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THE PRICING OF ASIAN OPTIONS IN HIGH VOLATILE MARKETS: A PDE APPROACH

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THE PRICING OF ASIAN OPTIONS IN HIGH VOLATILE
MARKETS: A PDE APPROACH

Nabil Kamal Riziq Al Farra

This thesis is submitted in partial fulfilment of the requirements for the degree of
Master of Science in Mathematics

Under the Supervision of Dr. Youssef El-Khatib

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Declaration of Original Work

I, Nabil Kamal Riziq Al Farra, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*the pricing of Asian options in high volatile markets (A PDE approach)*", hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Dr. Youssef El-Khatib, in the department of Mathematical Sciences at UAEU. This work has not been previously formed as the basis for the award of any academic degree, diploma or a similar title at this or any other university. The materials borrowed from other sources and included in my thesis have been properly cited and acknowledged.

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Abstract

Financial derivatives are very important tools in risk management since they decrease uncertainty. Moreover, if used effectively, they can grow the income and save the cost. There are many types of financial derivatives, for instance: futures/forwards, options, and swaps. The present thesis deals with the pricing problem for Asian options.

The main aim of the thesis is to generalize the Asian option pricing Partial Differential Equation (PDE) in order to handle post-crash markets where the volatility is high. In other words, we seek to extend the work on the Asian option pricing PDE under the well-known Black-Scholes model to a high volatility model. To this end, we first set up a model that accounts for high volatile situations and we solve the Stochastic Differential Equation (SDE) of the underlying asset price. Our illustrations confirm the high volatile behavior of the model. We then derive the Asian option PDE for the suggested model. The resulting PDE is reduced from two-dimensional space to one-dimensional space using a change of variable. Moreover, we derive a relationship between the Asian option prices of the Black-Scholes model and our high volatility model where the increase in volatility is a deterministic function of the interest rate.

Keywords: PDE; stochastic calculus; financial derivatives; Asian option; financial crisis.

Title and Abstract (in Arabic)

تسعير عقود الخيارات المالية الآسيوية بواسطة المعادلات التفاضلية الجزئية

الملخص

تعد المشتقات المالية أداة هامة جدا في إدارة المخاطر المالية. يوجد العديد من المشتقات المالية على سبيل المثال: عقود المستقبل/ الآجلة، الخيارات، والمبادلة. هذا البحث يتعامل مع مشاكل التسعير لنوع معين من الخيارات الغريبة: الخيارات الآسيوية للأسواق التي تتعرض للانهيار. إحدى الطرق لتقييم الخيارات المالية هي طريقة المعادلات التفاضلية الجزئية. من ناحية أخرى، فإنه من المعلوم أن التذبذب في الاسعار يكون عاليا خلال الازمات المالية. الهدف من هذا العمل هو التوسع في المعادلة التفاضلية الجزئية لتقييم الخيارات الآسيوية والتي اشتقت باستخدام نموذج بلاك-شولز لتشمل الاسواق المالية ذات التذبذب العالي. لكي نصل لهذا العمل، اولا نضع نموذجنا المعدل من بلاك-شولز والذي يوائم مع التذبذب في الاسواق ذات الأزمة المالية. ثم نتبع نفس طريقة المتبعة في اشتقاق المعادلة التفاضلية الجزئية للخيارات الآسيوية. وبالتالي نوجد المعادلة التفاضلية الجزئية لنموذجنا المعدل. في الجزء الثاني من البحث نخفض المعادلة التفاضلية الجزئية الجديدة من ثلاثة متغيرات إلى متغيرين. في النهاية نحصل على حل رقمي للمعادلة المخفضة. يتم توفير رسوم الحلول الرقمية لأسعار الأصول وهي مواتية لنموذجنا. وعلاوة على ذلك، نوجد علاقة بين أسعار الخيارات الآسيوية للنموذجين، نموذج بلاك شولز، ونموذجنا ولكن حيث الزيادة في التقلب هي سعر معدل الفائدة.

الكلمات المفتاحية: معادلات تفاضلية جزئية; التفاضل والتكامل العشوائي; المشتقات المالية; المشتقات الآسيوية; الأزمات المالية.

Acknowledgements

Dedication

To my beloved parents, wife and family

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Chapter 1:   Introduction

Asian options are options in which the underlying variable is the average price over a period of time. They were originally used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts; and hence the name "Asian" option, see [25].

Asian options are very important on products with low trading volumes (e.g. crude oil). The price of an Asian option depends essentially on the "average" of the underlying asset prices. Therefore, we need to find a "good" model to the underlying asset price in order to obtain a "fair" price to the Asian option. Since the pioneer work of ([1]), many asset price models have been suggested in the literature. One of the most popular model is the Black-Scholes model ([3]) which suffers from several shortcomings. Other types of models[1] were suggested to improve the Black-Scholes model, such as: stochastic volatility models (see [8]), and jump-diffusion models (see [16]). However, most of the existing asset pricing models do not reflect the financial crises aspects. In this thesis, we intend to consider a model that accounts for high volatility periods for Asian options. There are many empirical studies on markets under stress. For instance, in [22], the author concludes by experiment that asset prices follow a converging oscillatory motion during post-crash. On the other hand, in [15], the authors prove that financial markets follow power-law relaxation decay. According to [7], a closed form pricing formula of the option for the high volatile model is obtained for European Options. Here in, we address the pricing problem of Asian

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[1] The list is not exhaustive.

options for high volatile market, where the model is similar to the model presented in [7].

Since there is no available closed form solution to the pricing of the arithmetic Asian Option, this pricing problem becomes highly interesting. In general, to find the price of an option, we can use one of the following methods:

- Probabilistic method, where the price of an option can be expressed as an expected value of the discounted payoff.

- Partial Differential Equation (PDE) approach, where we use the Ito formula and the martingale properties of the discounted prices to derive the PDE of the option.

If a closed solution for the option-pricing problem is not possible then researchers suggest numerical solutions using different numerical methods. For instance, many studies utilize Monte Carlo simulations to find the price of an arithmetic Asian option (see [11]).

Most of the research works on pricing Asian options considers the PDE approach. The price of an Asian option can be expressed as solution of a PDE in two space dimensions (see [10]). However, [21] reduced the Asian option PDE to a one-space dimension PDE. The thesis objective is to find the price of the Asian option in high volatile market by deriving its PDE. We obtain the option price PDE and we reduce it. On other words, this work aims at generalizing the "existing" Asian option PDE to a situation where we have an increased volatility. The thesis intends to:

1. Suggest a model for high volatile situations.
2. Solve the Stochastic Differential Equation (SDE) of the underlying asset price of the suggested model.

3. Derive the PDE of the Asian option price under the suggested model.

4. Reduce the obtained PDE from two space-dimensions to one-space dimension.

5. Compare the Black-Scholes and the modified Asian option prices when the volatility increase is a deterministic function of the interest rate.

The thesis is organized as follows, in chapter 2, we provide an introduction to the financial products, and markets. Additionally, we present a brief discussion about financial derivatives, options and Asian Options. Chapter 3 is devoted to the stochastic calculus. The highlight is on the brownian motion, and Itô formula. Several applications to modeling financial asset prices are provided. In chapter 4, the reader can find our main contribution and results. We start the chapter by several important results from the literature on Black and Scholes model and the derivation of the PDE for European and Asian options. Then, we propose the high volatile model by adding a parameter $\alpha > 0$ to the volatility part of the stochastic differential equation of the underlying asset price. We derive the PDE for the Asian option price under the suggested model. We, then, use the separation of variables to reduce the PDE from two space dimensions to one-space dimension. Moreover, we derive a relation between the Asian option prices of the Black-Scholes and the high volatile models. We conclude the thesis by some remarks and propositions for further research.

Chapter 2:   Financial derivatives

Financial derivatives play an essential role in the risk management. "*Risk management*" does not mean necessarily the complete elimination of the risk. Instead, it means *managing the risk* by choosing the "acceptable" risks and reducing the undesirable ones. Actually, financial derivatives decrease the uncertainty but do not eliminate it completely. Nevertheless, if "properly" handled, derivatives can help controlling the risk in a better way. Moreover, they can serve the accomplishment of the companies risk-management goals. In general, they are utilized in two main ways: to hedge the risks or to speculate by taking a position in anticipation of a market movement. If they are used in the right way they help growing the income and also saving money.

There exists several types of financial derivatives among others: futures, options and swaps. Our study focuses on options, more precisely on the Asian ones, which are part of the exotic options. In this chapter, we provide a brief introduction on financial markets, products and derivatives. Moreover, a presentation on the different types of Asian options is given.

2.1 Financial markets, underlying assets, and derivatives

Financial markets are very important to the economy of any nation. They exist almost in every country in the world.

A financial market is a place where buyers and sellers meet to exchange goods and services. The location of a financial market can be physical or virtual (for example, the Internet). Financial products are the goods for sale (sellers) or for purchase (buyers).

They can be equities such as stocks (shares), commodities (crude oil), currencies (foreign currency) or derivatives (futures and options), etc.

2.1.1 Financial derivatives

In this context, the word "derivatives" does not follow the traditional meaning in Mathematics. A formal definition of a financial derivative is given below:

Definition 2.1 ([18]) *A financial derivative is an instrument derived from the value of some other financial instruments called the underlying assets.*

In general, the underlying assets can be:

- Stocks-bonds.

- Commodities: meat, wheat, oil, etc.

- Currencies: these are liabilities of governments or sometimes banks. They are not direct claims on real assets e.g. exchange between Euro, Dollar, and Dirham.

Moreover, derivatives with other types of underlying assets exist and are traded in derivatives markets. For instance, we can find derivatives that are built on the weather.

Historically, financial derivatives were designed to manage the risk, to speculate, to gain from arbitrage between markets, or to change the nature of a liability, for more details on financial derivatives we refer the reader to [9].

2.2 Major categories of the financial derivatives

2.2.1 Futures and forwards

Definition 2.2 ([9]) *A forward, or a forward contract, is an agreement between a buyer and a seller to exchange a commodity or a financial instrument for a prespecified amount of cash on a prearranged future date.*

An example of forward contract is the interest rate forwards. If a forward purchase is made, then the holder of such a contract is said to be long in the underlying asset. If at expiration the cash price is higher than the forward price, the long position makes a profit, otherwise there is a loss. For more details (see [18]).

Definition 2.3 ([20]) *A futures contract is an agreement between two parties to buy or sell at a certain time in the future for a certain price. Unlike forward contracts, futures contracts are normally traded on an exchange.*

To make trading possible, the exchange specifies certain standardized features of the contract. As the two parties to the contract do not necessarily know each other, the exchange also provides a mechanism that gives the two parties a guarantee that the contract will be honored.

2.2.2 Swap

A swap represents another important type of derivatives. It is used to exchange two financial instruments between two different organizations.

Definition 2.4 ([9]) *A swap is a derivative in which counterparties exchange cash flows of one party's financial instrument for those of the other party's financial instrument.*

Currency and interest rate are two basic types of swaps. Swaps on commodities are also available. For example, a company that consumes 200,000 barrels of oil per annum may pay $12,000,000 per year for the next five years and in return receive $200,000S$, where S is the prevailing market price of oil per barrel. This transaction locks in the price for oil at $60 per barrel (see [12]).

2.2.3 Options

The meaning of options comes from the right for the option holder (buyer), but not the obligation, to either buy or sell a specified quantity of an underlying asset, on or before an agreed date at a predetermined price from or to the writer (seller).

Both futures and options allow the holder to buy or sell the underlying asset in the future. The main difference between the two is that the option buyer has the right to buy or sell the underlying asset (e.g. the equity), but no obligation to do so. With a future, both parties are obliged to participate in the final movement of the underlying asset, unless they trade out of their position beforehand. Investors use derivatives for speculation, arbitrage and/ or for hedging an existing position (e.g. if they own a share). Investors should be sure they fully understand a derivative before trading. As leveraged products derivatives can offer high rewards, but also high risks; for example it is possible to lose a greater amount of money on a derivatives contract than is initially spent to enter into it.

Definition 2.5 ([9]) *Options are contracts giving the right to buy (or to sell) a certain financial asset (with price S) for a pre-specified price (Strike price K) at a predetermined time (Maturity T).*

The literature reveals several types of options:

- European: give the right to exercise at the maturity only.

- American: can be exercised before or at the expiration date.

- Exotic, such as path-dependent options.

2.3 Asian options

In brief, the Asian option is a contract giving the holder the right to buy or sell an underlying asset for its average price over a prescribed period of time. In other words, they are options where the payoff depends on the average price of the underlying asset during at least some part of the life of the option.

Let K denote the strike and $\bar{S}$ is the average price of the underlying asset. Then, the payoff from the average call is $\max(0, \bar{S} - K)$, and that from an average price put is $\max(0, K - \bar{S})$. The average price options are less expensive than regular options and are arguably more appropriate than regular options for meeting some of the needs of corporate treasurers.

Another type of Asian option is an average strike option. An average strike call payoff is $\max(0, S_T - \bar{S})$, and an average strike put payoff is $\max(0, \bar{S} - S_T)$, where S_T is the underlying asset price at expiration date. Notice that, average strike options can guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Alternatively, it can guarantee that the average price received for an asset in frequent trading over a period of time is not less than the final price. It is well-known that using Asian options has many advantages like:

1) Insurance against average price changes.
2) Counterpart has no incentive to influence prices at expiration.

Asian options are similar to European options but with different strike. While the strike in the European options is constant, the strike in an Asian option depends on the underlying asset price trajectory. There are several types of Asian options depending on the method used to compute the average of the stock prices or the final condition (the payoff). An example of the Asian call option strike is the continuous arithmetic average

$$A(T) = \frac{1}{T} \int_0^T S_t dt,$$

where $(S_t)_{t\epsilon[0,T]}$ represents the underlying asset price. In this case the final condition (the payoff) takes the form, $\max(S_T - A(T), 0)$. Below are some other examples:

1) Discrete arithmetic average:

$$A = \frac{1}{n} \sum_{i=1}^n S_{t_i}.$$

2) Discrete Geometric average:

$$G = \left(\prod_{i=1}^n S_{t_i}\right)^{\frac{1}{n}}.$$

3) Continuous Geometric average:

$$G = \exp\left(\frac{1}{T} \int_0^T \ln S_t dt\right).$$

The price of any option is given by:

$$C(S_T, K, T) = E[H(S_T)]e^{-rT}.$$

where S_T represents the underlying asset price, $C(S_T, K, T)$ is the Asian option price, $H(S_T)$ is the payoff, K is the strike, and r is the interest rate.

Note that the pricing problem for an arithmetic Asian option does not have an analytical solution since $A(T)$ does not have a known density. Many works on pricing

arithmetic Asian options use numerical methods such as Monte Carlo simulations or the binomial methods. Nevertheless, an important approach is to find the PDE for the price of an arithmetic Asian option.

In this thesis, we extend the PDE approach from the Black-Scholes model to a model with an increased volatility for an arithmetic Asian option with payoff given by:

$$H(S_T) = \max(A(T) - K, 0).$$

Thus the price of the Asian option is given by:

$$C(S_T, K, T) = E[\max(A(T) - K, 0)]e^{-rT}.$$

Chapter 3: Stochastic Calculus

Stochastic calculus is the branch of mathematics that is most identified with financial engineering and mathematical finance. We work on a probability space $(\Omega, \mathcal{F}, P)$ where P is the probability measure, Ω is the universe of possible outcomes. We use $\omega \in \Omega$ to represent a generic outcome, and the set $\mathcal{F}$ is a σ-Algebra and it represents the set of possible events where an event is a subset of Ω. There is also a filtration, $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$, that models the evolution of information through time. If we are working with a finite horizon, $[0, T]$, then we can take $\mathcal{F} = \mathcal{F}_T$.

We also say that a stochastic process, $(X_t)_{t \in \mathbb{R}_+}$, is $\mathcal{F}_t$-adapted if the value of X_t is known at time t when the information represented by $\mathcal{F}_t$ is known. All the processes we consider will be $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-adapted so we will not bother to state this in the consequence.

In the continuous-time models that we are study, it is understood that the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is the filtration generated by the stochastic processes (usually a Brownian motion, $(W_t)_{t \in \mathbb{R}_+}$, that are specified in the model description). For more details refer to [13].

3.1 Martingales and Brownian motion

3.1.1 Brownian motion

Robert Brown, a botanist, first observed the movement of pollen particles as described in his 1827 paper "A brief account of microscopical observations" (see [20]).

Brownian motion is the random motion of particles suspended in a fluid (a liquid or a gas) resulting from their collision with the quick atoms or molecules in the gas or liquid (see Figure 1, below). The term "Brownian motion" can also refer to the mathematical

model used to describe such random movements, which is often called a particle theory. For more details refer to [17].
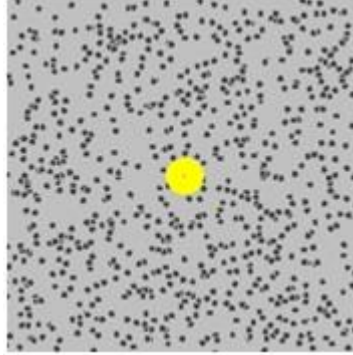
Figure 1: Pollen particles movement

Definition 3.1 ([19]) *The standard Brownian motion is a stochastic process* $(W_t)_{t\in\mathbb{R}_+}$ *satisfying the following properties:*

i. $W_0 = 0$ *almost surly.*

ii.   *The sample trajectories* $t \mapsto W$ *are continuous, with probability 1.*

iii.   *For any finite sequence of times* $t_0 < t_1 < \cdots < t_n$, *the increments* $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \dots, W_{t_n} - W_{t_{n-1}}$ *are independent.*

iv.   *For any times* $0 \le s < t$, $W_t - W_s$ *is normally distributed with mean zero and variance* $t - s$, *(i.e* $W_t - W_s \sim N(0, t - s)$), *thus* $E[W_t - W_0] = 0 \Rightarrow E[W_t] = 0$ *for any t.*

3.1.2 Martingale

In probability theory, a martingale is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. In particular, a martingale is a sequence of random variables (i.e. a stochastic process) for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values.

Definition 3.2 ([18]) Let $(M_t)_{0 \le t \le T}$ be a *stochastic process, then M_t is said to be a martingale if :*

$$M_t = M_0 + \int_0^t U_s dW_s.$$

3.2 Stochastic Integral

Consider a partition of the time interval, $[0, T]$ given by:

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T.$$

Let X_t be a Brownian motion and consider the sum of squared changes

$$Q_n(T) := \sum_{i=1}^n \left[\Delta X_{t_i} \right]^2,$$

where $\Delta X_{t_i} := X_{t_i} - X_{t_{i-1}}$.

Definition 3.3 ([19]) (Quadratic Variation) The quadratic variation of a stochastic process, X_t, is equal to the limit of $Q_n(T)$ as $\Delta t := max_i(t_i - t_{i-1}) \to 0$.

Definition 3.4 ([19]) *We say a process, $h_t(\omega)$, is elementary if it is piece-wise constant so that there exist a sequence of stopping times $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, and a set of $\mathcal{F}_{t_i}$-measurable (a function $f(\omega)$ is $\mathcal{F}_{t_i}$-measurable if its value is known by time t) functions, $e_i(\omega)$, such that:*

$$h_t(\omega) = \sum_i e_i(\omega) I_{[t_i, t_{i+1})}(t),$$

where $I_{[t_i, t_{i+1})}(t) = 1$ if $t \in I_{[t_i, t_{i+1})}$ and 0 otherwise.

Definition 3.5 *The stochastic integral of an elementary function, $h_t(\omega)$, with respect to a Brownian motion, W_t, is defined as:*

$$\int_0^T h_t(\omega)dW_t(\omega) := \sum_{i=0}^{n-1} e_i(\omega)\left(W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\right).$$

For a more general process, $X_t(\omega)$, we have:

$$\int_0^T X_t(\omega)dW_t(\omega) := \lim_{n\to\infty}\int_0^T X_{n,t}(\omega)dW_t(\omega),$$

where $X_{n,t}$ is a sequence of elementary processes that converges (in an appropriate manner) to X_t.

Example 3.1 We want to compute $\int_0^T W_t dW_t$. To reach this end, let

$$X_{n,t} := \sum_{i=0}^{n-1} W_{t_{n,i}} I_{[t_{n,i},t_{n,i+1})}(t),$$

where $0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,n} = T$ and $I_{[t_{n,i},t_{n,i+1})}(t) = 1$ if

$t \in [t_{n,i}, t_{n,i+1})$ and 0 otherwise. Then $X_{n,t}$ is an adapted elementary process and, by

continuity of Brownian motion, satisfies $\lim_{n\to\infty} X_{n,t} = W_t$ almost surely as

$max_i|t_{n,i}, t_{n,i+1}| \to 0$. The Itô integral of $X_{n,t}$ is given by:

$$\int_0^T X_{n,t}dW_t = \sum_{i=0}^{n-1} W_{t_{n,i}}\left(W_{t_{n,i+1}} - W_{t_{n,i}}\right)$$

$$= \frac{1}{2}\sum_{i=0}^{n-1}\left(W_{t_{n,i+1}}^2 - W_{t_{n,i}}^2 - \left(W_{t_{n,i+1}} - W_{t_{n,i}}\right)^2\right).$$

Thus we have:

$$\int_0^T X_{n,t}dW_t = \frac{1}{2}W_T^2 - \frac{1}{2}W_0^2 - \frac{1}{2}\sum_{i=0}^{n-1}\left(W_{t_{n,i+1}} - W_{t_{n,i}}\right)^2. \qquad (3.1)$$

By the definition of quadratic variation the sum on the right-hand-side of equation (3.1) converges in probability to T. And since $W_0 = 0$ we obtain:

$$\int_0^T W_t dW_t = \lim_{n \to \infty} \int_0^T X_{n,t} dW_t = \frac{1}{2} W_T^2 - \frac{1}{2} T. \ \blacksquare$$

Definition 3.6 ([19]) *We define the space $L^2[0,T]$ to be the space of processes, $X_t(\omega)$, such that:*

$$E\left[\int_0^T X_t^2 dt\right] < \infty.$$

Theorem 3.1([19]) *(Itô's Isometry) For any $X_t(\omega) \in L^2[0,T]$ we have*

$$E\left[\left(\int_0^T X_t dW_t\right)^2\right] = E\left[\int_0^T X_t^2 dt\right].$$

3.3 Stochastic Differential Equations

Definition 3.7 ([20]) *An n-dimensional Itô process, X_t, is a process that can be represented as:*

$$X_t = X_0 + \int_0^t V_t ds + \int_0^t U_t dW_s; \quad t \in \mathbb{R}_+, \tag{3.2}$$

where W is an $m-$dimensional standard Brownian motion, and V and U are $n-$dimensional and $n \times m-$dimensional $\mathcal{F}_t-$adapted processes, respectively. We often use the following notation as shorthand of equation (3.2):

$$dX_t = V_t dt + U_t dW_t, \tag{3.3}$$

where $(U_t)_{t \in \mathbb{R}_+}$ and $(V_t)_{t \in \mathbb{R}_+}$ are square-integrable adapted processes

An n-dimensional Stochastic Differential Equation (SDE) has the form:

$$dX_t = V(X_t, t)dt + U(X_t, t)dW_t; \quad X_0 = x, \tag{3.4}$$

where as stated before, W_t is an $m-$dimensional standard Brownian motion, and V and U are $n-$dimensional and $n \times m-$dimensional $\mathcal{F}_t-$adapted processes, respectively. Equation (3.4) is shorthand for:

$$X_t = x + \int_0^t V(X_s, s)ds + \int_0^t U(X_s, s)dW_s . \tag{3.5}$$

While we do not discuss the issue here, various conditions exist to guarantee existence and uniqueness of solutions to equation (3.5). A useful tool for solving SDE's is Itôs' lemma, which we discuss in the next section:

3.3.1 Itô's formula and its applications

Theorem 3.2 ([18]) *For any Itô process $(X_t)_{t \in \mathbb{R}_+}$ of the form of equation (3.2) and any $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ we have:*

$$\left(f(t, X_t) = f(0, X_0) + \int_0^t v_s \frac{\partial f}{\partial x}(s, X_s)ds + \int_0^t u_s \frac{\partial f}{\partial x}(s, X_s)dW_s \right.$$

$$\left. + \int_0^t \frac{\partial f}{\partial s}(s, X_s)ds + \frac{1}{2} \int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)ds \right), \tag{3.6}$$

where $f(t, X_t)$ is a smooth function of two variables.

If f is function of two variables $f(t, x, y)$, then the Itô formula in differential form is given by:

$$df = f_t dt + f_x dx + f_y dy + \frac{1}{2}\left[f_{xx}d\langle x, x\rangle + 2f_{xy}d\langle x, y\rangle + f_{yy}d\langle y, y\rangle \right]. \tag{3.7}$$

Consider, two processes $(X_t)_{t \in \mathbb{R}_+}$ and $(Y_t)_{t \in \mathbb{R}_+}$ following the dynamics of:

$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dW_s$, $t \in \mathbb{R}_+$, and $Y_t = Y_0 + \int_0^t b_s ds + \int_0^t a_s dW_s$,

then Itô formula leads to:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t,$$

where the product $dX_t dY_t$ is computed according to the Itô rule as follow:

$$(dt)^2 = 0, dt \cdot dW_t = 0, (dW_t)^2 = dt. \tag{3.8}$$

Example 3.2 Consider a martingale $Z = (Z_t)_{t\in[0,T]}$ and process $Y = (Y_t)_{t\in[0,T]}$ satisfying $dY_t = 3tdt + Y_t dW_t$, and a function $f(t,y)$. If $Z_t = f(t, Y_t)$, then f is a solution to the following PDE:

$$\frac{\partial f}{\partial t} + \frac{1}{2}y^2 \frac{\partial^2 f}{\partial y^2} + 3t\frac{\partial f}{\partial y} = 0. \tag{3.9}$$

Proof: By applying Itô's formula (3.6) to the process $f(t, Y_t)$, we obtain:

$$Z_t = f(t, Y_t) = f(0, Y_0) + \int_0^t 3\tau \frac{\partial f}{\partial y} d\tau + \int_0^t Y_t \frac{\partial f}{\partial y} dW_\tau + \int_0^t \frac{\partial f}{\partial \tau} d\tau$$

$$+ \frac{1}{2} \int_0^t Y_\tau{}^2 \frac{\partial^2 f}{\partial y^2} d\tau.$$

After simplyfying terms, we get

$$Z_t = f(0, Y_0) + \left[\int_0^t (3\tau\frac{\partial f}{\partial y} + \frac{\partial f}{\partial \tau} + \frac{1}{2}Y_\tau{}^2 \frac{\partial^2 f}{\partial y^2})d\tau\right] + \int_0^t Y_\tau \frac{\partial f}{\partial y} dW_\tau.$$

Since $(Z_t)_{t\geq 0}$ is a martingale, thus the drift process in the above equation is a null process. Hence, If we let $y := Y_\tau$, we obtain the PDE in (3.9). ∎

Example 3.3 (Black-Scholes Model) Suppose a stock price process, $S = (St)_{t\in[0,T]}$, satisfies the Stochastic Differential Equation (SDE):

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t,$$

then the process S is given by

$$S_t = S_0 \exp\left(\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dW_s\right). \tag{3.10}$$

Proof: We apply Itô's formula to $Y_t = \ln(S_t)$, we derive:

$$dY_t = \frac{1}{S_t} dS_t - \frac{1}{2}\frac{1}{S_t^2} d(S_t)$$

$$dY_t = \mu_t dt + \sigma_t dW_t - \frac{\sigma_t^2}{2S_t^2} S_t^2 dt$$

$$dY_t = \left(\mu_t - \frac{\sigma_t^2}{2}\right) dt + \sigma_t dW_t.$$

As a result, we get the explicit solution to the SDE in equation (3.10).

In particular if $\sigma_s = \sigma$ and $\mu_s = \mu$, then we obtain

$$S_t = \mu_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma dWt\right), \tag{3.11}$$

and hence $\ln(S_t) \sim N\left((\mu - \sigma^2/2)t, \sigma^2 t\right).\blacksquare$

Example 3.4 (Vasicek model)

Another application of Itô's formula lies in solving the Vasicek's model. The Vasicek model was designed to predict the trajectory of interest rates. It was introduced in 1977

by Oldřich Vašíček and can be also seen as a stochastic investment model, for more information refer to [23].

Consider the Vasick Stochastic Differential Equation (VSDE):

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

Here W_t is the Brownien motion, then the process r is given by

$$r_t = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s.$$

Proof: By applying Itô's formula to $Y_t = e^{at} r_t$ and using (VSDE), we derive:

$$dY_t = e^{at} dr_t + a r_t e^{at} dt$$

$$= bae^{at} dt + e^{at} \sigma dW_t$$

$$= bd(e^{at}) + e^{at} \sigma dW_t.$$

Thus, we get

$$Y_t = r_0 + b(e^{at} - 1) + \sigma \int_0^t e^{as} dW_s.$$

Hence and after simplifications,

$$r_t = b + (r_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s. \blacksquare$$

Chapter 4:   Derivation of the Asian Options PDE under the high volatility model

Asian options are path dependent options. Their payoff depends on either geometric or arithmetic average of the underlying asset price. They have several advantages. For instance, they are often cheaper than European options. In addition, an Asian option on a commodity reduces the price risk nearby the maturity. However, the downside is that they are in general difficult to value since the distribution of the payoff is usually unknown. In fact, at the present, no closed-form solution is available for pricing the arithmetic Asian options.

This chapter constitutes our main contribution in which we address the Asian options PDE for model with high volatility. This extends the case where model follows a Black-Scholes model.

The chapter is organized as follows, we first introduce the Black Sholes Model and the valuation of European options by PDE approach. Section 2 provides details on deriving the PDE for Asian options under the Black-Scholes framework. Our contribution starts in section 3 where we present the suggested high volatility model. We then solve the model, we perform numerical simulations, and we give some figures that are supportive to the model. In section 4, we find the PDE for the Asian option price. Then, we reduce it from two space-dimensions to one-space dimension. In addition, we show that one can obtain the modified Asian option price from the Black-Scholes Asian option value when the parameter of increase is equal to the interest rate.

4.1 The Black-Scholes Model

We work on a probability space $(\Omega, \mathcal{F}, P)$ where P is the probability measure, Ω is the universe of possible outcomes. The set $\mathcal{F}$ is a σ-Algebra and it represents the set of

possible events where an event is a subset of Ω. The time is varying from $t = 0$ to the expiration date T. In addition, we consider a standard Brownian motion $(W_t)_{t\in[0,T]}$, its natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that models the evolution of information through time and we suppose that $\mathcal{F} = \mathcal{F}_T$.

Let us consider an option and assume the market has two assets, a risky underlying asset with price $(S_t)_{t\in[0,T]}$, and a risk-free asset with price $(A_t)_{t\in[0,T]}$. The Black - Schols model supposes that the percentage change in the stock price in a short period of time are normally distributed. Moreover, it imposes the following conditions:

- The stock price follows a process with $N(\mu, \sigma)$ where μ and σ are constants.

- The short selling of securities with full use of proceeds is permitted.

- There are no transactions costs or taxes. All securities are perfectly divisible.

- There are no dividends during the life of the derivative.

- There are no riskless arbitrage opportunities.

- Security trading is continuous.

- The risk-free rate of interest, r, is constant and the same for all maturities. The reader can refer to [9] for more details.

From now on, we denote by $V(S_t, t)$ the option price. The dynamic of the underlying asset price is given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where $t \in [0, T]$, and S_0 is a positive constant. It should noticed that, $W_t \sim N(0, t - s)$, so S_t is lognormal. The Black-Scholes formula for European call options is given in the following theorem.

Theroem 1 ([3]) *The price $V(S_t, t)$ of an European call option on a stock S_t with strike K and maturity $T$is given by*

$$V(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2),$$

 where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx$ is the standard normal distribution function, σ is the volatility and r is the interest rate.

The previous formula is the solution of a partial differential equation as it is stated in the following proposition.

Proposition 1 ([3]) *The price $V(S_t, t)$ of an European call option on a stock S_t with strike K and maturity T satisfies the following PDE*

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS \frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0,$$

$$S \in]0, \infty[, \quad 0 \le t \le T,$$

with the terminal condition $V(S, T) = max(S - K, 0)$.

Proof: Define π as the value of the portfolio, $\pi = V_t - \Delta S_t$ where Δ is the hedging factor. For shortly, we will use the notation S instead of S_t and V instead of V_t

$$d\pi = dV - \Delta dS. \qquad (4.1)$$

Apply Itô's lemma to V (noting that V is a function of S and t):

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt.$$

Substitute in (4.1) gives:

$$d\pi = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS.$$

Combine the like terms:

$$d\pi = \left(\frac{\partial V}{\partial S} - \Delta\right) ds + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

where $\left(\frac{\partial V}{\partial S} - \Delta\right)$ is the stochastic random part and $\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)$ is the

deterministic part. Choose $\Delta = \frac{\partial V}{\partial S}$. This implies:

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

The assumption (No arbitrage argument) is that we are trading in a perfectly liquid market with no transaction costs (i.e) if we have cash flows in the bank , then they have to be equal. Pick the money in the bank we get:

$$d\pi = r\pi dt = r(V - \Delta S)dt = r\left(V - S\frac{\partial V}{\partial S}\right) dt.$$

Comparing the last two equations yields:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = \left(rV - rS\frac{\partial V}{\partial S}\right) dt.$$

Hence

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}.$$

Therefore the Black–Schols PDE for the European option price is as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad (4.2)$$

where $0 \leq S$, and $0 \leq t \leq T$. ∎

The above proof can be found in many references, we cite for instance [18].

4.2 Deriving the PDE for Asian Option under the Black and Scholes model

In this section we are going to derive the Asian Option PDE under the Black Sholes

Model. An Asian option depends on two things S_t and Y_t where:

$$Y_t = \int_0^t S_\tau d\tau.$$

We consider an arithmetic Asian call option with payoff given by

$$H(S_t) = \left(\frac{1}{T} \int_0^T S_\tau d\tau - K \right)^+. \tag{4.3}$$

Next, we define the self financing condition which is needed in deriving the PDE of

the Asian Option.

Definition 4.1 ([2]) *A portfolio is self-financing if there is no exogenous infusion or*

withdrawal of money; the purchase of a new asset must be financed by the sale of an

old one.

Let V be the value of the portfolio, $\Delta(t)$ be the number of units invested at time t, r be

the interest rate, and S_t be the price of the underlying asset at time t. Then the self

financing condition can be represented by the following equation:

$$dV = \Delta(t)dS_t + (V - \Delta(t)S_t)rdt.$$

Throughout we work within the Black-Sholes model

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $t \in [0, T]$, and S_0 is a positive constant. We assume that the Asian call is

replicable (attainable). Denote by $V(t, x, y)$, the time t option value/replicating

portfolio value under the assumption that $S_t = x$, and $Y_t = y$. The following proposition gives the PDE of the Asian call option price.

Proposition 2 ([21]) *The price* $V(S_t, t)$ *of a Asian call option on a stock* S_t *with strike K and maturity T satisfies the following PDE*

$$V_t(t, x, y) + xV_y(t, x, y) + rxV_x(t, x, y) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t, x, y) - rV(t, x, y) = 0$$

$$0 \leq t \leq T, \qquad x > 0, \qquad y \in \mathbb{R}.$$

with the terminal condition $V(S, T) = \left(\frac{1}{T}\int_0^T S_\tau d\tau - K\right)^+$.

Proof. For shortly, from now we will use the notation V instead of $V(t, x, y)$.

By applying Itô formula Equation (3.3) to V implies

$$dV = V_t dt + V_x dx + V_y dy + \frac{1}{2}\left[V_{xx}d\langle x, x\rangle + 2V_{xy}d\langle x, y\rangle + V_{yy}d\langle y, y\rangle\right].$$

Simplifying yields

$$dV = V_t dt + V_x dx + V_y dy + \frac{1}{2}V_{xx}d\langle x, x\rangle.$$

Substituting by the values of dx, dy, and $d\langle x, x\rangle$ gives

$$dV = V_t dt + V_x(\mu x dt + \sigma x dW_t) + xV_y dt + \frac{1}{2}\sigma^2 x^2 V_{xx} dt,$$

and

$$dV = \left[V_t + \mu x V_x + xV_y + \frac{1}{2}\sigma^2 x^2 V_{xx}\right]dt + \sigma x V_x dW_t$$

The self-financing condition yields

$$dV = \Delta(t)dS_t + (V - \Delta(t)S_t)rdt.$$

Therefore

$$dV = \Delta(t)(\mu x dt + \sigma x dW_t) + (V - \Delta(t)x)rdt.$$

and

$$dV = [\Delta(t)\mu x + Vr - \Delta(t)xr]dt + \Delta(t)\sigma x dW_t.$$

Matching the coefficients, we obtain

$$V_x = \Delta(t)$$

and

$$V_t + \mu x V_x + x V_y + \frac{1}{2}\sigma^2 x^2 V_{xx} = \mu x V_x + Vr - xr V_x.$$

Therefore the PDE of the Asian option is:

$$V_t(t,x,y) + x V_y(t,x,y) + rx V_x(t,x,y) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t,x,y) - rV(t,x,y) = 0,$$

To not stuck in the notation I'll re-write the last equation as follow:

$$V_t + x V_y + rx V_x + \frac{1}{2}\sigma^2 x^2 V_{xx} - rV = 0. \tag{4.4}$$

This is a second-order partial differential equation (PDE) with two space variables and one time variable. Moveover, the second order partial derivative with respect to x is degenerate. Numerical solutions of this partial differential equation is possible but cumbersome as well as time-consuming. We now introduce the variable reduction method which transforms equation (4.4) into a PDE with only one state variable and one time variable.

Proposition 3 ([5]) *The PDE of the Asian call option price on a stock $S_t = x$ with strike K and maturity T given by (4.4) can be reduced to*

$$f_t - \left(\frac{1}{T} + rx\right)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} = 0, \qquad 0 \le t \le T, \qquad x > 0,$$

with the terminal condition $f(S,T) = \left(\frac{1}{T}\int_0^T S_\tau d\tau - K\right)^+$.

Proof. Let $\eta = \frac{K - y/_T}{x}$, and $V(t,x,y) = xf(t,\eta)$, K is the strike price and T is the maturity time [5]. Now deriving V with respect to t: $V_t = xf_t + 0 \cdot f$. Thus $V_t = xf_t$.

We derive V with respect to x: $V_x = f(t,\eta) + xf_x(t,\eta)$.

$$V_x = f(t,\eta) + x\frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = f(t,\eta) + x\left(f_\eta \cdot \frac{-(K-y/T)}{x^2}\right),$$

notice that $\eta_x = \frac{-\eta}{x}$, therefore $V_x = f(t,\eta) - \eta f_\eta(t,\eta)$. Now we find the second

derivative of V $V_{xx} = \frac{\partial(V_x)}{\partial x} = \frac{\partial}{\partial x}\left[f(t,\eta) - \eta f_\eta(t,\eta)\right].$

$$V_{xx} = \frac{\partial}{\partial \eta}\left[f(t,\eta) - \eta f_\eta(t,\eta)\right] \cdot \frac{\partial \eta}{\partial x} = \left(f_\eta(t,\eta) - f_\eta(t,\eta) - \eta f_{\eta\eta}(t,\eta)\right) \cdot \frac{-\eta}{x} = \frac{\eta^2}{x}f_{\eta\eta}.$$

Moreover, we derive V with respect to y:

$$V_y = \frac{\partial V}{\partial y} = xf_y(t,\eta) = x \cdot \frac{\partial f(t,\eta)}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = x \cdot f_\eta(t,\eta) \cdot \frac{-1}{Tx} = \frac{-1}{T}f_\eta(t,\eta).$$

 Now we substitute in equation (4.4):

$$xf_t + x\left(\frac{-1}{T}f_\eta\right) + rx(f - \eta f_\eta + \frac{1}{2}\sigma^2 x^2\left(\frac{\eta^2}{x}f_{\eta\eta}\right) - rxf = 0.$$

Then

$$f_t - \frac{1}{T}f_\eta + r(f - \eta f_\eta) + \frac{1}{2}\sigma^2\eta^2 f_{\eta\eta} - rf = 0.$$

Therefore $f_t - \frac{1}{T}f_\eta + rf - r\eta f_\eta + \frac{1}{2}\sigma^2\eta^2 f_{\eta\eta} - rf = 0.$

Thus $f_t - \left(\frac{1}{T} + r\eta\right)f_\eta + \frac{1}{2}\sigma^2\eta^2 f_{\eta\eta} = 0.$

For sake of clarity, we use x instead of η:

$$f_t - \left(\frac{1}{T} + rx\right)f_x + \frac{1}{2}\sigma^2 x^2 f_{xx} = 0. \quad \blacksquare \tag{4.5}$$

4.3 The high volatility model

We assume that the market is suffering from an increased volatility. We suggest the stock price to be driven by the SDE:

$$dS_t = \mu S_t dt + (\sigma S_t + \alpha)dW_t, \qquad 0 \le t \le T, \qquad x > 0. \tag{4.6}$$

It is a stylized fact that during financial crisis the volatility is higher than normal situations. The suggested model can be seen as a stochastic volatility model in a complete market (contingent claims are attainable). The above model is a generalization of the Black-Scholes model.

Proposition 4 ([7]) *Consider a time $t \in [0,T]$ and the process $(\xi_t)_{0 \le t \le T}$ defined by*

$\xi_t := exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$. *Then the solution of the stochastic differential equation (4.6) of the high volatile model is given by:*

$$S_t = \left(x + \frac{\alpha}{\sigma}\right)\xi_t - \frac{\alpha}{\sigma}\left(1 + \mu \int_0^t \frac{\xi_t}{\xi_s}ds\right).$$

Proof: The solution is obtained by finding first a particular solution when $\alpha = 0$, then using the variation of the constants method. For more information see [7].$\blacksquare$

4.4 Simulation of price trajectories for the high volatility model

In this section, we perform numerical simulations for the asset price trajectories for our high volatility model. Several figures are provided below showing that our model accounts for high volatility situations. We first introduce to the methods for simulating stochastic differential equations.

4.4.1 Numerical methods for solving a stochastic differential equation

Most models of asset can be expressed in terms of a stochastic differential equation (SDE). Consider the following SDE:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad t_0 \le t \le T \quad (4.7)$$

$$X(t_0) = X_0,$$

where $W = (Wt)_{t \in [0,T]}$ is a standard Brownian motion, μ (the drift) and σ (the diffusion coefficient) are defined and measurable.

We have according to the Itô- Taylor expansion

$$X_t = X_{t_0} + \mu(X_{t_0}) \int_{t_0}^t ds + \sigma(X_{t_0}) \int_{t_0}^t dW(s)$$

$$+ \frac{1}{2}\sigma(X_{t_0})\sigma'(X_{t_0})\left([W(t) - W(t_0)]^2 - (t - t_0)\right) + R, \quad (4.8)$$

where R is the remainder. We simulate the SDE (4.7) by generating samples of the discretized version at a finite number of point: $X_{\Delta t}, X_{2\Delta t}, \dots, X_{m\Delta t}$, where m is the number of time steps and Δt is the time step assuming equidistant subinterval, $\Delta t = \frac{T}{m}$. To write it more formally:

$$\hat{X}_{t_1}, \hat{X}_{t_2}, \dots, \hat{X}_{t_j}, \dots, \hat{X}_{t_m}.$$

Where $t_j = t_0 + j\Delta t = j\Delta t$ for $j = 1, \dots, m$.

$$\Delta t = t_{j+1} - t_j, \quad \Delta W_j = W(t_{j+1}) - W(t_j).$$

We get the following expression for equation (4.8):

$$X_{t_{j+1}} = X_{t_j} + \mu\left(X_{t_j}\right)\Delta t + \sigma\left(X_{t_j}\right)\Delta W_j + \frac{1}{2}\sigma\left(X_{t_j}\right)\sigma'\left(X_{t_j}\right)\left((\Delta W_j)^2 - \Delta t\right) + R.$$

There are several schemes for simulating SDEs of this form. The most common schemes are: Euler-Maruyama, Milstein, and Runge-Kutta schemes. Below, we present the algorithm of Euler-Maruyama scheme [14]

Algorithm. *Let* $\Delta t := \frac{T}{m}$ *for a given* m. *Then approximate the SDE (4.7) via:*

1. *Set* $Y_m(0) = X(0) = x_0$.

2. *For* $j = 0$ *to* $m - 1$ *do*

    a. *Simulate a standard normally distributed random number* Z_j.

    b. *Set* $\Delta W(j\Delta t) = \sqrt{\Delta t} Z_j$ *and*

$$Y_m\big((j+1)\Delta t\big) = Y_m(j\Delta t) + \mu(Y_m(j\Delta t), j\Delta t)\Delta t$$

$$+ \sigma(Y_m(j\Delta t), j\Delta t)\Delta W(j\Delta t).$$

4.4.2 Illustrations of the underlying asset prices and the high volatility

We provide here some figures for the underlying asset price with high volatility. We have used Euler scheme to simulate the trajectory of the asset price for the both cases: high volatility and Black-Scholes. We provide additional figures when we take $S_0 = 7, \sigma = 0.5, \mu = 0.02, T = 1$, and the time step $\Delta t = 0.001$.
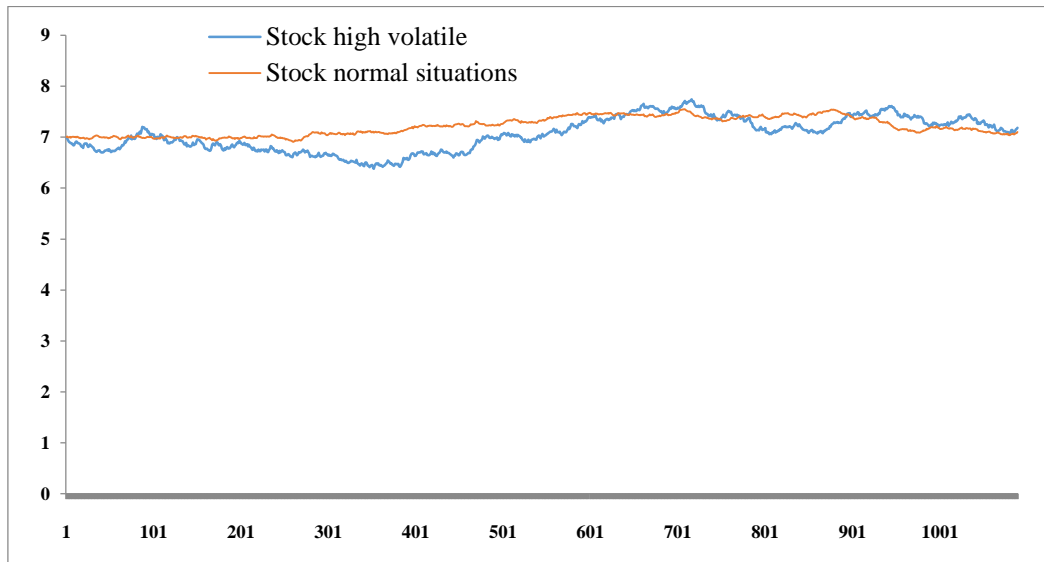

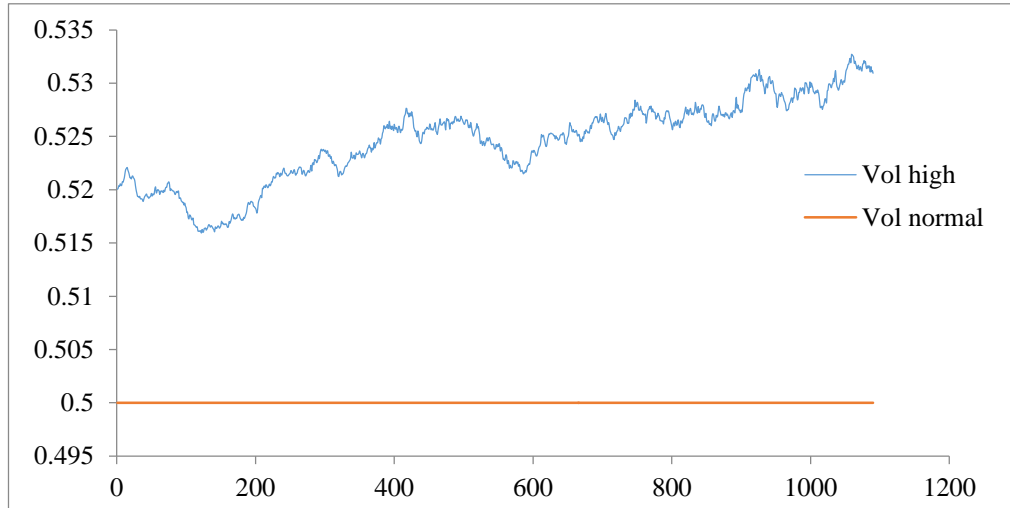Figure 2: Underlying asset price in high volatility and Black-Scholes for α=0.5

Figure 3: The increased in the volatility for our suggested model (α=1).

4.5 The modified Asian option: The PDE approach

In this section, we will consider the modified model described in equation (4.6). We believe that this model catch the financial crisis features. For this model, we will characterize the Asian option price using PDEs. To this end, for the reader's convenience, we recall the dynamic of the modified model given in equation (4.6):

$$dS_t = \mu S_t dt + (\sigma S_t + \alpha)dW_t,$$

where α is a constant arising from an increase in the volatility.

Proposition 5 *Suppose that the price process for the Asian option is given by* $V(t, S_t, Y_t)$. *Here* $Y_t = \int_0^t S_\tau d\tau$ *and* $V(t, x, y)$ *is a function. Then* $V(t, x, y)$ *is a solution to the following PDE*

$$V_t + xV_y + rxV_x + \frac{1}{2}(\sigma x + \alpha)^2 V_{xx} - rV = 0.$$

Proof: We can wrie $S_t = x$ and $Y_t = y$. Hence equation (4.6) becomes:

$$dx = \mu x dt + (\sigma x + \alpha)dW_t.$$

V is an $It\hat{o}$ Process in two variables. $It\hat{o}$ formula equation (3.7) implies:

$$dV(t, x, y) = V_t dt + V_x dx + V_y dy + \frac{1}{2}\left[V_{xx} d\langle x, x\rangle + 2V_{xy} d\langle x, y\rangle + V_{yy} d\langle y, y\rangle\right].$$

Notice that $d\langle x, y\rangle = d\langle y, y\rangle = 0$, $d\langle x, x\rangle = (\sigma x + \alpha)^2 dt$, *and* $dy = xdt$.

Thus $dV = V_t dt + V_x[\mu x dt + (\sigma x + \alpha)dW_t] + V_y x dt + \frac{1}{2}V_{xx}(\sigma x + \alpha)^2 dt$.

Therefore we get

$$dV = \left[V_t + \mu x V_x + x V_y + \frac{1}{2}V_{xx}(\sigma x + \alpha)^2\right] dt + (\sigma x + \alpha)V_x dW_t. \qquad (4.9)$$

The Self financing condition yields:

$$dV(t, x, y) = \Delta(t)dx + (V(t, x, y) - \Delta(t)x)rdt.$$

Substituting by the value of dx:

$$dV = \Delta(t)(\mu x dt + (\sigma x + \alpha)dW_t) + (V - \Delta(t)x)rdt.$$

Combining the like terms gives:

$$dV = [\Delta(t)\mu x + Vr - \Delta(t)xr]dt + [\Delta(t)\sigma x + \Delta(t)\alpha]dW_t. \qquad (4.10)$$

Matching the coefficients in equations (4.9) and (4.10) gives:

$$V_x = \Delta(t),$$

and

$$V_t + \mu x V_x + x V_y + \frac{1}{2}(\sigma x + \alpha)^2 V_{xx} = \mu x V_x + Vr - xr V_x.$$

Thus we get the PDE for the modified Asian Option.

$$V_t + x V_y + rx V_x + \frac{1}{2}(\sigma x + \alpha)^2 V_{xx} - rV = 0. \ \blacksquare \qquad (4.11)$$

The previous equation can be reduced to two variables.

4.5.1 Reduction of the PDE

The following proposition gives the reduced PDE.

Proposition 6 *The PDE for the Modified Asian Options, which is given by*

$$V_t + xV_y + rxV_x + \frac{1}{2}(\sigma x + \alpha)^2 V_{xx} - rV = 0,$$

where $0 \leq t \leq T$, and $V(t, x, y) = V$ is the price of the Asian option, can be reduced

to

$$f_t(t, x) + rxf_x(t, x) + \frac{(\sigma x + \alpha)^2}{2} f_{xx}(t, x) - (r + ax)f(t, x) = 0,$$

where $V(t, x, y) = cf(t, x)e^{by}$, $a, b,$ and c are constant.

Proof: We have from proposition 5:

$$V_t + xV_y + rxV_x + \frac{1}{2}(\sigma x + \alpha)^2 V_{xx} - rV = 0. \qquad (4.11)$$

We use a separation of variables method. Assume $V(t, x, y) = f(t, x)\Phi(y)$

Thus $V_t = \frac{\partial f(t,x)}{\partial t}\Phi(y), \quad V_y = f(t, x)\Phi'(y) , \qquad V_x = f_x(t, x)\Phi(y).$

And $V_{xx} = f_{xx}(t, x)\Phi(y).$

Substituting in (4.11) yields:

$$\left(\Phi(y)f_t(t, x) + xf(t, x)\Phi'(y) + rxf_x(t, x)\Phi(y) + \frac{1}{2}(\sigma x + \alpha)^2\Phi(y)f_{xx}(t, x) - \right.$$

$$\left. rf(t, x)\Phi(y) = 0\right).$$

Dividing by $xf(t,x)\Phi(y)$:

$$\frac{f_t(t,x)}{xf(t,x)} + \frac{\Phi'(y)}{\Phi(y)} + \frac{rf_x(t,x)}{f(t,x)} + \frac{(\sigma x + \alpha)^2}{2xf(t,x)} f_{xx}(t,x) - \frac{r}{x} = 0,$$

and

$$-\frac{\Phi'(y)}{\Phi(y)} = \frac{f_t(t,x)}{xf(t,x)} + \frac{rf_x(t,x)}{f(t,x)} + \frac{(\sigma x + \alpha)^2}{2xf(t,x)} f_{xx}(t,x) - \frac{r}{x} = a.$$

We know that $\Phi'(y) = \frac{d\Phi(y)}{dy}$, so $\frac{d\Phi(y)}{\Phi(y)dy} = -a \Rightarrow \frac{d\Phi(y)}{\Phi(y)} = -ady$. Then

$$\ln|\Phi(y)| = -ay + c_1.$$

So $\Phi(y) = ce^{-ay}$, where $c = e^{c_1}$. The solution of the equation (4.11) is:

$$V(t,x,y) = f(t,x)ce^{-ay}.$$

Where $f(t,x)$ satisfies:

$$\frac{f_t(t,x)}{xf(t,x)} + \frac{rf_x(t,x)}{f(t,x)} + \frac{(\sigma x + \alpha)^2}{2xf(t,x)} f_{xx}(t,x) - \frac{r}{x} = a.$$

Multiplying by $xf(t,x)$ gives:

$$f_t(t,x) + rxf_x(t,x) + \frac{(\sigma x + \alpha)^2}{2} f_{xx}(t,x) - (r + ax)f(t,x) = 0. \blacksquare \qquad (4.12)$$

4.6 Asian option with an increased volatility is deterministic $g(t) = \beta e^{rt}$

In this subsection we assume that the increase in the volatility is a deterministic function of the interest rate r. The underlying asset price SDE is given by

$$dX_t = rX_t dt + (\sigma X_t + \beta e^{rt})dW_t, \qquad t \in [0,T], \qquad X_0 > 0. \qquad (4.13)$$

where $t \in [0,T]$, and X_0 is a positive constant. The solution of the above equation is

then given by

$$X_t = \left(X_0 + \frac{\beta}{\sigma}\right) e^{\left[\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t\right]} - \frac{\beta}{\sigma} e^{rt}. \tag{4.14}$$

Proposition 7 ([7]) *Consider a time $t \in [0,T]$ and let β be a real number such that*

$\beta \le x\sigma e^{-\left(\frac{\sigma^2}{2}T+3\sigma\sqrt{T}\right)}$. *Then the value of X_t given by equation (4.14) is stricktly*

positive with 99% probability.

For the proof see [7].

Now, we can state the relationship between the modified and the Black-Scholes

Asian option prices.

Proposition 8 *Let $C(S,K,T)$ be the price of an arithmetic Asian call option with strike*

K, and we assume that the underlying asset price follows the Black-Scholes dynamic

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad t \in [0,T], \quad S_0 > 0.$$

And let $C^\beta(X,K,T)$ be the price of an arithmetic Asian call option with strike K and

where the underlying asset follows the SDE (4.13), with $X_0 = S_0 - \frac{\beta}{\sigma}$ then

$$C^\beta(X,K,T) = C\left(S, K - \frac{\beta}{Tr\sigma} e^{rT}, T\right).$$

Proof. Notice that the price of the Asian call option with underlying asset price X is

$$C^\beta(X,K,T) = e^{-rT} E\left[max\left(\frac{1}{T}\int_o^T X_t dt - K, 0\right)\right]$$

$$= e^{-rT} E\left[max\left(\frac{1}{T}\int_o^T \left[\left(X_0 + \frac{\beta}{\sigma}\right) e^{\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t\right]} - \frac{\beta}{\sigma} e^{rt}\right] dt - K, 0\right)\right]$$

$$= e^{-rT} E\left[max\left(\frac{1}{T}\int_o^T S_0 e^{\left[\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t\right]} dt - \frac{1}{T}\int_0^T \frac{\beta}{\sigma} e^{rt} dt - K, 0\right)\right]$$

$$= e^{-rT} E\left[max\left(\frac{1}{T}\int_o^T S_t dt - \frac{\beta}{Tr\sigma}(e^{rT} - 1) - K, 0\right)\right]$$

$$= e^{-rT} E\left[max\left(\frac{1}{T}\int_o^T S_t dt - \left(\frac{\beta}{Tr\sigma}(e^{rT} - 1) + K\right), 0\right)\right]$$

Thus $C^\beta(X, K, T) = C\left(S, \frac{\beta}{Tr\sigma}(e^{rT} - 1) + K, T\right)$. ∎

The previous proposition is important since it provides a relation between the two Asian option prices: the option price under the Black-Scholes model and the option price under our high volatile model. If we know the premium of the Asian option in the Black-Scholes model for a given strike K, it is sufficient to replace K by $\frac{\beta}{Tr\sigma}(e^{rT} - 1) + K$ in order to obtain the price of the Asian option under the high volatile model.

Chapter 5: Conclusion

Asian options are interesting financial derivatives that are extensively utilized in commodity, currency, energy, interest rate, equity and insurance markets. There is no closed form solution for pricing arithmetic Asian options. Nevertheless, most of the work on valuing such products, proves the existence of a partial differential equation for the option price. In this thesis, we derived the partial differential equation for the price of an Asian option under a high volatile model. An abundance of literature on modeling the underlying asset price is available in various places, but one with high volatility is rare to find. In this work, we propose a modified Black-Scholes model for the underlying asset price that handles increased volatilities. The suggested model is important, since it can be used to describe asset prices during a crisis. Another interesting advantage of the suggested model, is that it is a stochastic volatility model, ensuring completeness of the market. We solve the stochastic differential equation of the underlying asset price. Moreover, numerical simulations and figures are provided, and are favorable for the suggested model. Then, we found the Asian option price PDE for the modified model. In addition, we prove a formula that allows us to find the modified Asian option price from the Black-Scholes option price when the increase in volatility is a deterministic function of the interest rate.

As a future direction of research, it would be interesting to investigate how we can implement the model to be useful in practice. We need to calibrate the model, in other words, to find a "good" value for the increased volatility parameter "α".

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