2c}. \tag{4.4}$$

Thus, the type of fixed points depends only on $(d+a)^2$. It is the motivation of the next two Definitions 4.1.1 and 4.1.2 of trace and types, and next Theorem 4.1.3 of fixed point characterization for transformations in $PSL(2, \mathbb{R})$.

Definition 4.1.1 *For each element $T \in PSL(2, \mathbb{R})$, define the trace of T to be*

$$tr(T) = |a + d|.$$

Definition 4.1.2 *Let T be a non-identity element in $PSL(2, \mathbb{R})$.*

(a) T is called an elliptic element if $tr(T) < 2$.

(b) T is called a parabolic element if $tr(T) = 2$.

(c) T is called a hyperbolic element if $tr(T) > 2$.

From the disccusion at the beginning of this section, we immediately have the following theorem.

Theorem 4.1.3 *Let T be a non-identity element in $PSL(2, \mathbb{R})$, then*

(a) T is an elliptic if and only if it fixes two complex points in $\mathbb{C}$, and hence one of them in $\mathbb{H}$.

(b) T is a parabolic if and only if it fixes exactly one point in $\overline{\mathbb{R}}$.

(c) T is a hyperbolic if and only if it fixes exactly two points in $\overline{\mathbb{R}}$.

Proof. Since T is a non-identity element in $PSL(2, \mathbb{R})$, then its fixed points are given by Formulae (4.2) and (4.4).

Case I: Suppose that $c \neq 0$, then $T(\infty) \neq \infty$.

(a) Since T is elliptic if and only if $tr(T) = |a + d| < 2$, i.e., the discriminant

$$\Delta = (d + a)^2 - 4 < 0.$$

Thus, T has two complex fixed points in $\mathbb{C}$, and hence one of them in $\mathbb{H}$ as two complex numbers are conjugate, one above x-axis, and the other below x-axis.

(b) Since T is a parabolic if and only if $tr(T) = |a + d| = 2$, i.e.the discriminant

$$\Delta = (d + a)^2 - 4 = 2.$$

Thus, T has exactly one point in $\mathbb{R}$.

(c) Since T is a hyperbolic if and only if $tr(T) = |a + d| > 2$, i.e.the discriminant

$$\Delta = (d + a)^2 - 4 > 2.$$

Thus, T has exactly two point in $\mathbb{R}$.

Case II: Suppose that $c = 0$, then $T(\infty) = \infty$. Notice that, we can assume $a > 0$ by the Remark 3.1.24, so $tr(T) = |a + d| = a + \frac{1}{a} \geq 2$, because that $a + \frac{1}{a} - 2 = (\sqrt{a} - \frac{1}{\sqrt{a}}) \geq 0$.

That is, if $c = 0$, then T has either one fixed point $\infty \in \overline{\mathbb{R}}$ when $a = 1$, or two fixed points ∞ and $\frac{ba}{1-a^2} \in \overline{\mathbb{R}}$ when $a \neq 1$.

Therefore, (a), (b) and (c) are held. ■

Remark: The identity element $T(z) = z$ likes a parabolic element in terms of its trace because $tr(T) = |a + d| = 2$. But in terms of fixed points, $T(z) = z$ fixes all points in $\overline{\mathbb{C}}$, and hence it is not parabolic. This is why we should give the charaterizations are for non-identity elements.

Example 4.1.4 *Let* $e(z) = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$, $\theta \in (0, \pi)$, *then*

(1) $e(z) \in PSL(2, \mathbb{R})$ *as*

$$ad - bc = cos^2\theta + \sin^2\theta = 1.$$

(2) $e(z)$ *is an elliptic element, because the trace of* $e(z)$ *is* $2|\cos\theta| < 2$, *if* $\theta \in (0, \pi)$.

Alternatively, $e(z)$ *is elliptic as it fixes one point only in* $\mathbb{H}$. *In fact, by the Formula (4.3) the fixed points are*

$$z = \frac{a - d \pm \sqrt{(d + a)^2 - 4}}{2c} = \frac{\pm\sqrt{-4\sin^2\theta}}{2\sin\theta} = \pm i \ ,$$

and hence only $i \in \mathbb{H}$.

Example 4.1.5 *Let* $p(z) = z + b, b \in \mathbb{R}$, *then* $p(z) \in PSL(2, \mathbb{R})$ *as* $ad - bc = 1$. *Furthermore,* $p(z)$ *is a parabolic element, because the trace of* $p(z)$ *is* $tr(p) = 2$. *Alternatively,* $p(z)$ *is parabolic as it fixes exactly one point in* $\overline{\mathbb{R}}$. *In fact,* $a = 1, c = 0$ *and* $d = 1$, *by the Formula (4.2), the fixed point is* $\infty \in \overline{\mathbb{R}}$.

Example 4.1.6 *Let* $h(z) = \lambda^2 z = \frac{\lambda z}{\frac{1}{\lambda}}, \lambda \in \mathbb{R}$ *and* $\lambda \neq 1$, *then*

(1) $h(z) \in PSL(2, \mathbb{R})$ *as* $ad - bc = ad = 1$.

(2) $h(z)$*is a hyperbolic element, because the trace of* $h(z)$*is* $tr(h) = \left|\lambda + \frac{1}{\lambda}\right| = \lambda + \frac{1}{\lambda}$, *by using the Remark (3.1.24) to assume* $\lambda > 0$. *Since* $\lambda + \frac{1}{\lambda} - 2 = (\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}})^2 \geq 0$. *In addition,* $\lambda \neq 1$, *so* $\sqrt{\lambda} \neq \frac{1}{\sqrt{\lambda}}$. *It follows that* $(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}})^2 > 0$, *then* $\lambda + \frac{1}{\lambda} > 2$.

Alternatively, $h(z)$ *is hyperbolic as it fixes two points in* $\overline{\mathbb{R}}$. *In fact,* $a = \lambda, b = 0, c = 1$ *and* $d = \frac{1}{\lambda}$. *By the Formula (4.2), the two fixed point are* $0 \in \mathbb{R}$ *and* $\infty \in \overline{\mathbb{R}}$.

We have classified all non-identity elements in $PSL(2, \mathbb{R})$. The next Corollary 4.1.7 gives the identification of the identity element $T(z) = z$.

Corollary 4.1.7 *If an element* $\in PSL(2,\mathbb{R})$ *fixes three distinct points in* $\mathbb{H}$*. Then* T *is the identity i.e.* $T(z) = z$*.*

Now we turn our attention to conjugation. Two elements S and $T \in PSL(2,\mathbb{R})$ are said to be conjugate in $PSL(2,\mathbb{R})$ if there exists a conjugator $C \in PSL(2,\mathbb{R})$ such that

$$S = CTC^{-1}.$$

Conjugates are very useful tools. Conjugations preserve many properties we are interested in, besides simplifying the objects we are studying. For example, the trace is preserved under each conjugator $C \in PSL(2,\mathbb{R})$:

$$tr(CTC^{-1}) = tr(T), T \in PSL(2,\mathbb{R}).$$

Since our three types ofbelements in $PSL(2,\mathbb{R})$ are characterized by trace, the conjugation preserves the type of each element non-identity. Not surprisingly, even conjugation also preserves the algebraic structure of a group. Let's see the next Lemma 4.1.8.

Lemma 4.1.8 *Suppose* G *is a subgroup of* $PSL(2,\mathbb{R})$ *and* $C \in PSL(2,\mathbb{R})$*. Then the subgroup* CGC^{-1}*is isomorphic to* G*.*

Proof. By Lemma 4.1.8. We directly exhibit the homomorphism

$$\phi : G \to CGC^{-1}$$

given by

$$\phi : T \to CTC^{-1}$$

Then $\forall S, T \in G$:

$$\begin{aligned} \phi(ST) &= C(ST)C^{-1} \\ &= (CSC^{-1})(CTC^{-1}) \\ &= \phi(S)\phi(T). \end{aligned}$$

Thus, $G \cong CGC^{-1}$ for each $C \in PSL(2,\mathbb{R})$. ■

If the conjugator is chosen from $PSL(2,\mathbb{R})$, then it preserves distances in $\mathbb{H}$ and hence will also preserves the topological properties of the topological group its applied to.

Lemma 4.1.9 *Suppose that G is discrete subgroup of $PSL(2,\mathbb{R})$, then*

$$CGC^{-1} = \{CTC^{-1} : T \in G\}$$

is discrete for each $C \in PSL(2,\mathbb{R})$.

A practical application of this is a subgroup of $PSL(2,\mathbb{R})$, all of whose elements fix x a given set of points, we can use conjugation to take this subgroup to another one which fixes a more convenient set of points.

Theorem 4.1.10 *Let a non-identity element $T \in PSL(2,\mathbb{R})$, then :*

(a) T is elliptic if and only if T is conjugate in $PSL(2,\mathbb{R})$ to $e(z) = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$.

(b) T is parabolic if and only if T is conjugate in $PSL(2,\mathbb{R})$ to $p(z) = z + b$.

(c) T is hyperbolic if and only if T is conjugate in $PSL(2,\mathbb{R})$ to $h(z) = \lambda^2 z$.

Proof. (a) Since T is elliptic, let Γ fixes some point z_0 in $\in \mathbb{H}$, then let a conjugator $C : i \to z_0$and $C \in PSL(2,\mathbb{R})$. Let $e = CTC^{-1} \in PSL(2,\mathbb{R})$ and fixes i. This means that

$$\begin{aligned} e(i) &= i \\ i &= \frac{ai+b}{ci+d} \\ -c+id &= ai+b \\ b &= -c \;\; \text{and } a = d. \end{aligned}$$

Therefore, we know

$$ad - bc = a^2 + c^2 = 1.$$

We suggestively write $a = cos\theta$ and $c = \sin\theta$. Then $e(z) = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$.

(b) Since T is parabolic, let Γ fixes some point $x \in \overline{\mathbb{R}}$, let $C = \frac{1}{x-z} \in PSL(2, \mathbb{R})$ and let $p(z) = CTC^{-1}$. Then $C(x) = \infty$, and hence $p(z)$ fixes $\infty \in \overline{\mathbb{R}}$. it follows from the argument in the proof of Theorem 4.1 that $p(z) = z + b$.

(c) Since T is hyperbolic, let Γ fixes r_1 and $r_2 \in \overline{\mathbb{R}}$, let $C = \frac{z-r_1}{z-r_2}$, and let $h(z) = CTC^{-1}$. Then $C(r_1) = 0$, $C(r_2) = \infty$, and $h(z)$ fixes 0 and ∞. $h(z) = az + b$. Since $h(0) = 0, b = 0$, so $h(z) = az$.

Let $a = \lambda^2$, then $h(z) = \lambda^2 z$. ■

4.2 Fuchsian Groups

In this section, we introduce the notion of Fuchsian groups and one of its characterizations .

Definition 4.2.1 *A discrete subgroup of $PSL\,(2, \mathbb{R})$ is called Fuchsian groups .*

A transformation of $\mathbb{H}$ onto itself is called *hyperbolic isometry* if it preseves the hyperbolic distance on $\mathbb{H}$. It is clear that the set of all isometries of $\mathbb{H}$ forms a group, we shall denote by Isom($\mathbb{H}$). It is well known that $PSL(2, \mathbb{R})$ is a subgroup of Isom($\mathbb{H}$) :

$$PSL\,(2, \mathbb{R}) \leq Isom(\mathbb{H}).$$

Thus, Fuchsian groups are discrete subgroups of Isom($\mathbb{H}$), that are acting in $\mathbb{H}$.

Theorem 4.2.2 *Every Fuchsian group is a properly discontinuous subgroup acting on $\mathbb{H}$.*

We need to show that subgroup Γ of $PSL(2, \mathbb{R})$ is discrete if and only if it acts properly discontinuously $\mathbb{H}$, we need two lemmas. These two lemma are from [11].

Lemma 4.2.3 *For any $x \in \mathbb{H}$ and any compact set $K \subset \mathbb{H}$, the set*

$$\{T \in PSL(2, \mathbb{R}) : T(x) \in K\}.$$

is compact in $PSL(2, \mathbb{R})$.

Lemma 4.2.4 *Let* Γ *be a subgroup of* $PSL(\ 2,\mathbb{R})$ *action properly discontinuously on* $\mathbb{H}$*. Let* $x \in \mathbb{H}$ *be fixed by* $T \in \Gamma$*. Then there is a neighborhood of* x*,* W*, such that no other point of* W *is fixed by any transformation from* Γ *other than the identity.*

The following proof is for Theorem 4.2.2.

Proof. ($\Longrightarrow$) Let $x \in \mathbb{H}$ and $K \subset \mathbb{H}$ be compact. Then

$$\{T \in \Gamma : T(x) \in K\} = \{T \in PSL(2,\mathbb{R}) : T(x) \in K\} \cap \Gamma.$$

The rest set is compact, by lemma 4.2.3, the second is discrete, by our assumption. Therefore, the intersection of the two sets has to be finite. Therefore, we have shown the Γ orbit of any x has to be locally finite, by Definition 3.5.6 of properly discontinuous.

($\Longleftarrow$) Let $x \in \mathbb{H}$ be such that $Tx \neq x$ for any T which is not the identity element. (Such x must exist by lemma 4.2.4). Now, suppose for contradiction that Γ is not discrete. Then there is a sequence of T_n which converge to the identity transformation. Therefore, $T_n x \to x$ but never equal x (otherwise,this contradicts our choice of x). Hence, any disc around x has infinitely many points from the orbit of Γx. This contradicts the proper discontinuity of Γ. ∎

Example 4.2.5 *The modular group is the subgroup of* $PSL(2,\mathbb{Z})$ *is an example of a Fuchsian group*

$$PSL(2,\mathbb{Z}) = \{\frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z}\}.$$

Note that each element in $PSL(2,\mathbb{R})$ can be generated a cyclic subgroup of $PSL(2,\mathbb{R})$. Most of them are Fuchsian groups, see the next Theorem 4.2.6 for more details.

Theorem 4.2.6 *The following cyclic subgroup of* $PSL(2,\mathbb{R})$ *are Fuchsian groups.*

(1) All hyperbolic and parabolic cyclic subgroups of $PSL(2,\mathbb{R})$ *are Fuchsian groups.*

(2) An elliptic cyclic subgroups of $PSL(2,\mathbb{R})$ *is a Fuchsian groups if and only if it is finite.*

4.3 Abelian Fuchsian Groups

As we know that every cyclic group is an abelian and the converse is not true in general. In this section, we show that every abelian Fuchsian group is cyclic. Thus, a Fuchsian group is abelian if and only if it is cyclic.

Lemma 4.3.1 *If $TS = ST$ for $T, S \in PSL(2,\mathbb{R})$, then S maps the set of all fixed points set of T to itself .*

Proof. Let X be the set of all fixed points of T. For all $p \in X : T(p) = p$. Let show that $S(p) \in X$.

$$T(S(p)) = ST(p) = S(p) \in X.$$

i.e. $S(p)$ is fixed by T, and hence $S(p) \in X$. ■

Theorem 4.3.2 *Two non-identity elements of $PSL(2,\mathbb{R})$commute if and only if they have the same fixed point set.*

Proof. ($\Rightarrow$) Suppose the two elements, T and S, commute. Then T maps the fixed point set of S to itself. Similarly, S maps the fixed point set of T to itself. Hence, T and S must have the same number of fixed points.

(a) Since T and S are non-identity elements, they can fixed up to two points in $\mathbb{H}\cup \overline{\mathbb{R}}$. If they only have one fixed point, then T sends the fixed point of S to itself. This means T also fixes S's fixed point. Similarly, S also fixes T's fixed point. Therefore, if S and T only have one fixed point, they must have the same fixed point.

(b) If S and T have two fixed points, then they are hyperbolic, and these two points in $\overline{\mathbb{R}}$. We can choose a conjugator C such that $h = CTC^{-1}$ fixes 0 and ∞, and $h(z) = \lambda^2 z, \lambda \neq 1$. Let $S' = CSC^{-1}$, the conjugate of S, then $S'h = hS'$. Infact, $ST = TS$, and $T = ChC^{-1}$. Therefore,

$$\begin{aligned} SC^{-1}hC &= C^{-1}hCS \\ CSC^{-1}h &= hCSC^{-1} \\ S'h &= hS' \end{aligned}$$

Let $S^{'} = \frac{az+b}{cz+d} \in \ PSL(2,\mathbb{R})$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda^{-1} \\ c\lambda & d\lambda^{-1} \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c\lambda^{-1} & d\lambda^{-1} \end{pmatrix}.$$

It follows that

$$\begin{aligned} b(1-\lambda^2) &= 0 \text{ and } c(1-\lambda^2) = 0 \\ b\lambda^{-1} &= b\lambda \text{ and } c\lambda = c\lambda^{-1}. \end{aligned}$$

Since $\lambda > 0$ and $\lambda \neq 1$, so the only way to satisfy these conditions is if $b = c = 0$. Thus, $S^{'} = a^2 z$, which fixes 0 and ∞. Since the conjugates of T and S fix the same points, T and S must fix the same points.

($\Leftarrow$) Suppose two elements have the same fixed point set. They are then of the same type. They are also mapped by the same conjugator to one of the following 3 forms:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

Each of these forms commute with other forms of their type. Since the conjugates of the elements commute, the elements themselves must also commute. ■

Theorem 4.3.3 *Every abelian Fuchsian group is cyclic.*

Proof. Suppose Γ is an abelian Fuchsian group. Then each element from Γ must commute with every other element of Γ. However, by the Theorem 4.3.2, we know two non-identity elements of $PSL(2,\mathbb{R})$ commute if and only if they have the same fixed points. Therefore, all the elements from Γ must have the same fixed point set, and must be of the same type. Since all the non-identity elements of Γ have the same fixed point set, we can choose a conjugator to send all of the elements to a more convenient conjugate group. Suppose all the elements are hyperbolic. Then we can choose a conjugator which will take them all to elements which fix 0 and ∞. Each of the elements will therefore have the form

$$\begin{pmatrix} \lambda_T & 0 \\ 0 & \lambda_T^{-1} \end{pmatrix}$$

where the values on the diagonal are indexed by $T \in \Gamma$. We can take this group to a discrete subgroup of ($\mathbb{R}^+, .$) by mapping the matrix above to λ_T^2. (This mapping is both an isomorphism and a continuous function). We showed in the (topological group section) that the only discrete subgroups of ($\mathbb{R}^+, .$) are cyclic. Therefore, the conjugate group must be cyclic. Since we noted earlier that conjugation preserves algebraic structure, the original group must also be cyclic. Suppose all the elements are parabolic. Then we can use a conjugator to send them to elements fixing ∞. Then they all have the form

$$\begin{pmatrix} 1 & x_T \\ 0 & 1 \end{pmatrix}$$

We can take these directly to a discrete subgroup of $(\mathbb{R}, +)$ by taking the element above to x_T. (This mapping is both an isomorphism and a continuous mapping). Since the only discrete subgroups of ($\mathbb{R}, +)$ are cyclic, the conjugate group, and therefore our original group must also be cyclic. Suppose all the elements are elliptic. Then we use a conjugator to make all the elements fix i . Then all the elements will have the form

$$\begin{pmatrix} \cos\theta_T & -\sin\theta_T \\ \sin\theta_T & \cos\theta_T \end{pmatrix}$$

We map this discrete subgroup to a discrete subgroup of $(S^1, \cdot_{\mathbb{C}})$ by taking the element above to $e^{i\theta T}$. The only discrete subgroups of $(S^1, \cdot_{\mathbb{C}})$ are cyclic, as we showed in the beginning, so our original group must also be cyclic. Since our group must be composed of hyperbolic, parabolic, or elliptic elements, and we have shown in each case that the group must then be cyclic, then group Γ is cyclic. $\blacksquare$

Theorem 4.3.4 *[11] Let Γ be a Fuchsian groups all of whose non-identity elements have the same fixed points set. Then Γ is cyclic.*

Corollary 4.3.5 *[11] No Fuchsian group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

4.4 Non-Elementary Fuchsian Groups

Definition 4.4.1 *Let Γ be a Fuchsian group, and $z \in \mathbb{H}$. The following set is called the $\Gamma-$orbit of z, denoted by $\Gamma(z)$.*

$$\Gamma(z) = \{T(z) : T \in \Gamma\}.$$

Γ is called elementary if there exists some $z \in \mathbb{H}$, such that its $\Gamma-$orbit is finite. Otherwise, Γ is called non-elementary.

Definition 4.4.2 *If $g; h$ are elements of a group, then the commutator is the element $ghg^{-1}h^{-1}$, denoted by $[g, h]$.*

Clearly, if g and h commute, the commutator is just the identity.

Theorem 4.4.3 *[11] Let Γ be a subgroup of $PSL(2, \mathbb{R})$ containing besides the identity only elliptic elements. Then all elements of Γ is cyclic group, abelian and elementary.*

Corollary 4.4.4 *Any Fuchsian group containing besides the identity only elliptic elements is a finite cyclic group.*

Theorem 4.4.5 *Let Γ be a Fuchsian group. Then Γ is elementary if Γ can fix at least one point.*

The next theorem describes all elementary Fuchsian groups.

Theorem 4.4.6 *Any elementary Fuchsian groups is either cyclic or is conjugate in $PSL(2, \mathbb{R})$ to a group generated by $g(z) = kz$ $(k > 1)$ and $h(z) = \frac{-1}{z}$.*

Proof. Consider an elementary Fuchsian group Γ. We prove this theorem considering the following possible cases:

(1) Γ has an orbit of order 1.

(2) Γ has an orbit of order 2 in $\overline{\mathbb{R}}$, or any other situation.

Case 1 : This is equivalent that all elements of Γ share one fixed point in the closure of $\mathbb{H}$. If this point is in $\mathbb{H}$, then all elements in Γ are elliptic (as parabolic and hyperbolic elements have no fixed points in $\mathbb{H}$), so by Theorem 4.4.3 shows us Γ is cyclic. If the fixed point is in $\overline{\mathbb{R}}$, then all elements are either parabolic or hyperbolic. We wish to show they must all be of the same type. Assume the contrary, and conjugate Γ, so that the fixed point is ∞. Then if we pick a hyperbolic element g and a parabolic element h, $g(z) = \lambda z$, $h(z) = z + k$ for $\lambda > 1$, $k \neq 0$. If we couldn't choose such a k, there would be no non-identity parabolic elements, and for the same reason we can choose a non-zero λ, and take the inverse if necessary to satisfy the inequality condition. Then the element $g^{-n} \circ h \circ g^n(z) = z + \lambda^{-n}k$. Then $\|g^{-n} \circ h \circ g^n\| = (2^2 + \lambda^{-n}k)2)^{\frac{1}{2}}$, which is bounded, since $\lambda > 1$. Then we can extract a converging subsequence of distinct terms which contradicts discreteness. So there can only be parabolic or hyperbolic elements. If there are only parabolic elements, then all elements of Γ have the same fixed points and so Γ is cyclic. If all elements are hyperbolic, we wish to show that the second fixed point is also shared by all of them, which once again would imply Γ is cyclic. Conjugate so that some non-identity element f is represented by the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, which fixes 0 and ∞, and some other parabolic element g is represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we want g to fix only 0, $b = 0$, $c \neq 0$, $a \neq 0$ and $d = \frac{1}{a}$, so that $[f, g] = \begin{pmatrix} 1 & 0 \\ \frac{c}{a(\frac{1}{\lambda^2}-1)} & 1 \end{pmatrix}$. We know $c \neq 0$, otherwise g fixes ∞, and $\lambda \neq 1$, otherwise f is the identity, both of which contradict our assumptions. But then if we let $t = \frac{c}{a(\frac{1}{\lambda^2}-1)}$, g has the form

$g(z) = \frac{z}{tz+1}$, which has a fixed point only when $tz = 0$. But since $t \neq 0$, this has only one fixed point, and thus is parabollic, a contradiction. So all hyperbolic elements in Γ have the same fixed points, and so the group is cyclic.

Case 2 : Now suppose Γ has an orbit of order 2 in $\overline{\mathbb{R}}$. Then an element in the group either fixes each of these points or interchanges them. Then there can be no parabolic elements, for if we conjugate in order to express such an element as $f(z) = z + k$ for $k \neq 0$, it's clear that any z is either a fixed point or has an infinite orbit, as $f^n(z) = z + nk$ is an element of the group for any n and only ∞ is fixed by such a group of transforms. If there are just hyperbolic elements, then the points in the orbit correspond to the fixed points of these elements, as hyperbolic elements can't interchange two points. To see why, express such an element as $f(z) = \lambda z$. Then f^2 fixes the two points, which means λ and 1, and f is just the identity. It follows that the group is cyclic. If all elements are elliptic, by Theorem 4.4.3. Now consider the case in which there are both elliptic and hyperbolic elements. The hyperbolic elements must fix the two points, and the elliptic must alternate them. We can see that by thinking of elliptic transformations as rotations, in which case fixing two points on the boundary are equivalent to the identity. Let us conjugate Γ so that the two points of the orbit are 0 and ∞. In order to fix the two points, a hyperbolic element must be of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, and in order to alternate them, an elliptic element must be of the form $\begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}$. We can certainly conjugate so that at least one elliptic element is of the form $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We can use this transform so that if $\begin{pmatrix} 0 & \beta \\ \beta^{-1} & 0 \end{pmatrix}$ is any elliptic element in the group, there is also some hyperbolic element $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ obtained by composition which in a sense corresponds to the elliptic element. By discreteness, all the hyperbolic elements are generated by some element $\begin{pmatrix} k^{\frac{1}{2}} & 0 \\ 0 & \frac{1}{k^{\frac{1}{2}}} \end{pmatrix}$, and since we can associate a hyperbolic element to each elliptic element by means of composition with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then the whole group is generated by these two transforms, which yields the desired result.

Case 3 : Let Γ has an orbit of order 2 in $\mathbb{H}$ or an orbit of order $k \geq 3$ in $\overline{\mathbb{H}}$. Since the parabolic and hyperbolic elements can have at most 2 fixed points on $\overline{\mathbb{R}}$, and all points in $\mathbb{H}$ of these transforms have infinite orbits, then this means Γ contains only elliptic elements, and so is cyclic. ■

Theorem 4.4.7 *[11] A non- elementary subgroup Γ of $PSL(2, \mathbb{R})$ must contain a hyperbolic element.*

Proof. Let Γ be a subgroup and suppose it contains no hyperbolic elements. Then by Theorem 4.4.3 it must contain a parabolic element. Let us conjugate Γ so that this element fixes ∞, and is of the form $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Pick any other $g \in \Gamma$, so that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $f^n g = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}$, and $tr^2 = (a+d+nc)^2$. Since there are no hyperbolic elements, by using Formula 4.1 this value is less than or equal to 4 for any n. But since we can take n to be as large as we want, this implies $c = 0$. Then g also fixes ∞, and since g is identity, every element of Γ fixes ∞, which means that Γ is elementary, which contradicts our assumptions. ■

Corollary 4.4.8 *Every non- elementary contains infinitely many hyperbolic elements with distinct fixed points.*

Theorem 4.4.9 *[11] If Γ is a subgroup $PSL(2, \mathbb{R})$, which contains no elliptic elements it is either elementary or discrete.*

Chapter 5: Fundamental Domains of Fuchsian Groups

In this chapter we are going to describe fundamental domains of mainly Fuchsian groups. We introduce the Dirichlet region and its structure in Section 2. Finally, we present some geometric properties of Fuchsian groups such as compact Fuchsian groups in the Sections 3.

5.1 Fundamental Domains

Definition 5.1.1 *Suppose that Γ is a subgroup of $PSL(2,\mathbb{R})$. A subset F of $\mathbb{H}$ is said to be a fundamental domain for Γ if F satisfies the following conditions:*

(1) F is a domain in $\mathbb{H}$, and hence $F^0 = F$.

(2) $\displaystyle\bigcup_{T\in\Gamma} T(\overline{F}) = X$, where $\overline{F}$ is the clouser of F^0.

(3) $F \cap T(F) = \phi$, for all $T \in \Gamma - \{Id\}$.

If F is a fundamental domain of a subgroup Γ of $PSL(2,\mathbb{R})$. Then the collection $\{T(F) : T \in \Gamma\}$ is called a *tessellation* of $\mathbb{H}$. A fundamental domain for Γ with its boundary σF is called a *fundamental region* for Γ.

Theorem 5.1.2 *Let Γ be the cyclic group generated by the transformation $T(z) = 2z \in PSL(2,\mathbb{R})$, $\Gamma = \langle 2z\rangle = \{2^n z : n \in \mathbb{Z}\}$. Then the semi-annulus shown below is easily seen to be a fundamental region for Γ.*

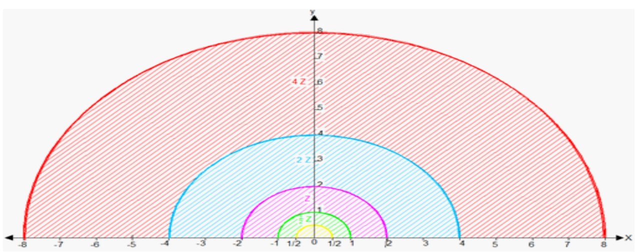

Figure 5.1.2. Fundamental region for (2z).

On the other hand, the hyperbolic area is infinite is

$$ha(R) = \iint \frac{dxdy}{y^2} = \int_0^\pi d\theta \int_1^2 \frac{r}{r^2 \sin^2\theta} dr = \int_0^\pi \csc^2\theta d\theta \int_1^2 \frac{1}{r} dr = -\left[\cot\right]_0^\pi \left[\ln|r|\right]_1^2 = \infty.$$

It is natural to ask a question: Do all fundamental regions for a certain subgroup of $PSL(2, \mathbb{R})$ have the same hyperbolic area. The next Theorem 5.1.3 states that the answer is for Fuchsian groups.

Theorem 5.1.3 *Let F_1 and F_2 be two fundamental regions for a Fuchsian group Γ, and the hyperbolic area of F is finite : $ha(F_1) < \infty$. Suppose that the boundaries of F_1 and F_2 have zero hyperbolic area.Then two fundamental regions F_1 and F_2 have the same hyperbolic area : $ha(\sigma F_1) = ha(\sigma F_2) = 0$.*

Proof. We have $\mu(F_i^\circ) = \mu(F_i), i = 1, 2$. Now

$$F_1 \supseteq F_1 \cap (\cup_{T\in\Gamma} T(F_2^\circ)) = \cup_{T\in\Gamma}(F_1 \cap T(F_2^\circ)).$$

Since F_2° is the interior of a fundamental region, the set $F_1 \cap T(F_2^\circ)$ are disjoint, and hence

$$\mu(F_1) \geq \Sigma_{T\in\Gamma}\mu(F_1 \cap T(F_2^\circ) = \Sigma_{T\in\Gamma}(T^{-1}(F_1) \cap F_2^\circ = \Sigma_{T\in\Gamma}(T(F_1) \cap F_2^\circ.$$

Since F_1 is a fundamental region

$$\cup_{T\in\Gamma} T(F_1) = \mathbb{H}.$$

and therefore

$$\cup_{T\in\Gamma}(T(F_1) \cap F_2^\circ) = F_2^\circ$$

Hence,

$$\Sigma_{T\in\Gamma}(T(F_1) \cap F_2^\circ \geq \mu(\cup_{T\in\Gamma}(T(F_1) \cap F_2^\circ) = \mu(F_2^\circ) = \mu(F_2)$$

■

Thus we have proved a very important fact: the area of a fundamental region, if it is finite, is a numerical invariant of the group.

Theorem 5.1.4 *Let Fuchsian group* Γ *with a fundamental region of infinite area is the group generated by* $T(z) = z + 1 \in PSL(2, \mathbb{R})$, $\Gamma = \langle z + 1 \rangle = \{z + n : n \in \mathbb{Z}\}$.

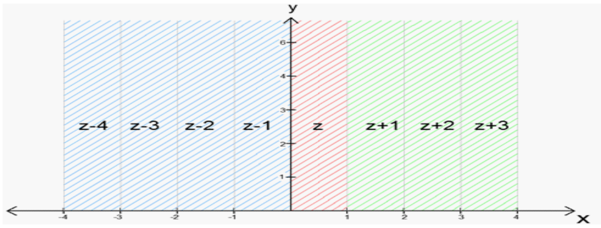

Figure 5.1.4. Fundamental region for (z+1).

The infinite hyperbolic area is

$$ha(F) = \iint \frac{dxdy}{y^2} = \int_0^\infty dy \int_0^1 \frac{1}{y^2} dx = \int_0^\infty \frac{1}{y^2} dy = -\left[\frac{1}{y}\right]_0^\infty = \infty.$$

If we ask a question, are there any fundamental region with finite hyperbolic area. The next Theorem 5.1.5 gives the answer.

Theorem 5.1.5 *(1) A compact fundamental region has a finite area.*

(2) If the $ha(F) < \infty$*, then* $ha(F_1) = ha(F_2)$ *which is invariant.*

To review a locall finite a collection of sets S_α is considered locally finite if and only if for any compact set K, $S_\alpha \cap K \neq \phi$ for only finitely many α.

Definition 5.1.6 *A fundamental region* F *for a Fuchsian group* Γ *is called locally finite if the tessellation* $\{T(F) : T \in \Gamma\}$ *is locally finite.*

Theorem 5.1.7 *Let* D *be any locally finite fundamental domain for a Fuchsian group* Γ*. Then*

$$\Gamma_0 = \{g \in \Gamma : g(\overline{D}) \cap \overline{D} \neq \phi\}$$

generates Γ.

Proof. Let Γ^* be the group generated by Γ_0. We may suppose that Γ acts in $\mathbb{H}$ so for any z in $\mathbb{H}$ there is some g in Γ with $g(z) \in \overline{D}$. Suppose also that the Fuchsian group $h(z) \in \overline{D}$. Then $h(z)$ is in both $\overline{D}$ and $hg^{-1}(\overline{D})$, so $hg^{-1} \in \Gamma_0$: thus we have equality of cosets, namely

$$\Gamma^* h = \Gamma^* g$$

This fact means that there is a properly defined map $\phi : \mathbb{H} \to \Gamma \diagup \Gamma^*$ given by $\phi(z) = \Gamma^* g$, where $g(z) \in \overline{D}$.

Consider any$z$ in $\mathbb{H}$. As D is locally finite, there exist a finite number of images

$$g_1(\overline{D}), ..., g_m(\overline{D})$$

each containing z and such that their union covers an open neighborhood N of z. If $w \in g_i(\overline{D})$ for some j and

$$\phi(w) = \Gamma^*(g_i)^{-1} = \phi(z)$$

We deduce that each z has an open neighborhood N on which ϕ is constant.

Now any function ϕ with this property is constant on $\mathbb{H}$ (give $\phi(\mathbb{H})$ the discrete topology : ϕ is continuous and $\phi(\mathbb{H})$ is connected, thus $\phi(\mathbb{H})$ contains only one point. This shows that

$$\phi(w) = \phi(z)$$

for all z and w in $\mathbb{H}$. Given any g in G we select z in D and w in $g^{-1}(D)$. Then as ϕ is constant

$$\Gamma^* = \phi(z) = \phi(w) = \Gamma^* g$$

and so $g \in \Gamma^*$. This proves that $\Gamma \subset \Gamma^*$. Clearly $\Gamma^* \subset \Gamma$ so $\Gamma^* = \Gamma$ and Γ_0 generates Γ. ■

5.2 The Dirichlet Region

Definition 5.2.1 *Let Γ be a Fuchsian group and let $p \in \mathbb{H}$ be not fixed by any element of $\Gamma - \{Id\}$. The Dirichlet region for Γ centered at p to be the set*

$$D_p(\Gamma) = \{z \in \mathbb{H} : \rho(z,p) \leq \rho(z,T(p))\} \text{ for all } T \in \Gamma$$

By the invariance of the hyperbolic distance$\rho(x,y) = d_H(x,y)$ under $PSL(2,\mathbb{R})$ this region can also be defined as

$$D_p(\Gamma) = \{z \in \mathbb{H} : \rho(z,p) \leq \rho(T(z),p)\} \text{ for all } T \in \Gamma$$

For each fixed $T_1 \in PSL(2,\mathbb{R})$,

$$\{z \in \mathbb{H} : \rho(z,p) \leq \rho(z,T_1(p))\}$$

is the set of points z which are closer in the hyperbolic distance to p than to $T_1(p)$. Clearly, $p \in D_p(\Gamma)$ and as the $\Gamma-$orbit of p is discrete, $D_p(\Gamma)$ contains a neighborhood of p. In order to describe the set $\{z \in \mathbb{H} : \rho(z,p) \leq \rho(z,T_1(p))\}$, we join the points p and $T_1(p)$ by a geodesic segment and construct a line given by the equation

$$\rho(z,p) = \rho(z,T_1(p))$$

The Dirichlet region for a Fuchsian group can be quite complicated. They are bounded by hyperbolic lines in $\mathbb{H}$ and possibly by segments of the real axis. If two such hyperbolic lines intersect in $\mathbb{H}$, their point of intersection is called *a vertex of Dirichlet region.* It can be shown that the vertices are isolated so that a Dirichlet region is bounded by a union of (possibly infinitely many) hyperbolic lines and possibly segments of the real axis.

Theorem 5.2.2 *The Dirichlet region F is a fundamental region.*

It is clear that every point $w \in \mathbb{H}$ fixed by an elliptic element $\acute{S}$ of Γ lies on the boundary of $T(F)$ for some $T \in \Gamma$. Hence $u = T^{-1}(w)$ lies on the boundary of F and is fixed by the elliptic elements $S = T^{-1}\acute{S}T$.

Definition 5.2.3 *The tessellation of $\mathbb{H}$ formed by a Dirichlet region F and all its images under Γ is called faces which is donated by $\{T(F) : T \in \Gamma\}$.*

The next Theorem 5.2.4 shows that the Dirichlet tessellation has a nice local properties.

Theorem 5.2.4 *A Dirichlet region is locally finite.*

Proof. Let $F = D_p(\Gamma)$, where p is not fixed by any element of $\Gamma - \{Id\}$. Let $a \in F$, let $K \subset \mathbb{H}$ be a compact neighborhood of a

Suppose that $K \cap T_i(F) \neq \phi$ for some infinite sequence $T_1, T_2, \ldots$ of distinct elements of Γ. Let $\sigma = \sup_{z \in K} \rho(z, p)$. Since $\sigma \leq \rho(p, a) + \rho(a, z)$, for all $z \in K$, and K is bounded , σ is finite. Let $w_j \in K \cap T_i(F)$. Then $w_j = T_i(z_j)$ for $z_j \in F$, and by triangular inequality,

$$\begin{aligned}
\rho(p, T_i(p)) &\leq \rho(p, w_j) + \rho(w_j, T_i(p)) \\
&= \rho(p, w_j) + \rho(z_j, p) \\
&\leq \rho(p, w_j) + \rho(w_j, p) \text{ (as } z_j \in D_p(\Gamma) \text{)} \\
&\leq 2\,\sigma
\end{aligned}$$

Thus, the infinite set of points $T_1(p), T_2(p), \ldots$belongs to the compact hyperbolic ball with center p and radius $2\,\sigma$, but this contradicts the properly discontinuous action of Γ. ■

We call the two points $u, v \in \mathbb{H}$ *congruent* if they belong to the same $\Gamma-$orbit. First, notice that two points in a fundamental region F may be congruent only if they belong to the boundary of F. Suppose now that F is a Dirichlet region for Γ, and let us consider congruent vertices of F.

The congruence is an equivalence relation on the vertices of F, while the equivalence classes are called *cycles*. If u is fixed by an elliptic element S, then $v = Tu$ is fixed by elliptic element such as $T^{-1}ST$

Remark 5.2.5 *If one vertex of the cycle is fixed by an elliptic element, then all vertices of that cycle are fixed by conjugate elliptic elements. Such a cycle is called an elliptic cycle and the vertices are called elliptic vertices.*

Corollary 5.2.6 *The number of elliptic cycles is equal to the number of non-congruent elliptic points in F.*

Theorem 5.2.7 *[11] If an elliptic element S has a finite order k with $k \geq 3$, then S is an isometry fixing u which maps hyperbolic lines to hyperbolic lines and u must be a vertex whose angle θ is at most $2\pi/k$.*

Remark 5.2.8 *The hyperbolically region F is bounded by a union of hyperbolic lines. The intersection of F with these hyperbolic lines is either a single point or a segment of a hyperbolic line. These segments are called sides of F.*

Corollary 5.2.9 *If an elliptic element S has order 2, then its fixed points lie on the interior of a side of F.*

In the above Corollary 5.2.9, S interchanges the two segments of this side separated by the fixed points. We will include such elliptic fixed points as vertices of F, the angle at such vertex being π. Thus a vertex of F is a point of intersection in $\mathbb{H}$ of two bounding hyperbolic lines of F or a fixed point of an elliptic element of order 2. (all the pervious definitions such as conjugate, elliptic cycles apply to this extended set of vertices). If a point in $\mathbb{H}$ has a nontrival stabilizer in Γ, then by Theorem 4.2.2, it is a maximal finite cyclic subgroup of Γ by Lemma 4.3.1. Conversely, every maximal finite cyclic subgroup of Γ is stabilizer of a single point in $\mathbb{H}$. We can summarize the above as the next Theorem 5.2.10.

Theorem 5.2.10 *[11] There is a one-to-one correspondence between the elliptic cycles of F and the conjugacy classes of non-trivial maximal finite cyclic subgroup of Γ.*

Theorem 5.2.11 *[11] Let F be a Dirichlet region for Γ. Let $\theta_1, \theta_2, \cdots, \theta_t$ be the internal angles at all congruent vertices of F. Let m be the order of the stabilizer in $\Gamma$of one of these vertices. Then $\theta_1 + \cdots + \theta_t = 2\pi/m$.*

Remark 5.2.12 *(1) As F is locally finite, there are only finitely many vertices in congruent cycle.*

(2) As the stabilizers of two points in a congruent set are congugate subgroups of Γ, they have the same order.

(3) If a vertex is not a fixed point, we have $m = 1$ and $\theta_1 + \cdots + \theta_t = 2\pi$.

5.3 Compact Fuchsian Groups

We saw that a Dirichlet region in $\mathbb{H}$ of Fuchsian group Γ is bounded by a number of hyperbolic line segments and possibly segments of the real axis. The bounding hyperbolic line segments in $\mathbb{H}$ are called *sides of F*, and form the boundary of F, if an elliptic fixed point of order 2 belongs to a hyperbolic line segment we include this point in the set of vertices of F and regard the two subsegments as two sides.

Let F be a Dirichlet region which is locally finite, the quotient space $\Gamma/\mathbb{H}$ is homemorphic to Γ/F, hence by choosing F to be a Dirichlet region which is locally finite by Theorem 5.2.4, we can find the topological type of $\Gamma/\mathbb{H}$. If Γ has a compact Dirichlet region F, then the vertices of a Dirichlet region are isolated, that is every vertex of F has a neighborhood containg no other vertices of F, where a compact Dirichlet region has a finite number of vertices, then F has finitely many sides, and the quotient space $\Gamma/\mathbb{H}$ is compact. The next Corollary 5.3.1 shows that if one Dirichlet region for Γ is compact, then all Dirichlet regions are compact.

Corollary 5.3.1 *The quotient space of a Fuchsian group Γ, $\Gamma/\mathbb{H}$ is compact if and only if any Dirichlet region for Γ is compact.*

Definition 5.3.2 *A Fuchsian group is called cocompact if the quotient -space $\Gamma/\mathbb{H}$ is compact.*

The following results reveal the relationship between cocompactness of Γ and the absence of parabolic elements in Γ.

Theorem 5.3.3 *If a Fuchsian group Γ has a compact Dirichlet region, then Γ contains no parabolic elements.*

Proof. *Let F be a compact Dirichlet region, then Γ, and*

$$\eta(z) = \inf\{\rho(z, T(z)) : T \in \Gamma - \{Id\}, T \text{ not elliptic}\}$$

Since the $\Gamma-$orbit of each $z \in \mathbb{H}$ is a discrete set and $T(z)$ is continuous, $\eta(z)$ is a continuous function of z and $\eta(z)>0$. Therefore, as F is compact, $\eta(z) = \inf\{\eta(z) : z \in F\}$ is attained and $\eta > 0$. If

$z \in \mathbb{H}$, there exists $S \in \Gamma$ such that $w = S(z) \in F$. Hence, if $T_0 \in \Gamma - \{Id\}$ is not elliptic,

$$\rho(z, T_0(z)) = \rho(S(z), S(T_0(z)) = \rho(w, STS^{-1}(w)) \geq \eta$$

and therefore

$$\inf\{\rho(z, T_0(z)) : z \in \mathbb{H}, T_0 \text{ not elliptic}\} = \eta > 0$$

Now, suppose that Γ contains a parabolic element T_1. If for some $R \in PSL(2, \mathbb{R}), \Gamma_1 = R\Gamma R^{-1}$, then $R(F)$ will be a compact fundamental region for Γ_1. Thus by conjugating Γ in $PSL(2, \mathbb{R})$ we may assume that $T_1(z)$ or $T_1^{-1}(z)$ is the transformation $z \to z + 1$. However by $\rho(z, z + 1) \to 0$ as $Im(z) \to \infty$ a contradiction. ■

Theorem 5.3.4 *[11] If Γ has a non-compact Dirichlet region, then the quotient space $\Gamma \diagup \mathbb{H}$ is not compact.*

The area on the quotient space $\Gamma \diagup \mathbb{H}$ is induced by the hyperbolic area on $\mathbb{H}$, the hyperbolic area of $\Gamma \diagup \mathbb{H}$, denoted by $ha(\Gamma \diagup \mathbb{H})$, is well defined and equal to $ha(F)$ for any fundamental region.

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