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TORIC VARIETIES AND COBORDISM

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TORIC VARIETIES AND COBORDISM

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
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ABSTRACT OF DISSERTATION

TORIC VARIETIES AND COBORDISM

A long-standing problem in cobordism theory has been to find convenient manifolds to represent cobordism classes. For example, in the late 1950's, Hirzebruch asked which complex cobordism classes can be represented by smooth connected algebraic varieties. This question is still open. Progress can be made on this and related problems by studying certain convenient connected algebraic varieties, namely smooth projective toric varieties. The primary focus of this dissertation is to determine which complex cobordism classes can be represented by smooth projective toric varieties. A complete answer is given up to dimension six, and a partial answer is described in dimension eight. In addition, the role of smooth projective toric varieties in the polynomial ring structure of complex cobordism is examined. More specifically, smooth projective toric varieties are constructed as polynomial ring generators in most dimensions, and evidence is presented suggesting that a smooth projective toric variety can be chosen as a polynomial generator in every dimension. Finally, toric varieties with an additional fiber bundle structure are used to study some manifolds in oriented cobordism. In particular, manifolds with certain fiber bundle structures are shown to all be cobordant to zero in the oriented cobordism ring.

KEYWORDS: toric variety, cobordism, fan, polytope, blow-up

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Chapter 1 Introduction

A *toric variety* is a certain type of algebraic variety that contains a torus which acts on the variety in a prescribed manner. Surprisingly, there is a bijective correspondence between toric varieties and certain objects from convex geometry called *fans*. This correspondence allows one to study complicated algebraic, geometric, and topological properties of toric varieties by examining the often more easily understood combinatorial properties of the corresponding fans.

Since they were first studied in the 1970's, the convenient combinatorial structure of toric varieties has encouraged mathematicians to use them in numerous seemingly disparate branches of mathematics. They typically arise in any area which involves algebraic varieties or fans, including algebraic geometry, polytope theory, linear optimization, coding theory, and mathematical physics (see [11, Appendix A] for details). The combinatorial nature of toric varieties often facilitates performing computations with them. For this reason, they are commonly used as special cases when an understanding of more general algebraic varieties is not feasible. In many situations, a large amount of information can be gleaned from these special cases alone.

Algebraic varieties also appear in many topological problems. One example is cobordism theory, which studies a certain equivalence relation between manifolds that can be thought of as a generalization of diffeomorphism. Chapter 2 will provide a brief introduction to the aspects of complex and oriented cobordism theory that are pertinent to this work.

If we consider the equivalence classes under the cobordism relation, we may wonder when such a *cobordism class* contains an algebraic variety. In other words, when can a manifold be deformed through this cobordism relation to obtain a much "nicer" algebraic variety, which can then be studied using techniques from algebraic geometry? Milnor answered this question in the 1950's by proving that every cobordism class with positive dimension can be represented by a smooth algebraic variety [41, Chapter VII]. However, his nonconstructive proof involves taking disjoint unions of varieties, so the smooth algebraic variety representatives are not necessarily connected. Hirzebruch posed the following related question in 1958.

Problem. ([22]) Which complex cobordism classes can be represented by *connected* smooth algebraic varieties?

Very little progress has been made on this difficult problem. This makes it an excellent candidate for the utilization of toric varieties. Smooth projective toric varieties are examples of connected smooth algebraic varieties. Their combinatorial structure could allow us to approach a toric version of Hirzebruch's question. Chapter 3 provides an introduction to toric varieties from a combinatorial perspective and describes several constructions that are helpful when utilizing toric varieties in cobordism. This background will allow us to consider the following toric version of Hirzebruch's question in Chapter 4.

Problem. Which complex cobordism classes can be represented by smooth projective toric varieties?

Answering this would at least partially resolve Hirzebruch's original question, and it might reveal techniques which could be generalized to other nontoric algebraic varieties. Unfortunately, even this greatly simplified question seems to have a complicated answer. Only in the lowest two complex dimensions does the presence of toric varieties in cobordism classes depend strictly on their combinatorial structure. For higher dimensions, the representation of a cobordism class by a toric variety seems to be determined by an intricate interplay of the combinatorics *and* geometry of toric varieties.

In Chapter 4, the above question is completely answered in complex dimension three. The outcome depends on values of the cobordism-invariant Chern numbers c_1c_2, c_3, and c_1^3.

Theorem. *Let $[M]$ be a cobordism class of complex dimension three.*

1. *If $c_1c_2\,[M] \neq 24$ or $c_3\,[M] \notin \{4, 6, 8, \ldots\}$, then $[M]$ is not represented by a smooth projective toric variety.*
2. *Suppose $c_1c_2\,[M] = 24$ and $c_3\,[M] = 4$. Then $[M]$ is represented by a smooth projective toric variety if and only if $[M] = [\mathbb{C}P^3]$.*
3. *Suppose $c_1c_2\,[M] = 24$ and $c_3\,[M] = 6$. Then $[M]$ is represented by a smooth projective toric variety if and only if $c_1^3\,[M] = 2a^2 + 54$ for some $a \in \mathbb{Z}$.*
4. *If $c_1c_2\,[M] = 24$ and $c_3\,[M] \in \{8, 10, 12, \ldots\}$, then $[M]$ is represented by a smooth projective toric variety.*

The techniques used to prove this theorem only yield partial results in complex dimension four. In particular, note that if c_3 is sufficiently large (and even) in complex dimension 3, then it no longer provides an obstruction to a cobordism class containing a smooth projective toric variety. A similar asymptotic result holds in complex dimension four.

Theorem. *Let $[M]$ be a cobordism class of complex dimension four. Choose integers g_1 and g_2 such that $2 \leq g_2 \leq g_1 - 1$. Suppose the following conditions are satisfied.*

$$\begin{aligned} c_4[M] &= 5 + 3g_1 + g_2 \\ c_1c_3[M] &= 50 + 6g_1 - 2g_2 \\ c_1^4[M] &= 4c_1^2c_2[M] + 3c_2^2[M] + 3g_1 - 3g_2 - 675 \end{aligned}$$

Then $[M]$ is represented by a smooth projective toric variety.

The computational complexity of these techniques grows quickly with the dimension, so other methods will likely be needed to study toric varieties in higher-dimensional cobordism.

The role that toric varieties play in representing individual cobordism classes is quite complicated. However, toric varieties likely play a much simpler and quite useful role in the algebraic structure of this set of cobordism classes. The set of complex cobordism classes forms a polynomial ring with a generator in each even dimension [37, 44]. The standard method for describing such generators involves taking products and disjoint unions of complex projective spaces $\mathbb{C}P^i$ and *Milnor hypersurfaces* $\mathcal{H}_{i,j} \subset \mathbb{C}P^i \times \mathbb{C}P^j$ [38]. While this method proves the existence of smooth algebraic not necessarily connected generators, it does not give a useful way of explicitly describing these generators in each dimension.

In 1998, Buchstaber and Ray provided another set of generators for the polynomial ring of complex cobordism [5, 7]. They proved that polynomial ring generators can be constructed by taking products and connected sums of certain smooth projective toric varieties. Unfortunately, the operation of connected sum does not preserve algebraicity, so the resulting generators are not themselves algebraic varieties. It is currently unknown whether or not every polynomial generator can be represented by a smooth connected algebraic variety.

The purpose of Chapter 5 is to introduce a drastically different approach to constructing polynomial generators of complex cobordism. This new method involves calculating certain cobordism invariants of a specific class of fairly simple smooth projective toric varieties which were classified by Kleinschmidt [29]. Applying a sequence of blow-ups to these varieties eventually produces new smooth projective toric varieties that can be used as polynomial generators. In particular, we do not need to take products or connected sums of manifolds to produce connected generators. It seems likely that this method will provide a smooth projective toric variety for each polynomial generator of complex cobordism. This would verify that it is indeed possible to choose a smooth connected algebraic variety for each generator. The following theorem will be proven in Chapter 5.

Theorem. *If n is odd or n is one less than a power of a prime, then the cobordism class of a smooth projective toric variety can be chosen for the cobordism ring polynomial generator of dimension $2n$.*

It seems very likely that generators can be found in the remaining even dimensions as well using a similar strategy. In fact, this would be a consequence of a certain number theory conjecture. While a proof of this conjecture remains elusive, there is a significant amount of numerical evidence that supports it.

Theorem. *If $n \leq 100001$, then the cobordism class of a smooth projective toric variety can be chosen for the cobordism ring polynomial generator of dimension $2n$.*

Chapter 6 provides an example of how working with toric varieties can inspire techniques that can be applied to more general objects. At the beginning of this chapter, certain projective toric varieties called *Bott towers* are examined. These varieties have a very strict fiber bundle structure, which aids in computations in oriented cobordism. Techniques that work for Bott towers serve as motivation for proving that certain generalized manifolds with a similar fiber bundle structure vanish in oriented cobordism.

Proposition. *Let ξ be a two-dimensional complex vector bundle over a compact, stably complex manifold N. Let $\mathbb{C}P(\xi) = (M, \pi, N)$ be its projectivization. Then the cobordism class of M vanishes in oriented cobordism.*

A generalization of this proposition involving hypersurfaces will then be considered. In this case, the proof is much more involved.

Theorem. *Let ξ_1 be a two-dimensional complex vector bundle over a compact, almost complex manifold P of dimension $2(n-2)$. Let $\mathbb{C}P(\xi_1) = (N, \pi_1, P)$ denote its projectivization. Now let ξ_2 be a two-dimensional complex vector bundle over N. Projectivize ξ_2 to form the bundle $\mathbb{C}P(\xi_2) = (M, \pi_2, N)$. If $V \subset M$ is a string hypersurface, then V is cobordant to zero in oriented cobordism.*

Generalizing these results may prove to be of interest in differential geometry. For example, Stolz conjectured that the vanishing of a certain oriented cobordism invariant called the *Witten genus* is related to positivity of the *Ricci curvature* of a manifold. More specifically,

Conjecture. *([40]) Let M be a smooth closed string manifold with dimension divisible by four. If M admits a metric of positive Ricci curvature, then $\phi_W(M) = 0$, where ϕ_W is the Witten genus.*

Since the manifolds described in the above proposition and theorem vanish in oriented cobordism, their Witten genus clearly vanishes as well. Overall, Stolz's conjecture has only been verified for a very limited number of special cases [40, 15, 8], and these projectivized manifolds provide an additional class of examples for which the conjecture holds.

Chapter 7 discusses questions motivated by the results of this dissertation along with some possible directions for further research.

Chapter 2 Cobordism

The focus of this chapter is to give a brief overview of the topics in cobordism theory that are pertinent to later results. Generally speaking, the necessary facts involve understanding cobordism from a geometric perspective and exploring the algebraic structure of cobordism rings. For a more complete treatment of cobordism theory, see [42].

Definition 2.1. Two smooth compact n-dimensional manifolds M_1 and M_2 are *cobordant* if their disjoint union $M_1 \coprod M_2$ forms the boundary of an $(n+1)$-dimensional smooth compact manifold-with-boundary.

It is easy to see that cobordism is an equivalence relation. The equivalence classes of the cobordism relation are called *cobordism classes*. It is often useful to place additional restrictions on all of the manifolds involved in 2.1. For example,

Definition 2.2. Two smooth oriented n-dimensional manifolds M_1 and M_2 are called *oriented cobordant* if the disjoint union $M_1 \coprod -M_2$ forms the boundary of an $(n+1)$-dimensional smooth compact oriented manifold-with-boundary, where $-M_2$ is M_2 with the orientation reversed.

The main object of study in this dissertation is a certain type of complex variety. For this reason, it would also be useful to restrict the manifolds considered in cobordism to only complex manifolds. However, this does not work. Given two complex manifolds of dimension $2n$, there is no complex manifold of odd dimension $2n+1$ which they could bound. To overcome this problem, the condition of having a complex structure is weakened.

Definition 2.3. ([6, Section 5.3]) A *stably complex manifold* consists of a smooth manifold M and a real vector bundle isomorphism between a complex vector bundle ξ over M and $\tau M \oplus \mathbb{R}^k$, where τM is the tangent bundle, and $\mathbb{R}^k$ is the k-dimensional trivial bundle. Two stably complex n-dimensional manifolds (M_1, ξ_1) and (M_2, ξ_2) are *complex cobordant* if there is a stably complex $(n+1)$-dimensional manifold (W, ζ) such that $\partial W = M_1 \coprod -M_2$ and ζ induces ξ_1 and ξ_2 by identifying the inward normal bundle with the trivial real one-dimensional bundle on ∂W.

When the type of cobordism is clear from context, the adjectives (like oriented and complex) will be suppressed.

2.1 Complex cobordism

For a more in-depth coverage of complex cobordism, see [42, 38, 6]. All cohomology considered in this section will be integral cohomology.

The algebraic structure of Ω_*^U

The complex cobordism classes can be given a useful algebraic structure. The set of complex n-dimensional cobordism classes forms a *cobordism group* under disjoint union. This group is denoted Ω_n^U. The identity of Ω_{2n}^U can be represented by S^n with its standard stably complex structure, since S^n bounds the disc. The inverse of a cobordism class $[M, \xi]$ is obtained by taking the same manifold M and giving ξ the opposite complex structure. More specifically, suppose $f : \tau M \oplus \mathbb{R}^k \to \xi$ is the pertinent isomorphism. Then one can define another isomorphism

$$f \oplus c : \tau M \oplus \mathbb{R}^k \oplus \mathbb{R}^2 \to \xi \oplus \mathbb{C}$$

by setting $c(x, y) = x - iy$, where $\mathbb{C}$ is a one-dimensional trivial complex bundle. This yields the opposite complex structure. If M is a complex manifold, then $[M]$ will denote its complex cobordism class with the standard complex structure on the tangent bundle of M. The cobordism groups can be combined into a graded ring by using Cartesian product as multiplication. This ring is denoted Ω_*^U.

It is possible to determine exactly when two stably complex manifolds are cobordant by studying Chern numbers of manifolds. Recall that the Chern class

$$c(\xi) = c_0(\xi) + c_1(\xi) + \dots$$

of a vector bundle $\xi = (E, \pi, B)$ is a cohomology class which satisfies the following four properties.

1. $c_0(\xi) = 1$ and $c_k(\xi) \in H^{2k}(B)$ for all k
2. (naturality) Let $f : \xi \to \zeta$ be a bundle map. Then $c(\xi) = f^* c(\zeta)$, where f^* is the induced map in cohomology.
3. (Whitney sum formula) $c(\xi \oplus \zeta) = c(\xi) \cdot c(\zeta)$
4. If η is the tautological line bundle over $\mathbb{C}P^1$, then $c(\eta) = 1 + c_1(\eta)$, where $c_1(\eta)$ is the canonical generator of $H^2(\mathbb{C}P^1)$.

Refer to [36, 25] for background on vector bundles and characteristic classes. If τM denotes the tangent bundle of a manifold M, then the Chern class $c(\tau M)$ is written $c(M)$ and is called the Chern class of the manifold. Note that Chern classes in the top cohomology group of a manifold can be evaluated on the fundamental class of the manifold. Unless otherwise specified, all cohomology will be integral cohomology, so this operation will assign an integer to a manifold.

Definition 2.4. Let M be a stably complex $2n$-dimensional manifold, and let

$$I = \{i_1, \dots, i_t\}$$

be a partition of a nonnegative integer $m \leq n$. Consider the cohomology class $c_{i_1}(M) \cdots c_{i_t}(M) \in H^{2m}(M)$. Evaluating this class on the fundamental class μ_M of M gives an integer

$$\langle c_{i_1}(M) \cdots c_{i_t}(M), \mu_M \rangle$$

called a *Chern number* of M. This Chern number is denoted as $c_{i_1} \cdots c_{i_t} [M]$ or $c_I [M]$. Note that $c_I [M]$ is zero unless I is a partition of the integer n itself.

These Chern numbers determine exactly when two manifolds are cobordant.

Theorem 2.5. *([34, 37]) Two stably complex manifolds M_1^{2n} and M_2^{2n} are cobordant if and only all of their Chern numbers are equal, i.e. $c_I [M_1] = c_I [M_2]$ for every partition I of n.*

Note that each Chern class has an even degree. This means that manifolds with odd dimensions have vanishing Chern numbers. Thus $\Omega_n^U = 0$ whenever n is odd.

Certain linear combinations of Chern numbers are also useful in describing the algebraic structure of the cobordism ring Ω_*^U.

Definition 2.6. ([36, Section 16]) Two monomials in $x_1, \ldots, x_n$ are called *equivalent* if each one can be obtained from the other through a permutation of $x_1, \ldots, x_n$. Fix a nonnegative integer $k \leq n$, and consider a partition $I = \{i_1, \ldots, i_j\}$ of k. Consider $\sum x_1^{i_1} x_2^{i_2} \cdots x_j^{i_j}$, where the sum is taken over all distinct monomials that are equivalent to $x_1^{i_1} \cdots x_j^{i_j}$. This is a symmetric polynomial, so it can be written in terms of the elementary symmetric polynomials. Define $s_I (\sigma_1, \ldots, \sigma_k) = \sum x_1^{i_1} x_2^{i_2} \cdots x_j^{i_j}$ to be this sum of monomials written in terms of the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_k$.

Now consider a manifold M^{2n}. By the splitting principle (see [24, Section 4.4] for details), its Chern class can be formally written as

$$c(M) = (1 + x_1) \cdots (1 + x_n).$$

Then $c_k (M) = \sigma_k (x_1, \ldots, x_n)$ is the k^{th} elementary symmetric polynomial. In this situation, $s_I (c(M)) = s_I (c_1 (M), \ldots, c_k (M))$ is a cohomology class in $H^* (M)$. Given the partition $I = \{n\}$ of n, the characteristic number $\langle s_I (c(M)), \mu_M \rangle \in \mathbb{Z}$ is called the *Milnor number*[1] and is denoted $s_n [M]$.

Example 2.7. Consider a manifold M of complex dimension 3 with Chern class

$$c(M) = (1 + x_1)(1 + x_2)(1 + x_3).$$

Consider the partition $\{3\}$ of 3. This gives the corresponding characteristic class

$$\begin{aligned} s_3 (c(M)) &= x_1^3 + x_2^3 + x_3^3 \\ &= (x_1 + x_2 + x_3)^3 - 3(x_1 + x_2 + x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3) + 3 x_1 x_2 x_3. \end{aligned}$$

In terms of Chern classes, this means that

$$s_3 (c(M)) = c_1 (M)^3 - 3 c_1 (M) c_2 (M) + 3 c_3 (M).$$

The Milnor number $s_3 [M]$ is therefore given by a sum of Chern numbers, namely $s_3 [M] = c_1^3 [M] - 3 c_1 c_2 [M] + 3 c_3 [M]$.

[1] Although this number appears frequently in the study of complex cobordism, it does not have a well-established name. Milnor attributes these numbers to Thom [35, Introduction to Part 4], and Thom in turn attributes them to Pontrjagin [44]. I will call these numbers *Milnor numbers* because of Milnor's extensive use of them in his work on cobordism.

Example 2.8. Consider the smooth projective toric variety $\mathbb{C}P^n$ for $n \geq 1$. Its Chern class is

$$c\left(\mathbb{C}P^n\right) = (1+x)^{n+1} \in H^*\left(\mathbb{C}P^n\right) \cong \mathbb{Z}[x] / \left(x^{n+1}\right),$$

where x is a generator of $H^2\left(\mathbb{C}P^n\right)$ (see [36, Section 14] for details). Then

$$s_n\left(c\left(\mathbb{C}P^n\right)\right) = (n+1)\, x^n.$$

Since x^n is the generator of $H^{2n}\left(\mathbb{C}P^n\right)$,

$$s_n\left[\mathbb{C}P^n\right] = n+1. \tag{2.1.1}$$

Theorem 2.9. *([37, 44]) The complex cobordism ring is a polynomial ring with one generator in each even dimension. A cobordism class $[M^{2n}]$ can be chosen for the polynomial generator α_n of $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ if and only if*

$$s_n[M] = \begin{cases} \pm 1 & \text{if } n+1 \neq p^m \text{ for any prime } p \text{ and integer } m \\ \pm p & \text{if } n+1 = p^m \text{ for some prime } p \text{ and integer } m. \end{cases}$$

There are many other useful cobordism invariants that can be written as a linear combination of Chern classes. Of particular importance in cobordism theory is the Todd genus. This is an example of a more general object called a multiplicative genus, which is a ring homomorphism $\Omega_*^U \to R$ for some $\mathbb{Q}$-algebra R. These multiplicative genera can all be constructed from power series that have one as their constant term (see [24] for details).

Definition 2.10. ([31, Chapter III Section 11], [23, Section 1.7]) Consider the power series $\mathrm{td}(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \ldots$. Given a stably complex manifold M of real dimension $2n$, formally write its Chern class as $c(M) = \prod_{k=1}^{n}(1+x_k)$. Now consider the symmetric function $\prod_{k=1}^{n}\mathrm{td}(x_k)$. This series can be written in terms of the elementary symmetric polynomials (i.e. the Chern classes of M) as

$$\prod_{k=1}^{n}\mathrm{td}(x_k) = 1 + \sum_{k=1}^{\infty}\mathrm{Td}_k\left(c_1(M), \ldots, c_n(M)\right),$$

where each Td_k is a homogeneous polynomial of degree k. For example, one can compute

$$\mathrm{Td}_1(M) = \frac{1}{2}c_1(M)$$
$$\mathrm{Td}_2(M) = \frac{1}{12}\left(c_1(M)^2 + c_2(M)\right)$$
$$\mathrm{Td}_3(M) = \frac{1}{24}c_1(M)\, c_2(M)$$

(see [23, Section 1.7] for more). The cohomology class

$$1+\sum_{k=1}^{\infty}\mathrm{Td}_k\left(c_1\left(M\right),\ldots,c_n\left(M\right)\right)\in H^*\left(M\right)$$

is denoted $\mathrm{Td}\left(M\right)$ and is called the *Todd class* of M. Evaluating the Todd class on the fundamental class of M yields the *Todd genus* $\mathrm{Td}\left[M\right]$ of M, i.e.

$$\mathrm{Td}\left[M\right]=\left\langle \mathrm{Td}\left(M\right),\mu_M\right\rangle=\left\langle \mathrm{Td}_n\left(M\right),\mu_M\right\rangle .$$

Representatives of complex cobordism classes

Although the algebraic structure of complex cobordism is well understood, this does not give much information about individual cobordism classes. More specifically, it can be difficult to find a "convenient" representative of a given cobordism class. There are still long-standing open problems regarding what types of manifolds can be chosen to represent a cobordism class. The most well-known is the following question, which was posed by Hirzebruch in 1958.

Problem 2.11. ([22]) Which complex cobordism classes can be represented by connected smooth algebraic varieties?

Partial answers to this question have been given by considering less specific types of manifolds.

Theorem 2.12. *(Milnor, see [42, Chapter VII]) For $n>0$, every complex cobordism class $[M]\in\Omega_n^U$ can be represented by a smooth not necessarily connected algebraic variety.*

This answers Hirzebruch's question if the connectedness condition is ignored. Given the cobordism class $[M]\in\Omega_n^U$ of such a smooth algebraic variety M, one can take the connected sum of its components to obtain a cobordant variety that is still smooth, but now is also connected. However, the connected sum operation does not preserve algebraicity. Therefore, every complex cobordism class of positive dimension can be represented by a smooth connected not necessarily algebraic manifold.

Because of Theorem 2.5, questions about individual cobordism classes can be approached by studying the corresponding Chern numbers. While every cobordism class is uniquely identified by its list of Chern numbers, there is *not* a cobordism class that corresponds to every list of integers. The integers which correspond to the Chern numbers of a cobordism class can be described by using a construction from K-theory. (Refer to [31] for a good introduction to K-theory which includes the following material.)

Definition 2.13. Let ξ be complex n-dimensional vector bundle, and formally write its Chern class as $c\left(\xi\right)=\prod_{k=1}^{n}\left(1+x_k\right)$. The *Chern character* of ξ is the rational cohomology class

$$\mathrm{ch}\xi=e^{x_1}+\ldots+e^{x_n}=\sum_{k=1}^{n}\sum_{i=0}^{\infty}\frac{x_k^i}{i!}=n+c_1\left(\xi\right)+\frac{1}{2}\left(c_1\left(\xi\right)^2-2c_2\left(\xi\right)\right)+\ldots.$$

The Chern character is particularly useful because it is additive and multiplicative.

Proposition 2.14. *Given two vector bundles ξ and ζ over M,* $\mathrm{ch}(\xi \oplus \zeta) = \mathrm{ch}\xi + \mathrm{ch}\zeta$ *and* $\mathrm{ch}(\xi \otimes \zeta) = \mathrm{ch}\xi \cdot \mathrm{ch}\zeta$*.*

Corollary 2.15. *The Chern character gives a ring homomorphism*

$$\mathrm{ch} : K(M) \to H^*(M; \mathbb{Q})$$

from the K-theory of M to the cohomology of M.

This means that the K-theory of a manifold could reveal information about certain cohomology classes. For certain choices of virtual bundles in K-theory, this information regards Chern classes.

Definition 2.16. ([2]) Let ξ be a vector bundle over a manifold M of dimension n. Set $\lambda_t(\xi) = \sum_{k=0}^{\infty} \Lambda^k(\xi)\, t^k$, where $\Lambda^k(\xi)$ is the k^{th}exterior power of ξ. The *Atiyah γ-functions* are defined by the equation

$$\lambda_{t/(1-t)}\left(\xi - \mathbb{C}^{\dim \xi}\right) = \sum_{k=0}^{\infty} \gamma_k(\xi)\, t^k$$

where $\mathbb{C}^{\dim \xi}$ is the trivial complex bundle of dimension $\dim \xi$ and $\lambda_{t/(1-t)}\left(\xi - \mathbb{C}^{\dim \xi}\right)$ is given by $\dfrac{\lambda_{t/(1-t)}(\xi)}{\lambda_{t/(1-t)}\left(\mathbb{C}^{\dim \xi}\right)}$.

Now consider some partition ω of some nonnegative integer $m \leq n$, and write $\gamma_k = \gamma_k(\tau M)$ for each k. Consider the virtual bundle $s_\omega(\gamma_1, \ldots, \gamma_m)$, where s_ω is the symmetric polynomial defined in 2.6. Applying the Chern character to this bundle yields a cohomology class $\mathrm{ch} s_\omega(\gamma_1, \ldots, \gamma_m) \in H^*(M)$.

Definition 2.17. (compare to [10, Sections 13 and 14]) The *K-theory Chern number* $\kappa_\omega[M]$ of M is found by multiplying the above cohomology class by the Todd class (see 2.10) and then evaluating on the fundamental class of M, i.e.

$$\kappa_\omega[M] = \langle \mathrm{ch} s_\omega(\gamma_1, \ldots, \gamma_m) \cdot \mathrm{Td}(M), \mu_M \rangle .$$

Note that $\kappa_\omega[M]$ is a rational linear combination of the Chern numbers of M. This means that it only depends on the cobordism class of $[M]$ by Theorem 2.5. Hattori and Stong proved that possible Chern numbers in complex cobordism are completely determined by when these rational combinations have integer values [20, 41]. The following statement of their theorem comes from [10, Section 14].

Theorem 2.18 (Hattori-Stong Theorem)**.** *Let $[M] \in \Omega^U_{2n}$ be an arbitrary complex cobordism class. For each partition ω of a nonnegative integer $m \leq n$, one can write $\kappa_\omega[M] = \sum_{I \in \pi(n)} \beta_I(\omega)\, c_I[M]$ as a linear combination of Chern numbers, where $\beta_I(\omega) \in \mathbb{Q}$ and the sum ranges over the set $\pi(n)$ of partitions of n. Now consider a family of integers $\{b_I\}_{I \in \pi(n)}$. Then $c_I[M] = b_I$ are the Chern numbers of a complex cobordism class if and only if $\kappa_\omega[M] = \sum_{I \in \pi(n)} \beta_I(\omega) \cdot b_I$ is an integer for every ω.*

In practice, computing K-theory Chern numbers can be cumbersome. The following formula provides some assistance.

Proposition 2.19. *([33, Section 2.6]) Let ξ be an n-dimensional bundle over a manifold M. Formally write $c(\xi) = (1 + x_1) \cdots (1 + x_n)$. Then*

$$\mathrm{ch}\gamma_k(\xi) = \sigma_k(e^{x_1} - 1, \ldots, e^{x_n} - 1)$$

where σ_k is the k^{th} elementary symmetric polynomial and $e^{x_i} = \sum_{j=0}^{\infty} \frac{x_i^j}{j!}$.

2.2 Oriented cobordism

Recall that two oriented manifolds M_1 and M_2 are (oriented) cobordant if and only if the disjoint union $M_1 \coprod -M_2$ is the boundary of an oriented manifold of one dimension higher. The algebraic structure of the oriented cobordism ring Ω_*^{SO} is more complicated than that of Ω_*^U (see [42]). However, determining when two manifolds are oriented cobordant is still fairly straight-forward. As in the complex case, this is determined by values of certain characteristic numbers on the manifolds (compare to 2.5). Refer to [36, 25] for more in-depth coverage of the following.

Recall that the Stiefel-Whitney class $w(\xi)$ of a vector bundle $\xi = (E, \pi, B)$ is a cohomology class with coefficients modulo $\mathbb{Z}/2$ which satisfies the following four properties.

1. $w_0(\xi) = 1$ and $w_k(\xi) \in H^k(B; \mathbb{Z}/2)$ for all k
2. (naturality) Let $f : \xi \to \zeta$ be a bundle map. Then $w(\xi) = f^* w(\zeta)$.
3. (Whitney sum formula) $w(\xi \oplus \zeta) = w(\xi) \cdot w(\zeta)$
4. If η is the tautological line bundle over $\mathbb{R}P^1$, then $w(\eta) = 1 + w_1(\eta)$, where $w_1(\eta)$ is nonzero in $H^1(\mathbb{R}P^1; \mathbb{Z}/2) = \mathbb{Z}/2$

These properties of Stiefel-Whitney classes are very similar to the properties satisfied by the Chern classes, and there is a close relationship between these two characteristic classes.

Theorem 2.20. *Let $\xi = (E, \pi, B)$ be a complex vector bundle, and let $\xi_{\mathbb{R}}$ denote the real bundle obtained by ignoring the complex structure. Then $c(\xi) \mapsto w(\xi_{\mathbb{R}})$ under the coefficient map $H^*(B; \mathbb{Z}) \to H^*(B; \mathbb{Z}/2)$. In particular, $w_k(\xi_{\mathbb{R}}) = 0$ for each odd k.*

Definition 2.21. Consider a connected orientable closed n-dimensional manifold M, and let $I = \{i_1, \ldots, i_t\}$ be a partition of n. Consider the cohomology class $w_{i_1}(M) \cdots w_{i_t}(M) \in H^n(M; \mathbb{Z}/2)$. The number $\langle w_{i_1}(M) \cdots w_{i_t}(M), \mu_M \rangle \in \mathbb{Z}/2$ is called a *Stiefel-Whitney number* of M and is denoted as $w_{i_1} \cdots w_{i_t}[M]$ or $w_I[M]$.

The Stiefel-Whitney numbers are not enough to determine when two oriented manifolds are cobordant. Their Pontrjagin numbers must also be compared. Recall that the Pontrjagin class $p(\xi)$ of a real vector bundle $\xi = (E, \pi, B)$ is defined to be $p(\xi) = 1 + p_1(\xi) + p_2(\xi) + \ldots$, where each $p_k(\xi) \in H^{4k}(B; \mathbb{Z})$ is defined in terms of Chern classes by $p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathbb{C})$, where $\xi \otimes \mathbb{C}$ is the complexification of ξ. The Pontrjagin class of a complex vector bundle $\xi = (E, \pi, B)$ can also be defined by considering the corresponding real bundle $\xi_{\mathbb{R}}$ which ignores the complex structure of ξ. More specifically, for every k,

$$p_k(\xi) = p_k(\xi_{\mathbb{R}}) = (-1)^k c_{2k}(\xi_{\mathbb{R}} \otimes \mathbb{C}) = (-1)^k c_{2k}\left(\xi \oplus \overline{\xi}\right),$$

where $\overline{\xi}$ is the conjugate bundle. This means that the Pontrjagin class of a complex vector bundle ξ is given in terms of its Chern class by

$$\sum_{k=0}^{\infty} (-1)^k p_k(\xi) = (1 - c_1(\xi) + c_2(\xi) - \ldots)(1 + c_1(\xi) + c_2(\xi) + \ldots) \tag{2.2.1}$$

(compare to [36, Section 15]).

Definition 2.22. Consider a smooth compact oriented $4n$-dimensional manifold M, and let $I = \{i_1, \ldots, i_t\}$ be a partition of n. Consider the cohomology class

$$p_{i_1}(M) \cdots p_{i_t}(M) \in H^{4n}(M; \mathbb{Z}).$$

The number $\langle p_{i_1}(M) \cdots p_{i_t}(M), \mu_M \rangle \in \mathbb{Z}$ is called a *Pontrjagin number* of M and is denoted as $p_{i_1} \cdots p_{i_t}[M]$ or $p_I[M]$.

Theorem 2.23. *([46, Corollary 1]) Two oriented manifolds M_1 and M_2 are oriented cobordant if and only if all of their Stiefel-Whitney numbers and Pontrjagin numbers coincide.*

A toric variety is a normal variety that contains the torus as a dense open subset such that the action of the torus on itself extends to an action on the entire variety. Remarkably, these varieties are in one-to-one correspondence with objects from convex geometry called fans. Therefore, studying the combinatorial properties of objects like fans and polytopes can reveal a great deal of information about the corresponding toric varieties. This chapter will provide an introduction to these objects and a description of how fans and polytopes are related to toric varieties. In particular, certain topological properties of toric varieties will be described in relation to properties of the corresponding fans. Refer to [11, 16] for a more thorough treatment of this topic.

3.1 Constructing toric varieties from fans

Convex geometric objects called fans can be used as blueprints for constructing toric varieties. Fans are comprised of objects called cones (see Figure 3.1).

Definition 3.1. A *(strongly convex rational polyhedral) cone* σ spanned by *generating rays* $v_1, \ldots, v_m \in \mathbb{Z}^n$ is a set of points

$$\sigma = \mathrm{pos}\,(v_1, \ldots, v_m) = \left\{ \sum_{k=1}^{m} a_k v_k \in \mathbb{R}^n | a_k \geq 0 \right\}$$

such that σ does not contain any lines passing through the origin.

The *dual cone* of σ is $\sigma^\vee = \{u \in \mathbb{R}^n | u \cdot v \geq 0 \text{ for all } v \in \sigma\}$.

A cone in $\mathbb{R}^n$ can be used to construct an affine toric variety (compare to [16]). To construct the affine variety corresponding to a cone $\sigma \in \mathbb{R}^n$, first consider the commutative semigroup S_σ consisting of all lattice points in $\mathbb{Z}^n$ that are contained in $\sigma^\vee$. This semigroup in turn determines a $\mathbb{C}$-algebra $\mathbb{C}\,[S_\sigma]$. This algebra has a vector space basis $\{\chi^v | v \in S_\sigma\}$, and its multiplicative structure is determined by the additive structure of S_σ, i.e. $\chi^{v_1} \cdot \chi^{v_2} = \chi^{v_1+v_2}$. One can choose a set of multiplicative

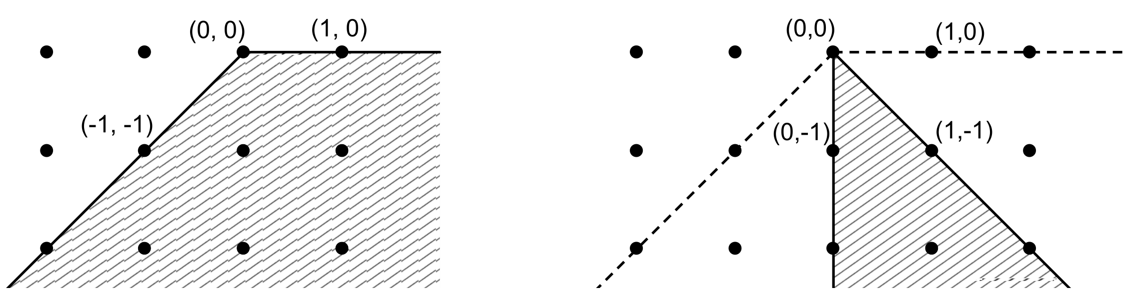

Figure 3.1: A cone (left) along with its dual cone (right)

generators and write $\mathbb{C}[S_\sigma] \cong \mathbb{C}[x_1, \ldots, x_q]/I$ for some ideal I. The corresponding affine variety is defined to be $U_\sigma = V(I)$, the zero locus of all polynomials in I.

Example 3.2. Consider the cone $\sigma = \{0\} \in \mathbb{R}^n$. Its dual cone is $\sigma^\vee = \mathbb{R}^n$. The corresponding semigroup is then $S_\sigma = \mathbb{Z}^n$, which is generated by the standard basis vectors $\{\pm e_k\}$. This defines a $\mathbb{C}$-algebra

$$\begin{aligned}\mathbb{C}[S_\sigma] &= \mathbb{C}\left[\chi^{e_1}, \chi^{-e_1}, \ldots, \chi^{e_n}, \chi^{-e_n}\right] \\ &\cong \mathbb{C}[x_1, x_2, \ldots, x_{2n}] / (x_1x_2 - 1, x_3x_4 - 1, \ldots, x_{2n-1}x_{2n} - 1).\end{aligned}$$

The corresponding affine variety U_σ is the zero locus of $x_1x_2 - 1, \ldots, x_{2n-1}x_{2n} - 1$. But this is easily seen to be isomorphic to the algebraic torus$(\mathbb{C}^*)^n$. Thus the algebraic torus is the toric variety corresponding to the fan $\{0\}$.

Note that every cone contains the cone $\{0\}$. This means that any affine variety constructed from a cone will contain the algebraic torus. One can use this result to prove that the affine varieties of all cones are in fact toric varieties (see [11] for details).

Example 3.3. Consider the cone $\sigma \in \mathbb{R}^2$ generated by $v_1 = (1, 0)$ and $v_2 = (-1, -1)$. Then $\sigma^\vee$ is generated by the vectors $(1, -1)$ and $(0, -1)$. That is,

$$\sigma^\vee = \left\{a_1(1, -1) + a_2(0, -1) \in \mathbb{R}^2 | a_1, a_2 \geq 0\right\}.$$

These cones are displayed in Figure 3.1. The corresponding semigroup of lattice points in $\mathbb{Z}^2$ is generated by $(1, -1)$ and $(0, -1)$. Then the $\mathbb{C}$-algebra for this cone is $S_\sigma = \mathbb{C}\left[\chi^{(1,-1)}, \chi^{(0,-1)}\right] \cong \mathbb{C}[x, y] = \mathbb{C}[x, y]/(0)$. Then the corresponding affine toric variety U_σ is the zero locus of the polynomial 0, i.e. $U_\sigma \cong \mathbb{C}^2$.

More general toric varieties can be constructed by gluing together affine toric varieties. On the level of fans, this process can be described by gluing together cones in a certain manner.

Definition 3.4. A *face* of a cone σ spanned by $v_1, \ldots, v_m \in \mathbb{Z}^n$ is a cone lying on the boundary of σ that is spanned by a subset of $\{v_1, \ldots, v_m\}$. The empty set corresponds to the face $\{0\}$ of a cone.

A *fan* Σ in $\mathbb{R}^n$ is a set of cones in $\mathbb{R}^n$ such that each face of a cone in Σ also belongs to Σ, and the intersection of any two cones in Σ is a face of each cone.

A fan Σ in $\mathbb{R}^n$ is *complete* if the union of its cones is $\mathbb{R}^n$.

A fan Σ in $\mathbb{R}^n$ is *regular* if every maximal n-dimensional cone of Σ is spanned by generating rays that form a basis of $\mathbb{Z}^n$.

Thus fans are constructed by gluing together multiple cones along their faces, being careful not to let the cones overlap except along their boundaries. Note that complete fans can be completely described by listing the sets of generating rays that span maximal cones. This information can be used to determine all cones of lower dimension.

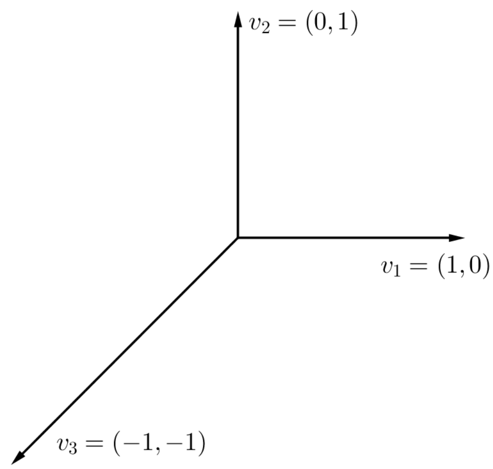

Figure 3.2: The fan corresponding to $\mathbb{C}P^2$

On the level of affine toric varieties, each face of a cone corresponds to a subvariety in the variety defined by the larger cone. If a face τ is contained in two cones σ_1 and σ_2 of a fan, then the affine varieties U_{σ_1} and U_{σ_2} of the two cones can be glued together along the subvariety U_τ associated to τ to produce a toric variety associated to the fan $\sigma_1 \cup \sigma_2$. Again, see [11, 16] for details.

Example 3.5. Consider the complete fan in $\mathbb{R}^2$ shown in Figure 3.2 with generating rays $v_1 = (1,0)$, $v_2 = (0,1)$, and $v_3 = (-1,-1)$. The maximal cones are $\operatorname{pos}(v_1, v_2)$, $\operatorname{pos}(v_1, v_3)$, and $\operatorname{pos}(v_2, v_3)$. As in Example 3.3, each of these two-dimensional cones corresponds to $\mathbb{C}^2$. It is also easy to see that each one-dimensional cone corresponds to $\mathbb{C} \times \mathbb{C}^*$. Gluing together the complex planes along the varieties $\mathbb{C} \times \mathbb{C}^*$ produces $\mathbb{C}P^2$. More specifically, one can write $\mathbb{C}P^2 = U_0 \cup U_1 \cup U_2$ where each $U_i = \{[z_0 : z_1 : z_2] \in \mathbb{C}P^2 | z_i \neq 0\}$ is isomorphic to $\mathbb{C}^2$ and corresponds to one of the maximal cones of the fan. The intersections of the U_i correspond to the intersections of the cones in the fan.

This construction demonstrates that every fan defines a corresponding toric variety. In fact, the converse is also true.

Theorem 3.6. *([11, Section 3.1]) There is a bijective correspondence between equivalence classes of fans in $\mathbb{R}^n$ under unimodular transformations and isomorphism classes of complex n-dimensional toric varieties.*

The fan corresponding to a variety X will be denoted Σ_X, and the variety corresponding to a fan Σ will be denoted X_Σ. This bijection can be proven by examining the orbits of a toric variety under the torus action. There is a bijective correspondence between these orbits and the cones of the associated fan.

Theorem 3.7. *([11, Section 3.2]) Consider a fan Σ in $\mathbb{R}^n$ and its associated complex-dimension n toric variety X_Σ. Every orbit of the torus action on X_Σ corresponds to a* *distinct cone in Σ. If such an orbit is a k-dimensional torus, then the corresponding cone will have dimension $n - k$.*

As a result of this correspondence between fans and toric varieties, many of the algebraic properties of toric varieties directly correspond to properties of the associated fans. Three such properties that will be particularly important in what follows are compactness, smoothness, and isomorphism of varieties.

Proposition 3.8. *([29]) Consider a fan Σ in $\mathbb{R}^n$.*

The toric variety X_Σ is compact if and only if Σ is a complete fan.

The variety is smooth if and only if Σ is regular.

The variety X_Σ is isomorphic to the variety $X_{\Sigma'}$ if and only if there is a unimodular transformation $\mathbb{Z}^n \to \mathbb{Z}^n$ which maps Σ into Σ' and preserves the simplicial structure of the fans.

If such a unimodular transformation exists, then Σ and Σ' will be called *isomorphic* fans.

3.2 Polytopes

A special class of toric varieties can be constructed from certain polytopes. The following material can be found in [6, 16]. For a more in-depth treatment of polytopes and toric varieties, refer to [11, 14, 47, 19].

Constructing toric varieties from polytopes

Definition 3.9. A *(convex) polytope* P in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$.

Let H be a hyperplane in $\mathbb{R}^n$, and suppose the affine hull of $H \cap P$ has dimension k. If all of P is contained in one of the closed half-spaces defined by H, then $H \cap P$ is called a *face* of P of dimension k. Faces of dimension zero are called *vertices*, one-dimensional faces are *edges*, and $(n-1)$-dimensional faces are called *facets.*

A *lattice* polytope is a polytope whose vertices lie in $\mathbb{Z}^n$.

All of the polytopes in this dissertation will be *full-dimensional*. That is, the affine hull of the vertices of a polytope in $\mathbb{R}^n$ will always be $\mathbb{R}^n$ itself.

A lattice polytope P in $\mathbb{R}^n$ can be used to construct a fan Σ_P called the *normal fan* to P (see Figure 3.3). This is done by choosing a vector for each facet of P that is normal to the facet pointing inwards. Because P is a lattice polytope, these vectors can be chosen to belong to $\mathbb{Z}^n$. Reposition these normal vectors at the origin to produce the generating rays of the normal fan Σ_P. The cones of Σ_P are determined from the simplicial structure of the polytope. More precisely, a set of generating rays spans a cone in the fan if and only if the intersection of the corresponding facets in P is nonempty. As a result, cones of dimension k in Σ_P correspond to faces of dimension $n - k$ in P. In particular, the vertices of P determine the maximal n-dimensional cones of Σ_P.

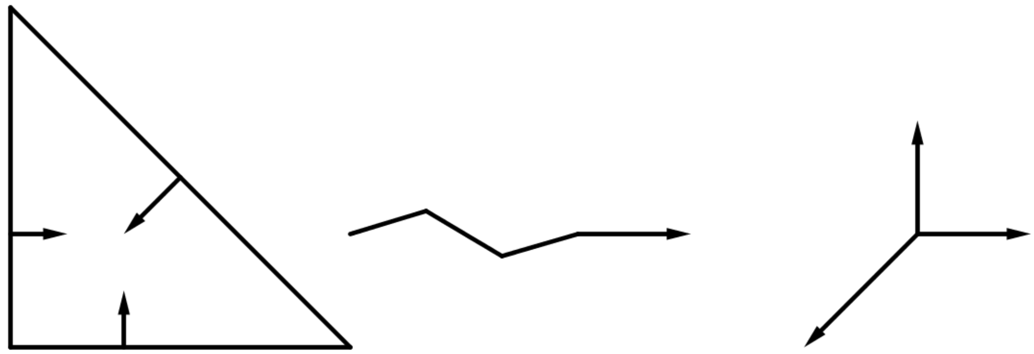

Figure 3.3: The polytope and associated normal fan that correspond to $\mathbb{C}P^2$

Since each fan in $\mathbb{R}^n$ determines a complex n-dimensional toric variety, each lattice polytope P in $\mathbb{R}^n$ also has an associated n-dimensional toric variety X_P. However, not all toric varieties correspond to polytopes.

Proposition 3.10. *X is a projective toric variety if and only if it corresponds to the normal fan of some lattice polytope.*

Some properties of projective toric varieties can easily be described in terms of the associated polytopes.

Proposition 3.11. *The toric variety X_P associated to the lattice polytope P in $\mathbb{R}^n$ is smooth if and only if n edges meet at every vertex of P, and the n edges emanating from each vertex define a basis of $\mathbb{Z}^n$. These polytopes are called* smooth *polytopes.*

The g-theorem

The combinatorial structure of a polytope is given by its *face poset*, the partially ordered set of faces of the polytope under inclusion.

Definition 3.12. Two polytopes are *combinatorially equivalent* if there is a bijection between their faces that preserves inclusion. A *combinatorial polytope* is an equivalence class of combinatorially equivalent polytopes.

The combinatorial structure of polytopes reveals a considerable amount of information about the topology of the associated projective toric varieties. In fact, a great deal of information can be extracted simply by counting the number of faces in each dimension. One can refer to [6] or most standard texts on polytopes for more details on the following.

Definition 3.13. A *simple polytope* P in $\mathbb{R}^n$ is a polytope in which exactly n facets meet at every vertex. Equivalently, P is simple if exactly n edges meet at every vertex.

The *f-vector*, or *face vector*, of a simple polytope P in $\mathbb{R}^n$ is $f(P) = (f_0, \ldots, f_{n-1})$, where f_k is the number of faces of dimension $n-k-1$ in P.

Also define $f_{-1} = 1$ to account for the interior of a simple polytope. Note that f_k counts the faces of *co*dimension $k+1$. For example, f_0 is the number of facets of P and f_{n-1} is the number of vertices of P. It is often more convenient to study different vectors that capture the same information about the number of faces of a polytope in a different way.

Definition 3.14. *The h-vector* $h(P) = (h_0, \ldots, h_n)$ *of an n-dimensional simple polytope P is defined by the following equation.*

$$\sum_{k=0}^{n} h_k t^{n-k} = \sum_{k=0}^{n} f_{k-1} (t-1)^{n-k}$$

Note in particular that $h_0 = f_{-1} = 1$. This definition can be used to derive the following useful formulas.

Proposition 3.15. *Let P be a simple n-polytope with f-vector $(f_0, \ldots, f_{n-1})$ and h-vector $(h_0, \ldots, h_n)$. Then for $k = 0, \ldots, n$,*

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{n-k} f_{i-1} \text{ and } f_{n-1-k} = \sum_{q=k}^{n} \binom{q}{k} h_{n-q}. \tag{3.2.1}$$

There are several advantages to using this h-vector to describe the faces of a polytope instead of the f-vector. For example, the h-vector has a geometric interpretation that will be useful. Let P be a simple polytope in $\mathbb{R}^n$. Choose a vector ν that is not perpendicular to any edge of P. This vector defines a directed graph on the vertices and edges on P. More specifically, choose the direction of an edge e so that $v \cdot e > 0$. Define the *index* of a vertex to be the number of edges pointing towards the vertex.

Proposition 3.16. *([6, Proof of Theorem 1.20]) Given P and ν as described above, the number of vertices of P with index q is equal to h_{n-q}. In particular, this number is independent of the choice of ν.*

Proof. Given a face F of P, the directed graph on the edges of P induces a directed graph on the edges of F. The graph determined by F has exactly one sink (highest vertex in relation to ν). Fix an integer k such that $0 \leq k \leq n$ and let q be an arbitrary integer such that $k \leq q \leq n$. Now consider a vertex v of P with index q. Each set of k edges out of the q that are directed towards v defines a k-dimensional face of P, and each of these faces has v as the sink of the corresponding graph. Thus there are $\binom{q}{k}$ -many distinct k-dimensional faces whose graph has v as the sink. This is true for each vertex of index q, and all of these k-dimensional faces are distinct since the corresponding graphs have a unique sink. Thus the number of k-dimensional faces of P is given by the formula

$$f_{n-k-1} = \sum_{q=k}^{n} \binom{q}{k} I_\nu(q)$$

where $I_\nu(q)$ denotes the number of vertices of P with index q. Comparing to (3.2.1) proves that $I_\nu(q) = h_{n-q}$. ∎

Another advantage of using the h-vector instead of the f-vector is that it is symmetric for simple polytopes.

Theorem 3.17 (Dehn-Sommerville relations)**.** *The h-vector of an n-dimensional simple polytope satisfies the relations $h_k = h_{n-k}$ for $k = 0, \ldots, n$.*

This means that all of the information about the number of face vectors of a simple polytope is actually contained in just the first half of its h-vector. This information can be written more compactly in terms of the g-vector.

Definition 3.18. The *g-vector* $g(P) = \left(g_0, g_1, \ldots, g_{\lfloor n/2 \rfloor}\right)$ is given by $g_0 = 1$ and $g_k = h_k - h_{k-1}$ for $k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

Note that the f-, g-, and h-vectors are different ways of expressing the exact same information about a simple polytope, namely how many faces of each dimension the polytope contains.

Example 3.19. The n-dimensional simplex Δ^n is the convex hull of the origin and the n standard basis vectors in $\mathbb{R}^n$. Note that any k vertices of Δ^n form a face of Δ^n which is itself combinatorially equivalent to a $(k-1)$-dimensional simplex. This means that $f(\Delta^n) = \left(\binom{n+1}{n}, \binom{n+1}{n-1}, \ldots, \binom{n+1}{1}\right)$. In particular,

$$\begin{aligned} f_{n-k-1}(\Delta^n) &= \binom{n+1}{k+1} \\ &= \binom{n}{k} + \binom{n}{k+1} \\ &= \binom{n}{k} + \binom{n-1}{k} + \binom{n-1}{k+1} \\ &= \vdots \\ &= \sum_{q=k}^{n} \binom{q}{k}. \end{aligned}$$

Comparing this with (3.2.1) proves that $h(\Delta^n) = (1, 1, \ldots, 1)$. Then by Definition 3.18, $g(\Delta^n) = (1, 0, \ldots, 0)$.

It is clear that the normal fan to Δ^n has generating rays $e_1, \ldots, e_n, (-1, \ldots, -1)$, where e_k is the k^{th} standard basis vector. The smooth projective toric variety corresponding to the polytope Δ^n is therefore $\mathbb{C}P^n$ (compare to Example 3.5).

A natural question to ask about polytopes is how many faces of different dimensions a polytope can have. In other words, describe all vectors that are the f-vectors (or g- or h-vectors) of polytopes. This classification question has been answered for simple polytopes with what is known as the *g-theorem*. This theorem describes a set of g-vectors satisfying certain conditions and states that these are exactly the g-vectors that correspond to simple polytopes. Billera and Lee [4] constructed simple polytopes for each of the g-vectors which satisfy the conditions, and Stanley [39] proved that these are the only g-vectors that could possibly correspond to a simple polytope.

Some additional notation is necessary to state the g-theorem (see [6, Section 1.3] for details). Let a and i be two positive integers. Then there is a unique binomial i-expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \ldots + \binom{a_j}{j}$$

such that $1 \leq j \leq a_j \leq \ldots \leq a_{i-1} \leq a_i$. Given this expansion, define the integer

$$a^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i} + \ldots + \binom{a_j + 1}{j+1}$$

and $0^{\langle i \rangle} = 0$.

Theorem 3.20. *(g-theorem) An integer vector $\left(g_0, g_1, \ldots, g_{\lfloor n/2 \rfloor}\right)$ is the g-vector of a simple n-dimensional polytope if and only if*

$$g_0 = 1, \;\; g_1 \geq 0, \quad \text{and } 0 \leq g_{k+1} \leq g_k^{\langle k \rangle} \text{ for } k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Example 3.21. Suppose $n = 3$. Then the only conditions provided by the g-theorem are $g_0 = 1$ and $g_1 \geq 0$. Thus the g-vectors that correspond to simple polytopes in $\mathbb{R}^3$ are those of the form $(1, g_1)$ where $g_1 \geq 0$.

Now suppose $n = 4$. Simple polytopes in this dimension are characterized by g-vectors (g_0, g_1, g_2). The g-theorem provides three conditions in this dimension. The simple four-dimensional polytopes obtain exactly the g-vectors satisfying these three conditions. Using the fact that $g_1^{\langle 1 \rangle} = \binom{g_1+1}{2}$, one can write these conditions as follows.

1. $g_0 = 1$
2. $g_1 \geq 0$
3. $0 \leq g_2 \leq \frac{1}{2} g_1 (g_1 + 1)$

3.3 Kleinschmidt's varieties

Generally speaking, very few classification results exist for fans, polytopes, or toric varieties. One exception to this is the classification of fans with relatively few generating rays.

Proposition 3.22. *Up to isomorphism, there is only one complete regular fan in $\mathbb{R}^n$ with $(n+1)$-many generating rays. The toric variety corresponding to this fan is $\mathbb{C}P^n$.*

Proof. Suppose Σ is a complete regular fan in $\mathbb{R}^n$ with generating rays $v_1, \ldots, v_{n+1}$. Without loss of generality, assume that $\operatorname{pos}(v_1, \ldots, v_n)$ is a maximal cone of Σ. Consider the unimodular transformation that sends each of these rays to the standard basis vectors of $\mathbb{R}^n$, i.e. $v_k \mapsto e_k$ for $k = 1, \ldots, n$. In order for the fan to be complete, the other maximal cones in the image of Σ under this transformation must be spanned by $n-1$ of the standard basis vectors and the image of v_{n+1}. But because the fan is regular and complete, the only possibility for the value of the image of v_{n+1} is $(-1, \ldots, -1)$. Then Σ is isomorphic to the fan with generating rays $u_1, \ldots u_{n+1}$ where $u_k = e_k$ for $k = 1, \ldots, n$ and $u_{n+1} = (-1, \ldots, -1)$. The maximal cones of this fan consist of all spans of subsets of n-many of these rays. By applying the same approach as in Example 3.5, it is easy to see that the toric variety corresponding to this fan is $\mathbb{C}P^n$. $\square$

Classifying complete regular fans in $\mathbb{R}^n$ with $n+2$ generating rays is already significantly more complicated. This classification was described in 1988 by Peter Kleinschmidt [29]. His results build on the earlier classification of simple polytopes with $n+2$ facets (see [19, Section 6.1], which classifies the *dual* simplicial polytopes with $n+2$ vertices). Kleinschmidt created a list of fans in $\mathbb{R}^n$ with $n+2$ generating rays so that any such fan would be isomorphic to a member of the list.

To describe these fans, fix a dimension $n \geq 2$. Choose $r \in \{1, 2, \ldots, n-1\}$ and select a weakly increasing set of integers $0 \leq a_1 \leq \ldots \leq a_r$. Now define two sets of vectors $U = \{u_1, \ldots, u_{r+1}\}$ and $V = \{v_1, \ldots, v_{n-r+1}\}$, where $u_k = e_k$ for $k = 1, \ldots, r$; $u_{r+1} = \left(-1, \overset{(r)}{\ldots}, -1, 0 \ldots, 0\right)$; $v_k = e_{k+r}$ for $k = 1, \ldots, n-r$; and $v_{n-r+1} = (a_1, \ldots, a_r, -1, \ldots, -1)$. Note that there are $n+2$ vectors in $U \cup V$. Define $\Sigma_n(a_1, \ldots, a_r)$ to be the fan whose generating rays are those in $U \cup V$, and its maximal cones consist of the spans of all sets of vectors consisting of r vectors from U and $n-r$ vectors from V. Denote the corresponding toric variety by $X_n(a_1, \ldots, a_r)$.

Example 3.23. Consider the fan $\Sigma_3(0,0)$ in $\mathbb{R}^3$. Since $r = 2$, the set of generating rays is $U \cup V = \{u_1, u_2, u_3\} \cup \{v_1, v_2\}$, where $u_1 = e_1$, $u_2 = e_2$, $u_3 = (-1, -1, 0)$, $v_1 = e_3$, and $v_2 = (0, 0, -1)$. The fan that is obtained is the product of the fans corresponding to $\mathbb{C}P^1$ and $\mathbb{C}P^2$, so it is not surprising that $X_3(0,0) \cong \mathbb{C}P^1 \times \mathbb{C}P^2$.

Consider the fan $\Sigma_4(1)$ in $\mathbb{R}^4$. The set of generating rays for the fan is

$$U \cup V = \{u_1, u_2\} \cup \{v_1, v_2, v_3, v_4\}$$

as shown in Figure 3.4. The fan can be visualized as the *join* of the fans of u-vectors and v-vectors. That is, a maximal cone in $\Sigma_4(1)$ is obtained by taking the combined span of a maximal cone in the left fan in Figure 3.4 and a maximal cone in the right fan. One can show that the corresponding toric variety $X_4(1)$ is a $\mathbb{C}P^1$-bundle over $\mathbb{C}P^3$.

Theorem 3.24. *([29]) Every compact smooth n-dimensional complex toric variety whose corresponding fan has $n+2$ generating rays is isomorphic to exactly one of the varieties $X_n(a_1, \ldots, a_r)$.*

In fact, all of the fans $\Sigma_n(a_1, \ldots, a_r)$ are normal fans to some polytopes in $\mathbb{R}^n$. Applying 3.10 yields the following

Theorem 3.25. *([29]) The toric varieties $X_n(a_1, \ldots, a_r)$ are all projective.*

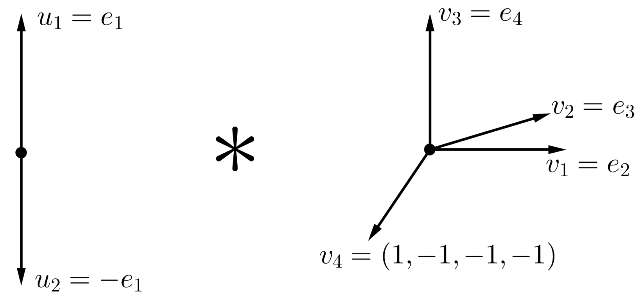

Figure 3.4: The fan $\Sigma_4(1)$

3.4 Cohomology of smooth toric varieties

Danilov [12] and Jurkiewicz [28] have computed the integral cohomology of smooth toric varieties by using information from the fans. Refer to [6, 11, 16] for a more thorough treatment of the cohomology of smooth toric varieties.

Consider a complete regular fan Σ in $\mathbb{R}^n$ with generating rays $v_1, \ldots, v_m$. Each of the rays v_k is a one-dimensional cone in Σ which corresponds to a codimension two subvariety X_k of X_Σ. Each of these subvarieties determines a cohomology class in $H^2(X_\Sigma)$ by taking the image of the fundamental class $[X_k]$ of X_k under the composition

$$H_{2n-2}(X_k) \hookrightarrow H_{2n-2}(X_\Sigma) \to H^2(X_\Sigma),$$

where the first map is induced from inclusion and the second is Poincaré duality. Denote the cohomology class in $H^2(X_\Sigma)$ corresponding to the ray v_k by v_k as well. It will be clear from context what the meaning of v_k is.

The cohomology ring $H^*(X_\Sigma)$ is isomorphic to the ring $\mathbb{Z}[v_1, \ldots, v_m]/I$, where I is an ideal which is determined by the geometry and combinatorial structure of Σ. More specifically, suppose that the coordinates of the generating rays are given by $v_j = (\lambda_{1j}, \ldots, \lambda_{nj})$. For $i = 1, \ldots, n$, set $\theta_i = \lambda_{i1}v_1 + \ldots + \lambda_{im}v_m \in \mathbb{Z}[v_1, \ldots, v_m]$. Define $L = (\theta_1, \ldots, \theta_n)$ to be the ideal generated by these linear polynomials. Also, define J to be the ideal generated by all square-free monomials $v_{i_1} \cdots v_{i_k}$ such that $v_{i_1}, \ldots, v_{i_k}$ do *not* span a cone in Σ. This ideal is the *Stanley-Reisner* ideal of Σ.

Theorem 3.26. *([12, 28]) Given a complete regular fan Σ in $\mathbb{R}^n$ with generating rays $v_1, \ldots, v_m$, the integral cohomology ring of the associated toric variety X_Σ is given by*

$$H^*(X_\Sigma) \cong \mathbb{Z}[v_1, \ldots, v_m]/(L + J).$$

Example 3.27. Consider the fan Σ in $\mathbb{R}^n$ associated to the toric variety $\mathbb{C}P^n$. As in Example 3.5, the generating rays of the fan are $v_k = e_k$ for $k = 1, \ldots, n$ and $v_{n+1} = (-1, \ldots, -1)$. Applying Theorem 3.26 should yield the well-known result $H^*(X_\Sigma) \cong \mathbb{Z}[x]/(x^{n+1})$. The linear ideal L consists of all $\theta_i = v_i - v_{n+1}$. The only subset of $\{v_1, \ldots, v_{n+1}\}$ that does not span a cone in Σ is $\{v_1, \ldots, v_{n+1}\}$ itself, so the Stanley-Reisner ideal for Σ is $J = (v_1 \cdots v_{n+1})$. The cohomology ring of X_Σ is

$\mathbb{Z}[v_1, \ldots, v_{n+1}] / (L + J)$. The relations in L imply that $v_i = v_{n+1}$ for $i = 1, \ldots, n$, so $J = (v_1 \cdots v_{n+1}) = \left(v_{n+1}^{n+1}\right)$. Then $H^*(X_\Sigma) \cong \mathbb{Z}[v_{n+1}] / \left(v_{n+1}^{n+1}\right)$ as expected.

Example 3.28. Consider the fan $\Sigma_3(a_1, a_2)$ for some integers $0 \leq a_1 \leq a_2$ (see Section 3.3). This fan in $\mathbb{R}^3$ has generating rays $u_1 = e_1$, $u_2 = e_2$, $u_3 = (-1, -1, 0)$, $v_1 = e_3$, and $v_2 = (a_1, a_2, -1)$. The linear ideal L is given by

$$L = (u_1 - u_3 + a_1 v_2, u_2 - u_3 + a_2 v_2, v_1 - v_2).$$

The Stanley-Reisner ideal is $J = (u_1 u_2 u_3, v_1 v_2)$. Combining these yields the ideal $L + J = (u_3^3 - (a_1 + a_2) u_3^2 v_2, v_2^2)$. Thus the cohomology ring of the corresponding smooth projective toric variety is given by

$$H^*(X_3(a_1, a_2)) \cong \mathbb{Z}[u_3, v_2] / \left(u_3^3 - (a_1 + a_2) u_3^2 v_2, v_2^2\right).$$

An understanding of the cohomology of smooth toric varieties allows certain characteristic numbers to be evaluated which play a key role in describing the cobordism of smooth toric varieties. In fact, evaluating cohomology classes associated to maximal cones on the fundamental class of a toric variety is straight-forward.

Proposition 3.29. *([16, Section 5.1]) Suppose $pos(v_1, \ldots, v_n)$ is a maximal cone of a complete regular fan Σ in $\mathbb{R}^n$. Let $v_k \in H^2(X_\Sigma)$ also denote the associated cohomology class of the generator. Then evaluating $v_1 \cdots v_n \in H^{2n}(X_\Sigma)$ on the fundamental class μ_{X_Σ} of the variety yields one, i.e.*

$$\langle v_1 \cdots v_n, \mu_{X_\Sigma} \rangle = 1.$$

The Chern class of a smooth toric variety can also be computed using combinatorial data. The complex structure of a smooth toric variety leads to a stable splitting of its tangent bundle, and this splitting is encoded in the fan associated to a toric variety.

Theorem 3.30. *(see [6, Section 5.3] for details) Given a complete regular fan Σ in $\mathbb{R}^n$ with generating rays $v_1, \ldots, v_m$, the total Chern class of X_Σ is given by*

$$c(X_\Sigma) = (1 + v_1)(1 + v_2) \cdots (1 + v_m) \in H^*(X_\Sigma).$$

3.5 Equivariant blow-ups

It will prove useful to explore operations on smooth projective toric varieties that produce new smooth projective toric varieties. One such operation is the equivariant blow-up.

Geometric description of blow-ups

A more detailed explanation of this construction can be found in Chapter 1 Section 4 of [18].

Definition 3.31. Consider the unit disc $D \subset \mathbb{C}^n$. The *blow-up* of D at the origin is $Bl_0 D = \{(z, L) \in D \times \mathbb{C}P^{n-1} | z_i L_j = z_j L_i \text{ for all } i, j\}$, where the z_i are complex coordinates of D, and the L_i are homogeneous coordinates in $\mathbb{C}P^{n-1}$.

It is easy to show that $Bl_0 D = \{(z, L) \in D \times \mathbb{C}P^{n-1} | z \in L\}$, where $L \in \mathbb{C}P^{n-1}$ is viewed as a line through the origin in $\mathbb{C}^n$. With this interpretation of $Bl_0 D$, the projection $\pi : Bl_0 D \to D$ onto the first coordinate is an isomorphism away from the origin, and $\pi^{-1}(0) \cong \mathbb{C}P^{n-1}$.

Definition 3.32. Given a complex manifold M^{2n}, the *blow-up* of M at $x \in M$, denoted $Bl_x M$, is found by applying the above construction locally to a neighborhood of x. There is a projection $\pi : Bl_x M \to M$ that is an isomorphism away from x, and $\pi^{-1}(x) \cong \mathbb{C}P^{n-1}$.

This construction of blowing up points can be generalized to blowing up submanifolds of higher dimension. It is again sufficient to understand how to blow up a subdisc of the unit disc in $\mathbb{C}^n$. This process can then be applied locally on a manifold (see [18, Chapter 4 Section 6] for details).

Definition 3.33. Let $D \subset \mathbb{C}^n$ be the unit disc. Let

$$V = \{(z_1, \ldots, z_n) \in D | z_{k+1} = \ldots = z_n = 0\}.$$

Consider $\mathbb{C}P^{n-k-1}$ with homogeneous coordinates $[L_{k+1} : \ldots : L_n]$. The *blow-up* of D along the submanifold V is

$$Bl_V D = \left\{(z, L) \in D \times \mathbb{C}P^{n-k-1} | z_i L_j = z_j L_i \text{ for } i, j = k+1, \ldots, n\right\}.$$

Now consider a complex manifold M^{2n} with submanifold V^{2k}. As in the case of blowing up at a point, there is a projection $Bl_V M \to M$ that is an isomorphism away from V, and $\pi^{-1}(v) \cong \mathbb{C}P^{n-k-1}$ for any $v \in V$.

Blow-ups of toric varieties

In some circumstances, blowing up along a subvariety of a toric variety produces another toric variety. This happens when the subvariety that is being blown up is an orbit of the torus action. Since the torus orbits correspond to the cones in the associated fan (see Theorem 3.7), the operation of blowing up on toric varieties can be described in terms of changes to the fans.

Example 3.34. ([16]) Consider $Bl_0 \mathbb{C}^2$, the blow-up of the complex plane at the origin. This variety is given explicitly by

$$Bl_0 \mathbb{C}^2 = \left\{(z_0, z_1) \times [L_0 : L_1] \in \mathbb{C}^2 \times \mathbb{C}P^1 | z_0 L_1 = z_1 L_0\right\}.$$

One can write $Bl_0 \mathbb{C}^2 = W_0 \cup W_1$ where

$$W_i = \left\{(z_0, z_1) \times [L_0 : L_1] \in \mathbb{C}^2 \times \mathbb{C}P^1 | z_0 L_1 = z_1 L_0 \text{ and } L_i \neq 0\right\}$$

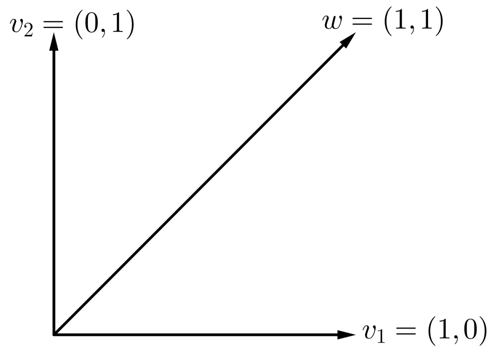

Figure 3.5: The fan associated to $Bl_0\mathbb{C}^2$

for $i = 1, 2$. Then each W_i is isomorphic to $\mathbb{C}^2$. More specifically, the isomorphism $\phi_0 : W_0 \to \mathbb{C}^2$ is given by $(z_0, z_1) \times [L_0 : L_1] \mapsto \left(z_0, \frac{L_1}{L_0}\right)$, and $\phi_1 : W_1 \to \mathbb{C}^2$ is given by $(z_0, z_1) \times [L_0 : L_1] \mapsto \left(\frac{L_0}{L_1}, z_1\right)$. The intersection of these two subvarieties is $\phi_0(W_0) \cap \phi_1(W_1) = \{(z_0, z_1) \in \mathbb{C}^2 | z_0 z_1 = 1\} \cong \mathbb{C} \times \mathbb{C}^*$.

Next consider the fan Σ shown in Figure 3.5 with maximal cones $\sigma_1 = \text{pos}(v_1, w)$ and $\sigma_2 = \text{pos}(v_2, w)$. The procedure in Section 3.1 can be used to find the toric variety associated to this fan.

The corresponding $\mathbb{C}$-algebras are $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}\left[\chi^{(0,1)}, \chi^{(1,-1)}\right] \cong \mathbb{C}[y, xy^{-1}]$ and $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}\left[\chi^{(-1,1)}, \chi^{(1,0)}\right] \cong \mathbb{C}[x^{-1}y, x]$. Then the affine varieties are $U_0 \cong U_1 \cong \mathbb{C}^2$ with intersection $U_0 \cap U_1 = \{(z_0, z_1) \in \mathbb{C}^2 | z_0 z_1 = 1\} \cong \mathbb{C} \times \mathbb{C}^*$. Note that the way in which U_0 and U_1 are glued together is exactly the same as the manner in which W_0 and W_1 are glued together. Thus the toric variety X_Σ is isomorphic to the blow-up $Bl_0\mathbb{C}^2$.

A similar procedure can be used to show that for any $n \in \mathbb{N}$, $Bl_0\mathbb{C}^n \cong X_\Sigma$, where Σ is the fan in $\mathbb{R}^n$ with generating rays $v_k = e_k$, $k = 1, \ldots, n$ and $w = (1, \ldots, 1)$ and maximal cones spanned by w and all but one of the v_k. In particular, the fan Σ corresponding to the blow-up can be obtained from the fan associated with $\mathbb{C}^n$ by inserting the generating ray $w = v_1 + \ldots + v_n$.

This process can be generalized to other fans. In general, consider a fan Σ in $\mathbb{R}^n$ which contains an n-dimensional cone $\sigma = \text{pos}(v_1, \ldots, v_n)$. Construct a new fan $Bl_\sigma\Sigma$ by inserting a new ray $w = v_1 + \ldots + v_n$ and replacing the cone σ with all cones spanned by w and all but one of the $v_1, \ldots, v_n$. There is a unimodular transformation of $Bl_\sigma\Sigma$ that maps these cones to the fan associated with $Bl_0\mathbb{C}^n$. Also note that no cones outside of σ were altered by the inclusion of w. This means that the associated variety $X_{Bl_\sigma\Sigma}$ is actually the blow-up of X_Σ at the torus-fixed point of X_Σ corresponding to σ. This operation can be extended to describing blow-ups along torus-fixed subvarieties of higher dimension.

Definition 3.35. ([11, Section 3.3]) Suppose Σ is a fan in $\mathbb{R}^n$. Let $\tau = \text{pos}(v_1, \ldots, v_k)$ be a cone in Σ for which all cones containing τ are regular (i.e. their generating rays

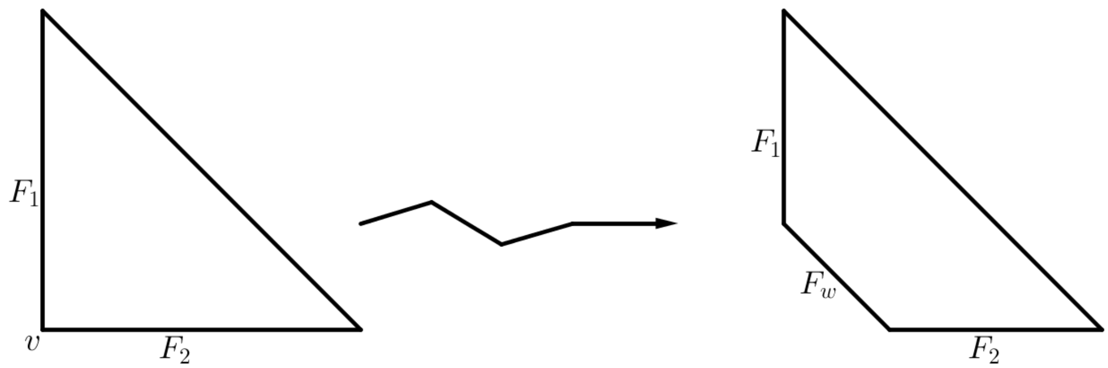

Figure 3.6: Truncation of a polytope at a vertex

form bases for $\mathbb{Z}^n$). Set $w = v_1 + \ldots + v_k$. The fan $Bl_\tau\Sigma$ is constructed by including w in the set of generating rays. The maximal cones of $Bl_\tau\Sigma$ are obtained by replacing every maximal σ of Σ that contains τ with k-many new maximal cones. Each of these new cones is spanned by w, $(k-1)$-many of the rays $v_1, \ldots, v_k$, and all of the rays spanning σ but not τ. The fan $Bl_\tau\Sigma$ is called the *star subdivision* of Σ relative to τ.

Proposition 3.36. *([11, Section 3.3]) Let Σ be a fan in $\mathbb{R}^n$. Let $\tau = pos\,(v_1, \ldots, v_k)$ be a cone in Σ for which all cones containing τ are regular. Let X_τ denote the $(n-k)$-dimensional toric subvariety of X_Σ which is associated to the cone τ. Then $X_{Bl_\tau\Sigma} = Bl_{X_\tau}X_\Sigma$. That is, the blow-up of X_Σ along the subvariety X_τ is a toric variety whose associated fan is the star subdivision of Σ relative to τ.*

The operation of blowing up along torus equivariant subvarieties preserves several key properties of toric varieties. For example, the blow-up of a smooth projective toric variety is itself smooth and projective. To understand why this is true, recall that a toric variety is smooth and projective if and only if its associated fan is the normal fan to a smooth polytope (See Propositions 3.11 and 3.10). The operation of blowing up can be understood in terms of changes to these polytopes.

Example 3.37. Consider the smooth projective toric variety $X_\Sigma = \mathbb{C}P^2$ (see Example 3.5). The associated fan Σ is normal to the smooth polytope in $\mathbb{R}^2$ shown on the left in Figure 3.6. Let $\sigma = \text{pos}\,(v_1, v_2)$ in the normal fan Σ, where $v_k = e_k$. Then σ corresponds to the vertex v of the polytope (as shown in the figure), and v_1 and v_2 correspond to the two edges F_1 and F_2 meeting at this vertex. Now consider what happens when $\mathbb{C}P^2$ is blown up at the torus-fixed point corresponding to σ. To create the new fan $Bl_\sigma\Sigma$, σ is removed. In the corresponding polytope P, the vertex v must be removed. The cones that are added to create $Bl_\sigma\Sigma$ consist of a new ray $w = (1,1)$ and two new maximal cones $\text{pos}\,(v_1, w)$ and $\text{pos}\,(v_2, w)$. In terms of the polytope, a new edge F_w must be inserted which meets each of F_1 and F_2 at a new vertex. The resulting polytope on the right in Figure 3.6 is the polytope associated to the equivariant blow-up of $\mathbb{C}P^2$ at a point.

This example can be generalized to higher dimensions. A polytope P in $\mathbb{R}^n$ can be *truncated* at an $(n-k)$-dimensional facet F by choosing a hyperplane that passes through P and separates F from all vertices of P that are not contained in F. The truncated polytope is the intersection of P and the half-space that does not contain F. It is easy to see that the simplicial structure of this new polytope matches the simplicial structure obtained from subdividing the cone associated to F in the normal fan. In fact, if P is smooth, then one can choose the hyperplane and dilate the new polytope so that the truncated polytope is also smooth.

Proposition 3.38. *Let X_Σ be a smooth projective toric variety whose associated fan Σ is the normal fan to a smooth polytope P in $\mathbb{R}^n$. Suppose τ is a cone in Σ corresponding to a facet F of P. Then $X_{Bl_\tau \Sigma}$ is a smooth projective toric variety whose associated polytope is obtained by truncating P at F.*

Define the g-vector of a smooth projective toric variety to be the g-vector of its associated polytope. Since blowing up changes the associated polytopes in predictable ways, one can study the change in the g-vector of smooth projective toric varieties during a blow-up. For example, the following theorem describes the change in g-vector for blow-ups of low-dimensional subvarieties.

Theorem 3.39. *Let X_P be a smooth projective toric variety with corresponding polytope P in $\mathbb{R}^n$. Suppose the g-vector of X_P is given by $\left(1, g_1, \ldots, g_{\lfloor n/2 \rfloor}\right)$.*

1. *The g-vector of a blow-up of X_P at any torus-fixed point is*

$$\left(1, g_1 + 1, g_2, g_3, \ldots, g_{\lfloor n/2 \rfloor}\right).$$

2. *Assume the real dimension of X_P is at least six. Suppose $Bl_{X_F} X_P$ is the blow-up of X_P along a two-dimensional subvariety corresponding to an edge F of P. Then the g-vector of $Bl_{X_F} X_P$ is given by $\left(1, g_1 + 1, g_2 + 1, g_3, g_4, \ldots, g_{\lfloor n/2 \rfloor}\right)$.*

Proof. Suppose $h(P) = (1, h_1, h_2, \ldots, h_{n-1}, 1)$.

First consider a blow-up at a torus-fixed point. This corresponds to truncating $P \subset \mathbb{R}^n$ at one of its vertices, call it v. The resulting truncated polytope is obtained by replacing this vertex with an $(n-1)$-dimensional polytope which is combinatorially equivalent to Δ^{n-1}, connecting each edge emanating from the original vertex to a distinct vertex of the simplex. Choose a vector $\nu \subset \mathbb{R}^n$ that is both not perpendicular to Δ^{n-1} and that makes v the source of the directed graph described in Proposition 3.16. The truncation removes v, which decreases $h_0 = 1$ by one. The truncation also adds n new vertices with the insertion of Δ^{n-1}. Since the h-vector of Δ^{n-1} is $\left(1, \overset{(n)}{\ldots}, 1\right)$ (see Example 3.19), the addition of Δ^{n-1} increases $h_0, \ldots, h_{n-1}$ each by one. Then the h-vector of the truncated polytope is $(1, h_1+1, h_2+1, \ldots, h_{n-1}+1, 1)$. If $g(P) = \left(1, g_1, \ldots, g_{\lfloor n/2 \rfloor}\right)$, then the g-vector of the truncated polytope is $\left(1, g_1+1, g_2, g_3, \ldots g_{\lfloor n/2 \rfloor}\right)$ by Definition 3.18.

Next consider a blow-up along a two-dimensional subvariety. This corresponds to truncating $P \subset \mathbb{R}^n$ at the associated edge F. The resulting truncated polytope is obtained by replacing F with $\Delta^{n-2} \times I$. This time, choose a vector $\nu \subset \mathbb{R}^n$ that is perpendicular to neither $\Delta^{n-2} \times \{0\}$ nor $\Delta^{n-2} \times \{1\}$ and that gives the vertices of F index zero and one, respectively, in the corresponding directed graph. Note that truncating P at F decreases both h_0 and h_1 by one. It is easy to show that $h\left(\Delta^{n-2} \times I\right) = \left(1, 2, \overset{(n-2)}{\ldots}, 2, 1\right)$. Then the inclusion of $\Delta^{n-2} \times I$ increases h_0 and h_{n-1} by one, and it increases $h_1, \ldots, h_{n-2}$ by two. Then the h-vector of the truncated polytope is $(1, h_1 + 1, h_2 + 2, h_3 + 2, \ldots, h_{n-2} + 2, h_{n-1} + 1, 1)$. Then the g-vector of the truncated polytope is $\left(1, g_1 + 1, g_2 + 1, g_3, g_4, \ldots, g_{\lfloor n/2 \rfloor}\right)$ by Definition 3.18. □

The change in complex cobordism during a blow-up can also be described. Since a blow-up results in a local change on a manifold, it is not surprising that the change in cobordism during a blow-up only depends on the part of the manifold that is being blown up. The following proposition follows from a calculation of Ustinovsky [45] (see [6, 21] for details as well).

Proposition 3.40. *Consider a cone τ in a regular fan Σ in $\mathbb{R}^n$. The change in cobordism class when blowing up the toric variety X_Σ along X_τ to obtain $Bl_{X_\tau} X_\Sigma$ is completely independent of all rays of Σ that do not belong to a cone containing τ.*

Chapter 4 Toric Varieties Representing Complex Cobordism

Toric varieties and complex cobordism interact in several interesting ways. Most importantly, there is a set of smooth projective toric varieties which multiplicatively generates Ω_*^U. By taking the connected sums of these, one can show that any complex cobordism class can be represented by a topological generalization of a toric variety called a *quasitoric manifold* [5, 7]. Unfortunately, taking connected sums of toric varieties does not preserve algebraicity. This property is sacrificed in order to represent every cobordism class with a quasitoric manifold.

Recall that an open problem of Hirzebruch is to figure out which complex cobordism classes can be represented by smooth connected algebraic varieties (see Problem 2.11). The combinatorial structure of toric varieties makes them very convenient to work with. This combinatorial structure could be exploited to approach a toric version of Hirzebruch's problem and to give a partial answer, since smooth toric varieties are connected and algebraic.

Problem 4.1. Which complex cobordism classes can be represented by smooth projective toric varieties?

4.1 Combinatorial obstructions

One well-known obstruction to a cobordism class containing a smooth toric variety is the Todd genus. For any smooth compact toric variety, the value of this genus is one [27, Theorem 3.3]. The contrapositive provides the following

Proposition 4.2. *Any cobordism class whose Todd genus is not one is not represented by a smooth projective toric variety.*

This is just one of a list of obstructions to a cobordism class containing a smooth toric variety that correspond to the combinatorial structure of toric varieties. In fact, this obstruction arises from the *Hodge structure* of a toric variety, and further examination of this structure leads to the other obstructions. As a Kähler manifold, a complex smooth projective toric variety X has a Hodge structure, a decomposition

$$H^r(X;\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X)$$

of its complex cohomology groups (see [18, Chapter 0 Section 7] for details). The *Hodge numbers* of such a variety X are defined by $h^{p,q} = h^{p,q}(X) = \dim H^{p,q}(X)$. For smooth projective toric varieties, the Hodge numbers are determined by the h-vector of the associated polytope.

Proposition 4.3. *([11, Section 9.4]) Let X_P be a smooth projective toric variety of real dimension $2n$, and let $h(P) = (h_0, \ldots, h_n)$ be the h-vector of the associated* *polytope. Then the Hodge numbers of X_P are given by*

$$h^{p,q} = \begin{cases} h_p & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}.$$

There is another way of encoding these Hodge numbers which happens to relate the Hodge structure to complex cobordism.

Definition 4.4. ([24, Section 5.4]) Given a complex smooth projective variety X of complex dimension n, set $\chi^p(X) = \sum_{q=0}^{n} (-1)^q h^{p,q}(X)$. The *$\chi_y$-genus* of X is defined to be the degree n polynomial

$$\chi_y(X) = \sum_{p=0}^{n} \chi^p(X) \cdot y^p. \tag{4.1.1}$$

The χ_y-genus displays several symmetries that arise from the underlying Hodge structure. Most importantly,

Proposition 4.5. *([23, Section 15.8]) $\chi^p(X) = (-1)^n \chi^{n-p}(X)$ for all $p = 0, \ldots, n$.*

Applying the definition of the χ_y-genus to Proposition 4.3 allows one to calculate the χ_y-genus of a smooth projective toric variety.

Corollary 4.6. *Let X_P be a smooth projective toric variety of real dimension $2n$, and let $h(P) = (h_0, \ldots, h_n)$ be the h-vector of its associated polytope. Then*

$$\chi^p(X_P) = (-1)^p h_p$$

and the χ_y-genus of X_P is given by $\chi_y(X_P) = \sum_{p=0}^{n} (-1)^p h_p \cdot y^p = \sum_{p=0}^{n} (-y)^p h_p$.

It will be useful to be able to state the χ^p in terms of the g-vector as well.

Corollary 4.7. *Let X_P be a smooth projective toric variety of real dimension $2n$, and write $g(P) = \left(1, g_1 \ldots, g_{\lfloor n/2 \rfloor}\right)$. Then for $p = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$,*

$$\chi^p(X_P) = (-1)^p \chi^{n-p}(X_P) = (-1)^p \sum_{k=0}^{p} g_k.$$

This means that the χ_y-genus of a smooth projective toric variety and its g-vector hold exactly the same information. On the other hand, the Hirzebruch-Riemann-Roch Theorem relates the χ_y-genus to cobordism invariants using a generalization of the Todd genus (see Definition 2.10).

Definition 4.8. ([23, Section 1.8]) Fix an indeterminate y, and consider the formal power series $Q(y,x) = \dfrac{x(y+1)}{1-e^{-x(y+1)}} - yx$. Given a stably complex manifold M^{2n}, formally write its Chern class with rational coefficients as $c(M) = \prod_{k=1}^{n}(1+x_k)$. Now consider the symmetric function $\prod_{k=1}^{n} Q(y,x_k)$. This series can be written in terms of the elementary symmetric polynomials (i.e. the Chern classes of M) as

$$\prod_{k=1}^{n} Q(y,x_k) = \sum_{n=0}^{\infty} T_n(y, c_1(M), \ldots, c_n(M)) \in H^*(M;\mathbb{Q})[y],$$

where each T_n is a homogeneous polynomial in the x_k of degree n. The sum of all of the T_n's is called the *generalized Todd class* of M. Each of the polynomials $T_n(y, c_1(M), \ldots, c_n(M))$ can be written as

$$T_n(y, c_1(M), \ldots, c_n(M)) = \sum_{p=0}^{n} T_n^p(c_1(M), \ldots, c_n(M))\, y^p. \tag{4.1.2}$$

Each $T_n^p(c_1(M), \ldots, c_n(M))$ is a cohomology class of degree $2n$ expressed in terms of Chern classes. Evaluating this cohomology class on the fundamental class of M therefore yields a cobordism invariant

$$T_n^p[M] = \langle T_n^p(c_1(M), \ldots, c_n(M)), \mu_M\rangle \in \mathbb{Q}.$$

If the entire polynomial $T_n(y, c_1(M), \ldots, c_n(M))$ is evaluated on the fundamental class, we get a polynomial in y called the *generalized Todd genus* of M. More specifically, this genus is defined to be

$$T[M] = \langle T_n(M), \mu_M\rangle \in \mathbb{Q}[y].$$

Note that just like the χ_y-genus, the generalized Todd genus of a manifold is a polynomial in the indeterminate y. These two constructions are in fact equivalent.

Theorem 4.9. *(Hirzebruch-Riemann-Roch Theorem, [23, Section 20]) If M^{2n} is a compact complex manifold, then $\chi^p(M^{2n}) = T_n^p[M^{2n}]$ for all p. In other words, $\chi_y(M) = T[M]$.*

In summary, the χ_y-genus of a smooth projective toric variety X_P is completely determined by the g-vector of P, and the generalized Todd genus can be written as a linear combination of Chern numbers with rational coefficients. The Hirzebruch-Riemann-Roch Theorem therefore provides a way of deducing information about the cobordism of a smooth projective toric variety from its g-vector. In fact, a distinct Chern number of a smooth projective toric variety can be determined from each coordinate of its g-vector.

Proposition 4.10. *([30, 32]) Let M^{2n} be a compact complex manifold. The numbers $\chi^p(M)$ provide exactly $\left\lfloor \frac{n+2}{2} \right\rfloor$-many linearly independent conditions on the Chern numbers of M. These conditions are given by*

$$\chi^p\left(M^{2n}\right) = T_n^p\left[M^{2n}\right]$$

for $p = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

This proposition along with Corollaries 4.6 and 4.7 give the following

Corollary 4.11. *Let X_P be a smooth projective toric variety of complex dimension n. Write the g-vector of P as $\left(1, g_1, \ldots, g_{\lfloor n/2 \rfloor}\right)$. Then each of these coordinates determines a distinct Chern number of the cobordism class $[X_P]$. This correspondence is described by*

$$(-1)^p \sum_{k=0}^{p} g_k = T_n^p\left[M^{2n}\right]$$

for $p = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$.

As an example, consider the relation corresponding to $p = 0$. Note that substituting $y = 0$ in the χ_y-genus yields $\chi_0 = \sum_{p=0}^{n} \chi^p \cdot 0^p = \chi^0$ (see (4.1.1)). By the Hirzebruch-Riemann-Roch Theorem and the definition of the generalized Todd genus, this means that χ_0 is the genus associated to the power series $Q(0, x) = \dfrac{x}{1 - e^{-x}}$. By Definition 2.10, $\chi^0 = \chi_0 = \mathrm{Td}$ is itself the Todd genus (i.e. the Todd genus is the constant term of the generalized Todd genus). Using the Hirzebruch-Riemann-Roch Theorem and the fact that $g_0 = 1$, we obtain the relation

$$1 = T_n^0\left[M^{2n}\right] = \chi^0(M) = \mathrm{Td}.$$

Thus the fact that the coordinate g_0 of the g-vector of a simple polytope is always one produces the well-known fact that the Todd genus of a smooth projective toric variety is always one (see Proposition 4.2).

Since the other g-vector coordinates of a smooth polytope do not have any fixed values, they result in much more complicated descriptions of Chern numbers. There is a more convenient way of deriving these conditions that follows from [32]. Consider the generalized Todd class $T_n(y; c_1, \ldots, c_n)$, and define a new class

$$t_n(y) = t_n(y; c_1, \ldots c_n) = T_n(y - 1; c_1, \ldots, c_n).$$

By the Hirzebruch-Riemann-Roch Theorem,

$$t_n(y)\left[M^{2n}\right] = \sum_{p=0}^{n} \chi^p(M)(y - 1)^p. \tag{4.1.3}$$

Corollary 4.12. *([32]) The $\left\lfloor \frac{n+2}{2} \right\rfloor$-many independent conditions on Chern numbers arising from the χ_y-genus (and thus the g-vector for smooth projective toric varieties) can be described by equating the coefficients of the even powers of y in (4.1.3).*

Recall that a cobordism class $[M^{2n}] \in \Omega_n^U$ is completely described by $|\pi(n)|$ integers, where $\pi(n)$ is the set of partitions of n (see Theorem 2.5). Thus for the cobordism class of a smooth projective toric variety, $|\pi(n)| - \lfloor \frac{n+2}{2} \rfloor$ of its Chern numbers are completely independent of the g-vector of the toric variety. Even in relatively low dimensions, these Chern numbers are determined by an intricate interplay of the combinatorics and geometry of the toric variety.

4.2 Smooth projective toric varieties in low-dimensional cobordism

Since $|\pi(n)| = \lfloor \frac{n+2}{2} \rfloor$ for $n = 1, 2$, the answer to Problem 4.1 in Ω_2^U and Ω_4^U can be completely described in terms of the combinatorial structure of smooth projective toric varieties. Unfortunately, these are the only two dimensions for which this is true (as will be seen in the next section).

Theorem 4.13. *The only cobordism class in Ω_2^U that is represented by a smooth projective toric variety is $[\mathbb{C}P^1]$.*

This is true simply because $\mathbb{C}P^1$ is the only smooth projective toric variety of dimension two. This follows from the fact that there is only one one-dimensional combinatorial polytope.

By Theorem 4.10, there are exactly $\lfloor \frac{2+2}{2} \rfloor = 2$ independent conditions on the Chern numbers determined by the χ^p in Ω_4^U. These can be computed by comparing the constant term and degree two term of $t_2(y)[M^4] = \sum_{p=0}^{2} \chi^p(M)(y-1)^p$ (see Corollary 4.12). In [32],

$$t_2(y) = c_2 - c_2 y + \frac{1}{12}\left(c_2 + c_1^2\right) y^2$$

is computed. Thus the conditions on Chern numbers in this dimension are given by $c_2 = \sum_{p=0}^{2} (-1)^p \chi^p = \chi_{-1}$ and $\frac{1}{12}(c_2 + c_1^2) = \chi^2 = \chi^0 = \mathrm{Td}$. If the cobordism class under consideration contains a smooth projective toric variety, then applying Corollary 4.7 yields

$$c_2 = 3 + g_1 \text{ and } \mathrm{Td} = 1, \tag{4.2.1}$$

where $(1, g_1)$ is the g-vector of the associated polytope.

Theorem 4.14. *A cobordism class $[M] \in \Omega_4^U$ can be represented by a smooth projective toric variety if and only if* $\mathrm{Td}(M) = 1$ *and* $c_2[M] \in \{3, 4, 5, \dots\}$.

Proof. By (4.2.1), if a cobordism class $[M] \in \Omega_4^U$ is represented by a smooth projective toric variety, then $\mathrm{Td}(M) = 1$ and $c_2[M] = 3 + g_1$ for some $(1, g_1)$ that is the g-vector of a smooth two-dimensional polytope. By the g-theorem 3.20, g-vectors that correspond to simple polytopes in $\mathbb{R}^2$ are given by $\{(1, g_1) \,|\, g_1 \geq 0\}$. Thus, if $[M] \in \Omega_4^U$ is represented by a smooth projective toric variety, then $\mathrm{Td}(M) = 1$ and $c_2[M] \in \{3, 4, 5, \dots\}$.

The converse can be proven by noting that a smooth polytope in $\mathbb{R}^2$ can be created for each of these g-vectors. For example, the g-vector of the triangle pictured in Figure 3.3 is $(1, 0)$. The corresponding smooth projective toric variety is $\mathbb{C}P^2$ (see Example 3.19). By Theorem 3.39, a four-dimensional smooth projective toric variety with g-vector $(1, g_1)$ can be obtained by applying a sequence of g_1-many blow-ups at torus fixed points, starting with $\mathbb{C}P^2$. On the level of polytopes, this corresponds to applying g_1-many truncations of vertices to the triangle in Figure 3.3 to produce a new polytope with g-vector $(1, g_1)$. This construction proves that each cobordism class in Ω_4^U satisfying $\mathrm{Td}(M) = 1$ and $c_2[M] \in \{3, 4, 5, \ldots\}$ can be represented by a smooth projective toric variety. □

This result can be generalized due to classification results in this small dimension. More specifically, all smooth compact toric varieties of dimension four have been classified. They are all obtained through a sequence of equivariant blow-ups at torus fixed points in either $\mathbb{C}P^2$ or a Hirzebruch surface [16, Section 2.5]. Since both $\mathbb{C}P^2$ and Hirzebruch surfaces are projective and equivariant blow-ups preserve projectivity, all of the smooth toric varieties of dimension four must be projective. As a consequence, the projectivity condition in Theorem 4.14 can be dropped.

Corollary 4.15. *A cobordism class $[M] \in \Omega_4^U$ can be represented by a smooth compact toric variety if and only if $Td(M) = 1$ and $c_2[M] \in \{3, 4, 5, \ldots\}$.*

4.3 Smooth projective toric varieties in six-dimensional cobordism

The answer to Problem 4.1 in Ω_6^U is already significantly more complicated than in lower dimensions. In this dimension, there are again $\left\lfloor \frac{3+2}{2} \right\rfloor = 2$ independent conditions on the Chern numbers determined by the χ_y-genus (see Theorem 4.10). As in Ω_4^U, these are given by equating the constant and degree two terms of

$$t_3(y)\left[M^6\right] = \sum_{p=0}^{3} \chi^p(M)(y-1)^p.$$

Again, the left-hand side is computed in [32] where is found to be

$$t_3 = c_3 + \frac{1}{2}(-3c_3)\,y + \frac{1}{12}(6c_3 + c_1c_2)\,y^2 + \frac{1}{24}(-c_1c_2)\,y^3.$$

Thus by Corollary 4.12, the conditions on the Chern numbers are

$$c_3 = \sum_{p=0}^{3}(-1)^p \chi^p = \chi_{-1} \text{ and } \frac{1}{12}(6c_3 + c_1c_2) = \chi^2 - 3\chi^3.$$

If $[M]$ contains a smooth projective toric variety, then Corollary 4.7 can be used to state the Chern number relations as

$$c_3[M] = 4 + 2g_1 \text{ and } \frac{1}{24}c_1c_2[M] = \mathrm{Td}(M) = 1. \tag{4.3.1}$$

The two Chern numbers $c_1c_2[M]$ and $c_3[M]$ in addition to a third Chern number $c_1^3[M]$ completely determine the cobordism class of M in Ω_6^U (see Theorem 2.5). However, $c_1^3[M]$ is not determined by the χ_y-genus.

Before considering this additional Chern number, we will first examine the results obtained by the above restrictions on c_3 and c_1c_2. According to the g-theorem, the g-vectors that correspond to simple polytopes in $\mathbb{R}^3$ are those with $g_1 \geq 0$. In fact, for each of these g-vectors, a smooth polytope can be constructed. This is done in the same way as in the proof of Theorem 4.14. The polytope associated to $\mathbb{C}P^3$ has g-vector $(1, 0)$. Apply a sequence of g_1-many blow-ups of torus-fixed points to obtain a smooth projective toric variety with g-vector $(1, g_1)$. Combining this with (4.3.1) implies that the only cobordism classes $[M] \in \Omega_6^U$ that could possibly contain smooth projective toric varieties are those that satisfy $\mathrm{Td}(M) = 1$ and $c_3[M] \in \{4, 6, 8, \ldots\}$.

Since the Chern number $c_1^3[X]$ of a smooth projective toric variety is not determined by its combinatorial structure, the answer to Question 4.1 becomes much more complicated.

Theorem 4.16. *Let* $[M] \in \Omega_6^U$.

1. *If* $c_1c_2[M] \neq 24$ *or* $c_3[M] \notin \{4, 6, 8, \ldots\}$, *then* $[M]$ *is not represented by a smooth projective toric variety.*

2. *Suppose* $c_1c_2[M] = 24$ *and* $c_3[M] = 4$. *Then* $[M]$ *is represented by a smooth projective toric variety if and only if* $[M] = [\mathbb{C}P^3]$.

3. *Suppose* $c_1c_2[M] = 24$ *and* $c_3[M] = 6$. *Then* $[M]$ *is represented by a smooth projective toric variety if and only if* $c_1^3[M] = 2a^2 + 54$ *for some* $a \in \mathbb{Z}$.

4. *If* $c_1c_2[M] = 24$ *and* $c_3[M] \in \{8, 10, 12, \ldots\}$, *then* $[M]$ *is represented by a smooth projective toric variety.*

Part 1 of Theorem 4.16 is proven by the preceding argument. In order to prove the remaining parts, one can consider all possible pairs $c_1c_2[M]$ and $c_3[M]$ given by (4.3.1) with $g_1 \geq 0$, and for each pair, find all values of $c_1^3[M]$ that result in a cobordism class containing a smooth projective toric variety. Before doing this, it is essential to know exactly which combinations of Chern numbers can represent cobordism classes in Ω_6^U in general. This can be accomplished by applying the Hattori-Stong Theorem 2.18 in this dimension.

K-theory Chern numbers and Ω_6^U

Before applying the Hattori-Stong Theorem 2.18, it will be useful to know several relations among symmetric polynomials and Chern numbers. Suppose $[M] \in \Omega_6^U$ and formally write $c(M) = (1 + x_1)(1 + x_2)(1 + x_3)$. To simplify notation, the Chern class $c_k(M)$ will be abbreviated as c_k. Then

$$\mathrm{Td}(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right) + \frac{1}{24}c_1c_2 + \ldots \qquad (4.3.2)$$

(see Definition 2.10). The following relations among symmetric polynomials are straight-forward to derive.

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= c_1^2 - 2c_2 \\ x_1^3 + x_2^3 + x_3^3 &= c_1^3 - 3c_1c_2 + 3c_3 \\ \sum_{i=1}^{3}\sum_{j\neq i} x_i^2 x_j &= c_1c_2 - 3c_3 \end{aligned} \tag{4.3.3}$$

The partitions that must be considered when using the Hattori-Stong Theorem in this dimension are $\omega \in \{\varnothing, 1, 11, 2, 111, 12, 3\}$. The K-theory Chern number corresponding to each of these partitions can be calculated by using the techniques described in Section 2.1. The classes s_ω are calculated in Section 16 of [36].

$\boldsymbol{\omega = \varnothing}$: $s_\varnothing() = 1$, so $\kappa_\varnothing[M] = \mathrm{Td}[M] = \frac{1}{24}c_1c_2$. This gives the first divisibility relation for Chern numbers in Ω_6^U:

$$c_1c_2[M] \equiv 0 \bmod 24 \tag{4.3.4}$$

$\boldsymbol{\omega = 1}$: $s_1(\gamma_1) = \gamma_1$, so

$$\begin{aligned} \kappa_1[M] &= \langle \mathrm{ch}\gamma_1 \cdot \mathrm{Td}(M), \mu_M\rangle \\ &= \langle \sigma_1(e^{x_1}-1, e^{x_2}-1, e^{x_3}-1)\cdot \mathrm{Td}(M), \mu_M\rangle \\ &= \left\langle \left((x_1+x_2+x_3) + \frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right) + \frac{1}{6}\left(x_1^3+x_2^3+x_3^3\right)\right)\cdot \right. \\ &\qquad \left. \cdot \mathrm{Td}(M), \mu_M \right\rangle \\ &= \left\langle \left(c_1 + \left(\frac{1}{2}c_1^2 - c_2\right) + \left(\frac{1}{6}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)\right)\cdot \right. \\ &\qquad \left. \cdot \left(1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2+c_2\right)\right), \mu_M \right\rangle \\ &= \left\langle \frac{1}{2}c_1^3 - \frac{11}{12}c_1c_2 + \frac{1}{2}c_3, \mu_M \right\rangle \\ &= \frac{1}{2}c_1^3[M] - \frac{11}{12}c_1c_2[M] + \frac{1}{2}c_3[M] \end{aligned}$$

This gives the second divisibility relation

$$6c_1^3[M] - 11c_1c_2[M] + 6c_3[M] \equiv 0 \bmod 12. \tag{4.3.5}$$

$\omega = 11$: $s_{11}(\gamma_1, \gamma_2) = \gamma_2$, so

$$
\begin{aligned}
\kappa_{11}[M] &= \langle \mathrm{ch}\gamma_2 \cdot \mathrm{Td}(M), \mu_M \rangle \\
&= \langle \sigma_2 (e^{x_1} - 1, e^{x_2} - 1, e^{x_3} - 1) \cdot \mathrm{Td}(M), \mu_M \rangle \\
&= \left\langle \left((x_1 x_2 + x_1 x_3 + x_2 x_3) + \frac{1}{2} \sum_{i=1}^{3} \sum_{j \neq i} x_i^2 x_j \right) \cdot \mathrm{Td}(M), \mu_M \right\rangle \\
&= \left\langle \left(c_2 + \frac{1}{2}(c_1 c_2 - 3c_3) \right) \cdot \left(1 + \frac{1}{2} c_1 \right), \mu_M \right\rangle \\
&= \left\langle c_1 c_2 - \frac{3}{2} c_3, \mu_M \right\rangle \\
&= c_1 c_2 [M] - \frac{3}{2} c_3 [M]
\end{aligned}
$$

This gives the third divisibility relation

$$2c_1 c_2 [M] - 3c_3 [M] \equiv 0 \text{ mod } 2. \tag{4.3.6}$$

$\omega = 2$: $s_2(\gamma_1, \gamma_2) = \gamma_1^2 - 2\gamma_2$, so $\mathrm{ch}s_2(\gamma_1, \gamma_2) = (\mathrm{ch}\gamma_1)^2 - 2\mathrm{ch}\gamma_2$. Then

$$
\begin{aligned}
\kappa_2[M] = &\left\langle \left(\left(c_1 + \left(\frac{1}{2} c_1^2 - c_2 \right) \right)^2 - 2 \left(c_2 + \frac{1}{2}(c_1 c_2 - 3c_3) \right) \right) \cdot \mathrm{Td}(M), \mu_M \right\rangle \\
&\hfill (\text{see } \omega = 1 \ \& \ \omega = 11) \\
= &\left\langle \left((c_1^2 - 2c_2) + (c_1^3 - 3c_1 c_2 + 3c_3) \right) \cdot \left(1 + \frac{1}{2} c_1 \right), \mu_M \right\rangle \\
= &\left\langle \frac{3}{2} c_1^3 - 4c_1 c_2 + 3c_3, \mu_M \right\rangle \\
= &\frac{3}{2} c_1^3 [M] - 4c_1 c_2 [M] + 3c_3 [M]
\end{aligned}
$$

This gives the fourth divisibility relation

$$3c_1^3 [M] - 8c_1 c_2 [M] + 6c_3 [M] \equiv 0 \text{ mod } 2. \tag{4.3.7}$$

$\omega \in \{111, 12, 3\}$: Since these are partitions of 3 itself, $\kappa_\omega [M] = \langle s_\omega (c_1, c_2, c_3), \mu_M \rangle$ for each of these partitions. Each of these is a linear combination of Chern numbers with integer coefficients, so their sums are always integers since the Chern numbers are themselves integers.

Simplifying and combining the relations (4.3.4), (4.3.5), (4.3.6), and (4.3.7) provides the following

Proposition 4.17. *A cobordism class $[M] \in \Omega_6^U$ can have Chern numbers $c_1^3[M]$, $c_1 c_2[M]$, and $c_3[M]$ if and only if the following divisibility relations hold.*

$$
\begin{aligned}
c_1^3 [M] &\equiv 0 \ mod \ 2 \\
c_1 c_2 [M] &\equiv 0 \ mod \ 24 \\
c_3 [M] &\equiv 0 \ mod \ 2
\end{aligned}
$$

Smooth projective toric varieties representing Ω_6^U

Now the remaining parts of Theorem 4.16 can be proven.

Proof of Theorem 4.16 part 2. Suppose $c_1c_2[M] = 24$ and $c_3[M] = 4$ for $[M] \in \Omega_6^U$. If $[M]$ is represented by a smooth projective toric variety, then it must have g-vector $(1, 0)$ according to (4.3.1). But there is only one smooth toric variety with this g-vector, namely $\mathbb{C}P^3$. This means that $[M]$ is represented by a smooth projective toric variety if and only if $[M] = [\mathbb{C}P^3]$. $\square$

This part of the theorem demonstrates that (4.3.1) are not the only obstructions to a cobordism class in Ω_6^U being represented by a smooth projective toric variety. Since $c_1^3[\mathbb{C}P^3] = 64$ (see [36, Section 14] for details), all the cobordism classes with $c_1c_2[M] = 24$, $c_3[M] = 4$, and $c_1^3[M] \neq 64$ do not contain any smooth projective toric varieties, even though these classes satisfy (4.3.1). Such cobordism classes exist by Proposition 4.17.

To prove part 3 of Theorem 4.16, consider a cobordism class $[M] \in \Omega_6^U$ with $c_1c_2[M] = 24$ and $c_3[M] = 6$. If $[M]$ were represented by a smooth projective toric variety, then its g-vector would be $(1, 1)$ by (4.3.1). This means that its h-vector would be $(1, 2, 2, 1)$. By (3.2.1), the f-vector of the associated polytope would be $(5, 9, 6)$. In particular, the polytope would have five facets. Then the normal fan would have $5 = 3 + 2$ generating rays. But these are exactly the fans that were classified by Kleinschmidt (see Theorem 3.24). In fact, it is easy to check that each of Kleinschmidt's varieties with real dimension six has $(5, 9, 6)$ as the f-vector of the associated polytope. Thus if $c_1c_2[M] = 24$ and $c_3[M] = 6$ for $[M] \in \Omega_6^U$, then the only possible smooth projective toric varieties that could represent $[M]$ are Kleinschmidt's varieties $X_3(a_1, a_2)$ and $X_3(a_1)$. In order to prove part 3 of Theorem 4.17, it suffices to calculate $c_1^3[X_3(a_1, a_2)]$ and $c_1^3[X_3(a_1)]$.

Lemma 4.18. *$c_1^3[X_3(a_1, a_2)] = 54$ for all integers $0 \leq a_1 \leq a_2$.*

Proof. Recall that $\Sigma_3(a_1, a_2)$ has five generating rays u_1, u_2, u_3, v_1, and v_2. From Example 3.28, $H^*(X_3(a_1, a_2)) \cong \mathbb{Z}[u_3, v_2] / (u_3^3 - (a_1 + a_2) u_3^2 v_2, v_2^2)$. In particular, the ideal L in Example 3.28 yields the following linear relations among the cohomology classes.

$$u_1 = u_3 - a_1 v_2$$
$$u_2 = u_3 - a_2 v_2$$
$$v_1 = v_2$$

Then applying Theorem 3.30 gives the first Chern class of $X_3(a_1, a_2)$ as

$$c_1(X_3(a_1, a_2)) = u_1 + u_2 + u_3 + v_1 + v_2 = 3u_3 + (2 - a_1 - a_2) v_2.$$

Then

$$\begin{aligned} c_1^3(X_3(a_1, a_2)) &= (3u_3 + (2 - a_1 - a_2) v_2)^3 \\ &= 27u_3^3 + 3 \cdot 9u_3^2 \cdot (2 - a_1 - a_2) v_2 \\ &= 54u_3^2 v_2. \end{aligned}$$

Note that $\text{pos}(u_1, u_3, v_2)$ is a maximal cone in $\Sigma_3(a_1, a_2)$. Using the above linear relation for u_1 and $v_2^2 = 0$ from the cohomology ring,

$$u_1 u_3 v_2 = u_3^2 v_2.$$

Thus by Proposition 3.29, $\langle u_3^2 v_2, \mu_{X_3(a_1,a_2)} \rangle = 1$. Then $c_1^3 [X_3(a_1, a_2)] = 54$. ☐

Lemma 4.19. *$c_1^3 [X_3(a)] = 2a^2 + 54$ for any $a \geq 0$.*

Proof. This is proven using a computation similar to that of the previous lemma. Again, start by calculating $H^*(X_3(a))$. The linear relations are

$$\begin{aligned} u_1 &= u_2 - a v_3 \\ v_1 &= v_3 \\ v_2 &= v_3 \end{aligned}$$

and the Stanley-Reisner ideal is $J = (u_1 u_2, v_1 v_2 v_3)$. Combining these yields

$$H^*(X_3(a)) = \mathbb{Z}[u_2, v_3] / \left(u_2^2 - a u_2 v_3, v_3^3\right).$$

The first Chern class of the variety is $c_1(X_3(a)) = 2u_2 + (3 - a) v_3$. Then

$$\begin{aligned} c_1^3(X_3(a)) &= 3 \cdot 2u_2 \cdot (3-a)^2 v_3^2 + 3 \cdot 4u_2^2 \cdot (3-a) v_3 + 8u_2^3 \\ &= \left(54 - 36a + 6a^2\right) u_2 v_3^2 + \left(36a - 12a^2\right) u_2 v_3^2 + 8a^2 u_2 v_3^2 \\ &= \left(54 + 2a^2\right) u_2 v_3^2. \end{aligned}$$

Since $\text{pos}(u_2, v_1, v_3)$ is a maximal cone and $u_2 v_1 v_3 = u_2 v_3^2$ in $H^*(X_3(a))$, the result follows from Proposition 3.29. ☐

Part 3 of Theorem 4.16 is an immediate consequence of the preceding two lemmas.

Part 4 of Theorem 4.16 states that if $c_3[M]$ is sufficiently large, then every cobordism class is represented by a smooth projective toric variety. This part will be proven in two steps. First, smooth projective toric variety representatives will be constructed for each cobordism class with $c_1 c_2 [M] = 24$ and $c_3 [M] = 8$. Next, a sequence of blow-ups will be applied to these varieties to prove the existence of smooth projective toric variety representatives for any higher possible $c_3[M]$.

Proposition 4.20. *If $c_1 c_2 [M] = 24$ and $c_3 [M] = 8$ for $[M] \in \Omega_6^U$, then $[M]$ can be represented by a smooth projective toric variety.*

Proof. Let $\tau_1 = \text{pos}(u_1, u_2)$ in $\Sigma_3(a_1, a_2)$. Consider $Bl_{\tau_1}\Sigma_3(a_1, a_2)$, the star subdivision with additional ray $z_1 = u_1 + u_2$. Denote the corresponding smooth projective toric variety as $X^1(a_1, a_2)$. To compute the cobordism class of $X^1(a_1, a_2)$, it suffices to compute its Chern numbers. By Theorem 3.39, the g-vector of the corresponding polytope is $(1, 2)$, so $c_1 c_2 [X^1(a_1, a_2)] = 24$ and $c_3 [X^1(a_1, a_2)] = 8$ by (4.3.1).

To compute $c_1^3 [X^1(a_1,a_2)]$, first calculate $H^*(X^1(a_1,a_2))$ (see Theorem 3.26). The linear relations in this cohomology ring are

$$\begin{aligned} u_1 &= u_3 - a_1 v_2 - z_1 \\ u_2 &= u_3 - a_2 v_2 - z_1 \\ v_1 &= v_2 \end{aligned}$$

and the Stanley-Reisner ideal is $J = (u_1u_2, v_1v_2, u_3z_1)$. Combining these yields

$$H^*\left(X^1(a_1,a_2)\right) = \mathbb{Z}[u_3,v_2,z_1] / \left(v_2^2, u_3z_1, u_3^2 - (a_1+a_2)u_3v_2 + (a_1+a_2)v_2z_1 + z_1^2\right).$$

Theorem 3.30 and the above linear relations can be used to calculate

$$c_1(X_{12}(a_1,a_2)) = 3u_3 + (2-a_1-a_2)v_2 - z.$$

Then

$$\begin{aligned} c_1\left(X^1(a_1,a_2)\right)^3 &= 27u_3^3 + 3\cdot 9u_3^2\cdot(2-a_1-a_2)v_2 + 3\cdot(2-a_1-a_2)v_2\cdot z_1^2 - z_1^3 \\ &= 27(a_1+a_2)u_3^2v_2 + 27(2-a_1-a_2)u_3^2v_2 + \\ &\qquad -3(2-a_1-a_2)u_3^2v_2 - (a_1+a_2)u_3^2v_2 \\ &= (48+2(a_1+a_2))u_3^2v_2 \end{aligned}$$

so $c_1^3[X^1(a_1,a_2)] = 48+2(a_1+a_2)$, as in the preceding lemmas. The cobordism class $[M]\in\Omega_6^U$ with $c_1c_2[M]=24$, $c_3[M]=8$, and $c_1^3[M]=48+2k\in\{48,50,52,\ldots\}$ is represented by the smooth projective toric variety $X^1(0,k)$.

Next define $\tau_2 = \operatorname{pos}(u_1,u_3)$ in $\Sigma_3(a_1,a_2)$. Consider $Bl_{\tau_2}\Sigma_3(a_1,a_2)$ with additional ray $z_2 = u_1+u_3$. Let $X^2(a_1,a_2)$ denote the corresponding smooth projective toric variety. The g-vector of the corresponding polytope is again $(1,2)$, so $c_1c_2[X^2(a_1,a_2)] = 24$ and $c_3[X^2(a_1,a_2)] = 8$. The linear relations in $H^*(X^2(a_1,a_2))$ are

$$\begin{aligned} u_1 &= u_2 - (a_1-a_2)v_2 - z_2 \\ u_3 &= u_2 + a_2v_2 - z_2 \\ v_1 &= v_2 \end{aligned}$$

and the Stanley-Reisner ideal is $J=(u_1u_3, v_1v_2, u_2z_2)$. Then

$$H^*\left(X^2(a_1,a_2)\right) = \mathbb{Z}[u_2,v_2,z_2]/\left(v_2^2, u_2z_2, u_2^2 + (2a_2-a_1)u_2v_2 - (2a_2-a_1)v_2z_2 + z_2^2\right).$$

The first Chern class of the variety is $c_1(X^2(a_1,a_2)) = 3u_2 + (2+2a_2-a_1)v_2 - z_2$. Then

$$\begin{aligned} c_1\left(X^2(a_1,a_2)\right)^3 &= 27u_2^3 + 3\cdot 9u_2^2\cdot(2+2a_2-a_1)v_2 + 3\cdot(2+2a_2-a_1)v_2\cdot z_2^2 - z_2^3 \\ &= -27(2a_2-a_1)u_2^2v_2 + 27(2+2a_2-a_1)u_2^2v_2 + \\ &\qquad -3(2+2a_2-a_1)u_2^2v_2 + (2a_2-a_1)u_2^2v_2 \\ &= (48+2a_1-4a_2)u_3^2v_2 \end{aligned}$$

so $c_1^3 [X^2 (a_1, a_2)] = 48 + 2a_1 - 4a_2$. Consider the cobordism class $[M] \in \Omega_6^U$ that satisfies $c_1c_2[M] = 24$, $c_3[M] = 8$, and $c_1^3[M] = 48 - 4k \in \{48, 44, 40, \ldots\}$. This cobordism class is represented by the smooth projective toric variety $X^2 (0, k)$. If instead, $c_1^3 [M] = 46 - 4k \in \{46, 42, 38, \ldots\}$, then $[M]$ is represented by the smooth projective toric variety $X^2 (1, k + 1)$.

Since $c_1^3 [M]$ must be even by Proposition 4.17, all possible cobordism classes with $c_1c_2 [M] = 24$ and $c_3 [M]$ have been obtained using the smooth projective toric varieties $X^1 (0, k)$, $X^2 (0, k)$, and $X^2 (1, k + 1)$. □

Recall from Proposition 3.40 that when a toric variety X_Σ is blown up along a subvariety X_τ given by a cone $\tau \subset \Sigma$, the change in cobordism is completely determined by the cones of Σ that contain τ. In the case of blowing up a point, τ is a maximal cone of Σ, so τ is the only cone that contains τ. A unimodular transformation can be chosen to send the generating rays of τ to the standard basis vectors. This produces an isomorphic toric variety. Thus the change in cobordism resulting from the blow-up of any toric variety at a point can be completely determined by computing the change in Chern numbers in just one convenient example.

Lemma 4.21. *Let X_1 be a smooth projective toric variety with fan Σ_{X_1} in $\mathbb{R}^3$. Suppose that Σ_{X_2} is obtained through a star subdivision of a maximal cone in Σ_{X_1}. Then $c_3 [X_2] = c_3 [X_1] + 2$ and $c_1^3 [X_2] = c_1^3 [X_1] - 8$.*

Proof. From the preceding argument, it suffices to verify this for any one example. Choose the fan $\Sigma_1 = \{x_1, x_2, x_3, y\}$ corresponding to $X_1 = \mathbb{C}P^3$, so $x_k = e_k$ is the standard basis vector and $y = (-1, -1, -1)$. Let $\Sigma_2 = \{x_1, x_2, x_3, y, z\}$ be the fan obtained by subdividing the maximal cone $\operatorname{pos}(x_1, x_2, x_3)$ with z, so $z = (1, 1, 1)$. Let X_2 be the smooth projective toric variety corresponding to this fan.

Equation (4.3.1) makes it easy to compute the change in c_3 during the blow-up. Note that $g (X_1) = (1, 0)$ and $g (X_2) = (1, 1)$ by Example 3.19 and Theorem 3.39. Then $c_3 [X_1] = 4$ and $c_3 [X_2] = 4 + 2 = c_3 [X_1] + 2$.

Recall that $c_1^3 [\mathbb{C}P^3] = 64$. To compute $c_1^3 [X_2]$, we must first compute its cohomology $H^* (X_2)$. The linear relations in $H^* (X_2)$ are $x_1 = x_2 = x_3 = y - z$, and the Stanley-Reisner ideal is $J = (x_1x_2x_3, yz)$. Combining these yields

$$H^* (X_2) = \mathbb{Z} [y, z] / \left(yz, y^3 - z^3\right).$$

The first Chern class of the variety is $c_1 (X_2) = x_1 + x_2 + x_3 + y + z = 4y - 2z$, so $c_1 (X_2)^3 = 64y^3 - 8z^3 = 56y^3$. Since $y^3 = x_1x_2y$ in $H^* (X_2)$ and $\operatorname{pos}(x_1, x_2, y)$ is a maximal cone in Σ_2, by Proposition 3.29, $c_1^3 [X_2] = 56 = c_1^3 [X_1] - 8$. □

Now a smooth projective toric variety can be constructed to represent any cobordism class $[M] \in \Omega_6^U$ such that $c_1c_2 [M] = 24$ and $c_3 [M] \in \{8, 10, 12, \ldots\}$.

Proof of part 4 of Theorem 4.16. Suppose $[M] \in \Omega_6^U$ is a cobordism class satisfying $c_1c_2 [M] = 24$, $c_3 [M] = 8 + 2B$ for some integer $B \geq 0$, and $c_1^3 [M] = 2k$ for some $k \in \mathbb{Z}$. (Recall that all cobordism classes in this dimension have even Chern number c_1^3 by Proposition 4.17.) By Proposition 4.20, there exists a smooth projective toric variety X_Σ satisfying $c_1c_2[X_\Sigma] = 24$, $c_3[X_\Sigma] = 8$, and $c_1^3[X_\Sigma] = 2k + 8B$. Apply a sequence of B-many blow-ups at torus-fixed points of X_Σ and the subsequent blown up varieties to obtain a new smooth projective toric variety $X_{\Sigma'}$. By Lemma 4.21, $c_1c_2[X_{\Sigma'}] = 24$, $c_3[X_{\Sigma'}] = 8+2B$, and $c_1^3[X_{\Sigma'}] = 2k$. Thus $[M] = [X_{\Sigma'}]$ is represented by a smooth projective toric variety. □

4.4 Smooth projective toric varieties in eight-dimensional cobordism

The techniques used to answer Question 4.1 in Ω_6^U can be applied to Ω_8^U as well. However, the computations that are involved are already significantly more complicated, meaning that extending the techniques to even higher dimensions is not practical. Also, studying Ω_8^U reveals some of the limitations of the methods that have been used so far. Unlike in lower dimensions, they can only provide a partial answer to Problem 4.1.

In this dimension, there are $\left\lfloor \frac{4+2}{2} \right\rfloor = 3$ independent conditions on the Chern numbers determined by the χ_y-genus (see Theorem 4.10). These are given by equating the constant, degree two, and degree four terms of $t_4(y)[M^8] = \sum_{p=0}^{4} \chi^p(M)(y-1)^p$ (see Corollary 4.12). The left-hand side is computed in [32] where it is found to be

$$t_4 = c_4 - 2c_4 y + \frac{1}{12}(14c_4 + c_1c_3)y^2 + \frac{1}{12}(-2c_4 - c_1c_3)y^3 + \\ + \frac{1}{720}\left(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4\right)y^4.$$

Thus the conditions on the Chern numbers are

$$c_4 = \sum_{p=0}^{4}(-1)^p \chi^p = \chi_{-1}$$

$$\frac{1}{12}(14c_4 + c_1c_3) = \chi^2 - 3\chi^3 + 6\chi^4$$

$$\frac{1}{720}\left(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4\right) = \chi^4 = \mathrm{Td}.$$

If $[M]$ contains a smooth projective toric variety, then Corollary 4.7 can be used to simplify these and write

$$\begin{aligned} c_4[M] &= 5 + 3g_1 + g_2 \\ c_1c_3[M] &= 50 + 6g_1 - 2g_2 \\ c_1^4[M] &= 4c_1^2c_2[M] + 3c_2^2[M] + 3g_1 - 3g_2 - 675 \end{aligned} \tag{4.4.1}$$

where $(1, g_1, g_2)$ is the g-vector of the associated polytope.

Now that some of the Chern numbers have been written in terms of g-vectors, it is important to know which g-vectors can correspond to smooth polytopes in $\mathbb{R}^4$. According to the g-theorem 3.20, a g-vector $(1, g_1, g_2)$ corresponds to a simple polytope in $\mathbb{R}^4$ if and only if

$$0 \le g_1 \text{ and } 0 \le g_2 \le \frac{1}{2}g_1(g_1+1). \tag{4.4.2}$$

This means that the only cobordism classes $[M] \in \Omega_8^U$ that can possibly be represented by a smooth projective toric variety are those satisfying (4.4.1) for some g-vector $(1, g_1, g_2)$ which satisfies (4.4.2).

Theorem 4.22. *Let $[M] \in \Omega_8^U$. If $[M]$ does not satisfy equations (4.4.1) for some g-vector $(1, g_1, g_2)$ satisfying (4.4.2), then $[M]$ does not contain a smooth projective toric variety.*

K-theory Chern numbers and Ω_8^U

As in Ω_6^U, it is next necessary to determine exactly which combinations of Chern numbers cobordism classes in Ω_8^U can have. There are $|\pi(4)| = 5$ Chern numbers in this dimension. Fix $[M] \in \Omega_8^U$, and abbreviate the Chern classes of M by $c_k(M) = c_k$. The first terms of the Todd class are

$$\mathrm{Td}(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right) + \frac{1}{24}c_1c_2 + \frac{1}{720}\left(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4\right)$$

(see [23, Section 1.7] for details). Formally write $c(M) = \prod_{k=1}^{4}(1 + x_k)$. The following relations among symmetric polynomials are also useful. The sums are taken over all monomials that are equivalent to the given monomial, i.e. all monomials obtained from a permutation of $x_1, \ldots, x_4$ (see Definition 2.6).

$$\begin{aligned}
\sum x_1^2 &= c_1^2 - 2c_2 \\
\sum x_1^3 &= c_1^3 - 3c_1c_2 + 3c_3 \\
\sum x_1^4 &= c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4 \\
\sum x_1^2x_2 &= c_1c_2 - 3c_3 \\
\sum x_1^3x_2 &= c_1^2c_2 - 2c_2^2 - c_1c_3 + 4c_4 \\
\sum x_1^2x_2^2 &= c_2^2 - 2c_1c_3 + 2c_4 \\
\sum x_1^2x_2x_3 &= c_1c_3 - 4c_4
\end{aligned}$$

The partitions that must be considered for the Hattori-Stong Theorem 2.18 in this dimension are

$$\omega \in \{\varnothing, 1, 11, 2, 111, 12, 3, 1111, 112, 22, 13, 4\}.$$

The computations of s_ω can again be found in Section 16 of [36].

$\boldsymbol{\omega = \varnothing}$**:** $s_\varnothing() = 1$, so

$$\kappa_\varnothing[M] = \mathrm{Td}[M] = \frac{1}{720}\left(-c_1^4[M] + 4c_1^2c_2[M] + 3c_2^2[M] + c_1c_3[M] - c_4[M]\right).$$

This gives the first divisibility relation for Chern numbers in Ω_8^U:

$$-c_1^4[M] + 4c_1^2c_2[M] + 3c_2^2[M] + c_1c_3[M] - c_4[M] \equiv 0 \bmod 720 \tag{4.4.3}$$

$\boldsymbol{\omega = 1}$**:** As in Ω_6^U,

$$\begin{aligned}
\kappa_1[M] &= \langle \sigma_1(e^{x_1}-1,\dots,e^{x_4}-1)\cdot \mathrm{Td}(M), \mu_M\rangle \\
&= \left\langle \left(\sum_{k=1}^4 x_k + \frac{1}{2}\sum_{k=1}^4 x_k^2 + \frac{1}{6}\sum_{k=1}^4 x_k^3 + \frac{1}{24}\sum_{k=1}^4 x_k^4\right)\cdot \mathrm{Td}(M), \mu_M \right\rangle \\
&= \left\langle \left(c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right)\right.\right. \\
&\qquad \left. + \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4\right)\right)\cdot \\
&\qquad \left. \cdot \left(1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right) + \frac{1}{24}c_1c_2\right), \mu_M \right\rangle \\
&= \left\langle \frac{1}{6}c_1^4 - \frac{5}{12}c_1^2c_2 + \frac{5}{12}c_1c_3 - \frac{1}{6}c_4, \mu_M \right\rangle \\
&= \frac{1}{6}c_1^4[M] - \frac{5}{12}c_1^2c_2[M] + \frac{5}{12}c_1c_3[M] - \frac{1}{6}c_4[M].
\end{aligned}$$

This gives the divisibility relation

$$2c_1^4[M] - 5c_1^2c_2[M] + 5c_1c_3[M] - 2c_4[M] \equiv 0 \bmod 12. \tag{4.4.4}$$

$\boldsymbol{\omega = 11}$**:** This computation is again similar to the one performed in Ω_6^U. The sums are taken over all monomials that are equivalent to the displayed monomial (see Definition 2.6).

$$\begin{aligned}
\kappa_{11}[M] &= \langle \sigma_2(e^{x_1}-1,\dots,e^{x_4}-1)\cdot \mathrm{Td}(M), \mu_M\rangle \\
&= \left\langle \left(c_2 + \frac{1}{2}\sum x_1^2x_2 + \frac{1}{6}\sum x_1^3x_2 + \frac{1}{4}\sum x_1^2x_2^2\right)\cdot \mathrm{Td}(M), \mu_M \right\rangle \\
&= \left\langle \left(c_2 + \frac{1}{2}\left(c_1c_2 - 3c_3\right) + \frac{1}{12}\left(2c_1^2c_2 - c_2^2 - 8c_1c_3 + 14c_4\right)\right)\cdot\right. \\
&\qquad \left. \cdot \left(1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right)\right), \mu_M \right\rangle \\
&= \left\langle \frac{1}{2}c_1^2c_2 - \frac{17}{12}c_1c_3 + \frac{7}{6}c_4, \mu_M \right\rangle \\
&= \frac{1}{2}c_1^2c_2[M] - \frac{17}{12}c_1c_3[M] + \frac{7}{6}c_4[M].
\end{aligned}$$

This gives the divisibility relation

$$6c_1^2c_2[M] - 17c_1c_3[M] + 14c_4[M] \equiv 0 \bmod 12. \tag{4.4.5}$$

$\boldsymbol{\omega}$ **= 2:** As in Ω_6^U,

$$\begin{aligned}
\kappa_2[M] = \Bigg\langle \Bigg(\bigg(c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^3 - 3c_1c_2 + 3c_3\right)\bigg)^2 + \\
- 2\left(c_2 + \frac{1}{2}\left(c_1c_2 - 3c_3\right) + \frac{1}{12}\left(2c_1^2c_2 - c_2^2 - 8c_1c_3 + 14c_4\right)\right)\Bigg) \cdot \\
\cdot \operatorname{Td}(M), \mu_M \Bigg\rangle \\
= \Big\langle \Big(\left(c_1^2 - 2c_2\right) + \left(c_1^3 - 3c_1c_2 + 3c_3\right) + \\
+ \frac{1}{12}\left(7c_1^4 - 28c_1^2c_2 + 14c_2^2 + 28c_1c_3 - 28c_4\right)\Big) \cdot \\
\cdot \left(1 + \frac{1}{2}c_1 + \frac{1}{12}\left(c_1^2 + c_2\right)\right), \mu_M \Big\rangle \\
= \left\langle \frac{7}{6}c_1^4 - \frac{47}{12}c_1^2c_2 + c_2^2 + \frac{23}{6}c_1c_3 - \frac{7}{3}c_4, \mu_M \right\rangle \\
\frac{7}{6}c_1^4[M] - \frac{47}{12}c_1^2c_2[M] + c_2^2[M] + \frac{23}{6}c_1c_3[M] - \frac{7}{3}c_4[M].
\end{aligned}$$

This gives the divisibility relation

$$14c_1^4[M] - 47c_1^2c_2[M] + 12c_2^2[M] + 46c_1c_3[M] - 28c_4[M] \equiv 0 \bmod 12. \quad (4.4.6)$$

$\boldsymbol{\omega}$ **= 111:** $s_{111}(\gamma_1, \gamma_2, \gamma_3) = \gamma_3$, so

$$\begin{aligned}
\kappa_{111}[M] &= \langle \mathrm{ch}\gamma_3 \cdot \operatorname{Td}(M), \mu_M \rangle \\
&= \langle \sigma_3\left(e^{x_1} - 1, \ldots, e^{x_4} - 1\right) \cdot \operatorname{Td}(M), \mu_M \rangle \\
&= \left\langle \left(c_3 + \frac{1}{2}\sum x_1^2x_2x_3\right) \cdot \operatorname{Td}(M), \mu_M \right\rangle \\
&= \left\langle \left(c_3 + \frac{1}{2}\left(c_1c_3 - 4c_4\right)\right) \cdot \left(1 + \frac{1}{2}c_1\right), \mu_M \right\rangle \\
&= \langle c_1c_3 - 2c_4, \mu_M \rangle \\
&= c_1c_3[M] - 2c_4[M].
\end{aligned}$$

Since the Chern numbers are all integers, $\kappa_{111}[M]$ is always an integer, so this partition does not give a new divisibility relation.

$\omega = 12$: $s_{12}(\gamma_1, \gamma_2, \gamma_3) = \gamma_1\gamma_2 - 3\gamma_3$, so

$$\begin{aligned}
\kappa_{12}[M] &= \langle(\mathrm{ch}\gamma_1 \cdot \mathrm{ch}\gamma_2 - 3\mathrm{ch}\gamma_3) \cdot \mathrm{Td}(M), \mu_M\rangle \\
&= \Bigg\langle \Bigg(\left(c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right)\right) \cdot \left(c_2 + \frac{1}{2}\left(c_1c_2 - 3c_3\right)\right) + \\
&\qquad - 3\left(c_3 + \frac{1}{2}\left(c_1c_3 - 4c_4\right)\right)\Bigg) \cdot \left(1 + \frac{1}{2}c_1\right), \mu_M \Bigg\rangle \\
&= \left\langle \frac{3}{2}c_1^2c_2 - c_2^2 - \frac{9}{2}c_1c_3 + 6c_4, \mu_M \right\rangle \\
&= \frac{3}{2}c_1^2c_2[M] - c_2^2[M] - \frac{9}{2}c_1c_3[M] + 6c_4[M].
\end{aligned}$$

This gives the divisibility relation

$$3c_1^2c_2[M] - 2c_2^2[M] - 9c_1c_3[M] + 12c_4[M] \equiv 0 \bmod 2. \tag{4.4.7}$$

$\omega = 3$: $s_3(\gamma_1, \gamma_2, \gamma_3) = \gamma_1^3 - 3\gamma_1\gamma_2 + 3\gamma_3$, so

$$\begin{aligned}
\kappa_3[M] &= \langle((\mathrm{ch}\gamma_1)^3 - 3\mathrm{ch}\gamma_1 \cdot \mathrm{ch}\gamma_2 + 3\mathrm{ch}\gamma_3) \cdot \mathrm{Td}(M), \mu_M\rangle \\
&= \Bigg\langle \Bigg(\left(c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right)\right)^3 - 3\left(c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right)\right) \cdot \\
&\qquad \left(c_2 + \frac{1}{2}\left(c_1c_2 - 3c_3\right)\right) + 3\left(c_3 + \frac{1}{2}\left(c_1c_3 - 4c_4\right)\right)\Bigg) \cdot \\
&\qquad \cdot \left(1 + \frac{1}{2}c_1\right), \mu_M \Bigg\rangle \\
&= \left\langle 2c_1^4 - \frac{15}{2}c_1^2c_2 + 3c_2^2 + \frac{15}{2}c_1c_3 - 6c_4, \mu_M \right\rangle \\
&= 2c_1^4[M] - \frac{15}{2}c_1^2c_2[M] + 3c_2^2[M] + \frac{15}{2}c_1c_3[M] - 6c_4[M].
\end{aligned}$$

This gives the divisibility relation

$$4c_1^4[M] - 15c_1^2c_2[M] + 6c_2^2[M] + 15c_1c_3[M] - 12c_4[M] \equiv 0 \bmod 2. \tag{4.4.8}$$

The remaining partitions are partitions of four itself. For each of these partitions, $\kappa_\omega[M] = \langle s_\omega(c_1, c_2, c_3, c_4), \mu_M\rangle$ is a linear combination of Chern numbers. Thus the $\kappa_\omega[M]$ do not introduce any new divisibility relations among the Chern numbers for these partitions ω.

Simplifying and combining (4.4.3), (4.4.4), (4.4.5), (4.4.6), (4.4.7), and (4.4.8) shows that a cobordism class $[M] \in \Omega_8^U$ can have Chern numbers $c_1^4[M]$, $c_1^2c_2[M]$,

$c_2^2[M]$, $c_1c_3[M]$, and $c_4[M]$ if and only if the following divisibility relations hold.

$$\begin{aligned}
-c_1^4[M] + 4c_1^2c_2[M] + 3c_2^2[M] + c_1c_3[M] - c_4[M] &\equiv 0 \text{ mod } 720 \\
2c_1^4[M] - 5c_1^2c_2[M] + 5c_1c_3[M] - 2c_4[M] &\equiv 0 \text{ mod } 12 \\
6c_1^2c_2[M] - 5c_1c_3[M] + 2c_4[M] &\equiv 0 \text{ mod } 12 \\
2c_1^4[M] + c_1^2c_2[M] - 2c_1c_3[M] - 4c_4[M] &\equiv 0 \text{ mod } 12 \\
c_1^2c_2[M] + c_1c_3[M] &\equiv 0 \text{ mod } 2
\end{aligned}$$

Smooth projective toric varieties representing Ω_8^U

The above congruences still do not give a very lucid idea of which combinations of Chern numbers are possible in Ω_8^U. Fortunately, to try to answer Question 4.1, we only need to consider the cobordism classes in Ω_8^U that could possibly contain a smooth projective toric variety. This means that the class $[M]$ must also satisfy the equations in (4.4.1) for some g-vector satisfying (4.4.2). Applying these relations to the above divisibility relations simplifies them considerably. This simple calculation yields the following

Theorem 4.23. *The only cobordism classes $[M] \in \Omega_8^U$ that can possibly contain a smooth projective toric variety are those whose Chern numbers satisfy (4.4.1) and the following divisibility relations for some g-vector satisfying (4.4.2).*

$$c_1^2c_2 \equiv 0 \textit{ mod } 2 \textit{ and } c_1^2c_2 + 2c_2^2 + c_1c_3 \equiv 0 \textit{ mod } 4 \tag{4.4.9}$$

This theorem does not say that every cobordism class satisfying these two relations contains a smooth projective toric variety. This is false, as will soon be seen. However, we do know that if a cobordism class does not satisfy the relations, then it cannot possibly contain a smooth projective toric variety.

As in Ω_6^U, the answer to Problem 4.1 in Ω_8^U is easy for "small" g-vectors. For example, the only smooth (projective) toric variety corresponding to the g-vector $(1, 0, 0)$ is $\mathbb{C}P^4$. Using (4.4.1), the conditions on the Chern numbers corresponding to this g-vector are

$$\begin{aligned}
c_4[M] &= 5 \\
c_1c_3[M] &= 50 \\
c_1^4[M] &= 4c_1^2c_2[M] + 3c_2^2[M] - 675.
\end{aligned} \tag{4.4.10}$$

Theorem 4.24. *Suppose the Chern numbers of $[M] \in \Omega_8^U$ satisfy the above three equations. Then $[M]$ can be represented by a smooth projective toric variety if and only if $[M] = [\mathbb{C}P^4]$.*

It is easy to find integers that satisfy (4.4.9) and (4.4.10) that are not the Chern numbers of $[\mathbb{C}P^4]$ (for example, $(c_1^4, c_1^2c_2, c_2^2, c_1c_3, c_4) = (-672, 0, 1, 50, 5)$). This means that (4.4.1) are not the only obstructions to a cobordism class in Ω_8^U containing a smooth projective toric variety.

Next consider g-vectors with $g_1 = 1$. By the g-theorem 3.20, the only such g-vectors that correspond to simple polytopes are those with $g_2 \in \{0, 1\}$ (see (4.4.2)). Writing the f-vector in terms of the g-vector in this dimension gives $f_0 = 5+g_1$. This is the number of facets of a polytope with g-vector $(1, g_1, g_2)$, or equivalently the number of rays of the corresponding fan. Then a complete smooth fan in $\mathbb{R}^4$ has 6 generating rays if and only if the corresponding normal polytope has g-vector $(1, 1, 0)$ or $(1, 1, 1)$. But all these fans have been classified by Kleinschmidt [29]. Thus, the cobordism classes in Ω_8^U corresponding to these g-vectors that contain smooth (projective) toric varieties are exactly those corresponding to the smooth toric varieties of dimension 8 given in Theorem 3.24, using $n = 4$.

Theorem 4.25. *Suppose $[M] \in \Omega_8^U$ satisfies (4.4.1) with $g_1 = 1$ and $g_2 \in \{0, 1\}$. Then $[M]$ can be represented by a smooth projective toric variety if and only if*

$$[M] \in \{[X_4(a_1, a_2, a_3)], [X_4(a_1, a_2)], [X_4(a_1)]\}$$

for some integers $0 \leq a_1 \leq a_2 \leq a_3$.

Calculating the Chern numbers of the above smooth toric varieties reveals that once again, they cannot represent all combinations of Chern numbers satisfying (4.4.1) with these g-vectors and (4.4.9). This means that (4.4.1) are not the only obstructions to a cobordism class containing a smooth projective toric variety for the g-vectors $(1, 1, 0)$ or $(1, 1, 1)$. However, by taking several equivariant blow-ups of $X_4(a_1, a_2, a_3)$, one can obtain enough smooth projective toric varieties with the g-vector $(1, 3, 2)$ to show that (4.4.1) are the only obstructions to containing a smooth projective toric variety for this g-vector.

Theorem 4.26. *Suppose $[M] \in \Omega_8^U$ satisfies (4.4.1) with g-vector $(1, 3, 2)$. That is,*

$$\begin{aligned} c_4[M] &= 16 \\ c_1c_3[M] &= 64 \\ c_1^4[M] &= 4c_1^2c_2[M] + 3c_2^2[M] - 672. \end{aligned} \tag{4.4.11}$$

Then $[M]$ can be represented by a smooth projective toric variety.

The following lemma will make the calculations in the proof of this theorem considerably simpler.

Lemma 4.27. *Let X^8 be a smooth projective toric variety with a corresponding regular complete fan Σ in $\mathbb{R}^4$. Suppose $\{u_1, u_2, u_3, v_1, v_2\} \subset G(\Sigma)$ are some of its generating rays, where $u_i = e_i$ for all i, $v_1 = e_4$, and $v_2 = (a_1, a_2, a_3, -1)$ for integers a_i. Assume that $pos(u_1, u_2, u_3, v_1)$ and $pos(u_1, u_2, u_3, v_2)$ are two cones in Σ. Now apply a star subdivision to $pos(u_1, u_2, u_3)$ in Σ to obtain a new fan Σ' with corresponding smooth projective toric variety $X_{\Sigma'}$. Let P and P' be the smooth polytopes corresponding to Σ and Σ', respectively.*

1. *If $g(P) = (1, g_1, g_2)$, then $g(P') = (1, g_1 + 1, g_2 + 1)$.*

2. *The changes in cobordism that rely on the combinatorial structure are described by* $c_1^2 c_2 [X_{\Sigma'}] = c_1^2 c_2 [X_\Sigma] + 4(a_1 + a_2 + a_3) - 16$ *and* $c_2^2 [X_{\Sigma'}] = c_2^2 [X_\Sigma]$.

Proof. The change in g-vector is given by Theorem 3.39 since the cone that is being blown up has dimension three and therefore corresponds to a face of codimension three, i.e. an edge, in the associated polytope.

By Proposition 3.40, the change in cobordism is completely independent of the unmentioned generating rays of Σ and Σ'. To prove the lemma, it therefore suffices to compute these Chern numbers for one example pair of suitable varieties X_Σ and $X_{\Sigma'}$.

Define $u_4 = (-1, -1, -1, 0)$, and let Σ be the fan whose generating rays are given by $\{u_1, u_2, u_3, u_4\} \cup \{v_1, v_2\}$ and whose maximal cones consist of three elements from the first set and one element from the second set (see Figure 4.1). Note that Σ is almost the same as $\Sigma_4(a_1, a_2, a_3)$, except we exclude the condition $0 \leq a_1 \leq a_2 \leq a_3$. It is easy to verify that Σ is still a complete regular fan. The classifying theorem 3.24 only states that this fan is isomorphic to some other fan $\Sigma_4(a_1, \ldots, a_r)$ for some $r \in \{1, 2, 3\}$ and integers $0 \leq a_1 \ldots \leq a_r$. The linear relations in $H^*(X_\Sigma)$ are

$$\begin{aligned} u_1 &= u_4 - a_1 v_2 \\ u_2 &= u_4 - a_2 v_2 \\ u_3 &= u_4 - a_3 v_2 \\ v_1 &= v_2 \end{aligned}$$

and the Stanley-Reisner ideal is $J = (u_1 u_2 u_3 u_4, v_1 v_2)$. Combining these relations yields

$$H^*(X_\Sigma) = \mathbb{Z}[u_4, v_2] / \left(v_2^2, u_4^4 - (a_1 + a_2 + a_3) u_4^3 v_2\right).$$

Using the above linear relations, the first Chern class of X_Σ is

$$c_1(X_\Sigma) = u_1 + u_2 + u_3 + u_4 + v_1 + v_2 = 4u_4 + (2 - a_1 - a_2 - a_3) v_2.$$

The second Chern class is given by $c_2(X_\Sigma) = \sigma_2(u_1, u_2, u_3, u_4, v_1, v_2)$, where σ_2 is the second elementary symmetric polynomial. It is straight-forward to simplify this using the relations in the cohomology ring to obtain

$$c_2(X_\Sigma) = 6u_4^2 + (8 - 3a_1 - 3a_2 - 3a_3) u_4 v_2.$$

Then

$$\begin{aligned} c_1^2(X_\Sigma) c_2(X_\Sigma) &= (4u_4 + (2 - a_1 - a_2 - a_3) v_2)^2 \left(6u_4^2 + (8 - 3a_1 - 3a_2 - 3a_3) u_4 v_2\right) \\ &= \left(16u_4^2 + (16 - 8a_1 - 8a_2 - 8a_3) u_4 v_2\right) \cdot \\ &\quad \cdot \left(6u_4^2 + (8 - 3a_1 - 3a_2 - 3a_3) u_4 v_2\right) \\ &= \Big(96(a_1 + a_2 + a_3) + 16(8 - 3a_1 - 3a_2 - 3a_3) + \\ &\quad + 6(16 - 8a_1 - 8a_2 - 8a_3)\Big) u_4^3 v_2 \\ &= 224 u_4^3 v_2. \end{aligned}$$

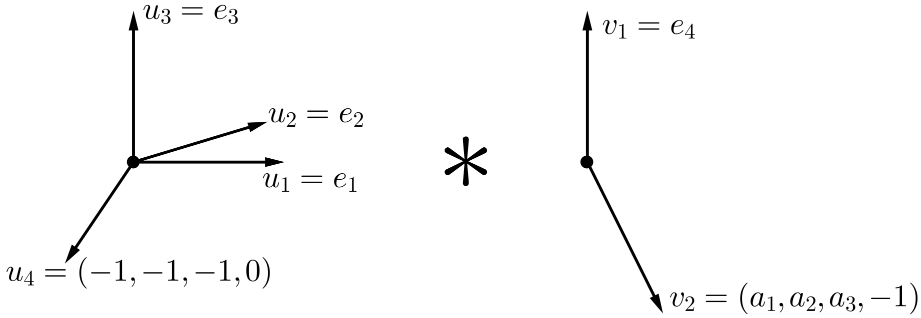

Figure 4.1: The fan Σ

Since $u_4^3 v_2 = u_2 u_3 u_4 v_2$ in $H^*(X_\Sigma)$ and $\text{pos}(u_2, u_3, u_4, v_2)$ is a maximal cone in Σ, $c_1^2 c_2 [X_\Sigma] = 224$ by Proposition 3.29. Similarly,

$$\begin{aligned} c_2(X_\Sigma)^2 &= \left(6u_4^2 + (8 - 3a_1 - 3a_2 - 3a_3) u_4 v_2\right)^2 \\ &= (36(a_1 + a_2 + a_3) + 12(8 - 3a_1 - 3a_2 - 3a_3)) u_4^3 v_2 \\ &= 96 u_4^3 v_2, \end{aligned}$$

so $c_2^2 [X_\Sigma] = 96$.

Now consider Σ' obtained by subdividing $\text{pos}(u_1, u_2, u_3)$ in Σ with $z = (1, 1, 1, 0)$ (see Figure 4.2). The linear relations in $H^*(X_{\Sigma'})$ are

$$\begin{aligned} u_1 &= u_4 - a_1 v_2 - z \\ u_2 &= u_4 - a_2 v_2 - z \\ u_3 &= u_4 - a_3 v_2 - z \\ v_1 &= v_2 \end{aligned}$$

and the Stanley-Reisner ideal is $J = (u_4 z, v_1 v_2, u_1 u_2 u_3)$. Then

$$H^*(X_{\Sigma'}) = \mathbb{Z}[u_4, v_2, z] / \left(u_4 z, v_2^2, u_4^3 - (a_1 + a_2 + a_3) u_4^2 v_2 - (a_1 + a_2 + a_3) v_2 z^2 - z^3\right).$$

The first Chern class of $X_{\Sigma'}$ is $c_1(X_{\Sigma'}) = 4u_4 + (2 - a_1 - a_2 - a_3) v_2 - 2z$. Its second Chern class is

$$\begin{aligned} c_2(X_{\Sigma'}) &= \sigma_2(u_1, u_2, u_3, u_4, v_1, v_2, z) \\ &= 6u_4^2 + (8 - 3a_1 - 3a_2 - 3a_3) u_4 v_2 - (4 - a_1 - a_2 - a_3) v_2 z. \end{aligned}$$

Then $c_1^2(X_{\Sigma'}) = 16u_4^2 + 8(2 - a_1 - a_2 - a_3) u_4 v_2 - 4(2 - a_1 - a_2 - a_3) v_2 z + 4z^2$, so

$$\begin{aligned} c_1^2(X_{\Sigma'}) \cdot c_2(X_{\Sigma'}) &= \Big(96(a_1 + a_2 + a_3) + 16(8 - 3a_1 - 3a_2 - 3a_3) + \\ &\quad + 48(2 - a_1 - a_2 - a_3) - 4(4 - a_1 - a_2 - a_3)\Big) v_2 z^3 \\ &= (208 + 4(a_1 + a_2 + a_3)) v_2 z^3. \end{aligned}$$

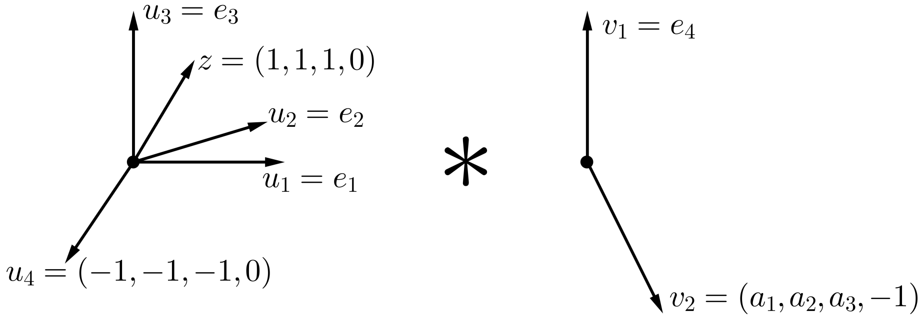

Figure 4.2: The fan Σ'

Since $v_2 z^3 = u_2 u_3 v_2 z$ in $H^*(X_{\Sigma'})$ and $\text{pos}(u_2, u_3, v_2, z)$ is a maximal cone in Σ', $c_1^2 c_2 [X_{\Sigma'}] = 208 + 4(a_1 + a_2 + a_3) = c_1^2 c_2 [X_\Sigma] + 4(a_1 + a_2 + a_3) - 16$. Similarly,

$$\begin{aligned} c_2 (X_{\Sigma'})^2 &= (36(a_1 + a_2 + a_3) + 12(8 - 3a_1 - 3a_2 - 3a_3)) v_2 z^3 \\ &= 96 v_2 z_3 \end{aligned}$$

Then $c_2^2 [X_{\Sigma'}] = 96 = c_2^2 [X_\Sigma]$.

□

This understanding of blow-ups of two-dimensional submanifolds will allow us to construct smooth projective toric varieties to represent all possible cobordism classes that satisfy the conditions in Theorem 4.26.

Proof of Theorem 4.26. Consider the complete regular fans $\Sigma_4(a_1, a_2, a_3)$ for integers $0 \leq a_1 \leq a_2 \leq a_3$ (see Section 3.3). Let $\Sigma_s(a_1, a_2, a_3)$ denote the fan obtained from $\Sigma_4(a_1, a_2, a_3)$ by applying a star subdivision to $\text{pos}(u_3, u_4)$ in $\Sigma_4(a_1, a_2, a_3)$. Let $z = u_3 + u_4 = (-1, -1, 0, 0)$ be the additional ray. The g-vector of the polytope corresponding to $\Sigma_4(a_1, a_2, a_3)$ is $(1, 1, 0)$, and one can calculate the g-vector of the polytope corresponding to $\Sigma_s(a_1, a_2, a_3)$ to be $(1, 2, 1)$. In order to compute the cobordism class of the corresponding smooth projective toric variety $X_s(a_1, a_2, a_3)$, it remains to compute its Chern numbers $c_1^2 c_2 [X_s(a_1, a_2, a_3)]$ and $c_2^2 [X_s(a_1, a_2, a_3)]$ by (4.4.1). The linear relations in $H^*(X_s(a_1, a_2, a_3))$ are

$$\begin{aligned} u_1 &= u_4 - a_1 v_2 + z \\ u_2 &= u_4 - a_2 v_2 + z \\ u_3 &= u_4 - a_3 v_2 \\ v_1 &= v_2 \end{aligned}$$

and the Stanley-Reisner ideal is $J = (u_3 u_4, v_1 v_2, u_1 u_2 z)$. Combining these yields

$$H^*(X_s(a_1, a_2, a_3)) = \mathbb{Z}[u_4, v_2, z]/I$$

where

$$I = \left(u_4^2 - a_3 u_4 v_2, v_2^2, \left(a_3 - a_1 - a_2\right) u_4 v_2 z + 2u_4 z^2 - \left(a_1 + a_2\right) v_2 z^2 + z^3\right).$$

The first Chern class of the variety is

$$c_1\left(X_s\left(a_1, a_2, a_3\right)\right) = 4u_4 + \left(2 - a_1 - a_2 - a_3\right) v_2 + 3z.$$

Its second Chern class is

$$\begin{aligned} c_2\left(X_s\left(a_1, a_2, a_3\right)\right) &= \sigma_2\left(u_1, u_2, u_3, u_4, v_1, v_2, z\right) \\ &= \left(8 - 3a_1 - 3a_2 + 3a_3\right) u_4 v_2 + 10 u_4 z + \\ &\quad + \left(6 - 2a_1 - 2a_2 - 3a_3\right) v_2 z + 3z^2. \end{aligned}$$

Then

$$c_1\left(X_s\left(a_1, a_2, a_3\right)\right)^2 = 8\left(2 - a_1 - a_2 + a_3\right) u_4 v_2 + 24 u_4 z + 6\left(2 - a_1 - a_2 - a_3\right) v_2 z + 9z^2,$$

so

$$\begin{aligned} &c_1\left(X_s\left(a_1, a_2, a_3\right)\right)^2 \cdot c_2\left(X_s\left(a_1, a_2, a_3\right)\right) = \\ &= \Big(24\left(2 - a_1 - a_2 + a_3\right) + 240 a_3 + 24\left(6 - 2a_1 - 2a_2 - 3a_3\right) + \\ &+72\left(a_1 + a_2 - 2a_3\right) + 60\left(2 - a_1 - a_2 - a_3\right) - 36\left(2 - a_1 - a_2 - a_3\right) + \\ &+9\left(8 - 3a_1 - 3a_2 + 3a_3\right) + 90\left(a_1 + a_2 - 2a_3\right) - 18\left(6 - 2a_1 - 2a_2 - 3a_3\right) + \\ &+81\left(a_3 - a_1 - a_2\right)\Big) u_4 v_2 z^2 \end{aligned}$$

This simplifies to

$$c_1\left(X_s\left(a_1, a_2, a_3\right)\right)^2 \cdot c_2\left(X_s\left(a_1, a_2, a_3\right)\right) = \left(204 - 6\left(a_1 + a_2 - a_3\right)\right) u_4 v_2 z^2.$$

Since $u_4 v_2 z^2 = u_1 u_4 v_2 z$ in $H^*\left(X_s\left(a_1, a_2, a_3\right)\right)$ and $\operatorname{pos}\left(u_1, u_4, v_2, z\right)$ is a maximal cone in the fan,

$$c_1^2 c_2\left[X_s\left(a_1, a_2, a_3\right)\right] = 204 - 6\left(a_1 + a_2 - a_3\right).$$

Similarly,

$$\begin{aligned} c_2\left(X_s\left(a_1, a_2, a_3\right)\right)^2 &= \Big(6\left(8 - 3a_1 - 3a_2 + 3a_3\right) + 100 a_3 + 20\left(6 - 2a_1 - 2a_2 - 3a_3\right) + \\ &\quad + 60\left(a_1 + a_2 - 2a_3\right) - 12\left(6 - 2a_1 - 2a_2 - 3a_3\right) \\ &\quad + 27\left(-a_1 - a_2 + a_3\right)\Big) u_4 v_2 z^2 \\ &= \left(96 - a_1 - a_2 + a_3\right) u_4 v_2 z^2, \end{aligned}$$

so

$$c_2^2\left[X_s\left(a_1, a_2, a_3\right)\right] = 96 - a_1 - a_2 + a_3.$$

Since $X_s(a_1, a_2, a_3)$ has g-vector $(1, 2, 1)$, (4.4.1) gives

$$\begin{aligned} c_1^4 [X_s(a_1, a_2, a_3)] &= 432 - 27(a_1 + a_2 - a_3) \\ c_1^2 c_2 [X_s(a_1, a_2, a_3)] &= 204 - 6(a_1 + a_2 - a_3) \\ c_2^2 [X_s(a_1, a_2, a_3)] &= 96 - a_1 - a_2 + a_3 \\ c_1 c_3 [X_s(a_1, a_2, a_3)] &= 60 \\ c_4 [X_s(a_1, a_2, a_3)] &= 12. \end{aligned}$$

As expected, this variety does not satisfy the conditions in (4.4.11) since it does not have the desired g-vector. To obtain varieties that do satisfy them, we need to apply another subdivision. First, consider the unimodular transformation $A : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$A(v) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot v.$$

Applying this transformation to the fan $\Sigma_s(a_1, a_2, a_3)$ gives a new fan $A(\Sigma_s(a_1, a_2, a_3))$ which corresponds to an isomorphic variety. In particular, consider $\tau = \text{pos}(u_1, u_3, z)$ in $\Sigma_s(a_1, a_2, a_3)$. This cone is contained in the two maximal cones $\text{pos}(u_1, u_3, z, v_1)$ and $\text{pos}(u_1, u_3, z, v_2)$. One can compute $A(\tau) = \text{pos}(A(u_1), A(u_3), A(z))$ to be given by $A(u_1) = e_3$, $A(u_3) = e_1$, and $A(z) = e_2$. We can also calculate $A(v_1) = e_4$ and $A(v_2) = (a_3, -a_2, a_1 - a_2, -1)$. Note in particular that the cones $A(\tau)$, $A(v_1)$, and $A(v_2)$ in $A(\Sigma_s(a_1, a_2, a_3))$ satisfy the conditions of Lemma 4.27. Now apply a star subdivision of $A(\tau)$ in $A(\Sigma_s(a_1, a_2, a_3))$ to obtain a new fan $\Sigma^{13}(a_1, a_2, a_3)$ with corresponding smooth projective toric variety $X^{13}(a_1, a_2, a_3)$. By Lemma 4.27, the g-vector of the polytope corresponding to $X^{13}(a_1, a_2, a_3)$ is $(1, 3, 2)$. By the same lemma,

$$\begin{aligned} c_1^2 c_2 \left[X^{13}(a_1, a_2, a_3)\right] &= 204 - 6(a_1 + a_2 - a_3) + 4(a_3 - a_2 + a_1 - a_2) - 16 \\ &= 188 - 2a_1 - 14a_2 + 10a_3 \end{aligned}$$

and

$$c_2^2 \left[X^{13}(a_1, a_2, a_3)\right] = 96 - a_1 - a_2 + a_3.$$

Next consider the unimodular transformation $B : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$B(v) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot v.$$

Again, $B(\Sigma_s(a_1, a_2, a_3))$ is a new fan whose corresponding variety is isomorphic to $X_s(a_1, a_2, a_3)$. Consider $\phi = \text{pos}(u_2, u_4, z)$ in $\Sigma_s(a_1, a_2, a_3)$. This cone is contained in the two maximal cones $\text{pos}(u_2, u_4, z, v_1)$ and $\text{pos}(u_2, u_4, z, v_2)$. One can compute $B(\phi) = \text{pos}(B(u_2), B(u_4), B(z))$ to be given by $B(u_2) = e_1$, $B(u_4) = e_2$, and

$B(z) = e_3$. Also, $B(v_1) = e_4$ and $B(v_2) = (-a_1 + a_2, -a_3, -a_1 + a_3, -1)$. Now apply a star subdivision of ϕ in $B(\Sigma_s(a_1, a_2, a_3))$ to obtain a new fan $\Sigma^{24}(a_1, a_2, a_3)$ with corresponding smooth projective toric variety $X^{24}(a_1, a_2, a_3)$. Applying Lemma 4.27, the g-vector of the polytope corresponding to $X^{24}(a_1, a_2, a_3)$ is $(1, 3, 2)$. By the same lemma,

$$\begin{aligned} c_1^2 c_2 \left[X^{24}(a_1, a_2, a_3)\right] &= 204 - 6(a_1 + a_2 - a_3) + 4(-a_1 + a_2 - a_3 - a_1 + a_3) - 16 \\ &= 188 - 14a_1 - 2a_2 + 6a_3 \end{aligned}$$

and

$$c_2^2 \left[X^{24}(a_1, a_2, a_3)\right] = 96 - a_1 - a_2 + a_3.$$

To complete the proof of the theorem, it suffices to show that any cobordism class in Ω_8^U that satisfies (4.4.11) can be represented by at least one of the smooth projective toric varieties $X^{13}(a_1, a_2, a_3)$ or $X^{24}(a_1, a_2, a_3)$, where $0 \leq a_1 \leq a_2 \leq a_3$ are integers. Note that the possible combinations of Chern numbers for such a cobordism class are exactly those that satisfy (4.4.9). But $c_1^2 c_2 [X^{13}(a_1, a_2, a_3)]$ and $c_1^2 c_2 [X^{24}(a_1, a_2, a_3)]$ are always even, and $c_1 c_3 [X^{13}(a_1, a_2, a_3)] = c_1 c_3 [X^{24}(a_1, a_2, a_3)] = 64$. The conditions in (4.4.9) therefore reduce to

$$c_1^2 c_2 + 2c_2^2 \equiv 0 \bmod 4$$

in this case. It remains to verify that every combination of $c_1^2 c_2$ and c_2^2 that satisfies this congruence is obtained by one of the $X^{13}(a_1, a_2, a_3)$ or $X^{24}(a_1, a_2, a_3)$.

Fix an integer $\alpha \geq 0$. First, assume $c_2^2[M] = 96 + \alpha$ for some $[M] \in \Omega_8^U$. The cobordism classes with this Chern number that could possibly contain a smooth projective toric variety can be described as those satisfying $c_1^2 c_2[M] = 188 + 6\alpha + 4\beta$ where β can be given any integer value.

Case 1. Suppose $\beta \leq \alpha$, so $0 \leq \alpha - \beta \leq 2\alpha - \beta$. Then

$$c_2^2 \left[X^{13}(0, \alpha - \beta, 2\alpha - \beta)\right] = 96 - 0 - (\alpha - \beta) + (2\alpha - \beta) = 96 + \alpha$$

and

$$c_1^2 c_2 \left[X^{13}(0, \alpha - \beta, 2\alpha - \beta)\right] = 188 - 0 - 14(\alpha - \beta) + 10(2\alpha - \beta) = 188 + 6\alpha + 4\beta.$$

Case 2. Suppose $\beta \geq 0$, so $0 \leq \beta \leq \alpha + \beta$. Then

$$c_2^2 \left[X^{24}(0, \beta, \alpha + \beta)\right] = 96 - 0 - \beta + \alpha + \beta = 96 + \alpha$$

and

$$c_1^2 c_2 \left[X^{24}(0, \beta, \alpha + \beta)\right] = 188 - 0 - 2\beta + 6(\alpha + \beta) = 188 + 6\alpha + 4\beta.$$

These two cases have shown that all cobordism classes $[M] \in \Omega_8^U$ satisfying (4.4.11) and $c_2^2[M] \geq 96$ can be represented by the smooth projective toric varieties

$X^{13}(a_1, a_2, a_3)$ and $X^{24}(a_1, a_2, a_3)$. In fact, the two cases overlap, so many classes can be represented by multiple examples of these varieties.

Next, assume $c_2^2[M] = 96 - \alpha$, where $\alpha \geq 0$ is still an arbitrary integer. The cobordism classes with this Chern number that could possibly contain a smooth projective toric variety can be described as those satisfying $c_1^2 c_2[M] = 188 - 6\alpha + 4\beta$ where β can be given any integer value.

Case 1. Suppose $\beta \leq 0$, so $0 \leq \alpha \leq \alpha - \beta$. Then

$$c_2^2\left[X^{13}(\alpha, \alpha - \beta, \alpha - \beta)\right] = 96 - \alpha - (\alpha - \beta) + (\alpha - \beta) = 96 - \alpha$$

and

$$c_1^2 c_2\left[X^{13}(\alpha, \alpha - \beta, \alpha - \beta)\right] = 188 - 2\alpha - 14(\alpha - \beta) + 10(\alpha - \beta) = 188 - 6\alpha + 4\beta.$$

Case 2. Suppose $\beta \geq -\alpha$, so $0 \leq \alpha \leq 2\alpha + \beta$. Then

$$c_2^2\left[X^{24}(\alpha, 2\alpha + \beta, 2\alpha + \beta)\right] = 96 - \alpha - (2\alpha + \beta) + (2\alpha + \beta) = 96 - \alpha$$

and

$$c_1^2 c_2\left[X^{24}(\alpha, 2\alpha + \beta, 2\alpha + \beta)\right] = 188 - 14\alpha - 2(2\alpha + \beta) + 6(2\alpha + \beta) = 188 - 6\alpha + 4\beta.$$

The previous two cases have shown that all cobordism classes $[M] \in \Omega_8^U$ satisfying (4.4.11) and $c_2^2[M] \leq 96$ can be represented by the smooth projective toric varieties $X^{13}(a_1, a_2, a_3)$ and $X^{24}(a_1, a_2, a_3)$, again with some overlap. Between the four cases, all possible cobordism classes that satisfy (4.4.11) have been represented by smooth projective toric varieties, which proves the theorem.

□

Now that all cobordism classes corresponding to one g-vector have been shown to have smooth projective toric varieties, induction can be used to say the same for other g-vectors. This is done by applying blow-ups along zero- and two-dimensional torus-equivariant subvarieties of the varieties $X^{13}(a_1, a_2, a_3)$ and $X^{24}(a_1, a_2, a_3)$.

Theorem 4.28. *Let $[M] \in \Omega_8^U$. Choose integers g_1 and g_2 such that $2 \leq g_2 \leq g_1 - 1$. Assume that the following conditions are satisfied.*

$$\begin{aligned} c_4[M] &= 5 + 3g_1 + g_2 \\ c_1 c_3[M] &= 50 + 6g_1 - 2g_2 \\ c_1^4[M] &= 4c_1^2 c_2[M] + 3c_2^2[M] + 3g_1 - 3g_2 - 675 \end{aligned}$$

Then $[M]$ is represented by a smooth projective toric variety.

Proof. Suppose $[M] \in \Omega_8^U$ satisfies the equations in the theorem (see (4.4.1)) with $2 \leq g_2 \leq g_1 - 1$. If $(1, g_1, g_2) = (1, 3, 2)$, then $[M]$ contains a smooth projective toric variety according to Theorem 4.26. Now fix a g-vector satisfying $2 \leq g_2 \leq g_1 - 1$. Recall that subdividing a maximal cone in a regular fan increases g_1 by one and subdividing a three-dimensional cone increases both g_1 and g_2 by one (see Theorem 3.39). This means that a regular fan with the fixed g-vector $(1, g_1, g_2)$ can be obtained through a sequence of star subdivisions starting with a regular fan with g-vector $(1, 3, 2)$. Start with one of the fans $\Sigma^{13}(a_1, a_2, a_3)$ or $\Sigma^{24}(a_1, a_2, a_3)$, whose g-vectors are all $(1, 3, 2)$ and fix such a sequence of subdivisions to obtain a fan with g-vector $(1, g_1, g_2)$. Choose these subdivisions so that any cone which intersects a cone containing the ray $v_2 = (a_1, a_2, a_3, -1)$ is never subdivided. By Proposition 3.40, the change in cobordism class through this sequence of subdivisions is independent of the values of a_1, a_2, and a_3. In other words, regardless of these values, the same constants are added to the Chern numbers of $X^{13}(a_1, a_2, a_3)$ and $X^{24}(a_1, a_2, a_3)$ to obtain the Chern numbers of the smooth projective toric varieties with g-vector $(1, g_1, g_2)$. But all possible combinations Chern numbers with g-vector $(1, 3, 2)$ are obtained using $X_{13}(a_1, a_2, a_3)$ and $X_{24}(a_1, a_2, a_3)$. This means that all possible combinations of Chern numbers can be obtained for a cobordism class corresponding to $(1, g_1, g_2)$, just by starting the sequence of subdivisions with the appropriate variety $X^{13}(a_1, a_2, a_3)$ or $X^{24}(a_1, a_2, a_3)$. □

Note that the same techniques that gave a complete answer to Problem 4.1 in Ω_6^U only give a partial answer in Ω_8^U. There are still infinitely many cobordism classes that satisfy (4.4.1) that may or may not contain a smooth projective toric variety. However, Theorem 4.28 gives a nice asymptotic result, which is displayed in Figure 4.3. The lattice points represent the values g_1 and g_2 of a g-vector $(1, g_1, g_2)$ that correspond to *simple* four-dimensional polytopes. Recall that each of these coordinates determines a distinct Chern number in complex cobordism (see Corollary 4.11). The shaded area in the figure represents the g-vectors for which the obstructions (4.4.9) on the corresponding Chern numbers are the only obstructions to a cobordism class containing a smooth projective toric variety. That is, each cobordism class with Chern numbers given by these g-vectors can be represented by a smooth projective toric variety.

Theorem 4.28 essentially says that if a g-vector allows for enough freedom in choices of smooth polytopes, such polytopes can be found so that the associated smooth projective toric varieties represent all possible cobordism classes. The remaining g-vectors that have not been addressed could possibly be approached by gaining a more thorough understanding of the smooth four-dimensional polytopes that can have these g-vectors.

4.5 Smooth projective toric varieties in higher-dimensional cobordism

As cobordism dimension increases, the number of Chern numbers $|\pi(n)|$ increases rapidly, and the previously used techniques quickly become impractical. Both Ω_6^U and Ω_8^U display asymptotic behavior regarding the g-vectors that correspond to cobordism classes with smooth projective toric varieties (see Theorems 4.16 and 4.28). It seems reasonable to expect that in any dimension, certain g-vectors will correspond to a large enough assortment of smooth polytopes to allow a similar asymptotic result to hold.

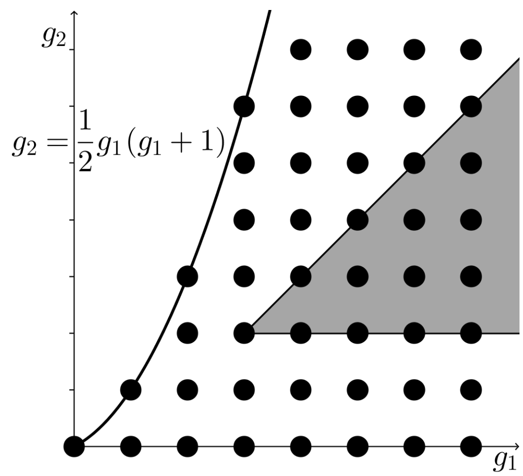

Figure 4.3: g-vectors satisfying the conditions of Theorem 4.28

Conjecture 4.29. *There is a set of g-vectors for which the corresponding $\left\lfloor \frac{n+2}{2} \right\rfloor$ independent conditions on Chern numbers are the only obstructions to a cobordism class in Ω_{2n}^U being represented by a smooth projective toric variety.*

Several facts about which cobordism classes contain smooth projective toric varieties can be proven by using the combinatorial structure of toric varieties and some basic classification theorems. For example, Corollary 4.11 can be thought of as the most generalized version of the fact that the Todd genus of a toric variety is one, and it describes all obstructions to a cobordism class containing a smooth projective toric variety that arise from the g-vector of the associated polytope.

As an example of the value of classification results on toric varieties, recall that the only n-dimensional smooth projective toric variety with g-vector $(1, 0, \ldots, 0)$ is $\mathbb{C}P^n$ (see Example 3.19). By Corollary 4.7, $\chi_y(\mathbb{C}P^n) = \sum_{k=0}^{n} (-y)^k$. This implies the following

Theorem 4.30. *Suppose $[M] \in \Omega_{2n}^U$ satisfies $\chi_y(M) = \sum_{k=0}^{n} (-y)^k$. Then $[M]$ can be represented by a smooth projective toric variety if and only if $[M] = [\mathbb{C}P^n]$.*

Kleinschmidt's classification theorem 3.24 of complete regular fans in $\mathbb{R}^n$ with $n+2$ generating rays also gives information about smooth projective toric varieties in certain dimensions.

Theorem 4.31. *Suppose $[M] \in \Omega_{2n}^U$ satisfies Corollary 4.7 for some g-vector corresponding to a smooth n-polytope, and assume $g_1 = 1$. Then $[M]$ is represented by a smooth projective toric variety if and only if M is cobordant to one of Kleinschmidt's varieties $X_n(a_1, \ldots, a_r)$, where $1 \le r \le n-1$ and $0 \le a_1 \le a_2 \le \ldots \le a_r$ are integers.*

Proof. If $g_1 = 1$, then the definition of the g- and h-vector (Definition 3.18 and (3.2.1)) yield $1 = g_1 = h_1 - 1 = f_0 - n - 1$. Then any smooth polytope with g-vector $(1, 1, g_2, \ldots)$ has $f_0 = n + 2$ facets. Then the normal fan to such a polytope has $n + 2$ generating rays. These are exactly the fans classified in Theorem 3.24. Since they account for every smooth variety whose fan has $n + 2$ generators, $[M]$ must be the cobordism class of one of these varieties if it is to contain a smooth projective toric variety with $g_1 = 1$. □

Chapter 5 Smooth Projective Toric Variety Polynomial Generators in Complex Cobordism

The previous chapter focused on examining which cobordism classes are represented by smooth projective toric varieties (see Problem 4.1). This was done by calculating certain cobordism invariants, i.e. the Chern numbers, of toric varieties. However, the answer to Problem 4.1 turns out to be quite complicated since the combinatorial structure of toric varieties does not yield much information about Chern numbers. This gives good reason to instead examine the Milnor numbers of toric varieties (refer to Definition 2.6). Like the Chern numbers, these are cobordism invariants. Recall that a cobordism class $[M] \in \Omega_{2n}^U$ is completely determined by its $|\pi(n)|$ Chern numbers (see 2.5). The Milnor numbers of a manifold hold much less information than this about cobordism. However, this loss of information is accompanied by a large decrease in computational difficulty, and the Milnor numbers still capture very useful information about complex cobordism. Recall that the value of Milnor numbers determines exactly when a cobordism class can be used as a polynomial generator of $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ (see Theorem 2.9). In particular, a cobordism class $[M^{2n}]$ can be chosen for the polynomial generator α_n of $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ if and only if its Milnor number $s_n[M]$ satisfies

$$s_n[M] = \begin{cases} \pm 1 & \text{if } n+1 \neq p^m \text{ for any prime } p \text{ and integer } m \\ \pm p & \text{if } n+1 = p^m \text{ for some prime } p \text{ and integer } m. \end{cases} \tag{5.0.1}$$

If the Milnor numbers of toric varieties could be computed, one could determine exactly when the cobordism class of a toric variety can be used as a polynomial generator α_n. This in turn may bring more understanding to the ring structure of complex cobordism.

Recall that a smooth algebraic variety (not necessarily connected) can be chosen to represent each α_n (see Theorem 2.12). By taking connected sums of algebraic varieties, smooth connected (not necessarily algebraic) manifolds can be chosen to represent each α_n. This raises the question of which polynomial generators can be represented by smooth connected algebraic varieties. Smooth projective toric varieties are particularly convenient examples of these varieties, and it seems likely that these can be used to represent any polynomial generator.

Conjecture 5.1. *For each $n \geq 1$, there exists a smooth projective toric variety whose cobordism class can be chosen for the polynomial generator α_n of $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$.*

Recall from (2.1.1) that $s_n[\mathbb{C}P^n] = n+1$ for $n \geq 1$. By applying (5.0.1), this means that the cobordism class of the smooth projective toric variety $\mathbb{C}P^n$ can be used for many of the polynomial generators.

Proposition 5.2. *The cobordism class $[\mathbb{C}P^n]$ can be chosen as the polynomial generator α_n of $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ if and only if $n+1$ is prime.*

Note in particular that one can choose $\alpha_1 = [\mathbb{C}P^1]$ and $\alpha_2 = [\mathbb{C}P^2]$.

5.1 Polynomial generators given by Kleinschmidt's varieties

In general, it is still very difficult to calculate the Milnor numbers of smooth projective toric varieties. The main difficulty in proving Conjecture 5.1 is finding smooth projective toric varieties that are simple enough to allow their Milnor numbers to be calculated yet complicated enough to yield many different possible values. Studying the Milnor numbers of Kleinschmidt's varieties (see Theorem 3.24) and some blow-ups of them could reach this balance.

Theorem 5.3. *Consider the smooth projective toric variety $X_n(a_1, \ldots, a_r)$ for some integers $n \geq 2$, $n - r \leq r \leq n-1$, and $0 \leq a_1 \leq a_2 \leq \ldots \leq a_r$ (see Theorem 3.24). The Milnor number of this variety is given by the formula*

$$s_n[X_n(a_1, \ldots, a_r)] = (r+1)\, h_{n-r}(a_1, \ldots, a_r) + \\ + \sum_{i=1}^{n-r} (-1)^i \binom{n}{i} s_i(a_1, \ldots, a_r)\, h_{n-r-i}(a_1, \ldots, a_r)$$

where $s_k(a_1, \ldots, a_r) = a_1^k + \ldots + a_r^k$ is the symmetric polynomial from Definition 2.6 and h_k is the complete homogeneous symmetric polynomial.

Proof. The cohomology of $X_n(a_1, \ldots, a_r)$ must be computed first (see Theorem 3.26). The set of generating rays of $\Sigma_n(a_1, \ldots, a_r)$ is $\{u_1, \ldots, u_{r+1}, v_1, \ldots, v_{n-r+1}\}$, where $u_k = e_k$ for $k = 1, \ldots, r$, $u_{r+1} = \left(-1, \overset{(r)}{\ldots}, -1, 0 \ldots, 0\right)$, $v_k = e_{r+k}$ for $k = 1, \ldots, n-r$, and $v_{n-r+1} = (a_1, \ldots, a_r, -1, \ldots, -1)$. Let u_k and v_k denote the cohomology classes corresponding to the rays u_k and v_k, respectively. Also, set $u = u_{r+1}$ and $v = v_{n-r+1}$ in $H^*(X_n(a_1, \ldots, a_r))$. The linear relations in cohomology are given by

$$\begin{aligned} u_k &= u - a_k v \text{ for } k = 1, \ldots, r \\ u_{r+1} &= u \\ v_k &= v \text{ for } k = 1, \ldots, n-r+1 \end{aligned}$$

and the Stanley-Reisner ideal is

$$J = (u_1 \cdots u_{r+1}, v_1 \cdots v_{n-r+1}).$$

Combining these yields

$$H^*(X_n(a_1, \ldots, a_r)) = \mathbb{Z}[u, v]/I,$$

where

$$I = \left(u \cdot \prod_{k=1}^{r} (u - a_k v), v^{n-r+1}\right).$$

Several relations in this cohomology ring will be particularly useful. One of these is

$$v^{n-r+1} = 0. \tag{5.1.1}$$

Another is given by

$$0 = u \cdot \prod_{k=1}^{r} (u - a_k v) = u^{r+1} + \sum_{i=1}^{n-r} (-1)^i \sigma_i (a_1, \ldots, a_r) u^{r-i+1} v^i,$$

where σ_i is the i^{th} elementary symmetric polynomial. Note that the terms with higher powers of v vanish since $v^{n-r+1} = 0$. This relation gives

$$u^{r+1} = \sum_{i=1}^{n-r} (-1)^{i+1} \sigma_i (a_1, \ldots, a_r) u^{r-i+1} v^i.$$

Then by (5.1.1),

$$u^{r+1} v^{n-r-1} = \sigma_1 (a_1, \ldots, a_r) u^r v^{n-r} = h_1 (a_1, \ldots, a_r) u^r v^{n-r}$$

where h_i is the i^{th} complete homogeneous symmetric polynomial. Similarly,

$$\begin{aligned} u^{r+2} v^{n-r-2} &= \sigma_1 (a_1, \ldots, a_r) u^{r+1} v^{n-r-1} - \sigma_2 (a_1, \ldots, a_r) u^r v^{n-r} \\ &= \sigma_1 (a_1, \ldots, a_r) h_1 (a_1, \ldots, a_r) u^r v^{n-r} - \sigma_2 (a_1, \ldots, a_r) u^r v^{n-r} \\ &= h_2 (a_1, \ldots, a_r) u^r v^{n-r} \end{aligned}$$

where the final equality follows from the well-known relation among symmetric polynomials

$$\sum_{i=0}^{j} (-1)^i \sigma_i (a_1, \ldots, a_r) h_{j-i} (a_1, \ldots, a_r) = 0 \tag{5.1.2}$$

for any integer $j \geq 1$ [17]. Using (5.1.2) and induction, for any integer $j = 0, \ldots, n-r$,

$$\begin{aligned} u^{r+j} v^{n-r-j} &= \sum_{i=1}^{j} (-1)^{i+1} \sigma_i (a_1, \ldots, a_r) u^{r+j-i} v^{n-r+i-j} \\ u^{r+j} v^{n-r-j} &= \sum_{i=1}^{j} (-1)^{i+1} \sigma_i (a_1, \ldots, a_r) h_{j-i} (a_1 \ldots a_r) u^r v^{n-r} \\ u^{r+j} v^{n-r-j} &= h_j (a_1, \ldots, a_r) u^r v^{n-r}. \end{aligned} \tag{5.1.3}$$

Now Theorem 3.30 can be applied to find the Chern class of $X_n (a_1, \ldots, a_r)$:

$$\begin{aligned} c (X_n (a_1, \ldots, a_r)) &= (1 + u_1) \cdots (1 + u_{r+1}) \cdot (1 + v_1) \cdots (1 + v_{n-r+1}) \\ &= (1 + u) \cdot \prod_{k=1}^{r} (1 + u - a_k v) \cdot (1 + v)^{n-r+1} \end{aligned}$$

Since this provides a splitting of the Chern class, Definition 2.6 gives a formula for the characteristic class

$$\begin{aligned}
s_n\left(c\left(X_n\left(a_1,\ldots,a_r\right)\right)\right) &= u^n + \sum_{k=1}^{r}\left(u-a_k v\right)^n + \left(n-r+1\right)v^n \\
&= u^n + \sum_{k=1}^{r}\sum_{i=0}^{n-r}\left(-1\right)^i\binom{n}{i}a_k^i u^{n-i}v^i \text{ by (5.1.1)} \\
&= u^n + \sum_{i=0}^{n-r}\left(-1\right)^i\binom{n}{i}\left(\sum_{k=1}^{r}a_k^i\right)u^{n-i}v^i \\
&= u^n + ru^n + \sum_{i=1}^{n-r}\left(-1\right)^i\binom{n}{i}s_i\left(a_1,\ldots,a_r\right)u^{n-i}v^i \\
&= \left(r+1\right)u^{r+(n-r)} + \\
&\quad + \sum_{i=1}^{n-r}\left(-1\right)^i\binom{n}{i}s_i\left(a_1,\ldots,a_r\right)u^{r+(n-i-r)}v^{n-r-(n-r-i)} \\
&= \left(r+1\right)h_{n-r}\left(a_1,\ldots,a_r\right)u^r v^{n-r} + \\
&\quad + \sum_{i=1}^{n-r}\left(-1\right)^i\binom{n}{i}s_i\left(a_1,\ldots,a_r\right)h_{n-r-i}\left(a_1,\ldots,a_r\right)u^r v^{n-r} \\
&\qquad \text{by (5.1.3).}
\end{aligned}$$

Note that in $H^*\left(X_n\left(a_1,\ldots,a_r\right)\right)$, $u_1\cdots u_r\cdot v_1\cdots v_{n-r} = v^{n-r}\cdot\prod_{k=1}^{r}\left(u-a_k v\right) = u^r v^{n-r}$ by (5.1.1). Then by Proposition 3.29,

$$\begin{aligned}
s_n\left[X_n\left(a_1,\ldots,a_r\right)\right] &= \left(r+1\right)h_{n-r}\left(a_1,\ldots,a_r\right) + \\
&\quad + \sum_{i=1}^{n-r}\left(-1\right)^i\binom{n}{i}s_i\left(a_1,\ldots,a_r\right)h_{n-r-i}\left(a_1,\ldots,a_r\right).
\end{aligned}$$

□

Note that even this fairly complicated formula does not give the Milnor number for every one of the varieties classified by Kleinschmidt whose associated fans have two more rays than the ambient dimension. It is only true when $n-r\le r$. If $n-r>r$, then (5.1.3) no longer holds, and it must be replaced with a much more complicated expression. On the other hand, the formula in Theorem 5.3 becomes much simpler for certain special choices of the a_k. For example,

Corollary 5.4. *Consider the smooth projective toric variety* $X_n^r\left(a\right) = X_n\left(0,\overset{(r-1)}{\ldots},0,a\right)$ *for some integers* $n\ge 2$, $n-r\le r\le n-1$, *and* $0\le a$. *Its Milnor number is*

$$s_n\left[X_n^r\left(a\right)\right] = a^\epsilon\left(r+\left(-1\right)^\epsilon\binom{n-1}{\epsilon}\right)$$

where $\epsilon = n-r$.

Proof. By Theorem 5.3,

$$\begin{aligned} s_n\left[X_n^r(a)\right] &= (r+1)\, h_{n-r}(0,\ldots,0,a) + \\ &\quad + \sum_{i=1}^{n-r} (-1)^i \binom{n}{i} s_i(0,\ldots,0,a)\, h_{n-r-i}(0,\ldots,0,a) \\ &= (r+1)\, a^{n-r} + \sum_{i=1}^{n-r} (-1)^i \binom{n}{i} a^i \cdot a^{n-r-i} \\ &= a^{n-r}\left(r+1+\sum_{i=1}^{n-r}(-1)^i\binom{n}{i}\right) \\ &= a^{n-r}\left(r+(-1)^{n-r}\binom{n-1}{n-r}\right). \end{aligned}$$

The last equality is a basic property of binomial coefficients [1]. □

This same formula happens to hold in this particular example even when $r < n-r$.

Proposition 5.5. *Consider the smooth projective toric variety*

$$X_n^r(a) = X_n\left(0, \overset{(r-1)}{\ldots}, 0, a\right)$$

for some integers $n \geq 2$, $1 \leq r \leq n-1$, and $0 \leq a$. Its Milnor number is

$$s_n\left[X_n^r(a)\right] = a^\epsilon \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$$

where $\epsilon = n-r$.

Proof. As in the proof of Theorem 5.3, one can calculate the cohomology of $X_n^r(a)$ to be $H^*\left(X_n^r(a)\right) = \mathbb{Z}[u,v]/I$, where $I = \left(u^r(u-av), v^{n-r+1}\right)$. In particular,

$$u^{r+1} = au^r v \text{ and } v^{n-r+1} = 0.$$

Then $u^{r+1}v^{n-r-1} = au^r v^{n-r}$. Similarly, for $j = 1, \ldots, n-r$,

$$\begin{aligned} u^{r+j}v^{n-r-j} &= au^{r+j-1}v^{n-r-j+1} \\ &= a^2 u^{r+j-2} v^{n-r-j+2} \\ &\ \ \vdots \\ &= a^j u^r v^{n-r}. \end{aligned}$$

The Chern class of $X_n^r(a)$ is given by $c(X_n^r(a)) = (1+u)^r(1+u-av)(1+v)^{n-r+1}$. Then

$$\begin{aligned}
s_n(c(X_n^r(a))) &= ru^n + (u-av)^n \\
&= ru^n + \sum_{k=0}^{n-r}(-1)^k\binom{n}{k}a^k u^{n-k}v^k \text{ since } v^{n-r+1}=0 \\
&= a^{n-r}ru^r v^{n-r} + \sum_{k=0}^{n-r}(-1)^k\binom{n}{k}a^k \cdot a^{n-r-k}u^r v^{n-r} \\
&= a^{n-r}\left(r + \sum_{k=0}^{n-r}(-1)^k\binom{n}{k}\right)u^r v^{n-r} \\
&= a^{n-r}\left(r + (-1)^{n-r}\binom{n-1}{n-r}\right)u^r v^{n-r}.
\end{aligned}$$

This produces the desired value for the Milnor number. □

Now that the Milnor number of many of Kleinschmidt's varieties has been calculated, one can begin exploring when this Milnor number satisfies (5.0.1). In other words, in which dimensions can the cobordism class of one of Kleinschmidt's varieties be used as a polynomial generator of complex cobordism?

Theorem 5.6. *If $n = 2^m - 1$ for some integer $m \geq 2$, then the cobordism class of a smooth projective toric variety can be chosen as the polynomial generator α_n of $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$. In particular, one can choose $\alpha_n = [X_n(1)]$.*

Proof. By Proposition 5.5, $s_n[X_n(1)] = s_n[X_n^1(1)] = 1 + (-1)^{n-1}\binom{n-1}{n-1} = 2$ since n is odd. Then by Theorem 2.9, $[X_n(1)]$ can be chosen for α_n. □

Recall that the cobordism class of $\mathbb{C}P^n$ can be used for the polynomial generator if n is one less than a prime. In fact, one can find more examples of polynomial generators in these dimensions among Kleinschmidt's varieties.

Proposition 5.7. *If $n = p - 1$ for some prime $p \geq 5$, then $[X_n^2(1)]$ can be chosen for the polynomial generator α_n.*

Proof. By Proposition 5.5, $s_n[X_n^2(1)] = 2 + (-1)^{n-2}\binom{n-1}{n-2} = n + 1 = p$ since n is even. Then by Theorem 2.9, $[X_n^2(1)]$ can be chosen for α_n. □

Note that in each of these dimensions $n = p-1$, where $p \geq 5$ is prime, there are two distinct choices for smooth projective toric varieties which can be used as polynomial generators. In fact, these varieties are not even cobordant. This can be seen by using Corollary 4.7 to calculate the Chern numbers $c_n[X_n^2(1)] = 3n-3 \neq n+1 = c_n[\mathbb{C}P^n]$.

5.2 Blow-ups of Kleinschmidt's varieties

While simple examples of Kleinschmidt's varieties have provided some examples of polynomial generators of $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ (see Theorem 5.6 and Proposition 5.7), there are still many dimensions in which examples have not been found. Calculating Milnor numbers of smooth projective toric varieties can be challenging in general. Instead of searching for more varieties which might have easily computable Milnor numbers, it may be easier to apply an operation which preserves smoothness and projectivity to a toric variety, tracking the change in Milnor number during this operation. One such operation which preserves these properties of toric varieties is the equivariant blow-up (see Section 3.5).

Proposition 5.8. *Consider a complex manifold M^{2n} and its blow-up Bl_xM at $x \in M$. The change in Milnor number is given by the following formula.*

$$s_n[Bl_xM] = \begin{cases} s_n[M] - (n+1) & \text{if } n \text{ is even} \\ s_n[M] - (n-1) & \text{if } n \text{ is odd} \end{cases}$$

Proof. This formula is a consequence of the well-known fact that Bl_xM is diffeomorphic to $M\#\overline{\mathbb{C}P^n}$ as an oriented differentiable manifold, where $\overline{\mathbb{C}P^n}$ is the complex projective space with the opposite of the standard orientation (see [26, Proposition 2.5.8] for details). Recall that the Chern class of $\mathbb{C}P^n$ with its standard complex structure is $c(\mathbb{C}P^n) = (1+x) \overset{(n+1)}{\cdots} (1+x)$, where $x \in H^2(\mathbb{C}P^2)$. The reversal of orientation in $\overline{\mathbb{C}P^n}$ changes the sign of one of the x's in this Chern class. That is,

$$c\left(\overline{\mathbb{C}P^n}\right) = (1+x)^n \cdot (1-x).$$

Then by Definition 2.6, $s_n\left(c\left(\overline{\mathbb{C}P^n}\right)\right) = nx^n + (-x)^n = (n + (-1)^n)\, x^n$. Then its Milnor number is

$$s_n\left[\overline{\mathbb{C}P^n}\right] = \langle (n + (-1)^n)\, x^n, \mu_{\overline{\mathbb{C}P^n}} \rangle = -(n + (-1)^n),$$

where the negative arises since x^n is dual to the fundamental class of $\mathbb{C}P^n$ with standard orientation. Then since $\left[M\#\overline{\mathbb{C}P^n}\right] = [M] + \left[\overline{\mathbb{C}P^n}\right]$ (see [41] for details),

$$s_n[Bl_xM] = s_n[M] + s_n\left[\overline{\mathbb{C}P^n}\right] = s_n[M] - (n + (-1)^n).$$

□

Now Proposition 5.7 can be generalized by examining blow-ups of some of Kleinschmidt's varieties when n is one less than a power of an odd prime, i.e. $n = p^m - 1$. In this situation, a cobordism class must have Milnor number $\pm p$ for it to be used as a polynomial generator in cobordism. But each blow-up at a point in this even complex dimension decreases the Milnor number by $n + 1 = p^m$. This means that in order to find a smooth projective toric variety with Milnor number p, it suffices to find one whose Milnor number is positive and is congruent to p modulo p^m. The extra multiples of p^m can then be removed by a sequence of blow-ups of points. By choosing these points to be torus-fixed points, each successive blow-up is itself a smooth projective toric variety. A technical lemma will be needed to show that there are smooth projective toric varieties in these dimensions that satisfy this congruence.

Lemma 5.9. *Let $n = p^m - 1$ for some odd prime p and integer $m \geq 2$. Then there exists an integer $\epsilon \in \{1, \dots, n-1\}$ such that*

1. *for any $x \in \mathbb{Z}$, if $p \nmid x$, then $x^\epsilon \equiv x \bmod p^{m-1}$ AND*
2. *$p \mid \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$, but $p^2 \nmid \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$, where $r = n - \epsilon$.*

Proof. Such an ϵ can be described explicitly. First, suppose $n = 3^2 - 1$, so $p = 3$ and $m = 2$. Set $\epsilon = 3$. Choose $x \in \mathbb{Z}$ that is not divisible by three. If $x \equiv 1 \bmod 3$, then $x = 3k + 1$ for some $k \in \mathbb{Z}$. Then $x^3 = 27k^3 + 27k^2 + 9k + 1 \equiv 1 \equiv x \bmod 3$. A similar calculation proves that $x^3 \equiv x \bmod 3$ when $x \equiv 2 \bmod 3$. Then this value for ϵ satisfies condition 1. Condition 2 is also satisfied since $r + (-1)^\epsilon \binom{n-1}{\epsilon} = 5 - \binom{7}{3} = -30$.

Now assume that $n \neq 8$. Set

$$\epsilon = p^{m-2} (p-1)^2 + 1. \tag{5.2.1}$$

Note that $\epsilon = p^m - p^{m-2}(2p-1) + 1$. Since $p \geq 3$ and $m \geq 2$, $3 \leq p^{m-2}(2p-1) \leq p^m$. Then

$$1 \leq p^m - p^{m-2}(2p-1) + 1 \leq p^m - 2.$$

In other words, ϵ satisfies $1 \leq \epsilon \leq p^m - 2 = n - 1$.

Let $\mathbb{Z}_k^\times = \{x \in \mathbb{Z}_k \mid \gcd(z, k) = 1\}$ be the multiplicative group of integers modulo k. Recall that $\mathbb{Z}_{p^{m-1}}^\times$ is a cyclic group of order $\phi(p^{m-1}) = p^{m-2}(p-1)$, where ϕ is the Euler ϕ function. This means that $x^{p^{m-2}(p-1)} \equiv 1 \bmod p^{m-1}$ for any $x \in \mathbb{Z}_{p^{m-1}}^\times$. Thus if $p \nmid x$, then $x \in \mathbb{Z}_{p^{m-1}}^\times$ and $x^{p-1} \in \mathbb{Z}_{p^{m-1}}^\times$, so

$$x^\epsilon = x^{p^{m-2}(p-1)^2+1} = \left(x^{p-1}\right)^{p^{m-2}(p-1)} \cdot x \equiv x \bmod p^{m-1}.$$

This verifies that ϵ satisfies the first condition.

To prove that the second condition is also satisfied, first consider $\binom{n-1}{\epsilon}$. Note that expanding ϵ in (5.2.1) gives $\epsilon = p^m - 2p^{m-1} + p^{m-2} + 1$. Then

$$r = n - \epsilon = p^m - 1 - \epsilon = 2p^{m-1} - p^{m-2} - 2. \tag{5.2.2}$$

Then

$$\begin{aligned} \binom{n-1}{\epsilon} &= \binom{n-1}{r-1} \\ &= \binom{p^m - 2}{2p^{m-1} - p^{m-2} - 3} \\ \binom{n-1}{\epsilon} &= \frac{(p^m-2)(p^m-3)\cdots(p^m - (2p^{m-1} - p^{m-2} - 2))}{(2p^{m-1} - p^{m-2} - 3)!}. \end{aligned} \tag{5.2.3}$$

In general, if $p \not| c$, then c has a multiplicative inverse c^{-1} modulo p^2. In this situation,

$$\frac{p^2 - c}{c} = c^{-1}\left(p^2 - c\right) \equiv -1 \bmod p^2. \tag{5.2.4}$$

If $p|c$, then this cancellation cannot be applied.

Case 1. Suppose $m = 2$. Since $p \geq 5$ (for otherwise $n = 8$, which was addressed above), (5.2.3) becomes

$$\begin{aligned}\binom{n-1}{\epsilon} &= \frac{(p^2-2)(p^2-3)\cdots(p^2-(2p-4))(p^2-(2p-3))}{(2p-4)!} \\ &= \frac{p^2-p}{p}\cdot\frac{p^2-2}{2}\cdots\frac{p^2-(p-1)}{p-1}\cdot\frac{p^2-(p+1)}{p+1}\cdots\frac{p^2-(2p-4)}{2p-4}\cdot \\ &\quad \cdot\left(p^2-(2p-3)\right) \\ &\equiv (p-1)\cdot(-1)^{p-2}\cdot(-1)^{p-4}\cdot(-2p+3) \bmod p^2 \\ &\equiv 5p-3 \bmod p^2.\end{aligned}$$

Then using (5.2.2) and the fact that $\epsilon = p^2 - 2p + 2$ is odd,

$$\begin{aligned}r + (-1)^{\epsilon}\binom{n-1}{\epsilon} &\equiv 2p-3-(5p-3) \bmod p^2 \\ &\equiv -3p \bmod p^2.\end{aligned}$$

Since $p \geq 5$, this shows that the second condition is satisfied for $m = 2$.

Case 2. Suppose $m \geq 3$. The factors of (5.2.3) can be separated into terms that can cancel as in (5.2.4) (Q_2 in the following) and terms that cannot cancel (Q_1 in the following), i.e.

$$\binom{n-1}{\epsilon} = Q_1 \cdot Q_2 \cdot \left(p^m - \left(2p^{m-1} - p^{m-2} - 2\right)\right).$$

Note that there is one extra term $p^m - (2p^{m-1} - p^{m-2} - 2)$ in the numerator that has no corresponding term in the denominator. More specifically, the terms that do not cancel are

$$\begin{aligned}Q_1 &= \frac{p^m-p}{p}\cdot\frac{p^m-2p}{2p}\cdots\frac{p^m-(2p^{m-1}-p^{m-2}-p)}{2p^{m-1}-p^{m-2}-p} \\ &= \frac{p^{m-1}-1}{1}\cdot\frac{p^{m-1}-2}{2}\cdots\frac{p^{m-1}-(2p^{m-2}-p^{m-3}-1)}{2p^{m-2}-p^{m-3}-1}.\end{aligned}$$

The terms that cancel can be written as $Q_2 = P_0 \cdot P_1 \cdots P_{2p^{m-2}-p^{m-3}-1}$ where

$$\begin{aligned}P_0 &= \frac{p^m-2}{2}\cdot\frac{p^m-3}{3}\cdots\frac{p^m-(p-1)}{p-1} \\ P_k &= \frac{p^m-(kp+1)}{kp+1}\cdot\frac{p^m-(kp+2)}{kp+2}\cdots\frac{p^m-(kp+p-1)}{kp+p-1} \\ &\quad \text{for } k = 1,\ldots,2p^{m-2}-p^{m-3}-2 \\ P_{2p^{m-2}-p^{m-3}-1} &= \frac{p^m-(2p^{m-1}-p^{m-2}-p+1)}{2p^{m-1}-p^{m-2}-p+1}\cdots\frac{p^m-(2p^{m-1}-p^{m-2}-3)}{2p^{m-1}-p^{m-2}-3}.\end{aligned}$$

Counting factors and applying the cancellations yields

$$\begin{aligned}\binom{n-1}{\epsilon} &= Q_1 \cdot Q_2 \cdot \left(p^m - \left(2p^{m-1} - p^{m-2} - 2\right)\right) \\ \binom{n-1}{\epsilon} &\equiv \frac{p^{m-1}-1}{1} \cdot \frac{p^{m-1}-2}{2} \cdots \frac{p^{m-1} - (2p^{m-2} - p^{m-3} - 1)}{2p^{m-2} - p^{m-3} - 1} \cdot \\ &\quad \cdot \left(2p^{m-1} - p^{m-2} - 2\right) \mod p^2. \end{aligned} \tag{5.2.5}$$

Now the same process of separating the terms that do cancel and those that do not cancel can be applied to (5.2.5). This yields

$$\begin{aligned}\binom{n-1}{\epsilon} &\equiv \frac{p^{m-2}-1}{1} \cdot \frac{p^{m-2}-2}{2} \cdots \frac{p^{m-2} - (2p^{m-3} - p^{m-4} - 1)}{2p^{m-3} - p^{m-4} - 1} \cdot \\ &\quad \cdot \left(2p^{m-1} - p^{m-2} - 2\right) \mod p^2. \end{aligned}$$

Applying this separation and cancellation process $m-2$ times (recall that $m \geq 3$) gives

$$\binom{n-1}{\epsilon} \equiv \frac{p^2-1}{1} \cdot \frac{p^2-2}{2} \cdots \frac{p^2 - (2p-2)}{2p-2} \cdot \left(2p^{m-1} - p^{m-2} - 2\right) \mod p^2.$$

Applying the separating and canceling procedure to this new expression is much simpler.

$$\begin{aligned}\binom{n-1}{\epsilon} &\equiv \frac{p^2-p}{p} \cdot \frac{p^2-1}{1} \cdots \frac{p^2-(p-1)}{p-1} \cdot \frac{p^2-(p+1)}{p+1} \cdots \frac{p^2-(2p-2)}{2p-2} \cdot \\ &\quad \cdot \left(2p^{m-1} - p^{m-2} - 2\right) \mod p^2 \\ &\equiv (p-1) \cdot (-1)^{p-1} \cdot (-1)^{p-2} \cdot \left(2p^{m-1} - p^{m-2} - 2\right) \mod p^2 \\ &\equiv -\left(2p^m - 3p^{m-1} + p^{m-2} - 2p + 2\right) \mod p^2 \end{aligned}$$

Then

$$\binom{n-1}{\epsilon} \equiv \begin{cases} p - 2 \mod p^2 & \text{if } m = 3 \\ 2p - 2 \mod p^2 & \text{if } m \geq 4 \end{cases}.$$

Thus, using (5.2.2) and the fact that $\epsilon = p^2 - 2p + 2$ is odd,

$$\begin{aligned} r + (-1)^\epsilon \binom{n-1}{\epsilon} &\equiv \begin{cases} 2p^{m-1} - p^{m-2} - 2 - (p-2) \mod p^2 & \text{if } m = 3 \\ 2p^{m-1} - p^{m-2} - 2 - (2p-2) \mod p^2 & \text{if } m \geq 4 \end{cases} \\ &\equiv -2p \mod p^2 \text{ if } m = 3 \text{ or } m \geq 4. \end{aligned}$$

This means that $p \mid \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$, but $p^2 \nmid \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$, which proves that ϵ also satisfies the second condition.

□

Theorem 5.10. *If $n = p^m - 1$ for some odd prime p and some integer $m \geq 2$, then there exists a smooth projective toric variety whose cobordism class can be chosen for the polynomial generator α_n of $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$.*

Proof. Choose an integer $\epsilon \in \{1, \ldots, n-1\}$ that satisfies the following conditions.

1. For any $x \in \mathbb{Z}$, if $p \not| x$, then $x^\epsilon \equiv x \bmod p^{m-1}$ AND
2. $p | \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$, but $p^2 \not| \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$, where $r = n - \epsilon$.

Such an integer exists by Lemma 5.9. Set $R = r + (-1)^\epsilon \binom{n-1}{\epsilon}$. Consider the congruence

$$Rz \equiv p \bmod p^m. \tag{5.2.6}$$

Since $p^2 \not| R$, this congruence has some integer solution $z = a$ such that $a < 0$. Since p divides R, p cannot divide a (for otherwise, $p^2 | Ra$, so a could not be a solution). Then by the first condition above, $a^\epsilon \equiv a \bmod p^{m-1}$. Then a^ϵ is also a solution to (5.2.6). That is,

$$a^\epsilon \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right) \equiv p \bmod p^m. \tag{5.2.7}$$

Now consider $X_n^r(a) = X_n\left(0, \overset{(r-1)}{\ldots}, 0, a\right)$. As in the proof of Lemma 4.27, $X_n^r(a)$ is a smooth projective toric variety, despite the fact that $a < 0$. Kleinschmidt's classification theorem 3.24 simply says that $X_n^r(a)$ is isomorphic to one of the varieties $X_n(a_1, \ldots a_s)$ for some integers $s \in \{1, \ldots, n-1\}$ and $0 \leq a_1 \leq a_2 \leq \ldots \leq a_s$. The positivity of a is also not needed in the proof of Proposition 5.5, so by this proposition, $s_n[X_n^r(a)] = a^\epsilon \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right)$. Thus $s_n[X_n^r(a)] \equiv p \bmod p^m$. In fact, the choice of $\epsilon = p^m - 2p^{m-1} + p^{m-2} + 1$ guarantees that $r + (-1)^\epsilon \binom{n-1}{\epsilon} < 0$. Then since $a < 0$ and ϵ is odd, $s_n[X_n^r(a)] = a^\epsilon \left(r + (-1)^\epsilon \binom{n-1}{\epsilon}\right) \geq p$.

Also recall that each blow-up at a point of a manifold of even complex dimension n decreases the Milnor number by $n+1$ (see Proposition 5.8). In this situation, this means that blowing up a smooth projective toric variety of dimension $n = p^m - 1$ at a torus-fixed point produces a new smooth projective toric variety, and the Milnor number decreases by $n+1 = p^m$. Thus by (5.2.7), applying sufficiently many of these blow-ups to $X_n^r(a)$ produces a smooth projective toric variety with Milnor number p. The cobordism class of this variety can be used as a polynomial generator of Ω_*^U by Theorem 2.9.

□

Example 5.11. Suppose $n = 5^2 - 1 = 24$, so $\epsilon = 5^0 \cdot (5-1)^2 + 1 = 17$ (see (5.2.1)). Then $R = 7 + (-1)^{17}\binom{23}{17} = -100940$. The congruence $-100940z \equiv 5 \bmod 5^2$ has a solution $z = -7$. Then $z = (-7)^{17} = -232630513987207$ is also a solution. Thus, $s_n[X_{24}^7(-7)] = (-7)^{17}\left(7 + (-1)^{17}\binom{23}{17}\right) = 23481724081868674580 \equiv 5 \bmod 5^2$. Each blow-up of a point in this dimension decreases the Milnor number by $n+1 = 25$. Thus, by applying a sequence of 939268963274746983 many blow-ups at torus-fixed points to $X_{24}^7(-7)$, one obtains a smooth projective toric variety with Milnor number $23481724081868674580 - 25 \cdot 939268963274746983 = 5$. The cobordism class of this variety can be used as the polynomial generator α_{24} of $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$ by Theorem 2.9.

This example demonstrates that although Theorem 5.10 gives the existence of smooth projective toric variety polynomial generators in certain dimensions, the theorem is not very useful in explicitly constructing such examples.

So far, the only odd complex dimensions which have been shown to have smooth projective toric variety representatives for polynomial generators are those that are one less than a power of two (see Theorem 5.6 and also recall that we can choose $\alpha_1 = [\mathbb{C}P^1]$). In fact, this theorem can be generalized, and Conjecture 5.1 is true for all odd n. Another number theory fact is needed to prove this.

Lemma 5.12. *Let n be a positive odd integer. If $n \neq 2^k - 1$ for any $k \in \mathbb{Z}$, then $n \equiv (2^m - 1) \mod 2^{m+1}$ for some integer $m \geq 1$.*

Proof. Suppose n is odd and $n \neq 2^k - 1$ for any integer k. Then $n + 1 = 2^m \cdot q$, where $m \geq 1$, $q > 1$, and $2 \nmid q$. Then $n + 1 - 2^m = 2^m \cdot q - 2^m = 2^m (q - 1)$, and $q - 1$ is even. Then $2^{m+1} | (n + 1 - 2^m)$, i.e. $n + 1 - 2^m = 2^{m+1} \cdot j$ for some integer j. Then $n = 2^m - 1 + 2^{m+1} \cdot j$, which is equivalent to the congruence given in the lemma. □

To prove Conjecture 5.1 for every odd n, blow-ups at points are not enough. The behavior of Milnor numbers during blow-ups of real-dimension two subvarieties must also be understood.

Lemma 5.13. *Let Σ be a complete regular fan in $\mathbb{R}^n$, where n is odd. Consider an $(n-1)$-dimensional cone τ. Without loss of generality (i.e. by applying a unimodular transformation to Σ to obtain an isomorphic fan), $\tau = pos(e_1, \ldots, e_{n-1})$. This $(n-1)$-dimensional cone is the intersection of two maximal cones. More specifically (again without loss of generality), $\tau = pos(e_1, \ldots, e_n) \cap pos(e_1, \ldots, e_{n-1}, v)$ where $v = (a_1, \ldots, a_{n-1}, -1)$ for some integers a_k. Then the change in the Milnor number upon blowing up X_Σ along the two-dimensional submanifold X_τ is given by*

$$s_n [Bl_{X_\tau} X_\Sigma] = s_n [X_\Sigma] + 2 (a_1 + \ldots + a_{n-1}).$$

Proof. By Proposition 3.40, the change in Milnor number is completely determined by the cones $\mathrm{pos}(e_1, \ldots, e_n)$ and $\mathrm{pos}(e_1, \ldots, e_{n-1}, v)$ that contain τ. Therefore, it suffices to consider one simple example containing such a cone τ and compute the change in Milnor number when τ is subdivided.

Let $\Sigma = \Sigma_n (a_1, \ldots, a_{n-1})$ as in Theorem 3.24, but allow the a_k to have any integer values (so Σ is isomorphic to some $\Sigma_n (b_1, \ldots, b_r)$ for some integers $r \in \{1, \ldots, n-1\}$ and $0 \leq b_1 \leq \ldots \leq b_r$). By Theorem 5.3,

$$s_n [X_\Sigma] = n \cdot h_1 (a_1, \ldots, a_{n-1}) - n \cdot s_1 (a_1, \ldots, a_{n-1}) = 0.$$

The generating rays of Σ are $\{u_1, \ldots, u_n, v_1, v_2\}$, where $u_k = e_k$ for $k = 1, \ldots, n-1$, $u_n = (-1, \ldots, -1, 0)$, $v_1 = e_n$, and $v_2 = (a_1, \ldots, a_{n-1}, -1)$. Consider the $(n-1)$-dimensional cone

$$\tau = \mathrm{pos}(u_1, \ldots, u_{n-1}) = \mathrm{pos}(u_1, \ldots, u_{n-1}, v_1) \cap \mathrm{pos}(u_1, \ldots, u_{n-1}, v_2)$$

in Σ. This cone satisfies the generality conditions specified in the lemma. Subdivide τ with a new ray $w = (1, \ldots, 1, 0)$ to obtain a new fan $Bl_\tau\Sigma$ with corresponding toric variety $Bl_{X_\tau}X_\Sigma$. The cohomology of this toric variety has linear relations

$$\begin{aligned} u_k &= u_n - a_k v_2 - w \text{ for } k = 1, \ldots, n-1 \\ v_1 &= v_2, \end{aligned}$$

and the Stanley-Reisner ring is $(u_n w, v_1 v_2, u_1 \cdots u_{n-1})$. Combining these relations and using the assumption that n is odd yields

$$\begin{aligned} 0 &= u_n w \\ 0 &= v_2^2 \\ 0 &= \prod_{k=1}^{n-1} (u_n - a_k v_2 - w) \qquad (5.2.8) \\ &= u_n^{n-1} + w^{n-1} - \left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-2} v_2 + \left(\sum_{k=1}^{n-1} a_k\right) v_2 w^{n-2}. \end{aligned}$$

These relations can be used to derive several other useful relations, namely

$$\begin{aligned} u_n^n &= u_n \left(-w^{n-1} + \left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-2} v_2 - \left(\sum_{k=1}^{n-1} a_k\right) v_2 w^{n-2}\right) \\ &= \left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-1} v_2 \end{aligned}$$

and

$$\begin{aligned} w^n &= w \left(-u_n^{n-1} + \left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-2} v_2 - \left(\sum_{k=1}^{n-1} a_k\right) v_2 w^{n-2}\right) \\ &= -\left(\sum_{k=1}^{n-1} a_k\right) v_2 w^{n-1}. \end{aligned}$$

Also,

$$u_n^{n-1} v_2 = v_2 \left(-w^{n-1} + \left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-2} v_2 - \left(\sum_{k=1}^{n-1} a_k\right) v_2 w^{n-2}\right) = -v_2 w^{n-1},$$

so

$$w^n = \left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-1} v_2.$$

Note that $\operatorname{pos}(u_2, \ldots, u_n, v_2)$ is a maximal cone in $Bl_\tau\Sigma$. Then by Proposition 3.29,

$$\left\langle u_2 \cdots u_n v_2, \mu_{Bl_{X_\tau}X_\Sigma} \right\rangle = 1.$$

But using the above cohomology relations yields

$$\begin{aligned}
u_2 \cdots u_n v_2 &= u_n v_2 \prod_{k=2}^{n-1} (u_n - a_k v_2 - w) \\
&= u_n v_2 \left(u_n^{n-2} - w^{n-2} - \left(\sum_{k=2}^{n-1} a_k \right) u_n^{n-3} v_2 - \left(\sum_{k=2}^{n-1} a_k \right) v_2 w^{n-3} \right) \\
&= u_n^{n-1} v_2.
\end{aligned}$$

Then $\langle u_n^{n-1} v_2, \mu_{Bl_{X_\tau} X_\Sigma} \rangle = 1$. By Theorem 3.30 and the linear relations in cohomology, the Chern class of $Bl_{X_\tau} X_\Sigma$ is given by

$$c\left(Bl_{X_\tau} X_\Sigma\right) = (1+u_1)\cdots(1+u_n)(1+v_1)(1+v_2)(1+w).$$

Then

$$s_n\left(c\left(Bl_{X_\tau} X_\Sigma\right)\right) = u_1^n + \ldots + u_{n-1}^n + u_n^n + v_1^n + v_2^n + w^n.$$

Using the cohomology relations and the fact that n is odd, for every $k = 1, \ldots, n-1$,

$$\begin{aligned}
u_k^n &= (u_n - a_k v_2 - w)^n \\
&= u_n^n - w^n - n a_k u_n^{n-1} v_2 - n a_k v_2 w^{n-1} \\
&= \left(\sum_{k=1}^{n-1} a_k \right) u_n^{n-1} v_2 - \left(\sum_{k=1}^{n-1} a_k \right) u_n^{n-1} v_2 - n a_k u_n^{n-1} v_2 + n a_k u_n^{n-1} v_2 \\
&= 0.
\end{aligned}$$

Then $s_n\left(c\left(Bl_{X_\tau} X_\Sigma\right)\right) = u_n^n + w^n = 2\left(\sum_{k=1}^{n-1} a_k\right) u_n^{n-1} v_2$, so

$$s_n\left[Bl_{X_\tau} X_\Sigma\right] = 2\sum_{k=1}^{n-1} a_k = s_n\left[X_\Sigma\right] + 2\left(a_1 + \ldots + a_{n-1}\right).$$

□

Theorem 5.14. *If n is odd, then there exists a smooth projective toric variety whose cobordism class can be chosen as the polynomial generator α_n of $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$.*

Proof. For $n = 1$, use $\alpha_1 = [\mathbb{C}P^1]$. If $n = 2^m - 1$ for some $m \geq 2$, then one can choose $\alpha_n = [X_n(1)]$ by Theorem 5.6. Now assume that $n \neq 2^k - 1$ for any integer k. Then by Lemma 5.12, there exists an integer $m \geq 1$ such that $n \equiv (2^m - 1) \mod 2^{m+1}$. Consider the smooth projective toric variety $X = X_n^{n-2^m}(2^m - 3)$. By Proposition 5.5,

$$s_n[X] = (2^m - 3)^{2^m} \left(n - 2^m + (-1)^{2^m} \binom{n-1}{2^m} \right). \tag{5.2.9}$$

The variety X will be used as a starting point to construct a smooth projective toric variety in this dimension with Milnor number ± 1. By Theorem 2.9, this is what the Milnor number needs to be in this dimension in order for the variety to be used as a polynomial generator. The variety X was chosen because it has two essential properties. Its Milnor number is positive and odd.

It is easy to see that $s_n[X]$ is positive, using (5.2.9) and the fact that $m \geq 1$ and $n > 2^m$. Demonstrating that $s_n[X]$ is odd is a more difficult task. Since by assumption $n-1 \equiv (2^m-2) \mod 2^{m+1}$, one can write $n-1 = 2^{m+1}K + 2^m - 2$, and K is a positive integer since $n \neq 2^m - 1$. Let $K = K_0 + 2K_1 + 2^2K_2 + \ldots$ be the binary expansion of K, so $K_i \in \{0,1\}$. Then $2^{m+1}K = 2^{m+1}K_0 + 2^{m+2}K_1 + 2^{m+3}K_2 + \ldots$. Since $m \geq 1$, the coefficient of 2 in the binary expansion of $2^{m+1}K$ is zero. Let i be the minimal index such that $K_i \neq 0$. Then the binary expansion of $2^{m+1}K - 2$ is $2 + 2^2 + \ldots + 2^{m+i} + 2^{m+i+1} \cdot 0 + 2^{m+i+2}K_{i+1} + \ldots$. Note that the coefficient of 2^m in this expansion is one regardless of the value of i. Then the coefficient of 2^m in the binary expansion of $2^{m+1}K + 2^m - 2$ is zero. Then

$$\begin{aligned}\binom{n-1}{2^m} &= \binom{2^{m+1}K + 2^m - 2}{2^m} \\ &\equiv \binom{0}{0}\binom{1}{0}\binom{1}{0}\cdots\binom{0}{1}\binom{1}{0}\cdots\binom{1}{0}\binom{K_{i+1}}{0}\binom{K_{i+2}}{0}\cdots \mod 2,\end{aligned}$$

where $\binom{0}{1}$ is the factor corresponding to the coefficients of 2^m in $n-1$ and 2^m. Since this factor is zero, $\binom{n-1}{2^m} \equiv 0 \mod 2$, i.e. $\binom{n-1}{2^m}$ is even. Since n is odd, this implies that

$$s_n[X] = (2^m-3)^{2^m}\left(n - 2^m + (-1)^{2^m}\binom{n-1}{2^m}\right)$$

is odd.

Now a smooth projective toric variety with odd, positive Milnor number has been constructed in each pertinent dimension. Next, an infinite sequence of blow-ups will be described with the property that each blow-up decreases the Milnor number by four. This means that applying sufficiently many of these blow-ups will produce a variety with the desired Milnor number ± 1 (see Theorem 2.9).

This construction begins with the $(n-1)$-dimensional cone $\tau_0 = \text{pos}(e_1, \ldots, e_{n-1})$ in the fan Σ corresponding to X. Note that in this fan,

$$\tau_0 = \text{pos}(e_1, \ldots, e_{n-1}, e_n) \cap \text{pos}(e_1, \ldots, e_{n-1}, v),$$

where $v = \left(0, \ldots, 0, 2^m - 3, -1, \overset{(2^m)}{\ldots}, -1\right)$. Define $X_1 = Bl_{X_{\tau_0}}X$ to be the blow-up along the real-dimension two subvariety corresponding to τ_0. In terms of fans, the fan of the blown up variety is obtained by a star subdivision of τ_0 in the original fan. Let $w = (1, \ldots, 1, 0)$ be the added generating ray. By Lemma 5.13,

$$s_n[X_1] = s_n[X] + 2(2^m - 3 + (2^m - 1)\cdot(-1)) = s_n[X] - 4.$$

That is, the Milnor number has decreased by four.

Now consider the new smooth projective toric variety X_1. Apply the unimodular transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$x \mapsto \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ -1 & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot x$$

to its associated fan to obtain an isomorphic fan denoted by $\phi(\Sigma_1)$. In particular, note that $\phi(w) = e_1$, $\phi(e_k) = e_k$ for $k = 2, \dots, n$, and $\phi(v) = v$. Then one of the $(n-1)$-dimensional cones in the blown up fan is $\tau_1 = \operatorname{pos}(w, e_2, \dots, e_{n-1})$. After applying ϕ, this cone in $\phi(\Sigma_1)$ is

$$\tau_1 = \operatorname{pos}(e_1, \dots, e_{n-1}) = \operatorname{pos}(e_1, \dots, e_{n-1}, e_n) \cap \operatorname{pos}(e_1, \dots, e_{n-1}, v).$$

Then locally, τ_1 in $\phi(\Sigma_1)$ is identical to τ in Σ. This means that when a star subdivision is applied to τ_1, the resulting fan is associated to a smooth projective toric variety for which the Milnor number has decreased by four more. One can apply ϕ to this newly subdivided fan to again create an $(n-1)$-dimensional cone that is locally identical to τ in Σ. This process can be continued indefinitely. Since $s_n[X]$ is odd and positive and each step in this process produces a new smooth projective toric variety with a Milnor number decreased by four, applying a certain number of these blow-ups will produce a smooth projective toric variety with Milnor number ± 1. The cobordism class of this variety can be used as the polynomial generator by Theorem 2.9. □

Example 5.15. Suppose $n = 5$. Then $n \equiv (2^1 - 1) \mod 2^2$. Let

$$X = X_5^3(-1) = X_5(0, 0, -1).$$

The generating rays of the corresponding fan Σ are $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_3$, $u_4 = (-1, -1, -1, 0, 0)$, $v_1 = e_4$, $v_2 = e_5$, and $v_3 = (0, 0, -1, -1, -1)$. By Proposition 5.5, $s_5[X] = (-1)^2 \left(3 + \binom{4}{2}\right) = 9$. Then two blow-ups of two-dimensional subvarieties as described in the proof of 5.14 must be applied to obtain a smooth projective toric variety with Milnor number $9 - 4 - 4 = 1$. The cobordism class of this variety can be chosen for the polynomial generator α_5 in $\Omega_*^U = \mathbb{Z}[\alpha_1, \alpha_2, \dots]$. More specifically, first apply a star subdivision to $\operatorname{pos}(u_1, u_2, u_3, v_1)$ in Σ, and let $w = (1, 1, 1, 1, 0)$ denote the added ray. Second, apply a star subdivision to the cone $\operatorname{pos}(w, u_2, u_3, v_1)$ in the new fan to obtain the desired fan.

5.3 Polynomial generators in the remaining dimensions

Conjecture 5.1 has now been verified in many dimensions. More specifically, the cobordism class of a smooth projective toric variety can be chosen as the polynomial generator α_n of $\Omega_n^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \dots]$ for any dimension n such that n is odd or n is one less than a power of a prime (see 5.6, 5.7, 5.10, and 5.14). The only dimensions in which Conjecture 5.1 has not yet been verified are those for which n is even and $n+1$ is not a prime power.

Towards a proof of Conjecture 5.1

While a proof of the conjecture in these dimensions remains elusive, there is overwhelming numerical evidence that suggests that the conjecture is true. In fact, it appears that a number-theoretic argument similar to the one used to prove Theorem 5.10 can be applied to the remaining dimensions. Unfortunately, the necessary number theory conjecture stated below has not yet been proven.

Conjecture 5.16. *Suppose n is a positive even integer such that $n+1$ is not a prime power. Then there exists an odd integer $\epsilon \in \{1, \ldots, n-1\}$ that satisfies the following conditions.*

1. $\gcd\left(\epsilon, \operatorname{ord}\left(\mathbb{Z}_{n+1}^{\times}\right)\right) = 1$
2. $\gcd\left(\epsilon + 1 + \binom{n-1}{\epsilon}, n+1\right) = 1$

Suppose that 5.16 is true. Write $n+1 = p_1^{m_1} \cdots p_t^{m_t}$, where the $p_1, \ldots, p_t$ are distinct primes, $m_1, \ldots, m_t \in \mathbb{N}$, and $t \geq 2$, and choose an integer ϵ that satisfies the conjecture. Condition 2 of the conjecture simply states that $p_k \not| \left(\epsilon + 1 + \binom{n-1}{\epsilon}\right)$ for every k. Since ϵ is odd, this means that $p_k \not| \left(-\epsilon - 1 + (-1)^{\epsilon}\binom{n-1}{\epsilon}\right)$. Since $p_k | (n+1)$, we get $p_k \not| \left(n - \epsilon + (-1)^{\epsilon}\binom{n-1}{\epsilon}\right)$. Set $R = r + (-1)^{\epsilon}\binom{n-1}{\epsilon}$, where $r = n - \epsilon$. Condition 2 of the conjecture then implies that R has an inverse R^{-1} in $\mathbb{Z}_{n+1}^{\times}$.

Now the first condition comes into play. Let $a, b \in \mathbb{Z}_{n+1}^{\times}$, and suppose

$$a^{\epsilon} \equiv b^{\epsilon} \bmod n+1.$$

Let $c = a^{-1}$ in $\mathbb{Z}_{n+1}^{\times}$. Then

$$1 \equiv (ac)^{\epsilon} \equiv a^{\epsilon}c^{\epsilon} \equiv b^{\epsilon}c^{\epsilon} \equiv (bc)^{\epsilon} \bmod n+1.$$

Since ϵ is relatively prime to the order of $\mathbb{Z}_{n+1}^{\times}$, the only element $x \in \mathbb{Z}_{n+1}^{\times}$ that satisfies $x^{\epsilon} \equiv 1 \bmod n+1$ is $x = 1$. Thus $bc \equiv 1 \bmod n+1$, so $c = b^{-1}$ in $\mathbb{Z}_{n+1}^{\times}$. Then $a \equiv b \bmod n+1$. This means that for integers ϵ that satisfy the first condition, $a \equiv b \bmod n+1 \iff a^{\epsilon} \equiv b^{\epsilon} \bmod n+1$. But this in turn implies that $\left\{a^{\epsilon} \bmod n+1 | a \in \mathbb{Z}_{n+1}^{\times}\right\} = \mathbb{Z}_{n+1}^{\times}$.

Consider again $R^{-1} \in \mathbb{Z}_{n+1}^{\times}$. By the above argument, $R^{-1} \equiv a^{\epsilon} \bmod n+1$ for some $a \in \mathbb{Z}_{n+1}^{\times}$. That is, $a^{\epsilon}\left(r + (-1)^{\epsilon}\binom{n-1}{\epsilon}\right) \equiv 1 \bmod n+1$. By Proposition 5.5,

$$s_n\left[X_n^r(a)\right] \equiv 1 \bmod n+1$$

for this choice of ϵ and a. Since ϵ is odd, one can also choose a to be either positive or negative to guarantee that $s_n\left[X_n^r(a)\right]$ is positive. Now apply a sequence of equivariant blow-ups at torus-fixed points of $X_n^r(a)$. By Proposition 5.8, each of these will decrease the Milnor number of the variety by $n+1$, since n is even. Applying sufficiently many of these blow-ups to $X_n^r(a)$ will produce a smooth projective toric variety with Milnor number equal to one. By Theorem 2.9, the cobordism class of this smooth projective toric variety can be chosen for the generator α_n of $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$.

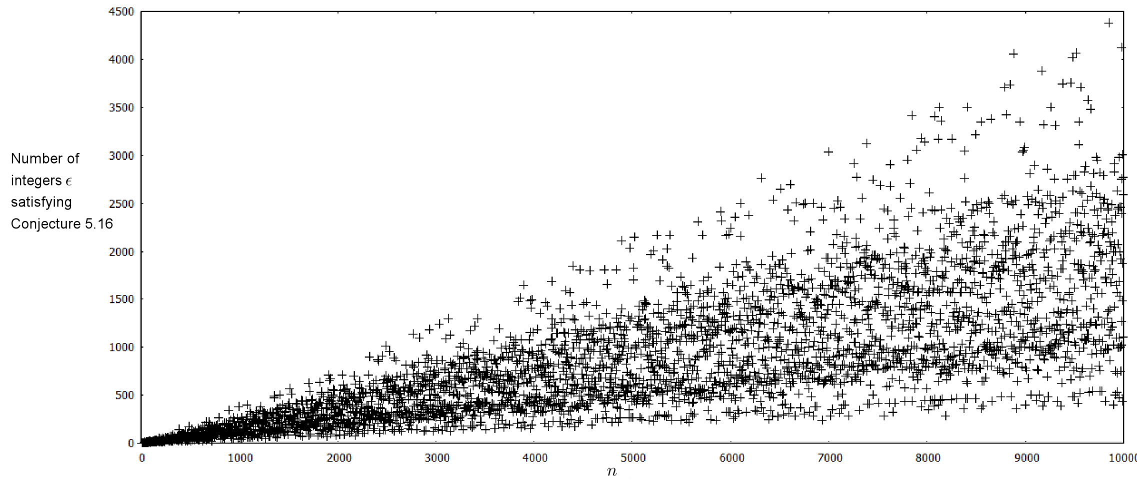

Figure 5.1: The number of integers ϵ satisfying Conjecture 5.16 for n up to 10000

Evidence supporting Conjecture 5.16

The main difficulty in proving Conjecture 5.16 involves the second condition. In general, it appears to be challenging to predict when the number $\epsilon + 1 + \binom{n-1}{\epsilon}$ is divisible by a prime factor of $n+1$ for a given value of ϵ. It is straight-forward to verify Conjecture 5.16 in a given dimension using a computer program. Doing so yields an overwhelming amount of numerical evidence that suggests that the conjecture is true in general.

Proposition 5.17. *Conjecture 5.16 is true for every pertinent* $n \leq 100000$*.*

Corollary 5.18. *For any complex dimension* $n \leq 100001$*, the cobordism class of a smooth projective toric variety can be chosen as the polynomial generator* α_n *of* $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$.

In fact, it appears that the number of integers ϵ that satisfy Conjecture 5.16 generally increases as n grows. In Figure 5.1, the number of ϵ satisfying the conditions of the conjecture is plotted against the dimension n for applicable n up to 10000. In order to verify the conjecture, only one such ϵ needs to exist for any given n. It seems likely that the trend in the graph would continue for larger n, making it doubtful that there exists some large complex dimension n for which there is no corresponding ϵ that satisfies the conjecture.

Since there are so many choices for ϵ as the dimension grows, it also seems reasonable to place additional convenient restrictions on ϵ in an attempt to simplify the conjecture. For example, given $n = p_1^{m_1} \cdots p_t^{m_t}$, we could try to find values for ϵ that are prime numbers such that $\max\{p_1, \ldots, p_t\} < \epsilon \leq n-1$. Such ϵ would automatically satisfy the first condition of Conjecture 5.16. Therefore, Conjecture 5.1 is true if the following number-theoretic conjecture holds.

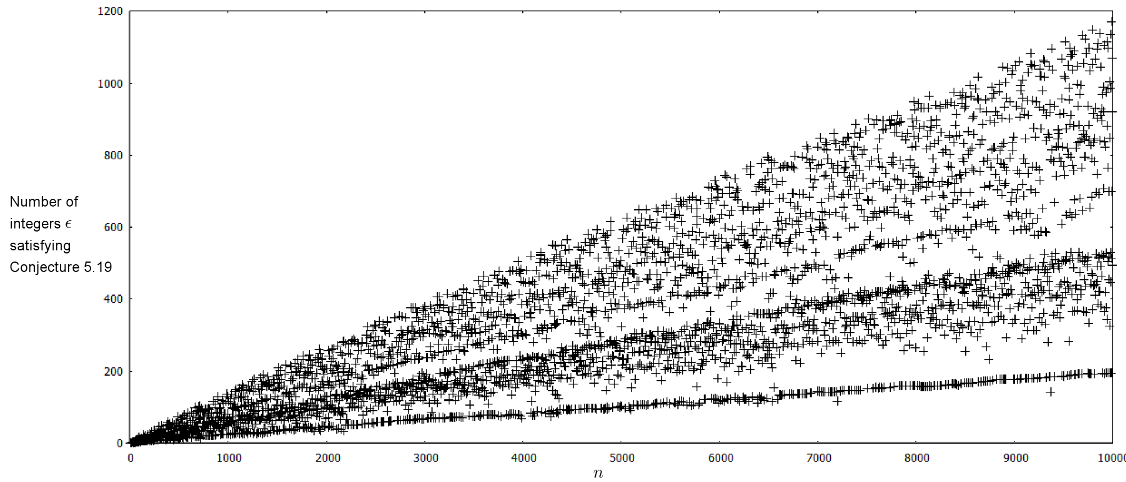

Figure 5.2: The number of integers ϵ satisfying Conjecture 5.19 for n up to 10000

Conjecture 5.19. *Assume n is a positive even integer such that $n+1 = p_1^{m_1} \cdots p_t^{m_t}$ is not a prime power. If $n \neq 20$ and $n \neq 50$, then there exists a prime ϵ in the interval $\max\{p_1, \ldots, p_t\} < \epsilon \leq n-1$ such that $\gcd\left(\epsilon + 1 + \binom{n-1}{\epsilon}, n+1\right) = 1$.*

Again, there is a significant amount of evidence supporting this conjecture. A simple program can compute examples of such ϵ for relatively small dimensions.

Proposition 5.20. *Conjecture 5.19 is true for every pertinent $n \leq 100000$.*

There are two complex dimensions, $n = 20 = 3 \cdot 7 - 1$ and $n = 50 = 3 \cdot 17 - 1$, for which no such prime ϵ exists. For $n = 20$, one can show that $\epsilon = 7$ is the only integer for which Conjecture 5.16 holds. For $n = 50$, the only suitable ϵ are $\epsilon \in \{21, 25, 27\}$, none of which is prime. Despite these low-dimensional anomalies, the number of ϵ that satisfy Conjecture 5.19 seems to grow as the dimension n increases. Figure 5.2 displays the number of ϵ satisfying Conjecture 5.19 for each pertinent dimension up to 10000. It seems unlikely for there to be a large n for which Conjecture 5.19 fails, assuming the trend in this graph continues.

Chapter 6 Vanishing Theorems in Oriented Cobordism

Although cohomology and characteristic classes of toric varieties are well-understood, these are typically quite difficult to compute in practice. Only in certain special situations can these calculations be carried out in great generality. One useful special case is the class of toric varieties called Bott manifolds (see [9] for details). While Bott manifolds are simple enough to facilitate topological computations, they are still varied enough to provide a useful testing ground for properties which may also hold for more general manifolds.

Definition 6.1. A *Bott tower of height* n is a collection of complex manifolds

$$\{N_k | 0 \leq k \leq n\}$$

which are constructed inductively as follows.

1. N_0 is a point.
2. For $1 \leq k \leq n$, $N_k = \mathbb{C}P\left(\mathbb{C} \oplus \xi_{k-1}\right)$ where ξ_{k-1} is a line bundle over the previous stage N_{k-1} of the Bott tower, and $\mathbb{C}$ is the trivial line bundle over N_{k-1}.

Each level of a Bott tower is called a *Bott manifold*.

It is not immediately obvious that Bott manifolds are indeed toric varieties. However, it can be shown that each Bott manifold of dimension n can be constructed as a toric variety from a polytope that is combinatorially equivalent to the hypercube I^n. The bundle structure of Bott manifolds makes their cohomology and characteristic classes easy to compute (again, see [9] for details).

Example 6.2. The simplest Bott tower is one in which $N_k = \mathbb{C}P\left(\mathbb{C} \oplus \mathbb{C}\right)$ for each k. In this situation, it is easy to see inductively that $N_k = \left(\mathbb{C}P^1\right)^k$. A more generic Bott tower can be thought of as a generalization of a product of complex projective lines.

Example 6.3. Now consider a Bott tower of height two. Since the only line bundle over a point is the trivial bundle, $N_1 = \mathbb{C}P\left(\mathbb{C} \oplus \mathbb{C}\right) = \mathbb{C}P^1$. The highest Bott manifold in the tower is $N_2 = \mathbb{C}P(\mathbb{C} \oplus \xi)$, where ξ is some line bundle over $\mathbb{C}P^1$. Since line bundles over $\mathbb{C}P^1$ are completely classified by their first Chern classes, this Bott tower is in fact completely determined by one integer $c_1\left(\xi\right) = a$. The manifold N_2 is often called a *Hirzebruch surface* and is denoted $\mathcal{H}_a$ (see [16, 11] for details). As a toric variety, the associated fan and polytope of $N_2 = \mathcal{H}_a$ are displayed in Figure 6.1.

Note that the fan of $\mathcal{H}_a$ has $4 = 2+2$ generating rays, so it is one of the toric varieties classified by Kleinschmidt in Theorem 3.24. By Kleinschmidt's construction, it is easy to see that $\mathcal{H}_a = X_2\left(a\right)$ if a is nonnegative. In general, the fans corresponding

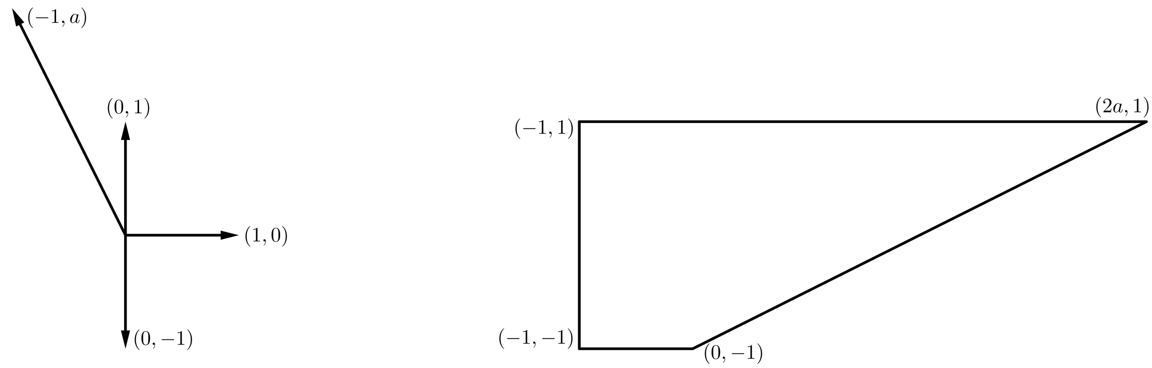

Figure 6.1: The fan of a Hirzebruch surface $\mathcal{H}_a$ and its associated polytope

to higher-dimensional Bott towers have too many generating rays to be one of the toric varieties classified by Kleinschmidt. However, both classes of toric varieties have similar projective bundle structures. While a Bott tower is a stack of $\mathbb{C}P^1$-bundles, each of the varieties classified by Kleinschmidt is a $\mathbb{C}P^i$-bundle over some $\mathbb{C}P^j$.

Bott towers and certain hypersurfaces within them provide further evidence supporting Stolz's conjecture involving the Witten genus (see [13] for details) and positive Ricci curvature (see [31] for details).

Conjecture 6.4. *([40]) Let M be a smooth closed string manifold with dimension divisible by four. If M admits a metric of positive Ricci curvature, then $\phi_W(M) = 0$, where ϕ_W is the Witten genus.*

In this chapter, Bott manifolds and string hypersurfaces within them will be shown to vanish in oriented cobordism. This implies that every multiplicative genus must be zero for these manifolds, so in particular their Witten genus is zero. Only a small portion of the structure present in Bott towers is needed to prove these results. In this situation, these toric varieties provide inspiration for vanishing theorems that hold for more generalized manifolds.

6.1 Vanishing of some projectivizations in oriented cobordism

Recall that an oriented manifold represents zero in Ω_*^{SO} if and only if all of its Stiefel-Whitney numbers and Pontrjagin numbers are zero (see Theorem 2.23). One can use the explicit computations of the Chern class and cohomology of Bott manifolds to prove that all of these manifolds represent zero in Ω_*^{SO}. This vanishing result can be extended to any manifold with a projective structure like the top projectivization of a Bott tower.

Proposition 6.5. *Let ξ be a two-dimensional complex vector bundle over a compact stably complex manifold N of dimension $2(n-1)$. Let $\mathbb{C}P(\xi) = (M, \pi, N)$ be its projectivization, so $\dim M = 2n$. Then $[M] = 0$ in Ω_{2n}^{SO}.*

This proposition will be proven after closer examination of the structure of this projectivization. By the splitting principle [24, Section 4.4], the induced bundle $\pi^*\xi = \lambda_1 \oplus \lambda_2$ splits into a sum of line bundles. Thus, we have the following diagram.

$$\begin{array}{ccc} E(\pi^*\xi) = E(\lambda_1 \oplus \lambda_2) & \longrightarrow & \mathbb{C}P(E(\xi)) = M \\ \downarrow & & \downarrow \pi \\ E(\xi) & \longrightarrow & N \end{array}$$

The tangent bundle of M splits as

$$TM = \pi^*TN \oplus T_F$$

where T_F is the bundle tangent to the fibers of $\mathbb{C}P(\xi)$. Then

$$c(M) = \pi^*c(N) \cdot (1 + \omega) \tag{6.1.1}$$

where $\omega = c_1(T_F)$. Note that $\omega \in H^2(M)$. However the bundle structure yields the following

Lemma 6.6. *Let ξ be a two-dimensional complex vector bundle over a compact, almost complex manifold N with real dimension $2(n-1)$. Let $\mathbb{C}P(\xi) = (M, \pi, N)$ be its projectivization, so $c(M) = \pi^*c(N) \cdot (1 + \omega)$ where $\omega = c_1(T_F)$. Then $\omega^2 \in \pi^*H^4(N)$.*

Proof. Set $l_1 = c_1(\lambda_1)$ and $l_2 = c_1(\lambda_2)$. Let $x = c_1(\eta^*)$, where η^* is dual to the tautological bundle over M. By [43], the bundle tangent to the fibers stably splits as

$$T_F \oplus \mathbb{C} = \eta^* \otimes \pi^*\xi.$$

Then $c(T_F) = c(\eta^* \otimes \pi^*\xi) = (1 + x + l_1)(1 + x + l_2)$ (compare to [36, Problem 7-C]). But T_F is a complex bundle of rank one, so $c_2(\eta^* \otimes \pi^*\xi) = 0$. That is,

$$x^2 + (l_1 + l_2)x + l_1 l_2 = 0. \tag{6.1.2}$$

Then $c(T_F) = 1 + 2x + l_1 + l_2 = 1 + \omega$, so $\omega = 2x + l_1 + l_2$. Then

$$\begin{aligned} \omega^2 &= 4x^2 + 4(l_1 + l_2)x + 2l_1 l_2 + l_1^2 + l_2^2 \\ &= -2l_1 l_2 + l_1^2 + l_2^2 \text{ by } (6.1.2) \\ &= (l_1 + l_2)^2 - 4l_1 l_2 \\ &= \pi^*c_1(\xi)^2 - 4\pi^*c_2(\xi) \end{aligned}$$

Then $\omega^2 \in \pi^*H^4(N)$. □

Several of the relations in the proof of this lemma will be useful later, so they will be summarized in the following

Corollary 6.7. *Let ξ be a two-dimensional complex vector bundle over a compact, almost complex manifold N with real dimension $2(n-1)$. Let $\mathbb{C}P(\xi) = (M, \pi, N)$ be its projectivization. Split $\pi^*\xi = \lambda_1 \oplus \lambda_2$ into a sum of line bundles. Set $l_1 = c_1(\lambda_1)$ and $l_2 = c_1(\lambda_2)$. Let $x = c_1(\eta^*)$, where η^* is dual to the tautological bundle over M. Then*

$$c(M) = \pi^* c(N) \cdot (1 + \omega)$$

where

$$\omega = 2x + l_1 + l_2.$$

Also,

$$x^2 + (l_1 + l_2)x + l_1 l_2 = 0.$$

Proof of Theorem 6.5. By Theorem 2.23, it suffices to prove that all of the Stiefel-Whitney numbers and Pontrjagin numbers of M^{2n} are zero. Let $I = i_1, \ldots, i_m$ be a partition of n and consider the cohomology class $\prod_{k=1}^{m} c_{i_k}(M) \in H^{2n}(M)$. By Corollary 6.7, $c_{i_k}(M) = \pi^* c_{i_k}(N) + \pi^* c_{i_k - 1}(N) \cdot \omega$. Since $\omega^2 \in \pi^* H^4(N)$ by Lemma 6.6,

$$\begin{aligned}\prod_{k=1}^{m} c_{i_k}(M) &= \prod_{k=1}^{m} \left(\pi^* c_{i_k}(N) + \pi^* c_{i_k - 1}(N) \cdot \omega\right) \\ &= \nu_1 + \omega \nu_2\end{aligned}$$

for some $\nu_1 \in \pi^* H^{2n}(N) = 0$ and $\nu_2 \in \pi^* H^{2(n-1)}(N)$. Then using Corollary 6.7,

$$\begin{aligned}\prod_{k=1}^{m} c_{i_k}(M) &= (2x + l_1 + l_2) \cdot \nu_2 \\ &= 2x\nu_2 + (l_1 + l_2)\nu_2 \\ &= 2x\nu_2 + c_1(\lambda_1 \oplus \lambda_2) \cdot \nu_2 \\ &= 2x\nu_2 + \pi^* c_1(\xi) \cdot \nu_2\end{aligned}$$

$$\prod_{k=1}^{m} c_{i_k}(M) = 2x\nu_2 \tag{6.1.3}$$

since $\pi^* c_1(\xi) \cdot \nu_2 \in \pi^* H^{2n}(N) = 0$. Since each of these cohomology classes is even, Theorem 2.20 implies that every Stiefel-Whitney number of M is zero.

If the dimension of M is not divisible by four, then all of its Pontrjagin numbers are zero by the definition of Pontrjagin classes. Now assume that the dimension $2n$ of M is divisible by four. Formally write the Chern class of N as $c(N) = \prod_{k=1}^{n-1}(1 + v_k)$. Then by Corollary 6.7,

$$c(M) = 1 + c_1(M) + c_2(M) + \ldots = (1 + \omega) \cdot \prod_{k=1}^{n-1}(1 + \pi^* v_k)$$

and

$$1 - c_1(M) + c_2(M) - \ldots = (1-\omega) \cdot \prod_{k=1}^{n-1} (1 - \pi^* v_k).$$

Then by (2.2.1),

$$1 - p_1(M) + p_2(M) - \ldots = \left(1-\omega^2\right) \cdot \prod_{k=1}^{n-1} \left(1 - \pi^* v_k^2\right),$$

so

$$p_k(M) = \sigma_k\left(\omega^2, \pi^* v_1^2, \ldots, \pi^* v_{n-1}^2\right),$$

where σ_k is the k^{th} elementary symmetric polynomial. In particular, for every k, $p_k(M) \in \pi^* H^{4k}(N)$ by Lemma 6.6. This means that every Pontrjagin number of M is obtained by evaluating a cohomology class in $H^{2n}(N) = 0$ on the fundamental class of M, so all of its Pontrjagin classes are zero. □

Theorem 6.5 can be considered in term of Stolz's Conjecture 6.4 as well.

Corollary 6.8. *Let ξ be a two-dimensional complex vector bundle over a compact, stably complex manifold N with dimension $4n-2$. Let $\mathbb{C}P(\xi) = (M, \pi, N)$ be its projectivization, so $\dim M = 4n$. If M is a string manifold that admits a metric of positive Ricci curvature, then the Witten genus of M is zero.*

Stolz has previously proven that his conjecture holds for total spaces of fiber bundles in which the structure group is a compact semi-simple Lie group [40, Theorem 3.1]. This corollary generalizes this class of manifolds by demonstrating that if the bundle is a real-dimension two projectivization, then these conditions on the structure group are not needed.

6.2 Cobordism vanishing theorem for hypersurfaces in projectivizations

A version of Theorem 6.5 also applies to certain hypersurfaces in the total space of projectivized bundles. However, several additional conditions must hold in order for the cobordism classes of these hypersurfaces to be zero. In particular, the hypersurface must lie in a manifold that is the top level of a stack of two projectivizations. The hypersurface must also be a *string* manifold, which places an extra condition on its characteristic numbers as described below.

Definition 6.9. An oriented manifold M is called a *spin* manifold if $w_2(M) = 0$ (refer to [31, Chapter II] for more details).

Now suppose that M is a spin manifold whose dimension is divisible by four. One can show that there is a unique cohomology class $\frac{1}{2}p_1(M)$ such that multiplying this class by two yields $p_1(M)$ (see [15, Chapter 1] for details). A spin manifold M is called a *string* manifold if $\frac{1}{2}p_1(M) = 0$ [13].

Theorem 6.10. *Let ξ_1 be a two-dimensional complex vector bundle over a compact, almost complex manifold P of dimension $2(n-2)$. Let $\mathbb{C}P(\xi_1) = (N, \pi_1, P)$ denote its projectivization, so N has dimension $2(n-1)$. Now let ξ_2 be a two-dimensional complex vector bundle over N. Projectivize ξ_2 to form the bundle $\mathbb{C}P(\xi_2) = (M, \pi_2, N)$, so M has dimension $2n$. If $V \subset M$ is a string hypersurface, then V represents zero in $\Omega^{SO}_{2(n-1)}$.*

The proof of Theorem 6.10 is quite technical, so it will be dealt with in several simpler cases. It is also important to have a thorough understanding of the cohomology structure of this construction, so this will be described in more detail before the proof is given.

By the splitting principle, $\pi_1^*\xi_1 = \lambda_{11} \oplus \lambda_{12}$ and $\pi_2^*\xi_2 = \lambda_{21} \oplus \lambda_{22}$ are sums of line bundles. The following diagram displays the structure that is present.

$$
\begin{array}{ccccc}
E(\pi_2^*\xi_2) = E(\lambda_{21} \oplus \lambda_{22}) & \longrightarrow & \mathbb{C}P(E(\xi_2)) = M & & \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} & & \\
E(\xi_2) & \longrightarrow & \mathbb{C}P(E(\xi_1)) = N & \longleftarrow & E(\pi_1^*\xi_1) = E(\lambda_{11} \oplus \lambda_{12}) \\
 & & \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow \\
 & & P & \longleftarrow & E(\xi_1)
\end{array}
$$

This bundle structure can be used to calculate the cohomology ring of M. Set $l_{ij} = c_1(\lambda_{ij})$ for $i, j \in \{1, 2\}$. Let η_1^* and η_2^* be dual to the tautological bundles over N and M, respectively. Let $x_1 = c_1(\eta_1^*)$ and $x_2 = c_1(\eta_2^*)$. By Corollary 6.7,

$$x_1^2 + (l_{11} + l_{12})x_1 + l_{11}l_{12} = 0 \tag{6.2.1}$$

and

$$x_2^2 + (l_{21} + l_{22})x_2 + l_{21}l_{22} = 0. \tag{6.2.2}$$

Then by the Leray-Hirsch Theorem,

$$\begin{aligned} H^*(M) &= \pi_2^*H^*(N) \oplus \pi_2^*H^*(N) \cdot x_2 \\ &= \pi_2^*\pi_1^*H^*(P) \oplus \pi_2^*\pi_1^*H^*(P) \cdot \pi_2^*x_1 \oplus \pi_2^*\pi_1^*H^*(P) \cdot x_2 \oplus \pi_2^*\pi_1^*H^*(P) \cdot \pi_2^*x_1 \cdot x_2 \end{aligned} \tag{6.2.3}$$

where the multiplicative structure is given by (6.2.1) and (6.2.2). In particular,

$$H^2(M) = \pi_2^*\pi_1^*H^2(P) \oplus \mathbb{Z}\pi_2^*x_1 \oplus \mathbb{Z}x_2.$$

Remark 6.11. To simplify notation, the maps π_1^* and π_2^* will be suppressed. This means for example that this cohomology group will be written as

$$H^2(M) = H^2(P) \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_2.$$

Now fix a cohomology class

$$z = u + a_1 x_1 + a_2 x_2 \in H^2(M) \tag{6.2.4}$$

where $u \in H^2(P)$ and $a_1, a_2 \in \mathbb{Z}$ are arbitrary. Let ν be the line bundle over M with $c_1(\nu) = z$. This determines a codimension two submanifold $V \overset{i}{\hookrightarrow} M$ whose tangent bundle TV satisfies $i^*TV \oplus \nu \cong TM$. This submanifold $V \subset M$ is called the *hypersurface dual to z* (see [15, Section 2.2] for details). In this situation, the Chern class of M splits as $c(M) = i^*c(V)(1+z)$. If the Chern class of P is formally written as $c(P) = \prod_{k=1}^{n-2}(1+v_k)$, then applying Corollary 6.7 twice yields

$$c(M) = (1+\omega_1)(1+\omega_2) \cdot \prod_{k=1}^{n-2}(1+v_k) \tag{6.2.5}$$

where

$$\omega_1 = 2x_1 + l_{11} + l_{12} \text{ and } \omega_2 = 2x_2 + l_{21} + l_{22}. \tag{6.2.6}$$

Remark 6.12. Applying Lemma 6.6 to each level of this projectivization shows that $\omega_2^2 \in H^4(N)$ and $\omega_1^2 \in H^4(P)$.

Knowing the formula for the Chern class of M make it possible to determine the Chern class and Pontrjagin class of V.

Proposition 6.13. *Suppose $V \overset{i}{\hookrightarrow} M$ is the hypersurface dual to $z = u + a_1x_1 + a_2x_2$ as described above. The inclusion i induces a map $i^* : H^*(M) \to H^*(V)$ on the level of cohomology. The k^{th} Chern class of V is given by*

$$c_k(V) = \sum_{j=0}^{k} (-1)^j \, i^*z^j \sigma_{k-j}\left(i^*\omega_1, i^*\omega_2, i^*v_1, \ldots, i^*v_{n-2}\right)$$

where σ_{k-j} is the $(k-j)^{th}$ elementary symmetric polynomial. The k^{th} Pontrjagin class of V is given by

$$p_k(V) = \sum_{j=0}^{k} (-1)^j \, i^*z^{2j} \sigma_{k-j}\left(i^*\omega_1^2, i^*\omega_2^2, i^*v_1^2, \ldots, i^*v_{n-2}^2\right).$$

Proof. The formula for the Chern class follows from $c(M) = c(V)(1+z)$ and (6.2.5). This splitting of the Chern class implies that the total Pontrjagin class of V is described by

$$p(V)\left(1+i^*z^2\right) = (1+i^*\omega_1^2)(1+i^*\omega_2^2) \cdot \prod_{k=1}^{n-2}(1+i^*v_k^2).$$

The formula for the Pontrjagin class is derived by expanding this. ∎

Remark 6.14. As with the maps π_1^* and π_2^*, i^* will be suppressed to make notation more manageable. This means that the k^{th} Chern class of V is given by

$$c_k(V) = \sum_{j=0}^{k} (-1)^j z^j \sigma_{k-j}(\omega_1, \omega_2, v_1, \ldots, v_{n-2}), \tag{6.2.7}$$

and its k^{th} Pontrjagin class will be written as

$$p_k(V) = \sum_{j=0}^{k} (-1)^j z^{2j} \sigma_{k-j}\left(\omega_1^2, \omega_2^2, v_1^2, \ldots, v_{n-2}^2\right). \tag{6.2.8}$$

If V is assumed to be a string manifold, then we obtain additional useful information about the cohomology class z^2.

Lemma 6.15. *Suppose $V \subset M$ is a string manifold dual to z as described above. Then $z^2 \in H^4(N)$.*

Proof. By (6.2.8), the first Pontrjagin class of V is given by

$$p_1(V) = \omega_1^2 + \omega_2^2 + v_1^2 + \ldots + v_{n-2}^2 - z^2.$$

Thus, if V is string, then

$$z^2 = \omega_1^2 + \omega_2^2 + v_1^2 + \ldots + v_{n-2}^2. \tag{6.2.9}$$

Then by Remark 6.12, $z^2 \in H^4(N)$. □

The string condition on V also places restrictions on the coefficients of the dual class $z = u + a_1x_1 + a_2x_2$. For example,

Lemma 6.16. *Suppose $V \subset M$ is a string manifold dual to $z = u + a_1x_1 + a_2x_2$ as described above. Then a_2 is even.*

Proof. Since V is string, it is also a spin manifold. By definition, this means that $c_1(V)$ must be even. By (6.2.7) and (6.2.6),

$$\begin{aligned} c_1(V) &= \omega_1 + \omega_2 + v_1 + \ldots + v_{n-2} - z \\ &= 2x_1 + l_{11} + l_{12} + 2x_2 + l_{21} + l_{22} + v_1 + \ldots + v_{n-2} - u - a_1x_1 - a_2x_2. \end{aligned}$$

In particular, the coefficient of x_2 in $c_1(V)$ is $2 - a_2$. Then in order for $c_1(V)$ to be even and V to be spin, a_2 must be even. □

Now we can begin to approach the proof of Theorem 6.10. In order to prove that the spin hypersurface V represents zero in Ω_*^{SO}, we must show that all of its Stiefel-Whitney numbers and Pontrjagin numbers are zero (see Theorem 2.23).

Proposition 6.17. *Suppose $V \subset M$ is a hypersurface dual to $z = u + a_1x_1 + a_2x_2$ with the bundle structure defined in 6.10. Then every Stiefel-Whitney number of V is zero.*

Proof. Consider a partition $I = i_1, \ldots, i_j$ of n. Using (6.2.7), the Chern number of V corresponding to I is

$$\begin{aligned}
\langle c_{i_1}(V)\cdots c_{i_t}(V), \mu_V\rangle &= \langle c_{i_1}(V)\cdots c_{i_t}(V)\cdot z, \mu_M\rangle \\
&= \left\langle z\prod_{k=1}^{t}\left(\sum_{j=0}^{i_k}(-1)^j z^j \sigma_{k-j}(\omega_1,\omega_2,v_1,\ldots,v_{n-2})\right), \mu_M\right\rangle.
\end{aligned}$$

Terms of degree n in the cohomology class that is being evaluated on μ_M have the form

$$\omega_1^{q_1}\omega_2^{q_2}v_1^{r_1}\cdots v_{n-2}^{r_{n-2}}z^s = \omega_1^{q_1}(2x_2+l_{21}+l_{22})^{q_2}v_1^{r_1}\cdots v_{n-2}^{r_{n-2}}(u+a_1x_1+a_2x_2)^s$$

where $q_1+q_2+r_1+\ldots+r_{n-2}+s=n$ (see (6.2.6)). After further expanding these terms, any term that lacks x_2 vanishes since it lies in $H^{2n}(N)=0$. The remaining nonzero terms containing x_2 must have even coefficients since a_2 is even by Lemma 6.16. Thus every Chern number of V is even, so every Stiefel-Whitney number is zero by Theorem 2.20. □

To prove Theorem 6.10, it only remains to demonstrate that all Pontrjagin numbers of the string hypersurface V vanish. This will be approached in several cases, depending on the nature of the dual cohomology class $z \in H^2(M)$.

Proposition 6.18. *Suppose $V \subset M$ is a hypersurface as in Theorem 6.10, where V is dual to $z = u + a_1x_1 + a_2x_2 \in H^2(M)$. If $a_2 = 0$, then $[V] = 0$ in $\Omega^{SO}_{2(n-1)}$.*

Proof. Consider a partition $I = i_1, \ldots, i_j$ of n. By (6.2.8), the Pontrjagin class corresponding to I is given by

$$\begin{aligned}
\langle p_{i_1}(V)\cdots p_{i_t}(V), \mu_V\rangle &= \langle p_{i_1}(V)\cdots p_{i_t}(V)\cdot z, \mu_M\rangle \\
&= \left\langle (u+a_1x_1)\prod_{k=1}^{t}\left(\sum_{j=0}^{i_k}(-1)^j z^{2j}\sigma_{k-j}\left(\omega_1^2,\omega_2^2,v_1^2,\ldots,v_{n-2}^2\right)\right), \mu_M\right\rangle
\end{aligned}$$

By Remark 6.12 and Lemma 6.15, the cohomology class that is being evaluated on μ_M in the last step belongs to $H^*(N)$. But this Pontrjagin number is found by evaluating the degree n terms of this cohomology class on μ_M, and $H^{2n}(N) = 0$. Thus this arbitrary Pontrjagin number is zero. □

Now assume that $V \subset M$ is dual to $z = u + a_1x_1 + a_2x_2$, where $a_2 \neq 0$. Since $z^2 \in H^4(N)$ by Lemma 6.15, it cannot include a term with x_2, since this class belongs to $H^*(M)\backslash H^*(N)$. But

$$\begin{aligned}
z^2 &= (u+a_1x_1+a_2x_2)^2 \\
&= a_2^2x_2^2 + 2a_2(u+a_1x_1)x_2 + (u+a_1x_1)^2 \\
&= a_2^2(-(l_{21}+l_{22})x_2 - l_{21}l_{22}) + 2a_2(u+a_1x_1)x_2 + (u+a_1x_1)^2 \quad \text{by (6.2.2)} \\
z^2 &= \left(2a_2u - a_2^2(l_{21}+l_{22}) + 2a_1a_2x_1\right)x_2 + (u+a_1x_1)^2 - a_2^2l_{21}l_{22}. \qquad (6.2.10)
\end{aligned}$$

This means that the term being multiplied by x_2 in this last expression must be zero. This can yield some useful information after the introduction of more notation.

Note that $l_{21} + l_{22} = c_1(\xi_2) \in H^2(N) = H^2(P) \oplus \mathbb{Z}x_1$ (by again suppressing the map π_2^*). Set

$$l_{21} + l_{22} = y_1 + bx_1 \tag{6.2.11}$$

where $y_1 \in H^2(P)$ and $b \in \mathbb{Z}$. Then (6.2.10) becomes

$$z^2 = \left(\left(2a_2u - a_2^2y_1\right) + \left(2a_1a_2 - a_2^2b\right)x_1\right)x_2 + \left(u + a_1x_1\right)^2 - a_2^2 l_{21}l_{22}. \tag{6.2.12}$$

Since the term in z^2 containing x_2 must be zero, this implies that $2a_2u - a_2^2y_1 = 0$ and $2a_1a_2 - a_2^2b = 0$. But $a_2 \neq 0$ by assumption, so

$$2u = a_2y_1 \text{ and } 2a_1 = a_2b. \tag{6.2.13}$$

By (6.2.9), the term in z^2 that contains x_1 must equal the term containing x_1 in $\omega_1^2 + \omega_2^2 + v_1^2 + \ldots + v_{n-2}^2$. By Remark 6.12, the only one of these squares which may contain a nonzero x_1 term is ω_2^2. Let C denote the coefficient of this term.

Proposition 6.19. *Suppose $V \subset M$ is a hypersurface as in Theorem 6.10, where V is dual to $z = u + a_1x_1 + a_2x_2$. If $a_2 \neq 0$ and $C = 0$, then $[V] = 0$ in $\Omega_{2(n-1)}^{SO}$.*

Proof. Again by Theorem 2.23 and Proposition 6.17, it suffices to prove that all Pontrjagin numbers of V vanish. By (6.2.8), The Pontrjagin number of V corresponding to I is

$$\begin{aligned}\langle p_{i_1}(V)\cdots p_{i_t}(V), \mu_V\rangle &= \langle p_{i_1}(V)\cdots p_{i_t}(V)\cdot z, \mu_M\rangle \\ &= \left\langle z\prod_{k=1}^{t}\left(\sum_{j=0}^{i_k}(-1)^j z^{2j}\sigma_{k-j}\left(\omega_1^2, \omega_2^2, v_1^2, \ldots, v_{n-2}^2\right)\right), \mu_M\right\rangle.\end{aligned}$$

Terms of degree n in the cohomology class that is being evaluated on μ_M have the form

$$\omega_1^{q_1}\omega_2^{q_2}v_1^{r_1}\cdots v_{n-2}^{r_{n-2}}z^{s-1}\cdot z$$

where $q_1 + q_2 + r_1 + \ldots + r_{n-2} + s = n$ and $q_1, q_2, r_1, \ldots, r_{n-2}, s-1$ are all even. But the coefficient C of x_1 in z^2 and ω_2^2 is zero by assumption. Thus by Lemma 6.15 and Remark 6.12, $z^{s-1}, \omega_2^{q_2} \in H^*(P)$. Since $\omega_1^{q_1}v_1^{r_1}\cdots v_{n-2}^{r_{n-2}} \in H^*(P)$ by Remark 6.12,

$$\omega_1^{q_1}\omega_2^{q_2}v_1^{r_1}\cdots v_{n-2}^{r_{n-2}}z^{s-1} \in H^{2(n-1)}(P) = 0.$$

Then each term of degree n in the above cohomology class is zero, so every Pontrjagin number of V is zero. □

Now assume the coefficient C of x_1 in z^2 and ω_2^2 is not zero. Since

$$l_{21}l_{22} = c_2(\xi_2) \in H^4(N) = H^4(P) \oplus H^2(P)\cdot x_1$$

(again suppressing the maps π_1^* and π_2^*), we can write

$$l_{21}l_{22} = y_{22} + y_{21}\cdot x_1 \tag{6.2.14}$$

for some $y_{22} \in H^4(P)$ and $y_{21} \in H^2(P)$. Since the coefficient of x_2 in z^2 is zero by Lemma 6.15, equation (6.2.12) becomes

$$\begin{aligned}
z^2 &= (u + a_1x_1)^2 - a_2^2 l_{21} l_{22} \\
&= a_1^2 x_1^2 + 2a_1 u x_1 + u^2 - a_2^2 y_{22} - a_2^2 y_{21} x_1 \\
&= a_1^2 (-(l_{11} + l_{12})x_1 - l_{11}l_{12}) + 2a_1 u x_1 + u^2 - a_2^2 y_{22} - a_2^2 y_{21} x_1 \text{ by (6.2.1)} \\
&= \left(2a_1 u - a_2^2 y_{21} - a_1^2 (l_{11} + l_{12})\right) x_1 + \text{terms lacking } x_1.
\end{aligned}$$

Then the coefficient C of x_1 in z^2 is

$$\begin{aligned}
C &= 2a_1 u - a_2^2 y_{21} - a_1^2 (l_{11} + l_{12}) \\
&= \frac{1}{2} \cdot 2u \cdot 2a_1 - a_2^2 y_{21} - \frac{1}{4} \cdot 2a_1 \cdot 2a_1 (l_{11} + l_{12}) \\
&= \frac{1}{2} a_2^2 b y_1 - a_2^2 y_{21} - \frac{1}{4} a_2^2 b^2 (l_{11} + l_{12}) \text{ by (6.2.13)} \\
C &= \frac{1}{4} a_2^2 \left(2by_1 - 4y_{21} - b^2 (l_{11} + l_{12})\right). \qquad (6.2.15)
\end{aligned}$$

But C is also the coefficient of the term containing x_1 in ω_2^2. Using (6.2.6),

$$\begin{aligned}
\omega_2^2 &= (2x_2 + l_{21} + l_{22})^2 \\
&= 4x_2^2 + 4x_2 (l_{21} + l_{22}) + (l_{21} + l_{22})^2 \\
&= 4\left(x_2^2 + x_2 (l_{21} + l_{22}) + l_{21} l_{22}\right) + (l_{21} + l_{22})^2 - 4l_{21} l_{22} \\
&= (l_{21} + l_{22})^2 - 4l_{21} l_{22} \text{ by (6.2.2)} \\
&= (y_1 + bx_1)^2 - 4(y_{22} + y_{21} x_1) \text{ by (6.2.11) and (6.2.14)} \\
&= y_1^2 + b^2 x_1^2 + 2by_1 x_1 - 4(y_{22} + y_{21} x_1) \\
&= y_1^2 + b^2 (-(l_{11} + l_{12})x_1 - l_{11} l_{12}) + 2by_1 x_1 - 4(y_{22} + y_{21} x_1) \text{ by (6.2.1)} \\
&= \left(2by_1 - 4y_{21} - b^2 (l_{11} + l_{12})\right) x_1 + \text{terms lacking } x_1.
\end{aligned}$$

Then $C = 2by_1 - 4y_{21} - b^2 (l_{11} + l_{12})$. Comparing this with (6.2.15) yields

$$C = \frac{1}{4} a_2^2 C.$$

Since $C \neq 0$ by assumption, this implies that $a_2 = \pm 2$.

Consider what happens to z as a result of this.

$$\begin{aligned}
z &= u + a_1 x_1 \pm 2x_2 \\
&= \frac{1}{2} (\pm 2) y_1 + \frac{1}{2} (\pm 2) bx_1 \pm 2x_2 \text{ by (6.2.13)} \\
&= \pm (y_1 + bx_1 + 2x_2) \\
&= \pm (l_{21} + l_{22} + 2x_2) \text{ by (6.2.11)} \\
z &= \pm \omega_2 \text{ by (6.2.6)} \qquad (6.2.16)
\end{aligned}$$

Proposition 6.20. *Suppose $V \subset M$ is a hypersurface as in Theorem 6.10, where V is dual to $z = u + a_1x_1 + a_2x_2$. If $a_2 \neq 0$ and $C \neq 0$, then $[V] = 0$ in $\Omega^{SO}_{2(n-1)}$.*

Proof. Consider a partition $I = \{i_1, \ldots, i_j\}$ of n. Using (6.2.8), the k^{th} Pontrjagin class of V is given by

$$\begin{aligned} p_k(V) &= \sum_{j=0}^{k} (-1)^j z^{2j} \sigma_{k-j}\left(\omega_1^2, \omega_2^2, v_1^2, \ldots, v_{n-2}^2\right) \\ &= \sum_{j=0}^{k} (-1)^j \omega_2^{2j} \sigma_{k-j}\left(\omega_1^2, \omega_2^2, v_1^2, \ldots, v_{n-2}^2\right) \text{ by (6.2.16).} \end{aligned}$$

Consider terms of degree $n-1$ in this cohomology class. By Remark 6.12, the only terms that do not automatically vanish for dimension reasons are those that contain ω_2^2. The terms that do contain ω_2^2 can be separated into those in which ω_2^2 appears in the monomial given by the symmetric polynomial, and those in which ω_2^2 does not appear there. That is,

$$\begin{aligned} p_k(V) &= \sum_{j=0}^{k-1} (-1)^j \omega_2^{2(j+1)} \sigma_{k-j-1}\left(\omega_1^2, v_1^2, \ldots, v_{n-2}^2\right) + \\ &\quad + \sum_{j=1}^{k} (-1)^j \omega_2^{2j} \sigma_{k-j}\left(\omega_1^2, v_1^2, \ldots, v_{n-2}^2\right) \\ &= -\sum_{j=1}^{k} (-1)^j \omega_2^{2j} \sigma_{k-j}\left(\omega_1^2, v_1^2, \ldots, v_{n-2}^2\right) + \sum_{j=1}^{k} (-1)^j \omega_2^{2j} \sigma_{k-j}\left(\omega_1^2, v_1^2, \ldots, v_{n-2}^2\right) \\ &= 0. \end{aligned}$$

For these string hypersurfaces satisfying $C \neq 0$ and $a_2 = \pm 2$, the entire Pontrjagin class vanishes. Thus all Pontrjagin numbers of the hypersurface are zero. Then $[V] = 0$ in $\Omega^{SO}_{2(n-1)}$ by Theorem 2.23. □

Now all possible string hypersurfaces in a stack of two projectivizations described in Theorem 6.10 have been considered. The results of Propositions 6.18, 6.19, 6.20 combine to prove Theorem 6.10, i.e. all such hypersurfaces are oriented cobordant to zero.

As in the simpler case involving one projectivization, this vanishing result on hypersurfaces gives additional evidence supporting Stolz's Conjecture 6.4.

Corollary 6.21. *Let ξ_1 be a two-dimensional complex vector bundle over a compact, almost complex manifold P of dimension $4k-2$. Let $\mathbb{C}P(\xi_1) = (N, \pi_1, P)$ denote its projectivization, so N has dimension $4k$. Now let ξ_2 be a two-dimensional complex vector bundle over N. Projectivize ξ_2 to form the bundle $\mathbb{C}P(\xi_2) = (M, \pi_2, N)$, so M has dimension $4k+2$. If $V \subset M$ is a string hypersurface which admits a metric of positive Ricci curvature, then the Witten genus of V is zero.*

This result can also be thought of as a generalization of cases for which Stolz's conjecture is known to be true. The conjecture holds for complete intersections in complex projective space [40] and also for complete intersections in products of complex projective spaces [15, 8]. The results of this section demonstrate that if the complete intersection has codimension two (i.e. it is a hypersurface), then a much weaker bundle structure suffices to satisfy Stolz's conjecture.

Chapter 7 Concluding Remarks

Each of Chapters 4, 5, and 6 displays a different problem relating toric varieties and cobordism. In the past, little has been studied regarding the interaction of toric varieties and cobordism. As a consequence, there are numerous opportunities to expand the results in each chapter and also to study other related questions.

In Chapter 4, the combinatorial structure of toric varieties is used to compute their χ_y-genus. This provides some information about when a cobordism class *cannot* be represented by a smooth projective toric variety. Unfortunately, this combinatorial structure only reveals a small proportion of the total information encoded in a cobordism class. Perhaps this means that the χ_y-genus is not the best cobordism invariant to use in this situation. Is there a different invariant that encodes more of the combinatorial and geometric information of a toric variety? This could in turn reveal more information about when they represent a given cobordism class.

Studying the χ_y-genus alone allows us to analyze thoroughly the presence of toric varieties in complex cobordism classes up to complex dimension three, and it also provides partial results in dimension four. Further results in dimension four may be obtained through a better understanding of smooth four-dimensional polytopes. If more of these polytopes were classified, then one may be able to directly compute their corresponding cobordism classes, as was the case with the varieties classified by Kleinschmidt.

As for Chapter 5, it is of course desirable to complete the proof of Conjecture 5.1, which asserts that the cobordism class of a smooth projective toric variety can be chosen for each polynomial generator α_n of $\Omega_*^U \cong \mathbb{Z}[\alpha_1, \alpha_2, \ldots]$. One way of accomplishing this would be to prove the sufficient number theory Conjecture 5.16. This conjecture involves considering blow-ups of a very special class of varieties that were classified by Kleinschmidt. There could also be many other smooth projective toric varieties that may be chosen for generators. Another approach to proving Conjecture 5.1 would be to find other smooth projective toric varieties that are complicated enough to produce a wide range of Milnor numbers, yet are simple enough to still allow these Milnor numbers to be calculated.

Theorems 5.10 and 5.14 prove that the generators α_n for n odd or n one less than a prime power can be chosen to be represented by smooth projective toric varieties. Unfortunately, this construction relies on applying an unspecified number of blow-ups to an initial toric variety. Thus there is no explicit description of each toric variety generator. However, it seems likely that there are in fact many distinct smooth projective toric varieties that can be chosen for each α_n (see the evidence in Section 5.3 for example). It may be worthwhile to seek explicit examples of smooth projective toric varieties that can be chosen for each generator. Is it possible to find a clear universal description of such toric variety generators? Being able to describe the cobordism ring so explicitly could be helpful in other unrelated cobordism calculations since toric varieties easily lend themselves to computational methods.

Chapter 6 provides several specific vanishing theorems for cobordism classes of manifolds in Ω_*^{SO} with certain fiber bundle structures. These results are proven by generalizing certain computations on Bott towers, which are toric varieties with a similar bundle structure. Since the manifolds described in Theorems 6.5 and 6.10 vanish in oriented cobordism, they provide additional examples that support Stolz's Conjecture 6.4.

Overall, Stolz's conjecture has only been verified for a very limited number of special cases. The proofs for all manifolds for which the conjecture is known to hold do not use positive Ricci curvature. That is, Stolz's conjecture has only been verified for classes of manifolds for which every manifold in the class automatically has positive Ricci curvature [40, 15]. One of these classes is complete intersections in products of projective spaces [15, 8]. These products of projective spaces are examples of smooth toric varieties, so it is reasonable to wonder if Stolz's conjecture can be generalized to complete intersections in toric varieties or to the toric varieties themselves. The convenient structure of toric varieties may help to facilitate computations in this case.

Toric varieties may be particularly interesting to study in terms of this conjecture since not all of them have positive Ricci curvature. Fortunately, positive Ricci curvature of toric varieties is already partially understood. In general a manifold has positive Ricci curvature if and only if it is a *Fano manifold* (see [3, Chapter 7] for details). The smooth Fano toric varieties are completely characterized by their associated polytopes. More specifically,

Proposition. *([11, Section 8.3]) Consider a lattice polytope which contains the origin in its interior. Suppose the vertices corresponding to each of its facets forms an integer basis. Then the dual polytope corresponds to a smooth Fano toric variety, and every smooth Fano toric variety can be constructed in this manner.*

Unfortunately, this condition is not always easy to work with in practice, and there is not yet a complete classification of the corresponding *Fano polytopes.* Further examination of this property could be applied to studying Stolz's conjecture for toric varieties and complete intersections within them. Exploring this problem may help to reveal the role that the positive Ricci curvature condition plays in this conjecture.

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