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HOMOGENIZATION IN PERFORATED DOMAINS AND WITH SOFT INCLUSIONS

Brandon C. Russell
University of Kentucky, brandon.russell700@uky.edu
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Brandon C. Russell, Student

Dr. Zhongwei Shen, Major Professor

Dr. Peter D. Hislop, Director of Graduate Studies

HOMOGENIZATION IN PERFORATED DOMAINS AND WITH SOFT INCLUSIONS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Brandon Chase Russell
Lexington, Kentucky

Director: Dr. Zhongwei Shen, Professor of Mathematics
Lexington, Kentucky

2018

ABSTRACT OF DISSERTATION

HOMOGENIZATION IN PERFORATED DOMAINS AND WITH SOFT INCLUSIONS

In this dissertation, we first provide a short introduction to qualitative homogenization of elliptic equations and systems. We collect relevant and known results regarding elliptic equations and systems with rapidly oscillating, periodic coefficients, which is the classical setting in homogenization of elliptic equations and systems. We extend several classical results to the so-called case of perforated domains and consider materials reinforced with soft inclusions. We establish quantitative H^1-convergence rates in both settings, and as a result deduce large-scale Lipschitz estimates and Liouville-type estimates for solutions to elliptic systems with rapidly oscillating periodic bounded and measurable coefficients. Finally, we connect these large-scale estimates with local regulartity results at the microscopic-level to achieve interior Lipschitz regularity at every scale.

KEYWORDS: Partial differential equations, elliptic equations, homogenization, linear elasticity, media with periodic structure

Author's signature: Brandon Chase Russell

Date: April 24, 2018

HOMOGENIZATION IN PERFORATED DOMAINS AND WITH SOFT INCLUSIONS

By
Brandon Chase Russell

Director of Dissertation: Zhongwei Shen

Director of Graduate Studies: Peter D. Hislop

Date: April 24, 2018

Dedicated to you, the reader.

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Chapter 1 Introduction

This dissertation contains two projects concerning quantitative homogenization of composite materials periodically perforated or periodically reinforced with soft inclusions and regularity results regarding the boundary value problems that model relatively small elastic deformations of these materials. In particular, for the case of perforated materials, we obtain the sharp convergence rate in H^1 and the optimal interior Lipschitz regularity. For the case of materials with embedded soft inclusions, we obtain a suboptimal H^1 convergence rate but derive still the optimal interior regularity.

In Chapter 2, we discuss results in the classical setting, and we derive the boundary value problem modeling relatively small deformations of elastic composite materials with a continuum mechanics approach. Variations of the derived boundary value problem will be studied in later chapters. These variations are explicitly stated in Section 2.2 following the discussion of the boundary value problem. In Section 2.3, we provide some history on the classical setting and the problem. In Section 2.4, we state relevant results that will be referenced throughout our analysis in Chapters 3 and 4. We prove some results, and we provide references for the other results.

In Chapter 3, we prove the quantitative convergence results. These results are new: in the case of perforated domains, the sharp rate is derived; in the case of soft inclusions, a suboptimal rate is derived, but the rate is uniform in the magnitude of the inclusions. Qualitative convergence in both cases has been known for some time. In Section 3.1, we consider the case of microscopically perforated materials. In Section 3.2, we consider the case of materials reinforced at the micrscopic level with soft inclusions. Also, we provide a new proof that the homogenized coefficients are uniformly elliptic uniformly in the magnitude of the embedded inclusions.

In Chapter 4, we acheive the optimal interior regularity for the boundary value problems corresponding to the case of perforated materials and materials reinforced with soft inclusions. That is, we show that the solutions to the boundary value problem derived in Chapter 2 are indeed Lipschitz. In Section 4.1, we use the results of Section 3.1 to obtain large-scale Lipschitz regularity in the case of perforated materials. Consequences of the results of Section 4.1 are stated and proved in Section 4.2. These include a Liouville-type estimate for unbounded perforated materials and Lipschitz estimates at every scale. In Section 4.3, we use the results of Section 3.2 to obtain Lipschitz regularity in the case of materials reinforced with soft inclusion. Consequences of the results in Section 3.2 appear next. In Section 4.4, we discuss sufficient conditions to obtain Lipcshitz estimates in interface problems and apply this to derive small-scale regularity estimates for materials reinforced with soft inclusions.

In Chapter 5, we discuss questions that remain unanswered. In particular, we give a short summary of what is known regarding $W^{1,p}$-estimates and regularity questions near the boundary of perforated domains and materials reinforced with inclusions.

Chapter 2 Preliminaries

In this chapter, we first introduce the concept of elastic deformations in materials with a complex, self-repeating microstructure, e.g., composite materials. Composite materials are materials consisting of two or more constituents with different attributes bound in some cementing matrix and well-mixed. The better the mixture, the more "homogeneous" the material appears. From filled resin in a dentist's office to fiberglass on an airplane wing, composite materials are used in a variety of ways. We derive the boundary value problem under consideration through a continuum mechanics approach.

Next, we introduce the mathematical formulation of the problem. We modify the derived boundary value problem to consider elastic deformations of materials reinforced at the microscopic levels with periodically placed perforations or soft inclusions. Perforations are vacuous holes in the material, and soft inclusions are constituents substantitally weaker than the surrounding, cementing matrix.

Next, we provide some history of the problems to be considered, and finally we discuss auxiliary results that will be referenced throughout our mathematical analysis. These include but are not limited to lemmas and theorems from classical texts on the subject. For each stated lemma or theorem, we provide either a thorough proof or a reference wherein a rigorous proof may be found.

2.1 Linear elasticity

Suppose a continuum $\Omega \subset \mathbb{R}^3$ undergoes a static deformation described by a mapping $x = (x_1, x_2, x_3) : \Omega \to \mathbb{R}^3$. That is, if $X = (X_1, X_2, X_3) \in \Omega$, then $x = x(X)$ denotes the location in space of the coordinate X after deformation as seen in Figure 2.1. The deformation gradient ∇x essentially describes the mapping experienced by infinitesimal line elements in the reference configuration. We define the relative displacement u of a point X by

$$u(X) = x(X) - X, \tag{2.1.1}$$

which explicitly depends on the deformation x (see Figure 2.2).

Given a deformation x, one can define the Eulerian (or referential) strain tensor e, which essentially describes the elongation undergone by infinitesimal line elements in the reference. In particular,

$$e = \frac{1}{2}(\nabla x^T \nabla x - \text{id.}), \tag{2.1.2}$$

where id. denotes the identity tensor with 1 on its diagonal and 0 elsewhere, ∇x^T denotes the transpose of the matrix valued function ∇x, and $\nabla \equiv \nabla_X$ denotes the gradient with respect to the referential variable X, i.e, $\nabla_X = (\partial/\partial X_1, \partial/\partial X_2, \partial/\partial X_3)$ If we consider an infinitesimal line element dX in the reference configuration, and let M denote the unit vector in the same direction, then

$$dX = M dS,$$

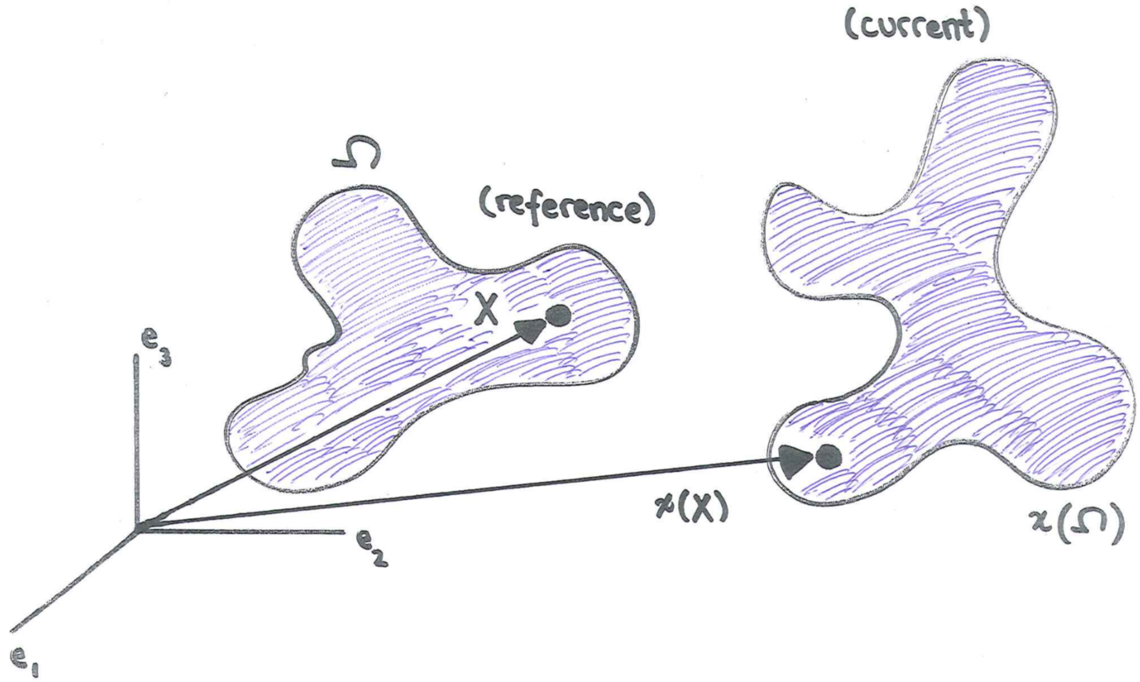


Figure 2.1: The set Ω is referred to as the reference configuration, and the set $x(\Omega)$ is referred to as the current configuration.

where dS denotes the infinitesimal length element of the vector dX. Suppose dX is mapped by the deformation gradient to $dx = mds$, where m is the unit vector in the same direction of dx and ds denotes the infinitesimal length of dx (see Figure 2.3). Note then

$$\begin{aligned}(ds)^2 - (dS)^2 &= mds \cdot mds - MdS \cdot MdS \\ &= dx \cdot dx - dX \cdot dX \\ &= \nabla x dX \cdot \nabla x dX - dX \cdot dX \\ &= dX \cdot (\nabla x^T \nabla x - \text{id.}) dX \\ &= dX \cdot 2e dX.\end{aligned}$$

In particular, if x denotes a rigid displacement (a rotation, translation, or a combination of these two), then $e = 0$. Recalling equation (2.1.1), we see

$$\nabla u = \nabla x - \text{id.},$$

and so another representation of the Eulerian strain tensor is given by

$$\begin{aligned}e &= \frac{1}{2}(\nabla x^T \nabla x - \text{id.}) \\ &= \frac{1}{2}([\nabla u + \text{id.}]^T [\nabla u + \text{id.}] - \text{id.}) \\ &= \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u),\end{aligned}$$

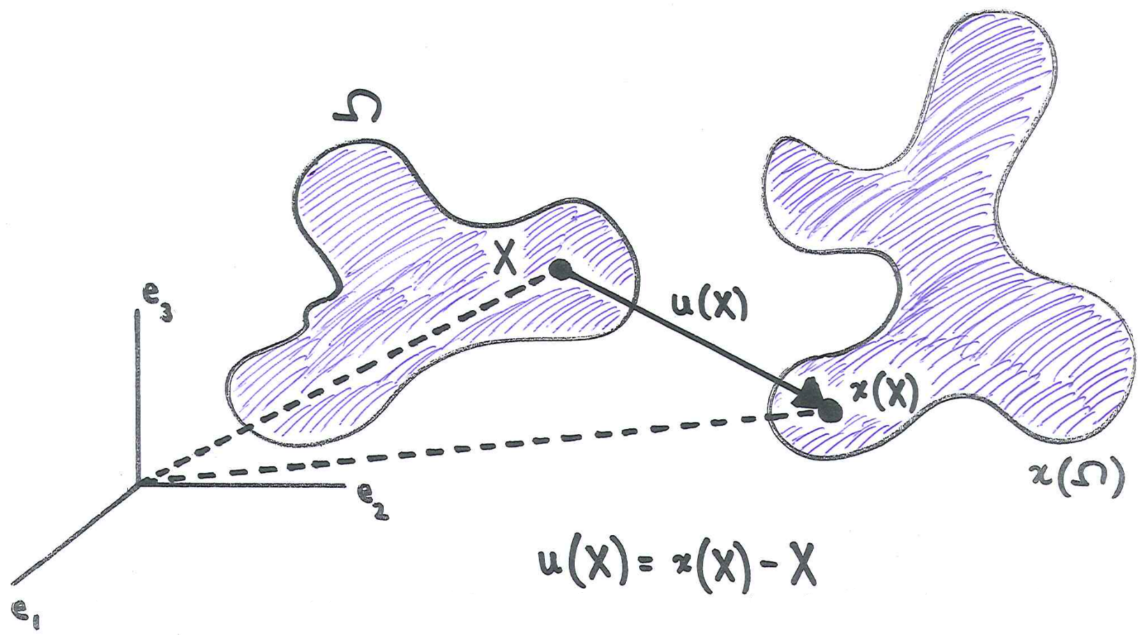


Figure 2.2: The displacement u measures the direction in which a coordinate is displaced and its distance relative to the original reference configuration.

where ∇u^T denotes the transpose of the tensor ∇u. Hence, the referential strain depends explicitly on the displacement u, i.e., $e = e(u; X)$.

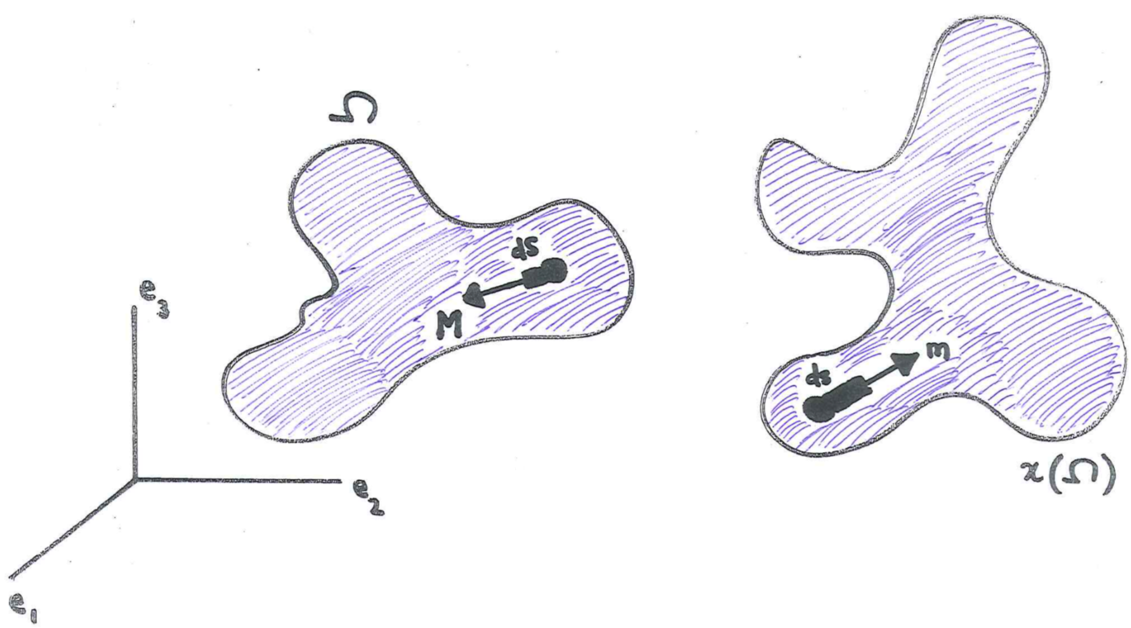


Figure 2.3: The Eulerian strain tensor e measures the difference of squares of line elements in the reference and their image under the deformation x.

For small displacements $|\nabla u| << 1$, the nonlinear quantity $\nabla u^T \nabla u$ is negligible relative to ∇u. Hence, for small displacements, we typically consider the linear Eulerian strain tensor

$$e = \frac{1}{2}(\nabla u + \nabla u^T). \tag{2.1.3}$$

Elastic materials that experience small displacements exhibit a linear relationship between the associated stress and strain. That is, for small displacements, the Cauchy stress tensor σ is linearly related to the referential strain, say

$$\sigma = Ae, \tag{2.1.4}$$

where the 4-tensor A is commonly referred to as the elasticity tensor for the continuum. The tensor A may or may not be isotropic and may change depending on which coordinate in the continuum is under consideration, i.e., $A \equiv A(X)$. One may consider each entry of the stress tensor $\sigma_i^\alpha(u; X)$ at a point X as internal forces in the direction e_i along the plane passing through X with normal vector e^α. The vectors e_i and e^α are the usual basis of $\mathbb{R}^3$ with 1 in the ith and αth positions, respectively, and 0 elsewhere.

It is known by conservation of momentum that the Cauchy stress tensor $\sigma = \{\sigma_i^\alpha\}_{i,\alpha}$ is symmetric, i.e.,

$$\sigma_i^\alpha = \sigma_\alpha^i, \quad \text{for any } 1 \leq i, \alpha, \leq 3.$$

Given definition (2.1.3), the linear strain tensor $e = \{e_i^\alpha\}_{i,\alpha}$ is also symmetric

$$e_i^\alpha = e_\alpha^i, \quad \text{for any } 1 \leq i, \alpha \leq 3.$$

As a consequence, the elasticity tensor $A = \{A_{ij}^{\alpha\beta}\}_{i,j,\alpha,\beta}$ defined by the relation (2.1.4) is symmetric in multiple ways

$$A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha} = A_{\alpha j}^{i\beta}, \quad \text{for any } 1 \leq i, j, \alpha, \beta \leq 3. \tag{2.1.5}$$

In three dimensions, this limits the number of independent entries to 36. In practice, however, the elasticity tensor A has more symmetries limiting the number of independent components to 21. Indeed, one defines the strain energy density of the deformed material as the product of stress and strain, substitutes relation (2.1.4), and relies on the symmetry of the elasticity tensor. However, in the chapters that follow and our mathematical analysis, we only use the symmetries shown in (2.1.5). The symmetries in (2.1.5) are typically referred to as the minor symmetries of the elasticity tensor, and the symmetries not listed and not used in our analysis are referred to as the major symmetries of the elasticity tensor.

Suppose the continuum Ω is a composite material with some microstructure at some fixed microscopic scale ε (see Figure 2.4), and suppose the elasticity tensor A encodes at each referential point the elastic properties of a certain constituent.

Generally composite materials are well-mixed, and the constituents are well-distributed throughout the continuum. For mathematical simplicity, we will assume that the constituents are periodically distributed and the period coincides with the microscopic scale. In particular, if A describes the elastic properties of each constituent, and if A is a periodic function, then $A(\cdot/\varepsilon)$ describes the elastic properties of the composite material.

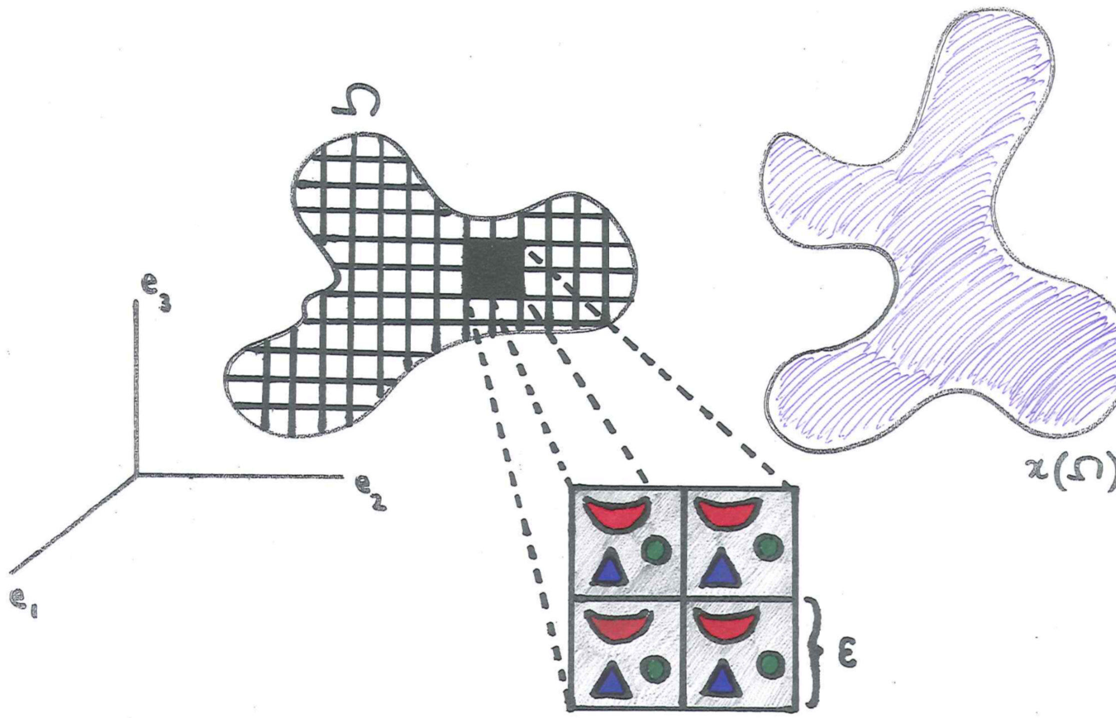


Figure 2.4: Composite materials contain multiple constituents that are bound together by some matrix material. Each constituent can have distinct elastic properties.

Ignoring inertial effects, given any body forces F and a prescribed boundary displacement f, there exists a unique displacement u_ε that balances the divergence of the Cauchy stress tensor with the body forces and has the prescribed boundary configuration. In particular, according to Cauchy's momentum equation for each $\varepsilon > 0$, there exists a unique displacement u_ε that solves the boundary value problem

$$\begin{cases} -\operatorname{div}\sigma(u_\varepsilon; X) = F(X) & \text{in } \Omega, \\ u_\varepsilon(X) = f(X) & \text{on } \partial\Omega, \end{cases} \tag{2.1.6}$$

where $\sigma(u_\varepsilon; X) = A(X/\varepsilon)e(u_\varepsilon; X)$. In this dissertation, we consider variations on the structure of the composite material Ω and discuss new results concerning boundary value problem (2.1.6) as $\varepsilon \to 0$.

2.2 The mathematical problem

In this dissertation, we consider variations of the boundary value problem (2.1.6). In particular, we consider the divergence-form operator

$$\mathcal{L}_{\varepsilon,\delta} = -\operatorname{div}\left(k_\delta^\varepsilon A^\varepsilon \nabla\right) = -\frac{\partial}{\partial x_i}\left(k_\delta\left(\frac{\cdot}{\varepsilon}\right) a_{ij}^{\alpha\beta}\left(\frac{\cdot}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right), \quad \varepsilon > 0, \tag{2.2.1}$$

where $A^\varepsilon = A(\cdot/\varepsilon)$, $A(y) = \{a_{ij}^{\alpha\beta}(y)\}_{1\leq i,j,\alpha,\beta\leq d}$ for $y \in \mathbb{R}^d$, $d \geq 2$, and $k_\delta^\varepsilon = k_\delta(\cdot/\varepsilon)$ denotes a weight depending on a parameter δ and the microscopic scale ε. Note the

Einstein summation convention will be used throughout. That is, repeated indices will be summed. Write

$$k_\delta(y) = \mathbf{1}_+(y) + \delta \mathbf{1}_-(y), \tag{2.2.2}$$

where $\mathbf{1}_+$ denotes the characteristic function of an unbounded Lipcshitz domain $\omega \subseteq \mathbb{R}^d$ and $\mathbf{1}_-$ denotes the characteristic function of $\mathbb{R}^d \backslash \omega$. The set ω plays the role of the cementing matrix discussed in Section 2.1. Note when $\delta = 1$, the operator (2.2.1) appears in the boundary value problem (2.1.6). For reference, consider ε to denote the size of the microstructure, A^ε as the elasticity tensor for some continuum with a self-repeating microstructure, and k_δ^ε as the ε-periodic displacement of inclusions in the continuum with some weight δ which is considered small relative to its surrounding cementing matrix ω and its elastic properties. When $\delta = 0$, the complement of the support of k_δ^ε denotes the ε-periodic displacement of perforations in the material.

We assume ω has a 1-periodic structure, i.e., we assume the characteristic function $\mathbf{1}_+$ of ω satisfies

$$\mathbf{1}_+(y + e_i) = \mathbf{1}_+(y), \;\; y \in \mathbb{R}^d, \, 1 \le i \le d \tag{2.2.3}$$

where e_i denotes a standard basis vector of $\mathbb{R}^d$ with 1 in the ith position and 0 elsewhere. We write $\varepsilon\omega$ to denote the ε-homothetic set

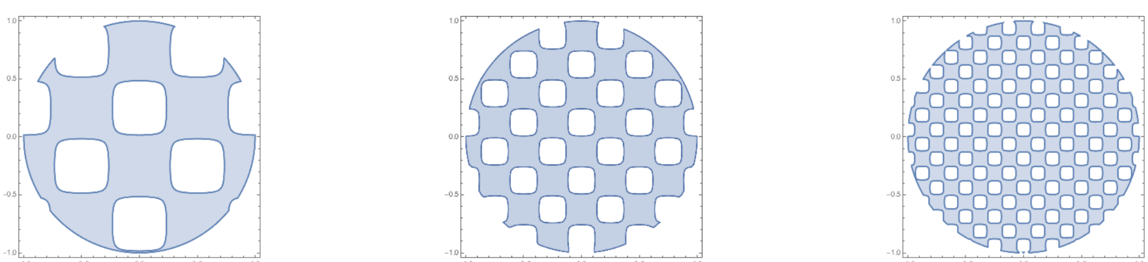
$$\varepsilon\omega = \{x \in \mathbb{R}^d \, : \, x/\varepsilon \in \omega\}.$$

We assume ω is connected and that any two connected components of $\mathbb{R}^d \backslash \omega$ are separated by some positive distance (to be quantified later). We also assume each individual connected component of $\mathbb{R}^d \backslash \omega$ is bounded. Recall that ω is an unbounded domain, and as $\varepsilon \to 0$ the volume of each connected component in $\mathbb{R}^d \backslash \varepsilon\omega$ decreases but the total number of components increases.

As an example, if $d = 2$, $\Omega = B(0, 1)$, and we define

$$\omega = \{(x_1, x_2) \, : \, \cos(2\pi x_1) \sin(2\pi x_2) < 0.1\},$$

then for various values of ε the set $\Omega_\varepsilon := \Omega \cap \varepsilon\omega$ looks as follows.



Ω_1 $\quad$ $\Omega_{0.5}$ $\quad$ $\Omega_{0.25}$

Figure 2.5: (left) The shaded area is the set ω inside the unit ball. (center) The shaded area is the set 0.5ω inside the unit ball. (right) The shaded area is the set 0.25ω inside the unit ball.

The case $\delta = 1$ is in fact the classical setting for homogenization of elliptic systems, and many authors have worked in this area [7, 12, 6, 24, 20, 11, 2, 26, 27, 28]. The case $\delta = 0$, which is considered in Sections 3.1 and 4.1, is referred to as the case of perforated materials. The set $\mathbb{R}^d\backslash\omega$ where $k_0(y) = 0$ for each y is commonly referred to as the "perforations" of the material. The case $\delta \in (0, 1)$, which is considered in Sections 3.2 and 4.3, is referred to as the case of materials reinforced with soft inclusions. The set $\mathbb{R}^d\backslash\omega$ where $k_\delta(y) = \delta$ for all y is commonly referred to as the "soft inclusions" embedded in the cementing matrix ω.

Throughout, we assume the coefficient matrix $A(y)$ is real, measurable, satisfies the elasticity symmetry conditions of equation (2.1.5) and an ellipticity condition. To be specific,

$$a_{ij}^{\alpha\beta}(y) = a_{ji}^{\alpha\beta}(y) = a_{\alpha j}^{i\beta}(y), \tag{2.2.4}$$

$$\kappa_1|\xi|^2 \leq a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \leq \kappa_2|\xi|^2, \tag{2.2.5}$$

for $y \in \mathbb{R}^d$, any $1 \leq i, j, \alpha, \beta \leq d$, and any symmetric matrix $\xi = \{\xi_i^\alpha\}_{1\leq i,\alpha\leq d}$, where $\kappa_1, \kappa_2 > 0$. Assumptions (2.2.4) and (2.2.5) are natural: the former is referred to in Section 2.1 as the minor symmetries of the elasticity tensor.

As an example, one may consider the tensor A given by

$$a_{ij}^{\alpha\beta}(y) = \lambda(y)\delta_i^\alpha\delta_j^\beta + \mu(y)\left\{\delta_{ij}\delta^{\alpha\beta} + \delta_i^\beta\delta_j^\alpha\right\}, \quad y \in \mathbb{R}^d \tag{2.2.6}$$

where $\lambda(y)$, $\mu(y)$ are scalar valued functions, and δ with subscripts and superscripts denotes the Kronecker delta indicator function, e.g., $\delta_i^\alpha = 1$ exaclty when $i = \alpha$ and $\delta_i^\alpha = 0$ otherwise. The functions λ and μ are typically referred to as the Lamé parameters, i.e., μ denotes the shear modulus of the continuum and together with λ defines the bulk modulus of the material. The thing to note, however, is that if $\lambda, \mu \geq C_1 > 0$ for some constant C_1 and are bounded, then (2.2.6) satisfies the symmetries (2.2.4) and the ellipticity condition (2.2.5) (one may take $\kappa_1 = C_1$ and $\kappa_2 = \sup_{y\in\mathbb{R}^d}\{3\lambda(y) + 2\mu(y)\}$).

Note by (2.2.5), $a_{ij}^{\alpha\beta} \in L^\infty(\mathbb{R}^d)$ for any given indices. That is, if we fix the indices i, j, α, β and choose the symmetric tensor ξ so that $\xi_i^\alpha = \xi_\alpha^i = \xi_j^\beta = \xi_\beta^j = 1$ and $\xi_k^\gamma = 0$ for any other indices k, γ, then (2.2.5) reads

$$a_{ij}^{\alpha\beta}(y) \leq 4\kappa_2$$

for each $y \in \mathbb{R}^d$. Hence, A is uniformly bounded in $\mathbb{R}^d$.

To ensure the constituents are well-distributed throughout the material (see our discussion in Section 2.1), we also assume A is 1-periodic in the sense of (2.2.3), i.e.,

$$A(y) = A(y + z) \quad \text{for } y \in \omega,\ z \in \mathbb{Z}^d. \tag{2.2.7}$$

If A satisfies (2.2.7), then the scaled matrix A^ε satsifies

$$A^\varepsilon(x + \varepsilon z) = A\left(\frac{x + \varepsilon z}{\varepsilon}\right) = A\left(\frac{x}{\varepsilon} + z\right) = A^\varepsilon(x),$$

i.e., A^ε is ε-periodic. Many authors have relaxed assumption (2.2.7) to almost-periodic [4, 28] or even considered random elastcity tensors [5, 3].

As detailed in Section 2.1, the coefficient matrix of the systems of linear elasticity describes the linear relation between the stress and strain a material experiences during relatively small elastic deformations. Consequently, the elasticity conditions (2.2.4), (2.2.5), and δ should be regarded as physical parameters of the system, whereas ε is clearly a geometric parameter.

Let Ω be a bounded domain. Boundary value problem (2.1.6) in the absence of body forces may be generalized as the following Dirichlet boundary value problem

$$\begin{cases} \mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \quad \text{in } \Omega, \\ u_{\varepsilon,\delta} = f \quad \text{on } \partial\Omega. \end{cases} \tag{2.2.8}$$

We say $u_{\varepsilon,\delta}$ is a weak solution to (2.2.8) if

$$\int_\Omega k_\delta^\varepsilon a_{ij}^{\alpha\beta\varepsilon} \frac{\partial u_{\varepsilon,\delta}^\beta}{\partial x_j} \frac{\partial w^\alpha}{\partial x_i} = 0 \quad \text{for any } w = \{w^\alpha\}_\alpha \in H_0^1(\Omega;\mathbb{R}^d) \tag{2.2.9}$$

and $u_{\varepsilon,\delta} - f \in H_0^1(\Omega;\mathbb{R}^d)$. Notice, in the case $\delta = 0$, Neumann boundary conditions are implicit on the boundaries of the perforations. Indeed, for any $w \in H_0^1(\Omega;\mathbb{R}^d)$, integrating by parts gives

$$\begin{aligned} 0 &= \int_\Omega \mathcal{L}_{\varepsilon,0}(u_{\varepsilon,0})w \\ &= \int_{\Omega\cap\partial(\varepsilon\omega)} -n_\varepsilon A^\varepsilon \nabla u_\varepsilon \cdot \nabla w + \int_{\Omega\cap\varepsilon\omega} A^\varepsilon \nabla u_\varepsilon \cdot \nabla w, \end{aligned}$$

and so to coincide with (2.2.9) we require $-n_\varepsilon A^\varepsilon \nabla u_\varepsilon = 0$ on $\Omega \cap \partial(\varepsilon\omega)$, where n_ε denotes the outward unit vector normal to $\varepsilon\omega$. This is the Neumann boundary condition on the perforations. That is, when $\delta = 0$, (2.2.8) is a mixed boundary value problem. This is important to note and is revisited in Chapter 5 when we discuss open problems regarding the boundary regularity of solutions $u_{\varepsilon,0}$. We should also note that our construction of $\Omega_\varepsilon = \Omega \cap \varepsilon\omega$ does not require the perforations or inclusions remain in the interior of Ω. Indeed, $\partial\Omega_\varepsilon$ may be fairly irregular.

As discussed in Section 2.1, the boundary value problem (2.2.8) models relatively small deformations of perforated elastic materials or elastic materials reinforced with soft inclusions subject to zero external body forces [12, 20, 24]. Soft inclusions are substantially "weaker" than the cementing matrix ω, but their embedding can be otherwise advantageous. For example, a material's compressive strength can be indirectly proportional with the increasing volume of soft inclusions but the thermal inertia and energy efficiency may be directly proportional [16].

For each $\delta \in (0,1]$, the existence and uniqueness of a weak solution $u_{\varepsilon,\delta} \in H^1(\Omega;\mathbb{R}^d)$ to (2.2.8) (that is, satisfying (2.2.9)) for $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$ follows from the Lax-Milgram theorem and Korn's inequality. Indeed, if we define the bilinear form $B_{\varepsilon,\delta}$ on $H_0^1(\Omega;\mathbb{R}^d)$ for a fixed ε, $\delta > 0$ by

$$B_{\varepsilon,\delta}[\phi, w] = \int_\Omega k_\delta^\varepsilon a_{ij}^{\alpha\beta\varepsilon} \frac{\partial \phi^\beta}{\partial x_j} \frac{\partial w^\alpha}{\partial x_i},$$

then the coercivity and boundedness of $B_{\varepsilon,\delta}$ follows from the ellipticity of A, the positivity of δ, and Korn's inequality. Hence, the Lax-Milgram theorem implies there exists a unique $\phi_{\varepsilon,\delta} \in H_0^1(\Omega;\mathbb{R}^d)$ satisfying $B_{\varepsilon,\delta}[\phi_{\varepsilon,\delta}, w] = 0$ for each $w \in H_0^1(\Omega;\mathbb{R}^d)$. For a fixed $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d))$, define $u_{\varepsilon,\delta} = \phi_{\varepsilon,\delta} + f'$, where $f' \in H^1(\Omega;\mathbb{R}^d)$ and $f' = f$ on $\partial\Omega$ (in the trace sense).

For $\delta = 0$, the existence and unqieness follows in a similar manner from the Lax-Milgram theorem but with Korn's inequality for perforated domains which is stated in Section 2.4 [11, 24]. In either case, it should be noted that the solution $u_{\varepsilon,\delta}$ is not bounded uniformly in $H^1(\Omega;\mathbb{R}^d)$ for $\delta < 1$, but rather

$$\|k_\delta^\varepsilon u_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^2(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)},$$

where C depends on κ_1, κ_2.

If $\delta > 0$, then the matrix $k_\delta^\varepsilon A^\varepsilon$ is uniformly elliptic in $\mathbb{R}^d$ (but not uniformly in δ), and so it can be shown that the weak solution to (2.2.8) converges weakly in $H^1(\Omega;\mathbb{R}^d)$ and consequently strongly in $L^2(\Omega;\mathbb{R}^d)$ as $\varepsilon \to 0$ to some $u_{0,\delta} \in H^1(\Omega;\mathbb{R}^d)$. This qualitative convergence is known. The function $u_{0,\delta}$ is a solution of a constant-coefficient equation in the domain Ω (see [9, 20, 24] and references therein). Indeed, we have the following theorem.

Theorem 2.2.1. *Suppose Ω is a bounded Lipschitz domain and that A satisfies* (2.2.4)*,* (2.2.5)*, and* (2.2.7)*. Let $u_{\varepsilon,\delta}$ denote a weak solution $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in Ω, and $u_{\varepsilon,\delta} = f$ on $\partial\Omega$ for some fixed $\delta > 0$. Then there exists a $u_{0,\delta} \in H^1(\Omega;\mathbb{R}^d)$ such that*

$$u_{\varepsilon,\delta} \rightharpoonup u_{0,\delta} \quad \text{weakly in } H^1(\Omega;\mathbb{R}^d).$$

Consequently, $u_{\varepsilon,\delta} \to u_{0,\delta}$ strongly in $L^2(\Omega;\mathbb{R}^d)$.

For a proof of the previous theorem, see any introdutory text of the subject, e.g., [11, Section 10.3]. The function $u_{0,\delta}$ is called the homogenized solution and the boundary value problem it solves is the homogenized system corresponding to (2.2.8). For an explicit discussion of the homogenized problem, see either Section 3.1 or Section 3.2.

2.3 History

In this section, we provide a short history.

In 1977, Luc Tartar essentially proved Theorem 2.2.1 for general elliptic boundary value problems by introducing his method of oscillating test functions [29]. This method relies on the variational formulation of 2.2.8 and the adjoint problem for the correctors. When $\delta = 1$, the adjoint problem for the correctors corresponding to (2.2.8) is given by

$$\begin{cases} \mathcal{L}_{1,1}^T(\widetilde{\chi}_j^\beta + P_j^\beta) = 0 & \text{in } Q := [0,1)^d, \\ \chi_j^\beta \text{ is 1-periodic}, & \int_Q \chi_j^\beta = 0, \end{cases} \tag{2.3.1}$$

where $P_j^\beta(y) = y_j e^\beta$ is a linear function, $\mathcal{L}_{1,1}^T = -\text{div}(A^T\nabla)$, and A^T denotes the transpose of A. Indeed, $w_\varepsilon = \varepsilon\widetilde{\chi}_j^\beta(\cdot/\varepsilon) + P_j^\beta$ satisfies $\mathcal{L}_{\varepsilon,1}^T(w_\varepsilon) = 0$ in $\mathbb{R}^d$ for each $1 \leq j, \beta \leq d$. Fixing $\varphi \in C_0^\infty(\Omega)$ and considering the weak formulations of (2.2.8) and (2.3.1),

$$\int_\Omega A^\varepsilon \nabla u_{\varepsilon,1} \cdot w_\varepsilon \nabla\varphi + \int_\Omega A^\varepsilon \nabla u_{\varepsilon,1} \cdot \varphi\nabla w_\varepsilon = 0 \tag{2.3.2}$$

and

$$\int_\Omega (A^\varepsilon)^T \nabla w_\varepsilon \cdot u_{\varepsilon,1}\nabla\varphi + \int_\Omega (A^\varepsilon)^T\nabla w_\varepsilon \cdot \varphi\nabla u_{\varepsilon,1} = 0. \tag{2.3.3}$$

It is fairly straightforward to show that solutions to (2.2.8) with $\delta = 1$ converge weakly, but the direct identification of the limiting function $u_{0,1}$ is not immediately obvious. Considering the difference of (2.3.3) and (2.3.2), the two terms containing only products of weakly convergent sequences cancel, allowing one to pass to the limit in the resulting expression. A limitation of Tartar's method is that it only produces the qualitative convergence of Theorem 2.2.1.

In 1979, Doina Cioranescu and Jeannine Saint Jean Paulin considered the elastic torsion of a perforated bar by examining cross sections of cylindrical bars with cylindrical cavities generated parallel to the generator of the bar, i.e., a variation of boundary value problem (2.2.8) with $\delta = 0$. The boundary values on the perforations they consider are $u_\varepsilon \equiv$ const. rather than the Neumann conditions implied by the weak formulation (2.2.9). Nevertheless, they deduce a qualitative convergence result similar to Theorem 2.2.1 by considering a trivial extension of $u_{\varepsilon,0}$ (simply take $\widetilde{u}_{\varepsilon,0} = u_{\varepsilon,0}$ in the material domain and $\widetilde{u}_{\varepsilon,0} \equiv$ const. throughout each perforation). To deduce the result, they essentially apply Tartar's method of oscillating test functions.

A breakthrough in the study of homogenization was achieved in 1987 by Marco Avellaneda and Fang-hua Lin when they introduced compactness methods to the theory [6]. In partiuclar, the qualitative convergence allowed the authors to derive interior and boundary regularity results for (2.2.8) when $\delta = 1$ that are uniform in the parameter ε. The method is essentially "proof by contradiction." For example, if $\mathcal{L}_{\varepsilon,1}(u_\varepsilon) = 0$ in $B(0,1)$, then u_ε is uniformly Hölder continuous with exponent τ if it satisfies the inequality

$$\sup_{\substack{|x|\leq 1/2\\ 0<r\leq 1/2}} r^{-\tau}\left(\fint_{B(x,r)} |u_\varepsilon - (u_\varepsilon)_{x,r}|^2\right)^{1/2} \leq C$$

for some C independent of ε. Suppose to the contrary that such is not true. Then for every radius r_k there exists an operator $\mathcal{L}_1^k$ with coefficients A^k satsifying (2.2.5), (2.2.7) and $\varepsilon_k > 0$ such that $\mathcal{L}_{\varepsilon_k}^k(u_{\varepsilon_k}) = 0$ in $B(0,1)$ but

$$\fint_{B(0,1)} |u_{\varepsilon_k} - (u_{\varepsilon_k})_{0,r_k}| > r_k^{2\tau}.$$

Given u_{ε_k} is bounded in $L^2(B(0,1))$, one may choose a weakly convergent subsequence and contradict the qualitative convergence of Theorem 2.2.1. Needless to say, the procedure is complicated, and the notation alone is quite cumbersome. Nevertheless, the authors were able to obtain new results.

In 2014, Scott Armstrong and Charles K. Smart introduced an argument for deducing regularity estimates on energy minimizers in stochastic homogenization [5]. This method applies to linear divergence form elliptic equations with random coefficients. In 2016, Armstrong and Zhongwei Shen modified the scheme of Armstrong and Smart to deduce optimal regularity estimates in the setting of linear divergence form elliptic equations with almost-periodic coefficients [4]. The scheme was further developed in 2017 by Shen [26]. The method is discussed in detail and essentially applied in Sections 4.1 and 4.3. We adapt the scheme to deduce interior regularity estimates for (2.2.8).

Many authors have included the study of periodically perforated domains and the case $\delta > 0$ in their manuscripts, e.g., [20, 24], and it has been of interest to many others. In particular, Li Ming Yeh in his many works has applied the compactness methods of Avellaneda and Lin to the cases $0 \leq \delta \leq 1$ and derived Hölder and Lipschitz estimates uniform in the parameters ε and δ [30, 32]. He has also derived uniform $W^{1,p}$ estimates for solutions in domains with perforations away from the boundary [31] and quantitative convergence results in L^∞ [32], both as consequences of the deduced regularity estimates.

We would like to point out that the results in this dissertation may be similar to some of the results obtained by Yeh, but the method by which they are obtained is drastically different. Our methods are direct and uniquely rely on derived quantitative convergence estimates.

2.4 Auxiliary results

In this section we collect relevant mathematical results that will be referenced throughout our analysis. We present each lemma and theorem fully and either provide proofs or references wherein proofs may be found.

Fix $\zeta \in C_0^\infty(B(0,1))$ so that $0 \leq \zeta \leq 1$ and $\int_{\mathbb{R}^d} \zeta = 1$. Define

$$K_\varepsilon(g)(x) = \int_{\mathbb{R}^d} g(x-y)\zeta_\varepsilon(y)\,dy, \quad g \in L^2(\mathbb{R}^d) \tag{2.4.1}$$

where $\zeta_\varepsilon(y) = \varepsilon^{-d}\zeta(y/\varepsilon)$. Note per the following lemma, K_ε is a continuous map from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. For any function g, set $g^\varepsilon(\cdot) = g(\cdot/\varepsilon)$.

Lemma 2.4.1. *For any $g \in L^2(\mathbb{R}^d)$,*

$$\|K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq \|g\|_{L^2(\mathbb{R}^d)},$$

Proof. Fix $g \in L^2(\mathbb{R}^d)$. Write $\zeta_\varepsilon = \zeta_\varepsilon^{1/2}\zeta_\varepsilon^{1/2}$. By Cauchy-Schwarz and since $\int_{\mathbb{R}^d} \zeta = 1$,

$$\begin{aligned}\|K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)}^2 &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g(y)\zeta_\varepsilon(x-y)\,dy \right|^2 dx \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(y)|^2 \zeta_\varepsilon(x-y)\,dydx.\end{aligned}$$

Applying Fubini's theorem gives the desired result. □

Lemma 2.4.2. *Let $g \in H^1(\mathbb{R}^d)$. Then*

$$\|g - K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon \|\nabla g\|_{L^2(\mathbb{R}^d)},$$

where C depends only on d.

Proof. Consider $g \in C_0^\infty(\mathbb{R}^d)$. Note that since $\text{supp}(\zeta) \subset B(0,1)$, we have

$$\text{supp}(\zeta_\varepsilon) \subset B(0,\varepsilon).$$

Hence, fixing $x \in \mathbb{R}^d$ and applying Cauchy-Schwarz as in the previous lemma gives

$$\begin{aligned}\left| g(x) - \int_{\mathbb{R}^d} g(x-y)\zeta_\varepsilon(y)\,dy \right| &\leq \int_{\mathbb{R}^d} |g(x) - g(x-y)|\,\zeta_\varepsilon(y)\,dy \\ &\leq \left(\int_{B(0,\varepsilon)} |g(x) - g(x-y)|^2 \zeta_\varepsilon(y)\,dy \right)^{1/2}.\end{aligned}$$

Therefore,

$$\|g - K_\varepsilon(g)\|^2_{L^2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \int_{B(0,\varepsilon)} |g(x) - g(x-y)|^2 \zeta_\varepsilon(y)\,dydx. \tag{2.4.2}$$

Write $G(t) = g(x + (t-1)y)$ for $t \in [0,1]$ and fixed $x, y \in \mathbb{R}^d$. Since $g \in C_0^\infty(\mathbb{R}^d)$, G is differentiable on (0,1) and continuous on [0,1]. By the Mean Value Theorem, there exists a $t' \in (0,1)$ such that

$$G(1) - G(0) = \frac{dG}{dt}(t'),$$

i.e.,

$$g(x) - g(x-y) = \nabla g(x + (t'-1)y) \cdot y$$

Thus, by Fubini's Theorem and (2.4.2),

$$\|g - K_\varepsilon(g)\|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon^2 \int_{B(0,\varepsilon)} \int_{\mathbb{R}^d} |\nabla g(x + (t'-1)y)|^2 \zeta_\varepsilon(y)\,dxdy \leq \varepsilon \|\nabla g\|_{L^2(\mathbb{R}^d)}.$$

By a density argument, the desired result follows. □

In the following lemma, let $L^2_{per}(\mathbb{R}^d)$ denote the closure of $C^\infty_{per}(\mathbb{R}^d)$ in the $L^2(\mathbb{R}^d)$ norm, where $C^\infty_{per}(\mathbb{R}^d)$ denotes the class of infinitely differentiable periodic functions.

Lemma 2.4.3. *Let $h \in L^2_{per}(\mathbb{R}^d)$. Then for any $g \in L^2(\mathbb{R}^d)$,*

$$\|h^\varepsilon K_\varepsilon(g)\|_{L^2(\mathbb{R}^d)} \leq C\|h\|_{L^2(Q)} \|g\|_{L^2(\mathbb{R}^d)},$$

where $Q = [0,1)^d$.

Proof. Consider $h \in C^\infty_{\text{per}}(\mathbb{R}^d)$ and $g \in C^\infty_0(\mathbb{R}^d)$. By Cauchy-Schwarz,

$$\begin{aligned} |h^\varepsilon(x)K_\varepsilon g(x)|^2 &\le \left| h\left(\frac{x}{\varepsilon}\right) \int_{\mathbb{R}^d} g(y)\zeta_\varepsilon(x-y)\,dy \right|^2 \\ &\le \left| \int_{\mathbb{R}^d} h\left(\frac{x}{\varepsilon}\right) g(y)[\zeta_\varepsilon(x-y)]^{1/2}[\zeta_\varepsilon(x-y)]^{1/2}\,dy \right|^2 \\ &\le \int_{\mathbb{R}^d} \left| h\left(\frac{x}{\varepsilon}\right) g(y) \right|^2 \zeta_\varepsilon(x-y)\,dy. \end{aligned}$$

Hence, by Fubini's theorem,

$$\begin{aligned} \|h^\varepsilon K_\varepsilon(g)\|^2_{L^2(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} |g(y)|^2 \int_{B(y,\varepsilon)} \left| h\left(\frac{x}{\varepsilon}\right) \right|^2 \zeta_\varepsilon(x-y)\,dxdy \\ &\le \varepsilon^{-d} \sup_{y\in\mathbb{R}^d} \int_{B(y,\varepsilon)} \left| h\left(\frac{x}{\varepsilon}\right) \right|^2 dx \cdot \int_{\mathbb{R}^d} |g(y)|^2\,dy, \\ &\le \sup_{y\in\mathbb{R}^d} \int_{B(y,1)} |h(x)|^2\,dx \cdot \int_{\mathbb{R}^d} |g(y)|^2\,dy, \end{aligned}$$

where we've made the chamge of variables $x \mapsto \varepsilon x$. Since h is periodic, this gives the desired result for smooth functions. For arbitrary $L^2_{per}(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ functions, consider a density argument. □

Lemma 2.4.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For any $g \in H^1(\Omega)$,*

$$\|g\|_{L^2(\mathcal{O}_r)} \le Cr^{1/2}\|g\|_{H^1(\Omega)},$$

where C depends on d and Ω, and $\mathcal{O}_r = \{x \in \Omega : dist(x, \partial\Omega) < r\}$.

Proof. See [24, Chapter 1.1, Lemma 1.5]. □

Lemma 2.4.5. *Suppose $B = \{b^{\alpha\beta}_{ij}\}_{1\le i,j,\alpha,\beta\le d}$ is 1-periodic and satisfies $b^{\alpha\beta}_{ij} \in L^2_{loc}(\mathbb{R}^d)$ with*

$$\frac{\partial}{\partial y_i} b^{\alpha\beta}_{ij} = 0, \quad \text{and} \quad \int_Q b^{\alpha\beta}_{ij} = 0.$$

There exists $\pi = \{\pi^{\alpha\beta}_{kij}\}_{1\le i,j,k,\alpha,\beta\le d}$ with $\pi^{\alpha\beta}_{kij} \in H^1_{loc}(\mathbb{R}^d)$ that is 1-periodic and satisfies

$$\frac{\partial}{\partial y_k}\pi^{\alpha\beta}_{kij} = b^{\alpha\beta}_{ij} \quad \text{and} \quad \pi^{\alpha\beta}_{kij} = -\pi^{\alpha\beta}_{ikj}.$$

Proof. Given $b^{\alpha\beta}_{ij} \in L^2(\mathbb{R}^d)$, there exist weak solutions $\phi^{\alpha\beta}_{ij} \in H^2(\mathbb{R}^d)$ to the boundary value problem

$$\begin{cases} -\Delta\phi^{\alpha\beta}_{ij} = b^{\alpha\beta}_{ij} & \text{in } Q = [0,1]^d \\ \phi^{\alpha\beta}_{ij} \text{ is 1-periodic,} & \int_Q \phi^{\alpha\beta}_{ij} = 0 \end{cases}$$

for $1 \le i, j, \alpha, \beta \le d$. Set

$$\pi^{\alpha\beta}_{kij} = \frac{\partial\phi^{\alpha\beta}_{kj}}{\partial\xi_i} - \frac{\partial\phi^{\alpha\beta}_{kj}}{\partial\xi_k},$$

and note the anti-symmetry of $\pi_{kij}^{\alpha\beta}$ follows. Moreover, we have

$$\begin{cases} -\Delta\left(\dfrac{\partial \phi_{ij}^{\alpha\beta}}{\partial \xi_i}\right) = 0 \quad \text{in } Q, \\ \dfrac{\partial \phi_{ij}^{\alpha\beta}}{\partial \xi_i} \text{ is 1-periodic,} \quad \displaystyle\int_Q \dfrac{\partial \phi_{ij}^{\alpha\beta}}{\partial \xi_i} = 0, \end{cases} \tag{2.4.3}$$

since

$$\frac{\partial}{\partial \xi_i}\left(-\Delta \phi_{ij}^{\alpha\beta}\right) = \frac{\partial b_{ij}^{\alpha\beta}}{\partial \xi_i}$$

It follows then from energy estimates of boundary value problem (2.4.3) that $\frac{\partial}{\partial \xi_i}\phi_{ij}^{\alpha\beta}$ is identically zero. Hence,

$$\frac{\partial \pi_{kij}^{\alpha\beta}}{\partial \xi_k} = \frac{\partial}{\partial \xi_k}\left(\frac{\partial \phi_{ij}^{\alpha\beta}}{\partial \xi_i}\right) - \Delta \phi_{ij}^{\alpha\beta} = b_{ij}^{\alpha\beta},$$

which completes the proof. □

Theorem 2.4.6 is a classical result in the study of periodically perforated domains. It can be used to prove Korn's first inequality in perforated domains (see Lemma 2.4.7), which is needed together with the Lax-Milgram theorem to prove the existence and uniqueness of weak solutions to (2.2.8) when $\delta = 0$. For a proof of Theorem 2.4.6 when $1 < p < \infty$, see the work of Acerbi, Chaidó Piat, Dal Maso, and Percivale [1]. For an alternative proof when $p = 2$, see [24].

Recall $\Omega_\varepsilon = \Omega \cap \varepsilon\omega$, where $\omega \subseteq \mathbb{R}^d$ is an unbounded, Lipschitz domain as described in Section 2.2. Let Γ_ε denote the material part of the boundary, i.e., $\Gamma_\varepsilon = \partial\Omega \cap \varepsilon\omega$.

Theorem 2.4.6. *Fix $0 < \varepsilon \leq 1$. Let Ω and Ω_0 be a bounded Lipschitz domains with $\overline{\Omega} \subset \Omega_0$ and dist$(\partial\Omega_0, \Omega) > 2$. For $1 \leq p < \infty$, there exists a linear extension operator $P_\varepsilon : W^{1,p}(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \to W_0^{1,p}(\Omega_0; \mathbb{R}^d)$ such that*

$$\|P_\varepsilon w\|_{W^{1,p}(\Omega_0)} \leq C_1 \|w\|_{W^{1,p}(\Omega_\varepsilon)}, \tag{2.4.4}$$

$$\|\nabla P_\varepsilon w\|_{L^p(\Omega_0)} \leq C_2 \|\nabla w\|_{L^p(\Omega_\varepsilon)}, \tag{2.4.5}$$

$$\|e(P_\varepsilon w)\|_{L^p(\Omega_0)} \leq C_3 \|e(w)\|_{L^p(\Omega_\varepsilon)}, \tag{2.4.6}$$

for some constants C_1, C_2, and C_3 depending on Ω and ω, where $e(w)$ denotes the symmetric part of ∇w, i.e.,

$$e(w) = \frac{1}{2}\left[\nabla w + (\nabla w)^T\right]. \tag{2.4.7}$$

Korn's inequalities are classical in the study of linear elasticity. The following lemma is essentially Korn's first inequality but formatted for periodically perforated domains. Lemma 2.4.7 follows from Theorem 2.4.6 and Korn's first inequality.

Lemma 2.4.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. There exists a constant C depending on Ω, ω, and d such that*

$$\|w\|_{H^1(\Omega_\varepsilon)} \leq C\|e(w)\|_{L^2(\Omega_\varepsilon)}$$

for any $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, where $e(w)$ is given by (2.4.7).

Proof. Fix $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, and let $P_\varepsilon : H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d) \to H^1_0(\Omega_0; \mathbb{R}^d)$ denote the extension operator in Theorem 2.4.6. By Korn's first inequality (see [11, Chapter 1.2, Theorem 2.1]) and (2.4.6) of Theorem 2.4.6,

$$\|w\|_{H^1(\Omega_\varepsilon)} \leq \|P_\varepsilon w\|_{H^1(\Omega_\varepsilon)} \leq C\|e(P_\varepsilon w)\|_{L^2(\Omega_0)} \leq C_1\|e(w)\|_{L^2(\Omega_\varepsilon)}.$$

□

Chapter 3 Quantitative homogenization

In Section 3.1, we consider weak solutions to the boundary value problem (2.2.8) with $\delta = 0$ and establish a quantitative convergece result for $u_{\varepsilon,0}$ and its gradient. In particular, we acheive the optimal $\mathcal{O}(\varepsilon^{1/2})$ convergence rate with an energy method and prove a theorem first presented by the author of this disseratation in [25]. Let K_ε denote the smoothing operator at the microscopic scale ε defined by (2.4.1). Let $\eta_\varepsilon \in C_0^\infty(\Omega)$ be the cut-off function defined by

$$\begin{cases} 0 \leq \eta_\varepsilon(x) \leq 1 \quad \text{for } x \in \Omega, \\ \operatorname{supp}(\eta_\varepsilon) \subset \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \geq 3\varepsilon\}, \\ \eta_\varepsilon = 1 \quad \text{on } \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) \geq 4\varepsilon\}, \\ |\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}. \end{cases} \tag{3.0.1}$$

for a bounded domain $\Omega \subset \mathbb{R}^d$. Let χ denote the first-order corrector associated with $\mathcal{L}_{\varepsilon,0}$ (see (3.1.3) in Section 3.1) and $u_{0,0}$ denote the solution to the homogenized boundary value problem associated with (2.2.8) (see (3.1.1) in Section 3.1).

Theorem 3.0.1. *Let Ω be a bounded Lipschitz domain and ω be an unbounded Lipschitz domain with 1-periodic structure, i.e., the characteristic function $\mathbf{1}_+$ of ω satisfies* (2.2.3). *Suppose A is real, measurable, and satisfies* (2.2.4), (2.2.5), *and* (2.2.7). *Fix $f \in H^1(\partial\Omega;\mathbb{R}^d)$. Let $u_{\varepsilon,0}$ denote the weak solution to $\mathcal{L}_{\varepsilon,0}(u_{\varepsilon,0}) = 0$ in Ω with $u_{\varepsilon,0} - f \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, where $\Gamma_\varepsilon = \partial\Omega \cap \varepsilon\omega$ and $\Omega_\varepsilon = \Omega \cap \varepsilon\omega$. There exists a constant C depending on d, Ω, ω, κ_1, and κ_2 such that*

$$\|u_{\varepsilon,0} - u_{0,0} - \varepsilon\chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}.$$

From Theorem 3.0.1 follows a sub-optimal L^2-convergence rate for $u_{\varepsilon,0}$. The following corollary will be used in Section 4.1 to prove large-scale Lipschitz estimates. We reiterate that the convergence rate is suboptimal in $L^2(\Omega)$.

Corollary 3.0.2. *Suppose Ω, ω, A, f, Γ_ε, and $u_{\varepsilon,0}$ are as stated in Theorem 3.0.1. There exists a constant C depending on d, Ω, ω, κ_1, and κ_2 such that*

$$\|u_{\varepsilon,0} - u_{0,0}\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}.$$

In Section 3.2, we consider the boundary value problem (2.2.8) with $0 < \delta \leq 1$ and establish a suboptimal quantitative convergence rate for $u_{\varepsilon,\delta}$ and its gradient. The novelty of the result is not necessarily in the rate itself, but rather that the rate is essentially uniform in δ. Indeed, for fixed $\delta > 0$, the coefficients $k_\delta^\varepsilon A^\varepsilon$ are uniformly elliptic in Ω (but not in δ), and so the optimal H^1-convergence rate remains $\mathcal{O}(\varepsilon^{1/2})$. However, as we will see in Section 4.3, the uniform rate presented is enough to establish large-scale Lipcshitz estimates for solutions to (2.2.8) with $\delta > 0$. Let χ_δ denote the first-order corrector associated with $\mathcal{L}_{\varepsilon,\delta}$ (see (3.2.1) in Section 3.2) and $u_{0,\delta}$ denote the weak solution to the homogenized boundary value problem associated with (2.2.8) (see (3.2.4) in Section 3.2).

Theorem 3.0.3. *Let Ω be a bounded Lipschitz domain and ω be an unbounded Lipschitz domain with 1-periodic structure, i.e., the characteristic function $\mathbf{1}_+$ of ω satisfies* (2.2.3) *Suppose A is real, measurable, and satisfies* (2.2.4), (2.2.5), *and* (2.2.7). *Fix $f \in H^1(\partial\Omega; \mathbb{R}^d)$. Let $u_{\varepsilon,\delta}$ denote a weak solution to $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in Ω with $u_{\varepsilon,\delta} - f \in H^1_0(\Omega; \mathbb{R}^d)$ for some $0 \leq \delta \leq 1$. There exists a constant C depending on κ_1, κ_2, d, Ω, and ω and a $\mu_0 > 0$ depending on κ_1, κ_2, d, and Ω such that*

$$\|k^\varepsilon_\delta r_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|k^\varepsilon_\delta \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} \leq C\varepsilon^{\mu(\delta)}\|f\|_{H^1(\partial\Omega)},$$

where $\mu(\delta) \geq \mu_0 > 0$ for any $0 \leq \delta \leq 1$, k^ε_δ is defined by (2.2.2), *and*

$$r_{\varepsilon,\delta} = u_{\varepsilon,\delta} - u_{0,\delta} - \varepsilon\chi^\varepsilon_\delta K^2_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon). \tag{3.0.2}$$

Note that Theorem 3.0.3 also gives an H^1-convergence rate for $u_{\varepsilon,0}$, but Theorem 3.0.1 is stronger for the case $\delta = 0$. Similar to before, from Theorem 3.0.3 follows a suboptimal L^2-convergence rate for $u_{\varepsilon,\delta}$. We reiterate that the novelty of the convergence rates presented are their uniformity in δ, i.e., in each statement, the constant C is completely independent of δ. The following corollary will be used in Section 4.3 to prove a large-scale Lipschitz estimate.

Corollary 3.0.4. *Suppose Ω, ω, A, f, and $u_{\varepsilon,\delta}$ are as stated in Theorem 3.0.3. There exists a constant C depending on d, Ω, ω, κ_1, and κ_2 and a $\mu_0 > 0$ depending on κ_1, κ_2, d, and Ω such that*

$$\|k^\varepsilon_\delta(u_{\varepsilon,\delta} - u_{0,\delta})\|_{L^2(\Omega)} \leq C\varepsilon^{\mu(\delta)}\|f\|_{H^1(\partial\Omega)},$$

where $\mu(\delta) \geq \mu_0 > 0$ for any $0 \leq \delta \leq 1$.

3.1 Perforated domains ($\delta = 0$)

In this section, we establish $H^1(\Omega_\varepsilon)$-convergence rates for solutions to (2.2.8) with $\delta = 0$ by proving Theorem 3.0.1. Throughout this section we use the notation

$$\Omega_\varepsilon = \Omega \cap \varepsilon\omega.$$

If $\omega = \mathbb{R}^d$, it can be shown that the weak solution to (2.2.8) with $\delta = 0$ converges weakly in $H^1(\Omega; \mathbb{R}^d)$ and consequently strongly in $L^2(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0$ to some $u_{0,0}$, which is a solution of a boundary value problem in the domain Ω [11, 20]. Indeed, we have the known qualitative convergence of Theorem 2.2.1. The function $u_{0,0}$ is called the homogenized solution and the boundary value problem it solves is the homogenized system corresponding to (2.2.8) when $\delta = 0$.

If $\omega \subsetneq \mathbb{R}^d$, then it is difficult to qualitatively discuss the convergence of $u_{\varepsilon,0}$, as $H^1(\Omega_\varepsilon; \mathbb{R}^d)$ and $L^2(\Omega_\varepsilon; \mathbb{R}^d)$ depend explicitly on ε. Qualitative convergence in this case is discussed by Allaire in [2] with the tools of two-scale convergence and Cioranescu and Saint Jean Paulin in [12] withinin the context of various extension theorems like Theorem 2.4.6. The homogenized system of elasticity corresponding to (2.2.8) when $\delta = 0$ and of which $u_{0,0}$ is a solution is given by

$$\begin{cases} \mathcal{L}_{0,0}\left(u_{0,0}\right) = 0 & \text{in } \Omega \\ u_{0,0} = f & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

where

$$\mathcal{L}_{0,0} = -\mathrm{div}(\widehat{A}\nabla) = -\frac{\partial}{\partial x_i}\left(\widehat{a}_{ij}^{\alpha\beta}\frac{\partial}{\partial x_j}\right),$$

$\widehat{A} = \{\widehat{a}_{ij}^{\alpha\beta}\}_{1\leq i,j,\alpha,\beta\leq d}$ denotes a constant matrix given by

$$\widehat{a}_{ij}^{\alpha\beta} = \fint_{Q\cap\omega} a_{ik}^{\alpha\gamma}\frac{\partial \mathbb{X}_j^{\gamma\beta}}{\partial y_k}, \tag{3.1.2}$$

and $\mathbb{X}_j^{\beta} = \{\mathbb{X}_j^{\gamma\beta}\}_{1\leq\gamma\leq d}$ denotes the weak solution to the boundary value problem

$$\begin{cases} \mathcal{L}_{1,0}(\mathbb{X}_j^{\beta}) = 0 & \text{in } Q \\ \chi_j^{\beta} := \mathbb{X}_j^{\beta} - P_j^{\beta} \text{ is 1-periodic}, & \int_{Q\cap\omega}\chi_j^{\beta} = 0. \end{cases} \tag{3.1.3}$$

In (3.1.3), $P_j^{\beta}(y) = y_j e^{\beta}$ and $e^{\beta}\in\mathbb{R}^d$ has a 1 in the βth position and 0 in the remaining positions. In particular, we say $\mathbb{X}_j^{\beta} = \chi_j^{\beta} + P_j^{\beta}$ is a weak solution to the boundary value problem (3.1.3) provided $\chi_j^{\beta}\in H^1_{per}(Q;\mathbb{R}^d)$ satisfies the integral equality

$$\int_{Q\cap\omega} a_{ik}^{\alpha\gamma}\frac{\partial}{\partial y_k}\left(\chi_j^{\gamma\beta} + P_j^{\gamma\beta}\right)\frac{\partial M^{\alpha}}{\partial y_i}\,dy = 0, \quad \text{for any } M = \{M^{\alpha}\}\in H^1_{per}(Q;\mathbb{R}^d).$$

For details on the existence of solutions to (3.1.3), see [24]. The functions χ_j^{β} are referred to as the first-order correctors for the system (2.2.8) when $\delta = 0$.

It is assumed throughout that any two connected components of $\mathbb{R}^d\backslash\omega$ are separated by some positive distance. Specifically, if $\mathbb{R}^d\backslash\omega = \cup_{k=1}^{\infty}H_k$ where H_k is connected and bounded for each k, then there exists a constant $\mathfrak{g}^{\omega}$ so that

$$0 < \mathfrak{g}^{\omega} \leq \inf_{i\neq j}\left\{\inf_{\substack{x_i\in H_i\\ x_j\in H_j}}|x_i - x_j|\right\}. \tag{3.1.4}$$

Such a constant is guaranteed to exist given the periodic structure of ω. With any weaker assumption, such as almost-periodic or random, the constant $\mathfrak{g}^{\omega}$ may not be positive.

It should be noted that if A satisfies (2.2.4) and (2.2.5), then $\widehat{A}$ defined by (3.1.2) satisfies conditions (2.2.4) and (2.2.5) but with possibly different constants $\widehat{\kappa}_1$ and $\widehat{\kappa}_2$ depending on κ_1 and κ_2. In particular, we have the following lemma. For a rigorous proof, see [24]. Note the following lemma also shows $\widehat{a}_{ij}^{\alpha\beta}\in L^{\infty}(\mathbb{R}^d)$ for each index $1\leq i,j,\alpha,\beta\leq d$.

Lemma 3.1.1. *Suppose A satisfies* (2.2.4), (2.2.5), *and* (2.2.7). *If $\mathbb{X}_j^\beta = \{\mathbb{X}_j^{\gamma\beta}\}_\gamma$ denote the weak solutions to* (3.1.3), *then $\widehat{A} = \{\widehat{a}_{ij}^{\alpha\beta}\}$ defined by* (3.1.2) *satisfies $\widehat{a}_{ij}^{\alpha\beta} = \widehat{a}_{ji}^{\beta\alpha} = \widehat{a}_{\alpha j}^{i\beta}$ and*

$$\widehat{\kappa}_1|\xi|^2 \leq \widehat{a}_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \leq \widehat{\kappa}_2|\xi|^2$$

for some $\widehat{\kappa}_1, \widehat{\kappa}_2 > 0$ depending κ_1, κ_2, and $|Q \cap \omega|$ and any symmetric matrix $\xi = \{\xi_i^\alpha\}_{i,\alpha}$.

Let K_ε be defined as in Section 2.4.1, and let η_ε be given by (3.0.1). If P_ε is the linear extension operator provided by Theorem 2.4.6, then we write $\widetilde{w} = P_\varepsilon w$ for $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$. Throughout, C denotes a harmless constant independent of ε that may change from line to line.

Lemma 3.1.2. *Let*

$$r_\varepsilon = u_{\varepsilon,0} - u_{0,0} - \varepsilon\chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon).$$

Then

$$\begin{aligned}
&\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w \\
&\quad = |Q\cap\omega| \int_\Omega \widehat{A}\nabla u_{0,0} \cdot \nabla\eta_\varepsilon \widetilde{w} - |Q\cap\omega| \int_\Omega (1-\eta_\varepsilon)\widehat{A}\nabla u_{0,0}\cdot\nabla\widetilde{w} \\
&\qquad + \int_\Omega \left[|Q\cap\omega|\widehat{A} - \boldsymbol{1}_+^\varepsilon A^\varepsilon\right]\left[\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right]\cdot\nabla\widetilde{w} \\
&\qquad + \int_\Omega \left[|Q\cap\omega|\widehat{A} - \boldsymbol{1}_+^\varepsilon A^\varepsilon \nabla\mathbb{X}^\varepsilon\right] K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\cdot\nabla\widetilde{w} \\
&\qquad - \varepsilon\int_{\Omega_\varepsilon} A^\varepsilon\chi^\varepsilon\nabla K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\cdot\nabla w
\end{aligned}$$

for any $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$.

Proof. Fix $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, where $\Omega_\varepsilon = \Omega \cap \varepsilon\omega$ and $\Gamma_\varepsilon = \partial\Omega \cap \varepsilon\omega$. If η_ε is defined by (3.0.1), then $\widetilde{w}\eta_\varepsilon \in H_0^1(\Omega; \mathbb{R}^d)$. Since $u_{\varepsilon,0}$ and $u_{0,0}$ are weak solutions to (2.2.8) with $\delta = 0$ and (3.1.1), respectively,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_{\varepsilon,0}\cdot\nabla w = 0$$

and

$$|Q\cap\omega| \int_\Omega \widehat{A}\nabla u_{0,0}\cdot\nabla(\widetilde{w}\eta_\varepsilon) = 0.$$

Therefore,

$$
\begin{aligned}
\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w &= \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_{\varepsilon,0} \cdot \nabla w - \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_{0,0} \cdot \nabla w \\
&\qquad - \int_{\Omega_\varepsilon} A^\varepsilon \nabla \left[\varepsilon \chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right] \cdot \nabla w \\
&= |Q \cap \omega| \int_\Omega \widehat{A} \nabla u_{0,0} \cdot \nabla(\widetilde{w}\eta_\varepsilon) - \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_{0,0} \cdot \nabla w \\
&\qquad - \int_{\Omega_\varepsilon} A^\varepsilon \nabla \left[\varepsilon \chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right] \cdot \nabla w
\end{aligned}
$$

Note the equalities

$$
\nabla(\widetilde{w}\eta_\varepsilon) = \nabla \eta_\varepsilon \widetilde{w} - \nabla \widetilde{w}(1-\eta_\varepsilon) + \nabla \widetilde{w}
$$

and

$$
\nabla \left[\varepsilon \chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right] = \nabla \chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) + \varepsilon \chi^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)
$$

and so

$$
\begin{aligned}
\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w &= |Q \cap \omega| \int_\Omega \widehat{A} \nabla u_{0,0} \cdot \nabla \eta_\varepsilon \widetilde{w} - |Q \cap \omega| \int_\Omega (1-\eta_\varepsilon) \widehat{A} \nabla u_{0,0} \cdot \nabla \widetilde{w} \\
&\qquad + |Q \cap \omega| \int_\Omega \widehat{A} \left[\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right] \cdot \nabla \widetilde{w} \\
&\qquad + |Q \cap \omega| \int_\Omega \widehat{A} K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \cdot \nabla \widetilde{w} - \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_{0,0} \cdot \nabla w \\
&\qquad - \int_{\Omega_\varepsilon} A^\varepsilon \nabla \left[\varepsilon \chi^\varepsilon K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right] \cdot \nabla w \\
&= |Q \cap \omega| \int_\Omega \widehat{A} \nabla u_{0,0} \cdot \nabla \eta_\varepsilon \widetilde{w} - |Q \cap \omega| \int_\Omega (1-\eta_\varepsilon) \widehat{A} \nabla u_{0,0} \cdot \nabla \widetilde{w} \\
&\qquad + \int_\Omega \left[|Q \cap \omega| \widehat{A} - \mathbf{1}_+^\varepsilon A^\varepsilon\right] \left[\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\right] \cdot \nabla \widetilde{w} \\
&\qquad + \int_\Omega \left[|Q \cap \omega| \widehat{A} - \mathbf{1}_+^\varepsilon A^\varepsilon - \mathbf{1}_+^\varepsilon A^\varepsilon \nabla \chi^\varepsilon\right] K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \cdot \nabla \widetilde{w} \\
&\qquad - \varepsilon \int_{\Omega_\varepsilon} A^\varepsilon \chi^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \cdot \nabla w,
\end{aligned}
$$

where $\mathbf{1}_+^\varepsilon = \mathbf{1}_+(\cdot/\varepsilon)$ denotes the characteristic function of $\varepsilon\omega$. This is the desired inequality. □

Lemma 3.1.3 presented below is used in the proof of Lemma 3.1.4, which establishes a Poincaré-type inequality for the perforated domain Ω_ε. We use the notation $\Delta(x,r) = B(x,r) \cap \partial\Omega$ to denote a surface ball of $\partial\Omega$. Essentially, the lemma shows that for any point along the boundary of Ω there exists a ball of fixed radius containing a consistent size of the material part of the boundary Γ_ε.

Lemma 3.1.3. *For sufficiently small ε, there exist $r_0, \rho_0 > 0$ depending only on ω such that for any $x \in \partial\Omega$,*

$$\Delta\,(y, \varepsilon\rho_0) \subset \Delta(x, \varepsilon r_0) \text{ and } \overline{\Delta\,(y, \varepsilon\rho_0)} \subset \Gamma_\varepsilon$$

for some $y \in \Gamma_\varepsilon = \partial\Omega \cap \varepsilon\omega$.

Proof. Write $\mathbb{R}^d \backslash \omega = \cup_{j=1}^{\infty} H_j$, where each H_j is connected and bounded by assumption. Since ω is 1-periodic, there exists a constant $M < \infty$ such that

$$\sup_{j\geq 1} \{\operatorname{diam} H_j\} \leq M.$$

Take

$$r_0 = 2 \max \{\mathfrak{g}^\omega, M\}, \tag{3.1.5}$$

where $\mathfrak{g}^\omega$ is defined by (3.1.4). Set $\rho_0 = \frac{1}{16}\mathfrak{g}^\omega$. Let

$$\widetilde{H}_j = \left\{ z \in \mathbb{R}^d \,:\, \operatorname{dist}(z, H_j) < \frac{1}{4}\mathfrak{g}^\omega \right\} \text{ for each } j,$$

and fix $x \in \partial\Omega$. If $x \in \partial\Omega \backslash (\cup_{j=1}^{\infty} \varepsilon\widetilde{H}_j)$, then take $y = x$. Indeed, for any $z \in \Delta(y, \varepsilon\rho_0) \subset \Delta(x, \varepsilon r_0)$ and any positive integer k,

$$\begin{aligned}
\operatorname{dist}(z, \varepsilon H_k) &\geq \operatorname{dist}(y, \varepsilon H_k) - |y - z| \\
&\geq \varepsilon \frac{1}{4}\mathfrak{g}^\omega - \varepsilon\rho_0 \\
&\geq \varepsilon \left\{ \frac{1}{4}\mathfrak{g}^\omega - \frac{1}{16}\mathfrak{g}^\omega \right\} \\
&\geq \varepsilon \frac{3}{16}\mathfrak{g}^\omega,
\end{aligned}$$

and so $\overline{\Delta(y, \varepsilon\rho_0)} \subset \Gamma_\varepsilon$.

Suppose $x \in \partial\Omega \cap (\cup_{j=1}^{\infty} \varepsilon\widetilde{H}_j)$. There exists a positive integer k such that $x \in \varepsilon\widetilde{H}_k$. Moreover, $\varepsilon\widetilde{H}_k \subset B(x, \varepsilon r_0)$ since for any $z \in \varepsilon\widetilde{H}_k$ we have

$$\begin{aligned}
|x - z| &\leq \operatorname{dist}(x, \varepsilon H_k) + \operatorname{diam}\,(\varepsilon H_k) + \operatorname{dist}(z, \varepsilon H_k) \\
&\leq \varepsilon \frac{1}{4}\mathfrak{g}^\omega + \varepsilon M + \varepsilon \frac{1}{4}\mathfrak{g}^\omega \\
&< \varepsilon\mathfrak{g}^\omega + \varepsilon M \\
&< \varepsilon r_0.
\end{aligned}$$

In this case, choose $y \in \varepsilon(\widetilde{H}_k \backslash H_k)$ so that $\operatorname{dist}(y, \varepsilon H_k) = \varepsilon(1/8)\mathfrak{g}^\omega$ and $y \in \partial\Omega$. Then for any $z \in \Delta(y, \varepsilon\rho_0) \subset [\partial\Omega \cap \varepsilon(\widetilde{H}_k \backslash H_k)] \subset \Delta(x, \varepsilon r_0)$,

$$\begin{aligned}
\operatorname{dist}(z, \varepsilon H_k) &\geq \operatorname{dist}(y, \varepsilon H_k) - |y - z| \\
&\geq \varepsilon \frac{1}{8}\mathfrak{g}^\omega - \varepsilon \frac{1}{16}\mathfrak{g}^\omega \\
&\geq \varepsilon \frac{1}{16}\mathfrak{g}^\omega,
\end{aligned}$$

and so $\overline{\Delta(y, \varepsilon\rho_0)} \subset \Gamma_\varepsilon$. □

Lemma 3.1.4. *For $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$,*

$$\|\widetilde{w}\|_{L^2(\mathcal{O}_{4\varepsilon})} \leq C\varepsilon \|\nabla \widetilde{w}\|_{L^2(\Omega)},$$

where $\mathcal{O}_{4\varepsilon} = \{x \in \Omega : dist(x, \partial\Omega) < 4\varepsilon\}$ and C depends on d, Ω, and ω.

Proof. We cover $\partial\Omega$ with the surface balls $\Delta(x, \varepsilon r_0)$ provided in Lemma 3.1.3 and partition the region $\mathcal{O}_{4\varepsilon}$. In particular, let r_0 denote the constant given by Lemma 3.1.3, and note $\cup_{x\in\partial\Omega}\Delta(x, \varepsilon r_0)$ covers $\partial\Omega$, which is compact. Then there exists $\{x_i\}_{i=1}^N$ with $\partial\Omega \subset \cup_{i=1}^N \Delta(x_i, \varepsilon r_0)$, where $N = N(\varepsilon)$. Write

$$\mathcal{O}_{4\varepsilon}^{(i)} = \{x \in \Omega : \text{dist}(x, \Delta_i) < 4\varepsilon\}, \text{ where } \Delta_i = \Delta(x_i, \varepsilon r_0).$$

Given that Ω is a Lipschitz domain, there exists a positive integer $M < \infty$ independent of ε such that $\mathcal{O}_{4\varepsilon}^{(i)} \cap \mathcal{O}_{4\varepsilon}^{(j)} \neq \emptyset$ for at most M positive integers j different from i.

Set $W(x) = \widetilde{w}(\varepsilon x)$. Note for each $1 \leq i \leq N$, by Lemma 3.1.4 there exists a $y_i \in \mathcal{O}_{4\varepsilon}^{(i)}$ such that $\widetilde{w} \equiv 0$ on $\Delta(y_i, \varepsilon\rho_0) \subset \Delta_i$. Hence, by Poincaré's inequality [23, Theorem 1],

$$\left(\int_{\mathcal{O}_{4\varepsilon}^{(i)}/\varepsilon} |W|^2\right)^{1/2} \leq C \left(\int_{\mathcal{O}_{4\varepsilon}^{(i)}/\varepsilon} |\nabla W|^2\right)^{1/2}, \tag{3.1.6}$$

where C depends on Ω, r_0, and ρ_0 but is independent of ε and i. Specifically,

$$\int_{\mathcal{O}_{4\varepsilon}} |\widetilde{w}(x)|^2\, dx \leq C\varepsilon^2 \sum_{i=1}^N \int_{\mathcal{O}_{4\varepsilon}^{(i)}} |\nabla \widetilde{w}(x)|^2\, dx \leq C_1\varepsilon^2 \int_{\mathcal{O}_{4\varepsilon}} |\nabla \widetilde{w}(x)|^2\, dx$$

where we've made the change of variables $\varepsilon x \mapsto x$ in (3.1.6) and C_1 is a constant depending on Ω, ω, and M but independent of ε.

□

Lemma 3.1.5. *For any $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$,*

$$\left|\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w\right| \leq C\left\{\|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \|(\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \right.$$
$$\left. + \varepsilon\|K_\varepsilon\big((\nabla^2 u_{0,0})\eta_\varepsilon\big)\|_{L^2(\Omega)}\right\} \|w\|_{H^1(\Omega_\varepsilon)}$$

Proof. By Lemma 3.1.2,

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.1.7}$$

where

$$
\begin{aligned}
I_1 &= |Q \cap \omega| \int_\Omega \widehat{A} \nabla u_{0,0} \cdot \nabla \eta_\varepsilon \widetilde{w}, \\
I_2 &= -|Q \cap \omega| \int_\Omega (1-\eta_\varepsilon) \widehat{A} \nabla u_{0,0} \cdot \nabla \widetilde{w}, \\
I_3 &= \int_\Omega \left[|Q \cap \omega| \widehat{A} - \mathbf{1}_+^\varepsilon A^\varepsilon \right] \left[\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \right] \cdot \nabla \widetilde{w}, \\
I_4 &= \int_\Omega \left[|Q \cap \omega| \widehat{A} - \mathbf{1}_+^\varepsilon A^\varepsilon \nabla \mathbb{X}^\varepsilon \right] K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \cdot \nabla \widetilde{w}, \\
I_5 &= -\varepsilon \int_{\Omega_\varepsilon} A^\varepsilon \chi^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \cdot \nabla w,
\end{aligned}
$$

and $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$. According to (3.0.1), $\operatorname{supp}(\nabla \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$, where $\mathcal{O}_{4\varepsilon}$ is defined in Lemma 3.1.4. Moreover, $|\nabla \eta_\varepsilon| \leq C\varepsilon^{-1}$. Hence, Lemma 3.1.4, Lemma 3.1.1, Cauchy-Schwarz, and (3.0.1) imply

$$
\begin{aligned}
|I_1| \leq C\varepsilon^{-1} \int_{\mathcal{O}_{4\varepsilon}} |\widehat{A} \nabla u_{0,0} \cdot \widetilde{w}| &\leq C\varepsilon^{-1} \|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} \|\widetilde{w}\|_{L^2(\mathcal{O}_{4\varepsilon})} \\
&\leq C \|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} \|\nabla \widetilde{w}\|_{L^2(\Omega)}
\end{aligned}
$$

Since $\operatorname{supp}(1-\eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$ and $\eta_\varepsilon \leq 1$, Lemma 3.1.1 implies

$$
|I_2| \leq C \int_{\mathcal{O}_{4\varepsilon}} \left| \widehat{A} \nabla u_{0,0} \cdot \nabla \widetilde{w} \right| \leq C \|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} \|\nabla \widetilde{w}\|_{L^2(\Omega)}.
$$

By Theorem 2.4.6,

$$
|I_1 + I_2| \leq C \|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} \|w\|_{H^1(\Omega_\varepsilon)}. \tag{3.1.8}
$$

Note

$$
\begin{aligned}
\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) &= (1-\eta_\varepsilon)\nabla u_{0,0} + (\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \\
&= (1-\eta_\varepsilon)\nabla u_{0,0} + (\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon) \\
&\quad - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon))
\end{aligned}
$$

Again, since $\operatorname{supp}(1-\eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$ (see (3.0.1)), Lemma 2.4.1 implies

$$
\begin{aligned}
&\|\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \\
&\qquad \leq \|(1-\eta_\varepsilon)\nabla u_{0,0}\|_{L^2(\Omega)} + \|(\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \\
&\qquad\qquad + \|K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon))\|_{L^2(\Omega)} \\
&\qquad \leq \|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} + C\|(\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)}.
\end{aligned}
$$

Therefore, by (2.2.5), Lemma 3.1.1, and Cauchy-Schwarz,

$$
\begin{aligned}
|I_3| &\leq C\|\nabla u_{0,0} - K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \|w\|_{H^1(\Omega_\varepsilon)} \\
&\leq C \left\{ \|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} \right. \\
&\qquad \left. + \|(\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|w\|_{H^1(\Omega_\varepsilon)}.
\end{aligned} \tag{3.1.9}
$$

Set $B = |Q \cap \omega|\widehat{A} - \mathbf{1}_+ A\nabla\mathbb{X}$. By (3.1.1) and (3.1.3), B satisfies the assumptions of Lemma 2.4.5. Therefore, there exists $\pi = \{\pi_{kij}^{\alpha\beta}\}$ that is 1-periodic with

$$\frac{\partial}{\partial y_k}\pi_{kij}^{\alpha\beta} = b_{ij}^{\alpha\beta} \quad \text{and} \quad \pi_{kij}^{\alpha\beta} = -\pi_{ikj}^{\alpha\beta},$$

where $y = x/\varepsilon$ and

$$b_{ij}^{\alpha\beta} = |Q \cap \omega|\widehat{a}_{ij}^{\alpha\beta} - \mathbf{1}_+ a_{ik}^{\alpha\gamma}\frac{\partial}{\partial y_k}\mathbb{X}_j^{\gamma\beta}.$$

Moreover, $\|\pi_{ij}^{\alpha\beta}\|_{H^1(Q)} \leq C$ for some constant C depending on κ_1, κ_2, and ω. Note for $x \in \mathbb{R}^d$,

$$b_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) = \frac{\partial}{\partial y_k}\pi_{kij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right) = \varepsilon\frac{\partial}{\partial x_k}\pi_{kij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right).$$

Recalling (3.0.1), also notice $K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon) \in H_0^1(\Omega;\mathbb{R}^d)$. Hence, integrating by parts gives

$$\begin{aligned}
\int_\Omega b_{ij}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_{0,0}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial \widetilde{w}^\alpha}{\partial x_i} &= \varepsilon\int_\Omega \frac{\partial}{\partial x_k}\pi_{kij}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_{0,0}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial \widetilde{w}^\alpha}{\partial x_i} \\
&= -\varepsilon\int_\Omega \pi_{kij}^{\alpha\beta\varepsilon}\frac{\partial}{\partial x_k}\left[K_\varepsilon^2\left(\frac{\partial u_{0,0}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial \widetilde{w}^\alpha}{\partial x_i}\right] \\
&= -\varepsilon\int_\Omega \pi_{kij}^{\alpha\beta\varepsilon}\frac{\partial}{\partial x_k}\left[K_\varepsilon^2\left(\frac{\partial u_{0,0}^\beta}{\partial x_j}\eta_\varepsilon\right)\right]\frac{\partial \widetilde{w}^\alpha}{\partial x_i},
\end{aligned}$$

since

$$\int_\Omega \pi_{kij}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_{0,0}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial^2 \widetilde{w}^\alpha}{\partial x_k \partial x_i} = 0$$

due to the anit-symmetry of π. Thus, by Lemmas 2.4.1, 2.4.3, and 2.4.5,

$$\begin{aligned}
|I_4| &\leq C\varepsilon\|\pi^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)}\|w\|_{H^1(\Omega_\varepsilon)} \\
&\leq C\varepsilon\|\pi\|_{L^2(Q)}\|\nabla K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)}\|w\|_{H^1(\Omega_\varepsilon)} \\
&\leq C\|K_\varepsilon\big((\nabla^2 u_{0,0})\eta_\varepsilon\big) + K_\varepsilon((\nabla u_{0,0})\nabla\eta_\varepsilon)\|_{L^2(\Omega)}\|w\|_{H^1(\Omega_\varepsilon)} \\
&\leq C\left\{\|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \varepsilon\|K_\varepsilon\big((\nabla^2 u_{0,0})\eta_\varepsilon\big)\|_{L^2(\Omega)}\right\}\|w\|_{H^1(\Omega_\varepsilon)}, \qquad (3.1.10)
\end{aligned}$$

where we've used (3.0.1) in each step.

Finally, by (2.2.5) and Lemmas 2.4.1, 2.4.2, and 2.4.3,

$$|I_5| \leq C\left\{\|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \varepsilon\|K_\varepsilon\big((\nabla^2 u_{0,0})\eta_\varepsilon\big)\|_{L^2(\Omega)}\right\}\|w\|_{H^1(\Omega_\varepsilon)} \qquad (3.1.11)$$

The desired estimate follows from (3.1.7)-(3.1.11). □

Lemma 3.1.6. *For $w \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$,*

$$\left|\int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla w\right| \leq C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}\|w\|_{H^1(\Omega_\varepsilon)}$$

Proof. Recall that $u_{0,0}$ satisfies $\mathcal{L}_{0,0}(u_{0,0}) = 0$ in Ω, and so it follows from estimates for solutions in Lipschitz domains for constant-coefficient equations that

$$\|(\nabla u_{0,0})^*\|_{L^2(\partial\Omega)} \le C\|f\|_{H^1(\partial\Omega)}, \tag{3.1.12}$$

where $(\nabla u_{0,0})^*$ denotes the nontangential maximal function of $\nabla u_{0,0}$ (see [14]). In particular,

$$(\nabla u_{0,0})^*(x) = \sup_{y\in T_\rho(x)} \nabla u_{0,0}(y), \;\; T_\rho(x) = \{y \in \Omega \,:\, |y-x| \le \rho\,\mathrm{dist}(y,\partial\Omega)\},$$

where $\rho > 0$ and $x \in \partial\Omega$. By the coarea formula,

$$\|\nabla u_{0,0}\|_{L^2(\mathcal{O}_{4\varepsilon})} \le C\varepsilon^{1/2}\|(\nabla u_{0,0})^*\|_{L^2(\partial\Omega)} \le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}. \tag{3.1.13}$$

Notice that if $u_{0,0}$ solves (2.2.8), then $\mathcal{L}_{0,0}(\nabla u_{0,0}) = 0$ in Ω, and so we may use an interior Lipschitz estimate for $\mathcal{L}_{0,0}$. That is,

$$|\nabla^2 u_{0,0}(x)| \le \frac{C}{\delta(x)}\left(\fint_{B(x,\delta(x)/8)} |\nabla u_{0,0}|^2\right)^{1/2}, \tag{3.1.14}$$

where $\delta(x) = \mathrm{dist}(x,\partial\Omega)$ (see [18]). By (3.0.1), we have

$$\begin{aligned}
\|(\nabla^2 u_{0,0})\eta_\varepsilon\|_{L^2(\Omega)} &\le \left(\int_{\Omega\setminus\mathcal{O}_{3\varepsilon}} |\nabla^2 u_{0,0}|^2\right)^{1/2} \\
&\le C\left(\int_{\Omega\setminus\mathcal{O}_{3\varepsilon}} \fint_{B(x,\delta(x)/8)} \left|\frac{\nabla u_{0,0}(y)}{\delta(x)}\right|^2 dy\,dx\right)^{1/2} \\
&\le C\left(\int_{3\varepsilon}^{C_0} t^{-2}\int_{\partial\mathcal{O}_t\cap\Omega}\fint_{B(x,t/8)} |\nabla u_{0,0}(y)|^2\,dy\,dS(x)\,dt\right)^{1/2} \\
&\qquad\qquad + C_1\left(\int_{\Omega\setminus\mathcal{O}_{C_0}} |\nabla u_{0,0}|^2\right)^{1/2} \\
&\le C\|(\nabla u_{0,0})^*\|_{L^2(\partial\Omega)}\left(\int_{3\varepsilon}^{C_0} t^{-2}\,dt\right)^{1/2} + C_1\|\nabla u_{0,0}\|_{L^2(\Omega)} \\
&\le C\left\{\varepsilon^{-1/2}\|f\|_{H^1(\partial\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}\right\} \\
&\le C\varepsilon^{-1/2}\|f\|_{H^1(\partial\Omega)}.
\end{aligned} \tag{3.1.15}$$

where C_0 is a constant depending on the diameter of Ω, and we've used (3.1.12), (3.1.13), the coarea formula, energy estimates, and (3.1.14). Hence, Lemma 2.4.1 implies

$$\varepsilon\|K_\varepsilon\big((\nabla^2 u_{0,0})\eta_\varepsilon\big)\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}. \tag{3.1.16}$$

Finally, by Lemma 2.4.2,

$$\|(\nabla u_{0,0})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}. \tag{3.1.17}$$

where the last inequality follows from (3.1.15) and Lemmas 2.4.4 and 2.4.2. Equations (3.1.13), (3.1.16), and (3.1.17) together with Lemma 3.1.5 give the desired estimate. □

Proof of Theorem 3.0.1. Note r_ε as defined in Lemma 3.1.2 satsifies $r_\varepsilon \in H^1(\Omega_\varepsilon, \Gamma_\varepsilon; \mathbb{R}^d)$, and so by Lemma 3.1.6 and (2.2.5),

$$\begin{aligned}\|e(r_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} &\le C \int_{\Omega_\varepsilon} A^\varepsilon \nabla r_\varepsilon \cdot \nabla r_\varepsilon \\ &\le C\varepsilon^{1/2} \|f\|_{H^1(\partial\Omega)} \|r_\varepsilon\|_{H^1(\Omega_\varepsilon)}.\end{aligned}$$

Lemma 2.4.7 gives the desired estimate. □

Proof of Corollary 3.0.2. With Theorem 3.0.1 at our disposal, Corollary 3.0.2 follows with the addition of Lemmas 2.4.3, 2.4.1 and energy estimates. Indeed,

$$\begin{aligned}\|u_{\varepsilon,0} - u_{0,0}\|_{L^2(\Omega)} &\le \|u_{\varepsilon,0} - u_{0,0} - \varepsilon\chi^\varepsilon K^2_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} + \varepsilon\|\chi^\varepsilon K^2_\varepsilon((\nabla u_{0,0})\eta_\varepsilon)\|_{L^2(\Omega)} \\ &\le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)} + C\varepsilon\|f\|_{H^{1/2}(\partial\Omega)} \\ &\le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}\end{aligned}$$

□

3.2 Soft inclusions ($0 < \delta \le 1$)

In this section, we quantitatively discuss the convergence of solutions to (2.2.8) as $\varepsilon \to 0$ but $\delta > 0$ by proving Theorem 3.0.3. First, we discuss the ellipticity of the homogenized coefficients $\widehat{A}_\delta$ which is shown to be uniform in δ. Next, we prove the theorem.

We now discuss the qualitative convergence of $u_{\varepsilon,\delta}$. Indeed, if $\delta > 0$, then the matrix of coefficients $k^\varepsilon_\delta A^\varepsilon$ is positive definite, i.e., satisfies (2.2.4). However, if one relies only on the positivity of δ and the ellipticity of A^ε, the lower-ellipticity constant (e.g., κ_1 in (2.2.4)) depends on δ. If δ is close to zero, then the coefficients $k^\varepsilon_\delta A^\varepsilon$ are ill-conditioned, which leads to very loose control on the solution $u_{\varepsilon,\delta}$ and its deriviatives. Nevertheless, we may deduce the existence of a homogenized solution $u_{0,\delta}$ and the convergence of $u_{\varepsilon,\delta}$ as $\varepsilon \to 0$. In particular, we have Theorem 2.2.1. In this section, we propose a quantitative convergence rate that is essentially uniform in δ.

For $\delta \ge 0$, let $\chi^\beta_{j,\delta} = \{\chi^{\gamma\beta}_{j,\delta}\}_{1\le\gamma\le d}$ denote the solution to the following variational problem

$$\begin{cases} \displaystyle\int_Q k_\delta a^{\alpha\gamma}_{ik} \frac{\partial \mathbb{X}^{\gamma\beta}_{j,\delta}}{\partial y_k} \frac{\partial \phi^\alpha}{\partial y_i}\, dy = 0, \quad \text{for any } \phi \in H^1_{\mathrm{per}}(Q;\mathbb{R}^d) \\ \chi^\beta_{j,\delta} := \mathbb{X}^\beta_{j,\delta} - y_j e^\beta \text{ is 1-periodic}, \quad \displaystyle\int_Q \chi^\beta_{j,\delta} = 0, \end{cases} \tag{3.2.1}$$

which coincides with (3.1.3) if $\delta = 0$, i.e., $\mathbb{X}^\beta_{j,0} \equiv \mathbb{X}^\beta_j$ and $\chi^\beta_{j,0} \equiv \chi^\beta_j$ where χ^β_j and $\mathbb{X}^\beta_j$ are defined in (3.1.3). To show the existence and uniqueness of the solutions $\chi^\beta_{j,\delta}$, we may apply the Lax-Milgram theorem to the space $H^1_{per}(Q;\mathbb{R}^d)$. As a consequence, with the correct choice of test functions one may obtain the bound

$$\|k_\delta \chi^\beta_{j,\delta}\|_{L^2(Q)} + \|k_\delta \nabla\chi^\beta_{j,\delta}\|_{L^2(Q)} \le C \tag{3.2.2}$$

for some constant C depending on κ_1, κ_2, and ω.

Define the constant matrix $\widehat{A}_\delta = \{a^{\alpha\beta}_{ij,\delta}\}$ by

$$\widehat{a}^{\alpha\beta}_{ij,\delta} = \fint_Q k_\delta a^{\alpha\gamma}_{ik} \frac{\partial \mathbb{X}^{\gamma\beta}_{j,\delta}}{\partial x_k}\, dy, \tag{3.2.3}$$

where $\mathbb{X}^\beta_{j,\delta}$ is defined in (3.2.1). Note the effective coefficients defined by (3.2.1) coincide with (3.1.2) when $\delta = 0$, i.e., $\widehat{A}_0 \equiv \widehat{A}$ where $\widehat{A}$ is defined by (3.1.2). The constant matrix $\widehat{A}_\delta$ is uniformly elliptic uniformly in δ, which is discussed below. Let $u_{0,\delta}$ denote the solution to the homogenized boundary value problem corresponding to (2.2.8) with $\delta \geq 0$, i.e., $u_{0,\delta}$ satisfies

$$\begin{cases} \mathcal{L}_{0,\delta}(u_{0,\delta}) = 0 & \text{in } \Omega \\ u_{0,\delta} = f & \text{on } \partial\Omega, \end{cases} \tag{3.2.4}$$

where $\mathcal{L}_{0,\delta} = -\text{div}(\widehat{A}_\delta \nabla)$ and $\widehat{A}_\delta$ is defined by (3.2.3).

If A satisfies (2.2.4) and (2.2.5), then $\widehat{A}_\delta$ defined by (3.2.3) satisfies conditions (2.2.4) and (2.2.5) but with possibly different constants $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ depending on κ_1 and κ_2 but not δ. In particular, we have the following lemma.

Lemma 3.2.1. *Let $\widehat{A}_\delta$ be defined by* (3.2.3) *for $0 \leq \delta \leq 1$. Then*

$$\widehat{a}^{\alpha\beta}_{ij,\delta}(y) = \widehat{a}^{\beta\alpha}_{ji,\delta}(y) = \widehat{a}^{i\beta}_{\alpha j,\delta}(y)$$
$$\widetilde{\kappa}_1 |\xi|^2 \leq a^{\alpha\beta}_{ij,\delta}(y) \xi^\alpha_i \xi^\beta_j \leq \widetilde{\kappa}_2 |\xi|^2$$

for any symmetric matrix $\xi = \{\xi^\alpha_i\}$, where $\widetilde{\kappa}_1, \widetilde{\kappa}_2 > 0$ depend on κ_1, κ_2, and $|Q \cap \omega|$.

Lemma 3.2.1 follows from the following two lemmas. The first discusses the convergence of $\chi^\beta_{j,\delta}$ for each $1 \leq j, \beta \leq d$ as $\delta \to 0$, and the second discusses the convergence of $\widehat{A}_\delta$ to $\widehat{A}_0$ as $\delta \to 0$. As $\widehat{A}_0$ is known to be uniformly elliptic (see Lemma 3.1.1 and note $\widehat{A}_0 \equiv \widehat{A}$), we obtain Lemma 3.2.1.

Lemma 3.2.2. *If $\mathbb{X}_0 = \{\mathbb{X}^\beta_{j,0}\}_{1 \leq j,\beta \leq d}$, $\mathbb{X}_\delta = \{\mathbb{X}^\beta_{j,\delta}\}_{1 \leq j,\beta \leq d}$ are defined by* (3.1.3) *and* (3.2.1)*, respectively, then for $\delta > 0$ we have the following estimates:*

(i) $\|\mathbf{1}_+ \nabla(\mathbb{X}_0 - \mathbb{X}_\delta)\|_{L^2(Q)} \leq C_1 \delta^{1/2}$,

(ii) $\|\mathbf{1}_- \nabla \mathbb{X}_\delta\|_{L^2(Q)} \leq C_2 \delta^{-1/4}$,

where C_1, C_2 depend on κ_1 and κ_2.

Proof. Set $\widetilde{\chi}^\beta_{j,0} = P\chi^\beta_{j,0} \in H^1_{per}(Q; \mathbb{R}^d)$ for each $1 \leq j, \beta \leq d$, where P is the linear extension operator given in [24]. Let

$$\widetilde{\mathbb{X}}^\beta_{j,0}(y) = y_j e^\beta + \widetilde{\chi}^\beta_{j,0}(y).$$

Recall that $\mathbf{1}_+\widetilde{\mathbb{X}}_0$ is a weak solution to (3.1.3) and $\mathbb{X}_\delta$ satisfies (3.2.1), and so for any $\phi \in H^1_{per}(Q;\mathbb{R}^d)$ we have

$$\int_Q k_\delta A\nabla(\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta)\cdot\nabla\phi = \delta\int_Q \mathbf{1}_-\nabla\widetilde{\mathbb{X}}_0\cdot\nabla\phi$$

Note

$$\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta = \widetilde{\chi}_0 - \chi_\delta \in H^1_{per}(Q;\mathbb{R}^d),$$

and so by the ellipticity of A (see (2.2.5)) and Cauchy-Schwarz,

$$\begin{aligned}\int_Q k_\delta|\nabla(\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta)|^2 &\le C\int_Q k_\delta A\nabla(\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta)\cdot\nabla(\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta)\\ &= C\delta\int_Q \mathbf{1}_-\nabla\widetilde{\mathbb{X}}_0\cdot\nabla(\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta)\\ &= C_1\delta\int_Q \mathbf{1}_+|\nabla\mathbb{X}_0|^2 + \delta\int_Q \mathbf{1}_-|\nabla(\widetilde{\mathbb{X}}_0 - \mathbb{X}_\delta)|^2,\end{aligned}$$

where C_1 only depends on κ_1 and κ_2. This gives (i). For (ii), note

$$\begin{aligned}\delta\int_Q \mathbf{1}_- A\nabla\mathbb{X}_\delta\cdot\nabla\chi_\delta &= -\int_Q \mathbf{1}_+ A\nabla(\mathbb{X}_0 - \mathbb{X}_\delta)\cdot\nabla\chi_\delta\\ &\le C\delta^{1/2}\|\mathbf{1}_+\nabla\mathbb{X}_0\|_{L^2(Q)}\|\mathbf{1}_+\nabla\chi_\delta\|_{L^2(Q)},\end{aligned}$$

where C only depends on κ_2. By (i),

$$\delta\int_Q \mathbf{1}_-|\nabla\mathbb{X}_\delta|^2 \le C\delta^{1/2}\|\mathbf{1}_+\nabla\mathbb{X}_0\|^2_{L^2(Q)},$$

where C depends on κ_1, κ_2. □

Lemma 3.2.3. *If $\widehat{A}_0$ and $\widehat{A}_\delta$ are defined by* (3.1.2) *and* (3.2.3), *then*

$$\Big||Q\cap\omega|\widehat{A}_0 - \widehat{A}_\delta\Big| \le C\delta^{1/2}\|\mathbf{1}_+\nabla\mathbb{X}_0\|_{L^2(Q)},$$

where C depends on κ_1 and κ_2.

Proof. Note

$$|Q\cap\omega|\widehat{A}_0 - \widehat{A}_\delta = \int_Q \mathbf{1}_+ A\nabla(\mathbb{X}_0 - \mathbb{X}_\delta) - \delta\int_Q \mathbf{1}_-\nabla\mathbb{X}_\delta,$$

from which the desired estimate follows by Lemma 3.2.2. □

Let K_ε be defined as in Section 2.4 by (2.4.1), and let $\eta_\varepsilon \in C_0^\infty(\Omega)$ be given by (3.0.1). Recall $\Gamma_\varepsilon = \partial\Omega\cap\varepsilon\omega$ and $H^1(\Omega,\Gamma_\varepsilon;\mathbb{R}^d)$ denotes the closure in $H^1(\Omega;\mathbb{R}^d)$ of $C^\infty(\mathbb{R}^d;\mathbb{R}^d)$ functions vanishing on Γ_ε. Let $u_{0,\delta}$ solve (3.2.4), χ_δ satisfy (3.2.1), and $u_{\varepsilon,\delta}$ denote a weak solution to (2.2.8) with $\delta \ge 0$.

Lemma 3.2.4. *Let*

$$r_{\varepsilon,\delta} = u_{\varepsilon,\delta} - u_{0,\delta} - \varepsilon\chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon).$$

Then

$$\begin{aligned}
&\int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta}\cdot\nabla w \\
&\quad = \int_\Omega (\eta_\varepsilon - 1)k_\delta^\varepsilon A^\varepsilon \nabla\left[u_{\varepsilon,\delta} - u_{0,\delta}\right]\cdot\nabla w + \int_\Omega k_\delta^\varepsilon A^\varepsilon\nabla\left[u_{\varepsilon,\delta}-u_{0,\delta}\right]\cdot\left[w\nabla\eta_\varepsilon\right] \\
&\qquad + \int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon\right]\left[\nabla u_{0,\delta} - K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\right]\cdot\nabla w \\
&\qquad - \int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon \nabla\mathbb{X}_\delta\right] K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\cdot\nabla w \\
&\qquad - \varepsilon\int_\Omega k_\delta^\varepsilon A^\varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\cdot\nabla w
\end{aligned}$$

for any $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$.

Proof. Fix $w \in H^1(\Omega,\Gamma_\varepsilon;\mathbb{R}^d)$, and note $w\eta_\varepsilon \in H_0^1(\Omega;\mathbb{R}^d)$. Since $u_{0,\delta}$ solves (3.2.4) and $u_{\varepsilon,\delta}$ satisfies (2.2.9),

$$\int_\Omega \widehat{A}_\delta \nabla u_{0,\delta}\cdot\nabla[w\eta_\varepsilon] = 0$$

and

$$\int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta}\cdot\nabla[w\eta_\varepsilon] = 0,$$

where η_ε denotes the cuttoff function defined by (3.0.1). Note

$$\nabla w = (1-\eta_\varepsilon)\nabla w - \eta_\varepsilon\nabla w = (1-\eta_\varepsilon)\nabla w - \nabla[w\eta_\varepsilon] + w\nabla\eta_\varepsilon, \tag{3.2.5}$$

and so

$$\begin{aligned}
\int_\Omega k_\delta^\varepsilon A^\varepsilon\nabla u_{\varepsilon,\delta}\cdot\nabla w &= \int_\Omega (1-\eta_\varepsilon)k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta}\cdot\nabla w - \int_\Omega k_\delta^\varepsilon A^\varepsilon\nabla u_{\varepsilon,\delta}\cdot\nabla[w\eta_\varepsilon] \\
&\quad + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta}\cdot[w\nabla\eta_\varepsilon] \\
&= \int_\Omega (1-\eta_\varepsilon)k_\delta^\varepsilon A^\varepsilon\nabla u_{\varepsilon,\delta}\cdot\nabla w + \int_\Omega k_\delta^\varepsilon A^\varepsilon\nabla u_{\varepsilon,\delta}\cdot[w\nabla\eta_\varepsilon].
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w \\
&\qquad = \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w - \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{0,\delta} \cdot \nabla w \\
&\qquad\qquad - \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \left[\varepsilon \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\right] \cdot \nabla w \\
&\qquad = \int_\Omega (1-\eta_\varepsilon) k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot [w \nabla \eta_\varepsilon] \\
&\qquad\qquad + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{0,\delta} \cdot \nabla w - \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \left[\varepsilon \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\right] \cdot \nabla w.
\end{aligned}
$$

Using the fact

$$
\nabla \left[\varepsilon \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\right] = \nabla \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) + \varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon),
$$

we continue with

$$
\begin{aligned}
&\int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w \\
&\qquad = \int_\Omega (1-\eta_\varepsilon) k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot [w \nabla \eta_\varepsilon] \\
&\qquad\qquad + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{0,\delta} \cdot \nabla w - \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w \\
&\qquad\qquad - \varepsilon \int_\Omega k_\delta^\varepsilon A^\varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w \\
&\qquad = \int_\Omega (1-\eta_\varepsilon) k_\delta^\varepsilon A^\varepsilon \nabla \left[u_{\varepsilon,\delta} - u_{0,\delta}\right] \cdot \nabla w + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \left[u_{\varepsilon,\delta} - u_{0,\delta}\right] \cdot [w \nabla \eta_\varepsilon] \\
&\qquad\qquad + \int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon\right] \nabla u_{0,\delta} \cdot \nabla w - \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w \\
&\qquad\qquad - \varepsilon \int_\Omega k_\delta^\varepsilon A^\varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w
\end{aligned}
$$

where we've also used the identity

$$
\begin{aligned}
\int_\Omega \widehat{A}_\delta \nabla u_{0,\delta} \cdot \nabla w &= \int_\Omega (1-\eta_\varepsilon) \widehat{A}_\delta \nabla u_{0,\delta} \cdot \nabla w - \int_\Omega \widehat{A}_\delta \nabla u_{0,\delta} \cdot \nabla [w \eta_\varepsilon] \\
&\qquad + \int_\Omega \widehat{A}_\delta \nabla u_{0,\delta} \cdot [w \nabla \eta_\varepsilon] \\
&= \int_\Omega (1-\eta_\varepsilon) \widehat{A}_\delta \nabla u_{0,\delta} \cdot \nabla w + \int_\Omega \widehat{A}_\delta \nabla u_{0,\delta} \cdot [w \nabla \eta_\varepsilon],
\end{aligned}
$$

which follows from (3.2.5).

Finally, continuing from above and combining similar terms,

$$\begin{aligned}
&\int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w \\
&\qquad = \int_\Omega (1-\eta_\varepsilon) k_\delta^\varepsilon A^\varepsilon \nabla \left[u_{\varepsilon,\delta} - u_{0,\delta}\right] \cdot \nabla w + \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \left[u_{\varepsilon,\delta} - u_{0,\delta}\right] \cdot \left[w \nabla \eta_\varepsilon\right] \\
&\qquad\quad + \int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon\right] \left[\nabla u_{0,\delta} - K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\right] \cdot \nabla w \\
&\qquad\quad - \int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon - k_\delta^\varepsilon A^\varepsilon \nabla \chi_\delta^\varepsilon\right] K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w \\
&\qquad\quad - \varepsilon \int_\Omega k_\delta^\varepsilon A^\varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w
\end{aligned}$$

which is the desired inequality. □

Lemma 3.2.5. *For* $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$,

$$\begin{aligned}
\left| \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w \right| \le C \Big\{ &\|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \|(\nabla u_{0,\delta})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)} \\
&+ \varepsilon \|K_\varepsilon\big((\nabla^2 u_{0,\delta})\eta_\varepsilon\big)\|_{L^2(\Omega)} + \|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} \Big\} \|\nabla w\|_{L^2(\Omega)}
\end{aligned}$$

Proof. By Lemma 3.2.4,

$$\int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.2.6}$$

where

$$\begin{aligned}
I_1 &= \int_\Omega (1-\eta_\varepsilon) k_\delta^\varepsilon A^\varepsilon \nabla \left[u_{\varepsilon,\delta} - u_{0,\delta}\right] \cdot \nabla w \\
I_2 &= \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla \left[u_{\varepsilon,\delta} - u_{0,\delta}\right] \cdot \left[w \nabla \eta_\varepsilon\right] \\
I_3 &= \int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon\right] \left[\nabla u_{0,\delta} - K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\right] \cdot \nabla w \\
I_4 &= -\int_\Omega \left[\widehat{A}_\delta - k_\delta^\varepsilon A^\varepsilon \nabla \mathbb{X}_\delta\right] K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w \\
I_5 &= -\varepsilon \int_\Omega k_\delta^\varepsilon A^\varepsilon \chi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon) \cdot \nabla w
\end{aligned}$$

and $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$.

Recall from (3.0.1) that $\operatorname{supp}(1-\eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$, where

$$\mathcal{O}_{4\varepsilon} = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) < 4\varepsilon\}.$$

By Cauchy-Schwarz, (2.2.5), and the assumption $\delta \le 1$, we have

$$\begin{aligned}
|I_1| &\le \int_{\mathcal{O}_{4\varepsilon}} |k_\delta^\varepsilon A^\varepsilon \nabla u_{0,\delta} \cdot \nabla w| + \int_{\mathcal{O}_{4\varepsilon}} |k_\delta^\varepsilon A^\varepsilon \nabla u_{\varepsilon,\delta} \cdot \nabla w| \\
&\le C \left\{ \|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} \right\} \|\nabla w\|_{L^2(\Omega)}.
\end{aligned} \tag{3.2.7}$$

Similarly, as $\operatorname{supp}(\nabla \eta_\varepsilon) \subset \mathcal{O}_{4\varepsilon}$, Cauchy-Schwarz, Lemma 3.1.6, and (3.0.1) imply

$$|I_2| \le C \left\{ \|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} \right\} \|\nabla w\|_{L^2(\Omega)}. \tag{3.2.8}$$

Using (3.0.1) again,

$$\begin{aligned}
&\|\nabla u_{0,\delta} - K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)} \\
&\quad \le \|(1-\eta_\varepsilon)\nabla u_{0,\delta}\|_{L^2(\Omega)} + \|(\nabla u_{0,\delta})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)} \\
&\qquad + \|K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon))\|_{L^2(\Omega)} \\
&\quad \le \|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} + C\|(\nabla u_{0,\delta})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, by Cauchy-Schwarz,

$$\begin{aligned}
|I_3| &\le C\|\nabla u_{0,\delta} - K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} \\
&\le C \left\{ \|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} \right. \\
&\qquad \left. + \|(\nabla u_{0,\delta})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)} \right\} \|\nabla w\|_{L^2(\Omega)}.
\end{aligned} \tag{3.2.9}$$

Set $B_\delta = \widehat{A}_\delta - k_\delta A \nabla \mathbb{X}$. By (3.2.3) and (3.2.2), $B_\delta = \{b_{ij,\delta}^{\alpha\beta}\}$ satisfies the assumptions of Lemma 2.4.5. Therefore, there exists $\pi_\delta = \{\pi_{kij,\delta}^{\alpha\beta}\}$ that is 1-periodic with

$$\frac{\partial}{\partial y_k}\pi_{kij,\delta}^{\alpha\beta} = b_{ij,\delta}^{\alpha\beta} \quad \text{and} \quad \pi_{kij,\delta}^{\alpha\beta} = -\pi_{ikj,\delta}^{\alpha\beta},$$

where

$$b_{ij,\delta}^{\alpha\beta} = \widehat{a}_{ij,\delta}^{\alpha\beta} - k_\delta a_{ik,\delta}^{\alpha\gamma}\frac{\partial}{\partial y_k}\mathbb{X}_{j,\delta}^{\gamma\beta}.$$

Moreover, $\|\pi_{ij,\delta}^{\alpha\beta}\|_{H^1(Q)} \le C$ for some constant C depending on κ_1, κ_2 but not δ (see Lemmas 2.4.5 and 3.2.2). Hence, integrating by parts gives

$$\begin{aligned}
\int_\Omega b_{ij,\delta}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_{0,\delta}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial \widetilde{w}^\alpha}{\partial x_i} &= -\varepsilon\int_\Omega \pi_{kij,\delta}^{\alpha\beta\varepsilon}\frac{\partial}{\partial x_k}\left[K_\varepsilon^2\left(\frac{\partial u_{0,\delta}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial w^\alpha}{\partial x_i}\right] \\
&= -\varepsilon\int_\Omega \pi_{kij,\delta}^{\alpha\beta\varepsilon}\frac{\partial}{\partial x_k}\left[K_\varepsilon^2\left(\frac{\partial u_{0,\delta}^\beta}{\partial x_j}\eta_\varepsilon\right)\right]\frac{\partial w^\alpha}{\partial x_i},
\end{aligned}$$

since

$$\int_\Omega \pi_{kij,\delta}^{\alpha\beta\varepsilon} K_\varepsilon^2\left(\frac{\partial u_{0,\delta}^\beta}{\partial x_j}\eta_\varepsilon\right)\frac{\partial^2 w^\alpha}{\partial x_k \partial x_i} = 0$$

due to the anit-symmetry of π_δ. Thus, by Lemma 2.4.3, and (3.0.1),

$$\begin{aligned}
|I_4| &\le C\varepsilon\|\pi_\delta^\varepsilon \nabla K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)} \\
&\le C\left\{\|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \varepsilon\|K_\varepsilon\big((\nabla^2 u_{0,\delta})\eta_\varepsilon\big)\|_{L^2(\Omega)}\right\}\|\nabla w\|_{L^2(\Omega)}.
\end{aligned} \tag{3.2.10}$$

Finally, by Lemma 2.4.3 and (3.0.1),

$$|I_5| \le C\left\{\|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} + \varepsilon\|K_\varepsilon\big((\nabla^2 u_{0,\delta})\eta_\varepsilon\big)\|_{L^2(\Omega)}\right\}\|\nabla w\|_{L^2(\Omega)} \tag{3.2.11}$$

The desired estimate follows from (3.2.7)–(3.2.11). □

Lemma 3.2.6. *For $w \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$,*

$$\left| \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla w \right| \leq C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)} \|w\|_{H^1(\Omega_\varepsilon)},$$

where $\mu > 0$ depends on d, κ_1, and κ_2.

Proof. Recall that $u_{0,\delta}$ satisfies $\mathcal{L}_{0,\delta}(u_{0,\delta}) = 0$ in Ω, and so it follows from estimates for solutions in Lipschitz domains for constant-coefficient equations that

$$\|(\nabla u_{0,\delta})^*\|_{L^2(\partial\Omega)} \leq C\|f\|_{H^1(\partial\Omega)}, \tag{3.2.12}$$

where $(\nabla u_{0,\delta})^*$ denotes the nontangential maximal function of $\nabla u_{0,\delta}$ (see [14]). By the coarea formula,

$$\|\nabla u_{0,\delta}\|_{L^2(\mathcal{O}_{4\varepsilon})} \leq C\varepsilon^{1/2} \|(\nabla u_{0,\delta})^*\|_{L^2(\partial\Omega)} \leq C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}. \tag{3.2.13}$$

Notice that if $u_{0,\delta}$ solves (3.2.4), then $\mathcal{L}_{0,\delta}(\nabla u_{0,\delta}) = 0$ in Ω, and so we may use the interior Lipschitz estimate for $\mathcal{L}_{0,\delta}$. That is,

$$|\nabla^2 u_{0,\delta}(x)| \leq \frac{C}{\rho(x)} \left(\fint_{B(x,\rho(x)/8)} |\nabla u_{0,\delta}|^2 \right)^{1/2}, \tag{3.2.14}$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ and C is uniform in δ given Lemma 3.2.1. In particular,

$$\begin{aligned}
\|(\nabla^2 u_{0,\delta})\eta_\varepsilon\|_{L^2(\Omega)} &\leq \left(\int_{\Omega\setminus\mathcal{O}_{3\varepsilon}} |\nabla^2 u_{0,\delta}|^2 \right)^{1/2} \\
&\leq C \left(\int_{\Omega\setminus\mathcal{O}_{3\varepsilon}} \fint_{B(x,\rho(x)/8)} \left| \frac{\nabla u_{0,\delta}(y)}{\rho(x)} \right|^2 dy\, dx \right)^{1/2} \\
&\leq C \left(\int_{3\varepsilon}^{C_0} t^{-2} \int_{\partial\mathcal{O}_t\cap\Omega} \fint_{B(x,t/8)} |\nabla u_{0,\delta}(y)|^2 \, dy\, dS(x)\, dt \right)^{1/2} \\
&\qquad\qquad + C_1 \left(\int_{\Omega\setminus\mathcal{O}_{C_0}} |\nabla u_{0,\delta}|^2 \right)^{1/2} \\
&\leq C\|(\nabla u_{0,\delta})^*\|_{L^2(\partial\Omega)} \left(\int_{3\varepsilon}^{C_0} t^{-2}\, dt \right)^{1/2} + C_1 \|\nabla u_{0,\delta}\|_{L^2(\Omega)} \\
&\leq C\left\{ \varepsilon^{-1/2}\|f\|_{H^1(\partial\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)} \right\} \\
&\leq C\varepsilon^{-1/2}\|f\|_{H^1(\partial\Omega)}.
\end{aligned}$$

where C_0 is a constant depending on Ω, and we've used (3.0.1), (3.2.12) (3.2.13), the coarea formula, energy estimates, and (3.2.14). Hence,

$$\varepsilon \|K_\varepsilon\big((\nabla^2 u_{0,\delta})\eta_\varepsilon\big)\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}. \tag{3.2.15}$$

By Lemma 2.4.2,

$$\|(\nabla u_{0,\delta})\eta_\varepsilon - K_\varepsilon((\nabla u_{0,\delta})\eta_\varepsilon)\|_{L^2(\Omega)} \le C\varepsilon^{1/2}\|f\|_{H^1(\partial\Omega)}. \tag{3.2.16}$$

where the last inequality follows from (3.2.15) and (3.0.1).

Finally, we establish a $W^{1,p}$-estimate for some $p > 2$ for $u_{\varepsilon,\delta}$ uniform in ε and δ by establishing a reverse Hölder inequality. Indeed, if there exists a $p > 2$ so that

$$\left(\int_\Omega |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^p\right)^{1/p} \le C\|f\|_{H^1(\partial\Omega)},$$

then Hölder's inequality implies

$$\int_{\mathcal{O}_{4\varepsilon}} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2 \le C\varepsilon^{(p-2)/p}\|f\|^2_{H^1(\partial\Omega)}. \tag{3.2.17}$$

The existence of such a p follows from the Lemma 3.2.7 below. Equations (3.2.13), (3.2.15), (3.2.16), and (3.2.17) together with Lemma 3.2.5 give the desired result. □

Lemma 3.2.7. *There exists a $p_0 > 2$ such that*

$$\left(\int_\Omega |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^{p_0}\right)^{1/p_0} \le C\|f\|_{H^1(\partial\Omega)}$$

for some constant C depending on κ_1, κ_2, d, p_0, and Ω.

Proof. The desired estimate essentially follows from Cacciopoli's inequality, the Poincaré-Sobolev inequality, and the self-improving property of reverse Hölder inequalities. We prove an interior estimate, and the boundary estimate follows with an analogous proof.

Take $B(x_0, 2r) \subset \Omega$, and note that Cacciopoli's inequality (see Lemma 4.3.1 in Section 4.3) implies

$$\begin{aligned}
&\left(\fint_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \\
&\quad \le \frac{C}{r}\left(\fint_{B(x_0,2r)} |k_\delta^\varepsilon u_{\varepsilon,\delta}|^2\right)^{1/2} \\
&\quad \le \frac{C}{r}\left\{\delta\left(\fint_{B(x_0,2r)} |u_{\varepsilon,\delta}|^2\right)^{1/2} + \left(\fint_{B(x_0,2r)} |P_\varepsilon(\mathbf{1}_+^\varepsilon u_{\varepsilon,\delta})|^2\right)^{1/2}\right\},
\end{aligned}$$

which is invariant if we subtract any constant vector from $u_{\varepsilon,\delta}$. If we consider the difference in $u_{\varepsilon,\delta}$ and the average value of $u_{\varepsilon,\delta}$ over the ball $B(x_0, 2r)$, then by the Poincaré-Sobolev ineqaulity

$$\begin{aligned}
&\left(\fint_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \\
&\quad \le \delta\left(\fint_{B(x_0,2r)} |\nabla u_{\varepsilon,\delta}|^s\right)^{1/s} + \frac{C}{r}\left(\fint_{B(x_0,2r)} |P_\varepsilon(\mathbf{1}_+^\varepsilon u_{\varepsilon,\delta})|^2\right)^{1/2},
\end{aligned}$$

where $s = 2d/(d+2)$. Similarly, by considering the difference of $P_\varepsilon(\mathbf{1}_+^\varepsilon u_{\varepsilon,\delta})$ and its average value over the ball $B(x_0, 2r)$ we can show

$$\left(\fint_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \leq \delta \left(\fint_{B(x_0,2r)} |\nabla u_{\varepsilon,\delta}|^s\right)^{1/s} + \left(\fint_{B(x_0,2r)} |\nabla P_\varepsilon(\mathbf{1}_+^\varepsilon u_{\varepsilon,\delta})|^s\right)^{1/s},$$

which by Lemma 2.4.6 shows

$$\left(\fint_{B(x_0,r)} w^q\right)^{1/q} \leq C \fint_{B(x_0,2r)} w,$$

where $w = |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^s$ and $q = 2/s$. By the self-improving property of reverse Hölder inequalities (see [17, Chapter V, Proposition 1.1]),

$$\left(\fint_{B(x_0,r)} w^t\right)^{1/t} \leq C \left(\fint_{B(x_0,2r)} w^q\right)^{1/q},$$

for any $t \in [q, q+\nu)$ for some $\nu > 0$ depending on κ_1, κ_2, and d. That is,

$$\left(\fint_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^p\right)^{1/p} \leq C \left(\fint_{B(x_0,2r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \tag{3.2.18}$$

for any $p \in [2, 2+\nu)$ and any $B(x_0, 2r) \subset \Omega$.

We may show a similar estimate for any ball $B(x_0, 2r)$ with $x_0 \in \partial\Omega$. That is, if $F = f$ on $\partial\Omega$ and $F \in H^{3/2}(\Omega)$, then the continuous injection $H^{3/2}(\Omega) \subseteq W^{1,q}(\Omega)$ for any $q \geq 2d/(d-1)$ gives the estimate

$$\left(\fint_{B(x_0,r)\cap\Omega} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^p\right)^{1/p} \leq C \left(\fint_{B(x_0,2r)\cap\Omega} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} + \left(\fint_{B(x_0,2r)\cap\Omega} |\nabla F|^q\right)^{1/q}. \tag{3.2.19}$$

Patching together inequalities (3.2.18) and (3.2.19) gives the desired estimate for some $p_0 > 2$. □

Proof of Theorem 3.0.3. Note $\delta r_{\varepsilon,\delta} \in H_0^1(\Omega; \mathbb{R}^d) \subset H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$, and so by Lemma 3.2.6 and (2.2.5),

$$\begin{aligned}\|\delta e(r_{\varepsilon,\delta})\|_{L^2(\Omega)}^2 &\leq C\delta \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla r_{\varepsilon,\delta} \\ &\leq C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)} \|\delta \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)},\end{aligned}$$

where $e(r_{\varepsilon,\delta})$ denotes the symmetric part of $\nabla r_{\varepsilon,\delta}$. Korn's first inequality then implies

$$\|\delta \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)}^2 \leq C\|\delta e(r_{\varepsilon,\delta})\|_{L^2(\Omega)}^2 \leq C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)} \|\delta \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)},$$

and so

$$\|\delta \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\varepsilon^{\mu} \|f\|_{H^1(\partial\Omega)}. \tag{3.2.20}$$

Note also $P_\varepsilon(\mathbf{1}_+^\varepsilon r_{\varepsilon,\delta}) \in H^1(\Omega, \Gamma_\varepsilon; \mathbb{R}^d)$, and so by Lemmas 3.2.6 and 2.4.6,

$$\begin{aligned}
&\|\mathbf{1}_+^\varepsilon e[P_\varepsilon(\mathbf{1}_+^\varepsilon r_{\varepsilon,\delta})]\|_{L^2(\Omega)}^2 \\
&\qquad \le C \int_\Omega k_\delta^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla P_\varepsilon(\mathbf{1}_+^\varepsilon r_{\varepsilon,\delta}) - \delta \int_\Omega \mathbf{1}_-^\varepsilon A^\varepsilon \nabla r_{\varepsilon,\delta} \cdot \nabla P_\varepsilon(\mathbf{1}_+^\varepsilon r_{\varepsilon,\delta}) \\
&\qquad \le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)} \|\mathbf{1}_+^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)},
\end{aligned}$$

where we've used (3.2.20). Korn's first inequality for periodically perforated domains, i.e., Lemma 2.4.7, implies

$$\|\mathbf{1}_+^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)}^2 \le C \|\mathbf{1}_+^\varepsilon e[P_\varepsilon(\mathbf{1}_+^\varepsilon r_{\varepsilon,\delta})]\|_{L^2(\Omega)}^2 \le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)} \|\mathbf{1}_+^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)},$$

and so

$$\|\mathbf{1}_+^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)}. \tag{3.2.21}$$

Equations (3.2.20) and (3.2.21) give the desired estimate. Indeed,

$$\|k_\delta^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} \le \|\mathbf{1}_+^\varepsilon \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} + \|\delta \nabla r_{\varepsilon,\delta}\|_{L^2(\Omega)} \le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)}$$

□

Proof of Corollary 3.0.4. Using Theorem 3.0.3, we have

$$\begin{aligned}
\|k_\delta^\varepsilon (u_{\varepsilon,\delta} - u_{0,\delta})\|_{L^2(\Omega)} &\le C \|k_\delta^\varepsilon (u_{\varepsilon,\delta} - u_{0,\delta} - \varepsilon \chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon))\|_{L^2(\Omega)} \\
&\qquad + C\varepsilon \|\chi_\delta^\varepsilon K_\varepsilon^2((\nabla u_{0,\delta})\eta_\varepsilon))\|_{L^2(\Omega)} \\
&\le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)} + C\varepsilon \|f\|_{H^{1/2}(\partial\Omega)} \\
&\le C\varepsilon^\mu \|f\|_{H^1(\partial\Omega)},
\end{aligned}$$

which is the desired result. □

The other main results of this dissertation are the following theorems which are basically large-scale interior Lipschitz estimates for solutions to (2.2.8). Theorem 4.0.1 concerns the boundary value problem with $\delta = 0$, and Theorem 4.0.3 concerns the boundary value problem with $\delta \geq 0$.

Theorem 4.0.1. *Suppose A satisfies* (2.2.4), (2.2.5), *and* (2.2.7). *Let $u_{\varepsilon,0}$ denote a weak solution to $\mathcal{L}_{\varepsilon,0}(u_{\varepsilon,0}) = 0$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$. For $\varepsilon \leq r \leq R$, there exists a constant C depending on d, ω, κ_1, and κ_2 such that*

$$\left(\fint_{B(x_0,r)\cap\varepsilon\omega} |\nabla u_{\varepsilon,0}|^2 \right)^{1/2} \leq C \left(\fint_{B(x_0,R)\cap\varepsilon\omega} |\nabla u_{\varepsilon,0}|^2 \right)^{1/2}. \tag{4.0.1}$$

The scale-invariant estimate in Theorem 4.0.1 should be regarded as a Lipschitz estimate for solutions $u_{\varepsilon,0}$, as under additional smoothness assumptions on the coefficients A we may deduce an interior Lipschitz estimate for solutions to (2.2.8) with $\delta = 0$ from local Lipschitz estimates for $\mathcal{L}_{1,0}$ and a "blow-up argument." In particular, if A is Hölder continuous, i.e., there exists a $\alpha \in (0, 1)$ with

$$|A(x) - A(y)| \leq C|x - y|^\alpha \quad \text{for } x, y \in \mathbb{R}^d \tag{4.0.2}$$

for some constant C uniform in x and y, we may deduce the following corollary.

Corollary 4.0.2. *Suppose A satisfies* (2.2.4), (2.2.5), (2.2.7), *and* (4.0.2), *and suppose ω is an unbounded $C^{1,\alpha}$ domain for some $\alpha > 0$. Let $u_{\varepsilon,0}$ denote a weak solution to $\mathcal{L}_{\varepsilon,0}(u_{\varepsilon,0}) = 0$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$. Then*

$$\|\nabla u_{\varepsilon,0}\|_{L^\infty(B(x_0,R/3)\cap\varepsilon\omega)} \leq C \left(\fint_{B(x_0,R)\cap\varepsilon\omega} |\nabla u_{\varepsilon,0}|^2 \right)^{1/2},$$

where C depends on d, ω, κ_1, κ_2, and α.

Interior Lipschitz estimates for the case $\omega = \mathbb{R}^d$ were first obtained *indirectly* through the method of compactness presented in [6]. Interior Lipschitz estimates for solutions to a single elliptic equation in the case $\omega \subsetneq \mathbb{R}^d$ were obtained indirectly in [32] through the same method of compactness. The method of compactness is essentially a "proof by contradiction" and relies on the qualitative convergence of solutions u_ε (see Theorem 2.2.1). The method relies on sequences of operators $\{\mathcal{L}^k_{\varepsilon_k}\}_k$ and sequences of functions $\{u_k\}_k$ satisfying $\mathcal{L}^k_{\varepsilon_k}(u_k) = 0$, where $\mathcal{L}^k_{\varepsilon_k} = -\text{div}(A^{\varepsilon_k}_k \nabla)$, $\{A^{\varepsilon_k}_k\}_k$ satisfies (2.2.4), (2.2.5), and (2.2.7) in $\omega + s_k$ for $s_k \in \mathbb{R}^d$. In the case $\omega = \mathbb{R}^d$, then $\omega + s_k = \mathbb{R}^d$ for any $s_k \in \mathbb{R}^d$, and so it is clear that estimate (4.0.1) is uniform in affine transformations of ω. In the case $\omega \subsetneq \mathbb{R}^d$, affine shifts of ω must be considered, which complicates the general scheme.

Interior Lipschitz estimates for the case $\omega = \mathbb{R}^d$ were obtained *directly* in [26] through a general scheme for establishing Lipschitz estimates at the macroscopic scale first presented in [5] and then modified for second-order elliptic systems in [4] and [26]. We emphasize that our result is unique in that Theorem 4.0.1 extends estimates presented in [26]—i.e., interior Lipschitz estimates for systems of linear elasticity—to the case $\omega \subsetneq \mathbb{R}^d$ while *completely avoiding the use of compactness methods*. The proof of Theorem 4.0.1 (see Section 4.1) relies on the quantitative convergence rates of the solutions $u_{\varepsilon,0}$.

Another main result of this dissertation is Theorem 4.0.3. We reiterate that no smoothness assumptions are required on the coefficients A, only the elasticity conditions (2.2.4), (2.2.5), and the periodicity condition (2.2.7). Theorem 4.0.3 should be considered as an analog to Theorem 4.0.1 for the case $\delta > 0$.

Theorem 4.0.3. *Suppose A satisfies (2.2.4), (2.2.5), and (2.2.7). Let $u_{\varepsilon,\delta}$ denote a weak solution to $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$. For $\varepsilon \leq r \leq R$, there exists a constant C depending on d, ω, κ_1, and κ_2 such that*

$$\left(\fint_{B(x_0,r)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \leq C \left(\fint_{B(x_0,R)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2}.$$

for any $0 \leq \delta \leq 1$.

The scale-invariant estimate in Theorem 4.0.3 should also be regarded as an interior Lipschitz estimate for (2.2.8) at the large scale but for any $\delta \geq 0$. Indeed, if Theorem 2.2.8 were to hold also for $0 < r < \varepsilon$, then we would be able to essentially bounded the gradient of $u_{\varepsilon,\delta}$ in the interior of Ω. However, Lipschitz estimates do not in general hold without more assumptions on the smoothness of the coefficients A and the domain ω. That is, the periodicity assumptions on A, ω, and the elasticity conditions (2.2.4), (2.2.5) alone contribute to the large-scale average behavior of the solution.

Under additional assumptions that A is Hölder continuous and the domain ω has a sufficiently regular boundary, an interior Lipschitz estimate at the microscopic scale for solutions to (2.2.8) follows from local $C^{1,\alpha}$-estimates for the operator $\mathcal{L}_{1,\delta^2}$. We modify a layer potential argument of Escaurazia, Fabes, and Verchota where nontangential estimates were obtained for single equation interface problems [15]. Yeh modified this same argument to obtain local $W^{1,p}$-estimates and Hölder estimates for (2.2.8) in the case of single equations with diagonal coefficients [30, 31]. This is discussed in Section 4.4.

Corollary 4.0.4. *Suppose A satisfies (2.2.4), (2.2.5), (2.2.7), and (4.0.2) for some $\alpha \in (0,1)$. Suppose ω is an unbounded $C^{1,\alpha}$ domain. Let $u_{\varepsilon,\delta}$ denote a weak solution to $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$. Then for $0 \leq \delta \leq 1$,*

$$\|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^\infty(B(x_0,R/3))} \leq C \left(\fint_{B(x_0,R)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \tag{4.0.3}$$

some constant C independent of ε and δ.

To prove Theorem 4.0.3, we use the sub-optimal quantitative convergence rates for solutions to (2.2.8) (see Section 3.2) and apply a general scheme of Armstrong and Smart that has since been adapted by Shen [5, 4, 26].

Hueristically, the scheme is a Campanato-type iteration verifying that on mesoscopic scales the solution $u_{\varepsilon,\delta}$ is "flatter." If P_1 denotes the space of affine functions in $\mathbb{R}^d$ and $H_{\varepsilon,\delta}(r)$ defined by

$$H_{\varepsilon,\delta}(r) = \frac{1}{r}\left(\inf_{p\in P_1} \fint_{B(r)} |k_\delta^\varepsilon(u_{\varepsilon,\delta} - p)|^2\right)^{1/2}$$

quantifies a weighted L^2-"flatness" of the solution in some ball $B(r)$ with radius r, then we show there exists a $\theta \in (0,1)$ such that

$$H_{\varepsilon,\delta}(\theta r) \leq CH_{\varepsilon,\delta}(r) + \text{error}, \tag{4.0.4}$$

where the "error" term is controllable whenever $\varepsilon \leq r$ and the constant $0 \leq C < 1$ indicates an improvement in "flatness." Indeed, (4.0.4) follows from the fact the $u_{\varepsilon,\delta}$—at least in the connected substrate—can be well-approximated in L^2 by a solution to a constant coefficient system. It is known from classical C^2 estimates that solutions to constant coefficient systems satisfy (4.0.4) with no error. In contrast to compactness methods, showing (4.0.4) relies on tractable L^2-convergence rates of $u_{\varepsilon,\delta}$, which we will see follows from new results regarding quantitative homogenization in H^1.

4.1 Large-scale interior Lipschitz estimates in perforated domains

In this section, we use Theorem 3.0.1 to investigate interior Lipschitz estimates down to the scale ε for the case $\delta = 0$. In particular, we prove Theorem 4.0.1. The proof of Theorem 4.0.1 is based on the scheme used in [26] to prove boundary Lipschitz estimates for solutions to (2.2.8) in the case $\omega = \mathbb{R}^d$, which in turn is based on a more general scheme for establishing Lipschitz estimates presented in [5] and adapted in [26] and [4]. For more details, see the introduction to this section.

The following Lemma is essentially Cacciopoli's inequality in a perforated ball. The proof is similar to a proof of the classical Cacciopoli's ineqaulity, but nevertheless we present a proof for completeness.

Throughout this section, let $B_\varepsilon(r)$ denote the perforated ball of radius r centered at some $x_0 \in \mathbb{R}^d$, i.e., $B_\varepsilon(r) = B(x_0,r)\cap\varepsilon\omega$. Let $S_\varepsilon(r) = \partial(\varepsilon\omega)\cap B(x_0,r)$ and $\Gamma_\varepsilon(r) = \varepsilon\omega\cap\partial B(x_0,r)$.

Lemma 4.1.1. *Suppose $\mathcal{L}_{\varepsilon,0}(u_\varepsilon) = 0$ in $B(2)$. There exists a constant C depending on κ_1 and κ_2 such that*

$$\left(\fint_{B_\varepsilon(1)} |\nabla u_\varepsilon|^2\right)^{1/2} \leq C\inf_{q\in\mathbb{R}^d}\left(\fint_{B_\varepsilon(2)} |u_\varepsilon - q|^2\right)^{1/2}$$

Proof. Let $\varphi \in C_0^\infty(B(2))$ satisfy $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B(1)$, $|\nabla\varphi| \leq C_1$ for some constant C_1. Let $q \in \mathbb{R}^d$, and set $w = (u_\varepsilon - q)\varphi^2$. Note

$$\nabla[(u_\varepsilon - q)\varphi^2] = \varphi^2\nabla u_\varepsilon + 2(u_\varepsilon - q)\varphi\nabla\varphi$$

and so by (2.2.9), (2.2.5),

$$\begin{aligned}0 &= \int_{B_\varepsilon(2)} A^\varepsilon \nabla u_\varepsilon \nabla w \\ &= \int_{B_\varepsilon(2)} A^\varepsilon \varphi^2 \nabla u_\varepsilon \nabla u_\varepsilon + 2\int_{B_\varepsilon(2)} A^\varepsilon (u_\varepsilon - q)\varphi \nabla u_\varepsilon \nabla \varphi \\ &\geq \kappa_1 \int_{B_\varepsilon(2)} \varphi^2 |e(u_\varepsilon)|^2 - C_1\kappa_2 \int_{B_\varepsilon(2)} |u_\varepsilon - q|\varphi \nabla u_\varepsilon \nabla \varphi \end{aligned} \tag{4.1.1}$$

By Cauchy-Schwarz,

$$\int_{B_\varepsilon(2)} |u_\varepsilon - q|\varphi \nabla u_\varepsilon \nabla\varphi \leq \gamma^{-1} \int_{B_\varepsilon(2)} |u_\varepsilon - q|^2 |\nabla \varphi|^2 + \gamma \int_{B_\varepsilon(2)} |e(u_\varepsilon)|^2 \varphi^2$$

for any $\gamma > 0$. Choose γ so that $\kappa_1 > C_1\kappa_2\gamma$, and see then

$$0 \geq C_2 \int_{B_\varepsilon(2)} |e(u_\varepsilon)|^2\varphi^2 - C_3 \int_{B_\varepsilon(2)} |\nabla\varphi|^2 |u_\varepsilon - q|^2 \tag{4.1.2}$$

for some constants C_2 and C_3 depending only κ_1 and κ_2. In particular,

$$\int_{B_\varepsilon(2)} |e(u_\varepsilon\varphi)|^2 \leq C\int_{B_\varepsilon(2)} |\nabla\varphi|^2 |u_\varepsilon - q|^2,$$

where C only depends on κ_1 and κ_2. Since $\varphi \equiv 1$ in $B(1)$ and $u_\varepsilon\varphi \in H^1(B_\varepsilon(2), \Gamma_\varepsilon(2); \mathbb{R}^d)$, equation (4.1.2) together with Lemma 2.4.7 gives the desired estimate. □

We extend Lemma 4.1.1 to hold for a ball $B_\varepsilon(r)$ with $r > 0$ by a convenient scaling technique—the so called "blow-up argument"—often used in the study of homogenization.

Lemma 4.1.2. *Suppose $\mathcal{L}_{\varepsilon,0}(u_\varepsilon) = 0$ in $B(2r)$ for some $r > 0$. There exists a constant C depending on κ_1 and κ_2 such that*

$$\left(\fint_{B_\varepsilon(r)} |\nabla u_\varepsilon|^2\right)^{1/2} \leq \frac{C}{r} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(2r)} |u_\varepsilon - q|^2\right)^{1/2}$$

Proof. Let $U_\varepsilon(x) = u_\varepsilon(rx)$, and note U_ε satisfies $\mathcal{L}_{\varepsilon/r,0}(U_\varepsilon) = 0$ in $B_\varepsilon(2)$. By Lemma 4.1.1,

$$\left(\fint_{B_{\varepsilon/r}(1)} |\nabla U_\varepsilon|^2\right)^{1/2} \leq C \inf_{q\in\mathbb{R}^d} \left(\fint_{B_{\varepsilon/r}(2)} |U_\varepsilon - q|^2\right)^{1/2}$$

for some C independent of ε and r. Note $\nabla U_\varepsilon = r\nabla u_\varepsilon$, and so

$$r^{1-d/2}\left(\fint_{B_\varepsilon(r)} |\nabla u_\varepsilon|^2\right)^{1/2} \leq Cr^{-d/2} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(2r)} |u_\varepsilon - q|^2\right)^{1/2},$$

where we've made the change of variables $rx \mapsto x$. The desired inequality follows. □

The following lemma is a key estimate in the proof of Theorem 4.0.1. Intrinsically, the following Lemma uses the convergence rate in Theorem 3.0.1 to approximate the solution u_ε with a "nice" function. Recall the operator $\mathcal{L}_{0,0}$ is introduced in Section 3.1 by (3.1.1).

Lemma 4.1.3. *Suppose $\mathcal{L}_{\varepsilon,0}(u_\varepsilon) = 0$ in $B_\varepsilon(3r)$ for some $r > 0$. There exists a $v \in H^1(B(r);\mathbb{R}^d)$ with $\mathcal{L}_{0,0}(v) = 0$ in $B(r)$ and*

$$\left(\fint_{B_\varepsilon(r)} |u_\varepsilon - v|\right)^{1/2} \leq C\left(\frac{\varepsilon}{r}\right)^{1/2}\left(\fint_{B_\varepsilon(3r)} |u_\varepsilon|^2\right)^{1/2}$$

for some constant C depending on d, ω, κ_1, and κ_2

Proof. With rescaling (see the proof of Lemma 4.1.2), we may assume $r = 1$. By Lemmas 4.1.2 and 2.4.6,

$$\begin{aligned}\left(\fint_{B(3/2)} |\widetilde{u}_\varepsilon|^2\right)^{1/2} + \left(\fint_{B(3/2)} |\nabla \widetilde{u}_\varepsilon|^2\right)^{1/2} &\leq \left(\fint_{B_\varepsilon(3/2)} |u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B_\varepsilon(3/2)} |\nabla u_\varepsilon|^2\right)^{1/2} \\ &\leq (1 + C_1)\left(\fint_{B_\varepsilon(3)} |u_\varepsilon|^2\right)^{1/2}\end{aligned}$$

where $\widetilde{u}_\varepsilon = P_\varepsilon u_\varepsilon \in H^1(B(3);\mathbb{R}^d)$, P_ε is the linear extension operator provided in Lemma 2.4.6, and C_1 depends only on κ_1 and κ_2. Note by the coarea formula

$$\int_0^{3/2} \|\nabla \widetilde{u}_\varepsilon\|_{L^2(\partial B(t))} + \|\widetilde{u}_\varepsilon\|_{L^2(\partial B(t))}\, dt \leq C\|u_\varepsilon\|_{L^2(B_\varepsilon(3))},$$

and so for a.e. $t \in [0, 3/2]$ we have

$$\|\nabla \widetilde{u}_\varepsilon\|_{L^2(\partial B(t))} + \|\widetilde{u}_\varepsilon\|_{L^2(\partial B(t))} \leq C\|u_\varepsilon\|_{L^2(B_\varepsilon(3))}. \tag{4.1.3}$$

Choose $t \in [1, 3/2]$ so that (4.1.3) holds. Let v denote the solution to the Dirichlet problem $\mathcal{L}_0(v) = 0$ in $B(t)$ and $v = \widetilde{u}_\varepsilon$ on $\partial B(t)$. Note that $v = u_\varepsilon = \widetilde{u}_\varepsilon$ on $\Gamma_\varepsilon(t)$. By Corollary 3.0.2,

$$\|u_\varepsilon - v\|_{L^2(B_\varepsilon(t))} \leq C\varepsilon^{1/2}\|\widetilde{u}_\varepsilon\|_{H^1(\partial B(t))} \tag{4.1.4}$$

Hence, (4.1.3) and (4.1.4) give

$$\|u_\varepsilon - v\|_{L^2(B_\varepsilon(1))} \leq \|u_\varepsilon - v\|_{L^2(B_\varepsilon(t))} \leq C\varepsilon^{1/2}\|u_\varepsilon\|_{L^2(B_\varepsilon(3))},$$

which is the desired result. □

Lemma 4.1.4. *Suppose $\mathcal{L}_{0,0}(v) = 0$ in $B(2r)$. For $r \geq \varepsilon$, there exists a constant C depending on $\omega, \kappa_1, \kappa_2$ and d such that*

$$\left(\fint_{B(r)} |v|^2\right)^{1/2} \leq C\left(\fint_{B_\varepsilon(2r)} |v|^2\right)^{1/2} \tag{4.1.5}$$

Proof. Let

$$T_\varepsilon = \{z \in \mathbb{Z}^d \,:\, \varepsilon(Q+z) \cap B(r) \neq \emptyset\},$$

and fix $z \in T_\varepsilon$. Let $\{H_k\}_{k=1}^N$ denote the bounded, connected components of $\mathbb{R}^d \backslash \omega$ with $H_k \cap (Q+z) \neq \emptyset$. Define $\varphi_k \in C_0^\infty(Q^*(z))$ by

$$\begin{cases} \varphi_k(x) = 1, & \text{if } x \in H_k, \\ \varphi_k(x) = 0, & \text{if } \operatorname{dist}(x, H_k) > \frac{1}{4}\mathfrak{g}^\omega, \\ |\nabla \varphi_k| \leq C, \end{cases}$$

where C depends on ω, $\mathfrak{g}^\omega > 0$ as defined in Section 3.1 by (3.1.4), and

$$Q^*(z) = \bigcup_{j=1}^{3^d} (Q + z_j), \quad z_j \in \mathbb{Z}^d \text{ and } |z - z_j| \leq \sqrt{d}.$$

Set $\varphi = \sum_{k=1}^N \varphi_k \in C_0^\infty(Q^*)$, where $Q^* = Q^*(z)$. Note by construction $\varphi \equiv 1$ in $Q^* \backslash \omega$.

Set $V(x) = v(\varepsilon x)$. Note $\mathcal{L}_{0,0}(V) = 0$ in $Q + z$. By Poincaré's and Cacciopoli's inequalities,

$$\int_{(Q+z)\backslash\omega} |V|^2 \leq \sum_{k=1}^N \int_{H_k} |V|^2 \leq C \int_{Q^*} |\nabla(V\varphi)|^2 \leq C \int_{Q^*} |V|^2 |\nabla\varphi|^2,$$

where C depends on ω, κ_1, κ_2, and d but is independent of z given the periodic structure of ω. Specifically, since $\nabla\varphi = 0$ in $Q^*\backslash\omega$ and $(Q+z) \subset Q^*$,

$$\int_{(Q+z)\cap\omega} |V|^2 + \int_{(Q+z)\backslash\omega} |V|^2 \leq C \int_{Q^*\cap\omega} |V|^2,$$

where C only depends on ω, κ_1, κ_2, and d. Making the change of variables $\varepsilon x \mapsto x$ gives

$$\int_{\varepsilon(Q+z)} |v|^2 \leq C \int_{\varepsilon(Q^*\cap\omega)} |v|^2.$$

Summing over all $z \in T_\varepsilon$ gives the desired inequality, since there is a constant $M < \infty$ depending only on d such that $Q^*(z_1) \cap Q^*(z_2) \neq \emptyset$ for at most M coordinates $z_2 \in \mathbb{Z}^d$ different from z_1. $\square$

For $w \in L^2(B_\varepsilon(r); \mathbb{R}^d)$ and $\varepsilon, r > 0$, set

$$H_\varepsilon(r; w) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d\times d} \\ q \in \mathbb{R}^d}} \left(\fint_{B_\varepsilon(r)} |w - Mx - q|^2 \right)^{1/2}, \tag{4.1.6}$$

and set

$$H_0(r; w) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d\times d} \\ q \in \mathbb{R}^d}} \left(\fint_{B(r)} |w - Mx - q|^2 \right)^{1/2}.$$

Lemma 4.1.5. *Let v be a solution of $\mathcal{L}_{0,0}(v) = 0$ in $B(r)$. For $r \geq \varepsilon$, there exists a $\theta \in (0, 1/4)$ such that*

$$H_\varepsilon(\theta r; v) \leq \frac{1}{2} H_\varepsilon(r; v).$$

The constant θ depends on d, κ_1, and κ_2.

Proof. By Lemma 3.1.1, there exists a constant C_1 depending only d, κ_1, and κ_2 such that

$$H_\varepsilon(r; v) \leq C_1 H_0(r; v)$$

for any $r > 0$. It follows from interior C^2-estimates for elasticity systems with constant coefficients that there exists $\theta \in (0, 1/4)$ with

$$H_0(\theta r; v) \leq \frac{1}{2C_2} H_0(r/2; v),$$

where $C_2 = C_3 C_1$ and C_3 is the constant in (4.1.5) given in Lemma 4.1.4. By Lemma 4.1.4, we have the desired inequality. □

Lemma 4.1.6. *Suppose $\mathcal{L}_{\varepsilon,0}(u_\varepsilon) = 0$ in $B_\varepsilon(2r)$. For $r \geq \varepsilon$,*

$$H_\varepsilon(\theta r; u_\varepsilon) \leq \frac{1}{2} H_\varepsilon(r; u_\varepsilon) + \frac{C}{r}\left(\frac{\varepsilon}{r}\right)^{1/2} \inf_{q \in \mathbb{R}^d} \left(\fint_{B_\varepsilon(3r)} |u_\varepsilon - q|^2 \right)^{1/2}$$

Proof. With r fixed, let $v_r \equiv v$ denote the function guaranteed in Lemma 4.1.3. Observe then

$$\begin{aligned} H_\varepsilon(\theta r; u_\varepsilon) &\leq \frac{1}{\theta r} \left(\fint_{B_\varepsilon(\theta r)} |u_\varepsilon - v|^2 \right)^{1/2} + H_\varepsilon(\theta r; v) \\ &\leq \frac{C}{r} \left(\fint_{B_\varepsilon(r)} |u_\varepsilon - v|^2 \right)^{1/2} + \frac{1}{2} H_\varepsilon(r; v) \\ &\leq \frac{C}{r} \left(\fint_{B_\varepsilon(r)} |u_\varepsilon - v|^2 \right)^{1/2} + \frac{1}{2} H_\varepsilon(r; u_\varepsilon), \end{aligned}$$

where we've used Lemma 4.1.5 and the constant C may depend on θ. By Lemma 4.1.3, we have

$$H_\varepsilon(\theta r; u_\varepsilon) \leq \frac{C}{r}\left(\frac{\varepsilon}{r}\right)^{1/2} \left(\fint_{B_\varepsilon(3r)} |u_\varepsilon|^2 \right)^{1/2} + \frac{1}{2} H_\varepsilon(r; u_\varepsilon).$$

Since H remains invariant if we subtract a constant from u_ε (i.e., the same argument holds for $u_\varepsilon - q$ with $q \in \mathbb{R}^d$), the desired inequality follows. □

Lemma 4.1.7. *Let $H(r)$ and $h(r)$ be two nonnegative continous functions on the interval* $(0, 1]$. *Let* $0 < \varepsilon < 1/6$. *Suppose that there exists a constant C_0 with*

$$\begin{cases} \max\limits_{r \leq t \leq 3r} H(t) \leq C_0 H(3r), \\ \max\limits_{r \leq t, s \leq 3r} |h(t) - h(s)| \leq C_0 H(3r), \end{cases}$$

for any $r \in [\varepsilon, 1/3]$. We further assume

$$H(\theta r) \leq \frac{1}{2}H(r) + C_0 \left(\frac{\varepsilon}{r}\right)^{1/2} \{H(3r) + h(3r)\}$$

for any $r \in [\varepsilon, 1/3]$, where $\theta \in (0, 1/4)$. Then

$$\max_{\varepsilon \leq r \leq 1} \{H(r) + h(r)\} \leq C\{H(1) + h(1)\},$$

where C depends on C_0 and θ.

Proof. See [26]. □

Proof of Theorem 4.0.2. By rescaling, we may assume $R = 1$. Indeed, we may consider $U_\varepsilon(x) = u_{\varepsilon,0}(x/R)$, which satisfies $\mathcal{L}_{R\varepsilon,0}(U_\varepsilon) = 0$ in $B_\varepsilon(x_0, 1)$. Moreover, we assume $\varepsilon \in (0, 1/6)$. If $\varepsilon \geq 1/6$, then the desired result follows classical results regarding interior estimates for systems of elasticity. Let $H(r) \equiv H_\varepsilon(r; u_{\varepsilon,0})$, where $H_\varepsilon(r; u_{\varepsilon,0})$ is defined above by (4.1.6). Let $h(r) = |M_r|$, where $M_r \in \mathbb{R}^{d\times d}$ satisfies

$$H(r) = \frac{1}{r} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(r)} |u_{\varepsilon,0} - M_r x - q|^2 \right)^{1/2}.$$

Note there exists a constant C independent of r so that

$$H(t) \leq CH(3r), \quad t \in [r, 3r]. \tag{4.1.7}$$

Indeed, for $t \in [r, 3r]$,

$$H(t) \leq \frac{1}{r} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(t)} |u_{\varepsilon,0} - Mx - q|^2 \right)^{1/2} \leq 3^{d+1} c_d^{1/2} H(3r),$$

where c_d denotes the volume of the unit ball in $\mathbb{R}^d$. Suppose $s, t \in [r, 3r]$. We have

$$\begin{aligned}
|h(t) - h(s)| &\leq \frac{C}{r} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(r)} |(M_t - M_s)x - q|^2 \right)^{1/2} \\
&\leq \frac{C}{t} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(t)} |u_{\varepsilon,0} - M_t x - q|^2 \right)^{1/2} \\
&\qquad + \frac{C}{s} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(s)} |u_{\varepsilon,0} - M_s x - q|^2 \right)^{1/2} \\
&\leq CH(3r),
\end{aligned}$$

where we've used (4.1.7) for the last inequality. Specifically,

$$\max_{r\leq t,s\leq 3r} |h(t) - h(s)| \leq CH(3r). \tag{4.1.8}$$

Clearly

$$\frac{1}{r} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(3r)} |u_{\varepsilon,0} - q|^2 \right)^{1/2} \leq H(3r) + h(3r), \tag{4.1.9}$$

and so Lemma 4.1.6 implies

$$H(\theta r) \leq \frac{1}{2} H(r) + C \left(\frac{\varepsilon}{r}\right)^{1/2} \{H(3r) + h(3r)\} \tag{4.1.10}$$

for any $r \in [\varepsilon, 1/3]$ and some $\theta \in (0, 1/4)$. Note equations (4.1.7), (4.1.8), and (4.1.10) show that $H(r)$ and $h(r)$ satisfy the assumptions of Lemma 4.1.7. Consequently, (4.1.9) and Lemmas 4.1.2 and 4.1.7 imply

$$\begin{aligned} \left(\fint_{B_\varepsilon(r)} |\nabla u_{\varepsilon,0}|^2\right)^{1/2} &\leq \frac{C}{r} \inf_{q\in\mathbb{R}^d} \left(\fint_{B_\varepsilon(3r)} |u_{\varepsilon,0} - q|^2\right)^{1/2} \\ &\leq C\{H(3r) + h(3r)\} \\ &\leq C\{H(1) + h(1)\} \\ &\leq C\left(\fint_{B_\varepsilon(1)} |u_{\varepsilon,0}|^2\right)^{1/2}. \end{aligned} \tag{4.1.11}$$

Since (4.1.11) remains invariant if we subtract a constant from $u_{\varepsilon,0}$, the desired estimate in Theorem 4.0.1 follows. □

4.2 Consequences of Theorem 4.0.1

In this section, we first combine the large-scale estimate Theorem 3.0.1 with local estimates for mixed boundary value problems to derive interior estimates at both the macroscopic and microscopic scale. In particular, we prove Corollary 4.0.2. The proof essentially uses the "blow-up" argument of Lemma 4.1.2 and classical results regarding elliptic systems in smooth domains with mixed boundary values.

Next, we prove a Liouville-type estimate for systems of elasticity with periodic coefficients in unbounded domains with periodic structure. The proof of this estimate requires only Theorem 3.0.1, and so it requires minimal regularity on the coefficients A, i.e., the estimate requires the coefficients to be measurable, bounded, and periodic.

Proof of Corollary 4.0.2. Under the Hölder continuous condition (4.0.2) and the assumption that ω is an unbounded $C^{1,\alpha}$ domain for some $\alpha > 0$, solutions to the systems of linear elasticity are known to be locally Lipschitz. That is, if $\mathcal{L}_{1,0}(u) = 0$ in $B(y, 1)$, then

$$\|\nabla u\|_{L^\infty(B(y,1/3)\cap\omega)} \leq C \left(\fint_{B(y,1)\cap\omega} |\nabla u|^2\right)^{1/2}, \tag{4.2.1}$$

where C depends on d, κ_1, κ_2, and ω.

By rescaling, we may assume $R = 1$. To prove the desired estimate, assume $\varepsilon \in (0, 1/6)$. Indeed, if $\varepsilon \geq 1/6$, then (4.0.2) follows from (4.2.1). From (4.2.1), a "blow-up argument," and Theorem 4.0.1 we deduce

$$\begin{aligned}\|\nabla u_{\varepsilon,0}\|_{L^\infty(B(y,\varepsilon)\cap\varepsilon\omega)} &\leq C\left(\fint_{B(y,3\varepsilon)\cap\varepsilon\omega}|\nabla u_{\varepsilon,0}|^2\right)^{1/2}\\ &\leq C\left(\fint_{B(x_0,1)\cap\varepsilon\omega}|\nabla u_{\varepsilon,0}\|^2\right)^{1/2}\end{aligned}$$

for any $y \in B(x_0, 1/3)$. The deisred esitmate readily follows by covering $B(x_0, 1/3)$ with balls $B(y, \varepsilon)$. □

Corollary 4.2.1. *Suppose A satisfies (2.2.4), (2.2.5), and (2.2.7), and suppose ω is an unbounded Lipschitz domain with 1-periodic structure. Let u denote a weak solution of $\mathcal{L}_{1,0}(u) = 0$ in ω. Assume*

$$\left(\fint_{B(0,R)\cap\omega}|u|^2\right)^{1/2} \leq CR^\nu, \tag{4.2.2}$$

for some $\nu \in (0, 1)$, some constant $C := C(u) > 0$, and for all $R > 1$. Then u is constant in ω.

Proof of Corollary 4.2.1. Fix $r > 0$ and let $R \geq r$. If u satisfies the growth condition (4.2.2), then by Lemma 4.1.1 and Theorem 3.0.1,

$$\left(\fint_{B(x_0,r)\cap\omega}|\nabla u|^2\right)^{1/2} \leq C\left(\fint_{B(x_0,R)\cap\omega}|\nabla u|^2\right)^{1/2} \leq CR^{\nu-1},$$

where C is independent of R. Take $R \to \infty$ and note $\nabla u = 0$ for arbitrarily large r. Since ω is connected, we conclude u is constant. □

4.3 Large-scale interior Lipschitz estimates in materials reinforced with soft inclusions

In this section, we discuss *a priori* interior estimates for the boundary value problem (2.2.8) with $\delta > 0$ at the macroscopic scale by proving Theorem 4.0.3. By macroscopic, we refer to the case when $\varepsilon/r \geq 1$. Throughout this section, let $B(r) \equiv B(x_0, r)$ denote the ball of radius $r > 0$ centered at some $x_0 \in \mathbb{R}^d$.

The following lemma is essentially Cacciopoli's inequality for the operator $\mathcal{L}_{\varepsilon,\delta}$ defined by (2.2.1). The proof is similar to a proof of the classical Cacciopoli's inequality, but nevertheless we present a proof for completeness.

Lemma 4.3.1. *Suppose $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $B(2)$. Then*

$$\left(\fint_{B(1)}|k_\delta^\varepsilon\nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \leq C\left(\fint_{B(2)}|k_\delta^\varepsilon u_{\varepsilon,\delta}|^2\right)^{1/2}$$

where C depends only on κ_1 and κ_2.

Proof. Let $\zeta \in C_0^\infty(B(2))$. Then

$$\begin{aligned} 0 &= \delta \int_{B(2)} k_\delta^\varepsilon A^\varepsilon \nabla u \cdot \nabla (u\zeta^2) \\ &= \delta \int_{B(2)} k_\delta^\varepsilon A^\varepsilon \nabla u \cdot (\nabla u)\zeta^2 + 2\delta \int_{B(2)} (u\zeta) k_\delta^\varepsilon A^\varepsilon \nabla u \cdot \nabla \zeta \end{aligned} \tag{4.3.1}$$

where $u \equiv u_{\varepsilon,\delta}$. Note since $\delta \leq 1$ and Cauchy-Schwarz,

$$2\delta \int_{B(2)} (u\zeta) k_\delta^\varepsilon A^\varepsilon \nabla u \cdot \nabla \zeta \leq C \left(\int_{B(2)} |k_\delta^\varepsilon \nabla u|^2 \zeta^2 \right)^{1/2} \left(\int_{B(2)} |k_\delta^\varepsilon u|^2 |\nabla \zeta^2| \right)^{1/2}.$$

Equation (4.3.1) then implies

$$0 \geq \delta \kappa_1 \int_{B(2)} k_\delta^\varepsilon |\nabla u|^2 \zeta^2 - C \left(\int_{B(2)} |k_\delta^\varepsilon \nabla u|^2 \zeta^2 \right)^{1/2} \left(\int_{B(2)} |k_\delta^\varepsilon u|^2 |\nabla \zeta^2| \right)^{1/2},$$

i.e.,

$$\int_{B(2)} \mathbf{1}_-^\varepsilon |\delta \nabla u|^2 \zeta^2 \leq \frac{C_1}{\gamma} \int_{B(2)} k_{\delta^2}^\varepsilon |\nabla u|^2 \zeta^2 + \gamma C_2 \int_{B(2)} k_{\delta^2}^\varepsilon u^2 |\nabla \zeta|^2$$

for any $\gamma > 0$. Similarly,

$$0 = \int_{B(2)} k_\delta^\varepsilon A^\varepsilon \nabla u \cdot \nabla (u\zeta^2)$$

implies

$$\int_{B(2)} \mathbf{1}_+^\varepsilon |\nabla u|^2 \zeta^2 \leq \frac{C_1}{\gamma} \int_{B(2)} k_{\delta^2}^\varepsilon |\nabla u|^2 \zeta^2 + \gamma C_2 \int_{B(2)} k_{\delta^2}^\varepsilon u^2 |\nabla \zeta|^2$$

Choosing γ large enough gives

$$\int_{B(2)} |k_\delta^\varepsilon \nabla u|^2 \zeta^2 \leq C \int_{B(2)} |k_\delta^\varepsilon u|^2 |\nabla \zeta^2|$$

for some constant C depending on κ_1 and κ_2. Choose ζ so that $\zeta \equiv 1$ in $B(1)$ and $|\nabla \zeta| \leq C$. The desired inequality follows. □

Lemma 4.3.2. *Suppose $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $B(3r)$. There exists $v \in H^1(B(r); \mathbb{R}^d)$ satisfying $\mathcal{L}_{0,\delta}(v) = 0$ in $B(r)$ and*

$$\left(\fint_{B(r)} |k_\delta^\varepsilon (u_{\varepsilon,\delta} - v)|^2 \right) \leq C \left(\frac{\varepsilon}{r} \right)^\mu \left(\fint_{B(3r)} |k_\delta^\varepsilon u_{\varepsilon,\delta}|^2 \right)^{1/2},$$

where C depends on κ_1, κ_2, and d and $\mu > 0$.

Proof. First we prove the lemma for $r = 1$. By Lemma 4.3.1 and Theorem 2.4.6 of Section 2.4,

$$\left(\fint_{B(3/2)} |\nabla P_\varepsilon(\mathbf{1}^\varepsilon_+ u)|^2\right)^{1/2} + \left(\fint_{B(3/2)} |\delta \nabla u|^2\right)^{1/2} \leq C\left(\fint_{B(3)} |k^\varepsilon_\delta u|^2\right)^{1/2},$$

where $u \equiv u_{\varepsilon,\delta}$. Specifically, there exists a $t \in [1, 5/4]$ such that

$$\|P_\varepsilon(\mathbf{1}^\varepsilon_+ u)\|_{H^1(\partial B(t))} + \delta\|u\|_{H^1(\partial B(t))} \leq C\|k^\varepsilon_\delta u\|_{L^2(B(3))}. \tag{4.3.2}$$

Let v denote the weak solution to the Dirichlet problem $\mathcal{L}_{0,\delta}(v) = 0$ in $B(t)$ and $v = P_\varepsilon(\mathbf{1}^\varepsilon_+ u)$ on $\partial B(t)$. Note $v = u = P_\varepsilon(\mathbf{1}^\varepsilon_+ u)$ on $\partial B(t) \cap \varepsilon\omega$. By Theorem 3.0.3,

$$\begin{aligned}\|k^\varepsilon_\delta(u - v)\|_{L^2(B(1))} \leq C\varepsilon^\mu \|P_\varepsilon(\mathbf{1}^\varepsilon_+ u)\|_{H^1(\partial B(t))} \\ + \delta\|\nabla P_\varepsilon(\mathbf{1}^\varepsilon_+ u) - \nabla u\|_{L^2(B(t))},\end{aligned} \tag{4.3.3}$$

since

$$\begin{aligned}\|k^\varepsilon_\delta \chi^\varepsilon_\delta K^2_\varepsilon((\nabla v)\eta_\varepsilon)\|_{L^2(B(t))} &\leq C\|\nabla v\|_{L^2(B(t))} \\ &\leq C\|P_\varepsilon(\mathbf{1}^\varepsilon_+ u)\|_{H^1(\partial B(t))},\end{aligned}$$

where we've used notation consistent with Theorem 3.0.3 and $\mu \geq \mu_0 > 0$ for some μ_0 indepedent of δ.

By Lemma 4.3.1,

$$\fint_{B(t)} |k^\varepsilon_\delta \nabla w|^2 \leq \fint_{B(t)} |\nabla P_\varepsilon(\mathbf{1}^\varepsilon_+ u)|^2 + C\fint_{B(2t)} |k^\varepsilon_\delta w|^2, \tag{4.3.4}$$

where $w = P_\varepsilon(\mathbf{1}^\varepsilon_+ u) - u$. Equation (4.3.4) follows from the fact that $\mathcal{L}_{\varepsilon,\delta}(w) = \mathcal{L}_{\varepsilon,\delta}(P_\varepsilon(\mathbf{1}^\varepsilon_+ u))$ in $B(3)$ and $t \in [1, 5/4]$. Note by Theorem 2.4.6, $w = 0$ a.e. in $B(3) \cap \varepsilon\omega$. Hence, Poincaré's inequality gives

$$\fint_{B(2t)} |k^\varepsilon_\delta w| = \delta^2 \fint_{B(2t)} \mathbf{1}^\varepsilon_- |w|^2 \leq \varepsilon^2 \fint_{B(3)} |k^\varepsilon_\delta \nabla w|^2. \tag{4.3.5}$$

Indeed, set $W(x) = w(\varepsilon x)$, and let $\{H_k\}_{k=1}^{N(\varepsilon)}$ denote the bounded, connected components of $\mathbb{R}^d \backslash \omega$ with $\varepsilon H_k \cap B(2t) \neq \emptyset$. Then $W = 0$ on ∂H_k for each k, and so

$$\int_{B(2t)} |w|^2 \leq \sum_{k=1}^N \int_{H_k} |W|^2 \leq C\varepsilon^2 \sum_{k=1}^N \int_{\varepsilon H_k} |\nabla w|^2 \leq C\varepsilon^2 \int_{B(3)} |\nabla w|^2,$$

where C is independent of ε since ω is periodic. Theorem 2.4.6 together with (4.3.2), (4.3.3) and (4.3.5) give the estimate for $r = 1$.

Now we prove the estimate for arbitrary $r > 0$. To this end, let $U(x) = u(rx)$, and note $\mathcal{L}_{\varepsilon/r,\delta}(U) = 0$ in $B(3)$. By the above, there exists a $V \in H^1(B(1); \mathbb{R}^d)$ satisfying $\mathcal{L}_{0,\delta}(V) = 0$ in $B(1)$ and

$$\left(\fint_{B(1)} |k^{\varepsilon/r}_\delta (U - V)|^2\right) \leq C\left(\frac{\varepsilon}{r}\right)^\mu \left(\fint_{B(3)} |k^{\varepsilon/r}_\delta U|^2\right)^{1/2},$$

The change of variables $rx \mapsto x$ gives the desired estimate.

□

Lemma 4.3.3. *Suppose $\mathcal{L}_{0,\delta}(v) = 0$ in $B(2r)$. Then for $r \geq \varepsilon$,*

$$\left(\fint_{B(r)} |v|^2 \right)^{1/2} \leq C \left(\fint_{B(2r)} |k_\delta^\varepsilon v|^2 \right)^{1/2} \tag{4.3.6}$$

for a constant C depending on ω, κ_1, κ_2, and d.

Proof. See Lemma 4.1.5 for a proof when $\delta = 0$. The case $\delta > 0$ follows with a similar argument as the coefficients $\widehat{A}_\delta$ are uniformly elliptic in $\mathbb{R}^d$ uniformly in δ (see Lemma 3.2.1). □

For $w \in L^2_{\text{loc}}(B(r); \mathbb{R}^d)$, $\delta \geq 0$, and $\varepsilon, r > 0$, set

$$H_{\varepsilon,\delta}(r; w) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{d\times d} \\ q \in \mathbb{R}^d}} \left(\fint_{B(r)} |k_\delta^\varepsilon (w - Mx - q)|^2 \right)^{1/2}. \tag{4.3.7}$$

Note this definition is consistent with (4.1.6).

Lemma 4.3.4. *Suppose v satisfies $\mathcal{L}_{0,\delta}(v) = 0$ in $B(1)$. For any $r \in [\varepsilon, 1]$, $\delta \in [0, 1]$, and $\theta \in (0, 1/4)$,*

$$H_{\varepsilon,\delta}(\theta r; v) \leq C\theta H_{\varepsilon,\delta}(r; v)$$

for some constant C depending on d, κ_1, κ_2, and ω.

Proof. It follows from interior C^2-estimates for elasticity systems with constant coefficients that for any $\theta \in (0, 1/4)$,

$$H_{\varepsilon,\delta}(\theta r; v) \leq H_{\varepsilon,1}(\theta r; v) \leq C_1 \theta H_{\varepsilon,1}(r/2; v),$$

where C_1 a constant depending on d, κ_1, κ_2. By Lemma 4.3.3, we have the desired estimate. □

Lemma 4.3.5. *Suppose $\mathcal{L}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $B(1)$. For any $\varepsilon \leq r \leq 1/3$,*

$$H_{\varepsilon,\delta}(\theta r; u_{\varepsilon,\delta}) \leq C_1 \theta H_{\varepsilon,\delta}(r; u_{\varepsilon,\delta}) + \frac{C_2}{r} \left(\frac{\varepsilon}{r} \right)^\mu \inf_{q \in \mathbb{R}^d} \left(\fint_{B(r)} |k_\delta^\varepsilon (u_{\varepsilon,\delta} - q)|^2 \right)^{1/2}$$

where $\theta \in (0, 1/4)$ and $\mu > 0$.

Proof. Fix $r \geq \varepsilon$, and let $v \equiv v_r$ denote the function given by Lemma 4.3.2. We have

$$\begin{aligned} H_{\varepsilon,\delta}(\theta r; u) &\leq \frac{1}{\theta r} \left(\fint_{B(\theta r)} |k_\delta^\varepsilon (u - v)|^2 \right)^{1/2} + H_{\varepsilon,\delta}(\theta r; v) \\ &\leq \frac{C}{r} \left(\fint_{B(r)} |k_\delta^\varepsilon (u - v)|^2 \right)^{1/2} + C_1 \theta H_{\varepsilon,\delta}(r; v) \\ &\leq \frac{C}{r} \left(\fint_{B(r)} |k_\delta^\varepsilon (u - v)|^2 \right)^{1/2} + C_1 \theta H_{\varepsilon,\delta}(r; u), \end{aligned}$$

where we've used Lemma 4.3.4 and $u \equiv u_{\varepsilon,\delta}$. By Lemma 4.3.2,

$$H_{\varepsilon,\delta}(\theta r; u) \leq \frac{C_2}{r}\left(\frac{\varepsilon}{r}\right)^{\mu}\left(\fint_{B(3r)} |k_\delta^\varepsilon u|^2\right)^{1/2} + C_1\theta H_{\varepsilon,\delta}(r; u). \tag{4.3.8}$$

Since (4.3.8) remains invariant if we subtract a constant from u, the desired estimate follows. □

Proof of Theorem 4.0.3. By rescaling, we may assume $R = 1$. We assume $\varepsilon \in (0, 1/6)$, and we let $H(r) \equiv H_{\varepsilon,\delta}(r; u)$, where $u \equiv u_{\varepsilon,\delta}$ and $H_{\varepsilon,\delta}(r; u)$ is defined above by (4.3.7). Let $h(r) = r^{-1}|M_r|$, where $M_r \in \mathbb{R}^{d\times d}$ satisfies

$$H(r) = \frac{1}{r}\inf_{q\in\mathbb{R}^d}\left(\fint_{B(r)} |k_\delta^\varepsilon(u - M_r x - q)|^2\right)^{1/2}.$$

Note there exists a constant C independent of r so that

$$H(t) \leq CH(3r), \quad t \in [r, 3r]. \tag{4.3.9}$$

Suppose $s, t \in [r, 3r]$. We have

$$\begin{aligned}
|h(t) - h(s)| &\leq \frac{C}{r}\inf_{q\in\mathbb{R}^d}\left(\fint_{B(r)} k_\delta^\varepsilon|(M_t - M_s)x - q|^2\right)^{1/2} \\
&\leq \frac{C}{t}\inf_{q\in\mathbb{R}^d}\left(\fint_{B(t)} k_\delta^\varepsilon|u - M_t x - q|^2\right)^{1/2} \\
&\qquad + \frac{C}{s}\inf_{q\in\mathbb{R}^d}\left(\fint_{B(s)} k_\delta^\varepsilon|u - M_s x - q|^2\right)^{1/2} \\
&\leq CH(3r),
\end{aligned}$$

where we've used (4.3.9) for the last inequality. Specifically,

$$\max_{r\leq t,s\leq 3r}|h(t) - h(s)| \leq CH(3r). \tag{4.3.10}$$

Clearly

$$\frac{1}{r}\inf_{q\in\mathbb{R}^d}\left(\fint_{B(3r)} |k_\delta^\varepsilon(u - q)|^2\right)^{1/2} \leq H(3r) + h(3r),$$

and so Lemma 4.3.5 implies

$$H(\theta r) \leq \frac{1}{2}H(r) + C\left(\frac{\varepsilon}{r}\right)^{\mu}\{H(3r) + h(3r)\} \tag{4.3.11}$$

for any $r \in [\varepsilon, 1/3]$ and some $\theta \in (0, 1/4)$. Note equations (4.3.9), (4.3.10), and (4.3.11) show that $H(r)$ and $h(r)$ satisfy the assumptions of Lemma 4.1.7. Consequently,

$$\begin{aligned} \left(\fint_{B(r)} |k_\delta^\varepsilon \nabla u|^2 \right)^{1/2} &\leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left(\fint_{B(3r)} |k_\delta^\varepsilon (u - q)|^2 \right)^{1/2} \\ &\leq C \{ H(3r) + h(3r) \} \\ &\leq C \{ H(1) + h(1) \} \\ &\leq C \left(\fint_{B(1)} |k_\delta^\varepsilon u|^2 \right)^{1/2}. \end{aligned} \tag{4.3.12}$$

Since (4.3.12) remains invariant if we subtract a constant from u, the desired estimate in Theorem 4.0.3 follows from Poincaré's inequality. □

4.4 Interior Lipschitz estimates in materials reinforces with soft inclusions at every scale

In this section, we combine the large-scale estimate Theorem 4.0.3 with $C^{1,\alpha}$ estimates for interface problems to derive interior estimates at both the macroscopic and microscopic scale. In particular, we prove Corollary 4.0.4.

To acheive our first goal, we prove the following lemma.

Lemma 4.4.1. *Suppose A satisfies* (2.2.4), (2.2.5)*, and is α-Hölder continuous for some $\alpha \in (0, 1)$, i.e., A satisfies* (4.0.2)*. Suppose ω is an unbounded $C^{1,\alpha}$ domain. Let $u_{1,\delta}$ denote a weak solution to $\mathcal{L}_{1,\delta}(u_\delta) = 0$ in $B(x_0, 1)$ for some $x_0 \in \mathbb{R}^d$. Then*

$$\|\nabla u_\delta\|_{C^{0,\alpha}(B(x_0,r) \cap \varepsilon\omega)} + \delta \|\nabla u_\delta\|_{C^{0,\alpha}(B(x_0,r) \setminus \varepsilon\omega)} \leq C \|k_\delta \nabla u_\delta\|_{L^2(B(x_0,1))},$$

for a constant C independent of δ and $0 < r \leq 1/3$. In particular,

$$\|k_\delta \nabla u_\delta\|_{L^\infty(B(x_0,r))} \leq C \|k_\delta \nabla u_\delta\|_{L^2(B(x_0,1))}$$

for $0 < r \leq 1/3$.

Lemma 4.4.1 was proved for scalar equations with diagonal coeffcients in smooth domains in [15, 30, 31]. Lemma 4.4.1 continues to hold for elliptic systems with coefficients and domains satisfying the given assumptions. Together, Lemma 4.4.1 and Theorem 4.0.3 give interior Lipschitz estimates for $\mathcal{L}_{\varepsilon,\delta}$ at every scale.

Let $\Gamma(\cdot, x)$ denote the matrix-valued fundamental solution associated with $\mathcal{L}_{1,1}$ in $\mathbb{R}^d$. That is, $\Gamma(\cdot, x) = \{\Gamma^{\alpha\beta}(\cdot, x)\}_{1 \leq \alpha, \beta \leq d}$ satisfies

$$f^\beta(x) = \int_{\mathbb{R}^d} a_{ij}^{\alpha\gamma}(\xi) \frac{\partial \Gamma^{\gamma\beta}}{\partial x_j}(\xi, x) \frac{\partial f^\alpha}{\partial x_i}(\xi) d\sigma(\xi)$$

for $f = \{f^\beta\}_{1 \leq \beta \leq d} \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$. Indeed, if A is VMO, i.e.,

$$\sup_{\substack{x \in \mathbb{R}^d \\ 0 < r < R}} \fint_{B(x,r)} \left| A(y) - \fint_{B(x,r)} A \right| dy \to 0 \quad \text{as } R \to 0^+ \tag{4.4.1}$$

then $\Gamma(\cdot, x) \in W^{1,1}_{\text{loc}}(\mathbb{R}^d \backslash \{x\}; \mathbb{R}^{d\times d})$ exists uniquely for each $x \in \mathbb{R}^d$ (see work of Hofmann and Kim [19] for $d \geq 3$ and work of Brown, Kim, and Taylor [8] for $d = 2$). If A satisfies (4.0.2), then A satisfies (4.4.1).

For a bounded, simply-connected domain H and $g \in L^2(\partial H; \mathbb{R}^d)$, the single-layer potential $\mathcal{S}g = \{(\mathcal{S}g)^\alpha\}_{1\leq\alpha\leq d}$ is given by

$$(\mathcal{S}g)^\alpha(x) = \int_{\partial H} \Gamma^{\alpha\beta}(x, \xi) g^\beta(\xi)\, d\sigma(\xi), \;\; x \in \mathbb{R}^d \backslash \partial H \tag{4.4.2}$$

and the double-layer potential $\mathcal{D}g = \{(\mathcal{D}g)^\alpha\}_{1\leq\alpha\leq d}$ is given by

$$(\mathcal{D}g)^\alpha(x) = \int_{\partial H} n_i(\xi) a^{\alpha\beta}_{ij}(\xi) \frac{\partial \Gamma^{\beta\gamma}}{\partial x_j}(\xi, x) g^\gamma(\xi)\, d\sigma(\xi), \;\; x \in \mathbb{R}^d \backslash \partial H \tag{4.4.3}$$

where $n(\xi) = \{n_i(\xi)\}_{1\leq i\leq d}$ denotes the unit vector outward normal to H at $\xi \in \partial H$.

It is known (see [21, Theorem 4.6]) that if $g \in L^2(\partial H; \mathbb{R}^d)$, then

$$\mathcal{D}g^\pm = \pm\frac{1}{2} g + \mathcal{K}g \quad \text{on } \partial H, \tag{4.4.4}$$

where $\mathcal{K}$ is given by

$$\mathcal{K}g(x) = \text{p.v.} \int_{\partial H} n_i(\xi) a^{\alpha\beta}_{ij}(\xi) \frac{\partial \Gamma^{\beta\gamma}}{\partial x_j}(\xi, x) g^\gamma(\xi)\, d\sigma(\xi), \;\; x \in \partial H$$

and

$$\mathcal{D}g^\pm(x) = \lim_{h\to 0^+} \mathcal{D}g(x \pm hn),$$

i.e., $\mathcal{D}g^+$ and $\mathcal{D}g^-$ denote the traces of $\mathcal{D}g$ on ∂H from the exterior of $\overline{H}$ and the interior of H, respectively. In particular, $w = \mathcal{D}g$ satisfies $\mathcal{L}_{1,1}(w) = 0$ in H and $w = (-\frac{1}{2} + \mathcal{K})g$ on ∂H. It is also known (see [21, Lemma 5.7]) that if H is Lipschitz and A satisfies (2.2.4) and (2.2.5), then

$$-\frac{1}{2} + \mathcal{K} : L^2(\partial H; \mathbb{R}^d) \to L^2(\partial H; \mathbb{R}^d) \tag{4.4.5}$$

is bounded and continuously invertible. For single equations, this follows from the compactness of $\mathcal{K}$ and Fredholm theory (see the argument of Yeh in [30, Lemma 3.2]). For systems with variable coefficients, the operator $\mathcal{K}$ is not compact (see the work of Kenig and Shen [21]). However, the operator is still boundedly invertible; for a proof of its invertibility on L^2 in this case, see [21]. The following lemma, however, is more or less known.

Lemma 4.4.2. *Suppose A satisfies* (2.2.4), (2.2.5) *and is α-Hölder continous, i.e., satisfies* (4.0.2), *for some $\alpha \in (0, 1)$. Suppose H is a bounded $C^{1,\alpha}$ domain. The operators*

$$\mathcal{S} : C^{0,\alpha}(\partial H; \mathbb{R}^d) \mapsto C^{1,\alpha}(\partial H; \mathbb{R}^d)$$

and

$$\mathcal{D} : C^{1,\alpha}(\partial H; \mathbb{R}^d) \mapsto C^{1,\alpha}(\partial H; \mathbb{R}^d)$$

defined by (4.4.2) *and* (4.4.3), *respectively, are bounded.*

From (4.4.4), we have the jump relations

$$g = w^+ - w^- \quad \text{and} \quad \left(\frac{\partial w}{\partial n}\right)^+ = \left(\frac{\partial w}{\partial n}\right)^-, \tag{4.4.6}$$

where $w = \mathcal{D}g$ and $\partial w/\partial n = n \cdot \nabla w$ denotes the normal derivative of w.

The following lemma essentially follows from the jump relations (4.4.6) and the regularity problems in $C^{0,\alpha}$ for the exterior Neumann and interior Dirichlet problems.

Lemma 4.4.3. *There exists a constant C depending on κ_1, κ_2 and H such that*

$$\|g\|_{1,\alpha} \leq C \left\| \left(-\frac{1}{2} + \mathcal{K}\right) g \right\|_{1,\alpha}$$

for any $g \in C^{1,\alpha}(\partial H; \mathbb{R}^d)$, where $\|\cdot\|_{1,\alpha} = \|\cdot\|_{C^{1,\alpha}(\partial H)}$.

For a proof of the following lemma, consult any introductory functional analysis text (e.g., [13]).

Lemma 4.4.4. *Let $T : X \to Y$ be a bounded linear operator between two Banach spaces X and Y. If $\|T\|_{X \to Y} < 1$, then there exists a constant C such that*

$$\|x\|_X \leq C\|(1 - T)x\|_Y$$

for any $x \in X$.

As mentioned in Section 3.1, any two connected components of $\mathbb{R}^d \backslash \omega$ are separated by some positive distance $\mathfrak{g}^\omega$. If $\mathbb{R}^d \backslash \omega = \cup_{k=1}^{\infty} H_k$, write H_k^* to denote the set

$$H_k^* = \{x \in \mathbb{R}^d : \operatorname{dist}(x, H_k) \leq \mathfrak{g}^\omega/4\}.$$

To prove Lemma 4.4.1, it suffices to show the result holds in each H_k^*. Indeed, if A satisfies (4.0.2), the boundedness of $\nabla u_{1,\delta}$ in the interior of ω follows from classical results regarding elliptic systems with Hölder continuous coefficients.

Lemma 4.4.5. *Suppose A satisfies* (2.2.4), (2.2.5)*, and is α-Hölder continuous for some $\alpha \in (0, 1)$. Suppose ω is an unbounded $C^{1,\alpha}$ domain. If $\mathcal{L}_{1,\delta}(u_{1,\delta}) = div(f)$ in H_k^* and $u_{1,\delta} = 0$ on ∂H_k^*, then*

$$\|k_\delta \nabla u_{1,\delta}\|_{C^{0,\alpha}(H_k^*)} \leq C \left\{ \|k_\delta \nabla u_{1,\delta}\|_{L^2(H_k^*)} + \|k_{\delta^{-1}} f\|_{C^{0,\alpha}(H_k^*)} \right\},$$

where C depends only on $\|A\|_{C^\alpha}$, α, ω, κ_1, κ_2.

Proof. Note that if $\delta_0 \leq \delta \leq 1$, then the result follows from general results regarding divergence form elliptic equations with α-Hölder continuous coefficients in $C^{1,\alpha}$ domains. Hence, we may assume $0 \leq \delta \leq \delta_0$ for some δ_0 to be determined.

Let u_1 satisfy the boundary value problem

$$\begin{cases} -\operatorname{div}(\delta^2 A \nabla u_1) = \operatorname{div}(f) & \text{in } H_k \\ -\operatorname{div}(A \nabla u_1) = \operatorname{div}(f) & \text{in } H_k^* \backslash H_k \\ u_1 = 0 & \text{on } \partial H_k \cup \partial H_k^* \end{cases}$$

By $C^{1,\alpha}$ estimates for elliptic systems with α-Hölder continuous coefficients in $C^{1,\alpha}$ domains (see [10, Chapter 9, Theorem 2.7]), we have

$$\|k_\delta \nabla u_1\|_{C^{0,\alpha}(H_k^*)} \leq C\|k_{\delta^{-1}} f\|_{C^{0,\alpha}(H_k^*)}. \tag{4.4.7}$$

Set $u_2 = u - u_1$, where $u \equiv u_{1,\delta}$. Note then u_2 satisfies the equation and jump conditions

$$\begin{cases} -\text{div}(k_{\delta^2} A\nabla u_2) = 0 \quad \text{in } H_k^* \\ \lfloor k_{\delta^2} A\nabla u_2 \rfloor_{\partial H_k} \cdot n = -\lfloor k_{\delta^2} A\nabla u_1 \rfloor_{\partial H_k} \cdot n \\ \lfloor u_2 \rfloor_{\partial H_k} = 0, \\ u_2 = 0 \quad \text{on } \partial H_k^* \end{cases} \tag{4.4.8}$$

where $\lfloor g \rfloor_{\partial H_k} = g^+ - g^-$, $g^\pm = \lim_{t\to 0^+} g(\cdot \pm tn)$, and n denotes the unit vector outward normal to H_k. Hence, for $x \in H_k$,

$$u_2(x) = -\int_{\partial H_k} \frac{\partial \Gamma}{\partial n_A}(x,y)u(y)\, d\sigma(y) + \Gamma(x,y)\frac{\partial u_2}{\partial n_A}(y)\, d\sigma(y), \tag{4.4.9}$$

where $\partial g/\partial n_A = A\nabla g \cdot n$. For $x \in H_k^* \backslash H_k$,

$$\begin{aligned} u_2(x) = &\int_{\partial H_k^*} \Gamma(x,y)\frac{\partial u_2}{\partial n_A^*}(y)\, d\sigma(y) \\ &- \int_{\partial H_k} \Gamma(x,y)\frac{\partial u_2}{\partial n_A}(y)\, d\sigma(y) + \frac{\partial \Gamma}{\partial n_A}(x,y)u(y)\, d\sigma(y), \end{aligned} \tag{4.4.10}$$

where n^* denotes the unit vector outward normal to H_k^*. Then (4.4.9) and (4.4.10) imply

$$u(x) = \left\{\frac{1}{2}u(x) - \mathcal{D}u(x)\right\} + \int_{\partial H_k} \Gamma(x,y)\frac{\partial u_2^-}{\partial n_A}(y)\, d\sigma(y), \tag{4.4.11}$$

and

$$\begin{aligned} u(x) = &\int_{\partial H_k^*} \Gamma(x,y)\frac{\partial u_2}{\partial n_A^*}(y)\, d\sigma(y) \\ &- \int_{\partial H_k} \Gamma(x,y)\frac{\partial u_2^+}{\partial n_A}(y)\, d\sigma(y) - \left\{-\frac{1}{2}u(x) - \mathcal{D}u(x)\right\}, \end{aligned} \tag{4.4.12}$$

for $x \in \partial H_k$ (see [22, Chapter 7]), where $\mathcal{D} \equiv \mathcal{D}_{\partial H_k}$. Equations (4.4.8), (5.1.1), and (4.4.12) then imply

$$\begin{aligned} &\left\{\frac{1}{2} + \mathcal{D}\right\} u(x) = \mathcal{S}\left(\frac{\partial u_2^-}{\partial n_A}\right)(x) \\ &\left\{\frac{1}{2} - \mathcal{D}\right\} u(x) = \int_{\partial H_k^*} \Gamma(x,y)\frac{\partial u_2}{\partial n_A^*}(y)\, d\sigma(y) - \mathcal{S}\left(\frac{\partial u_2^+}{\partial n_A}\right)(x), \end{aligned}$$

where $\mathcal{S} \equiv \mathcal{S}_{\partial H_k}$. Finally, by (4.4.8),

$$\begin{aligned}&\left[1 - 2\left(\frac{1-\delta^2}{1+\delta^2}\right)\mathcal{D}\right] u(x) \\ &\qquad = \frac{2}{1+\delta^2}\left\{\int_{\partial H_k^*} \Gamma(x,y)\frac{\partial u_2}{\partial n_A^*}(y)\, d\sigma(y) + \mathcal{S}\left(\lfloor k_{\delta^2} A\nabla u_1\rfloor_{\partial H_k}\cdot n\right)\right\}.\end{aligned}$$

Choose δ_0 small enough so that by Lemma 4.4.4 we have

$$\|\nabla u_2\|_{C^{0,\alpha}(\partial H_k)} \le C\left\{\|\lfloor k_{\delta^2} A\nabla u_1\rfloor_{\partial H_k}\|_{C^{0,\alpha}(\partial H_k)} + \|\partial u_2/\partial n_A\|_{C^{0,\alpha}(\partial H_k^*)}\right\},$$

for $0 \le \delta \le \delta_0$, where C is some constant independent of δ. Indeed, it is sufficient to take δ_0 so that

$$4C\left(\frac{\delta^2}{1+\delta^2}\right) < 1 \quad \text{for } \delta \le \delta_0,$$

where C is a constant depending only on the operator norm of $\mathcal{D}$, which is finite by Lemma (4.4.2). By (4.4.8) and (4.4.7),

$$\|k_\delta \nabla u_2\|_{C^{0,\alpha}(H_k^*)} \le C\|k_{\delta^{-1}} f\|_{C^{0,\alpha}(H_k^*)}. \tag{4.4.13}$$

Equations (4.4.7) and (4.4.13) give the desired estimate. □

Now combine the local $C^{1,\alpha}$ estimates of Lemma 4.4.1 with the large-scale Lipcshitz estimates of Theorem 4.0.3, i.e., we prove Corollary 4.0.4.

Proof of Corollary 4.0.4. By rescaling, we may assume $R = 1$. To prove the desired estimate, assume $\varepsilon \in (0, 1/9)$. Indeed, if $\varepsilon \ge 1/9$, then (4.0.3) follows from Theorem 4.0.3. From Lemma 4.4.1, Theorem 4.0.3, and a "blow-up argument" (see the proof of Lemma 4.3.1), we deduce

$$\begin{aligned}\|k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}\|_{L^\infty(B(y,\varepsilon))} &\le C\left(\fint_{B(y,3\varepsilon)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2} \\ &\le C\left(\fint_{B(x_0,1)} |k_\delta^\varepsilon \nabla u_{\varepsilon,\delta}|^2\right)^{1/2}\end{aligned}$$

for any $y \in B(x_0, 1/3)$. The desired estimate follows by covering $B(x_0, 1/3)$ with balls $B(y, \varepsilon)$. □

Chapter 5 Open problems and future research

There are still many questions to be answered with regards to system (2.2.8). In this chapter, we discuss some open problems.

5.1 Large-scale boundary estimates

In the case $\delta = 1$, the solution $u_{\varepsilon,1}$ is known to be Lipschitz not only in the interior but also near the boundary [26]. A scheme similar to the one in Section 4.1 can be used to prove the necessary large-scale estimates. For the small-scale estimates, one again relies on known results regarding boundary regularity of divergence-form elliptic systems in smooth domains.

Question 5.1.1. *Can uniform Lipschitz estimates for* (2.2.8) *with* $\delta = 0$ *be established in* $C^{1,\alpha}$ *domains? In particular, what is the correct setting for boundary Lipschitz estimates in perforated domains?*

At the large-scale, with a few slight modifications boundary estimates should be clear. As mentioned above, the same scheme used for the interior should also apply for the boundary case. However, it is unclear that for systems an estimate such as (4.0.1) holds at every scale. For the case $\delta = 0$, a simple scaling gives rise to a system with mixed boundary values. Without further assumptions, mixed boundary value problems are not guaranteed to have solutions that are even $C^{0,\alpha}$ for each $0 < \alpha < 1$. Hence, one initially should not expect Lipschitz estimates near the boundary at every scale.

One alternative setup, however, is to consider domains of type II mentioned in [24]. Specifically, type II domains exclude perforations in an ε-size boundary layer of Ω. Then, upon scaling, one attains a Dirichlet problem near the boundary and one can again rely on know results regarding divergence-form elliptic operators in smooth domains.

Similarly, one can ask, "What is the correct setting in the case of materials reinforced with soft inclusions?" Issues regarding boundary estimates for interface problems arise. In particular, the optimal regularity for solutions to interface problems in smooth domains with interfaces near the boundary is unknown. Of course, by considering domains of type II, one can rely on known results for small-scale regularity results.

5.2 $W^{1,p}$ estimates

Also established for the case $\delta = 1$ are $W^{1,p}$ estimates uniform in ε provided $A \in VMO(\mathbb{R}^d)$, i.e., satisfies (4.4.1) [26]. The same L^p gradient estimates for single constant-coefficient elliptic equations in smooth domains were established by Yeh in [31] for $0 < \delta \leq 1$. When $\delta = 1$, the result follows from interior $W^{1,p}$-estimates at the scale ε, large-scale interior Lipschitz estimates, and boundary Hölder estimates. In particular, local microscopic $W^{1,p}$ estimates together with large-scale interior Lipschitz give interior $W^{1,p}$ estimates, and interior $W^{1,p}$ estimates together with large-scale boundary Hölder estimates establish boundary $W^{1,p}$ estimates.

Question 5.2.1. *For $p \in (2, \infty)$, if $\mathcal{L}_{1,\delta}(u_{1,\delta}) = 0$ in $B(x_0, 2)$, does the estimate*

$$\left(\fint_{B(x_0,1)} |k_\delta \nabla u_{1,\delta}|^p \right)^{1/p} \leq C \left(\fint_{B(x_0,2)} |k_\delta \nabla u_{1,\delta}|^2 \right)^{1/2} \tag{5.2.1}$$

hold for a constant $C = C(d, \kappa_1, \kappa_2, p, [A]_{VMO})$?

If such an estimate were to hold, then an affirmative answer to Question 5.1.1 would provide the expected L^p-gradient estimates for (2.2.8). Of course, a duality argument would provide the estimate for $p \in (1, 2)$. It should be noted that (5.2.1) is at the scale ε, and so this is really a question for interface problems arising in elasticity, i.e., elliptic systems with discontinuous coefficients having some piecewise regularity. If the coefficients are Hölder continuous, then estimate (5.2.1) follows from a layer potential argument due to Escauriaza, Fabes, and Verchota [15]. An estimate like (5.2.1) together with optimal boundary regularity allows one to establish Rellich type estimates, which are not known to be accessible through compactness methods.

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Vita

Education

- **University of Kentucky** — Lexington, KY
  M.A., Mathematics — *May 2015*
- **Western Kentucky University (WKU)** — Bowling Green, KY
  B.A., Mathematics, Summa cum Laude — *May 2013*

Publications

B. Chase Russell. "Homogenization in perforated domains and interior Lipschitz estimates." *Journal of Differential Equations*, 263:3396-3418, 2017.

B. Chase Russell. "Homogenization with soft inclusions and interior Lipschitz estimates at every scale." Under revision.

Awards/Financial Support

Supported in part by NSF grant DMS-1600520	2017
College of Arts and Sciences Certificate for Outstanding Teaching, UK	2016
Supported in part by NSF grant DMS-1161154	2015
Henry M. and Zula G. Yarbrough Award in Mathematics, WKU	2013
Hugh F. and Katherine A. Johnson Award in Mathematics, WKU	2012
Robert C. Bueker Award in Mathematics, WKU	2011

Professional Positions

(2013-2018) **Graduate Teaching Assistant**, Mathematics Department, University of Kentucky, Lexington, KY

Brandon Chase Russell
Birthplace: Bowling Green, KY