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An Efficient Algorithm Based On Fractional Legendre-Collocation Method for Solving Fractional Initial Value Problems

Ahmed Salameh Hussein Abu Orner

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United Arab Emirates University

College of Science

Department of Mathematical Sciences

AN EFFICIENT ALGORITHM BASED ON FRACTIONAL
LEGENDRE-COLLOCATION METHOD FOR SOLVING
FRACTIONAL INITIAL VALUE PROBLEMS

Ahmed Salameh Hussein Abu Omer

This thesis is submitted in partial fulfillment of the requirements for the degree of
Master of Science in Mathematics

Under the Supervision of Dr. Qasem M. Al-Mdallal

April 2016

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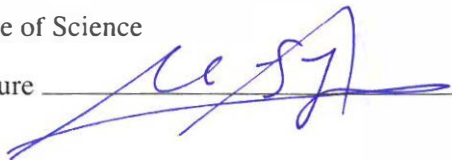
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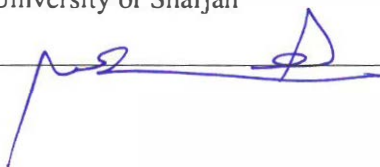
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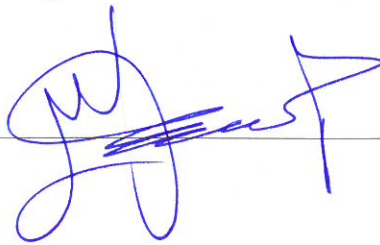
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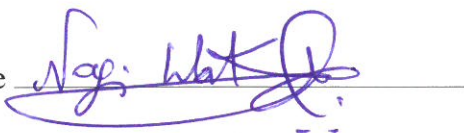
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Declaration of Original Work

I, Ahmed Salameh Hussein Abu Omer, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled "*An Efficient Algorithm Based on Fractional Legendre-Collocation Method for Solving Fractional Initial Value Problems*", hereby, solemnly declare that this thesis is an original research work that has been done and prepared by me under the supervision of Dr. Qasem M. Al-Mdallal, in the College of Science at UAEU. work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

In recent years, fractional calculus (the branch of calculus that generalizes the derivative of a function to non-integer order) has been a subject of numerous investigations by scientists from mathematics, physics and engineering communities. The interest in this area of research arises mainly from its applications to many models in the fields of fluid mechanics, electromagnetic, acoustics, chemistry, biology, physics and material sciences. In this thesis, we present a numerical algorithm for solving linear and non-linear fractional initial value problems. This numerical algorithm is based on the spectral method with fractional Legendre functions as basis. Then the collocation method is implemented to turn the original fractional initial value problem into algebraic system. Several examples are discussed to illustrate the efficiency and accuracy of the present scheme.

Keywords: Nonlinear Fractional Initial Value Problem, Caputo derivative, Spectral methods, Collocation method, Fractional Legendre functions, Multidimensional Newton's method.

Title and Abstract (in Arabic)

الخوارزمية الفعالة لحل مسائل تفاضلية كسرية ذات قيم ابتدائية باستخدام طريقة ليجندرا
كولوكيشن

الملخص

في السنوات القليلة الماضية كان التفاضل والتكامل الكسري (فرع من التفاضل والتكامل يعني بإيجاد المشتقة ذات الرتبة الكسرية للدوال عن طريق تعميم قواعد المشتقة ذات الرتبة الصحيحة) موضوع اهتمام لعدد كبير من الباحثين سواء في مجال الرياضيات، الفيزياء أو الهندسة. ازداد الاهتمام بهذا المجال نظراً لتطبيقاته المختلفة في عدة مجالات مثل علم ميكانيكية الموائع، علم الكهرومغناطيسية، علم الصوتيات، علوم الكيمياء والاحياء والفيزياء وعلوم أخرى. في هذه الأطروحة سوف نقدم خوارزمية عددية لحل مسائل تفاضلية كسرية خطية و لاخطية ذات القيم الابتدائية. كما تعتمد هذه الخوارزمية العددية على طريقة التجميع باستخدام دوال آليجنديراً كأساس لهذه الطريقة علماً بأن هذه الطريقة تعمل على تحويل المسألة الاصلية من مسألة تفاضلية كسرية ذات قيم ابتدائية الى نظام جبري. كما سوف نناقش عدة أمثلة لنين فعالية ودقة هذه الطريقة على هذه الأنواع من المسائل.

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Finally from the bottom of my heart I wish to thank my family for their patience and support throughout this work.

Dedication

To my family without whom I could not start my graduate degree

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Chapter 1: Introduction

Fractional calculus (the branch of calculus that generalizes the derivative of a function to non-integer order) isn't a new subject, it is a natural extension of the traditional calculus that deals with the integer derivative; i.e. $\frac{d^n y}{dx^n}$ for $n \in \mathbb{N}$. Historically, the idea of this subject appeared in a letter by Leibniz to L'Hospital in 1695 as a question: "What if n be $1/2$?" Since there, many theoretical works related to fractional calculus was reported in the literature to generalize the integer derivative to the fractional one, the reader is referred to [4] and [15]. It should be noted that, in 1819, Lacroix [3] wrote the first discussion of fractional derivatives by finding the n -th derivative of x^m :

$$\frac{d^n x^m}{dx^n} = \frac{m!}{(m-n)!} x^{m-n}, \quad m, n \in \mathbb{N}, \quad m \geq n$$

then setting $m = 1$ and $n = \frac{1}{2}$ to obtain the derivative of order $\frac{1}{2}$ of the function x , i.e.

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

But even with the results of Lacroix, still no clear definition of the fractional derivative. In fact, after several attempts by many notable mathematicians especially Riemann and Liouville, the modern scientists were able to define the so-called fractional derivative of arbitrary order as follows:

Definition 1.0.1. [19] The left sided Riemann-Liouville fractional integral operator of order α is defined by

$$I^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t) dt, \quad \alpha \in \mathbb{R}^+, \quad (1.1)$$

where $y \in L_1(a, b) := \left\{ z : [a, b] \rightarrow \mathbb{R} \mid \int_a^b z(t) dt < \infty \right\}$.

Notice that $\Gamma(x)$ generalizes the factorial $n!$ and allows n to take even non-integer and complex values. The Gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt,$$

for all $x \in \mathbb{R}^+$, provided that the integral exists. Common properties of the Gamma function are

1. $\Gamma(x+1) = x\Gamma(x)$,
2. $\Gamma(n) = (n-1)!$, where $n \in \mathbb{N}$

The properties of the operator I^α given in (1.1) are summarized in the following lemma (see [1] and [16]):

Lemma 1.0.1. *For any $f, g \in L_1[0, 1]$, $\alpha, \beta \geq 0$, $c_1, c_2 \in \mathbb{C}$ and $\gamma > -1$, the following properties hold:*

1. $I^0 f(x) = f(x)$
2. I^α exist for any $x \in [0, 1]$,
3. *The linearity property:* $I^\alpha(c_1 f(x) + c_2 g(x)) = c_1 I^\alpha f(x) + c_2 I^\alpha g(x)$,
4. *If f is continuous then* $I^\alpha(I^\beta f(x)) = I^{\alpha+\beta}(f(x)) = I^\beta(I^\alpha(f(x)))$, $x > 0$,
5. $I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

It is worth mentioning that there are several versions of fractional integral have been reported in the literature but we will consider the Riemann-Liouville fractional integral (1.1) in our study. Based on the definition of I^α given in (1.1), the left sided Caputo fractional derivative, $D^\alpha y(x)$ or $y^{(\alpha)}$ for $y \in L_1[a, b]$, is originally defined as follows [22]

$$D^\alpha y(x) = I^{n-\alpha} y^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} y^{(n)}(t) dt, \quad (1.2)$$

where $n = \lceil \alpha \rceil$ is the smallest integer greater than or equal to n and $\alpha \in \mathbb{R}^+$.

The Caputo fractional derivative satisfies the following properties for $f \in L_1[0, 1]$ $\alpha, \beta \geq 0$ and $n = [\alpha] + 1$:

1. $D^\alpha I^\alpha f(x) = f(x)$.
2. $I^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) (x^k/k!)$.
3. If f is continuous then $D^\alpha D^\beta f(x) = D^\beta D^\alpha f(x) = D^{\alpha+\beta} f(x), x > 0$,
4. $D^\alpha c = 0$, where c is a constant.
5. $D^\alpha x^\gamma = \begin{cases} 0 & , \gamma < \alpha, \gamma \in \{0, 1, 2, \dots\} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha} & , \text{otherwise} \end{cases}$.
6. $D^\alpha \left(\sum_{i=0}^m c_i f_i(x) \right) = \sum_{i=0}^m c_i D^\alpha f_i(x)$, where c_1, c_2, \dots, c_m are constants.

For the proof of these properties, the reader is referred to [19].

The first application on fractional calculus was presented by Niel Henrik Abel in 1823 to find the shape of a fractional wire laying in a vertical plane. Particulary, numerous researchers in the last few decades pointed out that the fractional derivatives and integrals are very suitable for describing the properties of various real material such as: polymers [2], the memory and hereditary properties [5], control engineering [6], [7], signal process [8], electromagnetism [9], fluid mechanics [10], the dynamics of viscoelastic material [11], pharmacokinetics [12], diffusion processes [13], and the discription of the rheological properties of rocks [14].

In this thesis, we develop an efficient numerical algorithm for solving a class of fractional initial value problems of the form:

$$D^\alpha u(x) + f(x, u, u') = 0 \quad x \in [0, 1], \quad 1 < \alpha \leq 2, \quad (1.3)$$

subject to

$$u(0) = u_0, \quad u'(0) = u_1. \quad (1.4)$$

where $u_0, u_1 \in \mathbb{R}$ and $u \in L_1(a, b)$.

The present work is motivated by the desire to find an approximate solution of the problem (1.3)-(1.4) using an efficient numerical technique based on the spectral method or more precisely fractional order Legendre Collocation method. The idea of this method is to write the solution of the (1.3) as a sum of fractional order Legendre functions "basis functions" and then to choose the coefficients in the sum in order to satisfy the differential equation as well as possible. It should be noted that this technique is used by several researchers to solve several types of Fractional ordinary differential equations or Fractional partial differential equations, see for example Kazem et al. [20], Klimek and Agrawal [24], Bhrawy and Alghamdi [25], Yiming et al. [23], Syam et al. [18], Syam and Al-Refai [26] and Bhrawy et al.[28].

Kazem et al. [20] constructed a fractional-order Legendre functions based on the well-known shifted Legendre polynomials. Moreover, they derived product operational matrices which together with the Tau method were utilized to reduce the solution of the fractional initial value problems to the solution of a system of algebraic equations. In the same year, Klimek and Agrawal [24] introduced a Fractional Legendre Equation and discussed its solution. They proved that the Legendre functions resulting from an fractional Legendre equation were the same as those obtained from the integer order Legendre equation. Bhrawy and Alghamdi [25] utilized the integer Legendre spectral Galerkin and pseudo-spectral approximations for fractional initial value problems. Yiming et al. [23] applied a series of fractional-order Legendre functions to discuss the numerical solution of fractional partial differential equations with variable coefficients given by:

$$a(x)D_x^\gamma u(x, t) + b(t)D_t^\gamma u(x, t) + c(x)u_t(x, t) + d(x)u(x, t) = g(x, t),$$

subject to

$$u(0,t) = u(x,0) = 0,$$

where $(x,t) \in \Omega := [0,h] \times [0,l]$ and D_x^ν, D_t^γ are the Caputo's derivatives with respect to x and t , respectively. They derived fractional differential operational and product matrices. These matrices were combined with Tau method to transform the fractional partial problem with variable coefficients to solve system of linear algebraic equations. Syam et al. [18] developed a numerical technique based on the Tau Legendre and path following methods to solve the fractional Riccati equation given by:

$$a(x)D^\alpha y(x) + b(x)y(x) + c(x)y^2(x) = g(x), \quad x \in (0,1), \text{ where } 0 < \alpha \leq 1.$$

subject to

$$y(0) = y_0,$$

Syam and Al-Refai [26] expanded the fractional shifted Legendre functions to solve the generalized time-fractional diffusion equation of the form

$$D_t^\alpha u(x,t) = a(x,t)D_x^2 u(x,t) + f(x,t), \quad x \in (-1,1), t \in (0,T),$$

subject to the initial and boundary conditions

$$u(-1,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u(x,0) = g(x).$$

where all functions are continuous on the required intervals, $T > 0$ and $0 < \alpha \leq 1$.

Bataineh et al. [27] derived Legendre operational matrix for obtaining the exact/approximate

solutions of the singular two-point boundary value problems of the type

$$\frac{1}{p(x)}u''(x) + \frac{1}{q(x)}u'(x) + \frac{1}{r(x)}(u(x))^n = g(x), \quad 0 < x \leq 1,$$

subject to boundary conditions

$$u(0) = \alpha_1, \quad u(1) = \beta, \quad \text{or} \quad u'(0) = \alpha_2, \quad u(1) = \beta$$

Bhrawy et al. [28] presented a review of the researchers who had constructed and used Legendre, Chebyshev, Jacobi and Bernstein operational matrices for obtaining the solution of fractional differential equations. In addition, they implemented a numerical technique for solving fractional differential equations on finite and semi infinite intervals by using various spectral methods depending on Laguerre polynomials.

In the next section we present some definitions and preliminary results about the fractional order Legendre functions that will be used in the entire study.

1.1 Properties of fractional order Legendre polynomials

Definition 1.1.1. The Legendre functions "polynomials" $\{P_n(x) : n = 0, 1, 2, \dots\}$ are the eigenfunctions of the singular Sturm-Liouville problem

$$((1-x^2)P_n'(x))' + n(n+1)P_n(x) = 0, \quad x \in [-1, 1].$$

They are given by

$$P_n(x) = 2^n \sum_{k=0}^n x^k \binom{n}{k} \binom{\frac{n+k-1}{2}}{n}.$$

Notice that the Legendre functions , P_n , are orthogonal with Legendre weight

function $w(x) = 1$ on the interval $(-1, 1)$; i.e.

$$\int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm},$$

where δ_{nm} is the Kronecker delta defined as

$$\delta_{nm} = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m. \end{cases}$$

In order to use Legendre polynomials on the interval $[0, 1]$, the so-called shifted Legendre polynomials can be defined by setting $x = 2t - 1$. By setting $P_n(2t - 1) = L_n(t)$, one may obtain the following orthogonality property:

$$\int_0^1 L_n(x)L_m(x)dx = \frac{1}{2n+1}\delta_{nm}.$$

The analytical closed form of the shifted Legendre polynomials of degree n is given by

$$L_n(t) = \sum_{k=0}^n (-1)^{n+k} \frac{(n+k)!}{(n-k)!(k!)^2} x^k, \quad x \in (0, 1). \quad (1.5)$$

In 2011, Rida and Yousef [21] generated a special type of fractional Legendre functions by replacing the integer order derivative in Rodrigues' formula for the Legendre function by fractional order derivatives; i.e.

$$P_n(x) = \frac{1}{2^n n!} \frac{d^\beta}{dx^\beta} [(x^2 - 1)^n].$$

In fact, the resultant functions were very difficult to be used for solving fractional differential equations. However, Kazem et al. [20] were able to generate an orthogonal set of fractional Legendre functions based on shifted Legendre polynomials (1.5) by setting

$x = t^\beta$ for $\beta > 0$; i.e.

$$F_n^\beta(t) = \sum_{k=0}^n (-1)^{n+k} \frac{(n+k)!}{(n-k)!(k!)^2} t^{k\beta}. \quad (1.6)$$

It can be easily verified that the functions (1.6) are particular solutions of the following singular Sturm-Liouville problem

$$\left((x - x^{1+\beta})(F_n^\beta)'(x) \right)' + \beta^2 n(n+1)x^{\beta-1} F_n^\beta(x) = 0, \quad x \in (0, 1).$$

Moreover, it can be easily seen that $F_n^\beta(0) = (-1)^n$ and $F_n^\beta(1) = 1$. One of the most interesting property of the fractional Legendre functions (FLFs) is the orthogonality with respect to the weight function $w(x) = x^{\beta-1}$ in the interval $(0, 1)$; i.e.

$$\int_0^1 F_n^\beta(x) F_m^\beta(x) w(x) dx = \frac{1}{2n+1} \delta_{nm}.$$

The graphs of the first five FLFs at $\beta = 1/2$ are displayed in Figure (1.1).

Using properties (4) and (5) of the Caputo fractional derivative; one can easily verify that

$$D^\beta F_n^\beta(x) = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!(k!)^2} \times \frac{\Gamma(k\beta+1)}{\Gamma((k-1)\beta+1)} x^{(k-1)\beta}.$$

For more details see [20].

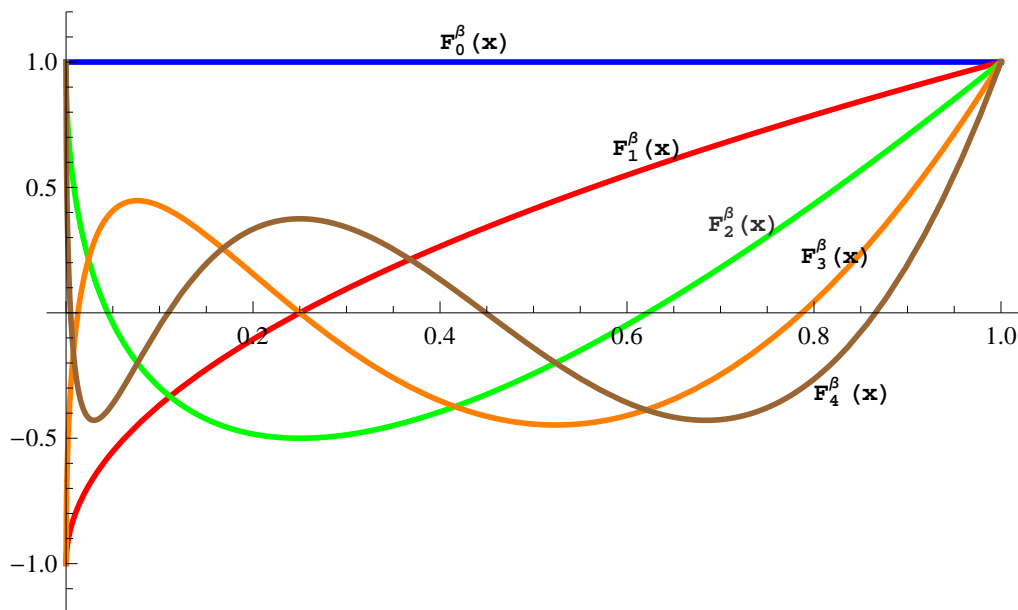


Figure 1.1: Graphs of the first five fractional Legendre functions at $\beta = 1/2$

1.1.1 Main Theorems

In this section, we present some results which are vital to the present work.

Theorem 1.1.1. *Let $u \in C[0, 1]$ and $u'(x)$ be a piecewise continuous function on $[0, 1]$.*

Then, $u(x)$ can be expressed as infinite series; i.e.

$$u(x) = \sum_{k=0}^{\infty} u_k F_k^\beta(x), \quad (1.7)$$

where

$$u_k = (2k + 1)\beta \int_0^1 u(x) F_k^\beta(x) w(x) dx, \quad (1.8)$$

and $w(x) = x^{\beta-1}$, represents the weight function.

Proof. See Kazem et al. [20] and Syam et al. [18]. □

Note that any function can be represented by the series (1.7). To illustrate the accuracy of theorem 1.1.1, we discuss the approximation of the function $u(x) = \sin(\pi x)$

for $x \in [0, 1]$. According to Theorem 1.1.1, one can approximate $u(x)$ in terms of finite sum of fractional Legendre functions, i.e.

$$u(x) \approx U_N(x) = \sum_{k=0}^N u_k F_k^\beta(x),$$

where u_k is defined in 1.8. The graphs of $\sin(\pi x)$ and its approximations, U_N for $N = 0, 2, 4, 6$ are displayed in Figure 1.2. Note the improvement in the approximation with increasing N .

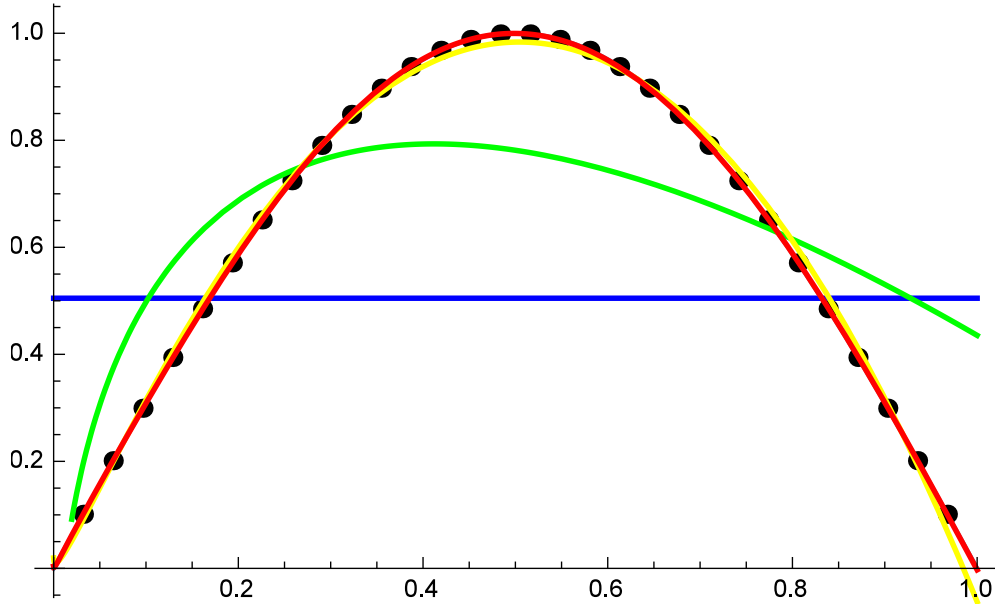


Figure 1.2: Graphs of $\sin(\pi x)$ (\bullet), $U_0(x)$ (blue), $U_2(x)$ (Green), $U_4(x)$ (Yellow), $U_6(x)$ (Red)

The next theorem gives the relation between the coefficients of the series solution of $D^\beta u(x)$ and the coefficients of the series expansion of $u(x)$.

Theorem 1.1.2. *Let $u \in C[0, 1]$ and $u''(x)$ be a piecewise continuous function on $[0, 1]$.*

Then, $\sum_{k=0}^{\infty} u_k^{(\beta)} F_k^\beta(x)$ converges uniformly on $[0, 1]$ to $D^\beta u(x)$, $0 < \beta < 1$, where

$$u_k^{(\beta)} = \sum_{j=k+1}^{\infty} a_{jk} u_j,$$

$$a_{jk} = (2k+1)\beta \int_0^1 D^\beta F_j^\beta(x) F_k^\beta(x) w(x) dx$$

for $k = 0, 1, 2, \dots, j = k + 1, k + 2, \dots$

Proof. See Syam et al. [18]. □

Since the previous theorem doesn't apply on the coefficients of the series solution of $u'(x)$, and since the target of our thesis is to use a finite term of the series (1.7), we may implement the following lemma.

Lemma 1.1.3. *Let $u \in C^2[0, 1]$ and $u'''(x)$ be a piecewise continuous function on $[0, 1]$.*

Then,

$$u_k^{(1)} = \sum_{j=k+1}^N c_{jk} u_j,$$

and

$$c_{j,k} = \left[a_{jk} \right]^{-1} \left[b_{jk} \right].$$

where $[a_{jk}]$ and $[b_{jk}]$ are square matrices defined, respectively, as

$$a_{jk} = \int_0^1 F_k^\beta(x) F_j^\beta(x) dx,$$

and

$$b_{jk} = \int_0^1 (F_k^\beta(x))' F_j^\beta(x) dx.$$

Proof. For the value of c_{jk} follows from the following definition

$$u'(x) = \sum_{k=0}^N u_k^{(1)} F_k^\beta(x) = \sum_{k=0}^N u_k (F_k^\beta(x))',$$

Multiply both sides by $F_j^\beta(x)$ and integral from 0 to 1, one obtains

$$\sum_{k=0}^N u_k^{(1)} \int_0^1 F_k^\beta(x) F_j^\beta(x) dx = \sum_{k=0}^N u_k \int_0^1 (F_k^\beta(x))' F_j^\beta(x) dx, \quad j = 0 : N \quad (1.9)$$

Thus, we can rewrite (1.9) by the matrix forms as:

$$\begin{bmatrix} a_{jk} \end{bmatrix}_{(N+1)(N+1)} U^{(1)} = \begin{bmatrix} b_{jk} \end{bmatrix}_{(N+1)(N+1)} U$$

where, $U^{(1)} = [u_0^{(1)}, u_1^{(1)}, \dots, u_N^{(1)}]^T$ and $U = [u_0, u_1, \dots, u_N]^T$

Hence $u_k^{(1)}$ is given by

$$u_k^{(1)} = \sum_{j=k+1}^{\infty} c_{jk} u_j,$$

and

$$c_{j,k} = \begin{bmatrix} a_{jk} \end{bmatrix}^{-1} \begin{bmatrix} b_{jk} \end{bmatrix}.$$

Since the original ordinary boundary problem has a unique solution, a_{jk} is a nonsingular.

□

The rest of the thesis is organized as follows: In chapter 2, we discuss the first and second order linear cases; i.e.

$$a(x)D^\alpha u(x) + b(x)u(x) = g(x), \quad u(0) = u_0,$$

and

$$a(x)D^\beta u(x) + b(x)u'(x) + c(x)u(x) = g(x), \quad u(0) = u_0, \quad u'(0) = u_1,$$

where $0 < \alpha \leq 1$ and $1 < \beta \leq 2$. In Chapter 3 we extend the study of Chapter two to the nonlinear first and second order cases; i.e.

$$D^\alpha u(x) = f(x, u), \quad u(0) = u_0, \tag{1.10}$$

and

$$D^\beta u(x) = f(x, u, u'), \quad u(0) = u_0, \quad u'(0) = u_1, \quad (1.11)$$

where $0 < \alpha \leq 1$ and $1 < \beta \leq 2$.

A summary and concluding remarks are given in Chapter 4.

It is an established fact that finding the exact solutions of the initial value problems of fractional order given by (1.10) and (1.11) remains far from trivial. Therefore, during the past years, several numerical algorithms have been developed to approximate such exact solutions. These algorithms include the Adomian's decomposition method [41], the homotopy perturbation method [42], [43], [44], the variational iteration method [45], [46], the fractional differential transform method [47], operational matrices techniques based on various orthogonal polynomials and wavelets [40], [39], [38], a nonstandard finite difference method (FDM) [37], [36], [35], [34], a predictor-corrector approach [33], spectral methods using fractional Laguerre orthogonal functions [32], collocation method [31], [30] and the method of lower and upper solutions [29].

Chapter 2: Numerical Technique for Solving Linear Fractional Initial Value Problems

In this chapter, we discuss the numerical solution of first and second orders linear fractional initial value problems using spectral method (fractional order Legendre Collocation method).

2.1 First-Order Linear Fractional Initial Value Problems

In this section, we focus on the numerical solution of the following first-order linear fractional initial value problem:

$$a(x)D^\alpha y(x) + b(x)y(x) = g(x), \quad 0 < \alpha \leq 1, x \in [0, 1], \quad (2.1)$$

subject to

$$y(0) = h_1, \quad (2.2)$$

where h_1 is a constant and $a(x), b(x)$ and $g(x)$ are continuous functions. To be able to apply the fractional-order Legendre -Collocation method to discretize problem (2.1) and (2.2), we approximate the solution $y(x)$ in terms of the fractional order Legendre functions as follows:

$$y(x) \approx Y(x) = \sum_{k=0}^{N+1} y_k F_k^\alpha(x), \quad (2.3)$$

where y_k is the undetermined Legendre coefficients. Particular, the α th derivative of (2.3) is given by

$$D^\alpha y(x) \approx D^\alpha Y(x) = \sum_{k=0}^N y_k^{(\alpha)} F_k^\alpha(x), \quad (2.4)$$

where the relation between $y_k^{(\alpha)}$ and y_k are given in Theorem 1.1.2 .

Inserting the series (2.3) in the main equation (2.1), we obtain the following residual

$$R(x) = a(x)D^\alpha Y(x) + b(x)Y(x) - g(x). \quad (2.5)$$

Orthogonalize the residual with respect to the Dirac delta function as follow:

$$\langle R(Y(x)), \delta(x - x_j) \rangle = \int_0^1 R(Y(x)) \delta(x - x_j) dx = 0, \quad \text{for } j = 0 : N + 1,$$

where x_j are the collocation points on the interval $[0, 1]$. Within our study, we choose the collocation points x_j to be the nodes $x_j = jh$ for $j = 0, 1, 2, \dots, N + 1$, where $h = \frac{1}{N+1}$. The unknown coefficients y_j are determined by making the residual $R(x)$ vanishes at the collocation points x_j for $j = 1, 2, \dots, N + 1$. Therefore, we obtain the following linear system which leads to the following elementwise equation

$$a(x_j)D^\alpha Y(x_j) + b(x_j)Y(x_j) - g(x_j) := 0, \quad j = 1 : N + 1. \quad (2.6)$$

Inserting the series representations (2.3) and (2.4) into (2.6), one obtains

$$a(x_j) \sum_{k=0}^N y_k^{(\alpha)} F_k^\alpha(x_j) + b(x_j) \sum_{k=0}^{N+1} y_k F_k^\alpha(x_j) - g(x_j) := 0, \quad (2.7)$$

for $j = 1 : N + 1$.

Let

$$\mathbf{V}^{(\alpha)} = \begin{bmatrix} y_0^{(\alpha)} \\ y_1^{(\alpha)} \\ \cdot \\ \cdot \\ y_N^{(\alpha)} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} y_0 \\ y_1 \\ \cdot \\ \cdot \\ y_{N+1} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \cdot \\ \cdot \\ g(x_{N+1}) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a(x_1) & 0 & \cdot & \cdot & 0 \\ 0 & a(x_2) & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & a(x_{N+1}) \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} b(x_1) & 0 & \cdot & \cdot & 0 \\ 0 & b(x_2) & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & b(x_{N+1}) \end{bmatrix}, \quad \mathbf{F}^* = \begin{bmatrix} F_0^\alpha(x_1) & F_1^\alpha(x_1) & \cdot & \cdot & F_N^\alpha(x_1) \\ F_0^\alpha(x_2) & F_1^\alpha(x_2) & \cdot & \cdot & F_N^\alpha(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_0^\alpha(x_{N+1}) & F_1^\alpha(x_{N+1}) & \cdot & \cdot & F_N^\alpha(x_{N+1}) \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} F_0^\alpha(x_1) & F_1^\alpha(x_1) & \cdot & \cdot & F_{N+1}^\alpha(x_1) \\ F_0^\alpha(x_2) & F_1^\alpha(x_2) & \cdot & \cdot & F_{N+1}^\alpha(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_0^\alpha(x_{N+1}) & F_1^\alpha(x_{N+1}) & \cdot & \cdot & F_{N+1}^\alpha(x_{N+1}) \end{bmatrix},$$

where \mathbf{F}^* is $(N+1) \times (N+1)$, \mathbf{F} is $(N+1) \times (N+2)$, \mathbf{V} is $(N+2) \times 1$, \mathbf{A} is $(N+1) \times (N+1)$, \mathbf{B} is $(N+1) \times (N+1)$, $\mathbf{V}^{(\alpha)}$ is $(N+1) \times 1$, and \mathbf{G} is $(N+1) \times 1$. Notice that we can rewrite (2.7) in the matrix form as:

$$\mathbf{A} \mathbf{F}^* \mathbf{V}^{(\alpha)} + \mathbf{B} \mathbf{F} \mathbf{V} - \mathbf{G} = 0. \quad (2.8)$$

It can be easily seen, from Theorem 1.1.2 and its results that

$$\mathbf{V}^{(\alpha)} = \mathbf{P} \mathbf{V},$$

where \mathbf{P} is $(N+1) \times (N+2)$ matrix. Consequently, the system (2.8) can be rewritten in the form

$$\mathbf{A} \mathbf{F}^* \mathbf{P} \mathbf{V} + \mathbf{B} \mathbf{F} \mathbf{V} - \mathbf{G} = 0. \quad (2.9)$$

It can be deduced from the initial conditions (2.2) that

$$Y(0) = h_1 = \sum_{k=0}^{N+1} (-1)^k y_k,$$

or in the following matrix representations form

$$\Lambda \mathbf{V} = h_1, \tag{2.10}$$

where $\Lambda = [1, -1, \dots, (-1)^{N+1}]$.

By combining the systems (2.9) and (2.10), we obtain the following $(N+2) \times (N+2)$ system

$$\Omega \mathbf{V} = \mathbf{G}_R \tag{2.11}$$

where

$$\Omega = \begin{bmatrix} \mathbf{A}\mathbf{F}^*\mathbf{P}^* + \mathbf{B}\mathbf{F} \\ \Lambda \end{bmatrix}, \text{ and } \mathbf{G}_R = \begin{bmatrix} \mathbf{G} \\ h_1 \end{bmatrix},$$

which can be solved easily using Gauss elimination method.

2.1.1 Numerical Results

In this section, the proposed numerical method is implemented to solve two examples in order to prove its efficiency and accuracy.

Example 2.1.1. Consider the following linear problems

$$D^{\frac{1}{2}}y(x) + xy(x) = g(x), \quad x \in [0, 1] \tag{2.12}$$

subject to

$$y(0) = 0$$

Notice that for $g(x) = \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} + x^3$, the exact solution is given by $y(x) = x^2$.

It can easily be seen that $\alpha = 1/2$, $a(x) = 1$ and $b(x) = x$. Note that the maximum number of terms in the Legendre series (2.3) is taken as $N = 6$, whereas, the step size in all examples is chosen to be $h = \frac{1}{N+1}$. Therefore, the collocation points are $x_j = jh$ for $j = 0 : N + 1$. Applying the above algorithm with $N = 6$ gives a (8×8) linear system, see (2.11). The graphs of the approximate solution, y , together with the exact solution $Y(x)$ are displayed in Figure 2.1.

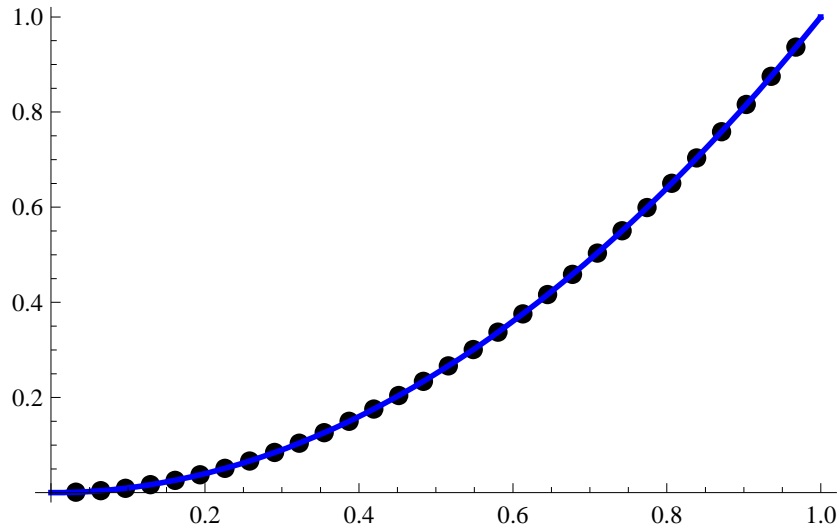


Figure 2.1: Graphs of the approximate solution, y (\bullet), and the exact solution, Y (solid), for Example 2.1.1

The error between the exact solution and computed one is shown in Figure 2.2. It is clearly seen that the two solutions are in excellent agreement. The computed L_2 error norm is given by

$$\|y(x) - Y(x)\| = \int_0^1 (y(x) - Y(x))^2 dx = 1.05807 \cdot 10^{-28},$$

which proves the high efficiency of our method.

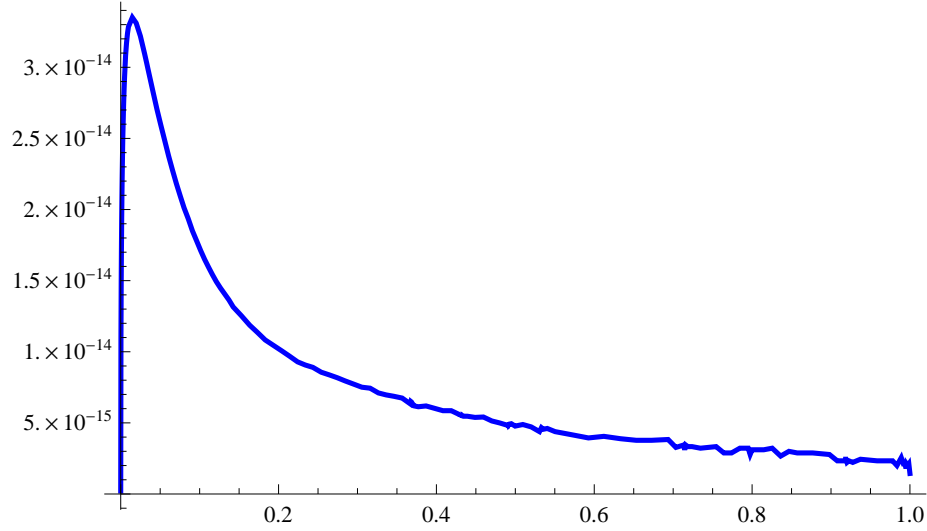


Figure 2.2: Computed absolute error between the exact and computed solutions for Example 2.1.1

Example 2.1.2. Consider the following linear problems

$$e^{2x}D^{\frac{1}{2}}y(x) + \frac{x}{4}y(x) = g(x), \quad x \in [0, 1], \quad (2.13)$$

subject to

$$y(0) = 0,$$

where $g(x) = \frac{2e^{2x}\sqrt{x}(9+4x)}{3\sqrt{\pi}} + \frac{1}{4}x(3x+x^2)$ and the exact solution is given by $y(x) = x^2 + 3x$.

Notice that $\alpha = 1/2$, $a(x) = e^{2x}$ and $b(x) = x/4$. Note that the maximum number of terms in the Legendre series (2.3) is taken as $N = 6$, whereas, the step size in all examples is chosen to be $h = \frac{1}{N+1}$. Therefore, the collocation points are $x_j = jh$ for $j = 0 : N + 1$. Following the methodology discussed in the previous example, we obtain the following approximate solution, Y :

$$\begin{aligned} Y(x) &= 1.22682 \times 10^{-16} + 4.16022 \times 10^{-12}\sqrt{x} + 3x + 9.68819 \times 10^{-11}x^{3/2} \\ &+ x^2 + 1.63156 \times 10^{-10}x^{5/2} - 8.43946 \times 10^{-11}x^3 + 1.80552 \times 10^{-11}x^{7/2}. \end{aligned}$$

which seems to have very close form to the exact solution $y(x)$. Table 2.1 presents the absolute error between the approximate solution, $Y(x)$ and the exact solution $y(x)$ over the interval $(0, 1)$ at the points $x_j = 0.1j$ for $j = 1 : 9$.

x_j	$ Y(x) - y(x) $
0.1	1.07116×10^{-13}
0.2	6.52117×10^{-14}
0.3	5.09089×10^{-14}
0.4	4.39579×10^{-14}
0.5	3.82125×10^{-14}
0.6	3.39243×10^{-14}
0.7	3.21097×10^{-14}
0.8	2.97321×10^{-14}
0.9	2.80470×10^{-14}

Table 2.1: Computed absolute error between the exact and approximate solutions for Example 2.1.2

These results presented in Table 2.1 ensures that the present technique is working very efficiently.

2.2 Second-Linear Fractional Initial Value Problems

In this section, we develop a numerical method for solving the following problem by using Legendre-Collocation method.

$$a(x)D^\beta y(x) + b(x)y'(x) + c(x)y(x) = g(x), \quad 1 < \beta \leq 2, \quad x \in [0, 1], \quad (2.14)$$

subject to

$$y(0) = h_1, \quad y'(0) = h_2, \quad (2.15)$$

where h_1, h_2 are constants and $a(x), b(x), c(x)$ and $g(x)$ are continuous functions and $a(x) > 0$ for all $x \in (0, 1)$.

In the following, we transform problems (2.14),(2.15) to a system of differential equations, consisting of a fractional and integer derivatives. Let $y_1 = y, y_2 = y'$ and $\alpha = \beta - 1$.

Using the fact that $D^\beta y(x) = D^\alpha y'(x)$, the system (2.14)-(2.15) is converted to

$$y_1' = y_2 \quad (2.16)$$

$$a(x)D^\alpha y_2(x) + b(x)y_2(x) + c(x)y_1(x) = g(x) \quad (2.17)$$

subject to

$$y_1(0) = h_1, \quad y_2(0) = h_2. \quad (2.18)$$

Approximate the solutions $y_1(x)$, $y_1'(x)$ and $y_2(x)$ in terms of the fractional order Legendre functions as follows:

$$\begin{aligned} y_1(x) &\approx u(x) = \sum_{k=0}^{N+1} u_k F_k^\alpha(x), \\ y_2(x) &\approx v(x) = \sum_{k=0}^{N+1} v_k F_k^\alpha(x) \\ y_1'(x) &\approx u'(x) = \sum_{k=0}^N u_k^{(1)} F_k^\alpha(x) \end{aligned} \quad (2.19)$$

where u_k , v_k and $u_k^{(1)}$ are the undetermined Legendre coefficients. The residuals for equations (2.16) and (2.17) are, respectively, given by

$$R_1(x) = u'(x) - v(x). \quad (2.20)$$

$$R_2(x) = a(x)D^\alpha v(x) + b(x)v(x) + c(x)u(x) - g(x). \quad (2.21)$$

Herein we choose the collocation points x_j to be the nodes $x_j = jh$ for $j = 0, 1, \dots, N+1$, where $h = \frac{1}{N+1}$. The unknown coefficients u_j and v_j are determined by making the residuals $R_1(x)$ and $R_2(x)$ vanish at the collocation points x_j for $j = 1, \dots, N+1$; i.e.

$$R_1(x_j) = u'(x_j) - v(x_j) := 0. \quad (2.22)$$

$$R_2(x_j) = a(x_j)D^\alpha v(x_j) + b(x_j)v(x_j) + c(x_j)u(x_j) - g(x_j) := 0. \quad (2.23)$$

Then, the elementwise equations by (2.19) will be

$$R_1(x_j) = \sum_{k=0}^N u_k^{(1)} F_k^\alpha(x_j) - \sum_{k=0}^{N+1} v_k F_k^\alpha(x_j) := 0. \quad (2.24)$$

$$R_2(x_j) = a(x_j) \sum_{k=0}^N v_k^{(\alpha)} F_k^\alpha(x_j) + b(x_j) \sum_{k=0}^{N+1} v_k F_k^\alpha(x_j) + c(x_j) \sum_{k=0}^{N+1} u_k F_k^\alpha(x_j) - g(x_j) := 0. \quad (2.25)$$

The expressions (2.24) and (2.25) can be rewritten, respectively, in the following matrices forms

$$\mathbf{F}^* \mathbf{U}^{(1)} - \mathbf{FV} = 0 \quad (2.26)$$

$$\mathbf{A}\mathbf{F}^*\mathbf{V}^{(\alpha)} + \mathbf{B}\mathbf{FV} + \mathbf{C}\mathbf{F}\mathbf{U} - \mathbf{G} = 0 \quad (2.27)$$

where \mathbf{F}^* is $(N+1) \times (N+1)$, $\mathbf{U}^{(1)}$ is $(N+1) \times 1$, \mathbf{F} is $(N+1) \times (N+2)$, \mathbf{V} is $(N+2) \times 1$, \mathbf{A} is $(N+1) \times (N+1)$, \mathbf{B} is $(N+1) \times (N+1)$, \mathbf{C} is $(N+1) \times (N+1)$, $\mathbf{V}^{(\alpha)}$ is $(N+2) \times 1$, \mathbf{U} is $(N+2) \times 1$ and \mathbf{G} is $(N+1) \times 1$. These matrices are given by

$$\mathbf{F}^* = \begin{bmatrix} F_0^\alpha(x_1) & F_1^\alpha(x_1) & \cdot & \cdot & F_N^\alpha(x_1) \\ F_0^\alpha(x_2) & F_1^\alpha(x_2) & \cdot & \cdot & F_N^\alpha(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_0^\alpha(x_{N+1}) & F_1^\alpha(x_{N+1}) & \cdot & \cdot & F_N^\alpha(x_{N+1}) \end{bmatrix}, \quad \mathbf{U}^{(1)} = \begin{bmatrix} u_0^{(1)} \\ u_1^{(1)} \\ \cdot \\ \cdot \\ u_N^{(1)} \end{bmatrix},$$

$$\begin{aligned}
\mathbf{F} &= \begin{bmatrix} F_0^\alpha(x_1) & F_1^\alpha(x_1) & \cdot & \cdot & F_{N+1}^\alpha(x_1) \\ F_0^\alpha(x_2) & F_1^\alpha(x_2) & \cdot & \cdot & F_{N+1}^\alpha(x_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ F_0^\alpha(x_{N+1}) & F_1^\alpha(x_{N+1}) & \cdot & \cdot & F_{N+1}^\alpha(x_{N+1}) \end{bmatrix}, \mathbf{V} = \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{N+1} \end{bmatrix}, \\
\mathbf{A} &= \begin{bmatrix} a(x_1) & 0 & \cdot & \cdot & 0 \\ 0 & a(x_2) & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & a(x_{N+1}) \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b(x_1) & 0 & \cdot & \cdot & 0 \\ 0 & b(x_2) & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & b(x_{N+1}) \end{bmatrix}, \\
\mathbf{C} &= \begin{bmatrix} c(x_1) & 0 & \cdot & \cdot & 0 \\ 0 & c(x_2) & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & c(x_{N+1}) \end{bmatrix}, \mathbf{V}^{(\alpha)} = \begin{bmatrix} v_0^{(\alpha)} \\ v_1^{(\alpha)} \\ \cdot \\ \cdot \\ v_{N+1}^{(\alpha)} \end{bmatrix}, \mathbf{U} = \begin{bmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ u_{N+1} \end{bmatrix}, \\
\mathbf{G} &= \begin{bmatrix} g(x_1) \\ g(x_2) \\ \cdot \\ \cdot \\ g(x_{N+1}) \end{bmatrix}.
\end{aligned}$$

It was shown by Lemma (1.1.3) that the fractional Legendre coefficient matrices, $\mathbf{U}^{(1)}$ and \mathbf{U} , are related via the following relation

$$\mathbf{U}^{(1)} = \mathbf{P}^{(1)}\mathbf{U}.$$

In addition, Theorem (1.1.2) relates that the coefficients, $\mathbf{V}^{(\alpha)}$ and \mathbf{V} , via the following relation

$$\mathbf{V}^{(\alpha)} = \mathbf{P}^{(\alpha)}\mathbf{V}.$$

Consequently, the systems (2.26) and (2.27) can be written, respectively, in the following forms

$$\mathbf{F}^* \mathbf{P}^{(1)} \mathbf{U} - \mathbf{FV} = 0 \quad (2.28)$$

$$\mathbf{AF}^* \mathbf{P}^{(\alpha)} \mathbf{V} + \mathbf{BFV} + \mathbf{CFU} - \mathbf{G} = 0. \quad (2.29)$$

It can be deduced from the initial conditions (2.18) that

$$u(0) = h_1 = \sum_{k=0}^{N+1} (-1)^k u_k,$$

$$v(0) = h_2 = \sum_{k=0}^{N+1} (-1)^k v_k,$$

or in the following matrix representations form

$$\Lambda \mathbf{U} = h_1, \quad \text{and} \quad \Lambda \mathbf{V} = h_2, \quad (2.30)$$

where $\Lambda = [1, -1, \dots, (-1)^{N+1}]$.

By combining the systems (2.28), (2.29) and (2.30), we obtain the following $(2N + 4) \times (2N + 4)$ linear system

$$\begin{bmatrix} \mathbf{CF} & \mathbf{AF}^* \mathbf{P}^{(\alpha)} + \mathbf{BF} \\ \mathbf{F}^* \mathbf{P}^{(1)} & -\mathbf{F} \\ \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ 0 \\ h_1 \\ h_2 \end{bmatrix},$$

which can be solved by using the Gauss elimination technique.

2.2.1 Numerical Results

In this section, the proposed numerical method is used to solve two examples in order to prove its efficiency and accuracy.

Example 2.2.1. Consider the following fractional initial value *singular* linear problem

$$2xD^{\frac{3}{2}}y(x) + e^x y'(x) + 3x^2 y(x) = g(x), \quad x \in [0, 1] \quad (2.31)$$

subject to

$$y(0) = 0, \quad y'(0) = 3.$$

where $g(x) = \frac{8x^{\frac{3}{2}}}{\sqrt{\pi}} + 3x^3(3+x) + e^x(3+2x)$. Notice that the exact solution is given by $y(x) = x^2 + 3x$.

Applying the above technique with $a(x) = 2x$, $b(x) = e^x$ and $c(x) = 3x^2$, we transform the initial value problem (2.31) into the following system of first order initial-value problems

$$y_1' = y_2 \quad (2.32)$$

$$2xD^{\frac{1}{2}}y_2(x) + e^x y_2(x) + 3x^2 y_1(x) = g(x) \quad (2.33)$$

subject to

$$y_1(0) = 0, \quad y_2(0) = 3. \quad (2.34)$$

The series solutions for $y_1(x)$ and $y_2(x)$ in terms of the fractional order Legendre functions are given by

$$y_1(x) \approx u(x) = \sum_{k=0}^{N+1} u_k F_k^{1/2}(x), \quad y_2(x) \approx v(x) = \sum_{k=0}^{N+1} v_k F_k^{1/2}(x), \quad (2.35)$$

where the number of terms in the Legendre series (2.35) is fixed at $N = 6$. Hence, the collocation points are $x_j = jh$ for $j = 0 : 7$. The graphs of the approximate solution, u , together with the exact solution $y(x)$ are displayed in Figure 2.3.

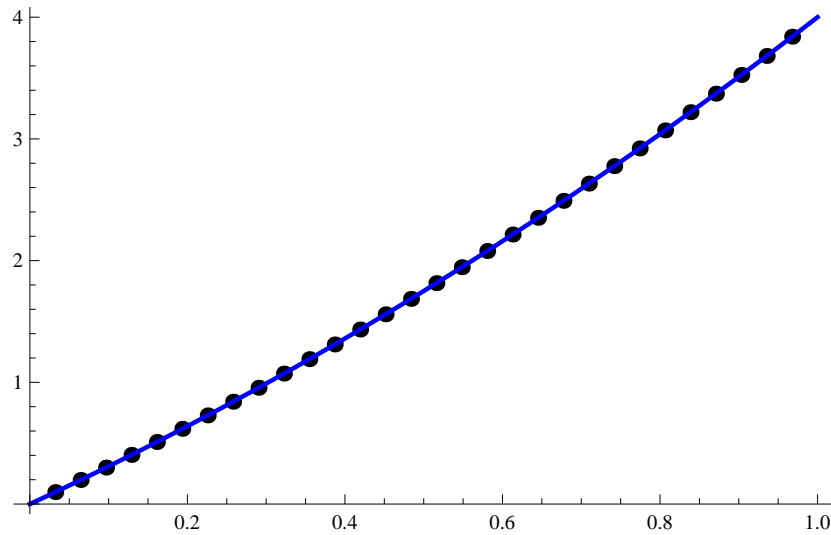


Figure 2.3: Graphs of the approximate solution, u (\bullet), and the exact solution, y (solid), for Example 2.2.1

The error between the exact solution and computed one is shown in Figure 2.4. It is clearly seen that the two solutions are in excellent agreement.

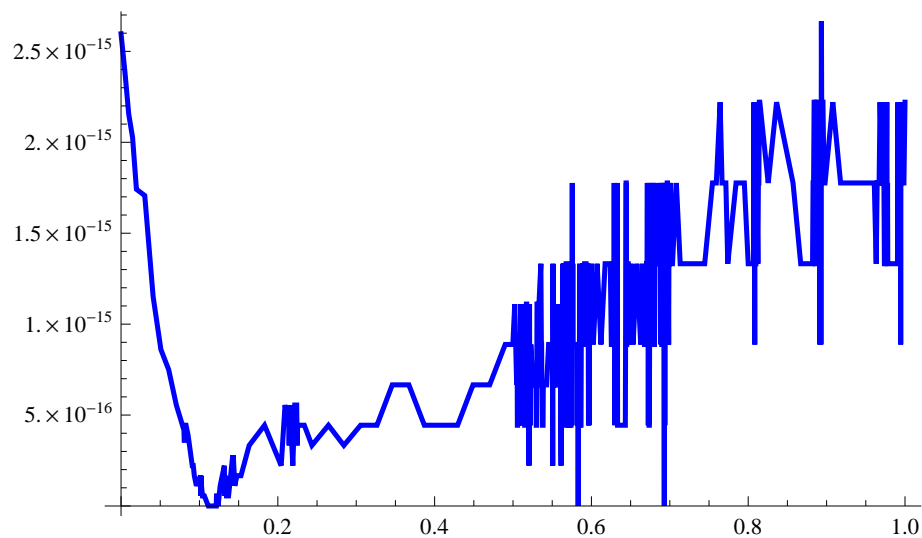


Figure 2.4: Computed absolute error between the exact and computed solutions for Example 2.2.1

Example 2.2.2. Consider the following fractional initial value linear problem

$$D^{1.3}y(x) + \cos(x)y'(x) + 2y(x) = g(x), \quad x \in [0, 1] \quad (2.36)$$

subject to

$$y(0) = 0, \quad y'(0) = 0.$$

where $g(x) = 2x^2 + 2x\cos(x) + \frac{\Gamma(3)}{\Gamma(1.7)}x^{0.7}$ and the exact solution is given by $y(x) = x^2$.

Applying the same methodology of solving the above example, we transform the initial value problem (2.36) into the following system of first order initial-value problems as follows

$$y_1' = y_2$$

$$D^{0.3}y_2(x) + \cos(x)y_2(x) + 2y_1(x) = g(x)$$

subject to

$$y_1(0) = 0, \quad y_2(0) = 0.$$

The series solutions for $y_1(x)$ and $y_2(x)$ in terms of the fractional order Legendre functions are given by

$$y_1(x) \approx u(x) = \sum_{k=0}^{N+1} u_k F_k^{0.3}(x), \quad y_2(x) \approx v(x) = \sum_{k=0}^{N+1} v_k F_k^{0.3}(x),$$

where N is chosen to be 10. It is worth mentioning that this example was discussed by Geng and Cui [48] by applying the Kernal method. Table (2.2) illustrates the absolute values of the error which obtained by the present technique and one which obtained by applying the Kernal method [48] for this example. It can be concluded that the present technique is accurate and efficient.

x_j	$ u(x) - y(x) $ (Present Method)	$ u(x) - y(x) $ ([48])
0.1	1.17279×10^{-6}	1.11×10^{-7}
0.2	9.73731×10^{-7}	2.87×10^{-7}
0.3	8.23202×10^{-7}	5.51×10^{-7}
0.4	6.97339×10^{-7}	9.23×10^{-7}
0.5	5.86355×10^{-7}	1.41×10^{-6}
0.6	4.87300×10^{-7}	2.03×10^{-6}
0.7	3.99126×10^{-7}	2.77×10^{-6}
0.8	3.20068×10^{-7}	3.64×10^{-6}
0.9	2.47957×10^{-7}	4.62×10^{-6}

Table 2.2: Comparison of the absolute errors for Example 2.2.2

Chapter 3: Numerical Technique for Solving Nonlinear Fractional Initial Value Problems

In this chapter, we discuss the numerical solution of first and second orders nonlinear fractional initial value problems using spectral method (fractional order Legendre Collocation method) on the company of Newton Methods.

3.1 First-Order Nonlinear Fractional Initial Value Problems

In this section, we focus on the numerical solution of the following first-order non linear fractional initial value problem:

$$D^\alpha y(x) = f(x, y), \quad 0 < \alpha \leq 1, \quad x \in [0, 1], \quad (3.1)$$

subject to

$$y(0) = h_1, \quad (3.2)$$

where h_1 is a constant. Following the same methodology presented in chapter two, we approximate the solution $y(x)$ in terms of the fractional order Legendre functions as follows

$$y(x) \approx Y(x) = \sum_{k=0}^{N+1} y_k F_k^\alpha(x), \quad (3.3)$$

where y_k is the undetermined Legendre coefficients. Particulary, the α th derivative of (3.3) is given by

$$D^\alpha y(x) \approx D^\alpha Y(x) = \sum_{k=0}^N y_k D^\alpha F_k^\alpha(x). \quad (3.4)$$

Therefore, we obtain the following residual function from the main equation (3.1)

$$R(x; y_k) = D^\alpha Y(x) - f(x, Y), \quad k = 0 : N + 1. \quad (3.5)$$

Orthogonalize the residual with respect to the Dirac delta function as follow

$$\langle R(x; y_0, \dots, y_{N+1}), \delta(x - x_j) \rangle = \int_0^1 R(x; y_0, \dots, y_{N+1}) \delta(x - x_j) dx = 0,$$

for $j = 1 : N + 1, k = 0 : N + 1$. Here x_j are the collocation points chosen to be the nodes $x_j = jh$ for $j = 0 : N + 1$ and $h = \frac{1}{N+1}$. The unknown coefficients y_k are determined by making the residual R vanishes at the collocation points x_j for $j = 1, 2, \dots, N + 1$. Therefore, we obtain the following linear system which leads to the following elementwise equation;

$$D^\alpha Y(x_j) - f(x_j, Y(x_j)) = 0, \quad \text{for } j = 1 : N + 1, \quad (3.6)$$

Inserting the series (3.3) into (3.6), we obtain the following elementwise equation;

$$\sum_{k=0}^N y_k D^\alpha F_k^\alpha(x_j) - f(x_j, \sum_{k=0}^{N+1} y_k F_k^\alpha(x_j)) := 0. \quad (3.7)$$

Notice that (3.7) gives $N + 1$ equations with $N + 2$ unknowns which requires one more equation which comes from the initial condition (3.2); i.e.

$$Y(0) = h_1 \implies \sum_{k=0}^{N+1} (-1)^k y_k - h_1 = 0. \quad (3.8)$$

combining (3.8) and (3.6) gives the following $N + 2$ nonlinear equations with $N + 2$ unknowns

$$\begin{aligned} F_0(y_0, y_1, \dots, y_{N+1}) &= \sum_{k=0}^{N+1} (-1)^k y_k - h_1 := 0, \\ F_j(y_0, y_1, \dots, y_{N+1}) &= \sum_{k=0}^N y_k D^\alpha F_k^\alpha(x_j) - f(x_j, \sum_{k=0}^{N+1} y_k F_k^\alpha(x_j)) := 0, \end{aligned} \quad (3.9)$$

for $j = 1 : N + 1$. Notice that, one may rewrite (3.9) in the following matrix form

$$\mathbf{F}(\mathbf{V}) = \begin{bmatrix} F_0(\mathbf{V}) \\ F_1(\mathbf{V}) \\ \cdot \\ \cdot \\ F_{N+1}(\mathbf{V}) \end{bmatrix},$$

where $\mathbf{V} = [y_0, y_1, \dots, y_{N+1}]^t$. Our main target now is to solve $\mathbf{F} = \mathbf{0}$ which can be solved by several techniques. In this thesis we used the multidimensional version of Newton's method which shows to be an efficient method as we will see later. To apply the multidimensional version of Newton's method, we differentiate equations (3.9) with respect to y_i for $i = 0 : N + 1$ to construct the Jacobian matrix, $\mathbf{J}(\mathbf{V})$, which is defined as

$$\mathbf{J}(\mathbf{V}) = \begin{bmatrix} \frac{\partial F_0}{\partial y_0}(\mathbf{V}) & \frac{\partial F_0}{\partial y_1}(\mathbf{V}) & \cdot & \cdot & \frac{\partial F_0}{\partial y_{N+1}}(\mathbf{V}) \\ \frac{\partial F_1}{\partial y_0}(\mathbf{V}) & \frac{\partial F_1}{\partial y_1}(\mathbf{V}) & \cdot & \cdot & \frac{\partial F_1}{\partial y_N}(\mathbf{V}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial F_{N+1}}{\partial y_0}(\mathbf{V}) & \frac{\partial F_{N+1}}{\partial y_1}(\mathbf{V}) & \cdot & \cdot & \frac{\partial F_{N+1}}{\partial y_N}(\mathbf{V}) \end{bmatrix},$$

where

$$\frac{\partial}{\partial y_i} F_j(y_0, y_1, \dots, y_N) = \frac{\partial}{\partial y_i} \left(\sum_{k=0}^N y_k D^\alpha F_k^\alpha(x_j) \right) - \frac{\partial}{\partial y_i} f(x_j, \sum_{k=0}^N y_k F_k^\alpha(x_j)) = 0, \quad (3.10)$$

for $i, j = 0 : N + 1$. Finally we arrived at the functional iteration procedure evolves from selecting \mathbf{V}^0 and generating, for $s \geq 1$,

$$\mathbf{V}^s = \mathbf{V}^{s-1} - \mathbf{J}(\mathbf{V}^{s-1})^{-1} \mathbf{F}(\mathbf{V}^{s-1}). \quad (3.11)$$

It is worth mentioning that this method generally expected to give quadratic convergence, provided that a sufficiently accurate starting value is known. To avoid computing $\mathbf{J}(\mathbf{V}^{s-1})^{-1}$ at each iteration which is time consuming, we first compute a vector \mathbf{H} that satisfies $\mathbf{J}(\mathbf{V}^{s-1})\mathbf{H} = -\mathbf{F}(\mathbf{V}^{s-1})$ using Gauss elimination method. Then the new approximation, \mathbf{V}^s , is calculated by $\mathbf{V}^s = \mathbf{V}^{s-1} + \mathbf{H}$. In addition, the iterations are repeat until a stopping criteria is satisfied. A stopping criteria could be $\|\mathbf{V}^s - \mathbf{V}^{s-1}\|_\infty < \varepsilon$ for some prescribed ε , where $\|\cdot\|_\infty$ is the infinity norm. It is important to mention here that the multidimensional Newton's method converges quadratically if

- (a) $\|\mathbf{J}_s^{-1}\| \leq M$ where $M > 0$; the norm of the inverse of the Jacobian at V^s is bounded.
- (b) $\|\mathbf{J}(\mathbf{z}_2) - \mathbf{J}(\mathbf{z}_1)\| \leq \|\mathbf{z}_2 - \mathbf{z}_1\|$; the Jacobian is Lipschitz continuous.

3.2 Numerical Results

In this section, the proposed numerical method is implemented to solve two examples in order to prove its efficiency and accuracy. In the proceeding examples, the stopping criteria for Newton method is chosen to be $\varepsilon = 10^{-10}$, and the number of iterations < 50 .

Example 3.2.1. Consider the following nonlinear problems

$$D^\alpha y(x) - xe^{y(x)} = g(x), \quad x \in [0, 1] \quad (3.12)$$

subject to

$$y(0) = 0$$

where $g(x) = -2xe^{x^2} + \frac{9x^{\frac{4}{3}}}{2\Gamma(\frac{1}{3})}$ and $\alpha = 2/3$. The exact solution for this problem is given by $y(x) = x^2$.

Note that the number of terms in the Legendre series (3.3) is taken as $N = 8$, whereas, the step size in all examples is chosen to be $h = \frac{1}{9}$. Therefore, the collocation points are $x_j = jh$ for $j = 0 : 9$. Applying the multidimensional version of Newton's method which is discussed above after 3 iterations and the initial guess values \mathbf{V}^0 equal

0, we obtain the following approximate solution, Y :

$$\begin{aligned} Y(x) &= -8.98177 \times 10^{-18} + 2.70972 \times 10^{-12}x^{\frac{2}{3}} - 2.88027 \times 10^{-11}x^{\frac{4}{3}} \\ &+ x^2 - 4.41651 \times 10^{-10}x^{\frac{8}{3}} - 8.31209 \times 10^{-10}x^{\frac{10}{3}} + 9.95315 \times 10^{-10}x^4 \\ &7.37241 \times 10^{-10}x^{\frac{14}{3}} - 3.08221 \times 10^{-10}x^{\frac{16}{3}} + 5.56399 \times 10^{-11}x^6 \end{aligned}$$

which seems to have very close form to the exact solution $y(x)$. Table 3.1 presents the absolute error between the approximate solution, $Y(x)$ and the exact solution $y(x)$ over the interval $(0, 1)$ at the points $x_j = 0.1j$ for $j = 1 : 9$.

x_j	$ Y(x) - y(x) $
0.1	6.83575×10^{-14}
0.2	5.05441×10^{-14}
0.3	5.05441×10^{-14}
0.4	3.95126×10^{-14}
0.5	3.66039×10^{-14}
0.6	3.44300×10^{-14}
0.7	3.26234×10^{-14}
0.8	3.12094×10^{-14}
0.9	3.00239×10^{-14}

Table 3.1: Computed absolute error between the exact and approximate solutions for Example 3.2.1

These results presented in table (3.1) ensures that the present technique is working very efficiently.

Example 3.2.2. Consider the following nonlinear problems

$$D^\alpha y(x) - 2y(x) + y^2(x) = g(x), \quad x \in [0, 1]$$

subject to

$$y(0) = 1,$$

where $g(x) = \frac{2\sqrt{x}(5+8x^2)}{5\sqrt{\pi}} - 2(x+x^3) + (x+x^3)^2$ and $\alpha = 1/2$. The exact solution is given by $y(x) = x^3 + x$.

Using the maximum number of terms in the Legendre series (3.3) as $N = 6$, and the step size $h = \frac{1}{7}$. Following the methodology discussed in the previous example, we obtain the following approximate solution, Y : The graphs of the approximate solution, Y , together with the exact solution $y(x)$ are displayed in Figure 3.1.

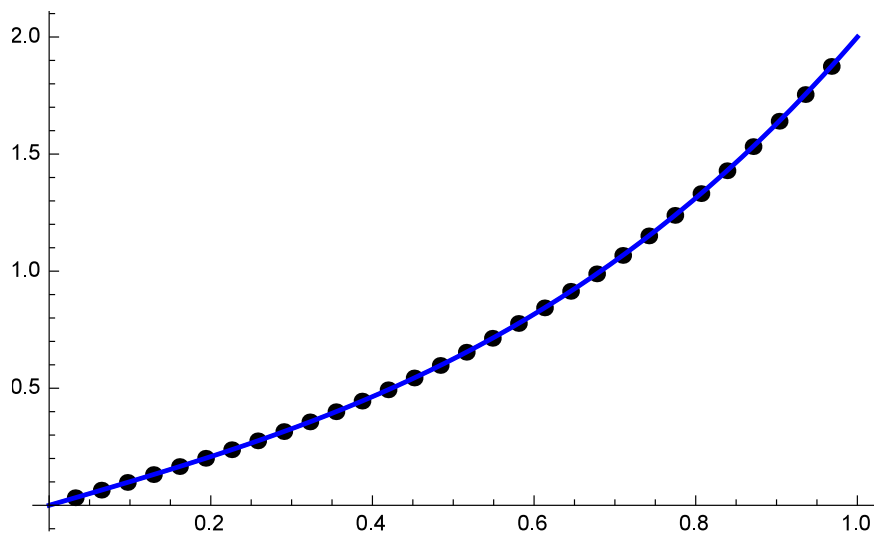


Figure 3.1: Graphs of the approximate solution, y (\bullet), and the exact solution, Y (solid), for Example 3.2.2

The error between the exact solution and the computed one is shown in Figure 3.2. It is clearly seen that the two solutions are in excellent agreement. The computed L_2 error norm is given by

$$\|y(x) - Y(x)\| = \int_0^1 (y(x) - Y(x))^2 dx = 1.10346 \times 10^{-23},$$

which proves the high efficiency of the present method.

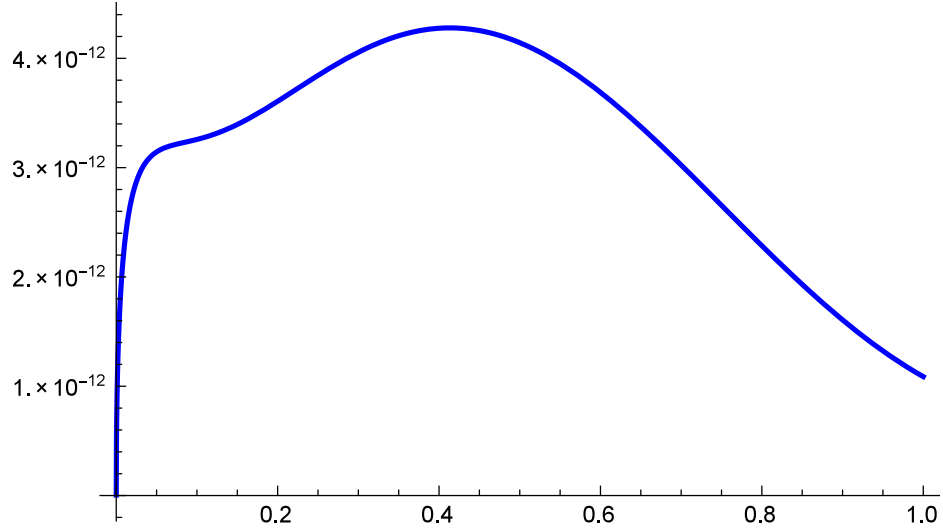


Figure 3.2: Computed absolute error between the exact and computed solutions for Example 3.2.2

3.3 Second-order Nonlinear Fractional Initial Value Problems

In this section, we discuss the numerical solution of the following second order nonlinear fractional initial value problem

$$D^{\beta}y(x) = f(x,y,y'), \quad 1 < \beta \leq 2, \quad x \in [0,1], \quad (3.13)$$

subject to

$$y(0) = h_1, \quad y'(0) = h_2 \quad (3.14)$$

where h_1, h_2 are constants. Following the same methodology used in Section 2.2, we transform (3.13)-(3.14) to a system of first order differential equations (consisting of a fractional and integer derivatives), given by

$$y_1' = y_2 \quad (3.15)$$

$$D^{\alpha}y_2(x) = f(x,y_1,y_2), \quad (3.16)$$

subject to

$$y_1(0) = h_1, \quad y_2(0) = h_2, \quad (3.17)$$

where $y_1 = y, y_2 = y'$ and $\alpha = \beta - 1$. Approximate the solutions $y_1(x)$ and $y_2(x)$ in terms of the fractional order Legendre functions; one obtains

$$y_1(x) \approx u(x) = \sum_{k=0}^{N+1} u_k F_k^\alpha(x), \quad (3.18)$$

$$y_2(x) \approx v(x) = \sum_{k=0}^{N+1} v_k F_k^\alpha(x).$$

where u_k, v_k are the undetermined Legendre coefficients. The associated residuals to equations (3.15) and (3.16) are, respectively, given by

$$R_1(x) = u'(x) - v(x), \quad (3.19)$$

$$R_2(x) = D^\alpha v(x) - f(x, u(x), v(x)). \quad (3.20)$$

The unknown coefficients u_k and v_k are determined by making the residuals $R_1(x)$ and $R_2(x)$ vanish at the collocation points x_j (for $j = 1 : N + 1$); i.e.

$$R_1(x_j) = u'(x_j) - v(x_j) := 0, \quad (3.21)$$

$$R_2(x_j) = a(x_j)D^\alpha v(x_j) - f(x_j, u(x_j), v(x_j)) := 0. \quad (3.22)$$

Inserting (3.18) into (3.21) and (3.22), we obtain the following elementwise equations

$$R_1(x_j) = \sum_{k=0}^N u_k (F_k^\alpha)'(x_j) - \sum_{k=0}^{N+1} v_k F_k^\alpha(x_j) := 0. \quad (3.23)$$

$$R_2(x_j) = \sum_{k=0}^{N+1} v_k D^\alpha F_k^\alpha(x_j) - f(x_j, \sum_{k=0}^{N+1} u_k F_k^\alpha(x_j), \sum_{k=0}^{N+1} v_k F_k^\alpha(x_j)) := 0. \quad (3.24)$$

for $j = 1 : N + 1$.

Notice that (3.23) and (3.24) give $2N + 2$ equations with $2N + 4$ unknowns which requires two more equations. Obviously, these two equations should be produced from the initial conditions (3.17) as follows

$$y_1(0) = h_1 \implies \sum_{k=0}^{N+1} (-1)^k u_k - h_1 := 0, \quad (3.25)$$

$$y_2(0) = h_2 \implies \sum_{k=0}^{N+1} (-1)^k v_k - h_2 := 0. \quad (3.26)$$

Combining (3.25), (3.26) together with (3.23), (3.24) gives the following $2N + 4$ nonlinear equations with $2N + 4$ unknowns

$$F_0(\mathbf{V}) = \sum_{k=0}^{N+1} (-1)^k u_k - h_1 := 0, \quad (3.27)$$

$$F_j(\mathbf{V}) = \sum_{k=0}^N v_k D^\alpha F_k^\alpha(x_j) - f(x_j, \sum_{k=0}^{N+1} u_k F_k^\alpha(x_j), \sum_{k=0}^{N+1} v_k F_k^\alpha(x_j)) := 0,$$

$$G_0(\mathbf{V}) = \sum_{k=0}^{N+1} (-1)^k v_k - h_2 := 0,$$

$$G_j(\mathbf{V}) = \sum_{k=0}^N u_k (F_k^\alpha)'(x_j) - \sum_{k=0}^{N+1} v_k F_k^\alpha(x_j) := 0.$$

for $j = 1 : N + 1$. Here the vector \mathbf{V} represents all the unknowns; i.e.

$$\mathbf{V} = [u_0, u_1, \dots, u_{N+1}, v_0, v_1, \dots, v_{N+1}]^t.$$

Notice that, one may rewrite (3.27) in the following matrix form

$$\mathbf{F}(\mathbf{V}) = \mathbf{0}_{2N+4 \times 1}, \quad (3.28)$$

where $\mathbf{F}(\mathbf{V}) = [F_0, \dots, F_{N+1}, G_0, \dots, G_{N+1}]^t$. Notice that we may use the multidimensional version of Newton's method (described in the previous section) to solve (3.28).

3.3.1 Numerical Results

In this section, the proposed numerical method is implemented to solve two examples in order to prove its efficiency and accuracy. In the proceeding examples, the stopping criteria for Newton method is chosen to be $\varepsilon = 10^{-10}$, and the number of iterations < 50 . The first example is presented for sake of comparison with the Legendre wavelet method [49].

Example 3.3.1. Consider the following nonlinear initial value problem

$$D^\beta y(x) + e^{-2\pi} y^2(x) = g(x), \quad x \in (0, 1), \quad (3.29)$$

subject to

$$y(0) = 0, \quad y'(0) = 0,$$

where $\beta = \frac{3}{2}$, $g(x) = \frac{105\sqrt{\pi}x^2}{32} + e^{-2\pi}x^7$ and the exact solution is given by $y(x) = x^{\frac{7}{2}}$.

Applying the above technique, we transform the initial value problem (3.29) into the following system of first order initial-value problems

$$y_1' = y_2 \quad (3.30)$$

$$D^{\frac{1}{2}} y_2(x) + e^{-2\pi} y_1^2(x) = g(x), \quad (3.31)$$

subject to

$$y_1(0) = 0, \quad y_2(0) = 0. \quad (3.32)$$

The series solutions for $y_1(x)$ and $y_2(x)$ in terms of the fractional order Legendre functions

are given by

$$y_1(x) \approx u(x) = \sum_{k=0}^{N+1} u_k F_k^{1/2}(x), \quad y_2(x) \approx v(x) = \sum_{k=0}^{N+1} v_k F_k^{1/2}(x). \quad (3.33)$$

In the present simulations for this example, we used $N = 6$, $h = \frac{1}{7}$. The numerical and exact solutions (after five steps of the multidimensional Newton's method) are presented in Figure 3.3.

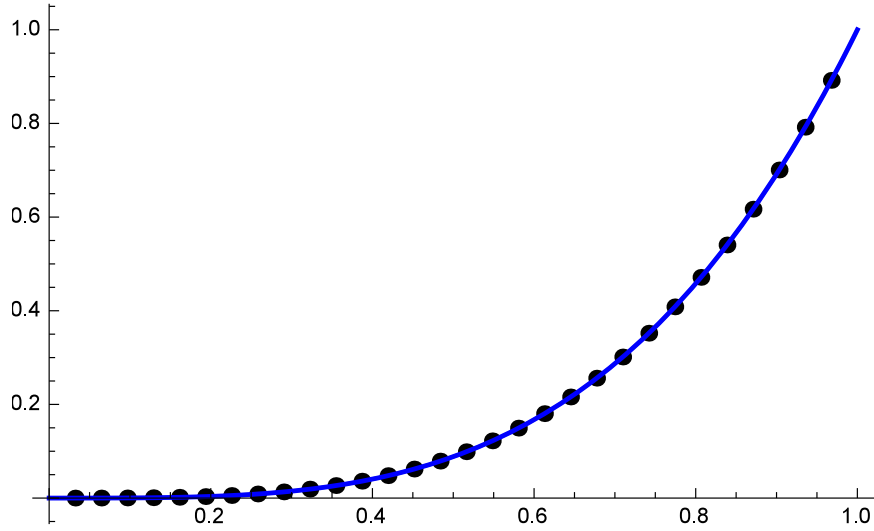


Figure 3.3: Graphs of the approximate solution, u (\bullet), and the exact solution, y (solid), for Example 3.3.1

In addition, to confirm the accuracy of the present results, Table (3.2) illustrates the absolute values of the error which obtained by the present technique and the one obtained by applying the Legendre wavelet [49] for this example. The advantages of our scheme is clearly seen through this table.

x_j	$ u(x) - y(x) $ (Present Method)	$ u(x) - y(x) $ [49]
0.1	6.33902×10^{-12}	9.6996×10^{-5}
0.2	6.41703×10^{-12}	9.3927×10^{-4}
0.3	6.46162×10^{-12}	1.5087×10^{-3}
0.4	6.50265×10^{-12}	3.3989×10^{-4}
0.5	6.53773×10^{-12}	2.4163×10^{-3}
0.6	6.56888×10^{-12}	3.1023×10^{-4}
0.7	6.59785×10^{-12}	1.4799×10^{-3}
0.8	6.62460×10^{-12}	6.3407×10^{-4}
0.9	6.64842×10^{-12}	4.6701×10^{-3}

Table 3.2: Comparison of the absolute errors for Example 3.3.1

Example 3.3.2. Consider the following nonlinear initial value problem

$$D^{1.8}y(x) - 2e^{y(x)} = g(x), \quad x \in (0, 1), \quad (3.34)$$

subject to

$$y(0) = -\frac{1}{8}, \quad y'(0) = \frac{3}{4},$$

and $g(x)$ is chosen so that the exact solution $y(x) = (x - \frac{1}{2})^3$.

Initially, We transform the initial value problem (3.34) into the following system of first order initial-value problems

$$y_1' = y_2 \quad (3.35)$$

$$D^{0.8}y_2(x) = 2e^{y_1(x)} + g(x), \quad 0 < \alpha < 1, \quad (3.36)$$

subject to

$$y_1(0) = -\frac{1}{8}, \quad y_2(0) = \frac{3}{4}. \quad (3.37)$$

The series solutions for $y_1(x)$ and $y_2(x)$ in terms of the fractional order Legendre functions are given by

$$y_1(x) \approx u(x) = \sum_{k=0}^{N+1} u_k F_k^{0.8}(x), \quad y_2(x) \approx v(x) = \sum_{k=0}^{N+1} v_k F_k^{0.8}(x). \quad (3.38)$$

Notice that we applied the present technique with $N = 8$ and $h = \frac{1}{9}$. The graphs of the approximate solutions u_s , for $s = 1, \dots, 4$, together with the exact solutions are displayed in Figure 3.4. It is clearly seen that the sequence u_s converges rapidly to the exact solution.

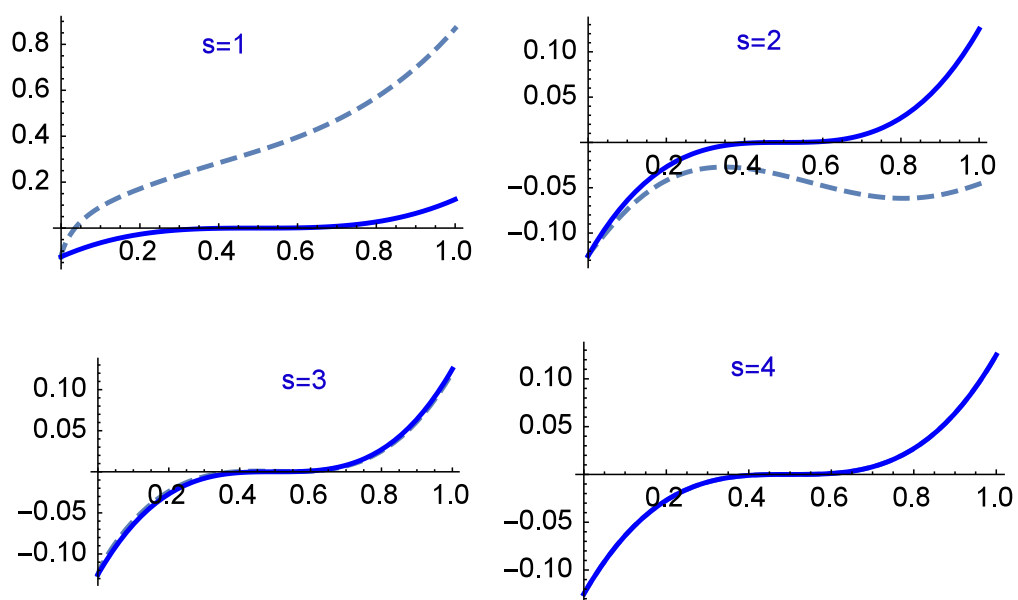


Figure 3.4: Graphs of the approximate solution, u (dashed), and the exact solution, y (solid), for Example 3.3.2

Chapter 4: Summary and Conclusions

The present thesis deals with numerical treatment of classes of linear and nonlinear fractional initial value problems (FIVP's). Based on the the spectral method with fractional Legendre functions as basis, we were able to represent the exact solution by a series solution with a finite sum. By a "good choice" of the collocation points, we converted the original fractional initial value problem into algebraic system which was solved numerically using the powerful multidimensional version of Newton's method. The present technique is applied to discuss the solution of first and second orders FIVP's. The efficiency and accuracy of the present scheme is discussed via solving several examples and compare with other researchers.

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