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Polyhedral Problems in Combinatorial Convex Geometry

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Dr. Peter Perry, Director of Graduate Studies

POLYHEDRAL PROBLEMS IN COMBINATORIAL CONVEX GEOMETRY

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Liam Solus
Lexington, Kentucky

Co-Directors: Dr. Benjamin Braun, Professor of Mathematics,
and Dr. Carl Lee, Professor of Mathematics
Lexington, Kentucky 2015

ABSTRACT OF DISSERTATION

POLYHEDRAL PROBLEMS IN COMBINATORIAL CONVEX GEOMETRY

In this dissertation, we exhibit two instances of polyhedra in combinatorial convex geometry. The first instance arises in the context of Ehrhart theory, and the polyhedra are the central objects of study. The second instance arises in algebraic statistics, and the polyhedra act as a conduit through which we study a nonpolyhedral problem.

In the first case, we examine combinatorial and algebraic properties of the Ehrhart h*-polynomial of the r-stable (n,k)-hypersimplices. These are a family of polytopes which form a nested chain of subpolytopes within the (n,k)-hypersimplex. We show that a well-studied unimodular triangulation of the (n,k)-hypersimplex restricts to a triangulation of each r-stable (n,k)-hypersimplex within. We then use this triangulation to compute the facet-defining inequalities of these polytopes. In the k=2 case, we use shelling techniques to devise a combinatorial interpretation of the coefficients of the h*-polynomials in terms of independent sets of certain graphs. From this, we then extract some results on unimodality. We also characterize the Gorenstein r-stable (n,k)-hypersimplices, and we conclude that these also have unimodal h*-polynomials.

In the second case, for a graph G on p vertices we consider the closure of the cone of concentration matrices of G. The extreme rays of this cone, and their associated ranks, have applications in maximum likelihood estimation for the undirected Gaussian graphical model associated to G. Consequently, the extreme ranks of this cone have been well-studied. Yet, there are few graph classes for which all the possible extreme ranks are known. We show that the facet-normals of the cut polytope of G can serve to identify extreme rays of this nonpolyhedral cone. We see that for graphs without K_5 minors each facet-normal of the cut polytope identifies an extreme ray in the cone, and we determine the rank of this extreme ray. When the graph is also series-parallel, we find that all possible extreme ranks arise in this fashion, thereby extending the collection of graph classes for which all the possible extreme ranks are known.

KEYWORDS: r-stable hypersimplices, hypersimplices, cut polytope, graphical models, facet-ray identification.

Author's signature: Liam Solus

Date: December 3, 2015

POLYHEDRAL PROBLEMS IN COMBINATORIAL CONVEX GEOMETRY

By
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Co-Director of Dissertation: Benjamin Braun

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Date: December 3, 2015

For Mom, Dad, and Dan.

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Chapter 1 Introduction

The notion of convexity will almost always be rediscovered by any curious student. A set is defined to be convex if for any two points in the set, every point on the straight line between these two points is also in the set. Intuitively put, a set is convex if the shortest path between any two points in your set will never leave your set. This definition roots itself in the question of accessibility of one location from another, and as such, presents itself naturally to even the earliest students of the universe. Following millennia of work, this intuitive concept of convexity has been shown to be rich with applications. Even now some of the most important computational questions of our time can be solved quickly and efficiently due to their convex structure.

While the notion of convexity is perhaps an intuitive one to discover, the invested student will learn that seemingly simple questions about convex sets can be surprisingly difficult to solve. An ancient and effective strategy for dealing with such problems is to identify combinatorial objects whose structure encode interesting data about the convex sets of interest. One then works to solve the problem using the data provided by these combinatorial objects. A combinatorial object is most simply described as an object defined by placing some structure on a finite set. Among the fundamental examples of combinatorial objects are graphs and permutations. For graphs, our finite set is the set of vertices and the structure placed upon the set is the relationship between vertices specified by the edges. Permutations are combinatorial objects defined on the finite set of numbers $1, 2, \ldots, n$ with the added structure being the order in which we list these numbers. Combinatorial convex geometry studies when a convex set admits a natural association with a combinatorial object and what data about the convex set can be extracted from these objects.

Some of the most fundamental convex sets studied in combinatorial convex geometry are a generalization of the convex polygons called polytopes. Polytopes are defined in terms of their finitely many vertices or facets. Thus, their geometry is inherently tied to a finite set, and we can have high hopes that they yield nice associations to interesting combinatorial objects.

The strategy of studying a convex set by studying the properties of its associated combinatorial structures has proven to be quite beneficial to the general study of convexity and its applications. Indeed, some of the most informative and enlightening examples within convex geometry are described in terms of their associated combinatorics. Simultaneously, problems in applied fields, such as convex optimization and statistics, can be solved by examining the combinatorics of various convex sets. Thus, important results in combinatorial convex geometry may come in the form of a solution to a problem in convex geometry or as an informative collection of examples. This dissertation aims to make contributions that fit both these important profiles.

Since polytopes are themselves objects defined by imposing structure on a finite set, namely their set of vertices or faces, they can be viewed both as convex sets as well as combinatorial objects. As such, they are doubly important in combinatorial convex geometry since they can be both the object of study as well as the combinatorial object that encodes data about another convex set. Thus, the goal of this dissertation is two-fold. In the coming chapters we will define a family of polytopes that serve as an interesting family of examples for problems in Ehrhart theory, a field of mathematics devoted to the study of enumerating the number of integer-points contained in a polytope. Ehrhart theory has connections to commutative algebra, and we will examine how this family of polytopes offers interesting examples in regards to fundamental questions about these connections. On the other hand, we will also work towards a polyhedral and combinatorial solution of the statistical problem of maximum likelihood estimation for Gaussian graphical models. Classic results for Gaussian graphical models tell us that the maximum likelihood estimation problem is a convex optimization problem with natural combinatorial interpretations.

This dissertation is organized as follows. In the remainder of this introduction, we will introduce to the basic concepts to be studied, namely polyhedra, spectrahedra, Ehrhart theory, and Gaussian graphical models. In chapter 2 we will then define and study the combinatorial geometry of the r-stable (n, k)-hypersimplices, a family of polytopes that generalize and refine the (n, k)-hypersimplices. In chapters 3 and 4 we will study the Ehrhart theory of these polytopes via their combinatorial geometry. Finally, in chapter 5, we will define the facet-ray identification property, a procedure for using a polytope to study the nonpolyhedral convex geometry of the space of canonical parameters of a Gaussian graphical model.

1.1 Polyhedra

A *polyhedron* $\mathcal{P}$ in $\mathbb{R}^d$ is the solution set to a finite collection of linear inequalities.

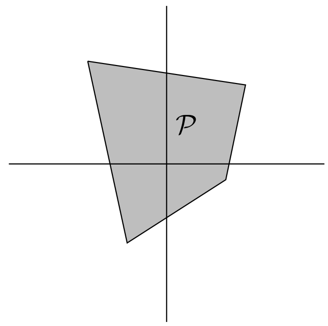
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d &\leq b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d &\leq b_n. \end{aligned}$$

To be concise, we write

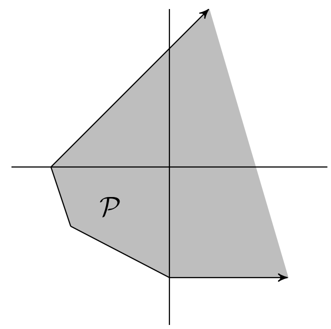
$$\mathcal{P} := \{x \in \mathbb{R}^d : Ax \leq b\},$$

where A is the $n \times d$ matrix with entries a_{ij} and $b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n$. We often write $x \in \mathbb{R}^d$ with the understanding that $x = (x_1, x_2, \ldots, x_d)^T$. Recall that a set $C \subset \mathbb{R}^d$ is *convex* if for every $x, y \in A$ and $0 \leq t \leq 1$ we have that $tx + (1-t)y \in A$. It is quick to check that polyhedra are convex sets. A polyhedron is always a closed set, however, it need not be bounded. When a polyhedron $\mathcal{P}$ in $\mathbb{R}^d$ is bounded we call it a *polytope*.

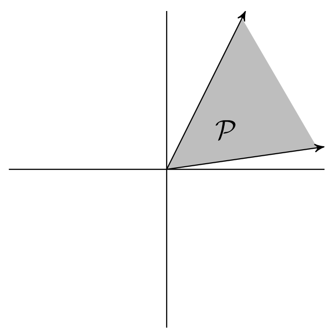
An important collection of unbounded polyhedra are the polyhedral cones. A subset $\mathcal{K} \in \mathbb{R}^d$ is called a *cone* if $0 \in \mathcal{K}$ and for every element x in $\mathcal{K}$ and every nonnegative real number $\lambda \geq 0$ we have that λx is also in $\mathcal{K}$. A *convex cone* is a cone $\mathcal{K}$ that is also a convex set. Equivalently, a set $\mathcal{K} \in \mathbb{R}^d$ is a convex cone if for every $x, y \in \mathcal{K}$ and $\lambda, \gamma \geq 0$ we have that $\lambda x + \gamma y \in \mathcal{K}$. If a convex cone $\mathcal{K} \in \mathbb{R}^d$ is also a polyhedron then we call $\mathcal{K}$ a *polyhedral cone*. Polyhedral cones are unbounded polyhedra, but not all unbounded polyhedra are polyhedral cones.



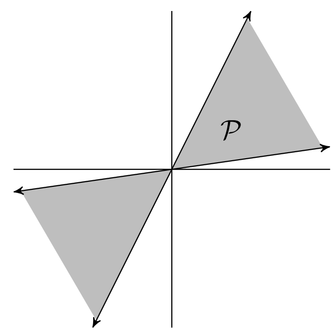
A polytope.



An unbounded polyhedron.



A convex cone.



A nonconvex cone.

Provided with a set of points $A \subset \mathbb{R}^d$ there is a natural way to produce a convex set containing A. The *convex hull of* A is the smallest convex set that contains the set A, and it is denoted $\mathrm{conv}(A)$. Similarly, we can consider the cone over the set A, which we call the *conic hull of* A:

$$\mathrm{co}(A) := \{\lambda x : x \in A, \lambda \geq 0\}.$$

The conic hull of a set A yields a natural way to construct a cone associated to a polytope. Given a polytope $\mathcal{P} \in \mathbb{R}^d$ we first lift $\mathcal{P}$ to height one in $\mathbb{R}^{d+1}$

$$\mathcal{P}^1 := \{(x, 1) \in \mathbb{R}^{d+1} : x \in \mathcal{P}\}.$$

Here, $(x, 1) := (x_1, x_2, \ldots, x_d, 1) \in \mathbb{R}^{d+1}$. We then define the *cone over* $\mathcal{P}$ to be the polyhedral cone

$$\mathrm{cone}(\mathcal{P}) := \mathrm{co}(\mathcal{P}^1).$$

Notice that we can recover $\mathcal{P}$ from $\mathrm{cone}(\mathcal{P})$ as the intersection of $\mathrm{cone}(\mathcal{P})$ with the hyperplane defined by the linear equation $x_{d+1} = 1$. The relationship between $\mathcal{P}$ and $\mathrm{cone}(\mathcal{P})$ generalizes nicely. A set $B \subset \mathbb{R}^d$ is called a *base* of a cone $\mathcal{K} \subset \mathbb{R}^d$ if $0 \notin B$ and every $x \in \mathcal{K}$ has a unique representation as $x = \lambda b$ for some $\lambda \geq 0$ and $b \in B$.

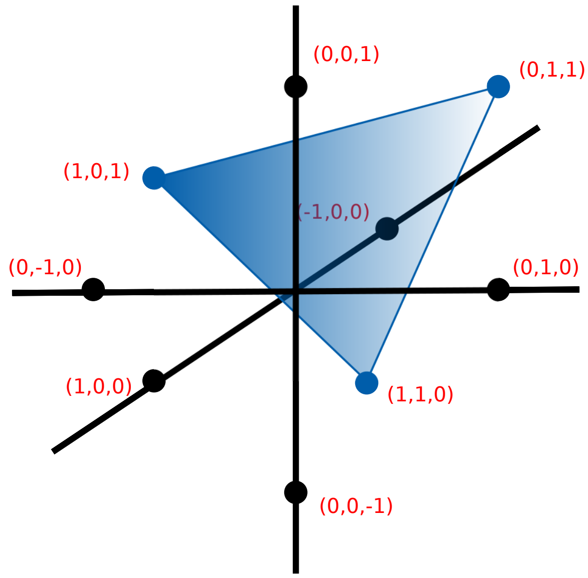


Figure 1.1: The $(3,2)$-hypersimplex is two-dimensional.

The Dimension of a Polyhedron

From our examples so far, it is easy to see that a polyhedron, or more generally, any convex set should have an associated dimension. Intuitively, we would like this dimension to be the smallest dimension of any ambient space containing our set. For example, the polytope in Figure 1.1 is two-dimensional, even though we draw it in dimension three. Recall that a set $C \subset \mathbb{R}^d$ is *convex* if for every $x, y \in A$ and $0 \leq t \leq 1$ we have that $tx + (1-t)y \in A$. The expression $tx + (1-t)y$ is an example of a convex combination of points. More generally, a *convex combination* of $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \mathbb{R}^d$ is an expression of the form

$$\lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_n x^{(n)}$$

where $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$ for all $0 \leq i \leq 1$. If we drop the requirement that $\lambda_i \geq 0$ then we call this expression an *affine combination* of $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$. The *affine hull* of $A \subset \mathbb{R}^d$ is the collection of points in $\mathbb{R}^d$ that can be expressed as an affine combination of finitely many points in A. That is, the *affine hull of* A is the set

$$\operatorname{aff}(A) := \left\{ \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_n x^{(n)} : x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in A, \sum_{j=1}^{n} \lambda_j = 1 \right\}.$$

The resulting set $\operatorname{aff}(A)$ is an *affine subspace* of $\mathbb{R}^d$, meaning that it is a translation of some linear subspace $\mathcal{L}$ of $\mathbb{R}^d$. The *dimension* of a convex set $C \subset \mathbb{R}^d$ is the dimension of this linear subspace $\mathcal{L}$.

The Faces of a Polyhedron

Recall, that an affine hyperplane in $\mathbb{R}^d$ is a set of the form

$$H = \{x \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\}$$

where $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$. Any hyperplane H defines two *closed halfspaces of* $\mathbb{R}^d$, and these are the sets

$$H^- = \{x \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b\}, \text{ and}$$
$$H^+ = \{x \in \mathbb{R}^d : a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b\}.$$

A *face* of a convex set $C \subset \mathbb{R}^d$ is a subset F of C with the property that for every $x, y \in C$ and $0 < t < 1$, $tx + (1-t)y \in F$ implies that $x, y \in F$. In other words, each open line segment in C that intersects F has the property that its closure lies in F. Notice that each face of a closed convex set is a closed convex set itself. A face F of a polyhedron $\mathcal{P}$ is always *exposed*, meaning there is some affine hyperplane H in $\mathbb{R}^d$ for which $F = H \cap C$ and C lies entirely in one of the closed halfspaces defined by H. In particular, each face of a polyhedron is a polyhedron. It is natural to categorize the faces of a convex set by dimension. For a d-dimensional polytope $\mathcal{P} \in \mathbb{R}^d$, the *vertices*, *edges*, and *facets* are the 0-dimensional, 1-dimensional, and $(d-1)$-dimensional faces, respectively. Notice that if we consider $\mathcal{P} \subset \mathbb{R}^{d+1}$, we can also think of $\mathcal{P}$ being a d-dimensional face of itself. We also consider the empty set to be a (-1)-dimensional face of $\mathcal{P}$.

A d-dimensional polyhedron $\mathcal{P} \subset \mathbb{R}^d$ has the wonderful property that it can be described in terms of either its highest dimensional faces or its lowest dimensional (nonempty) faces. To understand these two types of descriptions we first examine the case of polytopes. For a polytope $\mathcal{P} \subset \mathbb{R}^d$, the former description is commonly called an H-representation of $\mathcal{P}$, and the latter description is known as a V-representation of $\mathcal{P}$. An *H-representation* of $\mathcal{P}$ is a description of $\mathcal{P}$ as the intersection of finitely many closed halfspaces. Thus, the definition of a polytope provided at the start of this section is precisely an H-representation.

A V-representation is an alternate definition of a polytope that is given in terms of its set of vertices. Given a set $A \subset \mathbb{R}^d$ a point $x \in A$ is called an *extreme point* of A provided that for any two points $a, b \in A$ such that $x = \frac{a+b}{2}$ we have that $x = a = b$. For a polytope $\mathcal{P} \subset \mathbb{R}^d$, the extreme points of $\mathcal{P}$ are exactly its vertices. This observation becomes quite powerful in light of the following special case of a theorem due to M.G. Krein and D.P. Milman.

Theorem 1.1.1 (The Krein-Milman Theorem)**.** *Let $C \subset \mathbb{R}^d$ be a closed and bounded convex set. Then C is the convex hull of its set of extreme points.*

Thus, a polytope is exactly the convex hull of its set of vertices, and so we define a *V-representation* of $\mathcal{P}$ to be

$$\mathcal{P} = \mathrm{conv}(V)$$

where V denotes the set of vertices of $\mathcal{P}$.

The H and V-representations of polytopes serve to provide representations of polyhedral cones. Indeed, a polyhedral cone is also defined as the intersection of finitely many closed half-spaces. However, we can also describe a polyhedral cone in terms of its one dimensional faces. Let $\mathcal{K} \subset \mathbb{R}^d$ be a cone. Given a nonzero point $x \in \mathbb{R}^d$ the cone $\mathrm{co}(x)$ is called the *ray* spanned by x. A ray R contained in $\mathcal{K}$ is called an *extreme ray* of $\mathcal{K}$ if for every $x \in R$ and $a, b \in \mathcal{K}$ such that $x = \frac{a+b}{2}$ we have that $a, b \in \mathcal{K}$. The following proposition provides the intuitive connection between the extreme rays of a convex cone $\mathcal{K}$ and the extreme points of any base of $\mathcal{K}$.

Proposition 1.1.2. *Let $\mathcal{K} \subset \mathbb{R}^d$ be a convex cone with base B. A nonzero point $x \in \mathcal{K}$ spans an extreme ray of $\mathcal{K}$ if and only if $x = \lambda b$ for some $\lambda \geq 0$ and extreme point b of B.*

If $\mathcal{P}$ is a polyhedral cone with a closed and bounded base then this base is a polytope. Thus, it follows from the Krein-Milman Theorem that $\mathcal{P}$ is the smallest cone containing its extreme rays. That is, the cone $\mathcal{P}$ is completely described by its one-dimensional faces.

Subdivisions of Polyhedra

A useful technique for examining the geometry of a polyhedron is to decompose it into simpler polyhedra and study these pieces and how they fit together. A *polyhedral subdivision* of a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ is a collection of polyhedra $\mathcal{S} = \{Q_i\}_{i \in \Lambda}$ such that

(i) If $Q \in \mathcal{S}$ and F is a face of Q then $F \in \mathcal{S}$,

(ii) If $Q_1, Q_2 \in \mathcal{S}$ then $Q_1 \cap Q_2 \in \mathcal{S}$,

(iii) $\mathcal{P} = \bigcup_{i \in \Lambda} Q_i$.

Polyhedra admit numerous subdivisions, but some are certainly more useful than others. A common practice is to consider subdivisions of $\mathcal{P}$ consisting of relatively simple polyhedra that are easier to study.

Among the simplest polyhedra are the simplices. A set of points $v_0, v_1, \ldots, v_d \in \mathbb{R}^d$ are called *affinely independent* if the set of vectors $a_1 - a_0, a_2 - a_0, \ldots, a_d - a_0$ are linearly independent in $\mathbb{R}^d$. A *d-simplex* is the convex hull of $d+1$ *affinely independent* points. A 2-simplex is a triangle, and a 3-simplex is a tetrahedron. So a d-simplex is nothing more than d-dimensional generalization of a triangle. As such, the d-simplex is, in the sense of number of vertices or facets, the "simplest" d-dimensional polytope. Analogously, the "simplest" d-dimensional cones are the *simplicial cones.* A d-dimensional *simplicial cone* $\mathcal{P}$ is the conic hull of any $(d-1)$-simplex $\sigma \in \mathbb{R}^d$, i.e.

$$\mathcal{P} = \mathrm{co}(\sigma).$$

A *triangulation* of a polytope $\mathcal{P} \in \mathbb{R}^d$ is a polyhedral subdivision $\mathcal{S}$ of $\mathcal{P}$ consisting entirely of simplices. A *triangulation* of a polyhedral cone $\mathcal{P} \subset \mathbb{R}^d$ is a polyhedral subdivision $\mathcal{S}$ of $\mathcal{P}$ consisting of simplicial cones.

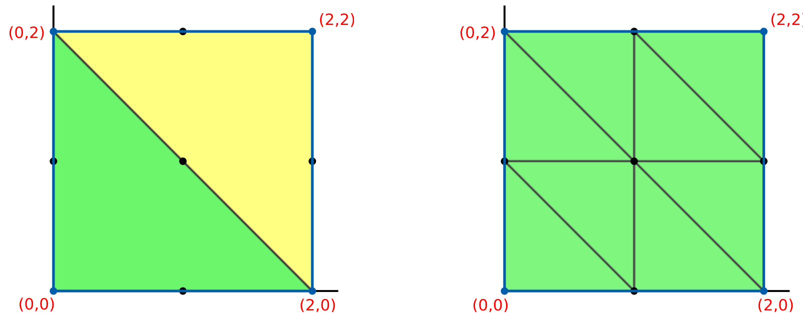
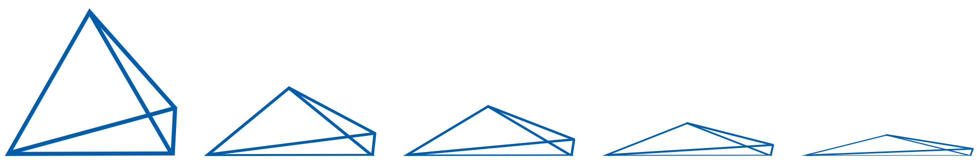


Figure 1.2: Two triangulations of the $2-\text{by}-2$ square $[0,2]^2$.

A d-simplex $\sigma \subset \mathbb{R}^d$ has the least number of vertices possible for a d-dimensional polytope, and in this sense is relatively simple to understand. However, the other geometric and combinatorial properties of σ could still be quite chaotic. For instance, we could choose the vertices of a simplex such that it has arbitrarily small volume.



This particular problem can be surpassed by appropriately restricting the choice of vertices for σ; the result will be an extraordinarily useful family of simplices.

A polytope $\mathcal{P} \subset \mathbb{R}^d$ is called a *lattice* or *integral* polytope if all of its vertices are points in $\mathbb{Z}^d$. A *unimodular d-simplex* is a lattice simplex with smallest possible volume. A polyhedral subdivision of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ is called a *lattice subdivision* if it consists entirely of lattice polytopes. A *unimodular triangulation* of $\mathcal{P}$ is a lattice subdivision of $\mathcal{P}$ consisting entirely of unimodular simplices. Since it consists only of simplices, we denote a unimodular triangulation of $\mathcal{P}$ by ∇. Figure 1.2 depicts a triangulation of the $2-\text{by}-2$ square first with nonunimodular simplices and then with unimodular simplices.

A unimodular triangulation ∇ of a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ can be useful for studying geometric, combinatorial, and algebraic properties of $\mathcal{P}$. Many such properties can be revealed if we are able to "build" $\mathcal{P}$ from the simplices in ∇ as if it were a puzzle. Given a triangulation ∇ of a d-dimensional polytope $\mathcal{P}$ let $\max\nabla$ denote the set of d-dimensional simplicies in ∇. We call an ordering of the simplices in $\max\nabla$, $(\sigma_1,\ldots,\sigma_s)$, a *shelling* of ∇ if for each $2 \leq i \leq s$, $\sigma_i \cap (\sigma_1 \cup \cdots \cup \sigma_{i-1})$ is a union of facets of σ_i. An equivalent condition for a shelling is that every σ_i has a unique minimal (with respect to dimension) face that is not a face of the previous simplices [46]. A triangulation with a shelling is called *shellable*. For a shelling and a maximal simplex σ in the triangulation define the *shelling number* of σ, denoted $\#(\sigma)$, to be the number of facets shared by σ and some previous simplex. Notice that $\#(\sigma)$ is also the dimension of the unique minimal new face of σ. In Figure 1.3 we order the simplices of our unimodular triangulation of the $2-\text{by}-2$ square from Figure 1.2 in two ways. The first way is not a shelling and the second way is a shelling.

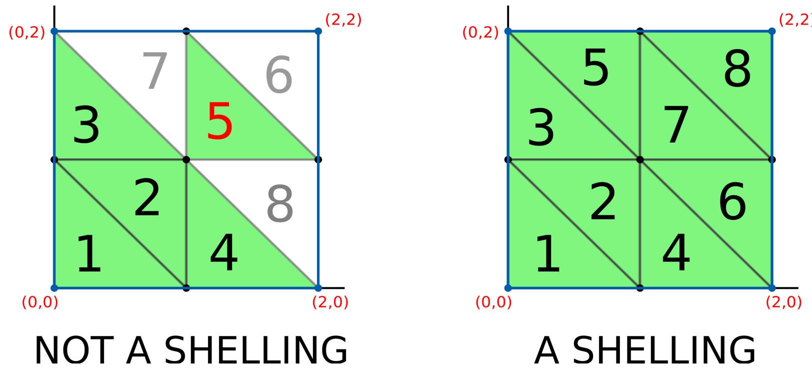


Figure 1.3: a non-shelling and a shelling of a triangulation of $[0,2]^2$.

1.2 Ehrhart Theory for Lattice Polytopes

In this section, we let $\mathcal{P} \subset \mathbb{R}^d$ be a d-dimensional lattice polytope. For a positive integer $t \in \mathbb{Z}_{>0}$ we let $t\mathcal{P} := \{tx \in \mathbb{R}^d : x \in \mathcal{P}\}$, and we call $t\mathcal{P}$ the t^{th} *dilate of* $\mathcal{P}$. The fundamental question of Ehrhart Theory is

How many lattice points, i.e. points in $\mathbb{Z}^d$, are contained in $t\mathcal{P}$ for a given t?

This question finds its motivation in the study of the volume of a lattice polytope. To see this, think of each lattice point $z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$ as the center of a d-cube

$$C(z) := \prod_{i=1}^{d} \left[z_i - \frac{1}{2}, z_i + \frac{1}{2} \right].$$

Then notice that, rather than dilating $\mathcal{P}$ by a factor of t, we can simply shrink our lattice by a factor of $\frac{1}{t}$, i.e.

$$\left(\frac{1}{t}\mathbb{Z}\right)^d := \left\{ \frac{1}{t} z : z \in \mathbb{Z}^d \right\}.$$

These two processes are the same since the number of lattice points in $t\mathcal{P}$ is the same as the number of points of $\left(\frac{1}{t}\right)\mathbb{Z}^d$ within $\mathcal{P}$, i.e.

$$|t\mathcal{P} \cap \mathbb{Z}^d| = \left| \mathcal{P} \cap \left(\frac{1}{t}\mathbb{Z}\right)^d \right|.$$

As we shrink our lattice, the boxes $C(z)$ also shrink. Since these shrinking boxes will have volume $\frac{1}{t^d}$ when we shrink by a factor of $\frac{1}{t}$ we see that

$$\lim_{t \longrightarrow \infty} \frac{1}{t^d} |t\mathcal{P} \cap \mathbb{Z}^d| = m(\mathcal{P}),$$

where $m(\mathcal{P})$ denotes the standard Euclidean volume (Lesbesgue measure) of $\mathcal{P}$. Thus, we study the value $|t\mathcal{P} \cap \mathbb{Z}^d|$ for a lattice polytope $\mathcal{P}$ as t grows large. The following theorem is due to E. Ehrhart [19].

Theorem 1.2.1 (Ehrhart's Theorem)**.** *The value $|t\mathcal{P} \cap \mathbb{Z}^d|$ is a polyomial in t of degree d.*

We call this polynomial the *Ehrhart polynomial of* $\mathcal{P}$ and denote it by $\mathcal{L}_{\mathcal{P}}(t)$. The Ehrhart polynomial has some remarkable properties. For one, it is quickly deduced from the above limit that leading coefficient of $\mathcal{L}_{\mathcal{P}}(t)$ is precisely the volume $m(\mathcal{P})$ of our polytope.

One way to prove Ehrhart's Theorem is to use properties of the *Ehrhart Series of* $\mathcal{P}$. This is the generating function

$$\mathrm{Ehr}_{\mathcal{P}}(x) := 1 + \sum_{t>0} |t\mathcal{P} \cap \mathbb{Z}^d| x^t.$$

A result of R. Stanley [44] states that, in closed form, $\mathrm{Ehr}_{\mathcal{P}}(x)$ is a rational function of the form

$$\mathrm{Ehr}_{\mathcal{P}}(x) = \frac{h_0^* + h_1^* x + h_2^* x^2 + \cdots + h_d^* x^d}{(1-x)^{d+1}},$$

where the coefficients $h_0^*, h_1^*, \ldots, h_d^*$ are all nonnegative integers. The polynomial $\mathrm{h}^*(\mathcal{P};\mathrm{x}) := h_0^* + h_1^* x + h_2^* x^2 + \cdots + h_d^* x^d$ is called the *(Ehrhart) h^*-polynomial* of $\mathcal{P}$. Ehrhart's Theorem then follows from an application of the following lemma.

Lemma 1.2.2. *Suppose that*

$$1 + \sum_{t>0} f(t) x^t = \frac{g(x)}{(1-x)^{d+1}}.$$

Then $f(t)$ is a polynomial of degree d for every t if and only if $g(x)$ is a polynomial degree at most d and $g(1) \neq 0$.

The polynomial $\mathrm{h}^*(\mathcal{P};\mathrm{x})$ is not only fundamental in this proof of Ehrhart's theorem, but is also interesting in its own right. Indeed, $\mathrm{h}^*(\mathcal{P};\mathrm{x})$ arises naturally through the enumeration of combinatorial data. Thus, the nonnegativity of its coefficients suggest that $\mathrm{h}^*(\mathcal{P};\mathrm{x})$ encodes combinatorial data associated to our polytope. Consequently, combinatorial interpretations of $\mathrm{h}^*(\mathcal{P};\mathrm{x})$ have been investigated for important families of lattice polytopes. For example, the h^*-polynomial of the d-cube can be computed using basic techniques from enumerative combinatorics.

Example 1.2.3 (h^*-polynomial of the d-cube)*.* For the d-cube $[0,1]^d \subset \mathbb{R}^d$ we have that $|t[0,1]^d \cap \mathbb{Z}^d| = (t+1)^d$. Thus,

$$\mathrm{Ehr}_{[0,1]^d}(x) = \sum_{t\geq 0} (t+1)^d x^t.$$

The formula on the right arises from repeated applications of the operator $x\frac{d}{dx}$ to the geometric series $\sum_{t\geq 0} x^t$. By applying this operator to the closed form as well we develop the following formula.

$$\left(x\frac{d}{dx}\right)^d\left(\frac{1}{1-x}\right) = \left(x\frac{d}{dx}\right)^d \sum_{t\geq 0} x^t,$$
$$\frac{\sum_{k=1}^{d} A(d,k)x^k}{(1-x)^{d+1}} = \sum_{t\geq 1} t^d x^t.$$

The polynomial $A_d := \sum_{k=1}^{d} A(d,k)x^k$ is called the d^{th} *Eulerian polynomial.* By this construction, A_d satisfies the following recursion

$$A_d = x(1-x)\frac{d}{dx}A_{d-1} + dxA_{d-1}.$$

Following some brief computations we can see that $\mathrm{h}^*\left([0,1]^{\mathrm{d}};\mathrm{x}\right) = \frac{1}{x}A_d$.

$$\frac{\sum_{k=1}^{d} A(d,k)x^{k-1}}{(1-x)^{d+1}} = \frac{1}{x}\sum_{t\geq 1} t^d x^t.$$
$$\frac{\sum_{k=1}^{d-1} A(d,k+1)x^k}{(1-x)^{d+1}} = \sum_{t\geq 0}(t+1)^d x^t.$$

The Eulerian polynomial can alternatively be defined in terms of permutation statistics. Consider the symmetric group S_d on the elements $[d] := \{1, 2, \ldots, d\}$. For a permutation $\pi := \pi_1\pi_2\cdots\pi_d \in S_d$ we say that $i \in [d]$ is a *descent* of π if $\pi_i > \pi_{i+1}$. We let $\mathrm{des}(\pi)$ denote the number of descents in a given permutation $\pi \in S_d$. Consider the polynomial

$$E_d(x) = \sum_{\pi\in S_d} x^{1+\mathrm{des}(\pi)}.$$

The coefficient $e_{d,k}$ of x^k in $E_d(x)$ is therefore the number of permutations $\pi \in S_d$ with $k-1$ descents. By restricting the recursion for A_d to a recursion for the coefficients $A(d,k)$ we see that

$$A(d,k) = kA(d-1,k) + (d-k+1)A(d-1,k-1).$$

Indeed, $e_{d,k}$ also satisfies this recursion for every d and k. To see this, simply count the number of permutations $\pi \in S_d$ with $k-1$ descents as follows. Every permutation $\pi \in S_d$ with $k-1$ descents arises from a permutation $\omega \in S_{d-1}$ in one of two ways. Either ω has $k-1$ descents and π is given by placing the letter d in immediately after a descent in ω (or at the end of ω), or ω has $k-2$ descents and π is given by placing d after any letter that is not located at a descent of π (or the beginning of ω). There are $ke_{d-1,k}$ ways to complete the former operation, and $(d-k+1)e_{d-1,k-1}$ ways to complete the latter. Thus, $A(d,k) = e_{d,k}$, and we conclude that

$$A_d = \sum_{\pi\in S_d} x^{1+\mathrm{des}(\pi)}.$$

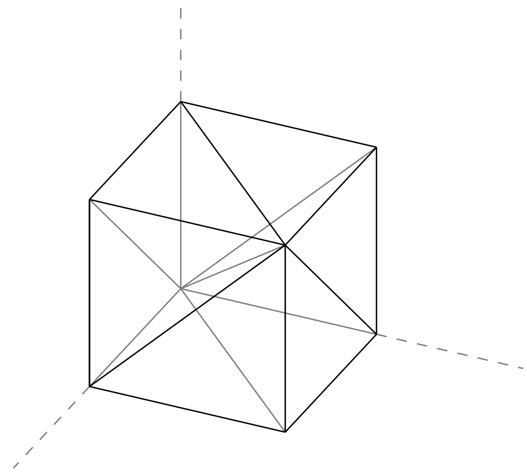


Figure 1.4: The 3-cube triangulated by the symmetric group S_3.

It is fascinating to note that the elements of the symmetric group S_d identify the maximal (with respect to dimension) simplices in a unimodular triangulation of $[0,1]^d$. The points $x \in \mathbb{R}^d$ lying in the relative interior of the simplex Δ_π corresponding to the permutation $\pi \in S_d$ satisfy

$$x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(d)}.$$

The existence of this triangulation was shown by R. Stanley in [43], and it is depicted for the 3-cube in Figure 1.4. We will now discuss a technique that allows us to compute $\mathrm{h}^*\left([0,1]^{\mathrm{d}};\mathrm{x}\right)$ using this triangulation.

A useful endeavor is to search for combinatorial interpretations of the coefficients of $\mathrm{h}^*(\mathcal{P};\mathrm{x})$ in terms of the geometric structure of $\mathcal{P}$. The following theorem of Stanley [44] provides a natural approach to this problem for any lattice polytope admitting a shellable unimodular triangulation.

Theorem 1.2.4. *Let ∇ be a unimodular shellable triangulation of a d-dimensional polytope $\mathcal{P}$. Then*

$$\sum_{j=0}^{d} h_j^* x^j = \sum_{\alpha \in \max \nabla} x^{\#(\alpha)+1}.$$

Example 1.2.5 (Shelling the octahedron). The octahedron depicted in Figure 1.5 is triangulated with eight tetrahedral faces whose edges are given by the green and blue lines. A shelling order on these tetrahedra is given by first placing the tetrahedra in the upper half in a clockwise fashion, and then placing the tetrahedra in the bottom half in a similar fashion. At each stage we record the dimension of the unique minimal new face, and then apply Theorem 1.2.4 to conclude that the h^*-polynomial of this polytope is $1 + 3x + 3x^2 + x^3$.

For a combinatorially defined polynomial with nonnegative coefficients, it is natural, from a probabilistic perspective, to investigate the distribution of these coefficients. Suppose that the coefficients of a polynomial $p(x) = a_0 + a_1x + a_2x^2 + a_dx^d$

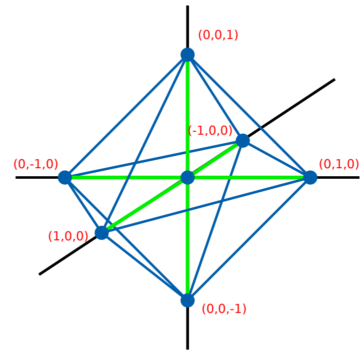


Figure 1.5: The triangulated octahedron.

have the property that a_k is the number of elements with property k in a collection of combinatorial objects Ω, and properties $k = 0, 1, \ldots, d$ are all mutually exclusive. Then the coefficients of $p(x)$ encode the probability distribution

$$\left(\frac{a_0}{\sum_{i=0}^d a_i}, \frac{a_1}{\sum_{i=0}^d a_i}, \ldots, \frac{a_d}{\sum_{i=0}^d a_i} \right),$$

and so it is reasonable to investigate the modality of the coefficients of $p(x)$. The polynomial $p(x)$, and its coefficient vector $(a_0, a_1, \ldots, a_d)$, is said to be *unimodal* if there exists some $0 \leq k \leq d$ such that $a_i \leq a_{i+1}$ for all $i < k$ and $a_{i-1} \geq a_i$ for all $i > k$. We end this subsection with an example summarizing these ideas.

Example 1.2.6 (The h^*-polynomial of a lattice simplex). Suppose we have a d-dimensional lattice simplex $\sigma = \mathrm{conv}(v_0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ for which we would like to compute $\mathrm{h}^*(\sigma; \mathrm{x})$. To do this, we will utilize a multivariate generating function for enumerating lattice points. Notice that we can encode any lattice point $z = (z_1, z_2, \ldots, z_d) \in \mathbb{Z}^d$ as a (laurent) monomial $x^z = x_1^{z_1} x_2^{z_2} \cdots x_d^{z_d}$. Given a set $A \subset \mathbb{R}^d$, the *integer-point transform* of A is the multivariate generating function

$$T_A(x) = T_A(x_1, \ldots, x_d) := \sum_{z \in A \cap \mathbb{Z}^d} x^z.$$

Notice that for a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, the integer-point transform of $\mathrm{cone}(\mathcal{P})$ specializes to the Ehrhart series of $\mathcal{P}$ when evaluated a (1,1,...,1,x). That is,

$$T_{\mathrm{cone}(\mathcal{P})}(1, 1, \ldots, 1, x) = \sum_{z \in \mathrm{cone}(\mathcal{P}) \cap \mathbb{Z}^d} 1^{z_1} 1^{z_2} \cdots 1^{z_d} x^{z_{d+1}} = \mathrm{Ehr}_{\mathcal{P}}(x).$$

In the special case of the simplex σ we can use this fact to quickly compute a formula for $\mathrm{h}^*(\sigma; \mathrm{x})$. The key to this computation is the fact that each integer point in $\mathrm{cone}(\sigma)$ is given by adding a nonnegative integer combination of the vectors $(v_0, 1), (v_1, 1), \ldots, (v_d, 1)$ to an integer point in the *fundamental parallelpiped*

$$\Pi := \left\{ \lambda_0(v_0, 1) + \lambda_1(v_1, 1) + \cdots + \lambda_d(v_d, 1) : 0 \leq \lambda_0, \lambda_1, \ldots, \lambda_d < 1 \right\}.$$

It then follows that

$$T_{\operatorname{cone}(\sigma)}(x) = \frac{T_\Pi(x)}{(1-x^{(v_0,1)})(1-x^{(v_1,1)})\cdots(1-x^{(v_d,1)})}.$$

Hence, $\mathrm{h}^*(\sigma; \mathrm{x}) = T_\Pi(1,1,\ldots,1,x)$. In particular, h_i^* is the number of lattice points $(z_1,\ldots,z_d,z_{d+1})$ in Π lying on the hyperplane $x_{d+1} = i$, and $\sum_{i=0}^d h_i^*$ is the total number of lattice points in Π. Thus, the distribution

$$\left(\frac{h_1^*}{\sum_{i=0}^d h_i^*}, \frac{h_1^*}{\sum_{i=0}^d h_i^*}, \ldots, \frac{h_d^*}{\sum_{i=0}^d h_i^*}\right),$$

is the probability distribution for selecting a lattice point in Π lying at height i for $i = 0,1,\ldots,d$. Indeed, this distribution may be unimodal. However in [39], S. Payne provides a method for constructing simplices whose h^*-polynomials achieve as many peaks and valleys (of arbitrary depth) as one could desire. While this is may not be surprising based simply on the algebraic structure of the polynomial, it is perhaps, from a geometric perspective, more shocking when one realizes these coefficients via the distribution of lattice points in Π across a family of parallel hyperplanes. Such geometric phenomena as this have resulted in a strong interest in lattice polytopes $\mathcal{P}$ for which $\mathrm{h}^*(\mathcal{P}; \mathrm{x})$ is unimodal.

1.3 Spectrahedra and Spectrahedral Shadows

In this subsection we introduce a particularly elegant and useful family of convex bodies called *spectrahedra*. Simply put, these convex bodies are defined by affine sections of the cone of positive semidefinite matrices. To formalize this, we let $\mathbb{S}^p$ denote the space of $p \times p$ real symmetric matrices, and $\mathbb{S}^p_{\succeq 0}$ denote the collection of positive semidefinite matrices within $\mathbb{S}^p$. When $A \in \mathbb{S}^p_{\succeq 0}$ we write $A \succeq 0$. Since a matrix $A \in \mathbb{S}^p$ is positive semidefinite if and only if $x^T A x \geq 0$ for every $x \in \mathbb{R}^p$, we can quickly verify that $\mathbb{S}^p_{\succeq 0}$ is a closed convex cone in $\mathbb{S}^p$. A *spectrahedron* is a closed convex set of the form

$$S = \left\{x \in \mathbb{R}^d : A(x) \succeq 0\right\},$$

where $A(x) := A_0 + \sum_{i=1}^d x_i A_i$ for some $A_0, A_1, \ldots, A_d \in \mathbb{S}^p$. The expression

$$A_0 + \sum_{i=1}^d x_i A_i \succeq 0$$

is called a *linear matrix inequality*, since it is essentially the linear inequality

$$a_1 x_1 + a_2 x_2 + \cdots + a_d x_d \geq b$$

where we have replaced the real numbers $a_1, a_2, \ldots, a_d, b$ with the symmetric $p \times p$ matrices $A_0, A_1, \ldots, A_d$. Indeed, the set of matrices

$$\mathcal{A} := \left\{A_0 + \sum_{i=1}^d x_i A_i : x \in \mathbb{R}^d\right\}$$

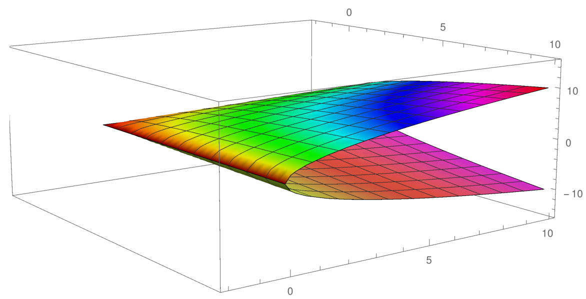


Figure 1.6: The PSD cone in $\mathbb{S}^2$.

is precisely a hyperplane in $\mathbb{S}^p$. In the case that $A_0, A_1, \ldots, A_d$ are linearly independent the sets $\mathbb{S}^p_{\succeq 0} \cap \mathcal{A}$ and S are affinely equivalent. Thus, it is common practice to refer to the set $\mathbb{S}^p_{\succeq 0} \cap \mathcal{A}$ as a spectrahedron as well.

Since a positive semidefinite matrix has all nonnegative eigenvalues, a spectrahedron S can be described in terms of scalar inequalities defined by the coefficients of the characteristic polynomial of the matrix $A(x)$, i.e. S is a *basic semialgebraic set*. That is, if I_p denotes the $p \times p$ identity matrix and

$$\det(A(x) - \lambda I_p) = \lambda^p + a_{p-1}\lambda^{p-1} + a_{p-2}\lambda^{p-2} + \cdots + a_1\lambda + a_0,$$

then $A(x)$ is positive semidefinite if and only if $(-1)^{p-j}a_j \geq 0$ for all $0 \leq j < p$. As a first example, notice the the positive semidefinite cone is itself a spectrahedron. To see this, let A_{ij} denote the $p \times p$ matrix consisting of all zeros except for entries $a_{ij} = a_{ji} = 1$. Then $\mathbb{S}^p_{\succeq 0}$ is affinely equivalent to the set

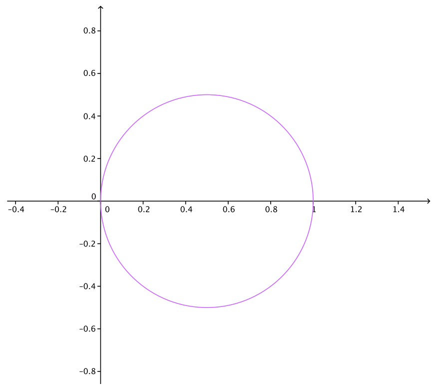
$$\left\{ x \in \mathbb{R}^{\binom{p+1}{2}} : \sum_{1 \leq i \leq j \leq 1} x_{ij} A_{ij} \succeq 0 \right\}.$$

When $p = 2$, we see that $\mathbb{S}^2_{\succeq 0}$ is defined by the inequalities

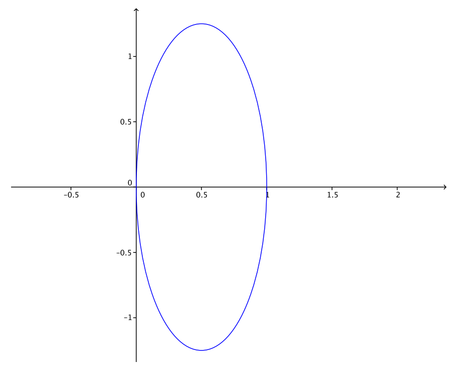
$$-(x+y) \geq 0, \text{ and}$$
$$xy - z^2 \geq 0.$$

This cone is depicted in Figure 1.6. Notice that the boundary of $\mathbb{S}^2_{\succeq 0}$ is given by $\det(A(x)) = 0$ and the inequality $-(x+y) \geq 0$ serves to isolate the desired region. Indeed, this will be the case for any spectrahedron S since we always have that $a_0 = \det(A(x))$.

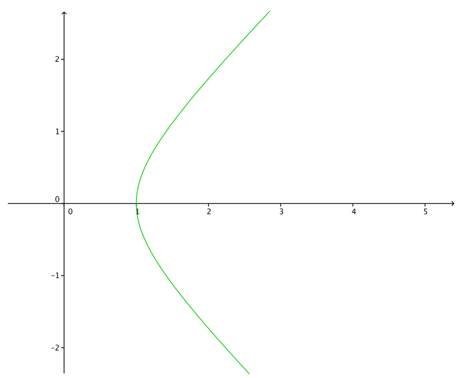
Classic algebraic sets can yield spectrahedra. For example, each of the conic sections bounds a spectrahedron defined using only 2×2 linear matrix inequalities.



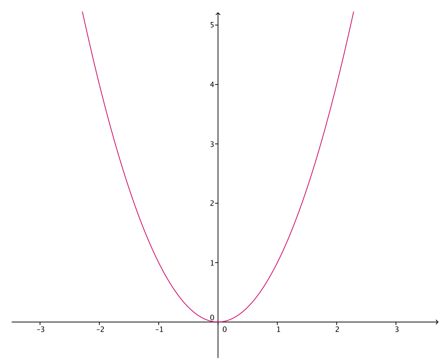
$$\begin{pmatrix} 1-x & y \\ y & x \end{pmatrix} \succeq 0.$$



$$\begin{pmatrix} 1-x & \frac{1}{3}y \\ \frac{1}{3}y & \frac{1}{4}x \end{pmatrix} \succeq 0.$$



$$\begin{pmatrix} x-y & 1 \\ 1 & x+y \end{pmatrix} \succeq 0.$$



$$\begin{pmatrix} y & x \\ x & 1 \end{pmatrix} \succeq 0.$$

Using a 3×3 linear matrix inequality we see that the convex hull of the bounded portion of the real graph of the elliptic curve

$$2y^2 + x^3 + 3x^2 - x - 3 = 0$$

is a spectrahedron [7]. This spectrahedron is depicted in Figure 1.7, and it is given by the linear matrix inequality

$$\begin{pmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{pmatrix} \succeq 0.$$

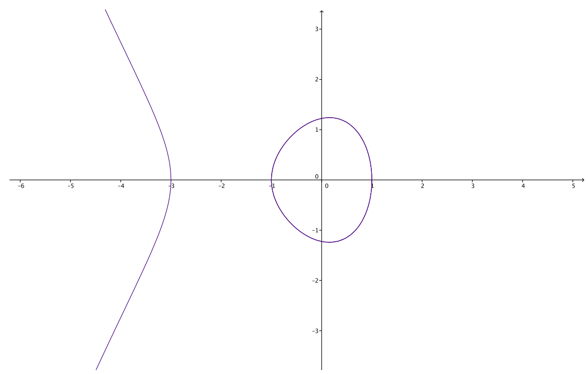


Figure 1.7: The shaded region is a spectrahedron.

Spectrahedra and Polyhedra

An important family of spectrahedra are the polyhedra. Recall that a polyhedron $\mathcal{P} \subset \mathbb{R}^d$ is defined by a finite collection of linear inequalities

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1d}x_d &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2d}x_d &\leq b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nd}x_d &\leq b_n. \end{aligned}$$

For convenience, we let $a_i = (a_{i1}, a_{i2}, \ldots, a_{id})^T$ and we write the inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{id}x_d \leq b_i$$

in the form $\langle a_i, x\rangle \leq b_i$. The polyhedron $\mathcal{P}$ then has the spectrahedral representation

$$\left\{ x \in \mathbb{R}^d : \begin{pmatrix} b_1 - \langle a_1, x\rangle & 0 & \cdots & 0 \\ 0 & b_2 - \langle a_2, x\rangle & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & b_n - \langle a_n, x\rangle \end{pmatrix} \succeq 0 \right\}.$$

Moreover, any linear matrix inequality defined by diagonal matrices yields a polyhedron.

From a convex optimization perspective, spectrahedra are perhaps the most natural relaxation of polyhedra. Recall that a linear program (LP) is a problem of the form

$$\begin{array}{ll} \text{minimize} & c^T x, \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where $c, x \in \mathbb{R}^d$, and A is an $n \times d$ matrix. The feasible region of this program is the polyhedron defined by the linear inequalities $x_i \geq 0$ for $0 \leq i \leq 1$, $Ax \leq b$, and

$-Ax \leq -b$. A semidefinite program (SDP) is the problem

$$\begin{array}{ll} \text{minimize} & \langle C, X\rangle, \\ \text{subject to} & \langle A, X\rangle = b_i, \text{ for } i = 1, \ldots, p \\ & X \succeq 0, \end{array}$$

where $C, A_1, A_2, \ldots, A_p \in \mathbb{S}^p$, $X \in \mathbb{S}^p_{\succeq 0}$, and $\langle X, Y\rangle = \mathrm{Tr}(X^T Y)$ is the Frobenius inner product. The feasible region of a semidefinite program is a spectrahedron since it is an affine section of the cone of positive semidefinite matrices. The program is linear exactly when the feasible region is a polyhedron. Thus, the SDP programs relax linear programs, and this is apparent in the geometry of the feasible regions.

Example 1.3.1 (The derivative of a simplex). Consider the linear program

$$\begin{array}{ll} \text{minimize} & c^T x, \\ \text{subject to} & x_1 + x_2 + \cdots + x_d = 1, \\ & x \geq 0. \end{array}$$

The feasible region of this linear program is the *standard* $(d-1)$*-simplex* or *probability simplex*, which is precisely the convex hull of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d \in \mathbb{R}^d$

$$\Delta_{d-1} = \mathrm{conv}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d).$$

To relax our feasible region to a (nonpolyhedral) spectrahedron we first consider the cone over Δ_{d-1}. Notice that

$$\mathrm{co}(\Delta_{d-1}) = \mathbb{R}^d_{\geq 0} = \left\{x \in \mathbb{R}^d : x_i \geq 0 \text{ for } 1 \leq i \leq d\right\}.$$

The boundary of this cone is given by the d^{th} *elementary symmetric polynomial* $E_{d,d}(x) = x_1 x_2 \cdots x_d$, and our feasible region Δ_{d-1} is returned by slicing this cone with the hyperplane $x_1 + x_2 + \cdots + x_d = 1$. To relax this cone we consider the hypersurface defined by the sum of all the partial derivatives of $E_{d,d}$, i.e. the $(d-1)^{th}$ elementary symmetric polynomial

$$E_{d,d-1}(x) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} E_{d,d}(x).$$

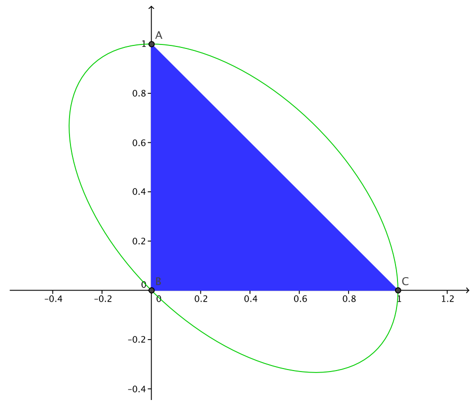
Our relaxed cone has boundary defined by the hypersurface $\{E_{d,d-1} = 0\}$ and its interior is the connected component of $\mathbb{R}^d \backslash \{E_{d,d-1} = 0\}$ containing the point $(1, 1, \ldots, 1)^T$. To return our relaxed simplex we intersect this cone with the hyperplane

$$x_1 + x_2 + \cdots + x_d = 1.$$

The result is the spectrahedron

$$S_{d-1} := \left\{ x \in \mathbb{R}^d : \begin{pmatrix} 1 - \sum_{i \neq 1} x_i & 1 - \sum_i x_i & \cdots & 1 - \sum_i x_i \\ 1 - \sum_i x_i & 1 - \sum_{i \neq 2} x_i & & \vdots \\ \vdots & 1 - \sum_i x_i & \ddots & 1 - \sum_i x_i \\ 1 - \sum_i x_i & \cdots & 1 - \sum_i x_i & 1 - \sum_{i \neq d-1} x_i \end{pmatrix} \succeq 0 \right\}.$$

When $d = 3$, our feasible region is a triangle and this process returns an ellipse containing our triangle.



$$\begin{pmatrix} 1-y & 1-x-y \\ 1-x-y & 1-y \end{pmatrix} \succeq 0$$

When $d = 4$, the feasible region Δ_3 is a tetrahedron, and our relaxation is the closed and bounded portion of the Cayley nodal cubic surface. Notice that the four nodes are precisely the vertices of Δ_3, and the edges of Δ_3 are still faces of S_3, whereas the facets of Δ_3 have been relaxed to families of regular extreme points.

$$\begin{pmatrix} 1-y-z & 1-x-y-z & 1-x-y-z \\ 1-x-y-z & 1-x-z & 1-x-y-z \\ 1-x-y-z & 1-x-y-z & 1-x-y \end{pmatrix} \succeq 0$$

These spectrahedral representations were first identified by R. Sanyal in [42]. In [8] P. Brändén generalizes this process to the higher derivatives of $\mathbb{R}^n_{\geq 0}$, and a spectrahedral representation of each of these derivatives is provided using the Matrix-Tree Theorem.

Spectrahedral Shadows and Polar Bodies

We have now seen a number of examples of convex sets that are spectrahedra but not polyhedra. However, we have yet to identify a nonspectrahedral convex set. A natural way to discern a nonspectrahedral convex set is by examining its facial structure. Recall that a subset F of a closed convex set $C \subset \mathbb{R}^d$ is a *face* of C if for

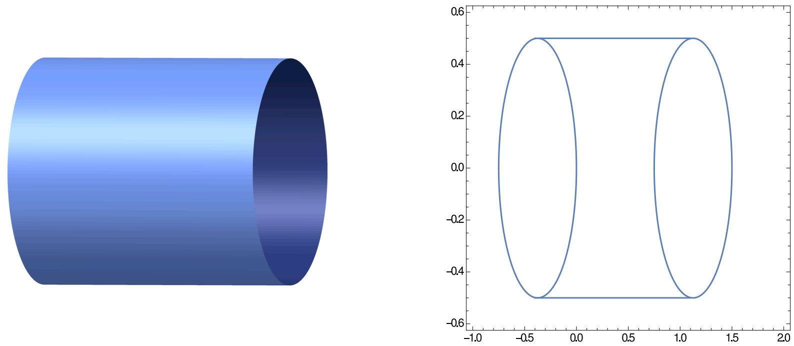


Figure 1.8: The convex hull of the two ellipses is the shadow of the cylinder.

every $x, y \in C$ and $0 < t < 1$, $tx + (1-t)y \in F$ implies that $x, y \in F$. In other words, each open line segment in C that intersects F has the property that its closure lies in F. A face F of C is called *exposed* if there is some affine hyperplane H in $\mathbb{R}^d$ for which $F = H \cap C$ and C lies entirely in one of the closed halfspaces defined by H. In [41], Ramana and Goldman show that all faces of a spectrahedron are exposed. So a quick way to identify a nonspectrahedral body is to show that it has some nonexposed faces. For example, the convex body depicted in Figure 1.8 has nonexposed faces and therefore is not a spectrahedron. However, it is a *projection* of a spectrahedron. A *spectrahedral shadow* is a closed convex set of the form

$$S = \left\{x \in \mathbb{R}^d : A(x, y) \succeq 0\right\},$$

where

$$A(x, y) := A_0 + \sum_{i=1}^{d} x_i A_i + \sum_{j=1}^{m} y_i B_i$$

for some $A_0, A_1, \ldots, A_d, B_1, B_2, \ldots, B_m \in \mathbb{S}^p$. That is, S is the projection of the spectrahedron in $\mathbb{R}^{d+m}$ defined by the linear matrix inequality $A(x, y) \succeq 0$. The convex body in Figure 1.8 is the spectrahedral shadow given by

$$A(x, y) = \begin{pmatrix} 1 - \frac{x+z}{\sqrt{2}} & y & 0 & 0 \\ y & \frac{x+y}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 - \frac{z-x}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 + \frac{z-x}{\sqrt{2}} \end{pmatrix}$$

This can be shown by constructing the ideal in $\mathbb{R}[x, y, z]$ generated by the determinant of $A(x, y)$ and its partial derivative with respect to z, and then computing the primary decomposition of the ideal given by eliminating z.

Spectrahedral shadows play an important role in the theory of spectrahedra. Not only do they generalize spectrahedra, but they offer a beautiful representation of the *polar body* of a spectrahedron. Recall that the *polar* of a subset $K \subset \mathbb{R}^d$ is

$$K^\circ := \{z \in \mathbb{R}^d : \langle x, z\rangle \leq 1 \text{ for all } x \in K\},$$

and the *dual* of K, denoted $K^\vee$ is the negative of the polar, i.e. $K^\vee = -K^\circ$. Let $\mathcal{W} := A_0 + \mathrm{span}_{\mathbb{R}}(A_1, \ldots, A_d)$ be the linear subspace of $\mathbb{S}^p$ defined by $A_0, \ldots, A_d \in \mathbb{S}^p$, and let

$$\pi_{\mathcal{W}} \,:\, \mathbb{S}^p \to \mathbb{S}^p/\mathcal{W}^\perp \,\simeq\, \mathbb{R}^d, \qquad X \mapsto (\langle X, A_1\rangle, \ldots, \langle X, A_d\rangle)$$

be the canonical projection. We define the $\binom{p+1}{2}$-dimensional spectrahedron

$$\mathcal{R} = \{X \in \mathbb{S}^p_{\succeq 0} : \langle X, A_0\rangle \leq 1\}.$$

In [41], Ramana and Goldman show that the polar of the spectrahedron S defined by $A_0 + \sum_{i=1}^d x_i A_i \succeq 0$ is always a spectrahedral shadow, namely the closure of the image of the spectrahedron $\mathcal{R}$ under the projection $\pi_{\mathcal{W}}$, i.e. $\mathcal{S}^\circ = \mathrm{cl}(\pi_{\mathcal{W}}(\mathcal{R}))$. As we see from this example, the polar of a spectrahedron is not always be a spectrahedron. This stands in stark contrast to the polarity conditions for polyhedra.

1.4 Gaussian Graphical Models

Many problems in statistics can be addressed using techniques from algebraic geometry, commutative algebra, combinatorics, and convex geometry. This is the focus of the field of study called *algebraic statistics*. Our interest is in the combinatorial convex geometry of the maximum likelihood problem for Gaussian graphical models. Recall that an p-dimensional random vector $X \in \mathbb{R}^p$ follows a *multivariate Gaussian (normal) distribution* $\mathcal{N}(\mu, \Sigma)$ if it has the probability density function

$$p_\theta = \frac{1}{(2\pi)^{\frac{p}{2}}(\det \Sigma)^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right\},$$

where $\theta = (\mu, \Sigma)$, $\mu \in \mathbb{R}^p$ is the *mean vector*, and $\Sigma \in \mathbb{S}^p_{\succ 0}$ is the associated *covariance matrix*. The Gaussian distributions are multivariate analogs of the standard normal distribution or "bell curve." Here, the mean vector serves as the center of the distibution, and the eigenvalues of Σ define the spread of the distribution.

A *statistical model* is a family of probability distributions defined on some state space $\mathcal{X}$ equipped with a base measure and a measurable function $T : \mathcal{X} \longrightarrow \mathbb{R}^d$. The image of this map $T(\mathcal{X})$ is called the *sufficient statistics*. In the case of the (saturated) multivariate Gaussian model we take $\mathcal{X} = \mathbb{R}^p$ and $T(x) = \frac{1}{2}xx^T \in \mathbb{S}^p$. Thus, the sufficient statistics is all $p \times p$ symmetric matrices of rank at most one. Since the covariance matrix Σ is positive definite it is expressible as a sum of the sufficient statistics of the Gaussian model. Thus, our probability distributions $\mathcal{N}(\mu, \Sigma)$ are defined by the choice of mean vector μ and an element in the interior of the *cone of sufficient statistics* $\mathrm{conv}(T(\mathcal{X}))$, which in this case is the entire positive semidefinite cone. More generally, a *Gaussian model* is a collection of probability distributions

$$P_\Theta := \{\mathcal{N}(\mu, \Sigma) : \theta = (\mu, \Sigma) \in \Theta\},$$

where Θ is a subset of $\mathbb{R}^p \times \mathbb{S}^p_{\succ 0}$.

For random vectors $X^{(1)}, X^{(2)}, \ldots, X^{(n)} \in \mathbb{R}^p$ that are independently and identically distributed according to some unknown probability distribution $p_\theta \in P_\Theta$, we define the *likelihood function*

$$L_n(\theta) := \prod_{i=1}^{n} p_\theta(X^{(i)}).$$

It is common to replace the likelihood function with its logarithm $\ell_n(\theta) = \log(L_n(\theta))$. The *maximum likelihood estimation problem* is the convex optimization problem

$$\begin{array}{ll} \text{maximize} & \ell_n(\theta), \\ \text{subject to} & \theta \in \Theta. \end{array}$$

The unknown parameter $\hat{\theta}$ that solves this problem is called the *maximum likelihood estimator* of θ. Provided with data $X^{(1)} = x^{(1)}, X^{(2)} = x^{(2)}, \ldots, X^{(n)} = x^{(n)}$, the *maximum likelihood estimate* of θ is the realization of the parameter $\hat{\theta}$ with respect to this data. The maximum likelihood estimate is thus the parameter value $\hat{\theta}$ that maximizes the likelihood of observing the given data. In the case of a Gaussian model, following some arithmetic simplification, the log-likelihood function is

$$\ell_n(\theta) = -\frac{n}{2} \log\det\Sigma - \frac{1}{2} \operatorname{Tr}\left(\Sigma^{-1} \sum_{i=1}^{n} (X^{(i)} - \mu)(X^{(i)} - \mu)^T \right),$$

up to a normalizing constant.

Many important Gaussian models are of the form $\Theta = \mathbb{R}^p \times \Theta_2$ for a subset $\Theta_2 \subset \mathbb{S}^p$. In these models, we do not put any assumption on our choice of mean vector μ, but we restrict the possible choices of covariance matrices. In this case, it is helpful to parametrize the model in terms of the inverse matrix $K := \Sigma^{-1}$, which we call the *concentration matrix*. Using the concentration matrix, the log-likelihood function becomes the strictly convex function

$$\ell_n(K) = \frac{n}{2} \log\det K - \frac{n}{2} \operatorname{Tr}(SK),$$

where

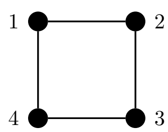
$$S := \frac{1}{n} \sum_{i=1}^{n} (X^{(i)} - \overline{X})(X^{(i)} - \overline{X})^T \qquad \text{and} \qquad \overline{X} := \frac{1}{n} \sum_{i=1}^{n} X^{(i)},$$

are the *sample covariance matrix* and *sample mean vector*, respectively.

We are interested in the *Gaussian graphical models*, which are nicely parametrized in terms of their concentration matrices. For a graph G the *cone of concentration matrices* of G, denoted $\mathcal{K}_G$, is the collection of all positive definite $p \times p$ matrices with zeros in all entries ij corresponding to nonedges of the graph G. For a graph G on $[p]$ vertices, the *(undirected) Gaussian graphical model* for G is the Gaussian model $\mathbb{R}^p \times \Theta_2$ where

$$\Theta_2 = \left\{ \Sigma : \Sigma^{-1} \in \mathcal{K}_G \right\}.$$

For the Gaussian graphical models, the maximum likelihood estimation problem can be rephrased as a *matrix completion problem.* Recall that a *(real) $p \times p$ partial matrix* $A = [a_{ij}]$ is a matrix in which some entries are specified real numbers and the remainder are unspecified. It is called *symmetric* if all the specified entries satisfy $a_{ij} = a_{ji}$, and it is called *PD-completable* if there exists a specification of the unknown entries of A that produces a matrix $\widetilde{A} \in \mathbb{S}^p$ that is positive definite. A partial sample covariance matrix is a symmetric partial $p \times p$ matrix in which a specified entry S_{ij} is the covariance between the i^{th} and j^{th} entries in a random vector. For example, the following matrix is a partial sample covariance matrix in which the specified covariance data is supported by the edges and vertices of the cycle on four vertices.



$$S = \begin{pmatrix} 10 & 12 & ? & -3 \\ 12 & 16 & 1 & ? \\ ? & 1 & 13 & 6 \\ -3 & ? & 6 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 10 & 12 & -6 & -3 \\ 12 & 16 & 1 & -4 \\ -6 & 1 & 13 & 6 \\ -3 & -4 & 6 & 10 \end{pmatrix} \in \mathbb{S}^4_{\succ 0}.$$

A positive definite completion of this partial sample covariance matrix amounts to a specification of the remaining entries that results in a matrix living in the cone $\mathbb{S}^4_{\succ 0}$. The following theorem proven in [18] reduces maximum likelihood estimation for Gaussian graphical models to identifying a specific PD-completion of a given partial sample covariance matrix.

Theorem 1.4.1. *Let G be a graph on vertex $[p]$ with edge set E. For the Gaussian graphical model Θ_2, the maximum likelihood estimate $\hat{\Sigma}$ for a given sample covariance matrix S is the unique positive definite matrix $\hat{\Sigma}$ for which $\hat{\Sigma}_{ij} = S_{ij}$ for every edge $ij \in E$ and for $i = j$ that satisfies $\hat{\Sigma}^{-1} \in \mathcal{K}_G$.*

Theorem 1.4.1 introduces an excellent opportunity to study this problem via combinatorial convex geometry. For a fixed graph G the collection of all PD-completable symmetric partial matrices $M \in \mathbb{R}^{E \cup V}$ forms a convex cone which we denote by $\mathcal{C}_G$. Indeed, $\mathcal{C}_G$ is precisely the image of the PD cone $\mathbb{S}^p_{\succ 0}$ under the projection $\pi_G : \mathbb{S}^p \longrightarrow \mathbb{R}^{E \cup V}$. Theorem 1.4.1 implies that we need to be able to identify when a given partial sample covariance matrix $S \in \mathbb{R}^{E \cup V}$ lies in $\mathcal{C}_G$. It is well-known that the dual cone to $\mathcal{C}_G$ is $\mathcal{K}_G$, the cone of concentration matrices of G. Thus, our partial sample covariance matrix S is PD-completable if and only if $\langle S, X \rangle > 0$ for every extreme matrix X in the topological closure of $\mathcal{K}_G$. In Chapter 5 we will study the combinatorial convex geometry of these extreme matrices.

Chapter 2 The r-stable Hypersimplices

2.1 Hypersimplices and r-stable Hypersimplices

We now turn our attention to a special collection of polytopes which we will call the *r-stable (n,k)-hypersimplices*. The (n,k)-hypersimplices are an important collection of integer polytopes arising naturally in the settings of convex optimization, matroid theory, combinatorics, and algebraic geometry. For integers $0 < k < n$ let $[n] := \{1, 2, \dots, n\}$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. The *characteristic vector* of a subset I of $[n]$ is the $(0,1)$-vector $\epsilon_I = (\epsilon_1, \dots, \epsilon_n)$ for which $\epsilon_i = 1$ for $i \in I$ and $\epsilon_i = 0$ for $i \notin I$. The *(n,k)-hypersimplex* is the convex hull in $\mathbb{R}^n$ of the collection of characteristic vectors $\{\epsilon_I : I \in \binom{[n]}{k}\}$, and it is denoted $\Delta_{n,k}$. Generalizing the standard $(n-1)$-simplex, the (n,k)-hypersimplices serve as a useful collection of examples in these various contexts. While these polytopes are well-studied, there remain interesting open questions about their properties in the field of Ehrhart theory, the study of integer point enumeration in dilations of rational polytopes (see for example [14]). We will now define a family of subpolytopes nested within the (n,k)-hypersimplex that share interesting geometric similarities with the hypersimplex in which they are contained.

Label the vertices of a regular n-gon embedded in $\mathbb{R}^2$ in a clockwise fashion from 1 to n. We define the *circular distance* between two elements i and j of $[n]$, denoted $\mathrm{cd}(i,j)$, to be the number of edges in the shortest path between the vertices i and j of the n-gon. We also denote the path of shortest length from i to j by $\mathrm{arc}(i,j)$. A subset $S \subset [n]$ is called *r-stable* if each pair $i, j \in S$ satisfies $\mathrm{cd}(i,j) \geq r$. The following polytopes will be the focus of our attention in chapters 2, 3, and 4.

Definition 2.1.1. The *r-stable (n,k)-hypersimplex*, denoted $\Delta_{n,k}^{\mathrm{stab}(r)}$, is the convex hull of the characteristic vectors of all r-stable k-subsets of $[n]$.

For fixed n and k, these polytopes form the nested chain

$$\Delta_{n,k} \supset \Delta_{n,k}^{stab(2)} \supset \Delta_{n,k}^{stab(3)} \supset \cdots \supset \Delta_{n,k}^{stab\left(\left\lfloor \frac{n}{k} \right\rfloor\right)}.$$

In this chapter, our goal is to investigate the geometry of these polytopes in detail. In chapters 3 and 4 we will then use this geometry to study the combinatorics of the Ehrhart h^*-polynomials of these polytopes. The contents of this chapter is in part joint work with Benjamin Braun and Takayuki Hibi.

2.2 A Regular Unimodular Triangulation

In [32], Lam and Postnikov compare four different triangulations of the hypersimplex, and show that they are identical. While these triangulations possess the same geometric structure the constructions are all quite different, and consequently each one reveals different information about the triangulation's geometry. Here, we utilize properties of two of these four constructions. The first is a construction given by Sturmfels in [47] using techniques from toric algebra. The second construction, known as the *circuit triangulation*, is introduced in [32] by Lam and Postnikov. We will show that this triangulation restricts to a triangulation of the r-stable hypersimplex.

Sturmfels' Triangulation.

We recall the description of this triangulation presented in [32]. Let I and J be two k-subsets of $[n]$ and consider their multi-union $I \cup J$. Let $\mathrm{sort}(I \cup J) = (a_1, a_2, \ldots, a_{2k})$ be the unique nondecreasing sequence obtained by ordering the elements of the multiset $I \cup J$ from least-to-greatest. Now let $U(I,J) := \{a_1, a_3, \ldots, a_{2k-1}\}$ and $V(I,J) := \{a_2, a_4, \ldots, a_{2k}\}$. As an example consider the 4-subsets of $[8]$, $I = \{1,3,4,6\}$ and $J = \{3,5,7,8\}$. For this pair of subsets we have that $\mathrm{sort}(I \cup J) = (1,3,3,4,5,6,7,8)$, $U(I,J) = \{1,3,5,7\}$, and $V(I,J) = \{3,4,6,8\}$. The ordered pair of k-subsets (I,J) is said to be *sorted* if $I = U(I,J)$ and $J = V(I,J)$. Moreover, an ordered d-collection $\mathcal{I} = (I_1, I_2, \ldots, I_d)$ of k-subsets is called *sorted* if each pair (I_i, I_j) is sorted for all $1 \leq i < j \leq d$. For a sorted d-collection $\mathcal{I}$ we let $\sigma_{\mathcal{I}}$ denote the $(d-1)$-dimensional simplex with vertices $\epsilon_{I_1}, \epsilon_{I_2}, \ldots, \epsilon_{I_d}$.

Theorem 2.2.1. *[47, Sturmfels] The collection of simplices $\sigma_{\mathcal{I}}$, where $\mathcal{I}$ varies over the sorted collections of k-element subsets of $[n]$, forms a triangulation of $\Delta_{n,k}$.*

Notice that the maximal simplices in this triangulation correspond to the maximal-by-inclusion sorted collections, which all have $d = n$.

This triangulation of $\Delta_{n,k}$ was identified by Sturmfels' via the correspondence between Gröbner bases for the toric ideal associated to $\Delta_{n,k}$ and regular triangulations of $\Delta_{n,k}$. To construct the toric ideal for $\Delta_{n,k}$ let $k[x_I]$ denote the polynomial ring in the $\binom{n}{k}$ variables x_I labeled by the k-subsets of $[n]$, and define the semigroup algebra homomorphism

$$\varphi : k[x_I] \longrightarrow k[z_1, z_2, \ldots, z_n]; \qquad \varphi : x_I \longmapsto z_{i_1} z_{i_2} \cdots z_{i_k}, \quad \text{for } I = \{i_1, i_2, \ldots, i_k\}.$$

The kernel of this homomorphism, $\ker\varphi$, is the toric ideal of $\Delta_{n,k}$. The correspondence between Gröbner bases for $\ker\varphi$ and regular triangulations of $\Delta_{n,k}$ is given as follows. Any sufficiently generic height vector induces a regular triangulation of $\Delta_{n,k}$. On the other hand, such a height vector induces a term order $<$ on the monomials in the polynomial ring $k[x_I]$. Thus, we may identify a Gröbner basis for $\ker\varphi$ with respect to this term order, say $G_<$. Moreover, the initial ideal associated to a Gröbner basis is square-free if and only if the corresponding regular triangulation is unimodular. The details of this correspondence are outlined nicely in [47].

Theorem 2.2.2. *[47, Sturmfels] The set of quadratic binomials*

$$\mathcal{G}_< := \left\{ \underline{x_I x_J} - x_{\mathrm{U}(I,J)} x_{\mathrm{V}(I,J)} : I, J \in \binom{[n]}{k} \right\}$$

$$\begin{tikzcd}
(0,1,0,0,1) \arrow[r, "2"] & (0,0,1,0,1) \arrow[r, "5"] & (1,0,1,0,0) \arrow[d, "3"] \\
(0,1,0,1,0) \arrow[u, "4"] & & (1,0,0,1,0) \arrow[ll, "1"']
\end{tikzcd}$$

Figure 2.1: Here is a minimal circuit in the directed graph $G_{5,2}$.

is a Gröbner basis for $\ker \varphi$ *under some term order* $<$ *on* $k[x_I]$ *such that the underlined term is the initial monomial. In particular, the initial ideal of* $G_<$ *is square-free, and the simplices of the corresponding unimodular triangulation are* $\sigma_{\mathcal{I}}$*, where* $\mathcal{I}$ *varies over the sorted collections of* k*-element subsets of* $[n]$*.*

We denote this triangulation of $\Delta_{n,k}$ by $\nabla_{n,k}$, and we let $\max \nabla_{n,k}$ denote the set of maximal simplices in $\nabla_{n,k}$. In [32], Lam and Postnikov prove a more general version of Theorem 2.2.2 which we will utilize to show that this triangulation restricts to a triangulation of the r-stable hypersimplex $\Delta_{n,k}^{\mathrm{stab}(r)}$.

The Circuit Triangulation.

The second construction of this triangulation that we will utilize first appeared in [32], and it arises from examining minimal length circuits in a particular direct graph with labeled edges. We construct this directed graph as follows. Let $G_{n,k}$ be the directed graph with vertices ϵ_I, where I varies over all k-subsets of $[n]$. For a vertex $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ of $G_{n,k}$ we think of the coordinate indices i as elements of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Hence, $\epsilon_{n+1} = \epsilon_1$. We construct the directed, labeled edges of $G_{n,k}$ as follows. Suppose $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and ϵ' are vertices of $G_{n,k}$ for which $(\epsilon_i, \epsilon_{i+1}) = (1, 0)$ and the vector ϵ' is obtained from ϵ by switching ϵ_i and ϵ_{i+1}. Then we include the directed labeled edge $\epsilon \xrightarrow{i} \epsilon'$ in $G_{n,k}$. Hence, each edge of $G_{n,k}$ is given by shifting a 1 in a vertex ϵ exactly one entry to the right (modulo n), and this can happen if and only if the next place is occupied by a 0.

We are interested in the circuits of minimal possible length in the graph $G_{n,k}$. We will call such a circuit *minimal*. A minimal circuit in $G_{n,k}$ containing the vertex ϵ is given by a sequence of edges moving each 1 in ϵ into the position of the 1 directly to its right. Hence, the length of such a circuit is precisely n. An example of a minimal circuit is given in Figure 2.1.

For a fixed initial vertex, the sequence of labels of edges in a minimal circuit forms a permutation $\omega = \omega_1\omega_2 \cdots \omega_n \in S_n$, the symmetric group on n elements. There is one such permutation for each choice of initial vertex in the minimal circuit. Hence, a minimal circuit in $G_{n,k}$ corresponds to an equivalence class of permutations in S_n where permutations are equivalent modulo cyclic shifts $\omega_1 \cdots \omega_n \sim \omega_n\omega_1 \cdots \omega_{n-1}$. In the following, we choose the representative ω of the class of permutations associated to the minimal circuit for which $\omega_n = n$. We remark that this corresponds to picking the initial vertex of the minimal circuit to be the lexicographically maximal $(0,1)$-vector in the circuit. For example, the lexicographic ordering on the $(0,1)$-vectors in the circuit depicted in Figure 2.1 is

$$(1,0,1,0,0) > (1,0,0,1,0) > (0,1,0,1,0) > (0,1,0,0,1) > (0,0,1,0,1),$$

and the permutation given by reading the edge labels of this circuit beginning at vertex $(1,0,1,0,0)$ is $\omega = 31425$. Thus, we see that $\omega_n = n$ as desired.

Theorem 2.2.3. *[32, Lam and Postnikov] A minimal circuit in the graph $G_{n,k}$ corresponds uniquely to a permutation $\omega \in S_n$ modulo cyclic shifts. Moreover, a permutation $\omega \in S_n$ with $\omega_n = n$ corresponds to a minimal circuit in $G_{n,k}$ if and only if the inverse permutation ω^{-1} has exactly $k-1$ descents.*

We label the minimal circuit in the graph $G_{n,k}$ corresponding to the permutation $\omega \in S_n$ with $\omega_n = n$ by (ω). Let $v_{(\omega)}$ denote the set of all vertices ϵ_I of $\Delta_{n,k}$ used by the circuit (ω), and let $\sigma_{(\omega)}$ denote the convex hull of $v_{(\omega)}$.

Theorem 2.2.4. *[32, Lam and Postnikov] The collection of simplices $\sigma_{(\omega)}$ corresponding to all minimal circuits in $G_{n,k}$ forms the collection of maximal simplices of a triangulation of the hypersimplex $\Delta_{n,k}$. This triangulation is identical to the triangulation $\nabla_{n,k}$.*

We call this construction of $\nabla_{n,k}$ the *circuit triangulation*. To simplify notation we will often write ω for the simplex $\sigma_{(\omega)} \in \nabla_{n,k}$.

The induced triangulation of the r-stable hypersimplex.

Let $\mathcal{M}$ be a collection of k-subsets of $[n]$, and let $\mathcal{P}_{\mathcal{M}}$ denote the convex hull in $\mathbb{R}^n$ of the $(0,1)$-vectors $\{\epsilon_I : I \in \mathcal{M}\}$. Notice that $\mathcal{P}_{\mathcal{M}}$ is a subpolytope of $\Delta_{n,k}$. The collection $\mathcal{M}$ is said to be *sort-closed* if for every pair of subsets I and J in $\mathcal{M}$ the subsets $U(I,J)$ and $V(I,J)$ are also in $\mathcal{M}$. In [32], Lam and Postnikov proved the following theorem.

Theorem 2.2.5. *[32, Lam and Postnikov] The triangulation $\nabla_{n,k}$ of the hypersimplex $\Delta_{n,k}$ induces a triangulation of the polytope $\mathcal{P}_{\mathcal{M}}$ if and only if $\mathcal{M}$ is sort-closed.*

We then have the following corollary to this theorem.

Corollary 2.2.6. *Fix an integer $0 < r \leq \left\lfloor \frac{n}{k} \right\rfloor$. Let $\mathcal{M}$ be the collection of r-stable k-subsets of $[n]$. The triangulation $\nabla_{n,k}$ induces a triangulation of the r-stable hypersimplex $\mathcal{P}_{\mathcal{M}} = \Delta_{n,k}^{\mathrm{stab}(r)}$.*

Proof. By Theorem 2.2.5, it suffices to show that the collection $\mathcal{M}$ is sort-closed. Let I and J be two elements of $\mathcal{M}$, and consider $\mathrm{sort}(I \cup J) = (a_1, a_2, \ldots, a_{2k})$. Suppose for the sake of contradiction that for some i, $a_{i+2} = a_i + t$ for some $t \in [r-1]$. Here we think of our indices and addition modulo n. We remark that t must be nonzero since the multiplicity of each element of $[n]$ in $\mathrm{sort}(I \cup J)$ is at most two. Without loss of generality, we assume that $a_i \in I$. Hence, $a_{i+2} \in J$ since I is r-stable and $\mathrm{cd}(a_i, a_{i+2}) < r$. Since $\mathrm{sort}(I \cup J)$ is nondecreasing it follows that $a_{i+1} = a_i + j$ for some $j \in \{0, 1, \ldots, t\}$. First consider the cases where $j = 0$ and $j = t$. In the former case we have that $a_{i+1} = a_i$, and in the latter case $a_{i+1} = a_{i+2}$. Hence, in the former case, the multiplicity of a_i in $\text{sort}(I \cup J)$ is two. Thus, a_i appeared in both I and J. Since $a_{i+2} \in J$, this contradicts the assumption that J is r-stable. Similarly, in the latter case the multiplicity of a_{i+2} in $\text{sort}(I \cup J)$ is two, so a_{i+2} must also appear in I, and this contradicts the assumption that I is r-stable. Now suppose that $0 < j < t$. Then since I is r-stable and $a_i \in I$, it must be that $a_{i+1} \in J$. But since J is r-stable and $a_{i+2} \in J$, then $a_{i+1} \in I$, a contradiction. $\square$

We let $\nabla^r_{n,k}$ denote the triangulation of $\Delta^{\text{stab}(r)}_{n,k}$ induced by $\nabla_{n,k}$. This gives the following nesting of triangulations

$$\nabla_{n,k} \supset \nabla^2_{n,k} \supset \nabla^3_{n,k} \supset \cdots \supset \nabla^{\left\lfloor \frac{n}{k} \right\rfloor}_{n,k}.$$

In Section 3.1, the following lemma will play a key role.

Lemma 2.2.7. *If* $n \equiv 1 \mod k$, *then* $\Delta^{\text{stab}(r)}_{n,k}$ *is* $(n-1)$*-dimensional for all* $r \in \left[\left\lfloor \frac{n}{k} \right\rfloor\right]$. *In particular,* $\Delta^{stab\left(\left\lfloor \frac{n}{k} \right\rfloor\right)}_{n,k}$ *is a unimodular* $(n-1)$*-simplex.*

Proof. Notice first that for $r = \left\lfloor \frac{n}{k} \right\rfloor$ there are precisely n r-stable k-subsets of $[n]$. Hence, $\Delta^{\text{stab}(r)}_{n,k}$ is an $(n-1)$-dimensional simplex. Now suppose ϵ is a vertex of this simplex. Then precisely k entries in ϵ are occupied by 1's, $k-1$ pairs of these 1's are separated by $r-1$ 0's, and the remaining pair is separated by r 0's. Hence, the only 1 that can be moved to the right and result in another r-stable vertex is the left-most 1 in the pair of 1's separated by r 0's. Making this move n times results in returning to the vertex ϵ, and produces a minimal circuit (ω) in $G_{n,k}$ using only r-stable vertices. Since there are only n such vertices it must be that $\sigma_{(\omega)} = \Delta^{\text{stab}(r)}_{n,k}$.

We may also prove this result using Sturmfels' construction of this triangulation. Simply notice that there are precisely n $\left\lfloor \frac{n}{k} \right\rfloor$-stable k-subsets of $[n]$, namely

$$\begin{gathered}
\{1, 1 + \left\lfloor \tfrac{n}{k} \right\rfloor, 1 + 2\left\lfloor \tfrac{n}{k} \right\rfloor, \ldots, 1 + (k-1)\left\lfloor \tfrac{n}{k} \right\rfloor\}, \\
\{2, 2 + \left\lfloor \tfrac{n}{k} \right\rfloor, 2 + 2\left\lfloor \tfrac{n}{k} \right\rfloor, \ldots, 2 + (k-1)\left\lfloor \tfrac{n}{k} \right\rfloor\}, \\
\vdots \\
\{n, n + \left\lfloor \tfrac{n}{k} \right\rfloor, n + 2\left\lfloor \tfrac{n}{k} \right\rfloor, \ldots, n + (k-1)\left\lfloor \tfrac{n}{k} \right\rfloor\}.
\end{gathered}$$

It is easy to see that these subsets form a sorted collection of k-subsets of $[n]$. Hence, they correspond to a unimodular $(n-1)$-simplex in the triangulation $\nabla_{n,k}$. $\square$

We now utilize the triangulation $\nabla^r_{n,k}$ to compute the facets of $\Delta^{\text{stab}(r)}_{n,k}$.

2.3 The Facets of the r-stable Hypersimplices

We first recall the definitions of the (n,k)-hypersimplices and the r-stable (n,k)-hypersimplices. For integers $0 < k < n$ let $[n] := \{1, 2, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. The *characteristic vector* of a subset I of $[n]$ is the $(0,1)$-vector $\epsilon_I = (\epsilon_1, \ldots, \epsilon_n)$ for which $\epsilon_i = 1$ for $i \in I$ and $\epsilon_i = 0$ for $i \notin I$. The

(n,k)-*hypersimplex* is the convex hull in $\mathbb{R}^n$ of the collection of characteristic vectors $\{\epsilon_I : I \in \binom{[n]}{k}\}$, and it is denoted $\Delta_{n,k}$. Label the vertices of a regular n-gon embedded in $\mathbb{R}^2$ in a clockwise fashion from 1 to n. Given a third integer $1 \leq r \leq \lfloor \frac{n}{k} \rfloor$, a subset $I \subset [n]$ (and its characteristic vector) is called r-*stable* if, for each pair $i, j \in I$, the path of shortest length from i to j about the n-gon uses at least r edges. The r-*stable* n,k-*hypersimplex*, denoted by $\Delta_{n,k}^{\mathrm{stab}(r)}$, is the convex polytope in $\mathbb{R}^n$ which is the convex hull of the characteristic vectors of all r-stable k-subsets of $[n]$. For a fixed n and k the r-stable (n,k)-hypersimplices form the nested chain of polytopes

$$\Delta_{n,k} \supset \Delta_{n,k}^{\mathrm{stab}(2)} \supset \Delta_{n,k}^{\mathrm{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\mathrm{stab}\left(\lfloor \frac{n}{k} \rfloor\right)}.$$

Notice that $\Delta_{n,k}$ is precisely the 1-stable (n,k)-hypersimplex.

The definitions of $\Delta_{n,k}$ and $\Delta_{n,k}^{\mathrm{stab}(r)}$ provided are V-representations of these polytopes. In this section we provide the minimal H-representation of $\Delta_{n,k}^{\mathrm{stab}(r)}$, i.e. its collection of facet-defining inequalities. It is well-known that the facet-defining inequalities of $\Delta_{n,k}$ are $\sum_{i=1}^n x_i = k$ together with $x_\ell \geq 0$ and $x_\ell \leq 1$ for all $\ell \in [n]$. Let H denote the hyperplane in $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^n x_i = k$. For $1 \leq r \leq \lfloor \frac{n}{k} \rfloor$ and $\ell \in [n]$ consider the closed convex subsets of $\mathbb{R}^n$

$$H_\ell^{(+)} := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_\ell \geq 0\} \cap H, \text{ and}$$
$$H_{\ell,r}^{(-)} := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=\ell}^{\ell+r-1} x_i \leq 1 \right\} \cap H.$$

In the definition of $H_{\ell,r}^{(-)}$ the indices i of the coordinates $x_1, \ldots, x_n$ are taken to be elements of $\mathbb{Z}/n\mathbb{Z}$. We also let H_ℓ and $H_{\ell,r}$ denote the $(n-2)$-flats given by strict equality in the above definitions. In the following we will say an $(n-2)$-flat is *facet-defining* (or *facet-supporting*) for $\Delta_{n,k}^{\mathrm{stab}(r)}$ if it contains a facet of $\Delta_{n,k}^{\mathrm{stab}(r)}$.

Theorem 2.3.1. *Let $1 < k < n-1$. For $1 \leq r < \lfloor \frac{n}{k} \rfloor$ the facet-defining inequalities for $\Delta_{n,k}^{\mathrm{stab}(r)}$ are $\sum_{i=1}^n x_i = k$ together with $\sum_{i=\ell}^{\ell+r-1} x_i \leq 1$ and $x_\ell \geq 0$ for $\ell \in [n]$. In particular,*

$$\Delta_{n,k}^{\mathrm{stab}(r)} = \bigcap_{\ell \in [n]} H_\ell^{(+)} \cap \bigcap_{\ell \in [n]} H_{\ell,r}^{(-)}.$$

The following is an immediate corollary to these results.

Corollary 2.3.2. *All but possibly the smallest polytope in the nested chain*

$$\Delta_{n,k} \supset \Delta_{n,k}^{\mathrm{stab}(2)} \supset \Delta_{n,k}^{\mathrm{stab}(3)} \supset \cdots \supset \Delta_{n,k}^{\mathrm{stab}\left(\lfloor \frac{n}{k} \rfloor\right)}$$

has $2n$ facets.

This is an interesting geometric property since the number of vertices of these polytopes strictly decreases down the chain. To prove Theorem 2.3.1 we will utilize the geometry of the circuit triangulation of $\Delta_{n,k}$, and the nesting of triangulations:

$$\nabla_{n,k} \supset \nabla_{n,k}^2 \supset \nabla_{n,k}^3 \supset \cdots \supset \nabla_{n,k}^{\lfloor \frac{n}{k} \rfloor}.$$

The method by which we will do this is outlined in the following remark.

Remark 2.3.3. To compute the facet-defining inequalities of $\Delta_{n,k}^{\mathrm{stab}(r)}$ we first consider the geometry of their associated facet-defining $(n-2)$-flats. Suppose that $\Delta_{n,k}^{\mathrm{stab}(r)}$ is $(n-1)$-dimensional. Since $\Delta_{n,k}^{\mathrm{stab}(r-1)} \supset \Delta_{n,k}^{\mathrm{stab}(r)}$ then a facet-defining $(n-2)$-flat of $\Delta_{n,k}^{\mathrm{stab}(r)}$ either also defines a facet of $\Delta_{n,k}^{\mathrm{stab}(r-1)}$ or it intersects $\operatorname{relint} \Delta_{n,k}^{\mathrm{stab}(r-1)}$, the relative interior of $\Delta_{n,k}^{\mathrm{stab}(r-1)}$. Therefore, to compute the facet-defining $(n-2)$-flats of $\Delta_{n,k}^{\mathrm{stab}(r)}$ it suffices to compute the former and latter collections of $(n-2)$-flats independently. To identify the former collection we will use an induction argument on r. To identify the latter collection we work with pairs of adjacent $(n-1)$-simplices in the set $\max \nabla_{n,k}^{r}$. Note that two simplices $u, \omega \in \max \nabla_{n,k}^{r}$ are adjacent (i.e. share a common facet) if and only if they differ by a single vertex. Therefore, their common vertices span an $(n-2)$-flat which we will denote by $H[u,\omega]$. Thus, we will identify adjacent pairs of simplices $u \in \max \nabla_{n,k}^{r-1}$ and
$\omega \in \max \nabla_{n,k}^{r-1} \setminus \max \nabla_{n,k}^{r}$ for which $H[u,\omega]$ is facet-defining.

Computing facet-defining inequalities via a nesting of triangulations.

Suppose $1 < k < n-1$. In order to prove Theorem 2.3.1 in the fashion outlined by Remark 2.3.3 we require a sequence of lemmas. Notice that $\Delta_{n,k}^{\mathrm{stab}(r)}$ is contained in $H_\ell^{(+)}$ and $H_{\ell,r}^{(-)}$ for all $\ell \in [n]$. So in the following we simply show that H_ℓ and $H_{\ell,r}$ form the complete set of facet-defining $(n-2)$-flats.

Lemma 2.3.4. *Let $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. For all $\ell \in [n]$, H_ℓ is facet-defining for $\Delta_{n,k}^{\mathrm{stab}(r)}$.*

Proof. First notice that the result clearly holds for $r = 1$. So we need only show that $n-1$ affinely independent vertices of $\Delta_{n,k}^{\mathrm{stab}(r)}$ lie in H_ℓ. Hence, to prove the claim it suffices to identify a simplex $\omega \in \max \nabla_{n,k}^{r}$ such that H_ℓ supports a facet of ω. Since $r \leq \left\lfloor \frac{n}{k} \right\rfloor - 1$ it also suffices to work with $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$.

Fix $\ell \in [n]$. For $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$ we construct a minimal circuit in the graph $G_{n,k}$ that corresponds to a simplex in $\max \nabla_{n,k}^{r}$ for which H_ℓ is facet-supporting. To this end, consider the characteristic vector of the k-subset $\{(\ell-1)-(s-1)r : s \in [k]\} \subset [n]$. Denote this characteristic vector by ϵ^ℓ, and think of its indices modulo n. Labeling the 1 in coordinate $(\ell-1)-(s-1)r$ of ϵ^ℓ as 1_s, we see that 1_s and 1_{s+1} are separated by $r-1$ zeros for $s \in [k-1]$. That is, the coordinate $\epsilon_i^\ell = 0$ for every $(\ell-1)-sr < i < (\ell-1)-(s-1)r$ (modulo n), and there are precisely $r-1$ such coordinates. Moreover, since $kr = k\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right) \leq n$ then there are at least $r-1$ zeros between 1_1 and 1_k. Hence, this vertex is r-stable. From ϵ^ℓ we can now construct an r-stable circuit (ω^ℓ) by moving the 1's in ϵ^ℓ one coordinate to the right (modulo n), one at a time, in the following pattern:

(1) Move 1_1.

(2) Move 1_1. Then move 1_2. Then move 1_3. … Then move 1_k.

(3) Repeat step (2) $r-1$ more times.

$$
\begin{tikzcd}
\epsilon^\ell = (0,1,0,1,0,0,0,0,1) \arrow[r, "4"] & (0,1,0,0,1,0,0,0,1) \arrow[r, "5"] & (0,1,0,0,0,1,0,0,1) \arrow[d, "2"] \\
 & & (0,0,1,0,0,1,0,0,1) \arrow[d, "9"] \\
(0,1,0,1,0,0,0,1,0) \arrow[uu, "8"] & & (1,0,1,0,0,1,0,0,0) \arrow[d, "6"] \\
(0,1,0,1,0,0,1,0,0) \arrow[u, "7"] & (1,0,0,1,0,0,1,0,0) \arrow[l, "1"'] & (1,0,1,0,0,0,1,0,0) \arrow[l, "3"']
\end{tikzcd}
$$

Figure 2.2: The minimal circuit (ω^ℓ) for $n = 9, k = 3$, and $\ell = 5$ constructed in Lemma 2.3.4.

(4) Move 1_1 until it rests in entry $\ell - 1$.

An example of (ω^ℓ) for $n = 9, k = 3$, and $\ell = 5$ is provided in Figure 2.2. This produces a minimal circuit in $G_{n,k}$ since each 1_s has moved precisely enough times to replace 1_{s+1}. Moreover, since $k > 1$ then $k\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right) \leq n - 2$. So there are at least $r + 1$ 0's between 1_1 and 1_k in ϵ^ℓ. From here, it is a straight-forward exercise to check that every vertex in (ω^ℓ) is r-stable. Therefore, $\omega^\ell \in \max \nabla^r_{n,k}$. Finally, since $r > 1$, the simplex ω^ℓ has only one vertex satisfying $x_\ell = 1$, and this is the vertex following ϵ^ℓ in the circuit (ω^ℓ). Hence, all other vertices of ω^ℓ satisfy $x_\ell = 0$. So H_ℓ supports a facet of ω^ℓ. Thus, we conclude that H_ℓ is facet-defining for $\Delta^{\mathrm{stab}(r)}_{n,k}$ for $r < \left\lfloor \frac{n}{k} \right\rfloor$. $\square$

The following theorem follows immediately from the construction of the $(n-1)$-simplex ω^ℓ in the proof of Lemma 2.3.4, and it justifies the assumption on the dimension of $\Delta^{\mathrm{stab}(r)}_{n,k}$ made in Remark 2.3.3.

Theorem 2.3.5. *The polytope $\Delta^{\mathrm{stab}(r)}_{n,k}$ is $(n-1)$-dimensional for all $r < \left\lfloor \frac{n}{k} \right\rfloor$.*

Lemma 2.3.6. *Suppose $r > 1$ and $\Delta^{\mathrm{stab}(r)}_{n,k}$ is $(n-1)$-dimensional. Then $H_{\ell,r-1}$ is not facet-defining for $\Delta^{\mathrm{stab}(r)}_{n,k}$.*

Proof. Suppose for the sake of contradiction that $H_{\ell,r-1}$ is facet-defining for $\Delta^{\mathrm{stab}(r)}_{n,k}$. Since $\Delta^{\mathrm{stab}(r)}_{n,k}$ is $(n-1)$-dimensional then there exists an $(n-1)$-simplex $\omega \in \max \nabla^r_{n,k}$ such that $H_{\ell,r-1}$ is facet-defining for ω. In other words, every vertex in (ω) satisfies $\sum_{i=\ell}^{\ell+r-2} x_i = 1$ except for exactly one vertex, say $\epsilon^\star$. Since all vertices in (ω) are $(0,1)$-vectors, this means all vertices other than $\epsilon^\star$ have exactly one coordinate in the subvector $(\epsilon_\ell, \epsilon_{\ell+1}, \ldots, \epsilon_{\ell+r-2})$ being 1 and all other coordinates are 0. Similarly, this subvector is the 0-vector for $\epsilon^\star$. Since (ω) is a minimal circuit this means that the move preceding the vertex $\epsilon^\star$ in (ω) results in the only 1 in $(\epsilon_\ell, \epsilon_{\ell+1}, \ldots, \epsilon_{\ell+r-2})$ exiting the subvector to the right. Similarly, the move following the vertex $\epsilon^\star$ in (ω) results in a single 1 entering the subvector on the left. Suppose that

$$\epsilon^\star = (\ldots, \epsilon^\star_{\ell-1}, \epsilon^\star_\ell, \epsilon^\star_{\ell+1}, \ldots, \epsilon^\star_{\ell+r-2}, \epsilon^\star_{\ell+r-1}, \ldots) = (\ldots, 1, 0, 0, \ldots, 0, 1, \ldots).$$

Then this situation looks like

$$\begin{tikzcd}[column sep=small]
 & \epsilon^\star = (\ldots,1,0,0,\ldots,0,1,\ldots) \arrow[dr, "\ell-1"] & \\
(\ldots,1,0,0,\ldots,1,0,\ldots) \arrow[ur, "\ell+r-2"] & & (\ldots,0,1,0,\ldots,0,1,\ldots)
\end{tikzcd}$$

Hence, neither the vertex preceding or following the vertex $\epsilon^\star$ is r-stable. For example, in the vertex following $\epsilon^\star$ there is a 1 in entries ℓ and $\ell + r - 1$. This contradicts the fact that $\omega \in \max \nabla^r_{n,k}$. $\square$

To see why Lemma 2.3.6 will be useful suppose that Theorem 2.3.1 holds for $\Delta^{\text{stab}(r-1)}_{n,k}$ for some $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. Then Lemmas 2.3.4 and 2.3.6 tell us that the collection of facet-defining $(n-2)$-flats for $\Delta^{\text{stab}(r-1)}_{n,k}$ that are also facet-defining for $\Delta^{\text{stab}(r)}_{n,k}$ is $\{H_\ell : \ell \in [n]\}$. This is the nature of the induction argument mentioned in Remark 2.3.3. To identify the facet-defining $(n-2)$-flats of $\Delta^{\text{stab}(r)}_{n,k}$ that intersect $\operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k}$ we will use the following definition.

Definition 2.3.7. Suppose u and ω are a pair of simplices in $\max \nabla_{n,k}$ satisfying

- $u \in \max \nabla^r_{n,k}$,
- $\omega \in \max \nabla^{r-1}_{n,k} \backslash \max \nabla^r_{n,k}$, and
- ω uses exactly one vertex that is not r-stable, called the *key vertex*, and this is the only vertex by which u and ω differ.

We say that the ordered pair of simplices (u, ω) is an *r-supporting pair* of $H[u, \omega]$, where $H[u, \omega]$ is the flat spanned by the common vertices of u and ω.

Lemma 2.3.8. *Suppose* $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$. *Suppose also that* H_F *is a* $(n-2)$*-flat defining a facet* F *of* $\Delta^{\text{stab}(r)}_{n,k}$ *such that* $H_F \cap \operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k} \neq \emptyset$. *Then* $H_F = H[u, \omega]$ *for some* r*-supporting pair of simplices* (u, ω).

Proof. Since $H_F \cap \operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k} \neq \emptyset$ and $\Delta^{\text{stab}(r)}_{n,k}$ is contained in $\Delta^{\text{stab}(r-1)}_{n,k}$ then $F \cap \operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k} \neq \emptyset$. That is, there exists some $\alpha \in F$ such that $\alpha \in \operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k}$. Recall that $\nabla^{r-1}_{n,k}$ is a triangulation of $\Delta^{\text{stab}(r-1)}_{n,k}$ that restricts to a triangulation $\nabla^r_{n,k}$ of $\Delta^{\text{stab}(r)}_{n,k}$. It follows that $\nabla^r_{n,k}$ and $\nabla^{r-1}_{n,k} \backslash \nabla^r_{n,k}$ give identical triangulations of $\partial \Delta^{\text{stab}(r)}_{n,k} \cap \operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k}$. Since $\Delta^{\text{stab}(r)}_{n,k}$ is $(n-1)$-dimensional we may assume, without loss of generality, that α lies in the relative interior of an $(n-2)$-dimensional simplex in the triangulation of $\partial \Delta^{\text{stab}(r)}_{n,k} \cap \operatorname{relint} \Delta^{\text{stab}(r-1)}_{n,k}$ induced by $\nabla^r_{n,k}$ and $\nabla^{r-1}_{n,k} \backslash \nabla^r_{n,k}$. Therefore, there exists some $u \in \max \nabla^r_{n,k}$ such that H_F is facet-defining for u and $\alpha \in u \cap H_F$, and there exists some $\omega \in \max \nabla^{r-1}_{n,k} \backslash \max \nabla^r_{n,k}$ such that $\alpha \in \omega \cap H_F$. Since $\nabla^{r-1}_{n,k}$ is a triangulation of $\Delta^{\text{stab}(r-1)}_{n,k}$ it follows that $u \cap H_F = \omega \cap H_F$. Hence, u

and ω are adjacent simplices that share the facet-defining $(n-2)$-flat H_F, and they form an r-supporting pair (u, ω) with $H[u, \omega] = H_F$. $\square$

It will be helpful to understand the key vertex of an r-supporting pair (u, ω). To do so, we will use the following definition.

Definition 2.3.9. Let $\epsilon \in \mathbb{R}^n$ be a vertex of $\Delta_{n,k}$. A *pair* of 1's in ϵ is an ordered pair of two coordinates of ϵ, (i, j), such that $\epsilon_i = \epsilon_j = 1$, and $\epsilon_t = 0$ for all $i < t < j$ (modulo n). A pair of 1's is called an *r-stable pair* if there are at least $r-1$ 0's separating the two 1's.

Lemma 2.3.10. *Suppose (u, ω) is an r-supporting pair, and let ϵ be the key vertex of this pair. Then ϵ contains precisely one $(r-1)$-stable but not r-stable pair, $(\ell, \ell+r-1)$. Moreover, $H[u, \omega] = H_{\ell,r}$.*

Proof. We first show that ϵ has precisely one $(r-1)$-stable but not r-stable pair, $(\ell, \ell+r-1)$. To see this, consider the minimal circuit (ω) in the graph $G_{n,k}$ associated with the simplex ω. Think of the key vertex ϵ as the initial vertex of this circuit, and recall that each edge of the circuit corresponds to a move of exactly one 1 to the right by exactly one entry. Hence, in the circuit (ω) the vertex following ϵ differs from ϵ by a single right move of a single 1. Since ϵ is the only vertex in (ω) that is $(r-1)$-stable but not r-stable then the move of this single 1 to the right by one entry must eliminate all pairs that are $(r-1)$-stable but not r-stable. Moreover, this move cannot introduce any new $(r-1)$-stable but not r-stable pairs. Since a single 1 can be in at most two pairs, and this 1 must move exactly one entry to the right, then this 1 must be in entry j in the pairs (i, j) and (j, t) where (i, j) is $(r-1)$-stable but not r-stable, and (j, t) is $(r+1)$-stable. Moreover, since the move of the 1 in entry j can only change the stability of the pairs (i, j) and (j, t) then it must be that all other pairs are r-stable.

Finally, since ω has the unique $(r-1)$-stable but not r-stable vertex ϵ, and since ϵ has the unique $(r-1)$-stable but not r-stable pair $(\ell, \ell+r-1)$ then all other vertices in ω satisfy $\sum_{i=\ell}^{\ell+r-1} x_i = 1$. Hence, $H[u, \omega] = H_{\ell,r}$. $\square$

Lemma 2.3.11. *Suppose $1 < r < \left\lfloor \frac{n}{k} \right\rfloor$. Suppose also that H_F is an $(n-2)$-flat defining a facet F of $\Delta_{n,k}^{\mathrm{stab}(r)}$ and $H_F \cap \operatorname{relint} \Delta_{n,k}^{\mathrm{stab}(r-1)} \neq \emptyset$. Then $H_F = H_{\ell,r}$ for some $\ell \in [n]$.*

Proof. By Lemma 2.3.8 $H_F = H[u, \omega]$ for some r-supporting pair (u, ω). By Lemma 2.3.10 ω has a unique vertex that is $(r-1)$-stable but not r-stable with a unique $(r-1)$-stable but not r-stable pair $(\ell, \ell+r-1)$ for some $\ell \in [n]$. Thus, $H_F = H[u, \omega] = H_{\ell,r}$. $\square$

We now show that $H_{\ell,r}$ is indeed facet-defining for $\Delta_{n,k}^{\mathrm{stab}(r)}$ for all $\ell \in [n]$.

Lemma 2.3.12. *Suppose $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ or $n = kr+1$. Then $H_{\ell,r}$ is facet-defining for $\Delta_{n,k}^{\mathrm{stab}(r)}$ for all $\ell \in [n]$.*

Proof. First we note that the result is clearly true for $r = 1$. So in the following we assume $r > 1$. To prove the claim we show that $H_{\ell,r}$ supports an $(n-1)$-simplex $\omega \in \max \nabla^r_{n,k}$.

To this end, consider the characteristic vector of the k-subset $\{(\ell-1)+(s-1)r : s \in [k]\} \subset [n]$. Denote this characteristic vector by ϵ^ℓ, and think of its indices modulo n. Labeling the 1 in coordinate $(\ell-1)+(s-1)r$ of ϵ^ℓ as 1_s, it is quick to see that 1_s and 1_{s+1} are separated by $r-1$ zeros for every $s \in [k]$. That is, $\epsilon^\ell_i = 0$ for every $(\ell-1)+(s-1)r < i < (\ell-1)+sr$ (modulo n), and there are precisely $r-1$ such coordinates. Moreover, since $r < \lfloor \frac{n}{k} \rfloor$ or $n = kr+1$ then $n \geq kr+1$. So there are at least r zeros between 1_1 and 1_k. Hence, this vertex is r-stable. From ϵ^ℓ we can now construct an r-stable circuit (ω^ℓ) by moving the 1's in ϵ^ℓ one coordinate to the right (modulo n), one at a time, in the following pattern:

(1) Move 1_k. Then move 1_{k-1}. Then move 1_{k-2}. ... Then move 1_1.

(2) Repeat step (1) $r-1$ more times.

(3) Move 1_k to entry ℓ.

Each move in this pattern produces a new r-stable vertex since there are always at least $r-1$ zeros between each pair of 1's. So $\omega^\ell \in \max \nabla^r_{n,k}$ and $H_{\ell,r}$ supports ω^ℓ since every vertex of (ω^ℓ) lies in $H_{\ell,r}$ except for the vertex preceding the first move of 1_1 in the circuit (ω^ℓ). □

Remark 2.3.13. When $n = kr+1$ then $\omega^\ell = \Delta^{\mathrm{stab}(r)}_{n,k}$ for all $\ell \in [n]$. So the facet-defining inequalities for $\omega^\ell = \Delta^{\mathrm{stab}(r)}_{n,k}$ are precisely $H^{(-)}_{\ell,r}$ for $\ell \in [n]$.

From Lemmas 2.3.11 and 2.3.12 we see that when $1 < r < \lfloor \frac{n}{k} \rfloor$ the facet-defining $(n-2)$-flats for $\Delta^{\mathrm{stab}(r)}_{n,k}$ that intersect relint $\Delta^{\mathrm{stab}(r-1)}_{n,k}$ are precisely $H_{\ell,r}$ for $\ell \in [n]$. We are now ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1.

First recall that Theorem 2.3.1 is known to be true for $r = 1$. Now let $1 < r < \lfloor \frac{n}{k} \rfloor$. By Theorem 2.3.5 we know that $\Delta^{\mathrm{stab}(r)}_{n,k}$ is $(n-1)$-dimensional. First let $r = 2$. By Lemma 2.3.4 we know that H_ℓ is facet-defining for $\Delta^{\mathrm{stab}(2)}_{n,k}$ for all $\ell \in [n]$. By Lemma 2.3.6 we know that for every $\ell \in [n]$ $H_{\ell,1}$ is not facet-defining for $\Delta^{\mathrm{stab}(2)}_{n,k}$. Thus, the collection of facet-defining $(n-2)$-flats for $\Delta_{n,k}$ that are also facet-defining for $\Delta^{\mathrm{stab}(2)}_{n,k}$ are $\{H_\ell : \ell \in [n]\}$, and all other facet-defining $(n-2)$-flats for $\Delta^{\mathrm{stab}(2)}_{n,k}$ must intersect the relative interior of $\Delta_{n,k}$. Therefore, by Lemmas 2.3.11 and 2.3.12 the remaining facet-defining $(n-2)$-flats for $\Delta^{\mathrm{stab}(2)}_{n,k}$ are $H_{\ell,2}$ for $\ell \in [n]$. Since $\Delta^{\mathrm{stab}(r)}_{n,k}$ is contained in $H^{(+)}_\ell$ and $H^{(-)}_{\ell,r}$, this proves the result for $r = 2$. Theorem 2.3.1 then follows by iterating this argument for $2 < r < \lfloor \frac{n}{k} \rfloor$.

2.4 The Facets for the r-stable Second Hypersimplices

In combinatorics, it is desirable to have more than one proof of the same result, as this often results into different insights into the underlying combinatorial, algebraic, or geometric structures. In section 2.3, we computed the facets of $\Delta_{n,k}^{\mathrm{stab}(r)}$ using the geometry of a unimodular triangulation, and this revealed much about the geometry of these polytopes. When $k = 2$ the polytope $\Delta_{n,2}^{\mathrm{stab}(r)}$ is an *edge polytope*, meaning that it can be defined in terms of the edges of a graph. In this section, we offer a second computation of the facets of $\Delta_{n,2}^{\mathrm{stab}(r)}$ that uses properties of this graph, thereby revealing the combinatorics associated to this geometric result. Indeed, the facets of $\Delta_{n,2}^{\mathrm{stab}(r)}$ demonstrate a high level of symmetry, and we will see that this symmetry is also apparent in their underlying graphs.

Let G be a finite connected graph on vertex set $V(G) = [n]$ having no loops or multiple edges (e.g. G is *simple*). Let $E(G) := \{\{i,j\} : i,j \in V(G)\}$ denote the set of edges of G. If $e = \{i,j\} \in E(G)$ then define $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$, where $\mathbf{e}_i$ denotes the i^{th} standard basis vector in $\mathbb{R}^n$. The *edge polytope of* G, denoted P_G, is the convex hull of the collection of vectors $\{\rho(e) : e \in E(G)\}$.

To compute the facets of $\Delta_{n,2}^{\mathrm{stab}(r)}$ we will use a well-known result of Ohsugi and Hibi which describes the defining hyperplanes of an edge polytope P_G [28]. To state this result we first recall a few definitions. For a vertex $i \in [n]$ a *neighbor of* i is a vertex $j \in [n]$ such that $\{i,j\} \in E(G)$. We write $N(G,i)$ for the set of all neighbors of i, and for a nonempty subset X of $[n]$ we write $N(G,X) := \bigcup_{i\in X} N(G,i)$. An *odd cycle* in a graph G is a cycle whose length is odd. We will also denote a *path* p *from* i *to* j *in* G by $p : i \rightsquigarrow j$. For a nonempty subset W of $[n]$ we let G_W denote the subgraph of G having vertex set W and edge set $\{\{i,j\} \in E(G) : i,j \in W\}$. We say that $i \in [n]$ is *regular in* G if every connected component of $G_{[n]\setminus\{i\}}$ has at least one odd cycle. A nonempty subset T of $[n]$ is called *independent in* G if $N(G,i) \cap T = \emptyset$ for every $i \in T$ (in other words, there is no edge $\{i,j\}$ in G such that $i,j \in T$). If T is independent in G, then the *bipartite graph induced by* T is the bipartite graph with vertex set $T \cup N(G,T)$ and edge set $\{\{i,j\} \in E(G) : i \in T, j \in N(G,T)\}$. If the graph G has at least one odd cycle we say that a nonempty subset T of $[n]$ is *fundamental in* G if

(i) T is independent in G and the bipartite graph induced by T is connected, and

(ii) either $[n] = T \cup N(G,T)$ or every connected component of the subgraph $G_{[n]\setminus(T\cup N(G,T))}$ contains at least one odd cycle.

For $i \in [n]$, let K_i denote the hyperplane

$$K_i := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i = 0\},$$

and for a nonempty subset $T \subset [n]$ let K_T denote the hyperplane

$$K_T := \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i\in T} x_i = \sum_{j\in N(G,T)} x_j \right\}.$$

With these definitions in hand we are ready to state the result of Hibi and Ohsugi.

Theorem 2.4.1. *[28, Theorem 1.7(a)] Let G be a finite connected graph on the vertex set $[n]$ allowing loops and having no multiple edges, and suppose that G contains at least one odd cycle (i.e.* $\dim P_G = n-1$*). Let Ψ denote the set of those hyperplanes K_i such that i is regular in G and K_T such that T is fundamental in G. Then the set of facets of P_G is*

$$\{K \cap P_G : K \in \Psi\}.$$

As noted in the introduction, the polytope $\Delta_{n,2}^{\mathrm{stab}(r)}$ is an edge polytope. In particular, the underlying graph for $\Delta_{n,2}^{\mathrm{stab}(r)}$ is the graph with vertex set $[n]$ and edges $\{i,j\}$ such that the set $\{i,j\}$ is an r-stable 2-subset of $[n]$. We call this graph the *r-stable graph on n elements* and denote it by $G_{n,r}$.

Proposition 2.4.2. *Fix a positive integer r. For every $n \geq 2r+1$ the r-stable graph on n elements $G_{n,r}$ is connected.*

Proof. Notice first that for all $i \in [n]$

$$\begin{aligned} N(G,i) &= \{i+r, i+r+1, i+r+2, \ldots, i-r-2, i-r-1, i-r\}, \text{ and} \\ N(G,i+1) &= \{i+r+1, i+r+2, i+r+3, \ldots, i-r-1, i-r, i-r+1\}. \end{aligned}$$

Since $n \geq 2r+1$ we have that $2r < n$. Hence $\#N(G,i) \geq 2$. Thus,

$$\{i+r, i+r+1\} \subset N(G,i).$$

So fix $v, w \in [n]$. Suppose, without loss of generality, that $v < w$. Then $w = v+s$ for some $s \in [n]$. It then follows that we have the path $p : v \rightsquigarrow w$ given by

$$\begin{aligned} p =& (\{v, v+r+1\}, \{v+r+1, v+1\}, \{v+1, v+r+2\}, \{v+r+2, v+2\}, \ldots \\ & \ldots, \{v+s-1, v+r+s\}, \{v+r+s, v+s=w\}). \end{aligned}$$

Hence, $G_{n,r}$ is connected. □

Remark 2.4.3. In $G_{n,r}$, we have that for every $i \in [n]$

$$\begin{aligned} N(G,i) &= \{i+r, i+r+1, i+r+2, \ldots, i-r-2, i-r-1, i-r\}, \text{ and} \\ N(G,i+1) &= \{i+r+1, i+r+2, i+r+3, \ldots, i-r-1, i-r, i-r+1\}. \end{aligned}$$

This relationship between the sets of neighbors of vertices in terms of their relative distance from one another on the n-gon will play an important role in the coming proofs.

Proposition 2.4.4. *Fix a positive integer r. For all $n \geq 2r+1$ the r-stable graph on n elements $G_{n,r}$ contains an odd cycle of length $2r+1$.*

Proof. Just as in Proposition 2.4.2, since $n \geq 2r+1$ then $2r < n$ so $\#N(G,i) \geq 2$. In particular,

$$\{i+r, i+r+1\} \subset N(G,i).$$

Fix $i \in [n]$. Using the same method as before we can construct the cycle $p : i \rightsquigarrow i$ given by

$$\begin{aligned} p =& (\{i, i+r+1\}, \{i+r+1, i+1\}, \{i+1, i+r+2\}, \{i+r+2, i+2\}, \ldots \\ & \ldots, \{i+r-1, (i+r-1)+r+1 = i+2r\}, \{i+2r, i+r\}, \{i+r, i\}). \end{aligned}$$

The cycle p uses precisely two edges to go from $i+s$ to $i+s+1$ for all $s \in \{0, 1, 2, \ldots, r-1\}$, and then finishes with the edge $\{i+r, i\}$. Hence, $G_{n,r}$ contains an odd cycle of length $2r+1$.

□

Proposition 2.4.5. *Fix a positive integer r and let $n \geq 2r+2$. Then the vertex i is regular in $G_{n,r}$ for every $i \in [n]$.*

Proof. First notice that $(G_{n,r})_{[n]\setminus\{i\}}$ always contains $G_{n-1,r}$ as a subgraph on the vertex set $[n]\setminus\{i\}$. Since $n \geq 2r+2$ then $n-1 \geq 2r+1$. So by Propositions 2.4.2 and 2.4.4, $G_{n-1,r}$ is connected and contains at least one odd cycle. Thus, i is regular in $G_{n,r}$.

□

We would next like to compute the fundamental sets in $G_{n,r}$. To do this, we first show that the cardinality of a fundamental set in $G_{n,r}$ is at most r. It will be helpful to have the following definition. Fix $i \in [n]$. Let B_i denote the bipartite graph with vertex set $U \cup V$ where

$$\begin{aligned} U &= \{i+1, i+2, \ldots, i+r-1\}, \\ V &= \{i-1, i-2, \ldots, i-r+1\}, \end{aligned}$$

and with edge set $\{\{i,j\} \in E(G) : i \in U, j \in V\}$. Notice that there is a perfect matching of U into V, which will denote by M_i, that is given by

$$M_i = \{\{i+s, i-(r-s)\} : s \in [r-1]\}.$$

Proposition 2.4.6. *Let T be fundamental in $G_{n,r}$. Then $\#T \leq r$.*

Proof. Since $T \neq \emptyset$ then there exists $i \in T$. Since we may have no $j \in T$ with $\{i,j\}$ an edge of $G_{n,r}$ it follows that

$$T \subset \{i-r+1, i-r+2, \ldots, i-1, i, i+1, \ldots, i+r-1\}.$$

Consider the bipartite graph B_i, and the perfect matching M_i. Since T is an independent set then T contains at most one vertex in each edge of M_i. Hence, $\#T \leq \#M_i + 1 = r$.

□

Remark 2.4.7. The matching M_i is in fact the only perfect matching in the bipartite graph B_i. To see this, suppose M' is a perfect matching of U into V. By the description of $N(G,i)$ given in Remark 2.4.3 it follows that for $i-s \in V$, where $s \in [r-1]$,

$$U \cap N(G, i-s) \subset \{i+r-1, i+r-2, \ldots, i+r-s\}.$$

Hence, $\{i+r-1, i-1\} \in M'$. But this implies $\{i+r-2, i-2\} \in M'$, which implies that $\{i+r-3, i-3\} \in M'$, and so on. Hence, $M_i \subset M'$, and we conclude that $M_i = M'$.

We next show that there are precisely n fundamental sets of size r in $G_{n,r}$.

Proposition 2.4.8. *There are precisely n fundamental sets of size r in $G_{n,r}$ and they are of the form*

$$\{i, i+1, i+2, \ldots, i+r-1\}$$

for $i \in [n]$.

Proof. We would like to construct a fundamental set T of size r. Hence, $T \neq \emptyset$, and so there exists some $i \in T$. Since T must also be an independent set it follows that

$$T \subset \{i-r+1, i-r+2, \ldots, i-1, i, i+1, \ldots, i+r-2, i+r-1\}.$$

In other words, with respect to the bipartite graph B_i, $T \subset U \cup V \cup \{i\}$. Since $\#T = r$ then T contains precisely one vertex of each element of the perfect matching M_i. Now, suppose that for some edge $\{i+s, i-(r-s)\} \in M_i$ we have that $i-(r-s) \in T$. Then $i+s+1 \notin T$ since

$$i+s+1 = i-(r-s)+r+1 \in N(G, i-(r-s)).$$

Hence, $i-(r-(s+1)) \in T$. It follows that $i-(r-s') \in T$ for all $s' > s$. Hence, T is of the form

$$\{i-r+s, i-r+s+1, i-r+s+2, \ldots, i-1, i, i+1, \ldots, i+s-1\}.$$

Note that there are exactly n such sets in $[n]$, and by reindexing, these sets are of the form

$$\{i, i+1, i+2, \ldots, i+r-1\}.$$

Since we have only used the facts that $\#T = r$ and T is independent then so far we have shown that these sets are the only independent sets in $G_{n,r}$ with cardinality r. To see that they are also fundamental recall the description of $N(G,i)$ given in Remark 2.4.3. From this description of $N(G,i)$ it is easy to see that

$$N(G,T) = \{i+r, i+r+1, \ldots, i-1\}.$$

Hence, $[n] = T \cup N(G,T)$.

Finally, to see that the bipartite graph induced by T is connected recall the argument used in Proposition 2.4.2 to show that $G_{n,r}$ is connected. Suppose that

$v, w \in T$. Without loss of generality assume $w = v + s$ for some positive integer s, and then apply the path $p : v \rightsquigarrow w$ from Proposition 2.4.2. Now suppose $v \in T$ and $w \in N(G,T)$. Then w is adjacent to some $u \in T$. So apply the path $p : v \rightsquigarrow u$ from Proposition 2.4.2 and then attach the edge $\{u, w\}$. Hence, T is fundamental in $G_{n,r}$. Hence, we conclude that the sets

$$\{i, i+1, i+2, \ldots, i+r-1\}$$

for $i \in [n]$ are the only fundamental sets of size r in $G_{n,r}$.

□

So far we have seen that for all $i \in [n]$ the vertex i is regular in $G_{n,r}$, there are precisely n fundamental sets with cardinality r, and there are no fundamental sets of cardinality greater than r. We next wish to show that there are no fundamental sets with cardinality less than r. To do this, we first need the following two lemmas.

Lemma 2.4.9. *Suppose $T \subset [n]$ is nonempty. Then $G_{[n]\backslash(T \cup N(G,T))}$ contains no odd cycle.*

Proof. Since $T \neq \emptyset$ then there exists some $i \in T$. Hence, $N(G,i) \subset N(G,T)$. Therefore, $[n]\backslash N(G,T) \subset [n]\backslash N(G,i)$. Thus,

$$\begin{aligned} [n]\backslash(T \cup N(G,T)) &= [n]\backslash T \cap [n]\backslash N(G,T), \\ &\subset [n]\backslash(N(G,T) \cup \{i\}), \\ &\subset [n]\backslash(N(G,i) \cup \{i\}). \end{aligned}$$

Hence, $G_{[n]\backslash(T \cup N(G,T))}$ is a subgraph of B_i, a bipartite graph. Therefore, it contains no odd cycle.

□

It then follows from Lemma 2.4.9 and (ii) of the definition of a fundamental set that if T is fundamental in $G_{n,r}$ then $[n] = T \cup N(G,T)$.

Lemma 2.4.10. *Suppose T is a fundamental set in $G_{n,r}$. Then T is a consecutive list of elements of $[n]$.*

Proof. Suppose for the sake of contradiction that T is fundamental in $G_{n,r}$, and T is not a consecutive list of elements of $[n]$. Then there exists some $i \in T$ and $s \in [r-1]$ such that $i, i+s \in T$, and for $0 < s' < s$, $i + s' \notin T$. Since T is fundamental in $G_{n,r}$ then by Lemma 2.4.9

$$[n] = T \cup N(G,T).$$

So $i + s' \in N(G,T)$ for $0 < s' < s$.

Now consider the bipartite graph induced by T, and call it X. Let $B = \{i + s' : 0 < s' < s\}$. We claim that no $b \in B$ is connected to $i \in T$. To see this, first notice that

$$N(X, N(X,B)) = B.$$

To prove this equivalence, we show containment in both directions. First let $w \in B$. Since $B \subset N(G,T)$ then there is some $v \in T$ such that v is a neighbor of w. So $v \in N(X,B)$. Hence, $w \in N(X,N(X,B))$. Conversely, suppose $w \in N(X,N(X,B))$. Then w is a neighbor of some neighbor $v \in T$ of some element $b \in B$. Since v is a neighbor of $b \in B$ then $N(G,v) \subset B$. This is because $N(G,v)$ is a consecutive list of elements of $[n]$ and T is independent with $i, i+s \in T$. So $w \in B$.

Next notice that since

$$B \subset \{i-r+1, i-r+2, \dots, i-1, i, i+1, \dots, i+r-2, i+r-1\}$$

then no element of B is adjacent to i.

Now fix $b \in B$ and suppose b is connected to i. Then there is a path in X

$$p = (\{v_0 = b, v_1\}, \{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{m-1}, v_m = i\}).$$

Since $b \in B$ and $N(X,N(X,B)) = B$ then each v_j, with $j \leq m-1$, is either an element of B or a neighbor of an element of B. However, v_{m-1} cannot be an element of B since no element of B is adjacent to i. So v_{m-1} is a neighbor of B, and thus $i \in N(X,N(X,B)) = B$, which is a contradiction. Hence, b is not connected to i. So the bipartite graph induced by T is not connected, which contradicts the assumption that T is fundamental.

□

We are now ready to show that no fundamental set in $G_{n,r}$ has cardinality less than r.

Proposition 2.4.11. *Suppose T is fundamental in $G_{n,r}$. Then $\#T \geq r$.*

Proof. Suppose for the sake of contradiction that T is fundamental in $G_{n,r}$ with $\#T < r$. Since $T \neq \emptyset$ then there exists some $i \in T$. Select i to be the "least element" of T in the following sense. Recall that $\#T < r$, $n \geq 2r+1$, and that we have constructed $G_{n,r}$ on a convex n-gon labeled in a clockwise fashion with the elements of $[n]$. Hence, by Lemma 2.4.10 the elements of T form an arc of length less than half the n-gon. Let i be the counterclockwise-most element of this arc. We claim $i-1 \notin T \cup N(G,T)$.

By the minimality of $i \in T$ we know that $i-1 \notin T$. So it suffices to show that $i-1 \notin N(G,T)$. To see this, suppose otherwise. Then there is some $j \in T$ such that $i-1 \in N(G,i)$. Since $j \in T$ and T is independent with $i \in T$ then

$$j \in \{i-r+1, i-r+2, \dots, i-1, i, i+1, \dots, i+r-1\}.$$

Since i is minimal in T then $T \subset \{i, i+1, \dots, i+r-2, i+r-1\}$. So

$$j \in \{i, i+1, \dots, i+r-2, i+r-1\}.$$

But since $i-1 \in N(G,j)$ then it must be that $j = i+r-1$ (otherwise j is not a neighbor of $i-1$). Hence, by Lemma 2.4.10, we have that

$$T = \{i, i+1, \dots, i+r-1\}.$$

So $\#T = r$, a contradiction. Hence $i-1 \notin N(G,T)$. Thus, $[n] \neq T \cup N(G,T)$. But, by Lemma 2.4.9, this contradicts the assumption that T is fundamental. Hence, it must be that $\#T \geq r$.

$\square$

Putting all of this together we arrive at the following theorem.

Theorem 2.4.12. *Fix a positive integer r. The number of facets of the r-stable $(n,2)$-hypersimplex $\Delta_{n,2}^{\mathrm{stab}(r)}$ is n for $n = 2r+1$ and $2n$ for $n > 2r+1$. Moreover, for $n = 2r+1$ the facets of $\Delta_{n,2}^{\mathrm{stab}(r)}$ are $K_T \cap P_G$, and for $n > 2r+1$ they are $K_i \cap P_G$ for $i \in [n]$ and $K_T \cap P_G$ for*

$$T = \{i, i+1, i+2, \ldots, i+r-1\},$$

for every $i \in [n]$.

Proof. For $n > 2r+1$, by Proposition 2.4.5 i is regular in $G_{n,r}$ for every $i \in [n]$, and by Propositions 2.4.6, 2.4.8, and 2.4.11 there are precisely n fundamental sets in $G_{n,r}$. Moreover, by Proposition 2.4.8 it is clear that the hyperplanes K_i for $i \in [n]$ and K_T for T fundamental in $G_{n,r}$ are all distinct. By Propositions 2.4.2 and 2.4.4 we may then apply Theorem 2.4.1 to conclude that the number of facets of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is $2n$. Moreover, these facets are precisely $K_i \cap P_G$ for $i \in [n]$ and $K_T \cap P_G$ for

$$T = \{i, i+1, i+2, \ldots, i+r-1\},$$

for every $i \in [n]$. For $n = 2r+1$, $\Delta_{n,2}^{\mathrm{stab}(r)}$ is a unimodular $(n-1)$-simplex by [**?**, Lemma 2.8]. By Propositions 2.4.6, 2.4.8, and 2.4.11 its facets are precisely those given by the fundamental sets in Proposition 2.4.8.

$\square$

Chapter 3 Ehrhart h^*-polynomials of r-stable Second Hypersimplices

In this chapter we study the r-stable $(n, 2)$-hypersimplices and their h^*-polynomials. We will identify a shelling of the triangulation $\nabla_{n,2}$ defined in Section 2.2 that first builds the r-stable $(n, 2)$-hypersimplex and then the $(r-1)$-stable $(n, 2)$-hypersimplex for every $1 < r < \lfloor \frac{n}{2} \rfloor$. We then apply Theorem 1.2.4 to compute the h^*-polynomials of these polytopes via this shelling order. In Sections 3.1 and 3.2 we identify the desired shelling and compute the associated h^*-polynomials in the case that n is odd. Then in Section 3.3 we extend the results to the case when n is even. Finally, in Section 3.4 we examine some unimodality results arising from these computations. The contents of this chapter are in part joint work with Benjamin Braun.

3.1 Shelling the r-stable Odd Second Hypersimplices

Fix $k = 2$ and n odd. In this section, we define a shelling of the triangulation $\nabla_{n,2}$ of the hypersimplex $\Delta_{n,2}$ that first shells the simplices within $\nabla^r_{n,2}$ and then extends this to a shelling of the $\nabla^{r-1}_{n,2}$ for every $1 < r < \lfloor \frac{n}{2} \rfloor$.

Theorem 3.1.1. *Let $k = 2$ and n odd. There exists a shelling of $\nabla_{n,2}$ that first builds $\Delta^{stab(r)}_{n,2}$ and then builds $\Delta^{stab(r-1)}_{n,2}$ for every $1 < r \leq \lfloor \frac{n}{2} \rfloor$. Hence, we say there exists a **stable shelling** of the odd second hypersimplex.*

Remark 3.1.2. Notice first that by Lemma 2.2.7 we can certainly shell $\nabla^r_{n,2}$ where $r = \lfloor \frac{n}{2} \rfloor$. Our goal is to inductively shell $\nabla^r_{n,2}$ with this as our base case. That is, assume we have previously shelled $\nabla^{r+1}_{n,2}$ for $0 < r < \lfloor \frac{n}{2} \rfloor$. In the following we describe a continuation of this shelling to $\nabla^r_{n,2}$. A summary of our method for describing this continuation is as follows.

Since we are assuming we have previously shelled the simplices in $\nabla^{r+1}_{n,2}$ we must extend this order to the set of simplices $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$. Each simplex ω in this set uses some vertices that are r-stable but not $(r + 1)$-stable. We will call these vertices the *r-adjacent vertices*. We first select a particular r-adjacent vertex of ω, and think of this as the initial vertex in the cycle (ω). Given this choice, we then associate to ω a composition of r into $n - r - 1$ parts that describes the cycle (ω) in terms of the selected initial vertex. Using this composition and its relationship with the vertices of ω we associate to ω a lattice path in a decorated ladder-shaped region of the plane. We then order the simplices in $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ by first collecting them into sets based on their initial vertex and the number of r-adjacent vertices they use, ordering these sets from least to most r-adjacent vertices used, and then ordering the elements within these sets via the colexicographic ordering applied to their associated compositions. We then utilize their associated lattice paths to identify the unique minimal new face for each simplex. In particular, we shell the simplices in terms of least r-adjacent vertices used to most r-adjacent vertices used.

r-adjacent vertices.

For $\ell \in [n]$, let $\mathrm{adj_r}(\ell)$ denote the vertex ϵ_I where $I = \{\ell, \ell + r\} \in \binom{[n]}{2})\}$. We call a vertex $\mathrm{adj_r}(\ell)$ an *r-adjacent vertex.* Let $\mathrm{Adj_r[n]} := \{\mathrm{adj_r}(\ell) : \ell \in [n]\}$. So $\mathrm{Adj}_r[\mathrm{n}]$ is precisely the set of vertices that are r-stable but not $(r+1)$-stable.

Lemma 3.1.3. *Let ϵ and ϵ' be two vertices in (ω) for some simplex $\omega \in \max \nabla_{n,2}$. Suppose that ϵ has entries $\epsilon_i = \epsilon_j = 1$ with $i < j$, and $\epsilon_t = 0$ for all $t \neq i, j$. Suppose also that ϵ' has entries $\epsilon'_k = \epsilon'_l = 1$ with $k < l$, and $\epsilon'_t = 0$ for all $t \neq k, l$. Then (modulo n) we have that*

$$i \leq k \leq j \leq l.$$

Proof. Since ϵ and ϵ' are vertices of a simplex in $\nabla_{n,2}$ they correspond to a sorted pair of 2-subsets of $[n]$. □

Corollary 3.1.4. *Let $\omega \in \max \nabla_{n,2}$, and suppose that $\mathrm{adj_r}(\ell)$ and $\mathrm{adj_r}(\ell')$ are vertices in $v_{(\omega)} \cap \mathrm{Adj_r[n]}$. Then $\mathrm{cd}(\ell, \ell') \leq r$.*

Proof. Lemma 3.1.3 indicates that

$$\ell \leq \ell' \leq \ell + r \leq \ell' + r, \text{ or}$$
$$\ell' \leq \ell \leq \ell' + r \leq \ell + r.$$

□

Remark 3.1.5. Consider a simplex $\omega \in \max \nabla_{n,2}$. Notice that for a fixed $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$ we may order the elements of the set $v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]$ as $\mathrm{adj_r}(\ell) <_{adj} \mathrm{adj_r}(\ell')$ if and only if $\ell < \ell'$. In this way, there exists a unique maximal element of the set $v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]$.

Associating a composition to $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$.

Fix $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$. Then ω uses at least one element of $\mathrm{Adj_r[n]}$. We may fix one such $\mathrm{adj_r}(\ell)$, and consider the circuit (ω) as having initial vertex $\mathrm{adj_r}(\ell)$. We refer to the 1 in entry ℓ of the vertex $\mathrm{adj_r}(\ell)$ as the *left* 1 and the 1 in entry $\ell + r$ as the *right* 1. Then, in (ω) each edge corresponds to a move of the left 1 or of the right 1. In particular,

($\star$) the left 1 makes r moves,

($\star$) the right 1 makes $n - r$ moves, and

($\star$) the left 1 cannot move first or last.

Note that the first two conditions are immediate from the definition of (ω) and the fact that $k = 2$. The third condition holds since ω uses only vertices that are r-stable. It follows that for a fixed $\mathrm{adj_r}(\ell) \in v_{(\omega)}$ we can think of the circuit (ω) as a sequence of moves of the left 1 and moves of the right 1 satisfying these conditions. Moreover, we may encode this as a composition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-r-1})$$

$$
\begin{tikzcd}
(*)\,\mathrm{adj}_4(15) = (0,0,0,1^R,0,0,0,0,0,0,0,0,0,0,1^L) \arrow[r, "R"] & (0,0,0,0,1^R,0,0,0,0,0,0,0,0,0,1^L) \arrow[r, "L"] & \mathrm{adj}_4(1) = (1^L,0,0,0,1^R,0,0,0,0,0 \arrow[d, "R"] \\
(0,0,0,1^L,0,0,0,0,0,0,0,0,0,1^R,0) \arrow[u, "R"] & & \\
\mathrm{adj}_4(14) = (0,0,1^L,0,0,0,0,0,0,0,0,0,0,1^R,0) \arrow[u, "L"] & & (1^L,0,0,0,0,1^R,0,0,0,0,0,0,0, \arrow[d, "R"] \\
(0,0,1^L,0,0,0,0,0,0,0,0,0,1^R,0,0) \arrow[u, "R"] & & (1^L,0,0,0,0,0,1^R,0,0,0,0,0,0, \arrow[d, "R"] \\
(0,0,1^L,0,0,0,0,0,0,0,0,1^R,0,0,0) \arrow[u, "R"] & & (1^L,0,0,0,0,0,0,1^R,0,0,0,0,0, \arrow[d, "L"] \\
(0,0,1^L,0,0,0,0,0,0,0,1^R,0,0,0,0) \arrow[u, "R"] & & (0,1^L,0,0,0,0,0,1^R,0,0,0,0,0, \arrow[d, "R"] \\
(0,0,1^L,0,0,0,0,0,0,1^R,0,0,0,0,0) \arrow[u, "R"] & (0,1^L,0,0,0,0,0,0,0,0,1^R,0,0,0,0,0) \arrow[l, "L"'] & (0,1^L,0,0,0,0,0,0,1^R,0,0,0, \arrow[l, "R"']
\end{tikzcd}
$$

Figure 3.1: The minimal circuit corresponding to the simplex $\omega = 5671892(10)(11)(12)(13)3(14)4(15)$.

$$
\begin{tikzcd}
(*)\,\mathrm{adj}_3(7) = (1^R,0,0,0,0,0,1^L,0,0) \arrow[r, "R"] & (0,1^R,0,0,0,0,1^L,0,0) \arrow[r, "R"] & (0,0,1^R,0,0,0,1^L,0,0) \arrow[d, "R"] \\
& & \mathrm{adj}_3(4) = (0,0,0,1^R,0,0,1^L,0,0) \arrow[d, "L"] \\
(1^L,0,0,0,0,1^R,0,0,0) \arrow[uu, "R"] & & (0,0,0,1^R,0,0,0,1^L,0) \arrow[d, "L"] \\
(1^L,0,0,0,1^R,0,0,0,0) \arrow[u, "R"] & \mathrm{adj}_3(1) = (1^L,0,0,1^R,0,0,0,0,0) \arrow[l, "R"'] & (0,0,0,1^R,0,0,0,0,1^L) \arrow[l, "L"']
\end{tikzcd}
$$

Figure 3.2: The minimal circuit corresponding to the simplex $\omega = 456123789$.

of r into $n-r-1$ parts, where part λ_i denotes the number of moves of the left 1 after the i^{th} move of the right 1 and before the $(i+1)^{st}$ move of the right 1.

Example 3.1.6. Consider the minimal circuit in $G_{15,2}$ depicted in Figure 3.1. This circuit corresponds to a simplex $\omega \in \max \nabla^4_{15,2} \setminus \max \nabla^5_{15,2}$. If we choose the initial vertex of this circuit to be the unique maximal element of the set $v_{(\omega)} \cap \mathrm{Adj}_4[15]$, namely $\mathrm{adj}_4(15)$, then this circuit has associated composition $\lambda = (1,0,0,1,0,1,0,0,0,1)$.

Example 3.1.7. Next consider the minimal circuit in $G_{9,2}$ depicted in Figure 3.2. This circuit corresponds to a simplex $\omega \in \max \nabla^3_{9,2} \setminus \max \nabla^4_{9,2}$. If we choose the initial vertex of this circuit to be the unique maximal element of the set $v_{(\omega)} \cap \mathrm{Adj}_3[9]$, namely $\mathrm{adj}_3(7)$, then this circuit has associated composition $\lambda = (0,0,3,0,0)$.

Proposition 3.1.8. *Fix* $\mathrm{adj_r}(\ell) \in \mathrm{Adj_r[n]}$. *Each simplex* $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ *that uses the vertex* $\mathrm{adj_r}(\ell)$ *corresponds uniquely to a composition*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-r-1})$$

of r into $n-r-1$ parts that satisfies

$$i+1+2r-n \leq \sum_{j=1}^{i} \lambda_j \leq i \tag{3.1.1}$$

for all $i = 1, 2, \ldots, n-r-1$.

Proof. Let $\omega \in \max \nabla^{r}_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ that uses vertex $\mathrm{adj_r}(\ell)$. Then (ω) is a minimal circuit in the directed graph $G_{n,2}$, one of whose vertices is $\mathrm{adj_r}(\ell)$. Thinking of $\mathrm{adj_r}(\ell)$ as the initial vertex we consider the 1 in place ℓ as the left 1 and the 1 in place $\ell + r$ as the right 1. By the above conditions it is clear that we may construct the partition λ of r into $n-r-1$ parts, where part λ_i denotes the number of moves of the left 1 after the i^{th} move of the right 1 and before the $(i+1)^{st}$ move of the right 1. Since $\omega \in \max \nabla^{r}_{n,2}$ the left 1 can never have made more moves that the right 1. This gives the upper bound on $\sum_{j=1}^{i} \lambda_j$ for each $i = 1, 2, \ldots, n-r-1$.

Similarly, since $\omega \in \max \nabla^{r}_{n,2}$ after the $(r+1)^{st}$-to-last move of the right 1 and before its r^{th}-to-last move we must have that the left 1 moved at least once. More generally, after the $n-r-t+1^{st}$ move of the right 1 we must have that the left 1 moved at least $r-t+2$ times for $t = r+1, r, r-1, \ldots, 2$. Hence, the number of left moves that occur after the i^{th} right move and before the $(i+1)^{st}$ right move is at least $i+1+2r-n$. This gives the lower bound on $\sum_{j=1}^{i} \lambda_j$.

Conversely, suppose that we have a composition λ satisfying the given conditions. We can construct a simplex $\omega(\lambda) \in \max \nabla^{r}_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ that uses the vertex $\mathrm{adj_r}(\ell)$ by constructing a minimal circuit in $G_{n,2}$ as follows. Starting with $\mathrm{adj_r}(\ell)$, and labeling the left 1 and right 1 as we have been, after the i^{th} move of the right 1 move the left 1 λ_i times. Once this has been done for all $i = 1, 2, \ldots, n-r-1$, move the right 1 once more. The upper bound ensures that the right distance between the 1s is always at least r. Similarly, the lower bound ensures that the left distance is always at least r. Since $\mathrm{adj_r}(\ell)$ is in the circuit $(\omega(\lambda))$ then this corresponds to a simplex $\omega(\lambda) \in \max \nabla^{r}_{n,2} \setminus \max \nabla^{r+1}_{n,2}$. □

Remark 3.1.9. By Remark 3.1.5 we can identify each simplex $\omega \in \max \nabla^{r}_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ with the unique maximal element of $v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]$, say $\mathrm{adj_r}(\ell)$. Let λ be the composition associated to ω via $\mathrm{adj_r}(\ell)$ by Proposition 3.1.8. Then we may uniquely label the simplex ω as $\omega_{\ell,\lambda}$.

Definition 3.1.10. For a simplex $\omega_{\ell,\lambda}$ recall that we think of the 1 in entry ℓ of $\mathrm{adj_r}(\ell)$ as the left 1, and the 1 in entry $\ell + r$ as the right 1.

- A **left move** in $(\omega_{\ell,\lambda})$ is an edge in $(\omega_{\ell,\lambda})$ corresponding to a move of the left 1, and
- A **right move** in $(\omega_{\ell,\lambda})$ is an edge in $(\omega_{\ell,\lambda})$ corresponding to a move of the right 1.
- The **parity** of an edge in $(\omega_{\ell,\lambda})$ is *left* if the edge is a left move, and *right* if the edge is a right move.

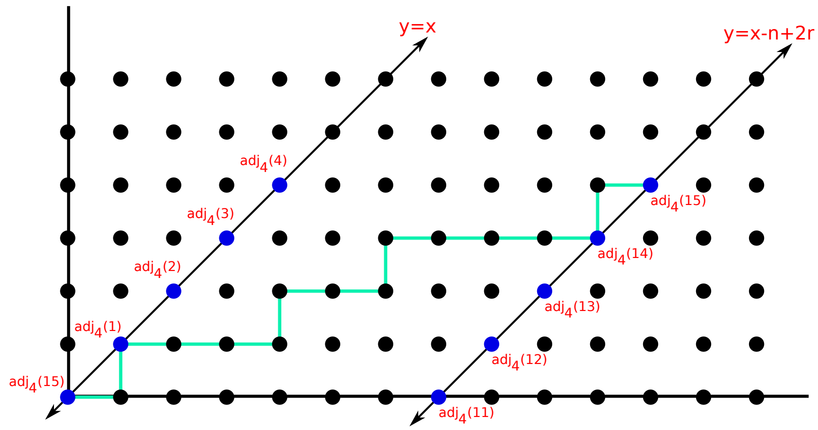


Figure 3.3: The lattice path $p(\omega_{15,\lambda})$, where $\lambda = (1, 0, 0, 1, 0, 1, 0, 0, 0, 1)$.

Remark 3.1.11 (**Lattice Path Correspondence**). Notice that each simplex $\omega_{\ell,\lambda}$ in the set $\max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ corresponds to a lattice path, $p(\omega_{\ell,\lambda})$, from $(0, 0)$ to $(n - r, r)$ that uses only North (0,1) and East (1,0) moves. Here, right moves in the circuit $(\omega_{\ell,\lambda})$ correspond to East moves in $p(\omega_{\ell,\lambda})$, and left moves in $(\omega_{\ell,\lambda})$ correspond to North moves in $p(\omega_{\ell,\lambda})$. By Proposition 3.1.8 the lattice path $p(\omega_{\ell,\lambda})$ is bounded between the lines $y = x$ and $y = x-n+2r$. Each vertex in $(\omega_{\ell,\lambda})$ corresponds uniquely to a lattice point on $p(\omega_{\ell,\lambda})$. In particular, for $0 \le t \le r$ the vertex $\mathrm{adj_r}(\ell + \mathrm{t})$ corresponds to the lattice point (t, t), and the vertex $\mathrm{adj_r}(\ell\text{-r} + \mathrm{t})$ corresponds to the lattice point $(t, n - 2r + t)$.

Here are some examples of simplices and their corresponding lattice paths.

Example 3.1.12. Let $n = 15$ and $r = 4$. Recall the simplex from Example 3.1.6

$$\omega = 5671892(10)(11)(12)(13)3(14)4(15) \in \max \nabla^4_{15,2} \setminus \max \nabla^5_{15,2}.$$

This simplex corresponds to the minimal circuit in the graph $G_{15,2}$ depicted in Figure 3.1.

From this, we can see that ω uses the vertices $\mathrm{adj_4}(15)$, $\mathrm{adj_4}(1)$, and $\mathrm{adj_4}(14)$. Hence, we label ω as $\omega_{15,\lambda}$, where

$$\lambda = (1, 0, 0, 1, 0, 1, 0, 0, 0, 1).$$

The lattice path corresponding to ω via this labeling is depicted in Figure 3.3.

Example 3.1.13. Let $n = 9$ and $r = 3$. Recall the simplex from Example 3.1.7

$$\omega = 456123789 \in \max \nabla^3_{9,2} \setminus \max \nabla^4_{9,2}.$$

This simplex corresponds to the minimal circuit in the graph $G_{9,2}$ depicted in Figure 3.2.

From this, we can see that ω uses the vertices $\mathrm{adj_3}(7)$, $\mathrm{adj_3}(4)$, and $\mathrm{adj_3}(1)$. Hence, we label ω as $\omega_{7,\lambda}$, where

$$\lambda = (0, 0, 3, 0, 0).$$

The lattice path corresponding to ω via this labeling is depicted in Figure 3.4.

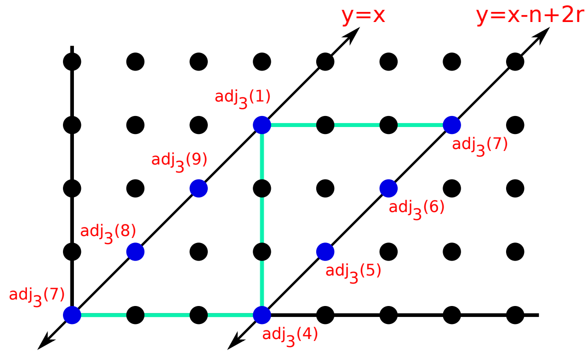


Figure 3.4: The lattice path $p(\omega_{7,\lambda})$, where $\lambda = (0, 0, 3, 0, 0)$.

The shelling order.

Recall that the *colexicographic* order on a pair of ordered m-tuples $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ is defined by $\mathbf{b} <_{colex} \mathbf{a}$ if and only if the right-most nonzero entry in $\mathbf{a} - \mathbf{b}$ is positive. Let $W_{\ell,s}$ denote the set of all simplices with label $\omega_{\ell,\lambda}$ that use precisely s elements of $\mathrm{Adj}_r[\mathrm{n}]$. Order the elements in each set $W_{\ell,s}$ with respect to the colexicographic ordering on their associated compositions (from least to greatest). We write $\omega_{\ell,\lambda} <_{colex} \omega_{\ell,\lambda'}$ if and only if $\lambda <_{colex} \lambda'$. Next order the sets $W_{\ell,s}$ (from least to greatest) with respect to the colexicographic ordering on the labels (ℓ, s). We then write $\omega_{\ell,\lambda} < \omega_{\ell',\lambda'}$ if and only $\omega_{\ell,\lambda} \in W_{\ell,s}$ and $\omega_{\ell',\lambda'} \in W_{\ell',s'}$ with $(\ell, s) <_{colex} (\ell', s')$ or if $(\ell, s) = (\ell', s')$ and $\omega_{\ell,\lambda} <_{colex} \omega_{\ell,\lambda'}$.

Theorem 3.1.14. *The order $<$ on the simplices $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ (from least to greatest) extends the shelling of $\nabla^{r+1}_{n,2}$ to a shelling of $\nabla^r_{n,2}$.*

Theorem 3.1.1 follows immediately from Theorem 3.1.14. To prove Theorem 3.1.14 it suffices to identify the unique minimal new face associated to each simplex in the shelling order. To do so, we first prove a sequence of lemmas.

Lemma 3.1.15. *Suppose the $\omega_{\ell,\lambda}$ uses $\mathrm{adj_r}(\ell')$ for $\ell' \neq \ell$. Then $\mathrm{adj_r}(\ell')$ is a vertex in $(\omega_{\ell,\lambda})$ that is either*

(i) produced by a right move for which the preceding number of left moves is minimal and not maximal with respect to equation (3.1.1), or

(ii) produced by a left move and followed by a right move for which the number of left moves preceding the right move is maximal and not minimal with respect to equation (3.1.1).

Proof. Since $\mathrm{adj_r}(\ell)$ is selected to be the greatest element of $v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]$ then each other $\mathrm{adj_r}(\ell')$ used by $\omega_{\ell,\lambda}$ is produced in $(\omega_{\ell,\lambda})$ by doing $n - 2r + t$ right moves for some number t of left moves, or $\mathrm{adj_r}(\ell')$ is produced by doing $0 < t < r$ right moves and the same number of left moves. In the former case, such a vertex corresponds to an entry λ_m in the composition λ with $m = n - 2r + t$ for which

$$t = m + 2r - n = (m-1) + 1 + 2r - n \leq \sum_{j=1}^{m-1} \lambda_j = t.$$

Hence, the number of left moves preceding the m^{th} right move is minimal. Moreover, the number of left moves preceding the m^{th} right move is maximal only if

$$m + 2r - n = t = \sum_{j=1}^{m-1} \lambda_j = m - 1.$$

Thus,

$$r = \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor.$$

But recall that since $\Delta_{n,2}^{stab(\lfloor \frac{n}{2} \rfloor)}$ is a unimodular $(n-1)$-simplex we are only completing the shelling of $\nabla_{n,2}^{r+1}$ to a shelling of $\nabla_{n,2}^{r}$ for $r < \lfloor \frac{n}{2} \rfloor$. So we conclude that the sum is minimal and not maximal.

In the latter case, the vertex $\mathrm{adj_r}(\ell_\mathrm{j})$ is produced by doing $0 < t < r$ right moves and the same number of left moves. Thus, following this vertex with another left move results in a vertex that is no longer r-stable. So the move following $\mathrm{adj_r}(\ell_\mathrm{j})$ must be a right move. Such a vertex corresponds to an entry λ_m in the composition λ for which $m = t$, and the right move following the vertex is the $(m+1)^{st}$ right move in the circuit. Thus,

$$m + 1 + 2r - n \leq \sum_{j=1}^{m} \lambda_j = t = m.$$

Hence, the number of left moves preceding the right move following the vertex is maximal. If this number is also minimal then it must be that

$$\begin{aligned} m + 1 + 2r - n &= m, \\ r &= \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor, \end{aligned}$$

and so we conclude that the sum is not minimal just as in the previous case. It remains to show that the move preceding the vertex $\mathrm{adj_r}(\ell_\mathrm{j})$ is a left move. Suppose for the sake of contradiction that $\mathrm{adj_r}(\ell_\mathrm{j})$ is preceded by a right move. Then $\lambda_m = 0$. Thus, since the number of left moves preceding the $(m+1)^{st}$ right move is maximal we have that

$$m = \sum_{j=1}^{m} \lambda_j = \sum_{j=1}^{m-1} \lambda_j \leq m - 1,$$

which is a contradiction. Thus, we conclude that $\mathrm{adj_r}(\ell_\mathrm{j})$ is produced by a left move and followed by a right move for which the number of left moves preceding the right move is not minimal. $\square$

Lemma 3.1.16. *Suppose that the simplex $\omega_{\ell,\lambda}$ uses the elements*

$$\mathrm{adj_r}(\ell_1) <_{adj} \mathrm{adj_r}(\ell_2) <_{adj} \cdots <_{adj} \mathrm{adj_r}(\ell_s) = \mathrm{adj_r}(\ell)$$

of $\mathrm{Adj_r[n]}$*. For $j \neq s$, the parities of the edges preceding* $\mathrm{adj_r}(\ell_j)$ *in* $(\omega_{\ell,\lambda})$ *and following* $\mathrm{adj_r}(\ell_j)$ *are opposite. Also, the parity of the edges about* $\mathrm{adj_r}(\ell)$ *is right.*

Proof. First recall that we have already noted that the first and last moves of $(\omega_{\ell,\lambda})$ must be right moves. Hence, the parity of the edges about $\mathrm{adj_r}(\ell)$ is right.

Now consider $\mathrm{adj_r}(\ell_j)$ for $j \neq s$. By Lemma 3.1.15 we have two cases. In case (ii), $\mathrm{adj_r}(\ell_j)$ is produced by a left move and followed by a right move for which the number of left moves preceding the right move is not minimal. Hence, the result is immediate.

In case (i), $\mathrm{adj_r}(\ell_j)$ is produced by a right move for which the preceding number of left moves is minimal and not maximal. Suppose for the sake of contradiction that the parities of the moves about $\mathrm{adj_r}(\ell_j)$ are the same. So if $\mathrm{adj_r}(\ell_j)$ is produced by the m^{th} right move we have that

$$m + 2r - n = (m-1) + 1 + 2r - n = \sum_{j=1}^{m-1} \lambda_j.$$

Since the parities of the edges about $\mathrm{adj_r}(\ell_j)$ are the same it is followed by a right move, and so it must be that $\lambda_m = 0$. Hence, by equation (3.1.1)

$$m + 1 + 2r - n \leq \sum_{j=1}^{m} \lambda_j = \sum_{j=1}^{m-1} \lambda_j = m + 2r - n,$$

which is a contradiction. □

Lemma 3.1.17. *Suppose that the simplex $\omega_{\ell,\lambda}$ uses the vertex* $\mathrm{adj_r}(\ell')$*. Then switching the parities of the moves about* $\mathrm{adj_r}(\ell')$ *does not replace* $\mathrm{adj_r}(\ell')$ *with another vertex in* $\mathrm{Adj_r[n]}$*.*

Proof. First consider the case where $\ell' \neq \ell$. By Remark 3.1.11 the simplex $\omega_{\ell,\lambda}$ corresponds to a lattice path $p(\omega_{\ell,\lambda})$ that is bounded between the lines $y = x$ and $y = x - n + 2r$, and the elements of $\mathrm{Adj}_r[\mathrm{n}]$ reachable from $\mathrm{adj_r}(\ell)$ all lie on these two lines. Suppose for the sake of contradiction that switching the parities of the moves about $\mathrm{adj_r}(\ell')$ replaced this vertex with another element of $\mathrm{Adj}_r[\mathrm{n}]$, say $\mathrm{adj_r}(\ell'')$. Then the resulting change in the lattice path $p(\omega_{\ell,\lambda})$ implies that $\mathrm{adj_r}(\ell'')$ lies on the opposite of these two lines from that on which $\mathrm{adj_r}(\ell')$ lies. It then follows that $n - 2r = 2$, or equivalently, $n = 2r + 2$. Since we have chosen n to be odd this is a contradiction.

Now consider the case where $\ell' = \ell$. Suppose for the sake of contradiction that switching the parities of the moves about $\mathrm{adj_r}(\ell)$ replaces $\mathrm{adj_r}(\ell)$ with another vertex $\mathrm{adj_r}(\ell'')$. Consider the vertex before the right move producing $\mathrm{adj_r}(\ell)$. Since this right move produces $\mathrm{adj_r}(\ell)$ then in this preceding vertex there must be precisely r 0's to the right of the right 1 and before the left 1. Similarly, since the left move produces the vertex $\mathrm{adj_r}(\ell'')$ it must be that there are r 0's to the right of the left 1 and before the right 1. Hence, $n = 2r + 2$, a contradiction. □

Lemma 3.1.18. *Suppose that the simplex $\omega_{\ell,\lambda}$ uses the elements*

$$\mathrm{adj_r}(\ell_1) <_{adj} \mathrm{adj_r}(\ell_2) <_{adj} \cdots <_{adj} \mathrm{adj_r}(\ell_\mathrm{s}) = \mathrm{adj_r}(\ell)$$

of $\mathrm{Adj_r[n]}$*. Switching the parity of the edges about* $\mathrm{adj_r}(\ell_\mathrm{j})$ *replaces the vertex* $\mathrm{adj_r}(\ell_\mathrm{j})$ *with an $(r+1)$-stable vertex, and leaves all other vertices in $(\omega_{\ell,\lambda})$ fixed.*

Proof. First fix $\mathrm{adj_r}(\ell_\mathrm{j})$ for $j \neq s$, and switch the parity of the moves directly before and after $\mathrm{adj_r}(\ell_\mathrm{j})$ in $(\omega_{\ell,\lambda})$. By Lemma 3.1.15 there are two cases. In case (i), Lemma 3.1.16 implies that the parity switch changes the move before from a right to a left, and the move after from a left to a right. Since each vertex in the circuit is determined by the number of left moves and right moves by which it differs from $\mathrm{adj_r}(\ell)$ this switching does not change any of the vertices preceding $\mathrm{adj_r}(\ell_\mathrm{j})$ in $(\omega_{\ell,\lambda})$. Similarly, it does not change any of the vertices following $\mathrm{adj_r}(\ell_\mathrm{j})$. The reader should also note that this switch changes the composition λ. However, Lemma 3.1.15 ensures that the resulting composition, say λ', still satisfies the bounds of equation (3.1.1). Hence, by Proposition 3.1.8 this switch produces a circuit $(\omega_{\ell,\lambda'})$ for which $\omega_{\ell,\lambda'} \in \max \nabla^r_{n,2} \backslash \max \nabla^{r+1}_{n,2}$. Moreover, the vertex which replaces $\mathrm{adj_r}(\ell_\mathrm{j})$ is not an element of $\mathrm{Adj}_r\mathrm{[n]}$ by Lemma 3.1.17. As an example, consider the following scenario for $r = 3$:

$$\begin{tikzcd}
(0,1^R,0,0,0,1^L,0,\dots,0) \arrow[d, "R"] \arrow[dr, "L"] & \\
(0,0,1^R,0,0,1^L,0,\dots,0) \in \mathrm{Adj_3[n]} \arrow[d, "L"] & (0,1^R,0,0,0,0,1^L,0,\dots,0) \notin \mathrm{Adj_3[n]} \arrow[dl, "R"] \\
(0,0,1^R,0,0,0,1^L,0,\dots,0) &
\end{tikzcd}$$

We remark that $\omega_{\ell,\lambda} \in W_{\ell,s}$, so $\omega_{\ell,\lambda'} \in W_{\ell,s-1}$.

In case (ii), Lemma 3.1.16 implies that the parity switch changes the move before $\mathrm{adj_r}(\ell_\mathrm{j})$ from a left to a right, and the move after $\mathrm{adj_r}(\ell_\mathrm{j})$ from a right to a left. Now apply the same argument as for case (i), and the result follows. We again remark that since $\omega_{\ell,\lambda} \in W_{\ell,s}$ then the parity switch results in a simplex $\omega_{\ell,\lambda'} \in W_{\ell,s-1}$.

Now consider $\mathrm{adj_r}(\ell)$. By the same argument as before, switching the parities of the moves about $\mathrm{adj_r}(\ell)$ replaces $\mathrm{adj_r}(\ell)$ with an $(r+1)$-stable vertex that is not in $\mathrm{Adj}_r\mathrm{[n]}$. This scenario is depicted in the following diagram for $r = 3$.

$$\begin{tikzcd}
(0,1,0,0,0,1,0,\dots,0) & \\
(0,1,0,0,1,0,\dots,0) = \mathrm{adj_3}(\ell) \arrow[u] & (1,0,0,0,0,1,0,\dots,0) \notin \mathrm{Adj_3[n]} \arrow[ul] \\
(1,0,0,0,1,0,\dots,0) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Notice that we omit the labels R and L. This is because removing $\mathrm{adj_r}(\ell)$, the vertex which defines the labels, demands a relabeling of the resulting circuit, and in general the new labels will not agree with the old. However, this is acceptable since

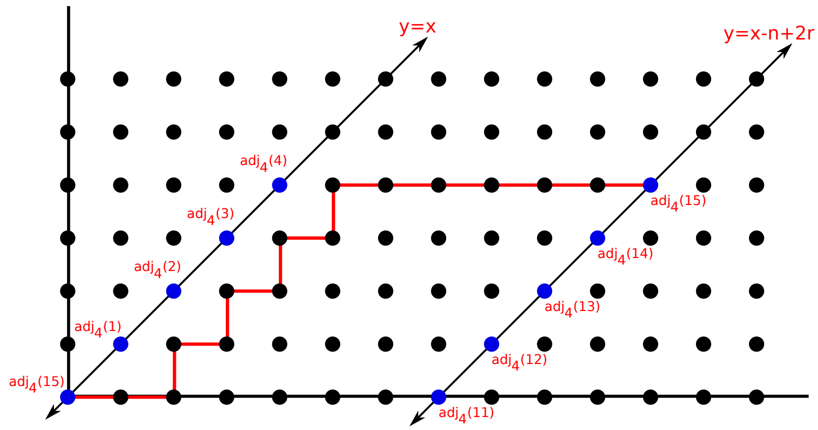


Figure 3.5: The lattice path $p(\omega_{15,\lambda^\star})$. Here, $\lambda^\star = (0, 1, 1, 1, 1, 0, 0, 0, 0, 0)$.

to switch the parities of the moves about $\mathrm{adj_r}(\ell)$ we simply note that the vertices directly before and after $\mathrm{adj_r}(\ell)$ in the circuit are completely determined by $\mathrm{adj_r}(\ell)$. Moreover, the edges before and after $\mathrm{adj_r}(\ell)$ each correspond to a move of a different 1 in $\mathrm{adj_r}(\ell)$. Hence, to switch the parities, starting at the vertex preceding $\mathrm{adj_r}(\ell)$ simply switch the order in which the 1's move.

□

Corollary 3.1.19. *For the simplex $\omega_{\ell,\lambda}$, switching the parities of the edges about $\mathrm{adj_r}(\ell_j)$, for $j \neq s$, reduces the simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$ to a simplex $\omega_{\ell,\lambda'} \in W_{\ell,s-1}$. For $s \neq 1$, switching the parities of the edges about $\mathrm{adj_r}(\ell)$ reduces the simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$ to a simplex $\omega_{\ell_{s-1},\lambda'} \in W_{\ell_{s-1},s-1}$. For $s = 1$, switching the parities of the edges about $\mathrm{adj_r}(\ell)$ reduces the simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$ to a simplex in $\max \nabla^{r+1}_{n,2}$.*

Proof of Theorem 3.1.14.

We are now ready to prove Theorem 3.1.14. Recall, to prove Theorem 3.1.14 it suffices to identify the unique minimal new face associated to each simplex in the shelling order. Given $\omega_{\ell,\lambda} \in W_{\ell,s}$ recall that we can associate to $\omega_{\ell,\lambda}$ a lattice path $p(\omega_{\ell,\lambda})$. Notice also that for each $\mathrm{adj_r}(\ell) \in \mathrm{Adj}_r[\mathrm{n}]$ the first simplex in our order that uses $\mathrm{adj_r}(\ell)$ is $\omega_{\ell,\lambda^\star}$ where

$$\lambda^\star = (0, 1, 1, \ldots, 1, 0, 0, \ldots, 0).$$

A picture of the lattice path $p(\omega_{15,\lambda^\star})$ corresponding to $\lambda^\star$ for $n = 15$, $r = 4$, and $\ell = 15$ is given in Figure 3.5.

We claim that the unique minimal new face of $\omega_{\ell,\lambda}$ is the collection of vertices

- $\mathrm{adj_r}(\ell)$,
- those vertices corresponding to lattice points on $p(\omega_{\ell,\lambda})$ that lie on $y = x$, and
- those vertices corresponding to lattice points on $p(\omega_{\ell,\lambda})$ that are
  - right-most in their row of the lattice,

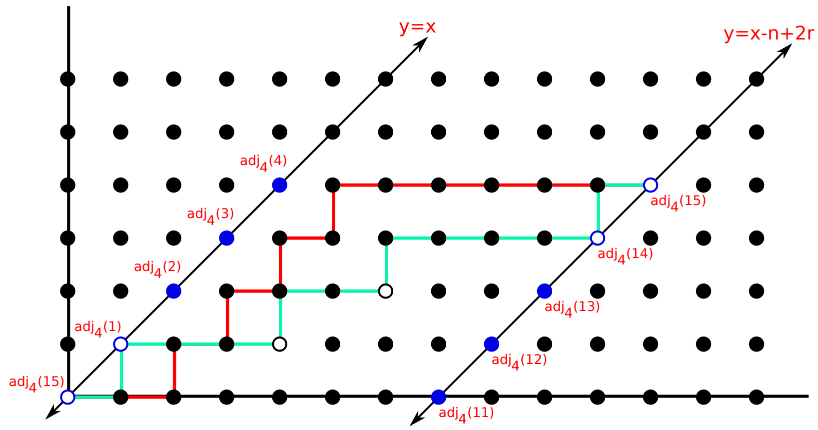


Figure 3.6: The lattice path $p(\omega_{15,\lambda})$ compared to $p(\omega_{15,\lambda^\star})$. Here, $\lambda = (1,0,0,1,0,1,0,0,0,1)$. The unique minimal new face of $\omega_{15,\lambda}$ is given by the open lattice points.

- corners of $p(\omega_{\ell,\lambda})$, and
- do not lie on $p(\omega_{\ell,\lambda^\star})$.

That is to say, the corners of $p(\omega_{\ell,\lambda})$ that are "furthest away" or "point away" from the path $p(\omega_{\ell,\lambda^\star})$.

Example 3.1.20. Let $n = 15$ and $r = 4$. Consider the simplex $\omega_{15,\lambda}$ from Example 3.1.12. The unique minimal new face for $\omega_{15,\lambda}$ is given by the vertices

$$\begin{gathered}(0,0,0,1,0,0,0,0,0,0,0,0,0,0,1)\\(1,0,0,0,1,0,0,0,0,0,0,0,0,0,0)\\(1,0,0,0,0,0,0,1,0,0,0,0,0,0,0)\\(0,1,0,0,0,0,0,0,0,1,0,0,0,0,0)\\(0,0,1,0,0,0,0,0,0,0,0,0,0,1,0)\end{gathered}$$

These vertices correspond to the open lattice points on the path $p(\omega_{15,\lambda})$ depicted in Figure 3.6. The reader should note the position of these points relative to the lattice path $p(\omega_{15,\lambda^\star})$.

To see these vertices form the unique minimal new face fix a simplex $\omega_{\ell,\lambda} \in W_{\ell,s}$, and suppose that this set of vertices is

$$G_\omega = \{v_0 := \mathrm{adj_r}(\ell), v_1, v_2, \ldots, v_q\}.$$

We will show that any face of $\omega_{\ell,\lambda}$ not using G_ω has previously appeared, and that G_ω is indeed a new face.

First consider a face F of $\omega_{\ell,\lambda}$ that does not use vertex $v_t \in G_\omega$ for $t \neq 0$. There are then two cases:

(1) $v_t \in \mathrm{Adj}_r[\mathrm{n}]$, or

(2) $v_t \notin \mathrm{Adj}_r[\mathrm{n}]$.

In case (1), we saw in Corollary 3.1.19 that $\omega_{\ell,\lambda}$ reduces to a previously shelled simplex that only differs for $\omega_{\ell,\lambda}$ by the vertex v_t. That is, to construct a simplex $\omega_{\ell,\lambda'}$ that uses the face F and was shelled before $\omega_{\ell,\lambda}$ switch the parities of the moves about v_t. This results in a simplex $\omega_{\ell,\lambda'} \in W_{\ell,s-1}$, which was therefore shelled before $\omega_{\ell,\lambda}$.

In case (2), we can identify a previously shelled simplex $\omega_{\ell,\lambda'} \in W_{\ell,s}$ that uses the face F and for which $\omega_{\ell,\lambda'} <_{colex} \omega_{\ell,\lambda}$ as follows. Since $v_t \notin \mathrm{Adj}_r[\mathrm{n}]$ then it must be that v_t corresponds to a lattice point on $p(\omega_{\ell,\lambda})$ that is a right-most vertex on a row of the lattice, that is a corner of the path, and is also not a point on the path $p(\omega_{\ell,\lambda^\star})$ (since all points on the line $y = x$ correspond to elements of $\mathrm{Adj}_r[\mathrm{n}]$). Hence, the vertex v_t is produced by a right move and followed by a left move in $(\omega_{\ell,\lambda})$. Switching the parities of these moves results in replacing v_t with a vertex v_t' which is produced by a left move and followed by a right move in the resulting cycle, say $(\omega_{\ell,\lambda'})$. Notice that the vertex v_t' is not an element of $\mathrm{Adj}_r[\mathrm{n}]$. To see this, assume otherwise. Then by Lemma 3.1.15 v_t' is a vertex produced by a left move and followed by a right move for which the number of left moves preceding the right move is maximal and not minimal with respect to equation (3.1.1). Hence, the lattice point corresponding to v_t' lies on the line $y = x$. But this implies that v_t is a vertex on $p(\omega_{\ell,\lambda^\star})$, which is a contradiction. Thus, v_t' is not an element of $\mathrm{Adj}_r[\mathrm{n}]$. Notice also that it is immediate from the parity switch of the moves about v_t that $\omega_{\ell,\lambda'} <_{colex} \omega_{\ell,\lambda}$. Hence, $\omega_{\ell,\lambda'} \in W_{\ell,s}$ with $\omega_{\ell,\lambda'} <_{colex} \omega_{\ell,\lambda}$. Moreover, since $\omega_{\ell,\lambda'}$ uses the face F since this simplex only differs from $\omega_{\ell,\lambda}$ by the vertex v_t, which is not used in F.

Now suppose $v_t = v_0 = \mathrm{adj_r}(\ell)$. By Corollary 3.1.19 we know that switching the parities about v_0 reduces to a previously shelled simplex, which only differs from the simplex $\omega_{\ell,\lambda}$ by the vertex v_0. Hence, the face F also appears in the previously shelled simplex.

We next show that G_ω is indeed a new face. Notice that by Remark 3.1.11 G_ω contains all the vertices in $v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]$. For the sake of contradiction, suppose that G_ω appeared in a previously shelled simplex, say $\omega_{\ell',\lambda'}$. That is, $\omega_{\ell',\lambda'} < \omega_{\ell,\lambda}$. Since $\omega_{\ell,\lambda} \in W_{\ell,s}$ and we are assuming $\omega_{\ell',\lambda'} < \omega_{\ell,\lambda}$ then $\omega_{\ell',\lambda'}$ uses at most s elements of $\mathrm{Adj}_r[\mathrm{n}]$. But since G_ω contains s elements of $\mathrm{Adj}_r[\mathrm{n}]$ we have that $\omega_{\ell',\lambda'} \in W_{\ell',s}$. In particular, $\omega_{\ell',\lambda'}$ uses precisely the same elements of $\mathrm{Adj}_r[\mathrm{n}]$ as $\omega_{\ell,\lambda}$, and so $\ell' = \ell$. Hence, $\omega_{\ell',\lambda'} = \omega_{\ell,\lambda'} \in W_{\ell,s}$. So it must be that $\lambda' <_{colex} \lambda$. That is, the right-most nonzero entry in $\lambda - \lambda'$ is positive, say $\lambda_m - \lambda_m' > 0$.

Consider the vertex in $\omega_{\ell,\lambda}$, say v_t, produced by the m^{th} right move in $(\omega_{\ell,\lambda})$. In $p(\omega_{\ell,\lambda})$ v_t corresponds to the right-most corner vertex in a row of the lattice since $\lambda_m > 0$. We then have two cases.

(1) The vertex v_t does not correspond to a point on $p(\omega_{\ell,\lambda^\star})$.

(2) The vertex v_t does correspond to a point on $p(\omega_{\ell,\lambda^\star})$.

In case (1), it follows that $v_t \in G_\omega$. Since $\omega_{\ell,\lambda}$ and $\omega_{\ell,\lambda'}$ both have the same largest element in the set $\mathrm{Adj}_r[\mathrm{n}]$, namely $\mathrm{adj_r}(\ell)$, every element of G_ω is uniquely

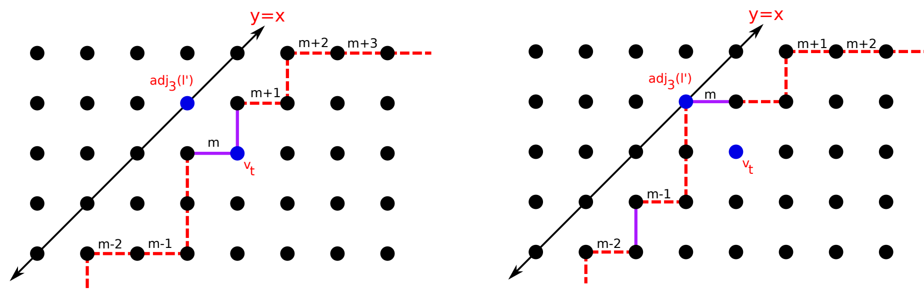


Figure 3.7: The case when vertex v_t is a point on $p(\omega_{\ell,\lambda^\star})$.

determined by the number of left and right moves needed to produce it from $\mathrm{adj}_\mathrm{r}(\ell)$. In particular, we have that v_t is produced by m right moves and $\sum_{j=1}^{m-1}\lambda_j$ left moves in $\omega_{\ell,\lambda}$, and v_t is produced by m right moves and $\sum_{j=1}^{m-1}\lambda'_j$ left moves in $\omega_{\ell,\lambda'}$. But since $\lambda_m - \lambda'_m > 0$ and $\lambda_j - \lambda'_j = 0$ for all $j > m$ we have that

$$\sum_{j=1}^{m-1}\lambda_j < \sum_{j=1}^{m-1}\lambda'_j,$$

a contradiction.

In case (2), it follows that $\omega_{\ell,\lambda}$ does not contain the vertex $\mathrm{adj}_\mathrm{r}(\ell')$ corresponding to the lattice point diagonally across from v_t on the line $y = x$. This scenario is depicted if Figure 3.7. However, since $\lambda_m - \lambda'_m$ is the right-most nonzero entry in $\lambda - \lambda'$ then $\omega_{\ell,\lambda'}$ does contain $\mathrm{adj}_\mathrm{r}(\ell')$, since one of the left moves accounted for by λ_m must now be accounted for in λ'_t for $t < m$. But this contradicts the fact that

$$v_{(\omega_{\ell,\lambda'})} \cap \mathrm{Adj}_r[\mathrm{n}] = v_{(\omega_{\ell,\lambda})} \cap \mathrm{Adj}_r[\mathrm{n}]\,.$$

Thus, G_ω is indeed a new face. This completes the proof of Theorem 3.1.14.

Some Corollaries of the Stable Shelling Theorem.

We first note that inductively Theorem 3.1.14 results in a shelling of the odd second hypersimplex $\Delta_{n,2}$, thereby proving Theorem 3.1.1. This shelling is interesting in the sense that it begins with simplices that use only the "most stable" vertices of the polytope and at each stage adds a simplex that uses more and more of the "less stable" vertices. We now give a few results that will be helpful in Section 3.2, where we examine the h^*-polynomials of these polytopes.

Corollary 3.1.21. *Let $\omega_{\ell,\lambda} \in \max\nabla^r_{n,2}\backslash\max\nabla^{r+1}_{n,2}$. The maximum dimension of the minimum new face G_ω is $r+1$.*

Proof. Consider the lattice path $p(\omega_{\ell,\lambda})$ and recall that the vertices of the minimal new face G_ω correspond to lattice points on $p(\omega_{\ell,\lambda})$ that lie on the line $y = x$ together with those that are the right-most such points in their row of the lattice, that are corners of $p(\omega_{\ell,\lambda})$, and are not on the path $p(\omega_{\ell,\lambda^\star})$. In particular, these are the lattice points (x, x) for $x \in [r]$ (which we will call *type 1*), and (x, y) where $y \leq x-3$, $y \in \{0, 1, \ldots, r-1\}$, and (x, y) is a corner of $p(\omega_{\ell,\lambda})$ that is not also on the path $p(\omega_{\ell,\lambda^\star})$ (which we will call *type 2*). The lattice path $p(\omega_{\ell,\lambda})$ can have at most one such point on each line $y = \alpha$ for $\alpha \in [r]$, and at most two such points on the line $y = 0$ (one of which is always $\mathrm{adj_r}(\ell)$). This gives a maximum possible dimension of $r + 2$ for G_ω.

Assume now that there exists $\omega_{\ell,\lambda}$ such that G_ω has dimension $r + 2$. Consider the vertices of G_ω corresponding to lattice points (x, x) for $x \in [r]$. Label this set of vertices by V, and for $v \in V$ label its corresponding lattice point by $p_v := (x_v, y_v)$. Notice that V is nonempty since by our assumption G_ω contains a vertex corresponding to a lattice point on the line $y = r$, and (since we do not wish to over-count the vertex $\mathrm{adj_r}(\ell)$) the only option is $\mathrm{adj_r}(\ell + \mathrm{r}) \in V$. So let $v \in V$. Then $p(\omega_{\ell,\lambda})$ cannot contain any type 2 points on the lines $y = y_v - 1$ or $y = y_v - 2$ (this is because $p(\omega_{\ell,\lambda})$ may only use North and East moves). But by our assumption the lines $y = y_v - 1$ and $y = y_v - 2$ must each contain a point corresponding to an element of G_ω. Hence, they are type 1 points, and therefore elements of V.

Beginning with $v = \mathrm{adj_r}(\ell + \mathrm{r})$, which has corresponding lattice point $p_v = (r, r)$, iterating this argument shows that for each line $y = \alpha$, $\alpha \in \{0, 1, 2, \ldots, r-1\}$, the path $p(\omega_{\ell,\lambda})$ uses the point (α, α), and no type 2 points on $y = \alpha$. Hence, $\#(G_\omega) = r + 1$, a contradiction. $\square$

For the case when $r = 1$, the next corollary is also a corollary to the algebraic formula given for the h^*-polynomial of $\Delta_{n,2}$ by Katzman in [**?**]. However, we are now able to give an entirely combinatorial proof of this result.

Corollary 3.1.22. *Let n be odd and $r < \left\lfloor \frac{n}{2} \right\rfloor$. The degree of the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is $\left\lfloor \frac{n}{2} \right\rfloor$, and it has leading coefficient n.*

Proof. Since $r < \left\lfloor \frac{n}{2} \right\rfloor$ we have shelled the simplices in $\max \nabla_{n,2}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \setminus \max \nabla_{n,2}^{\left\lfloor \frac{n}{2} \right\rfloor}$ in order to build the hypersimplex $\Delta_{n,2}^{\mathrm{stab}(r)}$. By Corollary 3.1.21 for a simplex $\omega \in \max \nabla_{n,2}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \setminus \max \nabla_{n,2}^{\left\lfloor \frac{n}{2} \right\rfloor}$ the maximum dimension of G_ω is $\left\lfloor \frac{n}{2} \right\rfloor$. It remains to show that this maximum dimension is achieved precisely n times.

Notice first that for $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$ we have that the lattice paths labeling simplices in $\max \nabla_{n,2}^{r} \setminus \max \nabla_{n,2}^{r+1}$ are bounded between the lines $y = x$ and $y = x - 3$. Also, for ω to satisfy $\#(G_\omega) = r + 1$ then there must be a total of r points of $p(\omega_{\ell,\lambda})$ on the lines $y = x$ and $y = x - 3$ other than $(0, 0)$ and $(n - r, r)$.

For $\max(v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]) = \mathrm{adj_r}(\ell)$ with $\ell \leq r$ this is impossible since there are less than r points on these lines that we may use without violating the choice of $\max(v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]) = \mathrm{adj_r}(\ell)$.

For $r < \ell < n$ consider the following. Suppose $p(\omega_{\ell,\lambda})$ uses a point (α, α) for $0 < \alpha \leq r$. Then by the same argument as in Corollary 3.1.21 this implies that $p(\omega_{\ell,\lambda})$ uses $(\alpha - 1, \alpha - 1)$. Iterating this just as before we get that $p(\omega_{\ell,\lambda})$ uses the points

$$\{(0, 0), (1, 1), (2, 2), \ldots, (\alpha, \alpha)\}.$$

However, this contradicts the fact that $r \leq \ell < n$ and $\max(v_{(\omega)} \cap \mathrm{Adj}_r[\mathrm{n}]) = \mathrm{adj_r}(\ell)$. Hence, the only point used by $p(\omega_{\ell,\lambda})$ on the line $y = x$ is $(0, 0)$. Therefore, $p(\omega_{\ell,\lambda})$ must be the path using all the points on the line $y = x - 3$.

For $\ell = n$ there are exactly $r+1$ paths such that $\#(G_\omega) = r+1$. This is also seen from the iterative argument we used in Corollary 3.1.21. Suppose the path $p(\omega_{\ell,\lambda})$ uses $s \leq r$ points of the set $\{(1,1), (2,2), \ldots, (r,r)\}$, and let (α, α) be the point in this collection for which the value of α is maximal. It then follows that $p(\omega_{\ell,\lambda})$ uses all the points

$$\{(0,0), (1,1), (2,2), \ldots, (\alpha, \alpha)\}.$$

Hence, it must be that $\alpha = s$. Since there is exactly one path that uses $r+1$ points on the lines $y = x$ and $y = x - 3$ and uses the points $\{(0,0), (1,1), (2,2), \ldots, (\alpha, \alpha)\}$, then we conclude that there are exactly $r+1$ simplices $\omega_{n,\lambda}$ with $\#(G_\omega) = r+1$.

Considering all of these cases together we conclude that there are exactly n simplices with $\#(G_\omega) = r+1$. $\square$

Remark 3.1.23. We remark that the proofs of the previous corollaries are intriguing since they point out that this shelling allows us to study the Ehrhart Theory of the odd second hypersimplices, as well as the r-stable odd second hypersimplices, by enumerating lattice paths in various ladder-shaped regions of the plane. However, this enumeration problem, in general, is not trivial as suggested by the work of Krattenthaler in [31].

3.2 The h^*-polynomials of the r-stable Odd Second Hypersimplices

In the following we compute the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ for n odd. For every $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$ we give a formula for the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ in terms of a sum of independence polynomials of certain graphs. In the case of $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$ we show that the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is precisely the independence polynomial of the cycle on n vertices, or equivalently, the n^{th} Lucas polynomial. Then, via Ehrhart-MacDonald Reciprocity, we demonstrate that the h^*-polynomial of the relative interior of this polytope is a univariate specialization of a polynomial that plays an important role in the theory of proper holomorphic mappings of complex balls in Euclidean space. Specifically, this polynomial is the squared Euclidean norm function of a well-studied CR mapping of the Lens space into the unit sphere within $\mathbb{C}^{r+3}$ [11, 12, 13].

The h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ via Independence Polynomials of Graphs.

It will be helpful to recall some basic facts about independence polynomials of graphs. Suppose that G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. An *independent* set in G is a subset of the vertices of G, $S \subset V(G)$, such that no two vertices in S are adjacent in G. Let s_i denote the number of independent sets in G with cardinality i, and let $\beta(G)$ denote the maximal size of an independent set in G.

The *independence polynomial* of G is the polynomial

$$I(G;x) := \sum_{i=0}^{\beta(G)} s_i x^i.$$

Independence polynomials of graphs are well-studied structures. Levit and Mandrescu nicely survey properties of these polynomials in [36]. Here, we restrict our attention only to those properties which we will use to compute h^*-polynomials. Suppose that G_1 and G_2 are two finite simple graphs, and let $G_1 \cup G_2$ denote their disjoint union. It is a well known fact (see for instance [22] or [36]) that

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x).$$

Let P_n and C_n denote the path and cycle on n vertices, respectively. In [2], Arocha showed that

$$I(P_n; x) = F_{n+1}(x) \qquad \text{and} \qquad I(C_n; x) = F_{n-1}(x) + 2xF_{n-2}(x),$$

where $F_n(x)$ denotes the n^{th} *Fibonacci polynomial*. The Fibonacci polynomials are defined for $n \geq 0$ by the recursion

$$F_0(x) = 1, \quad F_1(x) = 1, \quad \text{and} \quad F_n(x) = F_{n-1}(x) + xF_{n-2}(x).$$

A closely related class of polynomials are the *Lucas polynomials*, which are defined by the recursion

$$L_0(x) = 2, \quad L_1(x) = 1, \quad \text{and} \quad L_n(x) = L_{n-1}(x) + xL_{n-2}(x).$$

These collections of polynomials will play important roles in our computations of h^*-polynomials.

In the following we let $\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}};\mathrm{x}\right)$ denote the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$. Recall that Theorem 3.1.14 provides a shelling of the unimodular triangulation of $\Delta_{n,2}^{\mathrm{stab}(r)}$ induced by the circuit triangulation of $\Delta_{n,2}$. We let $\nabla_{n,2}^r$ denote this triangulation of $\Delta_{n,2}^{\mathrm{stab}(r)}$ and $\max \nabla_{n,2}^r$ denote the collection of maximal simplices in $\nabla_{n,2}^r$. By a theorem of Stanley [44], we may compute

$$\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}};\mathrm{x}\right) = \sum_{i=0}^{n-1} h_i^* x^i$$

where h_i^* equals the number of simplices in $\max \nabla_{n,2}^r$ with unique minimal new face of dimension $i-1$ with respect to the shelling described by Theorem 3.1.14. Also recall that by Lemma 2.2.7

$$\left\{\Delta_{n,2}^{stab\left(\lfloor \frac{n}{2} \rfloor\right)}\right\} = \max \nabla_{n,2}^{\lfloor \frac{n}{2} \rfloor},$$

and this simplex has unique minimal new face $\emptyset$. Moreover, every simplex in $\max \nabla^r_{n,2}$ other than this simplex has unique minimal new face of dimension at least 0. By the construction of the shelling, for a fixed $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$, the simplices in $\max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ correspond to lattice paths in decorated ladder-shaped regions of the plane. Since the shape of this region is fixed for fixed values of n and r, and there is one such region for each $\ell \in [n]$, we will refer to the decorated ladder-shaped region with origin label ℓ as an *ℓ-region*.

Suppose $\omega \in \max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ is a simplex with corresponding lattice path $\lambda(\omega)$. Then, for a fixed ℓ, $\lambda(\omega)$ is a lattice path in the corresponding ℓ-region if and only if $\mathrm{adj_r}(\ell)$ is the maximal r-adjacent vertex used by ω. Hence, any point on the boundary lines $y = x$ and $y = x - n + 2r$ labeled by s with $s > \ell$ cannot be used by $\lambda(\omega)$. In this way, these lattice points are *inaccessible* lattice points of the ℓ-region. We now formalize these definitions.

Definition 3.2.1. Fix an odd $n > 2$ and $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$. For each $\ell \in [n]$, we call the decorated ladder-shaped region of $\mathbb{Z}^2$ containing the lattice paths corresponding to simplices in $\max \nabla^r_{n,2}$ with maximal r-adjacent vertex $\mathrm{adj_r}(\ell)$, an **ℓ-region**. For a fixed $\ell \in [n]$, an **accessible** point in an ℓ-region is a lattice point that does not lie on the path $\lambda^\star$, but does lie on a path $\lambda(\omega)$ for some simplex $\omega \in \max \nabla^r_{n,2}$ with maximal r-adjacent vertex $\mathrm{adj_r}(\ell)$. Otherwise, it is called **inaccessible**.

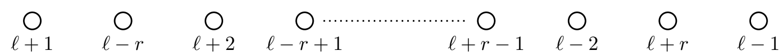
Recall that the vertices of the unique minimal new face of ω correspond to the lattice points labeled by ℓ and the corners of $\lambda(\omega)$ that "point away" from the path $\lambda^\star$. With these facts in hand, we are ready to compute the h^*-polynomials of $\Delta^{\mathrm{stab}(r)}_{n,2}$ for n odd.

We first examine the case where $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$. Then we will extend this result to the remaining values of r. So fix an odd $n > 2$ and set $r := \left\lfloor \frac{n}{2} \right\rfloor - 1$. Then the boundary lines of the ℓ-regions are given by $y = x$ and $y = x - 3$. Since the vertices of the unique minimal new face of ω correspond to the lattice points labeled by ℓ and the corners of $\lambda(\omega)$ that "point away" from the path $\lambda^\star$ then the dimension of the unique minimal new face of ω is the number of corners of $\lambda(\omega)$ that lie on the lines $y = x$ and $y = x - 3$. (Notice that the *cardinality* of the unique minimal new face is one more than this value, in which case we include the vertex corresponding to the origin in our count as well.) Suppose that (x, x) is an accessible lattice point in an ℓ-region. Then the lattice path $\lambda(\omega)$ may either use the lattice point in the set $\{(x, x)\}$ or it may use lattice points in the set $\{(x + 2, x - 1), (x + 1, x - 2)\}$, but it may not use points from both sets. This is an immediate consequence of the fact that $\lambda(\omega)$ only uses North and East moves. In this way, the lattice point (x, x) *inhibits* the lattice points in the set $\{(x+2, x-1), (x+1, x-2)\}$ and vice versa. For convenience, we make this a formal definition.

Definition 3.2.2. Fix odd $n > 2$, $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$, and $\ell \in [n]$. Suppose a and b are lattice points in the corresponding ℓ-region. We say that a and b **inhibit** one another if and only if the vertices corresponding to a and b cannot appear together in the unique minimal new face of any simplex in $\max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ with maximum

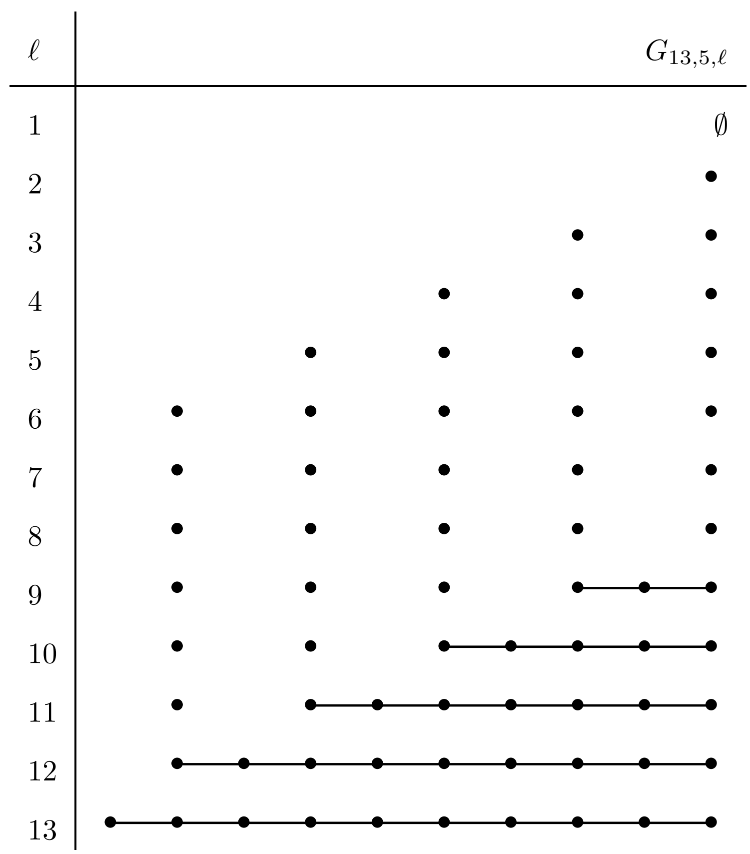
r-adjacent vertex $\mathrm{adj_r}(\ell)$. Moreover, if a and b possess labels ℓ_1 and ℓ_2, we also say that ℓ_1 and ℓ_2 **inhibit** one another.

Using the labels of these points in the ℓ-region this scenario can be represented in the following manner. Consider a line of $2r$ spots labeled as follows:



Notice that each label is adjacent to exactly those labels that it inhibits. For each $\ell \in [n]$ we will use this diagram to construct a graph $G_{n,r,\ell}$ whose independent sets (together with the origin) are precisely the unique minimal new faces of the simplices whose lattice paths reside in the ℓ-region. Fix $\ell \in [n]$, and let S denote the set of accessible vertices in the corresponding ℓ-region. Construct the graph $G_{n,r,\ell}$ by filling in each spot in the above diagram corresponding to a vertex in S. We think of these filled in spots as the vertices of $G_{n,r,\ell}$, and we place an edge between any two vertices that are not separated by a spot.

Example 3.2.3. Here are the graphs $G_{n,r,\ell}$ for each choice of ℓ when $n = 13$ and $r = \left\lfloor \frac{n}{2} \right\rfloor - 1 = 5$.



Proposition 3.2.4. *Fix odd $n > 2$ and let $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$. Then*

$$\mathrm{h}^*\left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}};\mathrm{x}\right) = 1 + x\sum_{\ell=1}^{n} I(G_{n,r,\ell};x).$$

Equivalently,

$$\mathrm{h}^*\left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}};\mathrm{x}\right) = 1+x\left(\sum_{i=0}^{r-1}(1+x)^i + 3(1+x)^r + \sum_{i=0}^{r-2}(1+x)^i F_{2r-2i}(x) + F_{2r+1}(x)\right).$$

Proof. Suppose

$$I(G_{n,r,\ell};x) = s_0 + s_1x + s_2x^2 + \cdots + s_{\beta(G_{n,r,\ell})}x^{\beta(G_{n,r,\ell})}.$$

Then s_i is equal to the number of independent sets of cardinality i in $G_{n,r,\ell}$. By the construction of $G_{n,r,\ell}$ this number is precisely the number of simplices in $\max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ that have unique minimal new face of dimension i and maximal r-adjacent vertex $\mathrm{adj_r}(\ell)$. Hence,

$$\mathrm{h}^*\left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}};\mathrm{x}\right) = 1 + x\sum_{\ell=1}^{n} I(G_{n,r,\ell};x).$$

To prove the second equality we must identify the graphs $G_{n,r,\ell}$. In order to do this, we must understand the accessible lattice points in an ℓ-region for each $\ell \in [n]$. In general, the set of accessible points for each ℓ-region is given by

ℓ	Set of accessible points in the ℓ-region
1	$\{\ell\}$
2	$\{\ell, \ell-1\}$
3	$\{\ell, \ell-1, \ell-2\}$
$\vdots$	$\vdots$
r	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1\}$
$r+1$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r\}$
$r+2$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r\}$
$r+3$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r\}$
$r+4$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r, \ell+r\}$
$r+5$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r, \ell+r, \ell+r-1\}$
$r+6$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r, \ell+r, \ell+r-1, \ell+r-2\}$
$\vdots$	$\vdots$
$n-1$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r, \ell+r, \ell+r-1, \ell+r-2, \ldots, \ell+2\}$
$n-1$	$\{\ell, \ell-1, \ell-2, \cdots, \ell-r+1, \ell-r, \ell+r, \ell+r-1, \ldots, \ell+2, \ell+1\}$

That is, as ℓ increases, we first gain points on the diagonal $y = x - 3$ from top-to-bottom, and then those on $y = x$ from top-to-bottom. This happens one point at a time except for $\ell = r+2$ and $\ell = r+3$, which have the same set of accessible points as $\ell = r+1$. This is because $n = 2r+3$ when $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$, and the labels on the diagonals of an ℓ-region are precisely those vertices of a convex n-gon labeled $1, 2, \ldots, n$ with circular distance at most r from ℓ.

Given this characterization of the accessible points in each ℓ-region we can determine the graphs $G_{n,r,\ell}$ as follows. For $\ell \leq r$ the graph $G_{n,r,\ell}$ is simply a collection of $\ell - 1$ disjoint vertices. For $\ell \in \{r+1, r+2, r+3\}$ the graph $G_{n,r,\ell}$ is a collection of r disjoint vertices. For $r+4 \leq \ell \leq n-1$ we begin to add edges. For convenience let $\ell = r+3+t$ for the suitable value of t. Then the graph $G_{n,r,\ell}$ includes the vertices

$$\ell + r, \ell + r - 1, \ell + r - 2, \ldots, \ell + r - (t-1).$$

Each of these vertices attaches to each vertex next to it in the diagram. This produces a path of length $2t+1$ and a collection of $r - (t+1)$ disjoint vertices. Finally, for $\ell = n$, all points are accessible so $G_{n,r,\ell} = P_{2r}$. Then, if we make the change of variables $i = r - (t+1)$, we have the following.

$$\begin{aligned}
\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right) &= 1 + x \sum_{\ell=1}^{n} I(G_{n,r,\ell}; x), \\
&= 1 + x \left(\sum_{i=0}^{r-1} (1+x)^i + 3(1+x)^r \right. \\
&\left. + \sum_{i=0}^{r-2} (1+x)^i I(P_{2r-2i-1}; x) + I(P_{2r}; x) \right), \\
&= 1 + x \left(\sum_{i=0}^{r-1} (1+x)^i + 3(1+x)^r + \sum_{i=0}^{r-2} (1+x)^i F_{2r-2i}(x) + F_{2r+1}(x) \right)
\end{aligned}$$

□

This formula for $\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right)$ is convenient since it allows us to compute this polynomial via rows and diagonals of Pascal's Triangle. In subsection 3.2, we will see that this formula is equal to the n^{th} Lucas polynomial. However, we first show how we may generalize this formula to $\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right)$ for $1 \leq r < \left\lfloor \frac{n}{2} \right\rfloor - 1$. To do so, we construct an inhibition diagram similar to the one used for $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Fix $1 \leq r < \left\lfloor \frac{n}{2} \right\rfloor - 1$. Construct a graph $G_{n,r}$ with vertex set consisting of all lattice points in the n-region that do not lie on the path $\lambda^\star$, and edge set $\{\{i,j\} : i \text{ inhibits } j\}$. Let $G_{n,r,\ell}$ denote the subgraph of $G_{n,r}$ that is induced by the set of accessible points in the ℓ-region. We refer to the graph $G_{n,r,\ell}$ as the *inhibition diagram* for its associated ℓ-region.

Proposition 3.2.5. *Fix odd $n > 2$ and $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Then*

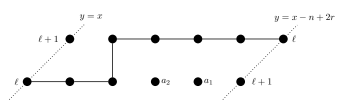
$$\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right) = 1 + x \left(\sum_{j=r}^{\left\lfloor \frac{n}{2} \right\rfloor - 1} \sum_{\ell=1}^{n} I(G_{n,j,\ell}; x) \right).$$

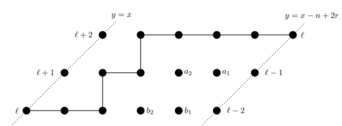
In particular, the h^*-polynomial of $\Delta_{n,2}$ is given by a sum of independence polynomials.

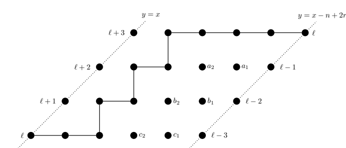
Proof. The proof of this formula is identical to the proof of the first formula given in Proposition 3.2.4. $\square$

Notice that Proposition 3.2.5 proves Theorem **??**. We end this subsection with a few examples of the graphs $G_{n,r,\ell}$ and the resulting h^*-polynomials.

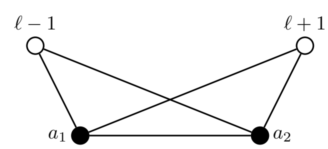
Example 3.2.6. In Example 3.2.3 we saw that the inhibition diagrams for ℓ-regions when $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$ correspond to subgraphs of a path of length $2r$. Suppose now that $r = \left\lfloor \frac{n}{2} \right\rfloor - 2$. Here are the general ℓ-regions for $n = 7, 8$, and 9, respectively.

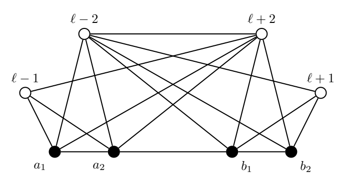


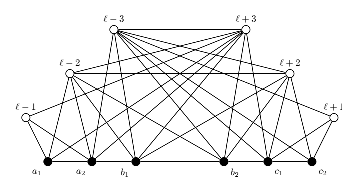




The corresponding inhibition diagrams are, respectively,



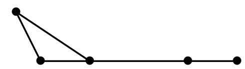
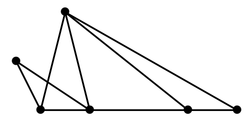
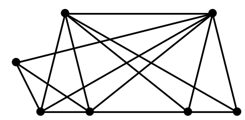
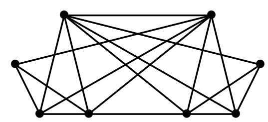




Using the formula given in Proposition 3.2.5, we can compute $\mathrm{h}^*\left(\Delta_{9,2}^{\mathrm{stab}(2)};\mathrm{x}\right)$. By Proposition 3.2.4 we know that

$$\mathrm{h}^*\left(\Delta_{9,2}^{\mathrm{stab}(3)};\mathrm{x}\right) = 1 + 9x + 27x^2 + 30x^3 + 9x^4.$$

Hence, it remains to compute the independence polynomials for $G_{9,2,\ell}$ for each $\ell \in [9]$. These polynomials are identified in the following table.

ℓ	$G_{9,2,\ell}$	$I(G_{9,2,\ell};x)$
1		$1+4x+3x^2$
2		$1+5x+5x^2$
$3-7$		$1+6x+6x^2$
8		$1+7x+6x^2$
9		$1+8x+10x^2$

It follows that

$$\begin{aligned}\mathrm{h}^*\left(\Delta_{9,2}^{\mathrm{stab}(2)};\mathrm{x}\right) &= \mathrm{h}^*\left(\Delta_{9,2}^{\mathrm{stab}(3)};\mathrm{x}\right) + x(9+54x+54x^2),\\ &= 1+18x+81x^2+84x^3+9x^4.\end{aligned}$$

Similarly, we can also compute that

$$\begin{aligned}\mathrm{h}^*\left(\Delta_{7,2}^{\mathrm{stab}(1)};\mathrm{x}\right) &= 1+14x+35x^2+7x^3,\\ \mathrm{h}^*\left(\Delta_{9,2}^{\mathrm{stab}(2)};\mathrm{x}\right) &= 1+18x+81x^2+84x^3+9x^4, \text{ and}\\ \mathrm{h}^*\left(\Delta_{11,2}^{\mathrm{stab}(3)};\mathrm{x}\right) &= 1+22x+143x^2+297x^3+165x^4+11x^5.\end{aligned}$$

The h^*-polynomial of $\Delta_{n,2}^{stab\left(\left\lfloor \frac{n}{2}\right\rfloor-1\right)}$ and CR Mappings of Lens Spaces.

Fix odd $n>2$ and $r := \left\lfloor \frac{n}{2}\right\rfloor - 1$. We begin this subsection by demonstrating that the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is the independence polynomial of the cycle on n vertices. As a corollary, we show that the h^*-polynomial of the relative interior of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is a univariate specialization of an important class of polynomials in CR geometry.

Theorem 3.2.7. *Fix odd $n>2$ and $r = \left\lfloor \frac{n}{2}\right\rfloor - 1$. Then*

$$\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab}(\mathrm{r})};\mathrm{x}\right) = I(C_n;x) = L_n(x).$$

To prove Theorem 3.2.7 we first need a lemma that relates Lucas polynomials and Fibonacci polynomials.

Lemma 3.2.8. *For $n \geq 2$, the n^{th} Lucas polynomial can be computed as*

$$L_n(x) = F_n(x) + xF_{n-2}(x).$$

Proof. In [37] it is noted that the diagonals of Pascal's Triangle given by

$$\binom{n}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \ldots, \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}$$

are precisely the coefficients of the n^{th} Fibonnaci polynomial

$$F_n(x) = \binom{n}{0} + \binom{n-1}{1}x + \binom{n-2}{2}x^2 + \cdots + \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor}x^{\lfloor \frac{n}{2} \rfloor}.$$

In Appendix A of [29] it is noted that the n^{th} Lucas polynomial is given by the same diagonal in the modified Pascal's Triangle

$$\begin{array}{ccccccccccc}
 & & & & & 2 & & & & & \\
 & & & & 1 & & 2 & & & & \\
 & & & 1 & & 3 & & 2 & & & \\
 & & 1 & & 4 & & 5 & & 2 & & \\
 & 1 & & 5 & & 9 & & 7 & & 2 & \\
1 & & 6 & & 14 & & 16 & & 9 & & 2
\end{array}$$

For convenience, we refer to this triangle as Lucas' Triangle. One way to produce Lucas' Triangle is to write the $2's$ on the right boundary as $2 = 1_b + 1_g$. In this way, we have a blue 1 and and green 1 summing to give 2. Imagine that the $1's$ on the left boundary are also blue $1's$. Now as we fill in the interior of the triangle using the standard Pascalian recursion write each entry as the sum of the blue $1's$ plus the sum of the green $1's$. This yields

$$\begin{array}{ccccccccc}
 & & & & 1_b + 1_g & & & & \\
 & & & 1_b & & 1_b + 1_g & & & \\
 & & 1_b & & (2)_b + (1)_g & & 1_b + 1_g & & \\
 & 1_b & & (3)_b + (1)_g & & (3)_b + (2)_g & & 1_b + 1_g & \\
1_b & & (4)_b + (1)_g & & (6)_b + (3)_g & & (4)_b + (3)_g & & 1_b + 1_g
\end{array}$$

With this decomposition we see that Lucas' Triangle can be produced by a term-by-term sum of Pascal's Triangle and the Pascal-like triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 0 & & 1 & & 2 & & 1 & & \\
 & 0 & & 1 & & 3 & & 3 & & 1 & \\
0 & & 1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

From this it is then easy to deduce the identity

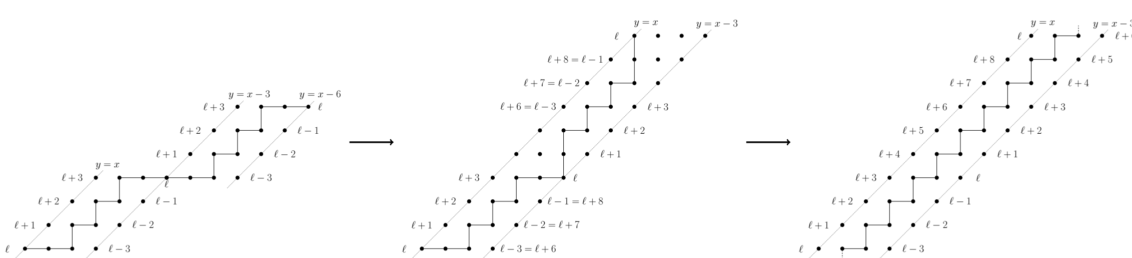
$$L_n(x) = F_n(x) + xF_{n-2}.$$

□

With this lemma in hand, we are ready to prove Theorem 3.2.7.

Proof of Theorem 3.2.7.

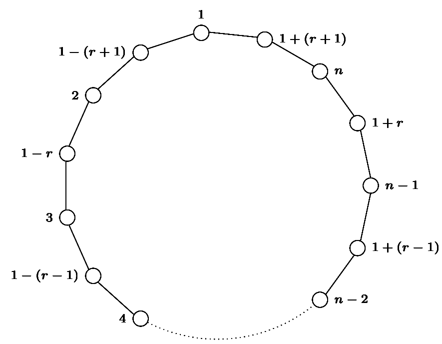
Recall that for a fixed $\ell \in [n]$ the number of simplices in $\max \nabla^r_{n,2} \setminus \max \nabla^{r+1}_{n,2}$ with maximum r-adjacent vertex $\mathrm{adj_r}(\ell)$ with unique minimal new face of dimension i is the number of ways to construct a lattice path in the ℓ-region that uses precisely i accessible lattice points on the lines $y = x$ and $y = x - 3$. Recall however, that in the ℓ-region the lattice points $(0, 0)$ and $(n - r, r)$ correspond to the same vertex. In particular, we can think of the ℓ-region as repeating itself in the region translated right $n-r$ and up r. Reflecting this translated copy of the ℓ-region about the line $y = x-3$ results in a strip of height n in the region between $y = x$ and $y = x - 3$ containing a lattice path, corresponding to $\lambda^\star$, that only touches the boundary diagonals at lattice points labeled by ℓ. Flip the corner of this path at the lattice point labeled by ℓ (also making the corresponding flips at the top and bottom of the diagram), and label the two points on the opposite boundary line that are inhibited by ℓ as $\ell - (r + 1)$ and $\ell + (r + 1)$. This results in a diagram with (t, t) labeled by $\ell + t$ and $(t, t - 3)$ labeled by $\ell - r + t$ for $t \in [n]$. The diagram below represents these manipulations of the ℓ-region for $n = 9$.



For this region, the inhibition diagram for the spots

$$X = \{1, 2, 3, \ldots, n\}$$

is exactly a cycle on n vertices labeled as



We remark that this labeling of C_n is precisely the underlying graph of the edge polytope $\Delta_{n,2}^{stab\left(\lfloor \frac{n}{2} \rfloor\right)}$, the next smallest r-stable hypersimplex in the chain.

Notice now that for a simplex with maximum r-adjacent vertex $\mathrm{adj_r}(\ell)$ and unique minimal new face of dimension i we have a unique independent set in this labeled copy of C_n of cardinality $i+1$ with maximal vertex label being ℓ. Conversely, if we pick an arbitrary independent $(i+1)$-set in this labeled copy of C_n it has a unique maximally labeled element, say the vertex labeled by ℓ. In the region for which this inhibition diagram arises we may then flip the corners of the lattice path corresponding to $\lambda^\star$ such that it touches the boundary diagonals $y = x$ and $y = x - 3$ precisely at the lattice points labeled by the elements of the independent set. Then translate the diagram so that ℓ is labeling the origin (and the first move is an East move). This lattice path in this ℓ-region gives the corresponding simplex in $\max \nabla_{n,2}^{r} \backslash \max \nabla_{n,2}^{r+1}$. This establishes a bijection between the independent $(i+1)$-sets in this labeled copy of C_n and the simplices in $\max \nabla_{n,2}^{r} \backslash \max \nabla_{n,2}^{r+1}$ with unique minimal new face of dimension i. Hence,

$$\mathrm{h}^* \left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}}; \mathrm{x} \right) = I(C_n; x).$$

Finally, to see that this polynomial is also the n^{th} Lucas polynomial we utilize a result of Arocha [2] which states that

$$I(C_n; x) = F_{n-1}(x) + 2xF_{n-2}(x).$$

From this fact, and the identity from Lemma 3.2.8, we see that

$$\begin{aligned}
\mathrm{h}^* \left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}}; \mathrm{x} \right) &= I(C_n; x), \\
&= F_{n-1}(x) + 2xF_{n-2}(x), \\
&= (F_{n-1}(x) + xF_{n-2}(x)) + xF_{n-2}(x), \\
&= F_n(x) + xF_{n-2}(x), \\
&= L_n(x).
\end{aligned}$$

This completes the proof of Theorem 3.2.7.

We now describe a connection between our polytopes and CR geometry. CR geometry is a fascinating field of study that examines properties of real hypersurfaces as submanifolds of $\mathbb{C}^n$ and their intrinsic complex structure induced by the ambient space $\mathbb{C}^n$. The interested reader should refer to [11] for a nice introduction to this theory. In [11, Chapter 5], D'Angelo describes the theory of proper holomorphic mappings between balls that are invariant under subgroups of the unitary group $U(n)$. These maps are particularly interesting as they induce maps from spherical space forms into spheres. In [13], D'Angelo, Kos, and Riehl define recursively the following collection of polynomials (the former-most author also defines this collection explicity in [11, 12]). Let

$$g_0(x,y) = x, \qquad g_1(x,y) = x^3 + 3xy,$$

and

$$g_n(x,y) = (x^2 + 2y)g_{n-1}(x,y) - y^2 g_{n-2}(x,y),$$

for $n \geq 2$. Then set

$$p_n(x,y) = g_n(x,y) + y^{2n+1}.$$

For odd $n > 2$ consider the group $\Gamma(n,2)$ of 2×2 complex matrices of the form

$$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^2 \end{pmatrix}^k$$

where γ is a primitive n^{th} root of unity and $k = 0, 1, 2, \ldots, n-1$. Recall that the Lens space, $L(n,2)$ is defined as the quotient $L(n,2) = S^3/\Gamma(n,2)$. In [11, 12] it is shown that the polynomials $p_n(x,y)$ correspond to proper holomorphic monomial maps between spheres (given by their monomial components) that are invariant under $\Gamma(n,2)$. In [13] it is shown that these polynomials are of highest degree with respect to this property. These polynomials share the following relationship with $\Delta_{n,2}^{\mathrm{stab}(r)}$ via Ehrhart-MacDonald reciprocity.

Theorem 3.2.9. *Fix odd $n > 2$ and let $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$. Then*

$$p_{r+1}(x,x) = \mathrm{h}^*\left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}^\circ}; \mathrm{x}\right) + x^n,$$

where $\Delta_{n,2}^{\mathrm{stab}(r)^\circ}$ denotes the relative interior of $\Delta_{n,2}^{\mathrm{stab}(r)}$. Hence, the h^*-polynomial of the relative interior of $\Delta_{n,2}^{\mathrm{stab}(r)}$ (plus an x^n term) is a univariate evaluation of the squared Euclidean norm function of a monomial CR mapping of the Lens space, $L(n,2)$, into the unit sphere of complex dimension $r+3$.

Proof. Consider the Lucas sequence defined by the recurrence

$$\mathcal{L}_0(x,y) = 2, \quad \mathcal{L}_1(x,y) = x, \quad \text{and} \quad \mathcal{L}_n(x,y) = x\mathcal{L}_{n-1}(x,y) + y\mathcal{L}_{n-2}(x,y)$$

for $n \geq 2$. Comparing this sequence to the Lucas polynomials $\{L_n\}$ we may deduce that

$$\mathcal{L}_n(x,y) = x^n L_n\left(\frac{y}{x^2}\right).$$

By applying the recurrence, we see that

$$\mathcal{L}_{n+2}(x,y) = \mathcal{L}_n(x,y)(x^2+2y) - y^2\mathcal{L}_{n-2}(x,y).$$

It then follows that the odd terms of the sequence $\{\mathcal{L}_n\}$ are precisely the terms of the sequence $\{g_n(x,y)\}$. That is, if $n = 2m+1$ then

$$\mathcal{L}_n(x,y) = g_m(x,y).$$

Recall now that $n = 2r+3 = 2(r+1)+1$. Applying these facts, together with Ehrhart-MacDonald reciprocity, we see that

$$\begin{aligned} p_{r+1}(x,x) &= g_{r+1}(x,x) + x^n, \\ &= \mathcal{L}_n(x,x) + x^n, \\ &= x^n L_n\left(\frac{x}{x^2}\right) + x^n, \\ &= x^n \,\mathrm{h}^*\left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}}; \frac{1}{\mathrm{x}}\right) + x^n, \\ &= \mathrm{h}^*\left(\Delta_{\mathrm{n,2}}^{\mathrm{stab(r)}^\circ}; \mathrm{x}\right) + x^n. \end{aligned}$$

The fact that $p_{r+1}(x,y)$ is the squared Euclidean norm function of a monomial CR mapping of the Lens space, $L(n,2)$, into a sphere of complex dimension $r+3$ is proven via the discussion in [11, pp.171-174]. $\square$

More generally, for $r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, the h^*-polynomial of the relative interior of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is a univariate specialization of the squared Euclidean norm function of certain polynomial maps that induce smooth immersions of the Lens space $L(n,2)$ into $\mathbb{C}^{\left\lfloor \frac{n}{2} \right\rfloor + 1}$. However, it is unclear that these maps are mapping $L(n,2)$ into a hypersurface with interesting structure. For this reason we pose the following question.

Question 3.2.10. For $r < \left\lfloor \frac{n}{2} \right\rfloor - 1$, does the h^*-polynomial of the relative interior of $\Delta_{n,2}^{\mathrm{stab}(r)}$ arise as a univariate specialization of a polynomial corresponding to maps between interesting manifolds?

3.3 The r-stable Even Second Hypersimplices

In this section we extend the results of sections 3.1 and 3.2 to the case of the even second hypersimplices, that is $\Delta_{n,k}^{\mathrm{stab}(r)}$ where n is even and $k = 2$. Unlike the case when n is odd, the smallest $(n-1)$-dimensional r-stable even second hypersimplices are not necessarily unimodular simplices. Thus, if we wish to parallel the results of sections 3.1 and 3.2 by providing a shelling of each triangulation $\nabla_{n,2}^r$ of $\Delta_{n,2}^{\mathrm{stab}(r)}$ for n even that inducts on r, we must first identify a base case and a shelling of this base case. By Theorem 2.3.5, we know that for n even $\Delta_{n,2}^{\mathrm{stab}\left(\frac{n}{2}-1\right)}$ is $(n-1)$-dimensional. As we will see in Chapter 4, these are the *Gorenstein* r-stable even second hypersimplices, and they will serve as our base case.

Shelling the r-stable Even Second Hypersimplices.

In the following, let n be even. Our first goal is to provide a shelling of $\nabla_{n,2}^{\frac{n}{2}-1}$. Following this, we will extend our inductive techniques of the previous sections to shell all r-stable even second hypersimplices, and compute their h^*-polynomials via sums of independence polynomials of graphs. Notice first that the labeling of the simplices in $\max \nabla_{n,2}^r$ given by Remarks 3.1.9 and 3.1.11 still apply. That is, $\omega \in \max \nabla_{n,2}^r$ is uniquely described by $\omega_{\ell,\lambda}$ where $\mathrm{adj_r}(\ell)$ is the unique maximal r-adjacent vertex in the simplex and $\lambda = (\lambda_1, \ldots, \lambda_{n-r-1})$ is a composition of r into $n-r-1$ parts for which

$$i - 1 \leq \sum_{j=1}^{i} \lambda_j \leq i$$

for $i = 1, \ldots, n-r-1$. Since $n = 2r+2$ when $r = \frac{n}{2} - 1$ we cannot implement the same shelling as in section 3.1 since Lemma 3.1.17, and thus Corollary 3.1.19, no longer hold. In fact, switching the parity about an r-adjacent vertex *always* produces another r-adjacent vertex when $n = 2r+2$. Thus, we need to modify our shelling to accommodate for this fact.

Recall, the order on the simplices for the n odd case first collects simplices into sets $W_{\ell,s}$ where $W_{\ell,s}$ consists of all simplices with maximal r-adjacent vertex $\mathrm{adj_r}(\ell)$ that use s elements of $\mathrm{Adj}_r[\mathrm{n}]$. We then order the elements in $W_{\ell,s}$ from least-to-greatest with respect to the colexicographic order on their associated compositions. Finally, we order the collection of sets $\{W_{\ell,s}\}$ from least-to-greatest with respect to the colexicographic order on the labels (ℓ, s). In the following remark, we define a new order on the simplices in $\nabla_{n,2}^r$ for $n = 2r+2$ which will allow us to shell our base case.

Remark 3.3.1. When $n = 2r+2$ every simplex in $\max \nabla_{n,2}^r$ uses precisely $r+1$ r-adjacent vertices. This is seen by noting that the lattice path corresponding to $\omega_{\ell,\lambda}$ lies in the region between the lines $y = x$ and $y = x - 2$. Thus, we no longer require the parameter s in the above formula. That is, for $n = 2r+2$, we denote the collection of simplices in $\max \nabla_{n,2}^r$ with unique maximal r-adjacent vertex $\mathrm{adj_r}(\ell)$ by W_ℓ. We now order the elements of each set W_ℓ from *greatest-to-least* with respect to the colexicographic order, and then order the sets W_ℓ as

$$W_1, W_2, \ldots, W_n.$$

Denote this order on the simplices in $\max \nabla_{n,2}^r$ by $<_e$.

Theorem 3.3.2. *Let $n = 2r+2$. The order $<_e$ on $\max \nabla_{n,2}^r$ is a shelling order.*

Proof. To show that this order is indeed a shelling it suffices to identify the unique minimal new face of each simplex in the order. We claim that the unique minimal new face of a simplex ω consists of all vertices whose corresponding lattice point in $p(\omega)$ lies on the line $y = x$. Denote this collection by $G_\omega = \{v_0 = \mathrm{adj_r}(\ell), v_1, v_2, \ldots, v_q\}$.

We must first show that any face F of ω consisting of $G_\omega \setminus \{v_t\}$ for some $t = 0, \ldots, q$ appears as a face of a previously shelled simplex. Consider first the case in which

$t \neq 0$. Let F be a face of $\omega_{\ell,\lambda}$ that does not use v_t. To construct a simplex $\omega_{\ell',\lambda'} <_e \omega_{\ell,\lambda}$ that uses F we do the following. Since $t \neq 0$ then v_t lies on $y = x$ and switching the parities of the moves about v_t replaces it with a vertex on $y = x - 2$. This lattice path corresponds to a simplex $\omega_{\ell',\lambda'}$ with $\ell' = \ell$. Moreover, $\lambda' >_{colex} \lambda$ since switching the parities of the moves about v_t amounts to shifting a move of 1 in λ one entry to the right. Thus, $\omega_{\ell,\lambda'} \in W_\ell$ such that $\omega_{\ell,\lambda'} <_e \omega_{\ell,\lambda}$.

To understand the case when $t = 0$ we first note that switching the parity about a point $\mathrm{adj_r}(\ell)$ replaces it with the point $\mathrm{adj_r}(\ell + \mathrm{r} + 1)$. This can be seen quickly from the labeling of the lattice region. Thus, switching the parity about v_0 replaces $v_0 = \mathrm{adj_r}(\ell)$ with $\mathrm{adj_r}(\ell + \mathrm{r} + 1)$. Note $\mathrm{adj_r}(\ell + \mathrm{r} + 1) < \mathrm{adj_r}(\ell)$ whenever $\ell > r + 1$. Since $W_1 = W_2 = \cdots = W_r = \emptyset$ and $W_{r+1} = \{\omega_{r+1,\lambda^\star}\}$ then $G_{\omega_{r+1,\lambda^\star}} = \emptyset$, as it is the first simplex in our shelling. Thus, it only remains to verify that G_ω is indeed a new face.

To see that G_ω is a new face, we simply note that every lattice path in the decorated region corresponding to $\mathrm{adj_r}(\ell)$ for $n = 2r + 2$ is uniquely determined by its vertices that lie on $y = x$. In other words, if G_ω were indeed used previously then the simplex that used it must be W_ℓ. However, the only simplex in W_ℓ that uses the vertices in G_ω is $\omega_{\ell,\lambda}$ itself. □

Corollary 3.3.3. *There exists a stable shelling of the even second hypersimplex.*

Proof. Recall that the stable shelling of the odd second hypersimplex given in section 3.1 happens inductively, and the base case shells the smallest $(n-1)$-dimensional r-stable $(n, 2)$-hypersimplex. The previous Theorem establishes the base case for n even. Moreover, the inductive step then applies here, just as in the odd case, since the only issue was in Lemma 3.1.17. In the proof of this lemma, a contradiction occurs for $n = 2r + 2$. However, we are not taking this r value in our inductive step. Thus, the same inductive shelling holds here with the Gorenstein r-stable second hypersimplices serving as the base case. □

Remark 3.3.4. One might be concerned that to shell the Gorenstein center of the even second hypersimplices we do almost the same shelling as in the odd case, but we reverse the order of the simplices in the sets W_ℓ. However, this choice is in fact consistent with our previous shelling in the sense that in both cases we always shell the simplices given by the lattice path $\lambda^\star$ first.

The h^*-polynomials of the r-stable Even Second Hypersimplices.

Provided with the inductive shelling of the r-stable even second hypersimplices described in Corollary 3.3.3 we may now apply Theorem 1.2.4 to compute the h^*-polynomials of these polytopes. We first compute the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ when $n = 2r + 2$ in terms of the shelling provided in Theorem 3.3.2.

Theorem 3.3.5. *Let $n = 2r + 2$. Then*

$$\mathrm{h}^* \left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x} \right) = (x + 1)^{r+1}.$$

Proof. We count the simplices contributing to the coefficient h_i^* by counting those arising from each set W_ℓ, $\ell = 1, \ldots, n$. For $\ell = 1, \ldots, r$, we have that $W_\ell = \emptyset$, so nothing is contributed the coefficient h_i^*. For $\ell = r + 1$, we have that $W_\ell = \{\omega_{\ell,\lambda^\star}\}$, and this simplex has the unique minimal new face $\emptyset$. Thus, $h_0 = 1$.

For $\ell > r + 1$, we have that $\ell = r + 2 + t$ for some $0 \leq t \leq r$. Moreover, the lattice point on the line $y = x$ in the ladder shaped region assoicated to $\mathrm{adj_r}(\ell)$ that is furthest from the origin is $\mathrm{adj_r}(\ell + \mathrm{r}) = \mathrm{adj_r}(2\mathrm{r} + 2 + \mathrm{t}) = \mathrm{adj_r}(\mathrm{t})$. Thus, for $\ell = r + 2 + t$, there are t available lattice points on $y = x$ (other than the origin). Since any choice of $s \leq t$ of these available points corresponds to a unique path in the region, which in turn corresponds to a unique simplex in $\max \nabla^r_{n,2}$ with unique minimal new face of dimension $s - 1$, then the polynomial

$$h_1^* x + h_2^* x^2 + \cdots + h_{n-1}^* x^{n-1} = x(x+1)^t.$$

Thus,

$$\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right) = 1 + x\sum_{t=0}^{r}(x+1)^t = (x+1)^{r+1}.$$

□

We may now extend the apply the computations of section 3.2 the the r-stable even second hypersimplices $\Delta_{n,2}^{\mathrm{stab}(r)}$ for any $1 \leq r < \left\lfloor \frac{n}{2} \right\rfloor$. Recall that for each ℓ-region, we can construct a graph $G_{n,r,\ell}$ called the inhibition diagram for the associated ℓ-region (see subsection 3.2). Since the inductive step for the stable shelling of the r-stable even second hypersimplices is the same as that of the odd ones we have the following corollary to Theorem 3.3.5.

Corollary 3.3.6. *Let $n > 0$ be even and $1 \leq r < \left\lfloor \frac{n}{2} \right\rfloor$. Then*

$$\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right) = (x+1)^{r+1} + x\left(\sum_{j=r}^{\frac{n}{2}-2}\sum_{\ell=1}^{n} I(G_{n,j,\ell}; x)\right).$$

Remark 3.3.7. Proposition 3.2.5 and Corollary 3.3.6 combine to show that the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ for any n and r for which this polytope is $(n-1)$-dimensional can be expressed as a sum of independence polynomials of certain graphs. This provides a combinatorial interpretation of the coefficients of $\mathrm{h}^*\left(\Delta_{\mathrm{n},2}^{\mathrm{stab(r)}}; \mathrm{x}\right)$, and in particular, this is true for the second hypersimplices $\Delta_{n,2}$.

3.4 Some Results on Unimodality of h^*-polynomials

A consequence of work by Katzman [30] is that the h^*-polynomial of $\Delta_{n,2}$ is unimodal. It appears that this is also true for the r-stable hypersimplices within. In this subsection, we utilize the shelling and our computations of h^*-polynomials to show that this observed unimodality does in fact hold in some specific cases. We begin with a two quick corollaries, one to Theorem 3.2.7 and the other to Theorem 3.3.5.

Corollary 3.4.1. *Fix odd $n > 2$ and let $r = \left\lfloor \frac{n}{2} \right\rfloor - 1$. The h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is log-concave and hence unimodal.*

Proof. Since the h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is $I(C_n; x)$ and C_n is a claw-free graph then by [23] it is log-concave and consequently unimodal. □

Corollary 3.4.2. *Let $n = 2r + 2$. The h^*-polynomial of $\Delta_{n,2}^{\mathrm{stab}(r)}$ is log-concave and unimodal.*

Proof. This corollary is immediate for the result of Theorem 3.3.5, which shows this h^*-polynomial is the generating polynomial for the binomial coefficiencts $\binom{n}{r+1}$. □

For $n = 2r + 2$, the r-stable hypersimplex $\Delta_{n,2}^{\mathrm{stab}(r)}$ is an example of a *Gorenstein* r-stable hypersimplex. In Chapter 4, we will use geometric techniques to extend the unimodality result of Corollary 3.4.2 to all Gorenstein r-stable hypersimplices. We also note that the polynomials describe in Corollaries 3.4.1 and 3.4.2 possess the much stronger property of *real-rootedness*, meaning that all zeros of these h^*-polynomials are real numbers.

The following results utilize the formula for the h^*-polynomial of $\Delta_{n,2}$ given by Katzman in [30]. Using this formula, we subtractively compute formulas for the h^*-polynomials of $\Delta_{n,2}^{\mathrm{stab}(2)}$ and $\Delta_{n,2}^{\mathrm{stab}(3)}$ by "undoing" our shelling.

Corollary 3.4.3. *The h^*-vector of $\Delta_{n,2}^{\mathrm{stab}(2)}$ is unimodal for n odd.*

Proof. Consider the completion of the shelling of $\nabla_{n,2}^2$ to a shelling of $\nabla_{n,2}$. Here, $r = 1$, and so the simplices $\omega_{\ell,\lambda} \in \max \nabla_{n,2} \setminus \max \nabla_{n,2}^2$ are labeled with compositions of length $n - 2$. The composition λ is a composition of 1 that must satisfy equation (3.1.1). We now determine which compositions are admissible for a fixed $\mathrm{adj}_1(\ell) = \max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}]$.

For $\ell \in \{2, 3, 4, \ldots, n-1\}$ the composition $(1, 0, 0, \ldots, 0)$ does not label a simplex with $\mathrm{adj}_1(\ell) = \max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}]$, since such a composition would necessarily have $\max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}] = \mathrm{adj}_1(\ell + 1)$. This is depicted in Figure 3.8. On the other hand, each other composition of 1 into $n - 2$ parts does correspond to a simplex with $\mathrm{adj}_1(\ell) = \max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}]$. By considering the associated lattice paths in Figure 3.9 it is easy to see that the simplex $\omega_{\ell,\lambda^\star}$ has $\#(G_\omega) = 1$, and the other $n-4$ simplices $\omega_{\ell,\lambda}$ with $\lambda = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ have $\#(G_\omega) = 2$.

For $\ell = 1$, the compositions $(1, 0, 0, \ldots, 0)$ and $(0, 0, \ldots, 0, 1)$ do not label a simplex with $\mathrm{adj}_1(1) = \max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}]$, since such a simplex necessarily has $\max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}] \in \{2, n\}$. This is depicted in Figure 3.9. Again, the simplex $\omega_{1,\lambda^\star}$ has $\#(G_\omega) = 1$, and the remaining $n-5$ simplices $\omega_{1,\lambda}$, for $\lambda = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ have $\#(G_\omega) = 2$.

Finally, for $\ell = n$ both the compositions $(1, 0, 0, \ldots, 0)$ and $(0, 0, \ldots, 0, 1)$ label a simplex with $\mathrm{adj}_1(1) = \max v_{(\omega)} \cap \mathrm{Adj}_1[\mathrm{n}]$, since $\mathrm{adj}_1(\mathrm{n}) = \max \mathrm{Adj}_1[\mathrm{n}]$. Hence there is one simplex, namely $\omega_{n,\lambda^\star}$, with $\#(G_\omega) = 1$, and the remaining $n - 3$ simplices have $\#(G_\omega) = 2$. Summarizing this analysis we have n simplices with $\#(G_\omega) = 1$, and $n(n - 4)$ simplices with $\#(G_\omega) = 2$.

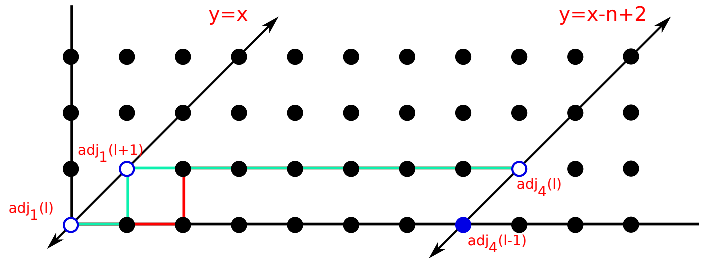


Figure 3.8: Here we let $n = 9$.

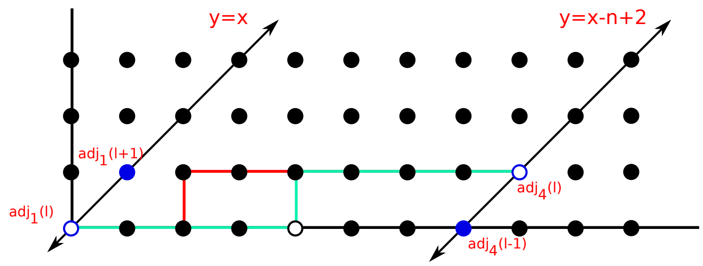


Figure 3.9: Here we let $n = 9$.

In [30], Katzman computed that for $\Delta_{n,2}$

$$h_i^* = \binom{n}{2i}$$

for $i \neq 1$, and

$$h_1^* = \binom{n}{2} - n.$$

Thus, since there are n elements in $\mathrm{Adj}_1[\mathrm{n}]$ we have that

$$(h_1^*)^{stab(2)} = h_1^* - n = \binom{n}{2} - 2n,$$
$$(h_2^*)^{stab(2)} = h_2^* - n(n-4) = \binom{n}{4} - n(n-4), \text{ and for } i \neq 1, 2$$
$$(h_i^*)^{stab(2)} = h_i^*.$$

It is then easy to verify that $(h_1^*)^{stab(2)} - (h_0^*)^{stab(2)} \geq 0$ and $(h_2^*)^{stab(2)} - (h_1^*)^{stab(2)} \geq 0$ for all $n \geq 0$. As well, $(h_3^*)^{stab(2)} - (h_2^*)^{stab(2)} \geq 0$ for all $n \neq 6, 7, 8$. But this is fine since the h^*-vector for $\Delta_{7,2}^{stab(2)}$ is

$$h^*\left(\Delta_{7,2}^{stab(2)}\right) = (1, 7, 14, 7, 0, 0).$$

We also remark that

$$h^*\left(\Delta_{6,2}^{stab(2)}\right) = (1, 3, 3, 1, 0), \text{ and}$$
$$h^*\left(\Delta_{8,2}^{stab(2)}\right) = (1, 12, 38, 28, 1, 0, 0).$$

$\square$

Notice that the result given by this subtractive formula for $\mathrm{h}^*\left(\Delta_{9,2}^{\mathrm{stab}(2)};\mathrm{x}\right)$ agrees with the result via independence polynomials computed in Example 3.2.6. It is possible to apply the same strategy used in the proof of Corollary 3.4.3 to show that the h^*-vector of $\Delta_{n,2}^{\mathrm{stab}(3)}$ is unimodal. In short, we count the lattice paths corresponding to simplices in the set $\max \nabla_{n,2}^2 \setminus \max \nabla_{n,2}^3$ with unique minimal new face of cardinality $i = 1, 2, 3$ for each choice of maximal $\mathrm{adj}_{\mathrm{r}}(\ell)$, $\ell \in [n]$. We then subtract these values from the corresponding coefficients in the h^*-vector of $\Delta_{n,2}^{\mathrm{stab}(2)}$, and check that the unimodality condition is satisfied for the resulting h^*-vector. However, the details of this computation are quite unpleasant, so we omit them.

Corollary 3.4.4. *Let n be odd. The h^*-vector of $\Delta_{n,2}^{\mathrm{stab}(3)}$ is given by*

$$
\begin{aligned}
(h_1^*)^{stab(3)} &= \binom{n}{2} - 3n, \\
(h_2^*)^{stab(3)} &= \binom{n}{4} - \frac{1}{2}(n(7n-55)+94), \\
(h_3^*)^{stab(3)} &= \binom{n}{6} - \frac{1}{2}(n^3 - 13n^2 + 40n + 16), \ \ \textit{and for } i \neq 1,2,3 \\
(h_i^*)^{stab(3)} &= \binom{n}{2i}.
\end{aligned}
$$

Moreover, $h^*\left(\Delta_{n,2}^{\mathrm{stab}(3)}\right)$ is unimodal.

We end this chapter with two conjectures.

Conjecture 3.4.5. The h^*-polynomials of the r-stable second hypersimplices $\Delta_{n,2}^{\mathrm{stab}(r)}$ are unimodal.

A natural first-step to validating Conjecture 3.4.5 would be to prove the following.

Conjecture 3.4.6. The independence polynomials $I(G_{n,r,\ell};x)$ are unimodal.

Indeed, computational evidence computed by the author suggests that these polynomials possess the much stronger property of real-rootedness, which would in turn imply log-concavity and unimodality. One approach to answering these conjectures would be to investigate when these polynomials are in fact real-rooted.

Chapter 4 The Gorenstein r-stable Hypersimplices

In Chapter 3, we saw that the r-stable hypersimplices appear to have unimodal Ehrhart h^*-vectors, and verified this observation for a collection of these polytopes in the $k = 2$ case. In [10], it is shown that a Gorenstein integer polytope with a regular unimodular triangulation has a unimodal h^*-vector. In section 2.2 it is shown that $\Delta_{n,k}^{\mathrm{stab}(r)}$ has a regular unimodular triangulation. One application for the equations of the facets of a rational convex polytope is to determine whether or not the polytope is Gorenstein [27]. We now utilize Theorem 2.3.1 to identify $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ for which $\Delta_{n,k}^{\mathrm{stab}(r)}$ is Gorenstein. We identify a collection of such polytopes for every $k \geq 2$, thereby expanding the collection of r-stable hypersimplices known to have unimodal h^*-vectors. The contents of this chapter are in part joint work with Takayuki Hibi.

In this section we let $1 < k < n-1$. This is because $\Delta_{n,1}$ and $\Delta_{n,n-1}$ are simply copies of the standard $(n-1)$-simplex, which are well-known to be Gorenstein [6, p.29]. We now recall the definition of a Gorenstein polytope. Let $P \subset \mathbb{R}^N$ be a rational convex polytope of dimension d, and for an integer $q \geq 1$ let $qP := \{q\alpha : \alpha \in P\}$. Let $x_1, x_2, \ldots, x_N$, and z be indeterminates over some field K. Given an integer $q \geq 1$, let $A(P)_q$ denote the vector space over K spanned by the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N} z^q$ for $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in qP \cap \mathbb{Z}^N$. Since P is convex we have that $A(P)_p A(P)_q \subset A(P)_{p+q}$ for all p and q. It then follows that the graded algebra

$$A(P) := \bigoplus_{q=0}^{\infty} A(P)_q$$

is finitely generated over $K = A(P)_0$. We call $A(P)$ the *Ehrhart Ring* of P, and we say that P is *Gorenstein* if $A(P)$ is Gorenstein.

We now recall the combinatorial criterion given in [16] for an integral convex polytope P to be Gorenstein. Let ∂P denote the boundary of P and let $\mathrm{relint}(P) = P - \partial P$. We say that P is of *standard type* if $d = N$ and the origin in $\mathbb{R}^d$ is contained in $\mathrm{relint}(P)$. When $P \subset \mathbb{R}^d$ is of standard type we define its polar set

$$P^\star = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d : \sum_{i=1}^{d} \alpha_i \beta_i \leq 1 \text{ for every } (\beta_1, \beta_2, \ldots, \beta_d) \in P \right\}.$$

The polar set $P^\star$ is again a convex polytope of standard type, and $(P^\star)^\star = P$. We call $P^\star$ the *dual polytope* of P. Suppose $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d$, and K is the hyperplane in $\mathbb{R}^d$ defined by the equation $\sum_{i=1}^{d} \alpha_i x_i = 1$. A well-known fact is that $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a vertex of $P^\star$ if and only if $K \cap P$ is a facet of P. It follows that the dual polytope of a rational polytope is always rational. However, it need not be that the dual of an integral polytope is always integral. If P is an integral polytope with integral dual we say that P is *reflexive*. This idea plays a key role in the following combinatorial characterization of Gorenstein polytopes.

Theorem 4.0.1. *[16, De Negri and Hibi] Let $P \subset \mathbb{R}^d$ be an integral polytope of dimension d, and let q denote the smallest positive integer for which*

$$q(\mathrm{relint}(P)) \cap \mathbb{Z}^d \neq \emptyset.$$

Fix an integer point $\alpha \in q(\mathrm{relint}(P)) \cap \mathbb{Z}^d$, and let Q denote the integral polytope $qP - \alpha \subset \mathbb{R}^d$. Then the polytope P is Gorenstein if and only if the polytope Q is reflexive.

Since Theorem 4.0.1 requires that the polytope be full-dimensional we consider $\varphi^{-1}\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right)$, where $\varphi : \mathbb{R}^{n-1} \longrightarrow H$ is the affine isomorphism

$$\varphi : (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \longmapsto \left(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, k - \left(\sum_{i=1}^{n-1} \alpha_i\right)\right).$$

Notice that φ is also a lattice isomorphism. Hence, we have the isomorphism of Ehrhart Rings as graded algebras

$$A\left(\varphi^{-1}\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right)\right) \cong A\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right).$$

Let $P_{n,k}^{\mathrm{stab}(r)} := \varphi^{-1}\left(\Delta_{n,k}^{\mathrm{stab}(r)}\right)$, and recall from Theorem 2.3.1 that

$$\Delta_{n,k}^{\mathrm{stab}(r)} = \left(\bigcap_{\ell=1}^{n} H_\ell^{(+)}\right) \cap \left(\bigcap_{\ell=1}^{n} H_{\ell,r}^{(-)}\right).$$

4.1 The H-representation for $P_{n,k}^{\mathrm{stab}(r)}$

We now give a description of the facet-defining inequalities for $P_{n,k}^{\mathrm{stab}(r)}$ in terms of those defining $\Delta_{n,k}^{\mathrm{stab}(r)}$. In the following, it will be convenient to let $T(\ell) = \{\ell, \ell+1, \ell+2, \ldots, \ell+r-1\}$ for $\ell \in [n]$. We also let $T(\ell)^c$ denote the complement of $T(\ell)$ in $[n]$. Notice that for a fixed $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ and $\ell \in [n]$, the set $T(\ell)$ is precisely the set of summands in the defining equation of the $(n-2)$-flat $H_{\ell,r}$. The defining inequalities of $P_{n,k}^{\mathrm{stab}(r)}$ corresponding to the $(n-2)$-flats $H_{\ell,r}$ come in two types, dependent on whether $n \notin T(\ell)$ or $n \in T(\ell)$. If $n \notin T(\ell)$ then

$$K_{\ell,r}^{(-)} := \varphi^{-1}\left(H_{\ell,r}^{(-)}\right) = \left\{(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i \in T(\ell)} x_i \leq 1\right\}.$$

If $n \in T(\ell)$ then

$$\widetilde{K}_{\ell,r}^{(+)} := \varphi^{-1}\left(H_{\ell,r}^{(-)}\right) = \left\{(x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i \in T(\ell)^c} x_i \geq k-1\right\}.$$

Similarly, if $\ell \neq n$ then

$$K_\ell^{(+)} := \varphi^{-1}\left(H_\ell^{(+)}\right) = \left\{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : x_\ell \geq 0\right\}.$$

Finally, if $\ell = n$ then

$$K_n^{(-)} := \varphi^{-1}\left(H_n^{(+)}\right) = \left\{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i \leq k\right\}.$$

Thus, we may write $P_{n,k}^{\mathrm{stab}(r)}$ as the intersection of closed halfspaces in $\mathbb{R}^{n-1}$

$$P_{n,k}^{\mathrm{stab}(r)} = \left(\bigcap_{n \notin T(\ell)} K_{\ell,r}^{(-)}\right) \cap \left(\bigcap_{n \in T(\ell)} \widetilde{K}_{\ell,r}^{(+)}\right) \cap \left(\bigcap_{i=1}^{n-1} K_\ell^{(+)}\right) \cap K_n^{(-)}.$$

To denote the supporting hyperplanes corresponding to these halfspaces we simply drop the superscripts $(+)$ and $(-)$.

4.2 The codegree of $P_{n,k}^{\mathrm{stab}(r)}$

Given the above description of $P_{n,k}^{\mathrm{stab}(r)}$, we would now like to determine the smallest positive integer q for which $qP_{n,k}^{\mathrm{stab}(r)}$ contains a lattice point in its relative interior. To do so, recall that for a lattice polytope P of dimension d we can define the *(Ehrhart) h^*-polynomial* of P. If we write this polynomial as

$$h_P^*(z) = h_0^* + h_1^* z + h_2^* z^2 + \cdots + h_d^* z^d$$

then we call the coefficient vector $h^*(P) = (h_0^*, h_1^*, h_2^*, \dots, h_d^*)$ the *h^*-vector* of P. We let s denote the degree of $h_P^*(z)$, and we call $q = (d+1) - s$ the *codegree* of P. It is a consequence of Ehrhart Reciprocity that q is the smallest positive integer such that qP contains a lattice point in its relative interior [6]. Hence, we would like to compute the codegree of $P_{n,k}^{\mathrm{stab}(r)}$. To do so requires that we first prove two lemmas. In the following let $q = \left\lceil \frac{n}{k} \right\rceil$. Our first goal is to show that there is at least one integer point in relint $\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ for $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. We then show that q is the smallest positive integer for which this is true. Recall that $q = \frac{n+\alpha}{k}$ for some $\alpha \in \{0, 1, \dots, k-1\}$. Also recall that for a fixed n and k we have the nesting of polytopes

$$P_{n,k} \supset P_{n,k}^{\mathrm{stab}(2)} \supset P_{n,k}^{\mathrm{stab}(3)} \supset \cdots \supset P_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right)} \supset P_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor\right)}.$$

Hence, if we identify an integer point inside relint $\left(qP_{n,k}^{\mathrm{stab}\left(\left\lfloor \frac{n}{k} \right\rfloor - 1\right)}\right)$ then this same integer point lives inside relint $\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ for every $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. Given these facts, we now prove two lemmas.

Lemma 4.2.1. *Suppose that* $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$ *where* $\alpha \in \{0,1\}$. *Then the integer point* $(1,1,\dots,1) \in \mathbb{R}^{n-1}$ *lies inside* relint $\left(qP_{n,k}^{\text{stab}(r)}\right)$ *for every* $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$.

Proof. It suffices to show that $(x_1, x_2, \dots, x_{n-1}) = (1,1,\dots,1)$ satisfies the set of inequalities

(i) $x_i > 0$, for $i \in [n-1]$,

(ii) $\sum_{i=1}^{n-1} x_i < kq$,

(iii) $\sum_{i \in T(\ell)} x_i < q$, for $n \notin T(\ell)$, and

(iv) $\sum_{i \in T(\ell)^c} x_i > (k-1)q$, for $n \in T(\ell)$.

We do this in two cases. First suppose that $\alpha = 0$. Then k divides n and $q = \frac{n}{k}$. Clearly, (i) is satisfied. To see that (ii) is also satisfied simply notice that $n-1 < kq$. To see that (iii) is satisfied recall that $\#T(\ell) = r$ and $r < \left\lfloor \frac{n}{k} \right\rfloor = q$. Finally, to see that (iv) is satisfied notice that $\#T(\ell)^c = n-r$. So we would like that $n-r > (k-1)q$. However, this follows quickly from the fact that $r < \frac{n}{k}$.

Now consider the case where $\alpha = 1$. Recall that it suffices to consider the case when $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$. Inequalities (i), (ii), and (iii) are all satisfied in the same fashion as the case when $\alpha = 0$. So we need only check that (iv) is also satisfied. Again we would like that $n - r > (k-1)q$. Notice since $\alpha = 1$ then k does not divide n, and so $\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n}{k} \right\rfloor + 1$. Hence, $q = r+2$. The desired inequality then follows from $n+2 > n+\alpha$. Thus, whenever $\alpha \in \{0,1\}$, the lattice point $(1,1,\dots,1) \in \text{relint}\left(qP_{n,k}^{\text{stab}(r)}\right)$ for every $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. □

Next we would like to identify an integer point in the relative interior of $qP_{n,k}^{\text{stab}(r)}$ for $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$ when $\alpha \geq 2$. In this case, the point $(1,1,\dots,1)$ does not always work, so we must identify another point. Recall that it suffices to identify such a point for $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$. To do so, we construct the desired point using the notions of r-stability. Fix n and k such that $q = \frac{n+\alpha}{k}$ for $\alpha \geq 2$, and let $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$. This also fixes the value $\alpha \in \{2,3,\dots,k-1\}$. Since $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$ we may construct an r-stable vertex in $\mathbb{R}^n$ as the characteristic vector of the set

$$\{n-r, n-2r, n-3r, \dots, n-(k-1)r\} \subset [n].$$

Notice that there are at least r 0's between the n^{th} coordinate of the vertex and the $n-(k-1)r^{th}$ coordinate (read from right-to-left modulo n). In particular, this implies that the n^{th} coordinate (and the 1^{st} coordinate) is occupied by a 0. To construct the desired vertex replace the 1's in coordinates

$$n-(\alpha+1)r, n-(\alpha+2)r, \dots, n-(k-1)r$$

with 0's. Now add 1 to each coordinate of this lattice point. If the resulting point is $(x_1, x_2, \dots, x_n)$ then replace $x_n = 1$ with the value $kq - \left(\sum_{i=1}^{n-1} x_i\right)$. Call the resulting vertex ϵ^α, and consider the isomorphism $\widetilde{\varphi} : \mathbb{R}^{n-1} \longrightarrow H_q$, defined analogously to φ, where H_q is the hyperplane in $\mathbb{R}^n$ defined by the equation $\sum_{i=1}^n x_i = kq$. Notice that by our construction of ϵ^α, the point $\widetilde{\varphi}^{-1}(\epsilon^\alpha)$ is simply ϵ^α with the last coordinate projected off.

Lemma 4.2.2. *Suppose that $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$ for $\alpha \in \{2, 3, \ldots, k-1\}$. Then the lattice point $\widetilde{\varphi}^{-1}(\epsilon^\alpha)$ lies inside* $\operatorname{relint}\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ *for every $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$.*

Proof. It suffices to show that when $r = \left\lfloor \frac{n}{k} \right\rfloor - 1$ the lattice point $(x_1, x_2, \ldots, x_{n-1}) = \widetilde{\varphi}^{-1}(\epsilon^\alpha)$ satisfies inequalities (i), (ii), (iii), and (iv) from the proof of Lemma 4.2.1. It is clear that (i) is satisfied. To see that (ii) is satisfied notice that $\sum_{i=1}^{n-1} x_i = n - 1 + \alpha$. This is because α coordinates of $\widetilde{\varphi}^{-1}$ are occupied by 2's and all other coordinates are occupied by 1's. Thus, inequality (ii) is satisfied since $n - 1 + \alpha < kq$. To see that (iii) is satisfied first notice that for $T(\ell)$ with $n \notin T(\ell)$

$$\sum_{i \in T(\ell)} x_i = \begin{cases} r & \text{if } T(\ell) \text{ contains no entry with value 2,} \\ r+1 & \text{otherwise.} \end{cases}$$

This is because we have chosen the 2's to be separated by at least $r - 1$ 0's. Thus, since k does not divide n we have that $\sum_{i \in T(\ell)} x_i \leq r + 1 = \left\lfloor \frac{n}{k} \right\rfloor < q$. Finally, to see that (iv) is satisfied first notice that for $T(\ell)$ with $n \in T(\ell)$

$$\sum_{i \in T(\ell)^c} x_i = \begin{cases} n - r + \alpha - 1 & \text{if } T(\ell) \text{ contains an entry with value 2,} \\ n - r + \alpha & \text{otherwise.} \end{cases}$$

Hence, we must show that $n - r + \alpha - 1 > (k-1)q$. However, since $\left\lceil \frac{n}{k} \right\rceil = \left\lfloor \frac{n}{k} \right\rfloor + 1$ then $r = q - 2$, and so the desired inequality follows from $n + \alpha + 1 > n + \alpha$. Therefore, $\widetilde{\varphi}^{-1}(\epsilon^\alpha) \in \operatorname{relint}\left(qP_{n,k}^{\mathrm{stab}(r)}\right)$ for every $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. □

Using Lemmas 4.2.1 and 4.2.2 we now show that $q = \left\lceil \frac{n}{k} \right\rceil$ is indeed the codegree of these polytopes.

Theorem 4.2.3. *Let $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$. The codegree of $P_{n,k}^{\mathrm{stab}(r)}$ is $q = \left\lceil \frac{n}{k} \right\rceil$.*

Proof. First recall that $P_{n,k}^{\mathrm{stab}(r)}$ is a subpolytope of $\Delta_{n,k}$. By a theorem of Stanley [45] it then follows that $h^*\left(P_{n,k}^{\mathrm{stab}(r)}\right) \leq h^*(\Delta_{n,k})$. Therefore, the codegree of $P_{n,k}^{\mathrm{stab}(r)}$ is no smaller than the codegree of $\Delta_{n,k}$. In [30, Corollary 2.6], Katzman determines that the codegree of $\Delta_{n,k}$ is $q = \left\lceil \frac{n}{k} \right\rceil$. Since Lemmas 4.2.1 and 4.2.2 imply that $qP_{n,k}^{\mathrm{stab}(r)}$ contains a lattice point inside its relative interior we conclude that the codegree of $P_{n,k}^{\mathrm{stab}(r)}$ is $q = \left\lceil \frac{n}{k} \right\rceil$. □

Recall that if an integral polytope P of dimension d with codegree q is Gorenstein then

$$\#\left(\operatorname{relint}(qP) \cap \mathbb{Z}^d\right) = 1.$$

With this fact in hand, we have the following corollary.

Corollary 4.2.4. *Suppose that* $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$, *where* $\alpha \in \{2, 3, \ldots, k-1\}$. *Then the polytope* $\Delta_{n,k}^{\mathrm{stab}(r)}$ *is not Gorenstein for every* $1 \le r < \left\lfloor \frac{n}{k} \right\rfloor$.

Proof. Recall the vertex $(x_1, x_2, \ldots, x_n)$ from which we produce ϵ^α. Since $x_1 = 1$ then cyclically shifting the entries of this vertex one entry to the left, and then applying the construction for ϵ^α results in a second vertex, say ζ^α, such that $\widetilde{\varphi}\left(\zeta^\alpha\right)^{-1}$ also lies in the relative interior of $qP_{n,k}^{\mathrm{stab}(r)}$. Thus, $\#\left(\mathrm{relint}\left(qP_{n,k}^{\mathrm{stab}(r)}\right) \cap \mathbb{Z}^d\right) > 1$, and we conclude that $\Delta_{n,k}^{\mathrm{stab}(r)}$ is not Gorenstein. □

4.3 Gorenstein r-stable hypersimplices and unimodal h^*-vectors

Notice that by Corollary 4.2.4 we need only consider those r-stable hypersimplices satisfying the conditions of Lemma 4.2.1. For these polytopes we now consider the translated integral polytope

$$Q := qP_{n,k}^{\mathrm{stab}(r)} - (1, 1, \ldots, 1).$$

From our H-representation of $P_{n,k}^{\mathrm{stab}(r)}$ we see that the facets of Q are supported by the hyperplanes

(a) $x_i = -1$, for $i \in [n-1]$,

(b) $\sum_{i=1}^{n-1} x_i = kq - (n-1)$,

(c) $\sum_{i \in T(\ell)} x_i = q - r$, for $n \notin T(\ell)$, and

(d) $\sum_{i \in T(\ell)^c} x_i = (k-1)q - (n-r)$, for $n \in T(\ell)$.

Given this collection of hyperplanes we may now prove the following theorem.

Theorem 4.3.1. *Let* $1 \le r < \left\lfloor \frac{n}{k} \right\rfloor$. *Then* $\Delta_{n,k}^{\mathrm{stab}(r)}$ *is Gorenstein if and only if* $n = kr + k$.

Proof. By Theorem 4.0.1 we must determine when all the vertices of $Q^\star$ are integral. We do so by means of the inclusion-reversing bijection between the faces of Q and the faces of $Q^\star$. It is immediate that the vertices of $Q^\star$ corresponding to hyperplanes given in (a) are integral. So consider the hyperplane given in (b). Recall that $q = \left\lceil \frac{n}{k} \right\rceil = \frac{n+\alpha}{k}$ for some $\alpha \in \{0, 1\}$. Hence, this hyperplane is equivalently expressed as

$$\sum_{i=1}^{n-1} \frac{1}{\alpha+1} x_i = 1.$$

Therefore, the corresponding vertex in $Q^\star$ is integral only if $\alpha = 0$. Notice next that the hyperplanes given in (c) will have corresponding vertex of $Q^\star$ integral only if $q - r = 1$. Since $\alpha = 0$ we have that $q = \frac{n}{k}$ where k divides n, and so it must be that $n = kr + k$. Finally, when $n = kr + k$ the hyperplanes given in (d) reduce to

$$\sum_{i \in T(\ell)^c} x_i = -1.$$

Hence, the corresponding vertex of $Q^\star$ is integral, and we conclude that, for $1 \leq r < \left\lfloor \frac{n}{k} \right\rfloor$, the polytope $\Delta_{n,k}^{\mathrm{stab}(r)}$ is Gorenstein if and only if $n = kr + k$. $\square$

Theorem 4.3.1 demonstrates that the Gorenstein property is quite rare amongst the r-stable hypersimplices. It also enables us to expand the collection of r-stable hypersimplices known to have unimodal h^*-vectors. Previously, this collection was limited to the case when $k = 2$ or when $\Delta_{n,k}^{\mathrm{stab}(r)}$ is a simplex [9]. Theorem 4.3.1 provides a collection of r-stable hypersimplices with unimodal h^*-vectors for every $k \geq 1$.

Corollary 4.3.2. *Let $k \geq 1$. The r-stable n, k-hypersimplices $\Delta_{n,k}^{\mathrm{stab}(r)}$ for $r \geq 1$ and $n = kr + k$ have unimodal h^*-vectors.*

Proof. By [9, Corollary 2.6] there exists a regular unimodular triangulation of $\Delta_{n,k}^{\mathrm{stab}(r)}$. By Theorem 4.3.1 the polytope $\Delta_{n,k}^{\mathrm{stab}(r)}$ is Gorenstein for $n = kr + k$ when $k > 1$. By [10, Theorem 1] we conclude that the h^*-vector of $\Delta_{n,k}^{\mathrm{stab}(r)}$ is unimodal. Finally, notice that when $k = 1$ these polytopes are just the standard $(n-1)$-simplices. $\square$

Chapter 5 The Facet-Ray Identification Property

5.1 Introduction

For a positive integer p let $[p] := \{1, 2, \ldots, p\}$, and let G be a graph with vertex set $V(G) = [p]$ and edge set $E := E(G)$. To the graph G we associate the closed convex cone $\mathbb{S}^p_{\succeq 0}(G)$ consisting of all real $p \times p$ positive semidefinite matrices with zeros in all entries corresponding to the nonedges of G. In this chapter, we study the problem of characterizing the possible ranks of the extremal matrices in $\mathbb{S}^p_{\succeq 0}(G)$. This problem has applications to the positive semidefinite completion problem, and consequently, maximum likelihood estimation for Gaussian graphical models. Thus, the extreme ranks of $\mathbb{S}^p_{\succeq 0}(G)$, and in particular the maximum extreme rank of $\mathbb{S}^p_{\succeq 0}(G)$, have been studied extensively [1, 21, 25, 34]. However, as noted in [1] the nonpolyhedrality of $\mathbb{S}^p_{\succeq 0}(G)$ makes this problem difficult, and as such there remain many graph classes for which the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ are not well-understood. Our main contribution to this area of study is to show that the polyhedral geometry of a second well-studied convex body, the cut polytope of G, serves to characterize the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ for new classes of graphs. This work is in part joint work with Caroline Uhler and Ruriko Yoshida.

The thrust of the research in this area has been focused on determining the *(sparsity) order of* G, i.e. the maximum rank of an extremal ray of $\mathbb{S}^p_{\succeq 0}(G)$. In [1] it is shown that the order of G is one if and only if G is a chordal graph, that is, a graph in which all induced cycles have at most three edges. Then in [34] all graphs with order two are characterized. In [25], it is shown that the order of G is at most $p-2$ with equality if and only if G is the cycle on p vertices, and in [21] the order of the complete bipartite graph is computed and it is shown that all possible extreme ranks are realized. However, beyond the chordal, order two, cycle, and complete bipartite graphs there are few graphs for which all extremal ranks are characterized. Our main goal is to demonstrate that the geometric relationship between $\mathbb{S}^p_{\succeq 0}(G)$ and the cut polytope of G can serve to expand this collection of graphs.

A *cut* of the graph G is a bipartition of the vertices, (U, U^c), and its associated *cutset* is the collection of edges $\delta(U) \subset E$ with one endpoint in each block of the bipartition. To each cutset we assign a (± 1)-vector in $\mathbb{R}^E$ with a -1 in coordinate e if and only if $e \in \delta(U)$. The (± 1)-cut polytope of G is the convex hull in $\mathbb{R}^E$ of all such vectors. The polytope $\mathrm{cut}^{\pm 1}(\mathrm{G})$ is affinely equivalent to the cut polytope of G defined in the variables 0 and 1, which is the feasible region of the *max-cut problem* in linear programming. Hence, maximizing over the polytope $\mathrm{cut}^{\pm 1}(\mathrm{G})$ is equivalent to solving the max-cut problem for G. The max-cut problem is known to be NP-hard [40], and thus the geometry of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ is of general interest. In particular, the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ have been well-studied [17, Part V], as well as a positive semidefinite relaxation of $\mathrm{cut}^{\pm 1}(\mathrm{G})$, known as the *elliptope of* G [4, 5, 33, 35].

Let $\mathbb{S}^p$ denote the real vector space of all real $p \times p$ symmetric matrices, and let $\mathbb{S}^p_{\succeq 0}$ denote the cone of all positive semidefinite matrices in $\mathbb{S}^p$. The *p-elliptope* is the collection of all $p \times p$ *correlation matrices*, i.e.

$$\mathcal{E}_p = \{X \in \mathbb{S}^p_{\succeq 0} : X_{ii} = 1 \text{ for all } i \in [p]\}.$$

The elliptope $\mathcal{E}_G$ is defined as the projection of $\mathcal{E}_p$ onto the edge set of G. That is,

$$\mathcal{E}_G = \{\mathbf{y} \in \mathbb{R}^E : \ \exists Y \in \mathcal{E}_p \text{ such that } Y_e = y_e \text{ for every } e \in E(G)\}.$$

The elliptope $\mathcal{E}_G$ is a positive semidefinite relaxation of the cut polytope $\mathrm{cut}^{\pm 1}(\mathrm{G})$ [35], and thus maximizing over $\mathcal{E}_G$ can provide an approximate solution to the max-cut problem.

In this chapter, we show that the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ identify extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$ for any graph G that has no K_5 minors. We will see in addition that the rank of the extreme ray identified by the facet with supporting hyperplane $\langle \alpha, x \rangle = b$ has rank b, and if G is also series-parallel (i.e. no K_4 minors), then all possible ranks of extremal rays are given in this fashion. The method by which we will make these identifications arises via the geometric relationship that exists between the three convex bodies $\mathrm{cut}^{\pm 1}(\mathrm{G})$, $\mathcal{E}_G$, and $\mathbb{S}^p_{\succeq 0}(G)$. A key component of this relationship is the following theorem which is to be proven in Section 5.3.

Theorem 5.1.1. *The polar of the elliptope $\mathcal{E}_G$ is*

$$\mathcal{E}_G^\circ = \{x \in \mathbb{R}^E : \exists\, X \in \mathbb{S}^p_{\succeq 0}(G) \textit{ such that } X_E = x \textit{ and } tr(X) = 2\}.$$

An immediate consequence of Theorem 5.1.1 is that the extreme points in $\mathcal{E}_G^\circ$ are projections of extreme matrices in $\mathbb{S}^p_{\succeq 0}(G)$ (recall that a subset F of a convex set K is called an *extreme set* of K if, for all $x \in F$ and $a, b \in K$, $x = (a+b)/2$ implies $a, b \in F$; so an *extreme point* is any point in the set that does not lie on the line segment between any two distinct points of K).

With Theorem 5.1.1 in hand, the identification of extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$ via facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ is guided by the following geometry. Since $\mathcal{E}_G$ is a positive semidefinite relaxation of $\mathrm{cut}^{\pm 1}(\mathrm{G})$, then $\mathrm{cut}^{\pm 1}(\mathrm{G}) \subset \mathcal{E}_G$. If all singular points on the boundary of $\mathcal{E}_G$ are also singular points on the boundary of $\mathrm{cut}^{\pm 1}(\mathrm{G})$, then the supporting hyperplanes of facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ will be translations of supporting hyperplanes of *regular extreme points* of $\mathcal{E}_G$ or *facets* of $\mathcal{E}_G$, i.e. extreme sets of $\mathcal{E}_G$ with positive Lebesgue measure in a codimension one affine subspace of the ambient space. It follows that the outward normal vectors to the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ generate the normal cones to these regular points and facets of $\mathcal{E}_G$. Dually, the facet-normal vectors of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ are then extreme points of $\mathcal{E}_G^\circ$, and consequently projections of extreme matrices of $\mathbb{S}^p_{\succeq 0}(G)$. Thus, we can expect to find extremal matrices in $\mathbb{S}^p_{\succeq 0}(G)$ whose off-diagonal entries are given by the facet-normal vectors of $\mathrm{cut}^{\pm 1}(\mathrm{G})$. This motivates the following definition.

Definition 5.1.2. Let G be a graph. For each facet F of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ let $\alpha^F \in \mathbb{R}^E$ denote the normal vector to the supporting hyperplane of F. We say that G has the **facet-ray identification property** (or FRIP) if for every facet F of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ there exists an extremal matrix $M = [m_{ij}]$ in $\mathbb{S}^p_{\succeq 0}(G)$ for which either $m_{ij} = \alpha^F_{ij}$ for every $\{i, j\} \in E(G)$ or $m_{ij} = -\alpha^F_{ij}$ for every $\{i, j\} \in E(G)$.

An explicit example of facet-ray identification and its geometry is presented in Section 5.3. With this example serving as motivation, our main goal is to identify interesting collections of graphs exhibiting the facet-ray identification property. Using the combinatorics of cutsets as well as the tools developed by Agler et al. in [1], we prove the following theorem.

Theorem 5.1.3. *Graphs without K_5 minors have the facet-ray identification property.*

Recall that a cycle subgraph of a graph G is called *chordless* if it is an induced subgraph of G. For graphs without K_5 minors the facet-defining hyperplanes of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ are of the form $\langle \alpha, x\rangle = b$, where $b = 1$ or $b = p-2$ for C_p a chordless cycle of G [3]. In [1], it is shown that the p-cycle C_p is a $(p-2)$-*block*, meaning that if C_p is an induced subgraph of G, then the sparsity order of G is at least $p-2$. Since the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ are given by the chordless cycles in G, then Theorem 5.1.3 demonstrates that this condition arises via the geometry of the cut polytope $\mathrm{cut}^{\pm 1}(\mathrm{G})$. That is, since the elliptope $\mathcal{E}_G$ is a positive semidefinite relaxation of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ we can translate the facet-supporting hyperplanes of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ to support points on $\mathcal{E}_G$. By Theorem 5.1.1 these supporting hyperplanes correspond to points in $\mathcal{E}_G^\circ$, and by Theorem 5.1.3 we see that these points are all extreme. In this way, the lower bound on sparsity order of G given by the chordless cycles is a consequence of the relationship between the chordless cycles and the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$. In the case that G is a series-parallel graph, we will prove that the facets determine all possible extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ in this fashion.

Theorem 5.1.4. *Let G be a series-parallel graph. The constant terms of the facet-defining hyperplanes of* $\mathrm{cut}^{\pm 1}(\mathrm{G})$ *characterize the ranks of extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$. These ranks are* 1 *and $p-2$ where C_p is any chordless cycle in G. Moreover, the sparsity order of G is p^*-2 where p^* is the length of the largest chordless cycle in G.*

The remainder of this chapter is organized as follows: In Section 5.2 we recall some of the previous results on sparsity order and cut polytopes that will be fundamental to our work. Then in Section 5.3, we describe the geometry underlying the facet-ray identification property. We begin this section with the motivating example of the 4-cycle, in which we explicitly illustrate the geometry described above. We then provide a proof of Theorem 5.1.1 and discuss how this result motivates the definition of the facet-ray identification property. In Section 5.4, we demonstrate that any graph without K_5 minors has the facet-ray identification property, thereby proving Theorem 5.1.3. We then identify the ranks of the corresponding extremal rays. In Section 5.5 we prove Theorem 5.1.4, showing that if G is also series-parallel then the facets are enough to characterize all extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$. Finally, in Section 5.6, we discuss how to identify graphs that do not have the facet-ray identification property.

5.2 Preliminaries

Graphs.

For a graph G with vertex set $[p]$ and edge set E we let $\overline{E}$ denote the set of *nonedges* of G, that is, all unordered pairs $\{i,j\}$ for which $i,j \in [p]$ but $\{i,j\} \notin E$. For such a graph G we define the *complement of* G to be the graph G^c on vertex set $[p]$ with edge set $\overline{E}$. Recall that a subgraph of G is any graph H whose vertex set is a subset of $[p]$ and edge set is a subset of E. A subgraph H of G with edge set E' is called *induced* if there exists a subset $V' \subset [p]$ such that the vertex set of H is V' and E' consists of all edges of G connecting any two vertices of V'. We let K_p denote the complete graph on p vertices, C_p denote the cycle on p vertices, and $K_{p,m}$ denote the complete bipartite graph with vertex set the disjoint union of $[p]$ and $[m]$. A subgraph H of G is called a *chordless cycle* if H is an induced cycle subgraph of G. A graph G is called *chordal* if every chordless cycle in G has at most three edges. We can *delete* an edge of G by removing it from the edge set E, and *contract* an edge $\{i,j\}$ of G by identifying the two vertices i and j and deleting any multiple edges introduced by this identification. Similarly, we *delete* a vertex of G by removing it from the vertex set of G as well as all edges of G attached to it. A graph H is called a *minor* of G if H can be obtained from G via a sequence of edge deletions, edge contractions, and vertex deletions.

Sparsity order of G.

We are interested in $\mathbb{S}^p_{\succeq 0}(G)$, the closed convex cone consisting of all $p \times p$ positive semidefinite matrices with zeros in the ij^{th} entry for all $\{i,j\} \in \overline{E}$. Recall that a matrix $X \in \mathbb{S}^p_{\succeq 0}(G)$ is *extremal* in $\mathbb{S}^p_{\succeq 0}(G)$ if it lies on an extreme ray of $\mathbb{S}^p_{\succeq 0}(G)$. The *(sparsity) order* of G, denoted $\operatorname{ord}(G)$, is the maximum rank of an extremal matrix in $\mathbb{S}^p_{\succeq 0}(G)$. In [1] the authors work to develop a general theory for studying graphs G with high sparsity order. Fundamental to their theory is the so-called *dimension theorem*, which is stated in terms of the expression of a positive semidefinite matrix as the Gram matrix for a collection of vectors. Recall that a (real) $p \times p$ matrix $X = [x_{ij}]$ is positive semidefinite if and only if there exist vectors $u_1, u_2, \ldots, u_p \in \mathbb{R}^k$ such that $x_{ij} = u_i^T u_j$. The sequence of vectors $(u_1, \ldots, u_p)$ is called a (k-dimensional) *Gram representation* of X, and if X has rank k this sequence of vectors is unique up to orthogonal transformation. Following the notation of [34], for a subset $A \subset E \cup \overline{E}$ define the set of $p \times p$ matrices

$$U_A := \{u_i u_j^T + u_j u_i^T : \{i,j\} \in A\}.$$

The real span of $U_{\overline{E}}$ is a subspace of the trace zero $k \times k$ real symmetric matrices that we call the *frame space of* X. The following theorem says that a matrix is extremal in $\mathbb{S}^p_{\succeq 0}(G)$ if and only if this frame space is the entire trace zero subspace of $\mathbb{S}^k$.

Theorem 5.2.1. *[1, Corollary 3.2] Let $X \in \mathbb{S}^p_{\succeq 0}(G)$ with rank k and k-dimensional*

Gram representation $(u_1, \ldots, u_p)$*. Then* X *is extremal if and only if*

$$\operatorname{rank}(U_{\overline{E}}) = \binom{k+1}{2} - 1.$$

In [1] it is shown that $\operatorname{ord}(G) = 1$ if and only if G is a chordal graph. Using Theorem 5.2.1, the authors then develop a general theory for detecting existence of higher rank extremals in $\mathbb{S}^p_{\succeq 0}(G)$ based on a fundamental collection of graphs. A graph G is called a *k-block* provided that G has order k and no proper induced subgraph of G has order k. The k-blocks are useful for identifying higher rank extremals since if H is an induced subgraph of G then $\operatorname{ord}(H) \leq \operatorname{ord}(G)$ [1]. In [1] it is shown that the cycle on p vertices is a $(p-2)$-block. A particularly important collection of k-blocks are the k-superblocks, the k-blocks with the maximum number of edges on a fixed vertex set. Formally, a *k-superblock* is a k-block whose complement has precisely $(k^2+k-2)/2$ edges. Understanding the k-blocks and k-superblocks is equivalent to understanding their complements. In [1, Theorem 1.5] the 3-blocks are characterized in terms of their complement graphs, and in [24, Theorem 0.2] the 4-superblocks are characterized in a similar fashion.

In related works the structure of the graph G is again used to describe the extreme ranks of G. In [25] it is shown that if G is a clique sum of two graphs G_1 and G_2 then $\operatorname{ord}(G) = \max\{\operatorname{ord}(G_1), \operatorname{ord}(G_2)\}$, and $\operatorname{ord}(G) \leq p-2$ with equality if and only if G is a p-cycle. Similarly, in [21] the order of the complete bipartite graph $K_{p,m}$ is determined and it is shown that all ranks $1, 2, \ldots, \operatorname{ord}(K_{p,m})$ are extremal.

The cut polytope of G.

First recall that to define the cut polytope in the variables 0 and 1 we assign to each cutset $\delta(U)$ a $(0,1)$-vector $x^{\delta(U)} \in \mathbb{R}^E$ with a 1 in coordinate x_e if and only if $e \in \delta(U)$. The polytope $\operatorname{cut}^{01}(\mathrm{G})$ is the convex hull of all such vectors, and it is affinely equivalent to $\operatorname{cut}^{\pm 1}(\mathrm{G})$ under the coordinate-wise transformation $x_e \mapsto 1 - 2x_e$ on $\mathbb{R}^E$. In order to prove that a graph G has the facet-ray identification property we need an explicit description of the facet-supporting hyperplanes of $\operatorname{cut}^{\pm 1}(\mathrm{G})$, or equivalently those of $\operatorname{cut}^{01}(\mathrm{G})$. For the complete graph K_p one of the most interesting classes of valid inequalities for $\operatorname{cut}^{\pm 1}(\mathrm{G})$ are the *hypermetric inequalities*. For an integer vector $b = (b_1, \ldots, b_p)$ satisfying $\sum_{i=1}^p b_i = 1$ we call

$$\sum_{1 \leq i < j \leq p} b_i b_j \leq \sum_{1 \leq i < j \leq p} b_i b_j x_{ij}$$

the *hypermetric inequality* defined by b. Notice that every facet-supporting hypermetric inequality identifies an extreme ray in $\mathbb{S}^p = \mathbb{S}^p_{\succeq 0}(K_p)$. Despite this large collection of inequalities having this property, not all complete graphs have the facet-ray identification property, as we will see in Section 5.6. Moreover, since the only extreme rank of $\mathbb{S}^p$ is 1, we are mainly interested in facet-defining inequalities that identify higher rank extreme rays for sparse graphs. The hypermetric inequalities generalize a collection of facet-defining inequalities of $\operatorname{cut}^{\pm 1}(\mathrm{G})$ called the *triangle inequalities*, i.e. the hypermetric inequalities defined by $b = (1, 1, -1)$. The triangle inequalities admit a second generalization to a collection of facet-defining inequalities for sparse graphs. Let C_m be a cycle in a graph G and let $F \subset E(C_m)$ be an odd cardinality subset of the edges of C_m. The inequality

$$\sum_{e \in E(C_m) \setminus F} x_e - \sum_{e \in F} x_e \leq p - 2$$

is called a *cycle inequality*. Using these inequalities Barahona and Mahjoub provide a linear description of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ for any graph without K_5 minors.

Theorem 5.2.2. *[3, Barahona and Mahjoub] Let G be a graph with no K_5 minor. Then $\mathrm{cut}^{\pm 1}(\mathrm{G})$ is defined by the collection of hyperplanes*

(1) $-1 \leq x_e \leq 1$ for all $e \in E(G)$, and

(2) $\sum_{e \in E(C_m) \setminus F} x_e - \sum_{e \in F} x_e \leq m - 2$ for all chordless cycles C_m of G and any odd cardinality subset $F \subset E(C_m)$.

Suppose that C_m is a chordless cycle in a graph G without K_5 minors. For an odd cardinality subset $F \subset E(C_m) \subset E(G)$ define the vector $v^F \in \mathbb{R}^E$, where

$$v_e^F = \begin{cases} -1 & \text{if } e \in F, \\ 1 & \text{if } e \in E(C_m) \setminus F, \\ 0 & \text{if } e \in E(G) \setminus E(C_m). \end{cases}$$

Similarly, let v^e denote the standard basis vector for coordinate $e \in E(G)$ for $\mathbb{R}^E$. Then by Proposition 5.2.2 we see that the facet-supporting hyperplanes of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ are

(1) $\langle \pm v^e, x \rangle = 1$ for all $e \in E$, and

(2) $\langle v^F, x \rangle = m - 2$ for all odd cardinality subsets $F \subset E(C_m)$ for all chordless cycles C_m in G.

In Section 5.4, we identify for each facet-supporting hyperplane $\langle \alpha, x \rangle = b$ of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ an extremal matrix $A = [a_{ij}]$ in $\mathbb{S}^p_{\succeq 0}(G)$ of rank b in which the off-diagonal nonzero entries a_{ij} are given by the coordinates α_e, for $e = \{i, j\}$, of the facet normal $\alpha \in \mathbb{R}^E$. In Section 5.5, we then show that the ranks b are *all* extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ as long as G is also series-parallel. To do so, it will be helpful to have the following lemma on the cut polytope of the cycle, the proof of which is left as an exercise.

Lemma 5.2.3. *The vertices of $\mathrm{cut}^{\pm 1}(\mathrm{C_m})$ are all (± 1)-vectors in $\mathbb{R}^E$ containing an even number of -1's.*

The polytope $\mathrm{cut}^{\pm 1}(\mathrm{C_m})$ appears in the literature as the *m-halfcube* or *demihypercube.*

5.3 The Geometry of Facet-Ray Identification

In this section, we examine the underlying geometry of the facet-ray identification property. Recall that the facet-ray identification property is defined to capture the following geometric picture. Since $\mathrm{cut}^{\pm 1}(\mathrm{G}) \subset \mathcal{E}_G$ then if all singular points on the boundary of $\mathcal{E}_G$ are also singular points on the boundary of $\mathrm{cut}^{\pm 1}(\mathrm{G})$, the supporting hyperplanes of facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ will be translations of supporting hyperplanes of regular extreme points of $\mathcal{E}_G$ or facets of $\mathcal{E}_G$. It follows that the outward normal vectors to the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ generate the normal cones to these regular points and facets of $\mathcal{E}_G$. In the polar, the facet-normal vectors of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ are then extreme points of $\mathcal{E}_G^\circ$, and consequently projections of extreme matrices of $\mathbb{S}^p_{\succeq 0}(G)$. Thus, we can expect to find extremal matrices in $\mathbb{S}^p_{\succeq 0}(G)$ whose off-diagonal entries are given by the facet-normal vectors of $\mathrm{cut}^{\pm 1}(\mathrm{G})$. Since the geometry of the elliptope is not at all generic this picture is, in general, difficult to describe from the perspective of real algebraic geometry. In subsection 5.3 we provide this geometric picture in the case of the cycle on four vertices. This serves to demonstrate the difficulty of the algebro-geometric approach for an arbitrary graph G, and consequently motivate the combinatorial work done in the coming sections. Following this example, we prove Theorem 5.1.1, the key to facet-ray identification.

Geometry of the 4-cycle: an example.

Consider the cycle on four vertices, C_4. For simplicity, we let $G := C_4$, and we identify $\mathbb{R}^{E(G)} \simeq \mathbb{R}^4$ by identifying edge $\{i, i+1\}$ with coordinate i for all $i = 1, 2, 3, 4$. Here we take the vertices of C_4 modulo 4. By Lemma 5.2.3, the cut polytope of G is the convex hull of all (± 1)-vectors in $\mathbb{R}^4$ containing precisely an even number of -1's. Equivalently, $\mathrm{cut}^{\pm 1}(\mathrm{G})$ is the 4-cube $[-1, 1]^4$ with truncations at the eight vertices containing an odd number of -1's. Thus, $\mathrm{cut}^{\pm 1}(\mathrm{G})$ has sixteen facets supported by the hyperplanes

$$\pm x_i = 1, \qquad \text{and} \qquad \langle v^F, x\rangle = 2,$$

where F is an odd cardinality subset of $\{1, 2, 3, 4\}$, and v^F is the corresponding vertex of the 4-cube $[-1, 1]^4$ with an odd number of -1's.

To see that the 4-cycle G has the facet-ray identification property amounts to identifying for each facet of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ an extremal matrix in $\mathbb{S}^p_{\succeq 0}(G)$ whose off-diagonal entries are given by the normal vector to the supporting hyperplane of the facet. For example, the facets supported by the hyperplanes $\pm x_1 = 1$ correspond to the rank 1 extremal matrices

$$Y = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, the facets $\langle v^F, x\rangle = 2$ for $v^F = (1, -1, 1, 1)$ and $v^F = (1, -1, -1, -1)$

respectively correspond to the rank 2 extremal matrices

$$Y = \tfrac{1}{3}\begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad Y = \tfrac{1}{3}\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

As indicated by Theorem 5.1.1, these four matrices respectively project to the four extreme points in $\mathcal{E}_G^\circ$

$$(1,0,0,0), \quad (-1,0,0,0), \qquad \frac{1}{3}(-1,1,-1,-1), \qquad \text{and} \qquad \frac{1}{3}(-1,1,1,1),$$

with the former two being vertices of $\mathcal{E}_G^\circ$ (extreme points with full-dimensional normal cones) and the latter two being regular extreme points on the rank 2 locus of $\mathcal{E}_G^\circ$. Indeed, all extreme points corresponding to the facets $\pm x_i = 1$ will be rank 1 vertices of $\mathcal{E}_G^\circ$, and all points corresponding to the facets $\langle v^F, x\rangle = 2$ will be rank 2 regular extreme points of $\mathcal{E}_G^\circ$. Consequently, all sixteen points arise as projections of extremal matrices of $\mathbb{S}^p_{\succeq 0}(G)$ of the corresponding ranks.

To see why these sixteen points in $\mathcal{E}_G^\circ$ are extreme points of the specified type we examine the relaxation of $\mathrm{cut}^{\pm 1}$ (G) to $\mathcal{E}_G$, and the stratification by rank of the spectrahedral shadow $\mathcal{E}_G^\circ$. We compute the algebraic boundary of $\mathcal{E}_G$ as follows. The 4-*elliptope* is the set of correlation matrices

$$\mathcal{E}_4 = \left\{ (x_1,x_2,x_3,x_4,u,v) \in \mathbb{R}^6 : X = \begin{pmatrix} 1 & x_1 & u & x_4 \\ x_1 & 1 & x_2 & v \\ u & x_2 & 1 & x_3 \\ x_4 & v & x_3 & 1 \end{pmatrix} \succeq 0 \right\}.$$

The algebraic boundary $\partial\mathcal{E}_4$ of $\mathcal{E}_4$ is defined by the vanishing of the determinant $D := \det(X)$. The elliptope of the 4-cycle is defined as $\mathcal{E}_G := \pi_G(\mathcal{E}_4)$ where

$$\pi_G : \mathbb{S}^4 \longrightarrow \mathbb{R}^{E(G)}; \qquad X_{i,i+1} \mapsto x_i.$$

To identify the algebraic boundary of $\mathcal{E}_G$ we form the ideal $I := \left(D, \frac{\partial}{\partial u}D, \frac{\partial}{\partial v}D\right)$, and eliminate the variables u and v to produce an ideal $J \subset \mathbb{R}[x_1,x_2,x_3,x_4]$. Using Macaulay2 we see that J is generated by the following product of eight linear terms corresponding to the rank 3 locus,

$$(x_1-1)(x_1+1)(x_2-1)(x_2+1)(x_3-1)(x_3+1)(x_4-1)(x_4+1),$$

and the following sextic polynomial (with multiplicity two) corresponding to the rank 2 locus,

$$\begin{aligned}&(4x_1^2x_2^2x_3^2 - 4x_1^3x_2x_3x_4 - 4x_1x_2^3x_3x_4 - 4x_1x_2x_3^3x_4 + 4x_1^2x_2^2x_4^2 + 4x_1^2x_3^2x_4^2 \\ &+ 4x_2^2x_3^2x_4^2 - 4x_1x_2x_3x_4^3 + x_1^4 - 2x_1^2x_2^2 + x_2^4 - 2x_1^2x_3^2 - 2x_2^2x_3^2 + x_3^4 \\ &+ 8x_1x_2x_3x_4 - 2x_1^2x_4^2 - 2x_2^2x_4^2 - 2x_3^2x_4^2 + x_4^4)^2.\end{aligned}$$

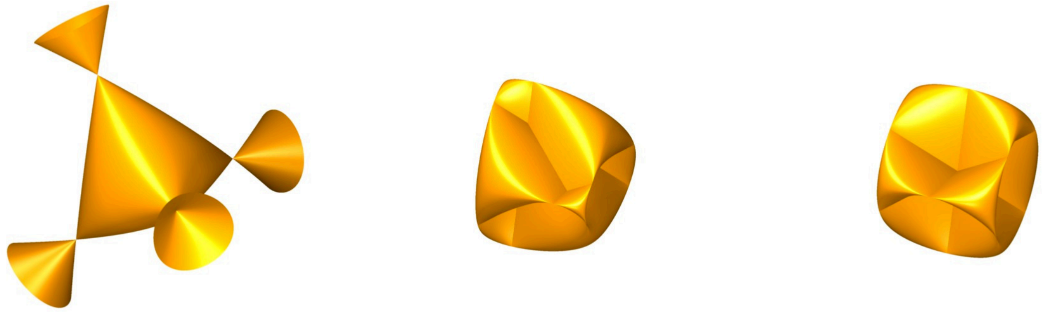


Figure 5.1: Level curves of the rank 2 locus of $\mathcal{E}_{C_4}$. The value of x_4 varies from 1 to 0 as we view the figures from left-to-right.

The sextic factor is the 4^{th} *cycle polynomial* Γ'_4 as defined in [48]. To visualize the portion of the elliptope cut out by this term we treat the variable x_4 as a parameter and vary it from 0 to 1. A few of these level curves (produced using Surfex) are presented in Figure 5.1. An interesting observation is that the level curve with $x_4 = 1$ is the Cayley nodal cubic surface, the bounded region of which is precisely the elliptope $\mathcal{E}_{C_3}$. We note that this holds more generally, i.e. the cut polytope of the p-cycle C_p is the p-halfcube, and the facets of this polytope that lie in the hyperplane $\pm x_i = 1$ are $(p-1)$-halfcubes. Thus, the elliptope $\mathcal{E}_G$ demonstrates the same recursive geometry exhibited by the polytope it relaxes.

The eight linear terms define the rank 3 locus as a hypersurface of degree eight. Since $\mathrm{cut}^{\pm 1}(\mathrm{G}) \subset \mathcal{E}_G \subset [-1,1]^p$, the eight linear terms of the polynomial p indicate that the facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ supported by the hyperplanes $\pm x_i = 1$ are also facets of $\mathcal{E}_G$. From this we can see that the eight hyperplanes $\pm x_i = 1$ correspond to vertices in $\mathcal{E}_G^\circ$. We can also see from this that only the simplicial facets of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ have been relaxed in $\mathcal{E}_G$, and this relaxation is defined by the hypersurface $\{\Gamma'_4 = 0\}$.

Recall that we would like the relaxation of the facets to be smooth in the sense that all singular points on $\partial\mathcal{E}_G$ are also singular points on $\partial\,\mathrm{cut}^{\pm 1}(\mathrm{G})$. If this is the case, then we may translate the supporting hyperplanes of the relaxed facets to support regular extreme points of $\mathcal{E}_G$. The normal vectors to these translated hyperplanes will then form regular extreme points in the polar body $\mathcal{E}_G^\circ$. To see that this is indeed the case, we check that the intersection of the singular locus of $\{\Gamma'_4 = 0\}$ with $\partial\mathcal{E}_G$ is restricted to the rank 3 locus of $\mathcal{E}_G$. With the help of Macaulay2, we compute that $\{\Gamma'_4 = 0\}$ is singular along the six planes given by the vanishing of the ideals

$$\begin{array}{lll} \langle x_3 - x_4, x_1 - x_2\rangle, & \langle x_3 + x_4, x_1 + x_2\rangle, & \langle x_2 - x_3, x_1 - x_4\rangle, \\ \langle x_2 + x_3, x_1 + x_4\rangle, & \langle x_2 + x_4, x_1 + x_3\rangle, & \langle x_2 - x_4, x_1 - x_3\rangle, \end{array}$$

and at eight points

$$\begin{array}{ll} \frac{1}{\sqrt{3}}\left(\mp 1, \pm 1, \pm 1, \pm 1\right), & \frac{1}{\sqrt{3}}\left(\pm 1, \mp 1, \pm 1, \pm 1\right), \\ \frac{1}{\sqrt{3}}\left(\pm 1, \pm 1, \mp 1, \pm 1\right), & \frac{1}{\sqrt{3}}\left(\mp 1, \mp 1, \mp 1, \pm 1\right). \end{array}$$

The six planes intersect $\partial\mathcal{E}_G$ only along the edges of $\text{cut}^{\pm1}(\text{G})$, and therefore do not introduce any new singular points that did not previously exist in $\text{cut}^{\pm1}(\text{G})$. The eight singular points sit just outside the cut polytope $\text{cut}^{\pm1}(\text{G})$ above the barycenter of each simplicial facet. However, these singular points lie in the interior of $\mathcal{E}_G$. This can be checked using the polyhedral description of $\mathcal{E}_G$ first studied by Barrett et al. [5]. The idea is that each point (x_1, x_2, x_3, x_4) of the elliptope $\mathcal{E}_G$ arises from a point (a_1, a_2, a_3, a_4) in the $(0,1)$-cut polytope, $\text{cut}^{01}(\text{G})$, by letting $x_i = cos(\pi a_i)$ for every $i \in [4]$. Since $\text{cut}^{01}(\text{G})$ is affinely equivalent to $\text{cut}^{\pm1}(\text{G})$ under the linear transformation $y_i = 1 - 2x_i$, we apply the arccosine transformation of Barrett et al. to the barycenter of each simplicial facet of $\text{cut}^{01}(\text{G})$ to produce the eight points on $\partial\mathcal{E}_G$:

$$\begin{array}{ll} \frac{1}{\sqrt{2}}(\mp1, \pm1, \pm1, \pm1), & \frac{1}{\sqrt{2}}(\pm1, \mp1, \pm1, \pm1), \\ \frac{1}{\sqrt{2}}(\pm1, \pm1, \mp1, \pm1), & \frac{1}{\sqrt{2}}(\mp1, \mp1, \mp1, \pm1). \end{array}$$

Thus, each of the eight singular points of Γ_4' lies in the interior of $\mathcal{E}_G$ on the line between the barycenter of a simplicial facet of $\text{cut}^{\pm1}(\text{G})$ and one of these eight points in $\partial\mathcal{E}_G$. From this we see that the relaxation of the simplicial facets of $\text{cut}^{\pm1}(\text{G})$ is smooth, and so we may translate the supporting hyperplanes $\langle v^F, x\rangle = 2$ away from $\text{cut}^{\pm1}(\text{G})$ until they support some regular extreme point on $\partial\mathcal{E}_G$.

In the polar $\mathcal{E}_G^\circ$ we check that the normal vectors to the hyperplanes $\pm x_i = 1$ form vertices of rank 1, and the normal vectors corresponding to the translated versions of the hyperplanes $\langle v^F, x\rangle = 2$ are regular points on the rank 2 strata of $\mathcal{E}_G^\circ$. The polar $\mathcal{E}_G^\circ$ is the spectrahedral shadow

$$\mathcal{E}_G^\circ = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \exists a, b, c \in \mathbb{R} : Y = \begin{pmatrix} a & x_1 & 0 & x_4 \\ x_1 & b & x_2 & 0 \\ 0 & x_2 & c & x_3 \\ x_4 & 0 & x_3 & 2-a-b-c \end{pmatrix} \succeq 0 \right\},$$

and the matrix Y is a trace two matrix living in the cone $\mathbb{S}^p_{\succeq 0}(G)$. The rank 3 locus of $\mathcal{E}_G^\circ$ can be computed by forming the ideal generated by the determinant of Y and its partials with respect to a, b, and c, and then eliminating the variables a, b, and c from the saturation of this ideal with respect to the 3×3 minors of Y. The result is a degree eight hypersurface that factors into eight linear forms:

$$\begin{array}{rr} x_1 - x_2 - x_3 + x_4 + 1, & -x_1 + x_2 - x_3 + x_4 + 1, \\ -x_1 - x_2 + x_3 + x_4 + 1, & x_1 + x_2 + x_3 + x_4 + 1, \\ x_1 - x_2 - x_3 + x_4 - 1, & -x_1 + x_2 - x_3 + x_4 - 1, \\ -x_1 - x_2 + x_3 + x_4 - 1, & x_1 + x_2 + x_3 + x_4 - 1. \end{array}$$

The eight points in $\mathcal{E}_G^\circ$ that are dual to the hyperplanes $\pm x_i = 1$ are vertices of the convex polytope whose H-representation is given by these linear forms. These vertices are projections of rank 1 matrices in $\mathbb{S}^p_{\succeq 0}(G)$. Our remaining eight hyperplanes supporting regular extreme points in $\mathcal{E}_G$ should correspond to rank 2 regular extreme points in $\mathcal{E}_G^\circ$. We check that the normal vectors to these hyperplanes don't lie on the singular locus of the rank 2 strata of $\mathcal{E}_G^\circ$. To compute the rank 2 strata of $\mathcal{E}_G^\circ$ we

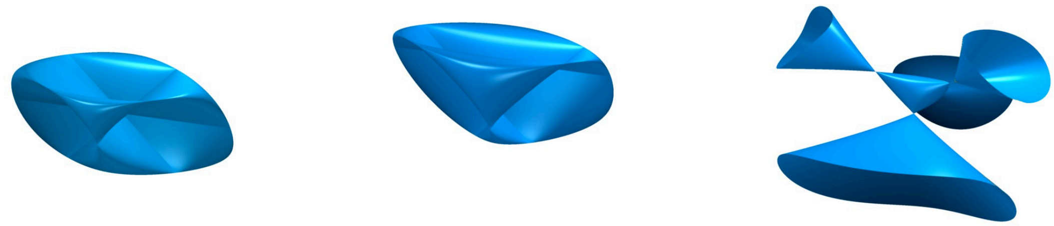


Figure 5.2: Level curves of the rank 2 locus of $\mathcal{E}_{C_4}^\circ$ defined by the hyperplane $x_1 + x_2 + x_3 + x_4 = b$. The value of b varies from 1 to 0 from left-to-right.

eliminate the variables a, b, and c from the ideal generated by the 3×3 minors of Y and all of their partial derivatives with respect to the variables a, b, and c. The result is a degree four hypersurface defined by the polynomial

$$x_1^2x_2^2x_3^2 - x_1^3x_2x_3x_4 - x_1x_2^3x_3x_4 - x_1x_2x_3^3x_4 + x_1^2x_2^2x_4^2 + x_1^2x_3^2x_4^2 + x_2^2x_3^2x_4^2 \\ - x_1x_2x_3x_4^3 + x_1x_2x_3x_4$$

that is singular along six planes defined by the vanishing of the ideals

$$\langle x_3, x_4\rangle, \quad \langle x_2, x_3\rangle, \quad \langle x_1, x_3\rangle, \quad \langle x_2, x_4\rangle, \quad \langle x_1, x_2\rangle, \quad \langle x_1, x_4\rangle.$$

To visualize the rank 2 locus of $\mathcal{E}_G^\circ$ we intersect this degree four hypersurface with the hyperplane $x_1 + x_2 + x_3 + x_4 = b$ and let b vary from 0 to 1. A sample of these level curves is presented in Figure 5.2. Since the normal vectors to our hyperplanes are nonzero in all coordinates, their corresponding points are regular points in the rank 2 locus of $\mathcal{E}_G^\circ$, and therefore arise as projections of extremal matrices of rank 2 in $\mathbb{S}^p_{\succeq 0}(G)$. The combinatorial work in Section 5.4 supports this geometry.

The polar of an elliptope

Recall that the polar of a subset $K \subset \mathbb{R}^d$ is

$$K^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1 \text{ for all } x \in K\}.$$

In this subsection we prove Theorem 5.1.1 via an application of spectrahedral polarity. We first review how to compute the polar for a spectrahedron via the methods of Ramana and Goldman described in [41].

Let $C, A_1, \ldots, A_d \in \mathbb{S}^p$, where $A_1, \ldots, A_d$ are linearly independent. A *spectrahedron* a closed convex set $\mathcal{S}$ of the form

$$\mathcal{S} = \left\{ x \in \mathbb{R}^d : A(x) = C + \sum_{i=1}^d x_i A_i \succeq 0 \right\},$$

where $A(x) \succeq 0$ indicates that $A(x)$ is positive semidefinite. Since the matrices $A_1, \ldots, A_d$ are linearly independent then $\mathcal{S}$ is affinely equivalent to the section of the positive semidefinite cone

$$\mathcal{A} \cap \mathbb{S}^p_{\succeq 0},$$

where $\mathcal{A} = C + \mathrm{span}_{\mathbb{R}}(A_1, \ldots, A_d)$. Thus, the affine section $\mathcal{A} \cap \mathbb{S}^p_{\succeq 0}$ is often also called a spectrahedron. Let $\mathcal{W} = \mathrm{span}_{\mathbb{R}}(A_1, \ldots, A_d)$ be the linear subspace defined by $A_1, \ldots, A_d$ and

$$\pi_{\mathcal{W}} \,:\, \mathbb{S}^p \to \mathbb{S}^p/\mathcal{W}^{\perp} \simeq \mathbb{R}^d, \qquad X \mapsto (\langle X, A_1 \rangle, \ldots, \langle X, A_d \rangle)$$

be the canonical projection. We define the $\binom{p+1}{2}$-dimensional spectrahedron

$$\mathcal{R} = \{X \in \mathbb{S}^p_{\succeq 0} : \langle X, C \rangle \leq 1\}.$$

Then the polar of the spectrahedron $\mathcal{S}$ is a *spectrahedral shadow*, namely the closure of the image of the spectrahedron $\mathcal{R}$ under the projection $\pi_{\mathcal{W}}$, i.e. $\mathcal{S}^{\circ} = \mathrm{cl}(\pi_{\mathcal{W}}(\mathcal{R}))$ [41].

Proof of Theorem 5.1.1

We first apply the general theory about spectrahedra to compute the polar of the set of correlation matrices

$$\mathcal{E}_p = \{X \in \mathbb{S}^p_{\succeq 0} : X_{ii} = 1 \text{ for all } i \in [p]\}.$$

Let $A_{ij} = [a_{ij}] \in \mathbb{S}^p$, $1 \leq i < j \leq p$ be the zero matrix except with $a_{ij} = a_{ji} = 1$. Then $\mathcal{E}_p$ is a spectrahedron

$$\mathcal{E}_p = \mathbb{S}^p_{\succeq 0} \cap \mathcal{A},$$

where $\mathcal{A}$ is the affine subspace

$$\mathcal{A} = \mathrm{I}_p + \mathrm{span}_{\mathbb{R}}(A_{ij} : 1 \leq i < j \leq p).$$

Notice that since $C = \mathrm{I}_p$, then $\mathcal{R} = \{X \in \mathbb{S}^p_{\succeq 0} : \langle X, C \rangle = 1\}$. Applying the above techniques we get that the polar of $\mathcal{E}_p$ is the spectrahedral shadow

$$\mathcal{E}_p^{\circ} = \{Y \in \mathbb{R}^{\binom{p}{2}} : \exists\, X \in \mathbb{S}^p_{\succeq 0} \text{ such that } X_{ij} = Y_{ij} \text{ for all } i < j \text{ and } \mathrm{tr}(X) = 2\}.$$

We now compute the polar of the elliptope

$$\mathcal{E}_G = \{y \in \mathbb{R}^E : \exists\, Y \in \mathcal{E}_p \text{ such that } Y_E = y\}.$$

Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^{\binom{p}{2}}$ defined by A_{ij}, $ij \in E$. We denote by $\mathcal{L}^{\perp}$ the orthogonal complement of $\mathcal{L}$ in $\mathbb{R}^{\binom{p}{2}}$. Then

$$((\mathcal{E}_p + \mathcal{L}^{\perp})/\mathcal{L}^{\perp})^{\circ} \;=\; \mathcal{E}_p^{\circ} \cap \mathcal{L},$$

which means that

$$\mathcal{E}_G^\circ = \{x \in \mathbb{R}^E : \exists\, X \in \mathbb{S}^p_{\succeq 0}(G) \text{ such that } X_E = x \text{ and } \operatorname{tr}(X) = 2\}.$$

This completes the proof of Theorem 5.1.1. $\square$

It is clear that the constraint $\operatorname{tr}(X) = 2$ is just a scaling. So the extreme points of the convex body $\mathcal{E}_G^\circ$ correspond to the extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$. Since an extreme point of $\mathcal{E}_G^\circ$ either has a full-dimensional normal cone without lineality or is a regular point of $\mathcal{E}_G^\circ$ we arrive at the following corollary.

Corollary 5.3.1. *The hyperplanes supporting facets of the elliptope $\mathcal{E}_G$ or regular extreme points of $\mathcal{E}_G$ correspond to extremal rays of the cone $\mathbb{S}^p_{\succeq 0}(G)$.*

A supporting hyperplane of $\mathcal{E}_G$ of the type described in Corollary 5.3.1 identifies an extremal ray of rank r if it corresponds to a point in the rank r strata of $\mathcal{E}_G^\circ$. This is the basis for the facet-ray identification property.

5.4 Facet-Ray Identification for graphs without K_5 minors

In this section, we show that all graphs without K_5 minors have the facet-ray identification property. We first demonstrate that the p-cycle C_p has the facet-ray identification property, and then generalize this result to all graphs without K_5 minors.

Facet-Ray Identification for the Cycle.

Let $G := C_p$ for $p \geq 3$. Here, we will make the identification $\mathbb{R}^E \simeq \mathbb{R}^p$ by identifying the coordinate $e = \{i, i+1\}$ in $\mathbb{R}^E$ with the coordinate i in $\mathbb{R}^p$. For an edge $e \in E$ we define two $p \times p$ matrices, X_e and X_e^- where

$$(X_e)_{s,t} := \begin{cases} 1 & \text{if } s,t \in e, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad (X_e^-)_{s,t} := \begin{cases} 1 & \text{if } s = t \text{ and } s,t \in e, \\ -1 & \text{if } s \neq t \text{ and } s,t \in e, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 5.4.1. *The matrices X_e and X_e^- are extremal in $\mathbb{S}^p_{\succeq 0}(G)$ of rank 1. Moreover, the off-diagonal entries of X_e and X_e^- are given by the normal vector to the hyperplane $\langle \pm v^e, x\rangle = 1$, respectively.*

Proof. These matrices are of rank 1 and have respective 1-dimensional Gram representations $(u_1^e, \dots, u_p^e)$ and $(w_1^e, \dots, w_p^e)$, where

$$u_t^e := \begin{cases} 1 & \text{if } t \in \{i,j\}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad w_t^e := \begin{cases} 1 & \text{if } t = i, \\ -1 & \text{if } t = j, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the collection $U_{\overline{E}}$ with respect to these Gram representations. If $\{s,t\} \in \overline{E}$ then either $s \notin \{i,j\}$ or $t \notin \{i,j\}$ (or both). Thus, $u_s = \overline{\mathbf{0}}$ ($w_s = \overline{\mathbf{0}}$) or $u_t = \overline{\mathbf{0}}$ ($w_t = \overline{\mathbf{0}}$) (or both). Hence, $U_{\overline{E}} = \{\overline{\mathbf{0}}\}$ and $\operatorname{rank}(U_{\overline{E}}) = 0$. So by Theorem 5.2.1 the matrices X_e and X_e^- are extremal in $\mathbb{S}^p_{\succeq 0}(G)$. Since for all $e \neq e' \in E$ the matrices X_e, $X_{e'}$, X_e^- and $X_{e'}^-$ are not scalar multiples of each other, each such matrix lies on a different extremal ray of $\mathbb{S}^p_{\succeq 0}(G)$. $\square$

Our next goal is to identify rank $p-2$ extremal matrices in $\mathbb{S}^p_{\succeq 0}(G)$ whose off-diagonal entries are determined by the normal vectors to the facet-supporting hyperplanes $\langle v^F, x\rangle = p-2$. Thus, we wish to prove the following theorem.

Theorem 5.4.2. *Let $F \subset [p]$ be a subset of odd cardinality. There exists a rank $p-2$ extremal matrix $\Delta_{p,F} \in \mathbb{S}^p_{\succeq 0}(G)$ such that $(\Delta_{p,F})_{i,i+1} = -v_i^F$ for all $i \in [p]$ (modulo p).*

To prove Theorem 5.4.2 we first construct the matrices $\Delta_{p,F}$ when F is a maximal odd cardinality subset of $[p]$, and then prove a lemma showing the existence of such matrices in all the remaining cases. Let F be a maximal odd cardinality subset of $[p]$, and let $e_1, e_2, \ldots, e_{p-2}$ denote the standard basis vectors for $\mathbb{R}^{p-2}$. Notice that for p even, $F = [p]\backslash\{i\}$ for some $i \in [p]$, and for p odd, $F = [p]$. For $i \in [p]$ we define the collection of vectors

$$\begin{aligned}
&u_i := e_{p-2},\\
&u_{i+1} := \textstyle\sum_{j=1}^{p-2}(-1)^{j+1}e_j,\\
&u_{i+2} := e_1,\\
&u_j := e_{j-i-2} + e_{j-i-1}, \quad \text{for } i+3 \leq j \leq i+p-1.
\end{aligned}$$

Here, we view the indices of these vectors modulo p, i.e. $p+1 = 1$. For p even and $F = [p]\backslash\{i\}$, let $\Delta_{p,F}$ denote the positive semidefinite matrix with Gram representation $(u_1, u_2, \ldots, u_p)$. Similarly, for p odd and $F = [p]$, let $i = 1$, and let $\Delta_{p,F}$ denote the matrix with Gram representation $(u_1, u_2, \ldots, u_p)$.

Remark 5.4.3. While independently discovered by the authors in terms of facets $\mathrm{cut}^{\pm 1}$ (G), the Gram representation $(u_1, \ldots, u_p)$ for $i = p-1$ was previously used in [1, Lemma 6.3] to demonstrate that the sparsity order of the p-cycle is larger than 1 for $p \geq 4$. Here, we verify that this representation is indeed extremal, and show that it arises as part of a collection of extremal representations given by the facets of the cut polytope $\mathrm{cut}^{\pm 1}$ (G).

Lemma 5.4.4. *Let F be a maximal odd cardinality subset of $[p]$. Then the matrix $\Delta_{p,F}$ is extremal in $\mathbb{S}^p_{\succeq 0}(G)$ with rank $p-2$.*

Proof. It is easy to check that all entries of $\Delta_{p,F}$ corresponding to nonedges of G will contain a zero. Notice also that all adjacent pairs u_j, u_{j+1} have inner product 1 except for the pair u_i, u_{i+1}, when p is even, whose inner product is -1. Moreover, $(u_1, u_2, \ldots, u_p)$ spans $\mathbb{R}^{p-2}$, and therefore $\mathrm{rank}(\Delta_{p,F}) = p-2$. Thus, by Theorem 5.2.1, it only remains to verify that $\mathrm{rank}(U_{\overline{E}}) = \binom{p-1}{2} - 1$. However, since $\#\overline{E} = \binom{p}{2} - p = \binom{p-1}{2} - 1$, it suffices to show that the collection of matrices $U_{\overline{E}}$ are a linearly independent set.

Without loss of generality, we set $i = 1$. First it is noted that the set $u_3, u_4, \ldots u_p$ are linearly independent in $\mathbb{R}^{p-2}$. Thus we can write u_1 and u_2 as the following:

$$\begin{array}{rcl} u_1 & = & \sum_{i=3}^{p} (-1)^{(p+i)} u_i, \\ u_2 & = & \sum_{i=3}^{p} (-1)^i (i-2) u_i + (-1)^{p-1}(p-1)u_1. \end{array}$$

Since $\{u_3, \ldots, u_p\}$ are linearly independent, we consider them as a basis of the vector space $\mathbb{R}^{(p-2)}$.

Since the graph G is a cycle of length p, $\overline{E}$ does not contain $\{i, i+1\}$ for $i = 1, \ldots p-1$ and $\{p, 1\}$. Thus, we consider the set of matrices

$$V := \{u_i \cdot u_j^T + u_j \cdot u_i^T : i = 3, \ldots, p,\ i < j,\ j \neq i+1\} \subset \mathbb{R}^{(p-2)\times(p-2)}.$$

Note that $u_i \cdot u_j^T + u_j \cdot u_i^T$ is a $(p-2) \times (p-2)$ matrix M whose $(i'j')^{th}$ element is

$$M_{i'j'} = \begin{cases} 1 & \text{if } i' = i,\ j' = j,\ j' \neq i'+1, \\ 1 & \text{if } i' = j,\ j' = i,\ j' \neq i'+1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the set

$$V = \{u_i \cdot u_j^T + u_j \cdot u_i^T | i = 3, \ldots p,\ i < j,\ j \neq i+1\}$$

is linearly independent.

Now we consider the matrix $u_1 \cdot u_k^T + u_k \cdot u_1^T$ for $k = 3, \ldots, p-1$. Note that

$$\begin{array}{rcl} u_1 \cdot u_k^T + u_k \cdot u_1^T & = & \left(\sum_{i=3}^{p} (-1)^{(p+i)} u_i\right) \cdot u_k^T + u_k \cdot \left(\sum_{i=3}^{p} (-1)^{(p+i)} u_i\right)^T \\ & = & \left(\sum_{i=3}^{p} (-1)^{(p+i)} u_i \cdot u_k^T\right) + \left(\sum_{i=3}^{p} (-1)^{(p+i)} u_k \cdot u_i^T\right) \\ & =: & \bar{M}^k, \end{array}$$

where

$$\bar{M}^k_{i'j'} = \begin{cases} (-1)^{(p+i')} \cdot 2 & \text{if } i' = j' = k, \\ (-1)^{(p+i')} & \text{if } i' \neq k,\ i' = 3, \ldots, (p-1) \text{ and } j' = k, \\ (-1)^{(p+j')} & \text{if } i' = k,\ j' \neq k, \text{ and } j' = 3, \ldots, (p-1), \\ 0 & \text{else.} \end{cases}$$

In addition, we consider the matrix $u_2 \cdot u_k^T + u_k \cdot u_2^T$ for $k = 4, \ldots, p$. Note that

$$\begin{array}{rcl} u_2 \cdot u_k^T + u_k \cdot u_2^T & = & \left(\sum_{i=3}^{p} (-1)^i (i-2) u_i\right) \cdot u_k^T + u_k \cdot \left(\sum_{i=3}^{p} (-1)^i (i-2) u_i\right)^T \\ & = & \left(\sum_{i=3}^{p} (-1)^i (i-2) u_i \cdot u_k^T\right) + \left(\sum_{i=3}^{p} (-1)^i (i-2) u_k \cdot u_i^T\right) \\ & =: & \tilde{M}^k, \end{array}$$

where

$$\tilde{M}^k_{i'j'} = \begin{cases} (-1)^{i'} \cdot 2 \cdot (i'-2) & \text{if } i' = j' = k, \\ (-1)^{i'} \cdot (i'-2) & \text{if } i' \neq k,\ i' = 3, \ldots, (p-1) \text{ and } j' = k, \\ (-1)^{j'} \cdot (j'-2) & \text{if } i' = k,\ j' \neq k, \text{ and } j' = 3, \ldots, (p-1), \\ 0 & \text{else.} \end{cases}$$

Since V does not contain the matrices $\hat{M}^i$ for $i = 3, \ldots p$ such that

$$\hat{M}^i_{i'j'} = \begin{cases} 1 & \text{if } i' = i,\, j' = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and the matrices M'^i for $i = 3, \ldots p$ such that

$$M'^i_{i'j'} = \begin{cases} 1 & \text{if } i' = i,\, j' = i, \\ 0 & \text{otherwise,} \end{cases}$$

we cannot write $\tilde{M}^k$ in terms of $\bar{M}^{k'}$ and elements of V (and also we cannot write $\bar{M}^k$ in terms of $\tilde{M}^{k'}$ and elements of V) for $k = 3, \ldots, p-1$ and $k' = 4, \ldots, p$. Hence, the matrices $\tilde{M}^k$ for $k = 4, \ldots, p$, $\bar{M}^k$ for $k' = 3, \ldots, p-1$, and the matrices in V are linearly independent. $\square$

To provide some intuition as to the construction of the remaining extremal matrices we note that a k-dimensional Gram representation of a graph G with vertex set $[p]$ is a map $Y : [p] \longrightarrow \mathbb{R}^k$ such that $\mathrm{span}_{\mathbb{R}}\{Y(i) | i \in [p]\} = \mathbb{R}^k$ and $Y(i)^T Y(j) = 0$ for all $\{i, j\} \in \overline{E}$. Hence, the Gram representation $(u_1, u_2, \ldots, u_p)$ is an inclusion of the graph G into the hypercube $[-1, 1]^{p-2}$. Here, the vertex i of G is identified with the vector $u_i \in \mathbb{R}^k$. In this way, the underlying cut U of a cutset $\delta(U)$ of G is now a collection of vectors as opposed to a collection of indices. We now consider the cutsets $\delta(U)$ of G with respect to the representation $(u_1, u_2, \ldots, u_p)$ for the maximal odd cardinality subsets F, and negate the vectors in the underlying cut U to produce the desired extremal matrices for lower cardinality odd subsets of $[p]$. This is the content of the following lemma.

Lemma 5.4.5. *Let $F \subset [p]$ be a subset of odd cardinality. There exists a rank $p-2$ extremal matrix $\Delta_{p,F}$ in $\mathbb{S}^p_{\succeq 0}(G)$ with off-diagonal entries satisfying*

$$(\Delta_{p,F})_{s,t} = \begin{cases} 0 & \text{if } \{s,t\} \in \overline{E}, \\ 1 & \text{if } t = s+1 \text{ and } s \in F, \\ -1 & \text{if } t = s+1 \text{ and } s \notin F. \end{cases}$$

Proof. We produce the desired matrices in two separate cases, when p is odd and when p is even. Suppose first that p is odd, and consider the $(p-2)$-dimensional Gram representation $(u_1, u_2, \ldots, u_p)$ defined above for the extremal matrix $\Delta_{p,[p]}$. This Gram representation includes G into the hypercube $[-1, 1]^{p-2}$ such that vertex i of G corresponds to u_i.

We now consider the cuts of G with respect to this inclusion. Recall from Section 5.2 that even subsets of E are the cutsets $\delta(U)$ of G, and they correspond to a unique cut (U, U^c) of G. For each $i \in [p]$ we can consider the edge $\{i, i+1\} \in E$. Let $F \subset [p]$ be of odd cardinality. Then F^c is of even cardinality and hence has an associated cut (U, U^c) such that $F^c = \delta(U)$. Now, thinking of $U \subset [p] = V(G)$, negate all vectors in $(u_1, u_2, \ldots, u_p)$ with indices in U to produce a new $(p-2)$-dimensional representation of G, say $(w_1, w_2, \ldots, w_p)$, where

$$w_t := \begin{cases} -u_t & \text{if } t \in U, \\ u_t & \text{if } t \notin U. \end{cases}$$

Let $\Delta_{p,F}$ denote the matrix with Gram representation $(w_1, w_2, \ldots, w_p)$. Since $F^c = \delta(U)$ is a cutset, negating all the vectors u_t with $t \in U$ results in $(\Delta_{p,F})_{i,i+1} = -1$ for every $i \in F^c$, and all other entries of $\Delta_{p,F}$ remain the same as those in $\Delta_{p,[p]}$. Moreover, $\operatorname{rank}(\Delta_{p,F}) = \operatorname{rank}(\Delta_{p,[p]})$ and $\operatorname{rank}(U_{\overline{E}}) = \binom{p-1}{2} - 1$. Thus, $\Delta_{p,F}$ is extremal in $\mathbb{S}^p_{\succeq 0}(G)$ with rank $p-2$.

Now suppose that p is even. Fix $i \in [p]$ and consider the $(p-2)$-dimensional Gram representation $(u_1, u_2, \ldots, u_p)$ defined above for the extremal matrix $\Delta_{p,[p]\backslash\{i\}}$. Partition the collection of odd subsets of $[p]$ into two blocks, A and B, where A consists of all odd subsets of $[p]$ containing i. Let $F \subset [p]$ be of odd cardinality, and suppose first that $F \in A$. Consider the even cardinality subset $M = F^c \cup \{i\}$. Thinking of each i in $[p]$ as corresponding to the edge $\{i, i+1\} \in E$, it follows that $M = \delta(U)$ for some cut (U, U^c) of G. Once more, thinking of $U \subset [p] = V(G)$, set

$$w_t := \begin{cases} -u_t & \text{if } t \in U, \\ u_t & \text{if } t \notin U, \end{cases}$$

and let $\Delta_{p,F}$ denote the matrix with Gram representation $(w_1, w_2, \ldots, w_p)$. Since M is a cutset, it follows that $(\Delta_{p,F})_{s,s+1} = -(\Delta_{p,[p]\backslash\{i\}})_{s,s+1}$ for every $s \in M$. In particular, $(\Delta_{p,F})_{i,i+1} = 1$.

Finally, suppose $F \in B$, and consider the even cardinality subset $M = F^c\backslash\{i\}$. Proceeding as in the previous case produces the desired matrix $\Delta_{p,F}$. Just as in the odd case, the matrices $\Delta_{p,F}$ for p even are extremal of rank $p-2$. $\square$

Example 5.4.6. We illustrate the construction in the proof of Lemma 5.4.5 by considering the case $p = 4$ and $i = 1$. The corresponding maximum cardinality subset is $\{2, 3, 4\}$. The $(p-2)$-dimensional Gram representation for this maximum cardinality odd subset is (u_1, u_2, u_3, u_4), where

$$u_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad u_2 := \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad u_3 := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u_4 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The resulting extremal matrix in $\mathbb{S}^p_{\succeq 0}(G)$ is

$$\Delta_{4,\{2,3,4\}} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Now consider the odd cardinality subset $F := \{2\} \subset [4]$. Then $M = F^c\backslash\{1\} = \{3, 4\} \simeq \{\{3, 4\}, \{4, 1\}\} \subset E(C_4)$. Thus, $M = \delta(U)$ where $U = \{4\}$. The Gram representation identified in the proof of Lemma 5.4.5 is $(w_1, w_2, w_3, w_4) := (u_1, u_2, u_3, -u_4)$.

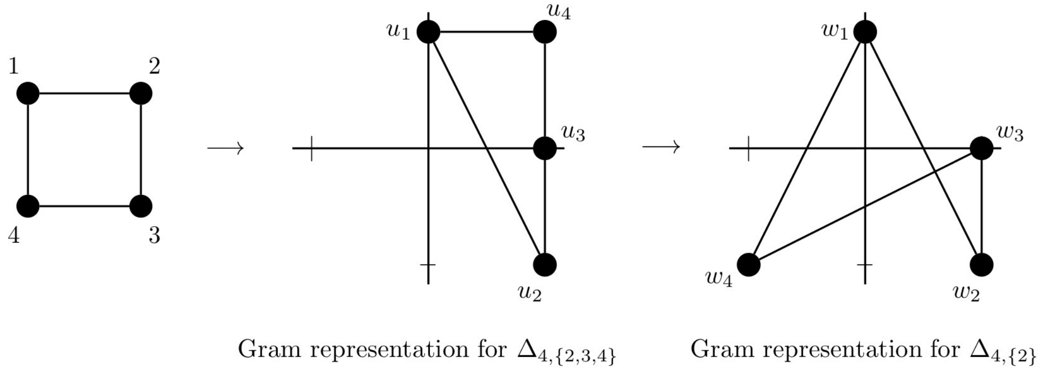


Figure 5.3: C_4 included into $[-1,1]^2$ via its Gram representation, and the Gram representations for Example 5.4.6.

Both of these Gram representations are depicted in Figure 5.3. The resulting extremal matrix associated to F is

$$\Delta_{4,\{2\}} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Notice that the off-diagonal entries corresponding to the edges of C_4 are given by

$$-v^F = (-1,1,-1,-1) \in \mathbb{R}^{E(C_4)},$$

the normal vector to the facet-supporting hyperplane $\langle v^F, x\rangle = 2$ of $\mathrm{cut}^{\pm 1}(\mathrm{C}_4)$. □

Lemmas 5.4.4 and 5.4.5 combined provide a proof of Theorem 5.4.2:

Proof of Theorem 5.4.2

Recall that we identify $\mathbb{R}^E \simeq \mathbb{R}^p$ by identifying the coordinate $e = \{i, i+1\}$ in $\mathbb{R}^E$ with the coordinate i in $\mathbb{R}^p$. Consider the projection map $\pi_E : \mathbb{S}^p \longrightarrow \mathbb{R}^E \simeq \mathbb{R}^p$ that projects a matrix onto its coordinates corresponding to the edges of G. For an odd cardinality subset F of $[p]$, the matrix $\Delta_{p,F}$ satisfies $\pi_E(\Delta_{p,F}) = -v^F$. This completes the proof of Theorem 5.4.2. □

Proposition 5.4.1 and Theorem 5.4.2 combine to prove that the p-cycle has the facet-ray identification property. We now use these results to provide a proof of Theorem 5.1.3.

Proof of Theorem 5.1.3

Let G be a graph without K_5 minors. To show that G has the facet-ray identification property, we must produce for every facet F of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ an extremal matrix $M \in \mathbb{S}^p_{\succeq 0}(G)$ whose off-diagonal entries are given by the normal vector to F. Recall from Section 5.2 that the supporting hyperplanes of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ are

(1) $\langle \pm v^e, x\rangle = 1$ for all $e \in E$, and

(2) $\langle v^F, x\rangle = m-2$ for all odd cardinality subsets $F \subset E(C_m)$ for all chordless cycles C_m in G.

In the case of the cycle C_m we have constructed the desired extremal matrices X_e, X_e^-, and $\Delta_{m,F}$ for each such hyperplane, and each such matrix possesses an underlying Gram representation $(u_1, \ldots, u_m)$. Thus, we define the $(m-2)$-dimensional Gram representation $(w_1, w_2, \ldots, w_{|E|})$ where

$$w_t := \begin{cases} u_t & \text{if } t \in [m] \simeq V(C_m) \subset [p] = V(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{X}_e$, $\widetilde{X}_e^-$, and $\widetilde{\Delta}_{m,F}$ denote the resulting matrices in $\mathbb{S}^p_{\succeq 0}(G)$ with Gram representation $(w_1, \ldots, w_{|E|})$. It follows from Proposition 5.4.1, Lemma 5.4.4 and Lemma 5.4.5 that these matrices are extremal in $\mathbb{S}^p_{\succeq 0}(G)$ of rank 1, 1, and $m-2$, respectively. This completes the proof of Theorem 5.1.3. $\square$

The Geometry of Facet-ray Identification Revisited

In Section 5.3 we saw that some distinguished extreme points of the polar $\mathcal{E}_G^\circ$ lift to matrices lying on extremal rays of the cone $\mathbb{S}^p_{\succeq 0}(G)$. Via polarity, this means that an extremal ray of $\mathbb{S}^p_{\succeq 0}(G)$ corresponds to a hyperplane supporting $\mathcal{E}_G$. Since the extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$ are the dimension 1 faces of the cone, we say that the *rank* of an extremal ray r of $\mathbb{S}^p_{\succeq 0}(G)$ is the rank of any nonzero matrix lying on r. Thus, the rank of an extremal ray of $\mathbb{S}^p_{\succeq 0}(G)$ is given by the rank of the corresponding vertex of $\mathcal{E}_G^\circ$. In the polar, the *rank* of a supporting hyperplane of $\mathcal{E}_G$ is the rank of the corresponding extremal ray in $\mathbb{S}^p_{\succeq 0}(G)$.

Let G be a graph without K_5 minors. For each facet of $\mathrm{cut}^{\pm 1}$ (G) we have identified an extremal matrix X_e, X_e^-, or $\Delta_{m,F}$, and each such matrix generates an extremal ray of $\mathbb{S}^p_{\succeq 0}(G)$:

$$r_e := \mathrm{span}_{\geq 0}(X_e), \qquad r_e^- := \mathrm{span}_{\geq 0}(X_e^-), \qquad \text{and} \qquad r_{m,F} := \mathrm{span}_{\geq 0}(\Delta_{m,F}),$$

respectively. Recall from Theorem 5.1.1 that $\mathcal{E}_G^\circ$ is a projection of the trace two affine section of the cone $\mathbb{S}^p_{\succeq 0}(G)$. Since $\mathrm{Tr}(X_e) = \mathrm{Tr}(X_e^-) = 2$ these matrices correspond to vertices of $\mathcal{E}_G^\circ$, which dually correspond to the facet-supporting hyperplanes $\langle \pm v^e, x\rangle = 1$ of the elliptope $\mathcal{E}_G$. On the other hand, $\mathrm{Tr}(\Delta_{m,F}) = \sum_{t=1}^m u_t^T u_t = 3m-6$. Thus, the matrix $Y_{m,F} := \frac{2}{3m-6}\Delta_{m,F}$ corresponds to the regular extreme point $\pi_E(Y_{m,F}) = -\frac{2}{3m-6}v^F$ of $\mathcal{E}_G^\circ$. Hence, the corresponding hyperplane in $\mathcal{E}_G$ is

$$\langle v^F, x\rangle = \frac{6-3m}{2}.$$

So the supporting hyperplane $\langle v^F, x\rangle = m-2$ of $\mathrm{cut}^{\pm 1}$ (G) is a translation by $\frac{5m-10}{2}$ of this rank $m-2$ hyperplane. This illustrates the geometry described in Section 5.3.

Remark 5.4.7. Note that the geometric correspondence between facets and extremal rays discussed in this section holds for any graph with the facet-ray identification property. Thus, while our proof of this property is combinatorial, the property itself is inherently geometric.

5.5 Characterizing Extremal Ranks

In this section, we discuss when facet-ray identification characterizes all extreme ranks of $\mathbb{S}^p_{\succeq 0}(G)$.

Series-parallel graphs.

Let G be a series-parallel graph. We show that the extremal ranks identified by the facets of $\mathrm{cut}^{\pm 1}$ (G) are *all* the possible extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$, thereby completing the proof of Theorem 5.1.4. To do so, we consider the dual cone of $\mathbb{S}^p_{\succeq 0}(G)$, namely the cone of all PSD-completable matrices, which we denote by $\mathcal{C}_G$. Recall that a *(real)* $p \times p$ *partial matrix* $A = [a_{ij}]$ is a matrix in which some entries are specified real numbers and the remainder are unspecified. It is called *symmetric* if all the specified entries satisfy $a_{ij} = a_{ji}$, and it is called *PSD-completable* if there exists a specification of the unknown entries of A that produces a matrix $\widetilde{A} \in \mathbb{S}^p$ that is positive semidefinite. It is well-known that the dual cone to $\mathbb{S}^p_{\succeq 0}(G)$ is the cone $\mathcal{C}_G$ of all PSD-completable matrices. Let H be an induced subgraph of G, and let $A[H]$ denote the submatrix of $A \in \mathbb{S}^m$ whose rows and columns are indexed by the vertices of H. A symmetric partial matrix A is called *(weakly) cycle-completable* if the submatrix $A[C_m] \in \mathbb{S}^m$ is PSD-completable for every chordless cycle C_m in G.

Proof of Theorem 5.1.4

By Theorem 5.1.3, G has the facet-ray identification property, and the extreme matrices in $\mathbb{S}^p_{\succeq 0}(G)$ identified by the facets of $\mathrm{cut}^{\pm 1}$ (G) are of rank 1 and $m-2$, where m varies over the length of all chordless cycles in G. So it only remains to show that these are all the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$. To do so, we consider the dual cone to $\mathbb{S}^p_{\succeq 0}(G)$.

In [4] it is shown that a symmetric partial matrix A is in the cone $\mathcal{C}_G$ if and only if A is cycle completable. Since $\mathcal{C}_G$ is the dual cone to $\mathbb{S}^p_{\succeq 0}(G)$ it follows that $A \in \mathcal{C}_G$ if and only $\langle A, X\rangle \geq 0$ for all $X \in \mathbb{S}^p_{\succeq 0}(G)$. Applying this duality, we see that the matrix A satisfies $\langle A, X\rangle \geq 0$ for all $X \in \mathbb{S}^p_{\succeq 0}(G)$ if and only if $\langle A[C_m], X\rangle \geq 0$ for all extremal matrices $X \in \mathbb{S}^m_{\succeq 0}(C_m)$ for all chordless cycles C_m in G. Here, we think of the matrices $A[C_m]$ and $X \in \mathbb{S}^m_{\succeq 0}(C_m)$ as living in $\mathbb{S}^p$ by extending the matrices $A[C_m]$ and X in $\mathbb{S}^{V(C_m)}$ by placing zeros in the entries corresponding to edges not in the chordless cycle C_m. It follows from this that the cone $\mathcal{C}_G$ is dual to $\mathbb{S}^p_{\succeq 0}(G)$ and the cone whose extremal rays are given by the chordless cycles in G. Thus, these two cones must be the same, and we conclude that the only possible ranks of the extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$ are those given by the ranks of $\mathbb{S}^m_{\succeq 0}(C_m)$ as C_m varies over all chordless cycles in G. This completes the proof of Theorem 5.1.4. $\square$

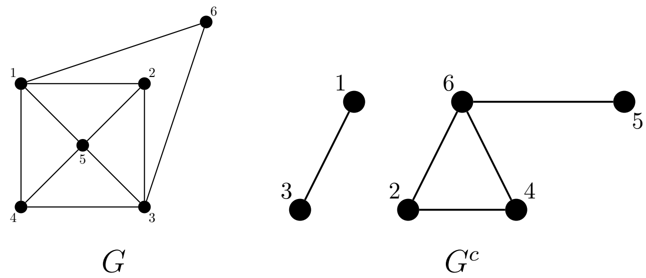


Figure 5.4: The graph G from Example 5.5.2 and its complement G^c.

Some further examples.

Theorem 5.1.4 provides a subcollection of the graphs with no K_5 minors for which the facets of $\mathrm{cut}^{\pm 1}$ (G) characterize all extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$, namely those which also have no K_4 minors. It is then natural to ask whether or not the extreme ranks of the graphs with K_4 minors but no K_5 minors are characterized by the facets as well. The following two examples address this issue. Example 5.5.1 is an example of a graph G with a K_4 minor but no K_5 minor for which the facets do not characterize all extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$, and Example 5.5.2 is an example of a graph G with a K_4 minor but no K_5 minor for which the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ are characterized by the facets of $\mathrm{cut}^{\pm 1}$ (G).

Example 5.5.1. Consider the complete bipartite graph $G := K_{3,3}$. In [21] the extremal rays of $\mathbb{S}^p_{\succeq 0}(G)$ are characterized, and it is shown that G has extremal rays of ranks $1, 2$, and 3. However, with the help of Polymake [20] we see that the facet-supporting hyperplanes of $\mathrm{cut}^{\pm 1}$ (G) are $x_e = \pm 1$ for each edge $e \in E(G)$ together with $\langle v^F, x \rangle = m - 2$ as C_m varies over the nine (chordless) 4-cycles within G. Thus, the constant terms of the facet-supporting hyperplanes only capture extreme ranks 1 and 2, but not 3.

Example 5.5.2. Consider the graph G depicted in Figure 5.5.3. Recall that a k-block is a graph P with order k that has no proper induced subgraph with order k. Agler et al. characterized all 3-blocks in [1, Theorem 1.5] in terms of their complements. It follows immediately from this theorem that G contains no induced 3-block. Thus, $\mathrm{ord}(G) \leq 2$, and since G is not a chordal graph we see that $\mathrm{ord}(G) = 2$. By Theorem 5.1.4 the facets of $\mathrm{cut}^{\pm 1}$ (G) identify extremal rays of rank 1 and 2. Thus, all possible extremal ranks of G are characterized by the facets of $\mathrm{cut}^{\pm 1}$ (G).

The reader may also notice that the graph G from Example 5.5.2 also has no $K_{3,3}$ minor, but the graph from Example 5.5.1 is $K_{3,3}$. Thus, it is natural to ask if the collection of graphs for which the facets characterize the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ are those with no $K_{3,3}$ minor. The following example says that this is not the case.

Example 5.5.3. Consider the following graph G and its complement G^c:

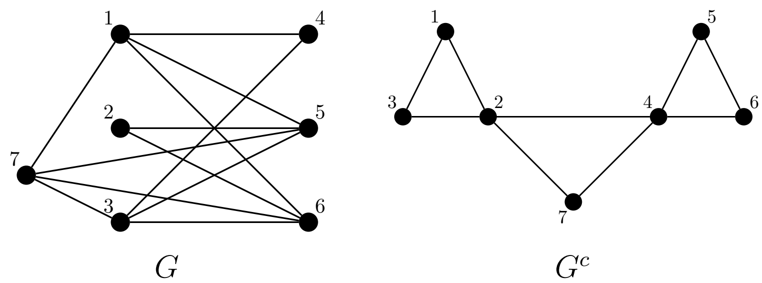
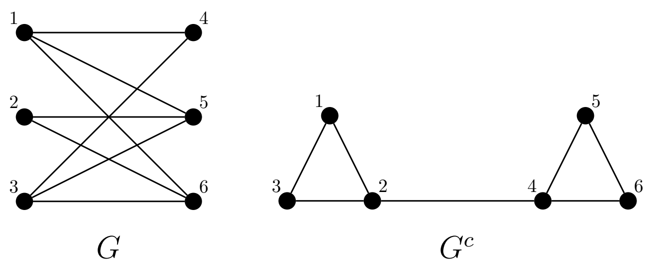


Figure 5.5: A graph with a K_5 minor whose facets characterize all extremal rays.



Notice that G contains no $K_{3,3}$ minor, but it does contain a K_4 minor. By [1, Theorem 1.5] G is a 3-block since its complement graph is two triangles connected by an edge. Thus, G has an extremal ray of rank 3, but by Theorem 5.1.3 they facets of $\text{cut}^{\pm 1}(\text{G})$ only detect extremal rays of ranks 1 and 2.

Examples 5.5.1, 5.5.2, and 5.5.3 collectively show that describing the collection of graphs for which the facets of $\text{cut}^{\pm 1}(\text{G})$ characterize the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ is more complicated that forbidding a particular minor. Indeed, the collection of graphs with this property is not even limited to the graphs with no K_5 minors, as demonstrated by Example 5.5.4.

Example 5.5.4. Consider the graph G depicted in Figure 5.5. To see that this graph has the facet-ray identification property we first compute the 114 facets of $\text{cut}^{\pm 1}(\text{G})$ using Polymake [20]. The resulting computation yields 72 cycle inequalities, 16 for the three 3-cycles, and 56 for the seven chordless 4-cycles, as well as eight inequalities for the four edges not in a 3-cycle. These 80 facets identify extremal rays of ranks 1 and 2 just as in the case of the graphs with no K_5 minors. The remaining 64 facet-supporting inequalities of $\text{cut}^{\pm 1}(\text{G})$ are given by applying the *switching operation* defined in [17, Chapter 27] to the inequality

$$x_{14} - x_{15} - x_{34} - x_{36} - x_{37} - x_{67} + x_{16} + x_{17} + x_{25} + x_{26} + x_{35} + x_{57} \leq 4.$$

This new collection of facets identifies extremal rays of rank 3. For example, the presented inequality specifies the off-diagonal entries of the following rank 3 matrix:

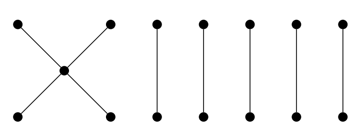
$$\begin{pmatrix} 2 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 1 & 0 & 0 & 2 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 1 \end{pmatrix}.$$

This matrix has the 3-dimensional Gram representation

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, u_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$
$$u_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, u_6 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, u_7 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

It follows via an application of Theorem 5.2.1 that this matrix is extremal in $\mathbb{S}^p_{\succeq 0}(G)$. Similar matrices can be constructed for each of the 64 facets of this type. Thus, G has the facet-ray identification property, and the facets identify extreme rays of rank $1, 2$, and 3.

To see that these are all of the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ recall from Section 5.2 that since G has 7 vertices then $\mathrm{ord}(G) \leq 5$ with equality if and only if G is the cycle on 7 vertices. Thus, it only remains to show that $\mathrm{ord}(G) \neq 4$. To see this, notice that the complement of G depicted in Figure 5.5. By [24, Theorem 0.2] G is not a 4-superblock since the complement of G can be obtained by identifying the vertices of the graphs



Thus, if G as rank 4 extremal rays then it must contain an induced 4-block. However, it can be checked that all induced subgraphs either have order 1, 2, or 3. Therefore, G is a graph with a K_5 minor that has the facet-ray identification property and for which the extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$ are characterized by the facets of $\mathrm{cut}^{\pm 1}$ (G). Moreover, this example shows that the types of facets which identify extreme rays of $\mathbb{S}^p_{\succeq 0}(G)$ are not limited to those arising from edges and chordless cycles.

We end this section with a problem presented by these various examples.

Problem 5.5.5. Determine all graphs G with the facet-ray identification property for which the facets of $\mathrm{cut}^{\pm 1}$ (G) characterize all extremal ranks of $\mathbb{S}^p_{\succeq 0}(G)$.

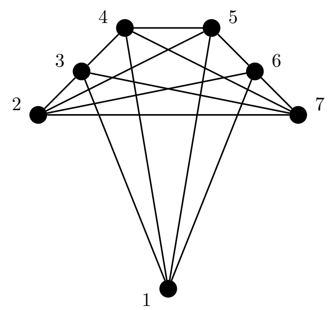


Figure 5.6: The parachute graph on 7 vertices.

5.6 Graphs Without the Facet-Ray Identification Property

In the previous sections we discussed various graphs G which have the facet-ray identification property. Here, we provide an explicit example showing that not all graphs admit the facet-ray identification property.

Example 5.6.1. Consider the *parachute graph* on 7 vertices depicted in Figure 5.6. Using Polymake [20] we compute the facets of $\mathrm{cut}^{\pm 1}$ (G) to see that

$$x_{13} + x_{14} + x_{15} + x_{16} + x_{25} + x_{26} + x_{27} + x_{37} + x_{47} - x_{23} - x_{34} - x_{45} - x_{56} - x_{67} \leq 4$$

is a facet-defining inequality. Thus, if G has the facet ray identification property there exists a filling of the partial matrix

$$M := \begin{pmatrix} x_1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & x_2 & -1 & 0 & 1 & 1 & 1 \\ 1 & -1 & x_3 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & x_4 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & x_5 & -1 & 0 \\ 1 & 1 & 0 & 0 & -1 & x_6 & -1 \\ 0 & 1 & 1 & 1 & 0 & -1 & x_7 \end{pmatrix}.$$

that results in a positive semidefinite matrix which is extremal in $\mathbb{S}^p_{\succeq 0}(G)$. Notice that the minimum rank of a positive semidefinite completion of M is 5. To see this, recall that if the rank$(M) < 5$ then the point $(x_1, x_2, \ldots, x_7)$ must lie on the variety of the ideal I generated by the 5×5 minors of M. Using Macaulay2, we see that the minimal generating set for the ideal I includes the generator $x_1 + x_2 + \ldots + x_7 + 10$. If M is positive semidefinite then $x_i \geq 0$ for all $1 \leq i \leq 7$, and so $(x_1, \ldots, x_7)$ cannot be a point in the variety of the ideal I.

On the other hand, the maximum dimension of the frame space

$$\mathrm{span}_{\mathbb{R}}\left(U_{\overline{E}}\right) = \mathrm{span}_{\mathbb{R}}(u_i u_j^T + u_j u_i^T : ij \in \overline{E})$$

for any k-dimensional Gram representation of G is at most the number of nonedges of G, which is seven. By Theorem 5.2.1, since $7 < \binom{5+1}{2} - 1$ no positive semidefinite completion of M can be extremal in $\mathbb{S}^p_{\succeq 0}(G)$. Thus, G does not have the facet-ray identification property.

The facet-defining inequality considered in Example 5.6.1 has been studied before as a facet-defining inequality of the cut polytope of the complete graph K_7 by Deza and Laurent [17], and is referred to as a *parachute inequality.* Thus, one consequence of the above example is that K_7 also does not have the facet-ray identification property, nor does any G for which the above inequality is facet-defining. This suggests that one way to determine the collection of graphs which have the facet-ray identification property is to study those facets which can never identify an extremal matrix in $\mathbb{S}^p_{\succeq 0}(G)$, i.e. determine "forbidden facets" as opposed to forbidden minors.

Problem 5.6.2. Determine facet-defining inequalities of $\mathrm{cut}^{\pm 1}(\mathrm{G})$ that can never identify extremal matrices in $\mathbb{S}^p_{\succeq 0}(G)$.

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Vita

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