2016

Inverse Scattering For The Zero-Energy Novikov-Veselov Equation

Michael Music
University of Kentucky, estill.music@gmail.com
Digital Object Identifier: http://dx.doi.org/10.13023/ETD.2016.342

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Michael Music, Student

Dr. Peter A. Perry, Major Professor

Dr. Peter Hislop, Director of Graduate Studies

Inverse Scattering For The Zero-Energy Novikov-Veselov Equation

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Michael Music
Lexington, Kentucky

Director: Dr. Peter A. Perry, Professor of Mathematics
Lexington, Kentucky 2016

ABSTRACT OF DISSERTATION

Inverse Scattering For The Zero-Energy Novikov-Veselov Equation

For certain initial data, we solve the Novikov-Veselov equation by the inverse scattering method. This is a (2+1)-dimensional completely integrable system that generalizes the (1+1)-dimensional Korteweg-de-Vries equation. The method used is the inverse scattering method. To study the direct and inverse scattering maps, we prove existence and uniqueness properties of exponentially growing solutions of the two-dimensional Schrödinger equation. For conductivity-type potentials, this was done by Nachman in his work on the inverse conductivity problem. Our work expands the set of potentials for which the analysis holds, completes the study of the inverse scattering map, and show that the inverse scattering method yields global in time solutions to the Novikov-Veselov equation. This is the first proof that the inverse scattering method yields classical solutions to the Novikov-Veselov equation for the class of potentials considered here.

KEYWORDS: inverse scattering, Novikov-Veselov equation, Schrödinger equation

Author's signature: Michael Music

Date: July 28, 2016

Inverse Scattering For The Zero-Energy Novikov-Veselov Equation

By
Michael Music

Director of Dissertation: Peter A. Perry

Director of Graduate Studies: Peter Hislop

Date: July 28, 2016

For my sister Mary Music Franey.

ACKNOWLEDGMENTS

I would like to thank my adviser, Dr. Perry, for all of the help and guidance he has given me and the great colleagues he has introduced me to. I also thank my committee members Drs. Russell Brown, John Lewis, James McDonough, and Alfred Shapere. I also thank Dr. Lewis for all of the times we have talked and the encouragement he has given me. Finally, I thank Dr. Samuli Siltanen for his wonderful support.

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Chapter 1 Introduction

The Novikov-Veselov equation at energy zero was first derived by Novikov and Veselov in 1984 as part of a larger hierarchy of completely integrable equations which generate families of explicit solutions to the two-dimensional Schrödinger equation. It is a (2+1)-dimensional generalization of the (1+1)-dimensional Korteweg-de Vries (KdV) equation

$$u_\tau + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

which generate isospectral flows for the one-dimensional Schrödinger equation. The KdV equation was the first example of an equation that admitted soliton solutions. Soliton solutions are traveling wave solutions where the dispersion, u_{xxx}, is exactly balanced by the nonlinearity, $6uu_x$. The simplest solutions of this kind are given by

$$u(x,\tau) = \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - c\tau)\right].$$

There are also multi-soliton solutions where the individual solitons traveling at different speeds pass through each other with only a small change in phase [10].

Associated with the KdV equation are two operators that form a Lax pair:

$$L = -\partial_x^2 + u(x,\tau), \qquad A = \partial_x^3 - \frac{3}{4}(u(x,\tau)\partial_x + \partial_x u(x,\tau)).$$

The KdV equation is equivalent to the compatibility condition

$$\partial_\tau L = [L, A] \tag{1.2}$$

where $[L, A] = LA - AL$ is the commutator. We may solve the KdV equation by examining the spectral data of the Schrödinger equation. Under the KdV flow, if we evolve a solution to $L\phi(x, 0) = 0$ according to the equation

$$\partial_\tau \phi(x,\tau) = -A\phi(x,\tau)$$

then $\phi(x,\tau)$ will continue to be solutions to $L\phi(x,\tau) = 0$ for all time. The evolution follows from taking derivatives of $L\phi(x,\tau)$ and applying the compatibility condition (1.2):

$$(L\phi)_\tau = L_\tau\phi + L\phi_\tau = L(A\phi + \phi_\tau) = 0.$$

The Novikov-Veselov equation generalizes KdV because the spectral problem is the 2-dimensional Schrödinger equation.

Manakov showed that the there are no nontrivial 2-dimensional equations described with a Lax pair [20]. The proper generalization is a Manakov triple. Novikov and Veselov derived the NV equation as the compatibility condition of a Manakov triple where the operator L is the 2-dimensional stationary Schrödinger operator.

The operators ∂_x and $\overline{\partial}_x$ are given by

$$\partial_x = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \qquad \text{and} \qquad \overline{\partial}_x = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

The Manakov triple for the NV equation is

$$L = -\partial_x\overline{\partial}_x + q + E, \qquad A = \partial_x^3 + \overline{\partial}_x^3 + u\partial + \bar{u}\overline{\partial}_x, \qquad B = \partial_x u + \overline{\partial}_x \bar{u},$$

and the NV equation is equivalent to the compatibility condition

$$\partial_\tau L = [L, A] + BL. \tag{1.3}$$

Written out, equation (1.3) gives the Novikov-Veselov equation at energy E:

$$\begin{cases} \partial_\tau q = \partial_x^3 q + \overline{\partial}_x^3 q + 4\partial(uq) + 4\overline{\partial}_x(\bar{u}q) - E(\partial_x q + \overline{\partial_x q}) \\ \partial_x q = \overline{\partial}_x u. \end{cases} \tag{1.4}$$

For the rest of the paper, we write $x \in \mathbb{R}^2$ as (x_1, x_2), but we also will freely treat it as a complex number $x = x_1 + ix_2$ when it is convenient. The function u in the Novikov-Veselov equation is not well defined since $\overline{\partial}_x$ has a kernel consisting of all analytic functions. We will choose u such that

$$\lim_{|x|\to\infty} u(x, \tau) = 0. \tag{1.5}$$

With the Manakov triple in mind, there should be a set of functions that solve the Schrödinger equation $L\phi = 0$ and evolve in time according to a prescribed equation as in the case of KdV. To see this, we take the derivative of $L\phi$

$$(L\phi)_\tau = L_\tau\phi + L\phi_\tau = L(\phi_\tau + A\phi) = 0.$$

Therefore, if q evolves according to the NV equation, then $\phi(x, \tau)$ should satisfy

$$\partial_\tau\phi = -A\phi.$$

The inverse scattering method involves finding a suitable set of solutions to the Schrödinger equation, evolving these solutions, and then finding a map from these solutions to the potential.

In 1987, Boiti, Leon, Manna, and Pempinelli developed the inverse scattering method for the Novikov-Veselov equation [3]. The first step of the inverse scattering method is to construct the Complex Geometric Optics (CGO) solutions to $L\phi = 0$.The next step is to use these solutions to construct the scattering data $\mathcal{T}[q] : \mathbb{R}^2 \mapsto \mathbb{C}$. The data $\mathcal{T}[q]$ has a linear evolution (1.15) when q evolves according to the nonlinear NV equation. The last step of the inverse scattering method is to take the evolved scattering data and reconstruct the CGO solutions and the evolved q.

In this paper, we will present a complete proof that the inverse scattering method yields classical solutions to the zero-energy ($E = 0$) Novikov-Veselov equation for a large set of potentials. At $E = 0$, the NV equation is

$$\begin{cases} \partial_\tau q = \partial_x^3 q + \overline{\partial}_x^3 q + 4\partial_x(uq) + 4\overline{\partial}_x(\bar{u}q) \\ \partial_x q = \overline{\partial}_x u. \end{cases} \tag{1.6}$$

For the forward scattering problem, we let $q(x) = q(x, 0)$ be the initial data for the NV equation. Let q be in $L^p(\mathbb{R}^2)$ for $p \in (1, 2)$. The set of solutions to the Schrödinger equation we will use are the Complex Geometric Optics (CGO) solutions, $\phi(x, k)$, originally described by Faddeev [9]. These functions satisfy

$$\begin{cases} (-\overline{\partial}_x \partial_x + q(x))\phi(x, k) = 0 \\ e^{-ikx}\phi(x, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2) \end{cases} \tag{1.7}$$

where $k \in \mathbb{C} \setminus \{0\}$ is a complex parameter, xk is calculated through complex multiplication, and $1/\tilde{p} = 1/p - 1/2$. This implies that $\exp(-ikx)$ is exponentially growing in some directions and oscillating in others. We call points k for which there do not exist unique solutions to (1.7) **exceptional points** and $\mathcal{E}$ is the set of exceptional points.

In 1996, Nachman proved the exceptional set is empty for a special set of potentials called conductivity-type [26]. Conductivity-type potentials are potentials in $L^p(\mathbb{R}^2)$ with the property that the equation $\overline{\partial}_x \partial_x \phi = q\phi$ has a bounded, strictly positive solution. We will generalize Nachman's results to the more general class of potentials with prescribed decay for which the associated Schrödinger operator is nonnegative. We say that $-\overline{\partial}_x \partial_x + q \geq 0$ when

$$\int_{\mathbb{R}^2} \frac{1}{4}|\nabla\psi|^2 + q|\psi|^2 dm \geq 0 \tag{1.8}$$

for every $\psi \in C_c^\infty(\mathbb{R}^2)$.

In order for unique solutions to (1.7) to exist, we require q to be in a weighted L^p space. We define $L^p_\rho(\mathbb{R}^2)$ by

$$L^p_\rho(\mathbb{R}^2) = \{f : \langle x\rangle^\rho f \in L^p(\mathbb{R}^2)\}.$$

Let q be in $L^p_\rho(\mathbb{R}^2)$. Murata, in 1986 [21], proved q satisfies (1.8) if and only if there exists some positive solution to the Schrödinger equation. Conductivity-type potentials, called critical by Murata, are the subset of $L^p_\rho(\mathbb{R}^2)$ potentials for which the the positive solutions are bounded. If the positive solution is unbounded, it is called a subcritical potential. We give more detail about the theory developed by Murata in Section 1.3.

We rescale solutions to (1.7) to have nicer behavior at infinity. Setting $\mu(x, k) = e^{-ikx}\phi(x, k)$ gives us the modified equation

$$\begin{cases} \overline{\partial}_x(\partial_x + ik)\mu(x, k) = q(x)\mu(x, k) \\ \mu(\,\cdot\,, k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2) \end{cases} \tag{1.9}$$

where

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}.$$

Faddeev's Green function, g_k is the fundamental solution kernel for (1.9) defined by

$$g_k(x) = \frac{1}{\pi^2} \int_{\mathbb{R}^2} \frac{e^{i(\xi x + \bar{\xi}\bar{x})}}{\bar{\xi}(\xi + k)} dm(\xi). \tag{1.10}$$

This is the Fourier transform of a function in $L^r(\mathbb{R}^2)$ for $r \in (1,2)$, and therefore $g_k \in L^{r'}(\mathbb{R}^2)$. We have

$$\overline{\partial}_x(\partial_x + ik)g_k(x) = \delta(x)$$

where the distribution $\delta(x)$ is the Dirac delta function.

Any solution of equation (1.9) will satisfy the integral equation

$$\mu(x,k) = 1 + g_k * (q\mu(x,k)). \tag{1.11}$$

Taking a large-x expansion of g_k (see [30, Theorem 3.11]), we find that

$$g_k(x) = -\frac{1}{\pi}\left(\frac{1}{ikx} + \frac{e_{-k}(x)}{i\bar{k}\bar{x}}\right) + O(|x|^{-2}) \tag{1.12}$$

where

$$e_k(x) = e^{i(kx + \bar{k}\bar{x})}.$$

Substituting (1.12) into (1.11), we formally have a large-x asymptotic for $\mu(x,k)$:

$$\mu(x,k) = 1 + \frac{\mathbf{h}(k)}{\pi ikx} - \frac{e_{-k}(x)\mathbf{t}(k)}{\pi i\bar{k}\bar{x}} + O(|x|^{-2}).$$

The functions $\mathbf{h}$ and $\mathbf{t}$ are given by the integrals

$$\mathbf{h}(k) = \int_{\mathbb{R}^2} q(x)\mu(x,k)dm(x) \tag{1.13}$$

and

$$\mathcal{T}[q](k) = \mathbf{t}(k) = \int_{\mathbb{R}^2} e_k(x)q(x)\mu(x,k)dm(x). \tag{1.14}$$

The function $\mathbf{t}(k)$ is the inverse scattering data, and the map $\mathcal{T}$ is the forward scattering map.

If $q(x,\tau)$ solves the NV equation, Boiti, Leon, Manna, and Pempinelli [3] showed that we get the following evolution equations for $\mathbf{h}$ and $\mathbf{t}$:

$$\partial_\tau \mathbf{t} = i(k^3 + \bar{k}^3)\mathbf{t}$$
$$\partial_\tau \mathbf{h} = 0.$$

We see that $\mathbf{h}$ is a conserved quantity under the NV flow, and $\mathbf{t}$ has the evolution

$$\mathbf{t}(k,\tau) = e^{i\tau(k^3 + \bar{k}^3)}\mathbf{t}(k,0). \tag{1.15}$$

Now that we have the inverse scattering data and the evolution of this data, we need a method of reconstructing $\mu(x,k,\tau)$ and $q(x,\tau)$ from this data. We have

$$\overline{\partial}_k \frac{1}{\xi - k} = \pi\delta(\xi),$$

so taking the $\overline{\partial}_k$ derivative of Faddeev's Green function gives us

$$\overline{\partial}_k g_k(x) = e_{-x}(k)\frac{1}{\pi\bar{k}}.$$

If we substitute this derivative into the integral equation (1.11) we get

$$\overline{\partial}_k \mu = e_{-x}(k)\frac{\mathbf{t}(k)}{\pi\bar{k}} + g_k * (q\overline{\partial}_k\mu). \tag{1.16}$$

We have the identity $g_k * (e_{-x}\bar{f}) = e_{-x}\overline{g_k * f}$. Taking the complex conjugate of equation (1.11) and multiplying by $e_{-x}(k)$ gives us

$$\begin{aligned} e_{-x}(k)\overline{\mu} - e_{-x}(k)\overline{g_k * q\mu} &= e_{-x}(k) \\ [I - g_k * (q\,\cdot)]e_{-x}\bar{\mu} &= e_{-x}(k). \end{aligned} \tag{1.17}$$

Using identity (1.17), the integral equation (1.16) becomes

$$\overline{\partial}_k \mu = e_{-x}(k)\frac{\mathbf{t}(k)}{\pi\bar{k}}\overline{\mu}.$$

Therefore, instead of evolving q under NV and solving the Schrödinger equation, we may solve the $\overline{\partial}_k$-equation

$$\begin{cases} \overline{\partial}_k \mu(x,k,\tau) = e_{-x}(k)e^{i\tau(k^3+\bar{k}^3)}\frac{\mathbf{t}(k)}{\pi\bar{k}}\overline{\mu} \\ \mu(x,\,\cdot\,,\tau) - 1 \in L^r(\mathbb{C}) \end{cases} \tag{1.18}$$

to recover μ. We will reconstruct q from large-k asymptotics of μ. Define

$$\mathbf{s}(k) = \frac{\mathbf{t}(k)}{\pi\bar{k}}.$$

The existence of solutions to (1.18) depends on the $L^p(\mathbb{C})$ space properties of the coefficient $\mathbf{s}$. For the critical potentials, Nachman proved that $\mathbf{s}$ is in $L^{p_1} \cap L^{p_2}(\mathbb{R}^2)$ for $p_1 \in (1,2)$ and $p_2 \in (2,\infty)$. This is the classical space needed to get uniqueness in the $\overline{\partial}_k$-problem [36]. For subcritical potentials, $\mathbf{s}(k)$ will be in $L^2(\mathbb{R}^2)$, but the theory still works out due to results of Brown and Uhlmann [5] that we present in Theorem 1.2.6.

We wish to write (1.18) as an integral equation. If $f \in L^p(\mathbb{R}^2)$ for $p \in (1,2)$, we may define the Cauchy integral, an inverse of the $\overline{\partial}_k$ operator, by

$$\overline{\partial}_k^{-1} f = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{1}{k - k'} f(k')\,dm(k'). \tag{1.19}$$

Using the $\overline{\partial}_k^{-1}$ operator, we rewrite the $\overline{\partial}_k$ equation as an integral equation

$$\mu(x,k,\tau) - 1 = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{k-k'} e_{-x}(k') e^{i\tau(k'^3+\bar{k}'^3)} \mathbf{s}(k') \overline{\mu(x,k',\tau)} dm(k'). \tag{1.20}$$

Assuming $|k|^n \mathbf{s}(k) \in L^{r_1} \cap L^{r_2}(\mathbb{R}^2)$ for $r_1 \in (1,2)$ and $r_2 \in (2,\infty)$, we may use the expansion

$$\frac{1}{k-k'} = \sum_{j=1}^{n} \frac{k'^{j-1}}{k^j} + \frac{k'^n}{k^n(k-k')} \tag{1.21}$$

in equation (1.20) to obtain the large-k expansion of $\mu(x,k,\tau)$:

$$\mu(x,k,\tau) = 1 + \sum_{j=1}^{n} \frac{a_j(x,\tau)}{k^j} + o(|k|^{-n}). \tag{1.22}$$

We can recover $q(x,\tau)$ by plugging (1.22) into the Schrödinger equation (1.9):

$$\begin{aligned} \lim_{|k|\to\infty} q\mu &= \lim_{|k|\to\infty} \overline{\partial}_x(\partial_x + ik)\mu \\ q &= i\overline{\partial}_x a_1. \end{aligned}$$

However, this assumes that $\mu(x,k,\tau)$ constructed from the $\overline{\partial}_k$-equation (1.18) solves the Schrödinger equation for all τ.

We would show that μ solves the Schrödinger equation by commuting the operator $-\overline{\partial}_x(\partial_x + ik) + q$ through (1.18):

$$\begin{aligned} [-\overline{\partial}_x(\partial_x + ik) + q]\overline{\partial}_k\mu &= [-\overline{\partial}_x(\partial_x + ik) + q] e_{-x}(k) \frac{\mathbf{t}(k,\tau)}{\pi\bar{k}} \overline{\mu} \\ \overline{\partial}_k[-\overline{\partial}_x(\partial_x + ik) + q]\mu &= e_{-x}(k) \frac{\mathbf{t}(k,\tau)}{\pi\bar{k}} \overline{[-\overline{\partial}_x(\partial_x + ik) + \bar{q}]\mu}. \end{aligned}$$

The Liouville theorem by Brown and Uhlmann [5] says that if $u \in L^r(\mathbb{C})$ for $r \in (2,\infty)$ is a solution to

$$\overline{\partial}_x u = au + b\bar{u}$$

then $u \equiv 0$. If we know a priori that $q(x,\tau) = \overline{q(x,\tau)}$, we would have μ solves $\overline{\partial}_x(\partial_x + ik)\mu = q\mu$.

Grinevich and Manakov worked out symmetries for the nonzero-energy scattering transform when the Schrödinger equation has a special form [13, Theorem 2]. They consider the problem of reconstructing the full Schrödinger operator with a magnetic field through the inverse scattering method. This operator is

$$-\overline{\partial}_x\partial_x + iA(x)\partial_x + B(x) + E,$$

and there are two symmetries that appear in the data assuming $A \equiv 0$ and $B = \bar{B}$. In the zero energy limit their symmetry becomes $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$. We prove in Lemma 4.3.1 that if the scattering data has the symmetry then $\mathcal{Q}[\mathbf{t}]$ is real.

To get an integral form for the $a_j(x,\tau)$ (and therefore $q(x,\tau)$), we plug the expansion (1.21) into (1.20) to obtain

$$a_j(x,\tau) = \frac{1}{\pi}\int_{\mathbb{C}} k'^{j-1} e_{-x}(k') e^{i\tau(k'^3+\bar{k}'^3)} \mathbf{s}(k')\overline{\mu(x,k',\tau)}dm(k'). \tag{1.23}$$

Therefore, a reconstruction formula for $q(x,\tau)$ is

$$\mathcal{Q}[\mathbf{t}](x,\tau) = q(x,\tau) = \frac{i}{\pi}\overline{\partial}_x \int e_{-x}(k)e^{i\tau(k^3+\bar{k}^3)}\mathbf{s}(k)\overline{\mu(x,k,\tau)}dm(k). \tag{1.24}$$

All these steps together make up the inverse scattering method for solving the Novikov-Veselov equation:

$$\begin{array}{ccc} \mathbf{t}(k,0) & \xrightarrow[e^{i\tau(k^3+\bar{k}^3)}]{\text{Multiplication by}} & \mathbf{t}(k,\tau) \\ \tau\Big\uparrow & & \Big\downarrow\mathcal{Q} \\ q(x,0) & \xrightarrow{\text{NV Evolution}} & q(x,\tau) \end{array} \tag{1.25}$$

We will prove that the inverse scattering method produces classical solutions to the NV equation for critical or subcritical initial data. The method here is from the paper of Music and Perry in [23] and is the first formal proof that uses the inverse scattering method directly for any of these potentials.

Theorem 1.0.1. *Given $q(x,0) \in W^{5,p}_\rho(\mathbb{R}^2)$ with $p \in (1,2)$, $\rho \in (2/p',\infty)$, and $-\overline{\partial}_x\partial_x + q(x,0) \geq 0$, $q(x,\tau)$ obtained from the inverse scattering method* (1.25) *is a global classical solution to the Novikov-Veselov equation with initial data $q(x,0)$.*

We prove the theorem using the large-k expansion (1.22) and finding a set of identities among the coefficients $a_j(x,\tau)$. We then find an evolution equation for $\mu(x,k,\tau)$ and use this to prove that the reconstructed $q = i\overline{\partial}_x a_1$ solves the Novikov-Veselov equation. The requirement that $q(x,0) \in W^{5,p}_\rho(\mathbb{R}^2)$ guarantees that $q(x,0) \in C^3(\mathbb{R}^2)$. This is necessary because we use classical derivatives in the method and not weak derivatives.

There are examples where it is known that inverse scattering method from diagram (1.25) fails completely. Taimanov and Tsarev constructed supercritical rational potentials with explicit formulas for the scattering data and CGO solutions using the Moutard transform [34, Section 3]. The solutions $q(x,\tau)$ to the NV equation with this initial data are smooth, have decay $q(x,0) \leq c\langle x\rangle^{-3}$, and blow up in finite time. To us, the most interesting feature is that the scattering transform of Taimanov and Tsarev's initial data is identically zero. R.G. Novikov and Grinevich calculated the scattering transforms for point potentials and found that their scattering transforms will sometimes have a circle of singularities [14]. Music, Perry, and Siltanen calculated perturbations of radially symmetric critical potentials and found the same circle of singularities when the perturbations are supercritical [24].

There are various groups who have worked on formalizing the zero-energy inverse scattering method for different potentials. Tsai in 1993 developed the small data theory for the inverse scattering method in an attempt to solve the zero-energy Novikov-Veselov equation rapidly decaying initial data [35] but did not know that the spectral condition (1.8) was required.

In 2007, Lassas, Mueller, and Siltanen continued from Nachman's work and showed that for certain smooth, compactly supported critical initial data the inverse scattering map $\mathcal{Q}[\mathbf{t}]$ is well-defined for all time and the reconstructed solution satisfies the estimate $|q_\tau(z)| \leq C(1+|x|)^{-2}$ [18]. Together with Stahel, they later showed that radially symmetric initial data in the same class stay critical for all time when the scattering data is evolved according to (1.15) [19].

In 2014, Perry [29] was the first to show that the inverse scattering method yields solutions of the NV equation at zero-energy for critical initial data. His method does not extend to subcritical initial data. As we shall see in Section 3.2, critical initial data is the boundary of the set of data we treat in the present work.

In addition, Angelopoulos used PDE techniques to show that the Novikov-Veselov equation is locally well-posed for initial data in $H^s(\mathbb{R}^2)$ with $s > 1/2$ [1].

For history and results on nonzero-energy inverse scattering, we direct the reader to the review articles by R.G. Novikov [27], Grinevich [12], and to the thesis of Kazeykina [17]. The review article of Croke, Mueller, Music, Perry, Siltanen, and Stahel contains a broad overview for the zero-energy Novikov-Veselov equation [7].

In the rest of this chapter, we gather some notation that will be used throughout the text, present theorems from Murata's work on positive solutions to the Schrödinger equation, and then discuss some key theorems from Nachman's work on the inverse scattering method.

In Chapter 2, we show that if q is subcritical, then the Schrödinger equation 1.9 has a unique solution for all $k \neq 0$ and these solutions solve the $\overline{\partial}_k$-equation (1.18). The uniqueness of the solutions in Section 2.1 is the largest technical hurdle here. The rest of the proofs in the chapter are adapted from Nachman [26] to deal with subcritical potentials. In some cases, we are able to prove the same results for $q \in L^p_\rho(\mathbb{R}^2)$ where ρ is smaller than was assumed by Nachman [26]. In Section 2.2, we prove the existence and decay of the CGO solutions for subcritical potentials. In Section 2.3, we show that the CGO solutions satisfy the $\overline{\partial}_k$ equation.

In Chapter 3, we encounter a difference in the behavior of the scattering data $\mathbf{t}(k)$ for subcritical potentials as opposed to critical potentials. For critical potentials, the inequality $\mathbf{t}(k) \leq c|k|^\epsilon$ holds near $k = 0$, and Nachman use this decay to prove that $\mathbf{s}$ belongs to $L^{p_1} \cap L^{p_2}(\mathbb{C})$ for $p_1 \in (1,2)$ and $p_2 \in (2,\infty)$. For subcritical potentials, we will show in Section 3.1 that near the origin $\mathbf{t}(k)$ satisfies

$$\mathbf{t}(k) = \frac{\pi a}{2(c_\infty - a(\gamma + \log|k|))} + O(|k|^\epsilon)$$

where γ is the Euler-Mascheroni constant and a and c_∞ are constants derived from the positive solution to the Schrödinger equation with potential q. This implies that $\mathbf{s}$ will only belong to $L^2(\mathbb{C})$. We are still able to solve the $\overline{\partial}_k$-equation even with these $L^2(\mathbb{C})$ coefficients. In Section 3.2, we prove that the subcritical potentials are an open set in the $L^p_\rho(\mathbb{R}^2)$ topology. In Section 3.3, we prove that if q has additional regularity then $\mathbf{s}$ has additional decay. Finally, in Section 3.4 we prove that if $q(x)$ is real than $\mathbf{t}$ has the symmetry $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$.

In Chapter 4 we analyze the inverse scattering transform $\mathcal{Q}$. In Section 4.1 we prove that, for suitable $\mathbf{s}$, the CGO solutions coming from the $\overline{\partial}_k$ equation (1.18) are differentiable in x and τ and have the large-k expansion (1.22). Section 4.2 proves continuity theorems for the map $\mathcal{Q}$ depending on the coefficient $\mathbf{s}$, so that in Section 4.3 we can prove that the reconstructed potentials are real.

Finally, in Chapter 5 we complete the proof of Theorem 1.0.1.

1.1 Notation

$dm(x)$ is 2-dimensional Lebesgue measure.

$d\sigma(x)$ is 1-dimensional surface measure.

$L^p(\Omega)$ is the space of all measurable functions on $\Omega \subset \mathbb{R}^2$ with finite $L^p(\Omega)$ norm given by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega f(x)^p \, dm(x) \right)^{1/p}$$

$W^{m,p}(\Omega)$ is the space of m times weakly differentiable functions with finite $W^{m,p}(\Omega)$ norm given by

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{\alpha \colon |\alpha| \le m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}.$$

$\alpha \in \mathbb{N}^n$ is a multiindex and $D^\alpha f$ is the derivative of f.

$L^p_w(\Omega)$ consists of all functions f with $fw \in L^p(\Omega)$. The norm is given by

$$\|f\|_{L^p_w(\Omega)} = \left(\int_\Omega [f(x)w(x)]^p \, dm(x) \right)^{1/p}.$$

$\langle x \rangle = (1 + |x|^2)^{1/2}$ and $L^p_\rho(\Omega)$ for $\rho \in \mathbb{R}$ is

$$L^p_\rho(\Omega) = L^p_{\langle \cdot \rangle^\rho}(\Omega).$$

$C(\Omega)$ is the space of continuous functions on Ω.

$C^n(\Omega)$ for $n \in \mathbb{N}$ is the space of n-times continuously differentiable functions on Ω.

$C(W, V)$ consists of continuous function from W to V.

$C^\alpha(\mathbb{R}^n)$ for $\alpha \in (0, 1)$ is the space of Hölder continuous functions with seminorm

$$\|f\|_{C^\alpha(\mathbb{R}^n)} = \sup_{x,y \in \mathbb{R}^n} |f(x) - f(y)|^\alpha.$$

$C_0(\mathbb{R}^n)$ is the space of continuous functions with limit zero at infinity.

$C_c^\infty(\Omega)$ consists of all infinitely differentiable functions with compact support on Ω.

$\mathcal{S}$ is the space of Schwartz class functions. f is in $\mathcal{S}$ if

$$|pD^\alpha f| \leq C(\alpha, p)$$

for all polynomials p and all multi-indices α.

$\mathcal{B}(W)$ is the set of bounded linear operators on Banach space W equipped with the norm

$$\|T\|_{\mathcal{B}(W)} = \sup_{\|f\|_W=1} \|Tf\|_W.$$

u_B is the average of u over the ball B.

BMO is the set of functions of Bounded Mean Oscillation with seminorm

$$\|u\|_{BMO(\Omega)} = \sup_{B\subset\Omega} \frac{1}{|B|}\int_B |u - u_B| dm < \infty.$$

VMO is the set of functions of Vanishing Mean Oscillation. For $\Omega = \mathbb{R}^2$, VMO is the closure of $C_0(\mathbb{R}^2)$ functions in the BMO seminorm.

$O(f(t))$ is Big-Oh notation for the set of functions growing asymptotically slower or the same as $f(t)$. For a large (small) t-limit, $g(t) = O(f(t))$ if $g(t) \leq cf(t)$ for **some** $c > 0$ and all t large (small) enough.

$o(f(t))$ is Little-Oh notation for the set of functions that grow strictly slower than $f(t)$. For a large (small) t-limit, $g(t) = O(f(t))$ if $g(t) \leq cf(t)$ for **all** $c > 0$ and all t large (small) enough.

1.2 Operator Properties

We will need some classical results from functional analysis. The most useful is the Fredholm alternative. This tells us when some operators of the form $\lambda I - K$ are invertible. In the same vein, the Rellich-Kondrachov theorem will allow use to show certain operators are compact. In addition to these, we will need sharp results on the Cauchy integral operator $\overline{\partial}_x^{-1}$, the Beurling transform $\partial_x\overline{\partial}_x^{-1}$, and the logarithmic potential $(-\Delta)^{-1}$. A reference for results on the logarithmic potential, the Beurling transform, and the Cauchy transform is the book by Astala, Iwaniec, and Martin [2].

Theorem 1.2.1 (Fredholm alternative)**.** *Let K be a compact operator from a Banach space W to itself. Then either*

1. *$\lambda I - K$ is invertible*

2. *λ is an eigenvalue of K.*

If λ is an eigenvalue of K then the kernel of $\lambda I - K$ is nontrivial. Therefore, we can prove that $\lambda I - K$ is invertible if $\lambda I - K$ has trivial kernel.

For operators on a bounded domain, the Rellich-Kondrachov theorem gives us an important compact embedding.

Theorem 1.2.2 (Rellich-Kondrachov)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. Then, the space $W^{1,p}(\Omega)$ continuously embeds into $L^{\tilde{p}}(\Omega)$ and compactly embeds into $L^r(\Omega)$ for $p \in [1, 2)$ and $r \in [1, \tilde{p})$.*

We say that $u \in L^1_{\text{loc}}(\mathbb{R}^2)$ satisfies the $\overline{\partial}_x$-equation

$$\overline{\partial}_x u = a$$

for $a \in L^1_{\text{loc}}(\mathbb{R}^2)$ if for any $\phi \in C_c^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} u(-\overline{\partial}_x \phi)\, dm(x) = \int_{\mathbb{R}^2} a\phi\, dm(x).$$

We may define an inverse of $\overline{\partial}_x$ for $L^p(\mathbb{R}^2)$ functions with $p \in (1, 2)$ by

$$[\overline{\partial}_x^{-1} f](x) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{x - x'} f(x') dm(x'). \tag{1.26}$$

Of course, the kernel of $\overline{\partial}_x$ is all analytic functions, so this inverse is not unique. For functions in $L^2(\mathbb{R}^2)$, we must be a little more careful and define the Cauchy transform by

$$[\overline{\partial}_x^{-1} f](x) = \frac{1}{\pi} \int_{\mathbb{R}^2} \left[\frac{1}{x - x'} + \frac{\chi_{\mathbb{C}\setminus B_1}(x')}{x'} \right] f(x') dm(x').$$

The $\overline{\partial}_x^{-1}$ operator is closely related to the singular integral operator $(-\overline{\partial}_x \partial_x)^{-1/2} = 2(-\Delta)^{1/2}$. The operator $(-\overline{\partial}_x \partial_x)^{\alpha/2}$ for $\alpha \in (0, 2)$ is convolution with the Riesz potential defined by

$$(-\overline{\partial}_x \partial_x)^{\alpha/2} f = \frac{1}{c_\alpha} \int_{\mathbb{R}^2} \frac{1}{|x - y|^{2-\alpha}} f(y)\, dm(y)$$

for $\alpha \neq 2$. For $\alpha = 2$, the integral kernel of $(-\overline{\partial}_x \partial_x)^{-1}$ is the logarithmic potential which we call G_0:

$$G_0(x) = -\frac{2}{\pi} \log |x| \tag{1.27}$$

We have the inequality

$$|\overline{\partial}_x^{-1} f| \leq c(-\Delta)^{1/2} |f|.$$

The Hardy-Littlewood-Sobolev inequality gives us the mapping properties of the fractional integral operator.

Theorem 1.2.3 (Hardy-Littlewood-Sobolev Inequality)**.** *Let $1 < p < q < \infty$ and let $q = 2p/(2 - \alpha p)$. Then*

$$\|(-\Delta)^{\alpha/2} f\|_{L^q(\mathbb{R}^2)} \leq c\|f\|_{L^p(\mathbb{R}^2)}$$

We gather more precise estimates on the $\overline{\partial}_x^{-1}$ operator from the book of Astala, Iwaniec, and Martin [2, Section 4.3.2].

Theorem 1.2.4. *Let $u = \overline{\partial}_x^{-1} f$ for $f \in L^p(\mathbb{R}^2)$ and let $\tilde{p} = 2p/(2-p)$.*

1. *If $p \in (2, \infty)$ then $\|u\|_{C^\alpha} \leq c\|f\|_{L^p(\mathbb{R}^2)}$ for $\alpha = 1 - 2/p$.*
2. *If $p \in (1, 2)$ then $\|u\|_{L^{\tilde{p}}} \leq c\|f\|_{L^p(\mathbb{R}^2)}$.*
3. *If $p = 2$ then $u \in VMO(\mathbb{R}^2)$ and $\|u\|_{BMO} \leq \|f\|_{L^2(\mathbb{R}^2)}$.*
4. *If $f \in L^p \cap L^q(\mathbb{R}^2)$ for $p < 2$ and $q > 2$ then*

$$\lim_{|x| \to \infty} f(x) = 0.$$

We will need to use the Fredholm alternative to show that the equation $\overline{\partial}_x u = au + b\bar{u}$ has unique solutions with $u - 1 \in L^r(\mathbb{R}^2)$ for $r \geq 2$. Therefore, we need to know that the kernel of $I - \overline{\partial}_x^{-1}[a(\,\cdot\,) + b\overline{(\,\cdot\,)}]$ is trivial. The Liouville type theorem that will show this for coefficients better than $L^2(\mathbb{R}^2)$ is due to Vekua [36].

Theorem 1.2.5. *Let $a, b \in L^p \cap L^q(\mathbb{R}^2)$ for $p \in [1, 2)$ and $q \in (2.\infty]$. If $u \in L^r(\mathbb{R}^2)$ for $r \in [2, \infty]$ satisfies*

$$\overline{\partial}_x u = au + b\bar{u}$$

in distribution sense, then $u \equiv 0$

Proof. We construct the function $f = ue^{-\bar{\partial}^{-1}(a + b\bar{u}/u)}$ where $\bar{u}/u$ is 1 when $u = 0$. We have

$$\overline{\partial}_x f = 0$$

and $\overline{\partial}_x^{-1}(a + b\bar{u}/u) \in L^\infty(\mathbb{R}^2)$ and goes to zero in the limit $|x| \to \infty$. Therefore, f is analytic and in $L^r(\mathbb{R}^2)$ so is identically zero. □

Brown and Uhlmann prove a stronger version of this theorem which only requires that the coefficients are in $L^2(\mathbb{R}^2)$ [5, Theorem 3.1].

Theorem 1.2.6. *Given u is in $L^p(\mathbb{R}^2)$ for $p \in [1, \infty]$ and in $L^2_{\text{loc}}(\mathbb{R}^2)$. If u satisfies*

$$\overline{\partial}_x u = au + b\bar{u}$$

in distribution sense, then $u \equiv 0$

We will modify their proof of the Liouville theorem to allow u to be in a negatively weighted L^p space. In the proof, we will need classical estimates for BMO functions. These can be found in Astala, Iwaniec, and Martin [2, Section 4.6.1].

Lemma 1.2.7. *Let $u \in BMO$ then*

1. *(John-Nirenberg Inequality)*

$$\frac{1}{|B|}\int_B \exp\left(2\delta\frac{|u-u_B|}{\|u\|_{BMO}}\right) dm \le 2 \tag{1.28}$$

2. *If $B^1 \cap B^2 \neq \emptyset$ and B^2 has a radius between half and twice the radius of B^1,*

$$|u_{B^1} - u_{B^2}| \le C\|u\|_{BMO} \tag{1.29}$$

Another important operator is the Beurling transform defined by $\partial_x \overline{\partial}_x^{-1}$. This may be viewed as a Fourier multiplier with symbol $\xi/\bar{\xi}$. A classical result for Calderón-Zygmund operators gives the following lemma (see for instance [2, Theorem 4.5.3]).

Lemma 1.2.8. *The operator $\partial_x \overline{\partial}_x^{-1}$ is bounded on $L^p(\mathbb{R}^2)$ for $p \in (1,\infty)$.*

Sometimes it is more convenient to decompose the inverse of $\overline{\partial}_x(\partial_x + ik)$ using the identity

$$[\overline{\partial}_x(\partial_x + ik)]^{-1} = \frac{1}{ik}\left[\partial_x^{-1} - (\partial_x + ik)^{-1}\right]\partial_x\overline{\partial}_x^{-1},$$

so we will need to treat the operator $(\partial_x + ik)^{-1}$ by itself. The operator $(\partial_x + ik)^{-1}$ can be written as $e_{-k}(x)\partial_x^{-1}e_k(\,\cdot\,)$, and we have the following estimates from Nachman [26, Lemma 1.2]:

Lemma 1.2.9. 1. *For $f \in L^p(\mathbb{R}^2)$ there is a unique solution to $(\partial_x + ik)u = f$ with $u \in L^{\tilde{p}}(\mathbb{R}^2)$.*

2. *If $v \in L^{\tilde{p}}(\mathbb{R}^2)$ and $\partial_x v \in L^p(\mathbb{R}^2)$ then there is a unique solution $w \in L^{\tilde{p}}(\mathbb{R}^2)$ to $(\partial_x + ik)w = v$. Additionally $w \in W^{1,\tilde{p}}(\mathbb{R}^2)$ and*

$$\|w\|_{L^{\tilde{p}}(\mathbb{R}^2)} \le \frac{c}{|k|}\left(\|v\|_{L^{\tilde{p}}(\mathbb{R}^2)} + \|\partial_x v\|_{L^p(\mathbb{R}^2)}\right)$$

We will also use the following estimates from Ben-Artzi, Koch, and Saut [4, Proposition 5.4] on the decay of the fundamental solution kernel for the linear part of the NV equation: $\partial_\tau v = \partial_x^3 v + \overline{\partial}_x^3 v$.

Lemma 1.2.10. *Let*

$$I_\tau(x) = \int e^{i\tau(k^3+\bar{k}^3) - i(kx+\bar{k}\bar{x})}\, dm(k). \tag{1.30}$$

Then:

$$I_\tau(x) = \tau^{-2/3} I_1\left(x\tau^{-1/3}\right)$$

and the estimates

$$\begin{aligned} |I_1(x)| &\le C\,(1+|x|)^{-1/2}, \\ |\nabla_x I_1(x)| &\le C \end{aligned}$$

hold.

1.3 Positive Solutions to the Schrödinger Equation

We saw in the introduction the importance of a potential having a nonnegative quadratic form (1.8). The inverse scattering method for critical potentials (called conductivity-type by other authors) has been extensively studied. The full classification of our potentials comes from Murata [21]:

Definition 1.3.1. *We say a potential $q \in L^1_{loc}(\mathbb{R}^2)$ is*

- ***Subcritical*** *if there exists a positive Green's function*
- ***Critical*** *if there is no positive Green's function and $-\overline{\partial}_x \partial_x + q \geq 0$*
- ***Supercritical*** *if $-\overline{\partial}_x \partial_x + q \ngeq 0$*

A positive Green's function $G(x)$ satisfies $(-\overline{\partial}_x \partial_x + q)G(x,y) = \delta(x-y)$ and $G(x,y) > 0$. The key result from Murata for the present work is the existence of special positive solutions to the Schrödinger equation for subcritical and critical potentials.

Lemma 1.3.2. *[21, Theorem 5.6] If $q \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho > 2/p'$ then*

- *q is subcritical if and only if there exists a positive solution ϕ with asymptotics $\phi = \log|x| + O(1)$*
- *q is critical if and only if there exists a positive solution ϕ with asymptotics $\phi = 1 + o(1)$*
- *q is supercritical if and only if there is no positive solution*

If we assume that a positive solution exists, the proof of the large-x asymptotics is straightforward for smooth compact q. Any positive solution will satisfy

$$\phi(x) = G_0 * (q\phi) + h(x)$$

for a harmonic $h(x)$ and G_0 given by (1.27). For x large enough we get the approximation

$$\phi(x) \approx -\frac{2\log|x|}{\pi} \int q\phi dm(x) + h(x).$$

Since ϕ is positive, $h(x) \geq -c\log|x|$ for large x and so $h(x)$ is constant. Proving a positive solution exists is harder: see [8, Theorem 2.12] for a proof.

Gesztesy and Zhao use Brownian motion techniques to obtain optimal criteria for a potential to be critical [11, Theorem 1.3]. If q satisfies

$$\lim_{\alpha \downarrow 0} \left\{ \sup_{x\in\mathbb{R}^2} \int_{|x-y|\leq\alpha} \ln\left(|x-y|^{-1}\right) |q(y)|\, dy \right\} = 0$$

and

$$\int_{|y|\geq 1} \ln|y||q(y)|\,dy < \infty$$

then q is critical if and only if there is a positive, bounded solution to $\overline{\partial}_x\partial_x\phi = q\phi$.

In Section 3.2, we will prove that subcritical potentials form an open set in the $L^p_\rho(\mathbb{R}^2)$ and the critical potentials are the boundary. To complete the proof, we will need Theorem 2.4 from Murata [21] which shows that the set of critical potentials is unstable under compact perturbations and the subcritical potentials are stable.

Theorem 1.3.3. *[21, Theorem 2.4] Let $W \in L^p_{loc}(\mathbb{R}^2)$ be a nonnegative function with compact support which is positive on a set of positive measure then*

- *V is critical if and only if $-\overline{\partial}_x\partial_x + V \geq 0$ and $-\overline{\partial}_x\partial_x + V - \epsilon W \not\geq 0$ for all $\epsilon > 0$.*
- *V is subcritical if and only if $-\overline{\partial}_x\partial_x + V - \epsilon W \geq 0$ for any sufficiently small $\epsilon > 0$.*

1.4 Nachman's Results for Critical Potentials

In his 1996 paper [26], Nachman used the inverse scattering method for critical potentials to solve the 2D inverse conductivity problem. This problem was formulated by Calderón in a 1980 paper [6]. Calderón asks if it is possible to reconstruct the interior conductivity, $\gamma(x)$, of an object by applying different voltages on the boundary and measuring the resulting currents.

Nachman takes a conductivity satisfying $0 < 1/c < \gamma < c$ with $\gamma \in W^{2,p}(\Omega)$. If ψ is a voltage then $\gamma\nabla\psi$ is a current which solves

$$\nabla \cdot \gamma\nabla\psi = 0. \tag{1.31}$$

If we induce a voltage, f, on the boundary then the voltage potential in Ω is a solution to (1.31) with boundary value $\psi|_{\partial\Omega} = f$. We may compute the current $\gamma\partial\psi/\partial\nu$ on the boundary to get the Dirichlet to Neumann map

$$\Lambda_\gamma f = \gamma\frac{\partial\psi}{\partial\nu}.$$

Reconstructing γ from Λ_γ is the goal of the inverse conductivity problem.

If $\gamma \in W^{2,p}(\Omega)$, we may use the change of variables $q = \overline{\partial}_x\partial_x(\gamma^{1/2})/\gamma^{1/2}$ and $\phi = \gamma^{-1/2}\psi$ to change the problem into the Schrödinger equation

$$\begin{cases} (\overline{\partial}_x\partial_x + q)\phi = 0 \\ \phi|_{\partial\Omega} = g. \end{cases} \tag{1.32}$$

Potentials formed in this way are critical by Lemma 1.3.2 because $\gamma^{1/2}$ is a bounded positive solution to the Schrödinger equation.

The Dirichlet to Neumann map for the problem (1.32) is

$$\Lambda_q = \left.\frac{\partial \phi}{\partial \nu}\right|_{\partial\Omega}.$$

The map Λ_q is related to map Λ_σ for the conductivity problem by the identity

$$\Lambda_q = \gamma^{-1/2}\left(\Lambda_\gamma + \frac{1}{2}\frac{\partial\gamma}{\partial\nu}\right)\gamma^{-1/2}.$$

The inverse scattering method is useful in solving the inverse conductivity problem because the scattering transform $\mathbf{t}$ of q may be calculated from Λ_q when q is compactly supported. Therefore, if we can reconstruct q from $\mathbf{t}$, then we can reconstruct q from Λ_q.

Let $\widetilde{\phi}(x,k)$ be the solution to Laplace's equation with the same boundary values as $\phi(x,k)$, the solutions to Schrödinger's equation (1.7). Using Green's Identity, the scattering transform may be calculated using the integral

$$\begin{aligned}
\int_{\partial\Omega} e^{i\bar{k}\bar{x}}\left(\Lambda_q - \Lambda_0\right)\phi(\,\cdot\,,k)d\gamma(x) &= \int_{\partial\Omega} e^{i\bar{k}\bar{x}}\left(\frac{\partial\phi}{\partial\nu}(x,k) - \frac{\partial\widetilde{\phi}}{\partial\nu}(x,k)\right)d\sigma(x)\\
&= \int_{\Omega}\nabla\left(e^{i\bar{k}\bar{x}}\right)\cdot\nabla\phi(x,k) + e^{i\bar{k}\bar{x}}\Delta\phi(x)dm(x)\\
&\qquad - \int_{\partial\Omega}\frac{\partial e^{i\bar{k}\bar{x}}}{\partial\nu}\widetilde{\phi}(x,k)d\sigma(x)\\
&= \int_{\Omega} e^{i\bar{k}\bar{x}}q(x)\phi(x,k)dm(x)\\
&= \mathbf{t}(k).
\end{aligned}$$

With this reconstruction in mind, Nachman proves the existence, uniqueness, and decay of the CGO solutions.

Theorem 1.4.1. *[26, Theorem 1.1] Let $q \in L^p(\mathbb{R}^2)$, $p \in (1,2)$ be such that there exists a real-valued $\phi_0 \in L^\infty(\mathbb{R}^2)$ with $q = (\overline{\partial}_x\partial_x\phi_0)/\phi_0$, $\phi_0(x) \geq c_0 > 0$ and $\nabla\phi_0 \in L^p(\mathbb{R}^2)$. Then for any $k \in \mathbb{C}\setminus 0$ there exists a unique solution $\phi(x,k)$ of*

$$-\overline{\partial}_x\partial_x\phi + q\phi = 0 \ \text{ in } \ \mathbb{R}^2$$

with $e^{-ikx}\phi(\,\cdot\,,k) - 1 \in L^{\tilde{p}} \cap L^\infty(\mathbb{R}^2)$. Furthermore, $e^{-ikx}\phi(\,\cdot\,,k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ and

$$\|e^{ikx}\phi(\,\cdot\,,k) - 1\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq c|k|^{s-1}\|q\|_{L^p(\mathbb{R}^2)}$$

for $s \in [0,1]$ and k sufficiently large.

We will prove the same theorem for subcritical potential in Theorem 2.2.2. However to guarantee a positive solution for the subcritical potentials, we will assume that $q \in L^p_\rho(\mathbb{R}^2)$. The only hard part in proving this theorem comes when we prove that there unique solutions to the Schrödinger equation (1.9). For critical potentials, Nachman uses the method of reduction of order and then the Liouville theorem 1.2.5 shows that there is a unique solution. For subcritical potentials, we will require a much more precise version of the Liouville theorem. Below, we reproduce Nachman's proof that equation (1.9) has unique solutions. In Remark 1.4.3, we discuss what differences arise when dealing with subcritical potentials.

Lemma 1.4.2. *[26, Lemma 1.5] Let $q \in L^p(\mathbb{R}^2)$ be such that there exists a real valued solution $\phi_0 \in L^\infty(\mathbb{R}^2)$ with $q = (\overline{\partial}_x \partial_x \phi_0)/\phi_0$, $\phi_0(x) \geq c_0 > 0$ and $\nabla \phi_0 \in L^p(\mathbb{R}^2)$. If h satisfies $(-\overline{\partial}_x \partial_x + q)h = 0$ in $\mathbb{R}^2$ and $he^{-ikx} \in W^{1,\tilde{p}}(\mathbb{R}^2)$ for some $k \in \mathbb{C}$, then $h \equiv 0$.*

Proof. Without loss of generality, assume that h is real. Following Nachman, we construct the function

$$v = (\phi_0 \partial_x h - h \partial_x \phi_0) e^{-ikx}.$$

The function v belongs to $L^{\tilde{p}}(\mathbb{R}^2)$.

By construction, v satisfies the $\overline{\partial}_x$-equation

$$\overline{\partial}_x v = \frac{\overline{\partial}_x \phi_0}{\phi_0} v - e_{-k} \frac{\partial_x \phi_0}{\phi_0} \bar{v}.$$

These coefficients $\overline{\partial}_x \phi_0/\phi_0$ and $\partial_x \phi_0/\phi_0$ are in $L^p \cap L^{\tilde{p}}(\mathbb{R}^2)$, so Vekua's Liouville theorem 1.2.5 implies that $v \equiv 0$.

Finally, since $v \equiv 0$, $\partial_x(h/\phi_0) = 0$ and so $h \equiv 0$ by the classical Liouville theorem for analytic functions. □

Remark 1.4.3. There are two problems with trying to apply this Lemma to the subcritical case. The first is that the function v will only be in the negatively weighted space $L^{\tilde{p}}_{-\epsilon}(\mathbb{R}^2)$ because of the logarithmic growth of the positive solution ϕ_0. The second problem is that the coefficients $\overline{\partial}_x \phi_0/\phi_0$ and $\partial_x \phi_0/\phi_0$ will be in $L^2(\mathbb{R}^2)$. To see this, we note that $\phi_0 = a \log|x| + O(1)$, so we should have $|\nabla \phi_0(x)| \approx a/|x|$ for large x. We have $c/(|x| \log|x|)$ is in $L^2(\mathbb{R}^2)$ away from the origin. Therefore, we will use the Liouville theorem of Brown and Uhlmann 1.2.6, but we must change the theorem to be able to deal with a negatively weighted $L^p(\mathbb{R}^2)$ space. We do this in Theorem 2.1.4.

The next result that Nachman proves is that the CGO solutions satisfy the $\overline{\partial}_k$-equation (1.18). Our proof will be essentially the same as his except we lower the weight required for the potential. Nachman requires $q \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho > 1$, and we will lower this to $\rho > 2/p'$.

Theorem 1.4.4. *[26, Theorem 2.1] Let q be real-valued and in $L^p_\rho(\mathbb{R}^2)$, $p \in (1,2)$ and $\rho > 1$. Then for any $k \in \mathbb{C} \setminus 0$ which is not an exceptional point, the equation (1.18) holds in the $W^{1,\tilde{p}}_{-\beta}(\mathbb{R}^2)$ topology, $\beta > 2/\tilde{p}$ with $\mathbf{t}$ the scattering transform defined by (1.14).*

In order to prove that $\mathbf{s}$ is in $L^{p_1} \cap L^{p_2}(\mathbb{C})$ for $p_1 \in (1,2)$ and $p_2 \in (2,\infty)$, Nachman examines the small-k behavior of $\mathbf{t}$ and proves that critical potentials have scattering transforms satisfying $|\mathbf{t}(k)| \leq c|k|^\epsilon$ for k small. In fact, he proves that small-k decay along with a lack of exceptional points characterizes critical potentials.

Theorem 1.4.5. *[26, Theorem 3] Let q be a real valued function in $L^p_\rho(\mathbb{R}^2)$ with $p \in (1,2)$ and $\rho > 1$. The following are equivalent:*

1. *$q = (\overline{\partial}_x \partial_x \phi_0)/\phi_0$ for some $\phi_0 \in L^\infty(\mathbb{R}^2)$ with $\phi_0 \geq c_0 > 0$ a.e.*

2. *There are no exceptional points $k \in \mathbb{C}$ and the scattering transform satisfies*

$$|\mathbf{t}(k)| \leq c|k|^\epsilon$$

for some $\epsilon > 0$ and all sufficiently small k.

Because we can prove that there are no exceptional points when the potential is subcritical, this theorem implies that the scattering transform of subcritical potentials must be more singular than for critical potentials. We calculate the exact behavior in Theorem 3.1.4.

Nachman continues with a method to reconstruct $\phi(x,k)$ on $\partial\Omega$ in order to calculate $\mathbf{t}$ from Λ_q. These results are not necessary to use the inverse scattering method to solve the NV equation, so we omit them.

Chapter 2 The Schrödinger Equation and Forward Scattering

We separate the inverse scattering method into two main parts. The first is solving the Schrödinger equation (1.9) and using the CGO solutions to describe properties of the scattering transform for a given potential. The second is using the scattering transform and equation (1.18) to reconstruct the original potential which will come in Chapter 4. The ideas and many of the proofs in this chapter are from [22].

2.1 Uniqueness of CGO Solutions

In this section, we will show that solutions to the Schrödinger equation (1.9) are unique for critical or subcritical potentials in $L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho > 2/p'$ for all $k \in \mathbb{C} \setminus \{0\}$. After proving a sharper result for pseudoanalytic functions, we prove this in the Liouville Theorem 2.1.4.

The results from Lemmas 1.2.4 and 1.2.7 allow us to prove a Liouville theorem for pseudo-analytic functions. This lemma is essentially the same as Theorem 3.1 from [5] except with a negatively weighted space for the function w. Our proof will rely more on the fact that $\overline{\partial}_x^{-1}$ applied to an $L^2(\mathbb{R}^2)$ function is in VMO.

Lemma 2.1.1. *Suppose f is in $L^2(\mathbb{R}^2)$, $w \in L^p_{-\rho}(\mathbb{R}^2)$ for $1 \leq p < \infty$, $\rho < \min(1, 2/p)$, and assume that $w \exp(-\bar{\partial}_x^{-1} f)$ is holomorphic. Then w is zero.*

Proof. Let $u = -\bar{\partial}_x^{-1} f$. By Theorem 1.2.4, $u \in$ VMO. For $x, y \in \mathbb{R}^2$ with $r > s$, we have

$$\left| u_{B_r(x)} - u_{B_s(y)} \right| \leq C \|u\|_{\text{BMO}} \log(|x-y|/r + r/s + 2). \tag{2.1}$$

This follows from iterating inequality (1.29) and counting the number of steps necessary to go from $B_r(x)$ to $B_s(y)$. Since u is in VMO, we may decompose $u = u^0 + u^1$ so that u^0 has small BMO norm depending on ϵ and $u^1 \in L^\infty(\mathbb{R}^2)$. We then get

$$\begin{aligned} |u_{B_r(x)} - u_{B_1(0)}| &\leq 2\|u^1\|_{L^\infty} + |u^0_{B_r(x)} - u^0_{B_r(0)}| + |u^0_{B_r(0)} - u^0_{B_1(0)}| \\ &\leq C(r, u^0, u^1) + \epsilon \log(|x|/r + r + 2). \end{aligned}$$

Taking the exponential of both sides, we find that for fixed r

$$\exp(u_{B_r(x)}) = O(|x|^\epsilon), \qquad \text{as } x \to \infty, \tag{2.2}$$

and for fixed $x = 0$

$$\exp(u_{B_r(0)}) = O(|r|^\epsilon), \qquad \text{as } r \to \infty. \tag{2.3}$$

We also have for given $p > 1$ there exists $C > 0$ and $r_0 = r_0(p, f)$ so that if $r < r_0$ then

$$\int_{B_r(x)} \exp(p'|u - u_{B_r(x)}|) dm(x) \leq C\mu(B_r(x)). \tag{2.4}$$

This follows from the John-Nirenberg inequality (1.28) using the decomposition $u = u^0 + u^1$ above.

Using (2.4) and Hölder's inequality we find

$$\begin{aligned}|(e^u w)_B| &\le |e^{u_B}|\mu(B)^{-1}\|\langle x\rangle^\rho\|_{L^\infty(B)}\|\exp(p'|u-u_B|)\|_{L^{p'}(B)}\|w\langle x\rangle^{-\rho}\|_{L^p(B)}\\ &\le |e^{u_B}|\mu(B)^{-1}\left(\sup_{x\in B}\langle x\rangle^\rho\right)\left(\int_B \exp(p'|u-u_B|)dm(x)\right)^{1/p'}\|w\|_{L^p_{-\rho}}.\end{aligned}$$

Using (2.2) and (2.4) we see that for fixed $r_0 = r_0(p,u)$, $\rho < 1$ and $\epsilon < 1-\rho$

$$|(e^u w)_B| = o(|x|), \qquad \text{as } x \to \infty$$

Since $e^u w$ is assumed holomorphic, this implies that $e^u w$ is constant. Fixing $x = 0$, we have the identity

$$\begin{aligned}|(e^u w)_B| &\le C r^\epsilon r^{-2} r^\rho r^{2/p'}\|w\|_{L^p_{-\rho}(\mathbb{R}^2)}\\ &\le C r^{\epsilon+\rho-2/p}\|w\|_{L^p_{-\rho}(\mathbb{R}^2)}.\end{aligned}$$

If we choose $\epsilon \in (0, 2/p-\rho)$ and let $r \to \infty$, we see $e^u w \equiv 0$ and therefore $w \equiv 0$. $\square$

This leads to a new Liouville theorem for pseudoanalytic functions. The same argument as in Lemma1.4.2 coupled with Lemma 2.1.1 now gives us the following corollary.

Corollary 2.1.2. *If the function $v \in L^p_{-\rho} \cap L^2_{\mathrm{loc}}(\mathbb{R}^2)$ for $p \in [1,\infty)$ and $\rho < min(1, 2/p)$ solves*

$$\bar\partial_x v = av + b\bar v$$

with coefficients $a, b \in L^2(\mathbb{C})$ then $v \equiv 0$.

In order to use Corollary 2.1.2 to prove Lemma 1.4.2, we will need to prove the coefficients $\partial_x\phi_0/\phi_0$ and $\overline{\partial}_x\phi_0/\phi_0$ are in $L^2(\mathbb{R}^2)$ where ϕ_0 is a positive solution to the Schrödinger equation.

Lemma 2.1.3. *Let $q(x) \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho > 2/p'$ be subcritical, and let ϕ_0 be a bounded positive solution of the Schrödinger equation guaranteed by Lemma 1.3.2. The functions $\bar\partial_x\phi_0(x)/\phi_0$ and $\partial_x\phi_0(x)/\phi_0$ are in $L^2(\mathbb{R}^2)$.*

Proof. Let $W(x) = \log(|x|+e)$. According to the asymptotics from Lemma 1.3.2, the statement of the Lemma is equivalent to showing $\partial_x\phi_0/W$, and $\overline{\partial}_x\phi_0/W$ are in $L^2(\mathbb{R}^2)$.

We write $\partial_x\phi_0(x) = f(x) + \frac{1}{4}\bar\partial_x^{-1}q\phi_0(x)$ for some analytic function $f(x)$.

We will show that $f(x) \neq 0$ is incompatible with the growth of ϕ_0. By Theorem 1.2.4, $\bar\partial_x^{-1}q\phi_0 \in L^r$ for all $r \in (\tilde p, \infty)$. We will integrate against a radially-symmetric non-negative bump function, $g \in C^\infty(\mathbb{R}^2)$, which is 1 for $|x| < 1$ and 0 for $|x| > 2$. For fixed $x \in \mathbb{R}^2$ and large $R > 0$ we have

$$\begin{aligned}\left| \int_{\mathbb{R}^2} g\left(\frac{|y-x|}{R}\right) \partial_x \phi_0(y) dy \right| \\ \geq \left| \int_{\mathbb{R}^2} g\left(\frac{|y-x|}{R}\right) f(y) dm(x) \right| - \left| \int_{\mathbb{R}^2} g\left(\frac{|y-x|}{R}\right) \frac{1}{4} \bar{\partial}_x^{-1} q \phi_0(y) dy \right|.\end{aligned}$$

Integrating by parts on the left and using the mean value property and Hölder's inequality on the right, we get

$$\left| \int_{\mathbb{R}^2} (a \log |R| + O(1)) \partial_x g\left(\frac{|y-x|}{R}\right) dy \right| \geq c_1 R^2 |f(x)| - c_2 R^{2/r'} \|\bar{\partial}_x^{-1} q \phi_0\|_{L^r}.$$

We have $\partial_x g(|x-y|/R) = O(1/R)$, so

$$c_3 R \geq c_1 R^2 |f(x)| - c_2 R^{2/r'} \|\bar{\partial}_x^{-1} q \phi_0\|_{L^r}$$

Taking R large shows $f(x) = 0$ for all x.

Using the equality $\partial_x \phi_0(x) = \frac{1}{4} \bar{\partial}_x^{-1} q \phi_0(x)$, we show $\bar{\partial}_x \phi_0 / W \in L^2(\mathbb{R}^2)$. Let $f(x) = q\phi_0 \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho > 2/p'$. First, we separate the L^2 norm into three regions.

$$\begin{aligned}\left\| \frac{\bar{\partial}_x^{-1} f(x)}{W(x)} \right\|_{L^2}^2 \leq & \frac{1}{\pi} \int_{|x|<1} \frac{1}{W(x)^2} \left(\int_{\mathbb{R}^2} \frac{f(y)}{x-y} dy \right)^2 dm(x) \\ & + \frac{1}{\pi} \int_{|x|>1} \frac{1}{W(x)^2} \left(\int_{|x-y|<|x|/2} \frac{f(y)}{x-y} dy \right)^2 dm(x) \\ & + \frac{1}{\pi} \int_{|x|>1} \frac{1}{W(x)^2} \left(\int_{|x-y|>|x|/2} \frac{f(y)}{x-y} dy \right)^2 dm(x) \\ = & \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

For the first two integrals we will not use the extra $W(x)$ weight. For integral I, we have since $f \in L^p(\mathbb{R}^2)$ that $\bar{\partial}_x^{-1} f \in L^{\tilde{p}}(\mathbb{R}^2) \subset L^2_{\mathrm{loc}}(\mathbb{R}^2)$ by Theorem 1.2.4.

For integral II , we use Hölder's inequality and then Theorem 1.2.4 on $\langle \cdot \rangle^\rho f \in L^p$ for $p \in (1,2)$ to get

$$\begin{aligned}\mathrm{II} &\leq c \int_{|x|>1} \frac{\langle x \rangle^{-2\rho}}{W(x)^2} \left(\int_{|x-y|<|x|/2} \frac{\langle y \rangle^\rho f(y)}{x-y} dy \right)^2 dm(x) \\ &\leq c \|\langle x \rangle^{-2\rho}\|_{L^{p'}} \|\bar{\partial}^{-1} \langle \cdot \rangle^\rho f(\,\cdot\,)\|^2_{L^{2p/(2-p)}} \\ &\leq c \|\langle x \rangle^{-2\rho}\|_{L^{p'}} \|\langle \,\cdot\, \rangle^\rho f(\,\cdot\,)\|^2_{L^p}.\end{aligned}$$

For integral III, we simply use Hölder's inequality, the embedding $L^p_\rho(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$, and the extra $(\log|x|)^2$ weight to get

$$\begin{aligned}
\mathrm{III} \leq & \frac{1}{\pi}\int_{|x|>1} \frac{1}{W(x)^2}\left(\int_{|x-y|>|x|/2} \frac{f(y)}{x-y}\,dw\right)^2 dm(x) \\
\leq & c\int_{|x|>1} \frac{1}{|x|^2 W(x)^2}\,dm(x)\left(\int_{\mathbb{C}} |f(y)|\,dy\right)^2 \\
\leq & c\|f\|^2_{L^1}
\end{aligned}$$

The last because for $|x| > 1$, the function $(|x|W(x))^{-2} \in L^1(\mathbb{R}^2)$. $\square$

We can now prove that solutions to the Schrödinger equation (1.9) are unique. See Lemma 1.4.2 for the proof in the critical case.

Theorem 2.1.4. *Let q be a critical or subcritical potential in $L^p_\rho(\mathbb{R}^2)$. If ϕ solves the Schrödinger equation*

$$(-\overline{\partial}_x\partial_x + q)\phi = 0$$

with $e^{-ikx}\phi(x) \in L^p(\mathbb{R}^2)$ for $p \in (2,\infty)$ then $\phi \equiv 0$.

Proof. Without loss of generality, assume ϕ is real. Let ϕ_0 be a positive solution to the Schrödinger equation, and let

$$v = (\phi_0\partial_x\phi - \phi\partial_x\phi_0)e^{-ikx}.$$

We have $v \in L^p_{-\epsilon}(\mathbb{R}^2)$, and we may calculate

$$\begin{aligned}
\overline{\partial}_x v &= (\overline{\partial}_x\phi_0\partial_x\phi - \overline{\partial}_x\phi\partial_x\phi_0)e^{ikx} \\
&= \frac{\overline{\partial}_x\phi_0}{\phi_0}v - e_{-k}(x)\frac{\partial_x\phi_0}{\phi_0}\bar{v}.
\end{aligned}$$

By Lemma 2.1.3, the coefficients are in $L^2(\mathbb{R}^2)$, so by Corollary 2.1.2 $\nu \equiv 0$. Therefore $\partial(\phi/\phi_0) = 0$. The function ϕ/ϕ_0 is in $L^p(\mathbb{R}^2)$ and analytic, so $\phi \equiv 0$. $\square$

2.2 Existence and Decay of CGO Solutions

We have proved that the solutions to the Schrödinger equation are unique, so we will use Fredholm theory to prove the existence of the CGO solutions. The CGO solutions, $\mu(x,k)$, to the Schrödinger equation (1.9) satisfy the integral equation

$$\mu(x,k) - 1 = g_k * (q\mu).$$

Define $T_k = g_k * (q\,\cdot)$. We will show that T_k is a compact operator and that $[I - T_k]$ has a trivial kernel. The Fredholm alternative tells us that $[I - T_k]$ is invertible, and we get the existence of the CGO solutions through the identity

$$[I - T_k]^{-1}T_k 1 = \mu(x,k) - 1. \tag{2.5}$$

We use the following lemma of Nachman that g_k maps $L^p(\mathbb{R}^2)$ to $W^{1,\tilde{p}}(\mathbb{R}^2)$.

Lemma 2.2.1. *[26, Lemma 1.3] For any $f \in L^p(\mathbb{R}^2)$ and $k \in \mathbb{C} \setminus \{0\}$ there is a unique $u \in L^{\tilde{p}}(\mathbb{R}^2)$ satisfying*

$$\overline{\partial}_x(\partial_x + ik)u = f.$$

Furthermore, $u \in W^{1,\tilde{p}}(\mathbb{R}^2)$ and

$$\|u\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq \frac{c}{|k|^{1-s}}\|f\|_{L^p(\mathbb{R}^2)} \tag{2.6}$$

for $|k| > k_0$ and $s \in [0,1]$. We may write

$$u = -\frac{1}{ik}\left[\partial_x^{-1} - (\partial_x + ik)^{-1}\right]\partial_x\overline{\partial}_x^{-1}]f = g_k * f \tag{2.7}$$

Theorem 2.2.2. *Given a critical or subcritical $q \in L^p_\rho(\mathbb{R}^2)$ with $p \in (1,2)$ and $\rho > 2/p'$, there exists a unique $\mu(\,\cdot\,,k) - 1 \in W^{1,\tilde{p}}(\mathbb{R}^2)$ which solves (1.9). We also have for all $|k| > k_0(q)$*

$$\|\mu(\,\cdot\,,k) - 1\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq \frac{c}{|k|^{1-s}}\|q\|_{L^p(\mathbb{R}^2)} \tag{2.8}$$

for $s \in [0,1]$.

Proof. By equation (2.7), $g_k*(\,\cdot\,)$ is bounded from $L^p(\mathbb{R}^2)$ to $W^{1,\tilde{p}}(\mathbb{R}^2)$. Multiplication by $q \in L^p(\mathbb{R}^2)$ is compact as a map from $W^{1,\tilde{p}}(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$. Therefore, T_k is compact.

By the Fredholm alternative, the operator $I - T_k$ is invertible if and only if its kernel is trivial. Take a solution $f = g_k * (gf)$ with $f \in W^{1,\tilde{p}}(\mathbb{R}^2)$. The function f solves

$$\overline{\partial}_x(\partial_x + ik)f = qf.$$

By Theorem 2.1.4, $f \equiv 0$, so $I - T_k$ is invertible.

By Lemma 2.2.1, the function $T_k 1 = g_k * q \in W^{1,\tilde{p}}(\mathbb{R}^2)$ and $\|T_k 1\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \leq c\|q\|_{L^p(\mathbb{R}^2)}$. Therefore, for $s_0 \in (2/\tilde{p}, 1]$

$$\|\mu - 1\|_{L^\infty(\mathbb{R}^2)} \leq \|\mu - 1\|_{W^{s_0,\tilde{p}}(\mathbb{R}^2)} \leq \frac{c}{|k|^{1-s_0}}\|q\|_{L^p(\mathbb{R}^2)}.$$

Writing

$$\mu - 1 = g_k * (q\mu)$$

and using Lemma 2.2.1 again gives

$$\begin{aligned}
\|\mu(\,\cdot\,,k) - 1\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} &\leq \frac{c}{|k|^{1-s}}\|q\mu\|_{L^p(\mathbb{R}^2)} \\
&\leq \frac{c}{|k|^{1-s}}\left(1 + \frac{c}{|k|^{1-s_0}}\|q\|_{L^p(\mathbb{R}^2)}\right)\|q\|_{L^p(\mathbb{R}^2)} \\
&\leq \frac{\tilde{c}}{|k|^{1-s}}\|q\|_{L^p(\mathbb{R}^2)}.
\end{aligned}$$

In the last line we chose k_0 large enough so that $c/|k|^{1-s_0}\|q\|$ would be small. $\square$

From equation (2.8), we immediately get decay in $\mathbf{t}(k)$.

Lemma 2.2.3. *Given $q \in L^p_\rho(\mathbb{R}^2)$ subcritical or critical with $p \in (1,2)$ and $\rho \in (2/p', \infty)$, the function $\mathbf{s}$ is in $L^r(|k| > k_0)$ for all $r \in (\tilde{p}', \infty)$ and k_0 large enough.*

Proof. We have

$$\left|\frac{\mathbf{t}(k)}{\bar{k}}\right| \leq \frac{1}{|k|}|\mathcal{F}q| + \frac{1}{|k|}\int |q(x)||\mu(x,k) - 1|dm(x).$$

The Fourier term satisfies

$$\|\mathcal{F}q/\bar{k}\|_{L^r(|k|>k_0)} \leq \|q\|_{L^p(\mathbb{R}^2)}\|1/\bar{k}\|_{L^\sigma(\mathbb{C})}$$

for $1/r = 1/p' + 1/\sigma$ and $\sigma \in (2, \infty]$. This is equivalent to requiring $r \in (\tilde{p}', \infty]$.

We have $\mu(\,\cdot\,, k) - 1 \in W^{s,\tilde{p}}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ if $s > 2/\tilde{p}$, so

$$\left\|\frac{1}{|k|}\int |q(\mu - 1)|dm(x)\right\|_{L^r(|k|>k_0} \leq \|q\|_{L^p(\mathbb{R}^2)}\|\mu - 1\|_{W^{s,\tilde{p}}(\mathbb{R}^2)}\|1/|k|^{2-s}\|_{L^r(|k|>k_0)}$$

which holds for $r(2-s) > 2$ or $r \in (\tilde{p}', \infty)$ □

2.3 CGO Solutions Solve the DBar Equation

The main theorem in this section is that μ solves the $\overline{\partial}_k$ equation (1.18) in the classical sense when $q \in L^p_\rho(\mathbb{R}^2)$ is critical or subcritical. To prove this, we will use Lemma 2.3 from Nachman concerning the Beurling transform between weighted L^p spaces [26].

Lemma 2.3.1. *[26, Lemma 2.3] If $\alpha \in (2/p', 1)$ and $\delta \in (\alpha + 1 - 2/p, 2/p')$ then*

$$\partial_x \overline{\partial}_x^{-1} L^p_\alpha(\mathbb{R}^2) \subset L^p_\alpha(\mathbb{R}^2) + \{u \in L^p_\delta(\mathbb{R}^2) : \partial_x u \in L^p_\alpha(\mathbb{R}^2)\}$$

Theorem 2.3.2. *Let $q \in L^p_\rho(\mathbb{R}^2)$ be subcritical or critical with $p \in (1,2)$ and $\rho > 2/p'$. The CGO solution $\mu(x,k)$ is differentiable for all $k \in \mathbb{C}\setminus\{0\}$ in the classical sense with*

$$\overline{\partial}_k \mu(x,k) = \frac{\mathbf{t}(k)}{\pi \bar{k}} e_{-x}(k)\overline{\mu(x,k)}.$$

Proof. We will prove the $\overline{\partial}_k$ equation holds for $q\mu$ with the derivative existing distributionally in $L^1 \cap L^p(\mathbb{R}^2)$. The proof follows Nachman [26, Theorem 2.1].

Let $D^1_h(f) = [f(k_1 + h + ik_2) - f(k_1 + ik_2)]/h$ and $D^2_h(f) = [f(k_1 + i(k_2 + h)) - f(k_1 + ik_2)]/h$. We have

$$D^1_h(q\mu) = \frac{q}{h}[g_{k+h} - g_k] * q\mu(\,\cdot\,, k+h) + g_k * [D^1_h q\mu],$$

and

$$D_h^1(q\mu) = [I - qg_k * (\,\cdot\,)]^{-1}\left(\frac{q}{h}[g_{k+h} - g_k] * q\mu(\,\cdot\,, k+h)\right).$$

We need the function

$$\frac{1}{h}[g_{k+h} - g_k] * q\mu(\,\cdot\,, k+h)$$

to converge in the $L^\infty(\mathbb{R}^2)$ norm. We write

$$g_k * f = \frac{1}{ik}\left[\overline{\partial}_x^{-1} - (\partial_x + ik)^{-1}\partial_x\overline{\partial}_x^{-1}\right] f.$$

The derivatives of $1/(ik)$ will converge for all $k \neq 0$, so we focus our attention on

$$(\partial_x + ik)^{-1} f = e_k(-x)\overline{\partial}_x^{-1} e_k(\,\cdot\,) f = \frac{1}{\pi}\int \frac{e_k(y-x)}{y-x} f(y)dm(y).$$

Taking finite differences of the above gives

$$D_h^1(\partial_x + ik)^{-1} f = \frac{1}{\pi}\int \frac{e_k(y-x)}{y-x}\left(\frac{e_h(y-x) - 1}{h}\right) f(y)dm(y).$$

Using the fact that $f(y) \in L^1(\mathbb{R}^2)$ and the fact that $|(e_h(y-x) - 1)/h| \leq 2|y_1 - x_1|$, we may use dominated convergence to have (uniformly for $k \in B_r(0)$)

$$\lim_{h\to 0} D_h^1(\partial_x + ik)^{-1} f = \frac{\partial}{\partial k_1}(\partial + ik)^{-1} f = \frac{2e_{-k}(x)}{\pi}\int e_k(y)\frac{y_1 - x_1}{y-x} f(y)dm(y).$$

For all $h \in (0,1)$ the above is in $L^\infty(\mathbb{R}^2)$ independent of h. We get a similar result for D_h^2 and adding the two

$$\overline{\partial}_k(\partial_x + ik)^{-1} f = \frac{e_{-k}(x)}{\pi}\int e_k(y) f(y)dm(y).$$

Now, a problem is that the Beurling transform does not map L^1 to itself, so we use the decomposition from Lemma 2.3.1 to write

$$(\partial_x + ik)^{-1}\partial_x\overline{\partial}_x^{-1} q\mu = (\partial_x + ik)^{-1} f_1 + \frac{1}{ik} f_2 - \frac{1}{ik}(\partial_x + ik)^{-1}\partial_x f_2$$

Putting everything together (with the fact that the Beurling transform has symbol $k/\bar{k}$), we get the following convergence in $L^p(\mathbb{R}^2)$:

$$q[\overline{\partial}_k g_k] * (q\mu(\,\cdot\,, k)) = q\frac{e_{-k}(x)}{\pi}\mathbf{t}(k).$$

Therefore,

$$\overline{\partial}_k(q\mu) = [I - qg_k * (\,\cdot\,)]^{-1}\left(q\frac{e_{-k}}{\pi}\mathbf{t}(k)\right).$$

Finally, using the identity (1.17) gives us

$$[I - qg_k * (\,\cdot\,)]^{-1} e_{-k} q = e_{-k}(x)\overline{q\mu},$$

so we have

$$\overline{\partial}_k q\mu(x,k) = \frac{\mathbf{t}(k)}{\pi\bar{k}} e_{-x}(k)\overline{q\mu(x,k)}$$

in the $L^1 \cap L^p(\mathbb{R}^2)$ norm.

To prove the result for μ, we write

$$\begin{aligned}\overline{\partial}_k \mu &= \overline{\partial}_k [g_k * (q\mu)] \\ &= [\overline{\partial}_k g_k] * (q\mu) + g_k * (\overline{\partial}_k q\mu) \\ &= e_{-k}\frac{\mathbf{t}(k)}{\pi\bar{k}} + e_{-k}\frac{\mathbf{t}(k)}{\pi\bar{k}}\overline{\mu - 1}.\end{aligned}$$

□

Chapter 3 Properties of the Scattering Transform

In the previous chapter we showed that, if our potential is critical or subcritical, the CGO solutions exist, are unique, and have decay $|k|^n \mathbf{s}(k) \in L^r(\mathbb{R}^2)$. When $q \in L^p_\rho(\mathbb{R}^2)$ with $p \in (1,2)$ and $\rho > 2/p'$, we used the decay to show that $\mathbf{s}$ belongs to $L^r(|k| > k_0)$ for some $k_0 > 0$ and $r \in (\tilde{p}', \infty)$ in Lemma 2.2.3. In this chapter, we will prove that $\mathbf{s}$ is in L^2 for k near the origin, and in fact we will get an explicit formula for the behavior of $\mathbf{t}$ as $k \to 0$. We will then show that the scattering transform behaves just as the Fourier transform by interchanging smoothness in the potential for decay in the scattering transform. The decay in the transform will come from a generalized version of estimate (2.8) that takes into account the regularity of q. Theorems in Sections 3.1 and 3.3 are from [22], and theorems in Section 3.2 and 3.4 are from [23].

3.1 Small-k Behavior of t(k)

Recall $G_0 = -\frac{2}{\pi} \log |x|$ is the fundamental solution to Laplace's equation and let

$$\tilde{g}_k = g_k + 2[\log |k| + \gamma]/\pi$$

where g_k is Faddeev's Green function and γ is the Euler-Mascheroni constant. Define

$$\tilde{T}_0 f = qG_0 * f$$

and

$$\tilde{T}_k f = q\tilde{g}_k * f.$$

By Nachman [26, Lemma 3.4], we have the estimate.

$$|\tilde{g}_k - G_0| \le C_\epsilon |k|^\epsilon \langle x \rangle^\epsilon \tag{3.1}$$

for all $\epsilon \in (0,1)$.

In Theorem 3.1.4, we will need to make reference to the positive solution ϕ_0 to Schrödinger's equation. When $q \in L^p_\rho(\mathbb{R}^2)$ with $p \in (1,2)$ and $\rho > 2/p'$, Nachman proves that ϕ_0 solves the integral equation

$$\phi_0 = c_\infty - G_0 * (q\phi_0) \tag{3.2}$$

for some real number c_∞ [26, Lemma 3.1]. When studying this equation, it will become necessary to have $c_\infty \neq 0$ to invert the operators in the proof. The following lemma shows how to scale the equation and be able to change the value of c_∞ when q is subcritical. This is needed for Theorem 3.1.4.

Lemma 3.1.1. *Consider a subcritical potential $q(x) \in L^p_\rho(\mathbb{R}^2)$ with $p \in (1,2)$, $\rho \in (2/p', \infty)$, and positive solution ϕ_0 which satisfies equation (3.2). If we rescale the potential $q_r(x) = r^2 q(rx)$, then the positive solution $\phi_0^r(x) = \phi_0(rx)$ solves $(-\overline{\partial}_x \partial_x + q_r)\phi_0^r = 0$ and satisfies the asymptotic $\phi_0^r(x) = a \log |x| + \mathcal{O}(1)$. The function ϕ_0^r also solves equation (3.2) with $c_\infty^r = c_\infty + a \log |r|$. The associated scattering transform $\mathbf{t}_r(k)$ satisfies $\mathbf{t}_r(k) = \mathbf{t}(k/r)$.*

Proof. The identity $\mathbf{t}_r(k) = \mathbf{t}(k/r)$ is proved in Siltanen [30, Theorem 3.19]. From Nachman [26, Lemma 3.1], by integrating $\phi_0 + G_0 * (q\phi_0)$ over a ball of radius R, we have

$$c_\infty \pi R^2 = \int_{|x|<R} \phi_0(x) dm(x) - \left(\frac{1}{2}R^2 \log R + \frac{1}{4}R^2\right) \int_{\mathbb{R}^2} q\phi_0 + O(R^{2/p'}). \tag{3.3}$$

Expanding the first integral with $\phi_0 = a \log |x| + O(1)$ and matching terms with $R^2 \log |R|$, we find

$$a = \frac{2}{\pi} \int q\phi_0.$$

Using this with equation (3.2) and the change of variables $y' = ry$ we find

$$\begin{aligned}
\phi_0^r(x) &= c_\infty^r + \frac{2}{\pi} \int_{\mathbb{R}^2} (\log |x-y|) q_r(y) \phi_0^r(y) dy \\
&= c_\infty^r + \frac{2}{\pi} \int_{\mathbb{R}^2} (\log |x-y|) r^2 q(ry) \phi_0(ry) dy \\
&= c_\infty^r + \frac{2}{\pi} \int_{\mathbb{R}^2} (\log |rx - y'| - \log |r|)\, q(y') \phi_0(y') dy' \\
&= c_\infty^r - a \log |r| - c_\infty + \phi_0(rx).
\end{aligned}$$

This proves $c_\infty^r = a \log |r| + c_\infty$. □

We now prove an elementary estimate on the logarithmic potential G_0 (compare to [26, Lemma 3.2]).

Lemma 3.1.2. *For $f \in L^p_\rho(\mathbb{R}^2)$ with $p \in (1, 2)$ and $\rho \in (2/p', \infty)$ we have*

$$\left\| G_0 * f + \frac{2}{\pi} \log(|x| + 1) \int f \right\|_{L^\infty(\mathbb{R}^2)} \le c \|f\|_{L^p_\rho(\mathbb{R}^2)}$$

Proof. The left hand side of the inequality is

$$\frac{2}{\pi} \left| \int \log \left(\frac{|x-y|}{|x|+1} \right) f(y) dm(y) \right|.$$

We split $\mathbb{R}^2 \times \mathbb{R}^2$ into three regions:

$$\begin{aligned}
\Omega_1 &= \{(x, y) : |x-y| < 1\} \\
\Omega_2 &= \{(x, y) : 1 < |x-y| < |x| + 1\} \\
\Omega_3 &= \{(x, y) : |x| + 1 < |x-y|\}.
\end{aligned}$$

In Ω_1, we have $\log(|x| + 1) < \log(|y| + 2)$ so

$$\left| \log(|x| + 1) \int f(y) dm(y) \right| \le \int \log(|y| + 2) |f(y)| dm(y) \le c \|f\|_{L^p_\rho(\mathbb{R}^2)}$$

and
$$\left|\int \log|x-y|f(y)dm(y)\right| \leq c\|\log|\cdot|\|_{L^{p'}(B_1(0))}\|f\|_{L^p(\mathbb{R}^2)}.$$

In Ω_2, we have
$$\begin{aligned}\frac{|x|+1}{|x-y|} &\leq 4\frac{\langle x\rangle}{\langle x-y\rangle}\\ &\leq 4\langle y\rangle,\end{aligned}$$

so we estimate the integral by
$$\left|\int \log\left(\frac{|x-y|}{|x|+1}\right)f(y)dy\right| \leq c\int \log(\langle y\rangle)|f(y)|dm(y) \leq c\|f\|_{L^p_\rho(\mathbb{R}^2)}.$$

A similar estimate holds in Ω_3. □

We are now ready to prove compactness and continuity of our maps $\tilde{T}_k$

Lemma 3.1.3. *Let $q \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$, $\rho > 2/p'$, and $\rho' \in (2/p', \rho)$ then*

1. *$\tilde{T}_0$ is bounded from $L^p_{\rho'}(\mathbb{R}^2) \to L^p_{\rho'}(\mathbb{R}^2)$*
2. *$\tilde{T}_0$ is compact from $L^p_{\rho'}(\mathbb{R}^2) \to L^p_{\rho'}(\mathbb{R}^2)$*
3. *If q is critical or subcritical and there is a ϕ_0 with $c_\infty \neq 0$ then $[I - \tilde{T}_0]$ is invertible*
4. *If q is critical or subcritical and there is a ϕ_0 with $c_\infty \neq 0$ then $[I - \tilde{T}_k]$ is invertible for small k and*
$$\|[I-\tilde{T}_0]^{-1}f - [I-\tilde{T}_k]^{-1}f\|_{L^p_{\rho'}(\mathbb{R}^2)} \leq C(q,\epsilon,\rho')|k|^\epsilon\|f\|_{L^p_{\rho'}(\mathbb{R}^2)}$$
for $\epsilon \in (0, \min[\rho-\rho', \rho'-2/p'])$.

Proof. (1) This follows from the operator estimate in Lemma 3.1.2 and the fact that multiplication by $q \in L^p_\rho(\mathbb{R}^2)$ maps $a\log(|x|+e) + L^\infty(\mathbb{R}^2) \to L^p_{\rho'}(\mathbb{R}^2)$.

(2) Let $q_j \in C^\infty_c(\mathbb{R}^2)$ and $q_j \to q \in L^p_\rho(\mathbb{R}^2)$. We have by the Hardy-Littlewood-Sobolev inequality that $\|\nabla G_0 * f\|_{L^{\tilde{p}}} \leq \|f\|_{L^p(\mathbb{R}^2)}$ so that $\chi_{B_R(0)}G_0 * f \in W^{1,\tilde{p}}(\mathbb{R}^2)$ which compactly embeds into $L^\infty(\mathbb{R}^2)$. Therefore the operator $q_jG_0 * [\,\cdot\,]$ is compact from $L^p_{\rho'}(\mathbb{R}^2) \to L^p_{\rho'}(\mathbb{R}^2)$. Taking the limit $j \to \infty$ we have
$$\lim_{j\to\infty}\|(q_j - q)G_0 * f\|_{L^p_{\rho'}(\mathbb{R}^2)} \leq \lim_{j\to\infty} c\|q_j - q\|_{L^p_\rho(\mathbb{R}^2)}\|f\|_{L^p_{\rho'}(\mathbb{R}^2)} = 0$$
so $\tilde{T}_0$ is a limit of compact operators and is compact.

(3) By the Fredholm alternative, we just need to show the kernel is empty. Take $f \in L^p_{\rho'}(\mathbb{R}^2)$ such that $[I - \tilde{T}_0]f = 0$. Then $h = G_0 * f$ satisfies
$$h = G_0 * [qG_0 * f] = G_0 * [qh].$$

We have $h \in a\log(|x|+e) + L^\infty(\mathbb{R}^2)$ and $\nabla h \in L^{\tilde{p}}(\mathbb{R}^2)$ satisfies

$$\overline{\partial}_x \partial_x h = qh.$$

Taking the function

$$v = h\partial_x \phi_0 - \phi_0 \partial_x h \in L^{\tilde{p}}_{-\epsilon}(\mathbb{R}^2),$$

we see that v satisfies

$$\overline{\partial}_x v = \frac{\overline{\partial}_x \phi_0}{\phi_0} v - \frac{\partial_x \phi_0}{\phi_0} \bar{v}.$$

By the Liouville theorem 2.1.2, $v \equiv 0$. Therefore h/ϕ_0 is antianalytic and by the Liouville theorem for analytic functions, must be a constant, c. Therefore

$$h = c\phi_0 = c_\infty c - cG_0 * [q\phi_0] = cc_\infty + G_0 * (h).$$

Since $c_\infty \neq 0$, we must have $c = 0$.

(4) This follows from the second resolvent identity. We have

$$\begin{aligned}
[I - \tilde{T}_0]^{-1} f - [I - \tilde{T}_k]^{-1} f &= [I - \tilde{T}_0]^{-1} [\tilde{T}_k - \tilde{T}_0][I - \tilde{T}_k]^{-1} f \\
&= [I - \tilde{T}_0]^{-1} \left(q \langle \cdot \rangle^\epsilon \langle \cdot \rangle^{-\epsilon} (\tilde{g}_k - G_0) * ([I - \tilde{T}_k]^{-1} f) \right).
\end{aligned}$$

And by equation (3.1)

$$\begin{aligned}
|(\tilde{g}_k - G_0) * g| &\leq C_\epsilon |k|^\epsilon \int_{\mathbb{R}^2} \langle x - y \rangle^\epsilon |g(y)| dy \\
&\leq C_\epsilon |k|^\epsilon \langle x \rangle^\epsilon \| \langle \cdot \rangle^\epsilon g \|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\| [I - \tilde{T}_0]^{-1} f - [I - \tilde{T}_k]^{-1} f \|_{L^p_{\rho'}(\mathbb{R}^2)} &\leq C_\epsilon |k|^\epsilon \| [I - \tilde{T}_0]^{-1} \|_{\mathcal{B}(L^p_{\rho'})} \| q \langle \cdot \rangle^{-\epsilon} \|_{L^p_{\rho'}} \| f \|_{L^p_{\rho'}} \\
&\leq C(q, \epsilon, \rho') \| f \|_{L^p_{\rho'}(\mathbb{R}^2)}.
\end{aligned}$$

□

We are now ready to describe the behavior of $\mathbf{t}(k)$ near $k = 0$.

Theorem 3.1.4. *Let $q \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1, 2)$, $\rho > 2/p'$ be critical or subcritical. The scattering transform satisfies*

$$\mathbf{t}(k) = \frac{\pi a}{2[c_\infty - a(\log|k| + \gamma)]} + O(|k|^\epsilon).$$

for some $\epsilon > 0$.

Proof. Let ϕ solve $\phi = 1 + G_0 * (q\phi)$ which if $c_\infty \neq 0$ is possible through scaling. Let $q\tilde{\mu} = [I - \tilde{T}_k]^{-1} q$. Now, we have

$$q\phi - q\tilde{\mu} = [I - \tilde{T}_0]^{-1} q - [I - \tilde{T}_k]^{-1} q.$$

So by the second resolvent identity and part (4) of Lemma 3.1.3 we have

$$\|q\phi - q\tilde{\mu}\|_{L^p_{\rho'}(\mathbb{R}^2)} \leq C(q,\epsilon,\rho')|k|^\epsilon \|q\|_{L^p_{\rho'}(\mathbb{R}^2)}.$$

Letting

$$\tilde{\mathbf{h}}(k) = \int q(x)\tilde{\mu}(x,k)\,dm(x)$$

and

$$a = \frac{2}{\pi}\int q(x)\phi(x)dm(x),$$

we have for small k

$$\tilde{\mathbf{h}}(k) - \frac{\pi}{2}a = O(|k|^\epsilon). \tag{3.4}$$

Another application of the second resolvent formula gives

$$\begin{aligned} q\mu - q\tilde{\mu} &= [I - qg_k * (\,\cdot\,)]^{-1} q(\tilde{g}_k - g_k) * [I - \tilde{T}_k]^{-1} q \\ &= [I - qg_k * (\,\cdot\,)]^{-1} q \left(\frac{2}{\pi}\log|k| + \gamma\right) \int q\tilde{\mu}\,dm(x) \\ &= q\mu \frac{2}{\pi}\left(\log|k| + \gamma\right)\tilde{\mathbf{h}}(k). \end{aligned}$$

Integrating and rearranging we have the identity

$$\mathbf{h}(k) = \frac{\tilde{\mathbf{h}}(k)}{1 - \tilde{\mathbf{h}}(k)2(\log|k| + \gamma)/\pi.}$$

Substituting equation (3.4) in the equality yields

$$\mathbf{h}(k) = \frac{\pi a}{2[1 - a(\log|k| + \gamma)]} + O(|k|^\epsilon).$$

For small k, $\mathbf{t}(k) = \mathbf{h}(k) + O(|k|)$, so

$$\mathbf{t}(k) = \frac{\pi a}{2[1 - a(\log|k| + \gamma)]} + O(|k|^\epsilon).$$

Lastly, we consider the case when $c_\infty \neq 1$. Combining the result for $c_\infty \neq 0$ with the scaling properties from Lemma 3.1.1, the asymptotics for an arbitrarily scaled potential $q_r(x)$ are

$$\mathbf{t}_r(k) = \frac{\pi a}{2[c^r_\infty - a(\log|k| + \gamma)]} + O(k^\epsilon).$$

Using the relation $\mathbf{t}(k) = \mathbf{t}_r(rk)$ we get

$$\begin{aligned} \mathbf{t}(k) &= \frac{\pi a}{2[c^r_\infty - a(\log|rk| + \gamma)]} + O(k^\epsilon) \\ &= \frac{\pi a}{2[(c^r_\infty - a\log|r|) - a(\log|k| + \gamma)]} + O(k^\epsilon) \\ &= \frac{\pi a}{2[c_\infty - a(\log|k| + \gamma)]} + O(k^\epsilon). \end{aligned}$$

$\square$

3.2 Topology of the Set of Subcritical Potentials

We now take a slight detour and prove an interesting result on the topology of subcritical potentials. From Theorem 1.3.3, we know that the set of critical potentials is not stable under compact perturbations. Here we go a step further and show the subcritical potentials are an open set in the topology of $L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho \in (2/p', \infty)$.

Theorem 3.2.1. *The set of subcritical potentials is open in the $L^p_\rho(\mathbb{R}^2)$ topology for $p \in (1,2)$ and $\rho \in (2/p', \infty)$ and the set of critical potentials is its boundary.*

Proof. We prove that the set of subcritical potentials is open by finding positive logarithmically growing solutions, ϕ_v, to $(-\overline{\partial}_x \partial_x + v)\phi_v = 0$ for all v close enough to any given subcritical potential q.

Let ϕ_q be a positive solution to the Schrödinger equation with $c_\infty = 1$ in (3.2). By Lemma 3.1.3 with $c_\infty \neq 0$, we have that $[I - \tilde{T}_0]$ is invertible on $L^p_{\rho'}(\mathbb{R}^2)$ for $\rho' \in (2/p', \rho)$. If $c_\infty = 0$, we may use the argument from 3.1.1 to scale q, and therefore ϕ_q, so that $c_\infty = 1$. For some $\epsilon > 0$ the operator $[I - vG_0 * (\,\cdot\,)]$ is invertible for all potentials v satisfying $\|q - v\|_{L^p_{\rho'}} < \epsilon$. Thus we have $v\phi_v = [I - vG_0 * (\,\cdot\,)]^{-1} v$ and $\phi_v = 1 + G_0 * (v\phi_v)$. Because q is subcritical, we have $\phi_q = a \log|x| + O(1)$ is a positive solution. The solutions ϕ_v satisfy the equation

$$\phi_v = 1 - G_0 * (v\phi_v).$$

Taking the difference between ϕ_q and ϕ_v we find

$$\begin{aligned}\phi_q - \phi_v = G_0 * (v\phi_v - q\phi_q) + \frac{2}{\pi} \log(|x| + e) \int (v\phi_v - q\phi_q) dm(x) \\ - \frac{2}{\pi} \log(|x| + e) \int (v\phi_v - q\phi_q) dm(x).\end{aligned}$$

While by Lemma 3.1.2,

$$|\phi_q - \phi_v| \leq c\|v\phi_v - q\phi_q\|_{L^p_{\rho'}(\mathbb{R}^2)} + \frac{1}{2\pi} \log(|x| + e)\|v\phi_v - q\phi_q\|_{L^1(\mathbb{R}^2)}.$$

By the inequality $\|f\|_{L^1(\mathbb{R}^2)} \leq c\|f\|_{L^p_\rho(\mathbb{R}^2)}$ when $\rho > 2/p'$, we get

$$\phi_v \geq \phi_q - c\|v\phi_v - q\phi_q\|_{L^p_{\rho'}} - \frac{1}{2\pi} \log(|x| + e)\|v\phi_v - q\phi_q\|_{L^p_\rho}.$$

The right hand side is positive and logarithmically growing for $\|v\phi_v - q\phi_q\|_{L^p_\rho}$ small. The function ϕ_v is a positive distributional solution to $(-\overline{\partial}_x \partial_x + v)\phi_v = 0$, so by Theorem 1.3.2 v is subcritical because the positive solutions for critical potentials in this weighted space have the asymptotics $\phi = c + o(1)$ whereas the positive solutions for subcritical potentials obey $\phi = a \log|x| + O(1)$. This proves the set of subcritical potentials is open.

From Theorem 1.3.3, we know that nonnegative compact perturbations of critical potentials are subcritical and nonpositive perturbations are supercritical. Thus critical potentials are a subset of the boundary of subcritical potentials. If a sequence of potentials q_j has a nonnegative quadratic form (1.8) then its limit also has a nonnegative quadratic form and must either be subcritical or critical. Therefore, critical potentials form the entire boundary. $\square$

3.3 Regularity and Decay

We will show that when q is n times weakly differentiable, $\mu(x,k)$ will be $(n+1)$ times differentiable in x and $(\mu(x,k)-1)$ will vanish in norm as $|k| \to \infty$. We then use this to show that $|k|^n \mathbf{s}(k) \in L^r(|k| > k_0)$ for $r \in (\tilde{p}', \infty)$. The decay in $\mathbf{s}$ is exactly comparable to the decay in the Fourier transform of a differentiable function.

The large-k behavior will allow us, in the next chapter, to show that $\mu(x,k)$ has a large-k expansion. Siltanen proved a variation of the following lemmas on the decay of μ and $\mathbf{t}$ for compactly supported, conductivity type potentials[30, Section 3.2.1].

Lemma 3.3.1. *If $q \in W^{n,p}(\mathbb{R}^2)$ for $p \in (1,2)$ has no exceptional points then $\mu(\,\cdot\,,k) - 1 \in W^{n+1,\tilde{p}}(\mathbb{R}^2)$ and*

$$\|D^\alpha(\mu(\,\cdot\,,k)-1)\|_{W^{s,\tilde{p}}(\mathbb{R}^2)} \leq \frac{c}{|k|^{1-s}} \sum_{m=0}^{n} \|q\|_{W^{m,p}(\mathbb{R}^2)} \|q\|_{L^2(\mathbb{R}^2)}^{n-m} \tag{3.5}$$

for $s \in [0,1]$, $|\alpha| \leq n$, c depending on n, and $k > k_0(\|q\|_{W^{n,p}}, n)$.

Proof. The case $n = 0$ is Theorem 2.2.2. We induct on this using Lemma 2.2.1. We take derivatives in equation (1.9) with $u = (\mu - 1)$, $f = q(\mu-1) + q$, and assume the result for all multi-indices less than α. We have $D^\alpha(\mu-1) \in L^{\tilde{p}}(\mathbb{R}^2)$ solves

$$\overline{\partial}_x(\partial_x + ik) D^\alpha \mu(x,k) = f$$

with

$$\begin{aligned} f &= \left[D^\alpha q + q D^\alpha(\mu-1) + \sum_{\beta : 0<\beta<\alpha} \binom{\alpha}{\beta} D^\beta q D^{\alpha-\beta}(\mu-1) \right] \\ &= I + II + III. \end{aligned} \tag{3.6}$$

We estimate the asymptotic behavior of the norms as $k \to \infty$ using the three parts in equation (3.6):

$$\begin{aligned} \|D^\alpha(\mu(\,\cdot\,,k,0)-1)\|_{W^{s,\tilde{p}}} &\leq \frac{\tilde{c}}{|k|^{1-s}} \|f\|_{L^p} \\ &\leq \frac{\tilde{c}}{4|k|^{1-s}} (\|I\|_{L^p} + \|II\|_{L^p} + \|III\|_{L^p}). \end{aligned}$$

The norm $\|I\|_{L^p} \leq \|q\|_{W^{n,p}}$ is independent of k and already accounted for in inequality (3.5). For the second norm we use the induction hypothesis with

$$\|II\|_{L^p} = \|q D^\alpha(\mu-1)\|_{L^p} \leq \|q\|_{L^2} \|D^\alpha(\mu-1)\|_{L^{\tilde{p}}} \leq c \sum_{m=0}^{n-1} \|q\|_{W^{m,p}} \|q\|_{L^2}^{n-m}.$$

Now we show the third norm decreases faster in k then the other terms, and therefore it will not contribute to the asymptotic behavior. Choosing $s_0 \in (2/\tilde{p}, 1)$,

$$\begin{aligned}\|III\| &\leq \sum_{\beta:0<\beta<\alpha} \binom{\alpha}{\beta} \|D^\beta q D^{\alpha-\beta}(\mu-1)\|_{L^p} \\ &\leq \sum_{\beta:0<\beta<\alpha} \binom{\alpha}{\beta} \|D^\beta q\|_{L^p} \|D^{\alpha-\beta}(\mu-1)\|_{W^{s_0,\tilde{p}}} \\ &\leq \frac{c}{|k|^{1-s_0}} \sum_{\beta:0<\beta<\alpha} \binom{\alpha}{\beta} \|D^\beta q\|_{L^p} \|q\|_{L^p}^n.\end{aligned}$$

Choosing k_0 large enough finishes the result. □

We are ready to prove the decay of $\mathbf{s}$.

Lemma 3.3.2. *If $q \in W^{n,p}_\rho(\mathbb{R}^2)$ with $\rho \in (2/p', \infty)$ and $p \in (1,2)$ has no exceptional points then $|k|^n \mathbf{s}(k) \in L^r(|k| > k_0)$ for all $r \in (\tilde{p}', \infty]$ and large enough k_0. Additionally, $\mathbf{s}(k)$ is continuous on $\mathbb{C} \setminus \{0\}$ and is in $L^2(\mathbb{C})$.*

Proof. First note that $W^{n,p}_\rho(\mathbb{R}^2) \subset W^{n,1} \cap W^{n,p}(\mathbb{R}^2)$ when $\rho > 2/p'$. Denote the Fourier transform by $\mathcal{F}(g)(k) = \int e_{-k}(x) g(x)\, dm(x)$. Then we may write

$$\bar{k}^{-1}\mathbf{t}(k) = \bar{k}^{-1}\mathcal{F}(q)(k) + \bar{k}^{-1} \int_{\mathbb{R}^2} e_{-k}(x) q(x)(\mu(x,k)-1)\, dm(x).$$

By the Hausdorff-Young inequality and the differentiability of $q \in W^{n,1} \cap W^{n,p}(\mathbb{R}^2)$, the first term satisfies $|k|^{n+1}\bar{k}^{-1}\mathcal{F}(q)(k) \in L^{p'} \cap L^\infty(\mathbb{R}^2)$. For the second term, we may integrate by parts because $e_{-k}(x) \in W^{n,\infty}(\mathbb{R}^2)$ for any n and the product $q(\mu-1) \in W^{n,1}(\mathbb{R}^2)$ by Lemma 3.3.1. Therefore we have

$$\begin{aligned}(i\bar{k})^n \int_{\mathbb{R}^2} & e_{-k}(x) q(x)(\mu(x,k)-1) dm(x) \\ &= \int_{\mathbb{R}^2} (-1)^n [\bar{\partial}^n e_{-k}(x)] q(x)(\mu(x,k)-1) dm(x) \\ &= \int_{\mathbb{R}^2} e_{-k}(x) \left(\sum_{j=0}^n \binom{n}{j} \bar{\partial}_x^j (\mu-1) \bar{\partial}_x^{n-j} q \right) dm(x).\end{aligned}$$

Using the decay estimate from Lemma 3.3.1 with $s_0 \in (2/\tilde{p}, \infty)$, we get for k large,

$$\begin{aligned}|k|^n |\bar{k}^{-1}(t(k) - \mathcal{F}(q)(k))| &\leq |k|^{-1} \left(\sum_{j=0}^n \binom{n}{j} \|\bar{\partial}_z^j(\mu-1)\|_{W^{s_0,\tilde{p}}} \|\bar{\partial}_z^{n-j} q\|_{L^1} \right) \\ &\leq \frac{c}{|k|^{2-s_0}} \left(\sum_{m=0}^n \|q\|_{W^{m,p}} \|q\|_{L^2}^{n-m} \right) \|q\|_{W^{n,1}}.\end{aligned}$$

With this $|k|^{s_0-2}$ decay, we get the second term is in $L^r(|k| > k_0)$ if $(2-s_0)r > 2$ which is $r > \frac{2}{2-2/\tilde{p}} = \tilde{p}'$. Continuity of $\mathbf{t}(k)$ follows in the same way as that for conductivity type potentials. Nachman proves this in [26, Theorem 4]. Using the continuity and the small k behavior of $\mathbf{t}$ from theorem 3.1.4 implies $\bar{k}^{-1}\mathbf{t}(k) \in L^2(|k| < \epsilon)$ for small enough ϵ. $\square$

3.4 Continuity in q and Symmetries of the Scattering Transform

We follow the ideas of Grinevich and Manakov [13, Theorem 2] where they prove similar symmetries in the case of nonzero-energy scattering. In the nonzero-energy case, there are two different symmetries coming for the scattering transform of the Schrödinger equation with an electromagnetic term

$$[-\overline{\partial}_x \partial_x + iA(x)\partial_x + B(x) + E]f(x) = 0.$$

If $A = 0$, Grivenich and Manakov prove one symmetry, and if f is real they prove a different symmetry. These symmetries could both be worked out for the case of zero-energy, but the $\overline{\partial}_k$-problem is a system when $A(x) \neq 0$ or $B(x)$ is not real. There would also be additional scattering data. Therefore, we will only deal with the Schrödinger equation with no magnetic term.

Given a real potential q the scattering transform has the symmetry $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$. This fact will be necessary to prove that $\mathcal{Q}\mathbf{s}(k,\tau)$ is real for all time. To prove this we will construct a particular closed differential form. The theory becomes easier if the form is smooth, so we will approximate q by $C_c^\infty(\mathbb{R}^2)$ functions. Therefore, to prove the symmetry we need to prove that $\mathcal{T}q(k)$ is continuous in q for each k.

In this section, we keep track of the explicit dependence of the operator T_k on q, so we write

$$T_k(q)f = g_k * (qf)$$

and let $\mu(x, k; q)$ be the solution to the Schrödinger equation (1.9) for a given potential q and $k \in \mathbb{C} \setminus \{0\}$ not an exceptional point. We have

$$\mu(x, k; q) - 1 = [I - T_k(q)]^{-1} T_k(q) 1. \tag{3.7}$$

Lemma 3.4.1. *Fix $p \in (1, 2)$, $k \in \mathbb{C} \setminus \{0\}$, and $q_0 \in L^p(\mathbb{R}^2)$. Suppose that*

$$R(q_0) := (I - T_k(q_0))^{-1}$$

exists as a bounded operator from $W^{1,\tilde{p}}(\mathbb{R}^2)$ to itself. There is a number $r > 0$ and a constant $c(p, k, q_0)$ so that for all $q \in L^p(\mathbb{R}^2)$ with $\|q_0 - q\|_{L^p(\mathbb{R}^2)} \leq r$, the estimate

$$\|\mu(\,\cdot\,, k, q) - \mu(\,\cdot\,, k, q_0)\|_{W^{1,\tilde{p}}} \leq c(p, k, q_0)\, \|q - q_0\|_p$$

holds.

Proof. In what follows c denotes a constant depending only on k, p, and q_0 whose value may vary from line to line. By Lemma 2.2.1, for a given $k \neq 0$

$$\|g_k * f\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \leq c\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.8}$$

In particular,

$$\|T_k(q)\|_{\mathcal{B}(W^{1,\tilde{p}}(\mathbb{R}^2))} \leq c\|q\|_{L^p(\mathbb{R}^2)}.$$

It also follows from (3.8) that

$$\|T_k(q)1 - T_k(q_0)1\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \leq c\|q - q_0\|_{L^p(\mathbb{R}^2)}. \tag{3.9}$$

By the second resolvent identity

$$R(q) - R(q_0) = R(q)[R(q_0) - R(q)]R(q_0).$$

Estimating the $\mathcal{B}(W^{1,\tilde{p}}(\mathbb{R}^2))$ norm of the result

$$\|R(q) - R(q_0)\|_{\mathcal{B}(W^{1,\tilde{p}}(\mathbb{R}^2))} \leq \|R(q)\|_{\mathcal{B}(W^{1,\tilde{p}})}\|R(q_0)\|_{\mathcal{B}(W^{1,\tilde{p}})}\|R(q_0) - R(q)\|_{\mathcal{B}(W^{1,\tilde{p}})} \tag{3.10}$$

$$\leq c\|q - q_0\|_{L^p(\mathbb{R}^2)}. \tag{3.11}$$

The last line follows since $\|q - q_0\|$ is small which implies that $\|R(q)\|$ is close to $\|R(q_0)\|$, and both then only depend on the norm of q and r. We absorb the dependence into the constant c. Using (3.7), we have

$$\begin{aligned}
\|\mu(x,k,q) - \mu(x,k,q_0)\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} &= \|R(q)T_k(q)1 - R(q_0)T_k(q_0)1\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \\
&\leq \|R(q)[T_k(q)1 - T_k(q_0)1]\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \\
&\qquad + \|[R(q) - R(q_0)]T_k(q_0)\|_{W^{1,\tilde{p}}(\mathbb{R}^2)} \\
&\leq c\|q - q_0\|_{L^p(\mathbb{R}^2)}
\end{aligned}$$

□

Lemma 3.4.1 allows us to prove continuity of $\mathbf{t}$ as a function of q.

Lemma 3.4.2. *Suppose that $p \in (1,2)$, $\rho \in (2/p', \infty)$, that $q \in L^p_\rho(\mathbb{R}^2)$, and $\{q_n\}$ is a sequence from $L^p_\rho(\mathbb{R}^2)$ with $q_n \to q$ in $L^p_\rho(\mathbb{R}^2)$. Finally, let $\mathbf{t}_n = \mathcal{T}(q_n)$ and $\mathbf{t} = \mathcal{T}(q)$. Then, for all non-exceptional nonzero k, $\mathbf{t}_n(k) \to \mathbf{t}(k)$ pointwise.*

Proof. Using Lemma 3.4.1 we estimate

$$\begin{aligned}
|\mathbf{t}_n(k) - \mathbf{t}(k)| &\leq \left|\int e_k(x)(q_n(x) - q(x))\mu_n(x)\,dm(x)\right| \\
&\quad + \left|\int e_k(x)q(x))(\mu_n(x) - \mu(x))\,dm(x)\right| \\
&\leq \|q_n - q\|_{L^1}\|\mu_n\|_{L^\infty} + \|q\|_{L^1}\|\mu_n - \mu\|_{L^\infty} \\
&\leq \|q_n - q\|_{L^p_\rho}(\|\mu_n - 1\|_{W^{1,\tilde{p}}} + 1) + \|q\|_{L^p_\rho}\|\mu_n - \mu\|_{W^{1,\tilde{p}}}
\end{aligned}$$

and conclude that $\mathbf{t}_n(k) \to \mathbf{t}(k)$. □

Theorem 3.4.3. *If $q \in L^p_\rho(\mathbb{R}^2)$ for $p \in (1,2)$ and $\rho > 2/p'$ is real and k and $-k$ are not exceptional points, then $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$.*

Proof. First assume that $q \in C_c^\infty(\mathbb{R}^2)$ so that the corresponding $\mu(x,k)$ is smooth in x. Consider the form

$$\omega = e_k \overline{\overline{\partial}_x \mu(x,-k)} dx + e_k \overline{\partial}_x \mu(x,k) d\bar{x}$$

where

$$dx = dx_1 + i dx_2, \quad d\bar{x} = dx_1 - i dx_2.$$

We have

$$\begin{aligned} d\omega &= e_k \left((\partial_x + ik)\overline{\partial}_x \mu(x,k) - \overline{(\partial_x - ik)\overline{\partial}_x \mu(x,-k)} \right) dx \wedge d\bar{x} \\ &= e_k \left(q(x)\mu(x,k) - q(x)\overline{\mu(x,-k)} \right) dx \wedge d\bar{x}. \end{aligned}$$

Hence, by Stokes' Theorem,

$$\mathbf{t}(k) - \overline{\mathbf{t}(-k)} = \int_{\mathbb{R}^2} e_k(x) q(x) \mu(x,k) - e_k(x) q(x) \overline{\mu(x,-k)}\, dm(x) = 0.$$

To obtain the result for general q, we approximate $q \in L^p_\rho(\mathbb{R}^2)$ by a sequence $\{q_n\}$ from $C_c^\infty(\mathbb{R}^2)$ and appeal to Lemma 3.4.2. $\square$

Chapter 4 The Inverse Scattering Transform

In the previous chapter we found the decay properties of the scattering transform $\mathbf{t}$. We can now use these properties in the $\overline{\partial}_k$-equation (1.18) to recover differentiability of $\mu(x,k,\tau)$ in x and τ and to write out the large-k expansion for $\mu(x,k,\tau)$ from equation (1.22). In Section 4.3, we will prove that the reconstructed $q = i\overline{\partial}_x a_1$ are real, but we use an argument with differential forms that relies on the coefficient $\mathbf{s}(k)$ being smooth. To get around the smoothness requirement, we prove continuity of $\mathcal{Q}$ in $\mathbf{t}$ in Section 4.2. After we prove q is real for smooth $\mathbf{s}$, we may take a limit $\mathbf{s}_n \to \mathbf{s}$ to prove the reality for q reconstructed from more general $\mathbf{s}$. Arguments in Section 4.1 are from [22], and arguments in the rest of the chapter are from [23].

4.1 Properties of Reconstructed CGO Solutions

In the next two lemmas, we only need the results for equation (1.18) without time included. However, under the flow for the Novikov-Veselov equation, the scattering transform for later times τ becomes $e^{i\tau(\bar{k}^3+k^3)}\mathbf{t}(k)$. The extra phase does not change the $L^p(\mathbb{C})$ space properties of $\mathbf{s}(k)$, but proving the results in the generality here will allow their use in our study of the Novikov-Veselov equation at times $\tau \neq 0$. We therefore define the exponent $S(x,k,\tau) = -(kx+\bar{k}\bar{x})/\tau + (k^3+\bar{k}^3)$. The evolved $\bar{\partial}_k$ equation (1.18) for $\mu(x,k,\tau)$ then can be written as

$$\begin{cases} \bar{\partial}_k \mu(x,k,\tau) = e^{i\tau S}\mathbf{s}(k)\overline{\mu(x,k,\tau)} \\ \mu(x,\,\cdot\,,\tau) - 1 \in L^r(\mathbb{C}) \end{cases}$$

for some $r \in (2,\infty)$. The operator we must study is then

$$T_x f(k) = \bar{\partial}_k^{-1}(e^{i\tau S}\mathbf{s}(k)\overline{f})(k).$$

The function $(\mu-1)$ solves the integral equation $(\mu-1) = T_x 1 + T_x(\mu-1)$. Inverting the operator yields

$$\mu - 1 = [I - T_x]^{-1} T_x 1. \tag{4.1}$$

We prove the results in this chapter using only the properties of $\mathbf{s}$ proved in Lemma 3.3.2 and Theorem 3.4.3. We gather these into the definition of the space $\mathcal{X}^{\epsilon}_{n,r}$

Definition 4.1.1. *The space $\mathcal{X}^{\epsilon}_{n,r}$ for $n \geq 1$, $r \in (1,2)$, and $\epsilon > 0$ is the closure of $C_c^\infty(\mathbb{C})$ functions which satisfy the relation $\bar{k}f(k) = -k\overline{f(-k)}$ in the norm*

$$\|f\|_{\mathcal{X}^{\epsilon}_{n,r}} = \|f\|_{L^2(\mathbb{C})} + \|(\,\cdot\,)^n f(\,\cdot\,)\|_{L^{r'+\epsilon}(\mathbb{C})} + \|(\,\cdot\,)^n f(\,\cdot\,)\|_{L^r(\mathbb{C})}.$$

For $n = 0$ the norm is

$$\|f\|_{\mathcal{X}^{\epsilon}_{0,r}} = \|f\|_{L^2(\mathbb{C})} + \|f(\,\cdot\,)\|_{L^{r'+\epsilon}(|k|>1)} + \|f(\,\cdot\,)\|_{L^r(\mathbb{C})}.$$

Remark 4.1.2. We saw in Theorem 3.4.3 that if q is real valued, then $\bar{k}\mathbf{s}(k) = -k\overline{\mathbf{s}(-k)}$, and we will see in Lemma 4.3.1 that this symmetry requirement guarantees that the reconstructed potential is real-valued. The conditions in this definition are preserved under the linearized NV flow (1.15). By Lemma 3.3.2, scattering transforms $\mathbf{s}(k)$ coming from potentials $q \in W^{n,p}_{\rho}(\mathbb{R}^2)$ with $p \in (1,2)$ and $\rho \in (2/p', \infty)$ are in $\mathcal{X}^{\epsilon}_{n,r}$ for all $r \in (\tilde{p}', \infty)$ and $\epsilon > 0$.

When $\mathbf{s} \in L^2(\mathbb{R}^2)$, the operator T_x is compact on L^p for $p \in (2, \infty)$. The proof can be found in the preprint to Nachman's 1996 paper [25, Lemma 4.2], but we reproduce it here for the reader's convenience.

Lemma 4.1.3. *If* $\mathbf{s}(k) \in L^2(\mathbb{C})$, *then the operator* $T = \bar{\partial}_k^{-1}(\mathbf{s}(k)\bar{\cdot})$ *is compact on* $L^p(\mathbb{C})$ *for all* $p \in (2, \infty)$.

Proof. We will prove the result for the dual operator $\mathbf{s}(k)\bar{\partial}_k^{-1}$ on $L^q(\mathbb{C})$ for $q \in (1, 2)$. First, we have that the operator is bounded on $L^q(\mathbb{C})$. Take $f \in L^q(\mathbb{C})$, then by Theorem 1.2.4 (ii) with $1/\tilde{q} = 1/q - 1/2$

$$\|\mathbf{s}(k)\bar{\partial}_k^{-1} f(\,\cdot\,))\|_{L^q(\mathbb{C})} \leq \|\mathbf{s}(\,\cdot\,)\|_{L^2(\mathbb{C})} \|\bar{\partial}_k^{-1} f(\,\cdot\,)\|_{L^{\tilde{q}}(\mathbb{C})} \leq c\|\mathbf{s}\|_{L^2(\mathbb{C})}\|f\|_{L^q(\mathbb{C})}. \tag{4.2}$$

Now assume that $\mathbf{s}(k)$ is continuous with compact support. We have $\partial_k\bar{\partial}_k^{-1} f \in L^q(\mathbb{C})$ by Lemma 1.2.8. We estimate $\nabla \mathbf{s}\bar{\partial}_k^{-1} f$ by taking the $\bar{\partial}_k$ and ∂_k derivatives separately. The estimate is

$$\|\partial_k \mathbf{s}\bar{\partial}_k^{-1} f\|_{L^q(\mathbb{C})} \leq \|\partial_k \mathbf{s}\|_{L^2(\mathbb{C})}\|\bar{\partial}_k^{-1} f\|_{L^{\tilde{q}}(\mathbb{C})} + \|\mathbf{s}\|_{L^\infty(\mathbb{C})}\|\partial_k\bar{\partial}_k^{-1} f\|_{L^q(\mathbb{C})} \leq c\|f\|_{L^q(\mathbb{C})}.$$

The same holds for the $\bar{\partial}_k$ derivative. Thus, we have the inequality $\|\mathbf{s}\bar{\partial}_k^{-1} f\|_{W^{1,q}} \leq c\|f\|_{L^q}$, and since $\mathbf{s}$ has compact support, we can use the Rellich-Kondrachov theorem 1.2.2 to show $\mathbf{s}\bar{\partial}_k^{-1}$ is a compact operator on $L^q(\mathbb{C})$. For $\mathbf{s}$ a general function in $L^2(\mathbb{C})$, we may approximate any T_x by compact operators and use inequality (4.2) to show T is compact on $L^q(\mathbb{C})$. □

By the Fredholm alternative, the compactness of T_x, and Lemma 2.1.2, $I - T_x$ is invertible.

Lemma 4.1.4. *If* $\mathbf{s}(k) \in L^r \cap L^2(\mathbb{C})$ *for some* $r \in (1, 2)$ *then the operator* $I - T_x$ *is invertible on* $L^{\tilde{r}}(\mathbb{C})$ *and there is a unique solution of equation* (1.18) *satisfying* $\mu(x, \,\cdot\,, \tau) - 1 \in L^{\tilde{r}}(\mathbb{C})$.

Proof. From Lemma 4.1.3, the operator T_x is compact on $L^{\tilde{r}}(\mathbb{C})$ so $I - T_x$ is Fredholm. Assume $h \in L^{\tilde{r}}(\mathbb{C})$ solves $(I - T_x)h = 0$, then $\bar{\partial}_k h = e^{i\tau S}\mathbf{s}(k)\bar{h}$. By Lemma 2.1.2 with the coefficient $e^{i\tau S}\mathbf{s}(k) \in L^2(\mathbb{C})$ we get $h \equiv 0$. Thus $I - T_x$ is invertible on $L^{\tilde{r}}(\mathbb{C})$.

To construct a solution, we note that formally

$$\mu(x, k, \tau) - 1 = [I - T_x]^{-1} T_x 1.$$

By definition $T_x 1 = \bar{\partial}^{-1}[e^{i\tau S}\mathbf{s}(k)]$, so using Lemma 1.2.4 we have $T_x 1$ is in $L^{\tilde{r}}(\mathbb{C})$. Thus $\mu = 1 + [I - T_x]^{-1} T_x 1$ solves (1.18). □

Equation (1.18) is conjugate linear. This makes it harder to prove differentiability of μ in the x variable by looking at the $\bar{\partial}_x$ and ∂_x derivatives. Instead we will take real derivatives in the $x = (x_1, x_2)$ variables written as

$$D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index.

Lemma 4.1.5. *If $\mathbf{s} \in \mathcal{X}_{n,r}^{\epsilon}$ then the unique solution $\mu(x, \cdot, \tau) - 1 \in L^{\tilde{r}}(\mathbb{C})$ of equation (1.18) is α times differentiable in x and m times differentiable in τ for $(3m + |\alpha|) \le n$. Additionally, the derivatives of the map $(x, \tau) \to \mu(x, \cdot, \tau) \in L^{\tilde{r}}(\mathbb{C})$ satisfy $\partial_\tau^m D_x^\alpha \mu(x, \cdot, \tau) \in L^r(\mathbb{C})$. The derivatives are given by*

$$\partial_\tau^m D_x^\alpha \mu(x, k, \tau) = [I - T_x]^{-1}\bar{\partial}^{-1}[\mathbf{s}(k) f(x, k, \tau)] \tag{4.3}$$

where

$$f(x, k, \tau) = \partial_\tau^m D_x^\alpha[e^{i\tau S}\overline{\mu(x, k, \tau)}] - e^{i\tau S}\partial_\tau^m D_x^\alpha \overline{\mu(x, k, \tau)}, \tag{4.4}$$

and $\bar{\partial}^{-1}[\mathbf{s}(k) f(x, k, \tau)] \in L^r(\mathbb{C})$.

Proof. We illustrate the case $\alpha = (1, 0)$, $m = 0$. In the following, let $h \in \mathbb{R}$ and therefore $x + h = (x_1 + h) + ix_2$. The function

$$D_h \mu(x + h, k, \tau) = \frac{\mu(x + h, k, \tau) - \mu(x, k, \tau)}{h}$$

is in $L^{\tilde{r}}(\mathbb{R}^2)$ and satisfies the equation

$$\bar{\partial}_k D_h \mu(x, k, \tau) = \mathbf{s}(k)\left(\overline{\mu(x + h, k, \tau)} D_h e^{i\tau S} + e^{i\tau S}\overline{D_h \mu(x, k, \tau)}\right).$$

By dominated convergence, we have $\mathbf{s}(k) D_h e^{i\tau S} \to i(k + \bar{k})\mathbf{s}(k) e^{i\tau S} \in L^2(\mathbb{C})$ because $|k|\mathbf{s}(k) \in L^2$. We also have $\mu(x + h, k, \tau) - 1 \to \mu(x, k, \tau) - 1 \in L^r(\mathbb{C})$ by continuity, and by the $L^r(\mathbb{C})$ continuity of the operators T_x and $[I - T_x]^{-1}$ we get

$$\frac{\partial}{\partial x_1}\mu(x, k, \tau) = [I - T_x]^{-1}\bar{\partial}^{-1}[i(k + \bar{k})e^{i\tau S}\mathbf{s}(k)\overline{\mu(x, k, \tau) - 1} + i(k + \bar{k})e^{i\tau S}\mathbf{s}(k)].$$

By the definition of $\mathcal{X}_{n,r}^{\epsilon}$, the function $i(k + \bar{k})e^{i\tau S}\mathbf{s}(k)$ is in $L^r(\mathbb{C})$ and $\mathbf{s}(k) \in L^2(\mathbb{C})$. Therefore we have $\mathbf{s}(k)(\mu(x, k, \tau) - 1)$ is in $L^r(\mathbb{C})$. Lemma 1.2.4 then shows

$$\bar{\partial}^{-1}[i(k + \bar{k})e^{i\tau S}\mathbf{s}(k)\overline{\mu(x, k, \tau) - 1} + i(k + \bar{k})e^{i\tau S}\mathbf{s}(k)] \in L^r(\mathbb{C}).$$

Derivatives in τ follow in the same manner except with factors of $(k^3 + \bar{k}^3)$ pulled down from the exponential. $\square$

Now that we know the derivatives of the reconstructed CGO solutions exist, we can show that the large-k expansion from equation (1.22) exists when $\mathbf{s}$ is in $\mathcal{X}_{n,r}^{\epsilon}$. In fact, we may differentiate some of the coefficients, $a_j(x, \tau)$, if n is large enough.

Lemma 4.1.6. *Suppose* $\mathbf{s} \in \mathcal{X}^{\epsilon}_{n,r}$. *If* μ *solves the equation* (1.18) *with* $\mu(x,\,\cdot\,)-1 \in L^{\tilde{r}}(\mathbb{C})$, *then* μ *admits the large-*k *expansion* (1.22) *for fixed* x *and* τ. *Moreover, we may take* α *spatial derivatives and* m *time derivatives with* $(|\alpha|+3m) \le n$ *of* μ *to get*

$$\partial_\tau^m D_x^\alpha(\mu(x,k,\tau)-1) = \sum_{j=1}^{n-|\alpha|-3m} \frac{\partial_\tau^m D_x^\alpha a_j(x,\tau)}{k^j} + o\left(|k|^{-n+|\alpha|+3m}\right).$$

Proof. Note that

$$\mu(x,k,\tau) = 1 + \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{k-k'} e^{i\tau S(x,k,\tau)} \mathbf{s}(k')\overline{\mu(x,k',\tau)}dm(k').$$

Expanding $(k-k')^{-1}$ as the sum

$$\frac{1}{k-k'} = \sum_{j=1}^{n} \frac{k'^{j-1}}{k^j} + \frac{k'^n}{k^n(k-k')}, \tag{4.5}$$

we have

$$a_j(x,\tau) = \frac{1}{\pi}\int_{\mathbb{C}} k'^{j-1} e^{i\tau S(x,k',\tau)} \mathbf{s}(k')\overline{\mu(x,k',\tau)}dm(k')$$

with remainder

$$R_n(x,k,\tau) = \frac{1}{\pi}k^{-n}\int_{\mathbb{C}} \frac{k'^n}{k-k'} e^{i\tau S(x,k',\tau)} \mathbf{s}(k')\overline{\mu(x,k',\tau)}d(k').$$

We look at $\Omega_1 = \{k' : |k'| \le 1\}$ and $\Omega_2 = \mathbb{C}\setminus\Omega_1$ separately. The function $\mathbf{s}(k)$ is in $L^2(\Omega_1)$, and $\mu(x,\,\cdot\,,\tau)$ is in $L^{\tilde{r}}(\Omega_1)$. The integral from R_n over this region decreases like $|k|^{-1}$, and the total contribution of to the integral over this region is therefore $|k|^{-(n+1)} = o(|k|^{-(n+1)})$.

In Ω_2, we have that $k^n\mathbf{s}(k)$ is in $L^2 \cap L^{r'+\epsilon}(\Omega_2)$ by definition. We also have $D^\alpha(\mu(x,\,\cdot\,,\tau)-1) \in L^{\tilde{r}}(\mathbb{C})$ by Lemma 4.1.5. Combining these two results shows that the product is in $L^{r_1} \cap L^{r_2}(\mathbb{C})$ for

$$0 < \frac{1}{r_1} = \frac{1}{r'+\epsilon} + \frac{1}{\tilde{r}} < \frac{1}{2} < \frac{1}{2} + \frac{1}{\tilde{r}} = \frac{1}{r_2} < 1.$$

Therefore, we have the inequality $1 < r_2 < 2 < r_1 < \infty$, and by Lemma 1.2.4, we have the asymptotic $R_n(x,k,\tau) = o(|k|^{-n})$.

To show decay of the derivatives, we look at the difference quotients. As in Lemma 4.1.5 we will take derivatives with respect to x_1. Let h be a real number, then we have

$$\begin{aligned} D_h a_j(x,\tau) =& \frac{1}{\pi}\int_{\mathbb{C}} k'^{j-1} e^{i\tau S(x,k',\tau)} \frac{e^{i(k'h+\bar{k}'h)}-1}{h} \mathbf{s}(k')\overline{\mu(x+h,k',\tau)}dm(k') \\ &+ \frac{1}{\pi}\int_{\mathbb{C}} k'^{j-1} e^{i\tau S(x,k',\tau)} \mathbf{s}(k')\overline{D_h\mu(x,k',\tau)}dm(k') \\ =& I + II. \end{aligned}$$

From Lemmas 4.1.5, we have the following limits in the $L^{\tilde{r}}(\mathbb{C})$ topology:

$$\begin{aligned} \mu(x+h,\,\cdot\,,\tau)-1 &\to \mu(x,\,\cdot\,,\tau)-1,\\ D_h\mu(x,\,\cdot\,,\tau) &\to \partial_{x_1}\mu(x,\,\cdot\,). \end{aligned} \tag{4.6}$$

By Lemma 3.3.2, we have the limit

$$k'^{j-1}\mathbf{s}(k')\frac{e^{i(k'h+\bar{k}'h)}-1}{h} \to k'^{j-1}i(k'+\bar{k}')\mathbf{s}(k) \tag{4.7}$$

in the $L^{r'}(\mathbb{C})$ topology when $j \le n$. Together, the limits (4.6) and (4.7) give convergence of the difference quotients.

Using the same methods, we may get the derivatives for $R_n(x,k,\tau)$, but because we get extra factors of k' (3 for every time derivative, 1 for every space derivative) we may only take the expansion to order $(n-|\alpha|-3m)$ in order for the derivatives of $R_{n-|\alpha|-3m}(x,k,\tau)$ to exist. □

Lemma 4.1.6 for $n=2$ allows us to prove that the inverse scattering transform recovers q at time zero.

Theorem 4.1.7. *If $q \in W^{2,p}_\rho(\mathbb{R}^2)$ with $p \in (1,2)$ and $\rho > 2/p'$ is subcritical or critical then $\mathcal{Q}[\mathcal{T}q] = q$.*

Proof. The solutions $\mu(x,k)$ from the equations (1.18) and (1.9) are the same by Theorem 2.3.2, so we may plug in the large k expansion of μ into equation (1.9). By our assumptions on q and Lemma 4.1.6 we may take 2 x-derivatives of $\mu(x,\,\cdot\,)-1 \in L^r(\mathbb{C})$. The large-$k$ expansion of μ gives us $|\bar{\partial}_x\partial_x\mu((x,k)| = o(1)$ as $|k| \to \infty$ since q has two derivatives. Thus, using equation (1.9) and formula 1.23 for $a_1(x)$, we write

$$\begin{aligned} q(x)\mu(x,k) =&\bar{\partial}_x(\partial_x+ik)\mu(x,k)\\ =&\frac{i}{\pi}\bar{\partial}_x\int_{\mathbb{C}} e_{-k}(x)s(k)\overline{\mu(x,k)}\,dm(k)+o(1). \end{aligned}$$

Taking the limit as $|k| \to \infty$, and using the fact that $\mu(x,k) \to 1$ point-wise as $|k| \to \infty$, we have

$$q(x) = \frac{i}{\pi}\bar{\partial}_x\int_{\mathbb{C}} e_{-k}(x)s(k)\overline{\mu(x,k)}\,dm(k).$$

□

4.2 Continuous Dependence of Reconstructed q on t(k)

In this section we develop the continuity theory surrounding the scattering map $\mathcal{Q}$. We use the continuity of $\mathcal{Q}$ when proving the reality of the reconstructed potential $q=\bar{q}$.

We note the following estimates.

Lemma 4.2.1. *Suppose that* $\mathbf{s} \in L^2(\mathbb{C})$ *and* $r \in (1,2)$. *Then* $(I - T_{x,\tau})^{-1}$ *exists as an operator in* $\mathcal{B}(L^{\tilde{r}}(\mathbb{R}^2))$ *and*

$$\sup_{(x,\tau)\in\mathbb{R}^2\times\mathbb{R}} \left\|(I - T_{x,\tau})^{-1}\right\|_{\mathcal{B}(L^{\tilde{r}})} < \infty. \tag{4.8}$$

Proof. The operator $T_{x,\tau}$ is continuous in its parameters in the strong operator topology. To see this, note that for $f \in L^{\tilde{r}}(\mathbb{R}^2)$

$$T_{x',\tau'}f - T_{x,\tau}f = \overline{\partial}_k^{-1}\left[\left(e^{i\tau' S(\,\cdot\,,x',\tau')} - e^{i\tau S(\,\cdot\,,x,\tau)}\right)\mathbf{s}f\right].$$

The function $\mathbf{s}f$ is in $L^r(\mathbb{C})$, and the multiplication operator $e^{i\tau' S(\,\cdot\,,x',\tau')} - e^{i\tau S(\,\cdot\,,x,\tau)}$ converges to zero in the strong operator topology on $L^r(\mathbb{C})$. Since $\overline{\partial}_l^{-1}$ is a bounded map from $L^r(\mathbb{C})$ to $L^{\tilde{r}}(\mathbb{C})$, $T_{x,\tau}$ is continuous in x and τ in the strong operator topology. By Lemma 4.1.4, $(I - T_{x,\tau})$ is always invertible, so estimate (4.8) holds for (x,τ) in any compact set.

To handle the region outside the compact set, we will prove the estimates:

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}^2,\ |\tau|\geq T} \|T_{x,\tau}\|_{\mathcal{B}(L^{\tilde{r}})} = 0 \tag{4.9}$$

and for each $T > 0$

$$\lim_{R\to\infty} \sup_{|\tau|\leq T,\ |x|\geq R} \|T_{x,\tau}\|_{\mathcal{B}(L^{\tilde{r}})} = 0. \tag{4.10}$$

Instead of a direct proof, we will prove the result for the dual operator $T'_{x,\tau} = e^{i\tau S}\mathbf{s}(k)\overline{\partial}_k^{-1}$. First, note that $T'_{x,\tau} = e^{i\tau S}W$ where $W = \mathbf{s}\overline{\partial}_k^{-1}$ is a compact operator independent of (x,τ). We have the estimate $\left\|T'_{x,\tau}\right\|_{\mathcal{B}(L^{\tilde{r}'})} \leq \|W\|_{\mathcal{B}(L^{\tilde{r}'})}$ uniform in (x,τ). For any $\epsilon > 0$ there is a finite-rank operator F on $L^{\tilde{r}'}(\mathbb{C})$ so that $\|W - F\|_{\mathcal{B}(L^{\tilde{r}'})} < \epsilon$. The operator F is a finite sum $\sum_j \langle \varphi_j, \,\cdot\,\rangle \psi_j$ where $\varphi_j \in L^{\tilde{r}'}(\mathbb{C})$, $\psi_j \in L^{\tilde{r}}(\mathbb{C})$, and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual dual pairing of $L^{\tilde{r}}(\mathbb{C})$ and $L^{\tilde{r}'}(\mathbb{C})$. Since $\mathcal{S}(\mathbb{C})$ is dense in $L^p(\mathbb{R}^2)$, we may take ψ_j and φ_j in $\mathcal{S}(\mathbb{C})$ without loss. Thus, it suffices to show that

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}^2,\ |\tau|\geq T} \left|\langle \varphi, e^{i\tau S}\psi\rangle\right| = 0 \tag{4.11}$$

and that, for each fixed $T > 0$,

$$\lim_{R\to\infty} \sup_{|\tau|\leq T,\ |x|\geq R} \left|\langle \varphi, e^{i\tau S}\psi\rangle\right| = 0 \tag{4.12}$$

where φ and ψ belong to $\mathcal{S}(\mathbb{C})$.

Let $f(k) = \psi(k)\varphi(k)$. A short computation shows that

$$\langle \varphi, e^{i\tau S}\psi\rangle = \int I_\tau(x-y)\left(\mathcal{F}^{-1}f\right)(y)\,dy$$

where I_τ is given by (1.30) and

$$\left(\mathcal{F}^{-1}f\right)(x) = \frac{1}{\pi}\int e_{-k}(x)f(k)\,dm(k).$$

We now appeal to Lemma 1.2.10 to estimate

$$\left|\langle \varphi, e^{i\tau S}\psi\rangle\right| \leq \int \tau^{-2/3}(1+|(x-y)/\tau^{1/3}|)^{-1/2}\left|\left(\mathcal{F}^{-1}f\right)(y)\right|\, dm(y). \tag{4.13}$$

To prove (4.11), we estimate the right-hand side of (4.13) by $T^{-2/3}\left\|\mathcal{F}^{-1}f\right\|_{L^1}$. This gives (4.9). To prove (4.12) we make the change of variables $\xi = (x-y)/\tau^{1/3}$ and obtain

$$\begin{aligned}\left|\langle \varphi, e^{i\tau S}\psi\rangle\right| &\leq \int (1+|\xi|)^{-1/2}\left|\left(\mathcal{F}^{-1}f\right)(x-\xi\tau^{1/3})\right|\, dm(\xi)\\ &\leq C(f,N)(I_1(x,\tau)+I_2(x,\tau))\end{aligned}$$

where $C(f,N)$ depends on the Schwartz seminorms of f and

$$\begin{aligned}I_1(x,\tau) &= \int_{|\xi|\leq|x|/(2T^{1/3})}(1+|\xi|)^{-1/2}(1+|x-\xi\tau^{1/3}|)^{-N}\, dm(\xi)\\ I_2(x,\tau) &= \int_{|\xi|\geq|x|/(2T^{1/3})}(1+|\xi|)^{-1/2}(1+|x-\xi\tau^{1/3}|)^{-N}\, dm(\xi).\end{aligned}$$

For $|\tau|>1$, equation (4.12) now follows from the preceding arguments and the elementary estimates

$$\begin{aligned}|I_1(x,\tau)| &\leq c(N)\left(\frac{|x|}{2T^{1/3}}\right)^2(1+|x|)^{-N}\\ |I_2(x,\tau)| &\leq c(N)\left(1+\frac{|x|}{2T^{1/3}}\right)^{-1/2}.\end{aligned}$$

For $|\tau|<1$, we note that the we can rewrite the operator $T'_{x,\tau}$ as

$$T'_{x,\tau} = e^{i(\tau-3)S(x,k,\tau-3)}\left[e^{3i(k^3+\bar{k}^3)}\mathbf{s}(k)P_k\right].$$

Equation (4.12) then holds for $|\tau-3|>1$ as well. This implies (4.10). □

Given the uniform resolvent bounds, we can use the second resolvent formula and the continuous dependence of $T_{x,\tau}$ in $\mathcal{B}(L^{\tilde{r}}(\mathbb{C}))$ on its parameters to prove various continuity results about the resolvent.

Lemma 4.2.2. *Suppose that* $\mathbf{s}\in L^2(\mathbb{C})$. *For any* $\tilde{r}\in(2,\infty)$,

(i) The mapping

$$\begin{aligned}\mathbb{R}^2\times\mathbb{R} &\longrightarrow \mathcal{B}(L^{\tilde{r}})\\ (x,\tau) &\mapsto (I-T_{x,\tau})^{-1}\end{aligned}$$

is continuous.

(ii) The mapping

$$\begin{aligned}L^2(\mathbb{C}) &\longrightarrow C_0(\mathbb{R}^2\times\mathbb{R},\mathcal{B}(L^{\tilde{r}}))\\ \mathbf{s} &\mapsto \left((x,\tau)\mapsto(I-T_{x,\tau})^{-1}\right)\end{aligned}$$

is continuous.

Proof. (i) This was already proved in the proof of Lemma 4.2.1.

(ii) Fix (x,τ) and write $T_{x,\tau} = T_{x,\tau}(\mathbf{s})$ to emphasize the dependence on $\mathbf{s}$. From the second resolvent formula

$$\begin{aligned}(I - T_{x,\tau}(\mathbf{s}_1))^{-1} - (I - T_{x,\tau}(\mathbf{s}_2))^{-1} \\ = (I - T_{x,\tau}(\mathbf{s}_2))^{-1}\left[T_{x,\tau}(\mathbf{s}_1) - T_{x,\tau}(\mathbf{s}_2)\right](I - T_{x,\tau}(\mathbf{s}_1))^{-1}\end{aligned}$$

and the fact that

$$\sup_{(x,\tau)\in\mathbb{R}^2\times\mathbb{R}} \|T_{x,\tau}(\mathbf{s}_1) - T_{x,\tau}(\mathbf{s}_2)\|_{\mathcal{B}(L^{\tilde{r}})} \leq C_{\tilde{r}}\|\mathbf{s}_1 - \mathbf{s}_2\|_2$$

we easily deduce that $\left\|(I - T_{x,\tau}(\mathbf{s}))^{-1}\right\|_{\mathcal{B}(L^{\tilde{r}})}$ is bounded uniformly in (x,τ) for $\mathbf{s}$ in a small metric ball in L^2 whose radius depends on the center but is uniform in (x,τ). For $\mathbf{s}_1$ and $\mathbf{s}_2$ in such a metric ball B we may estimate

$$\sup_{(x,\tau)\in\mathbb{R}^2\times\mathbb{R}} \left\|(I - T_{x,\tau}(\mathbf{s}_1))^{-1} - (I - T_{x,\tau}(\mathbf{s}_2))^{-1}\right\|_{\mathcal{B}(L^{\tilde{r}})} \leq C(B)\|\mathbf{s}_1 - \mathbf{s}_2\|_2.$$

This gives the claimed continuity. □

We have already established in Lemma (4.1.5) that, for $\mathbf{s} \in \mathcal{X}^{\epsilon}_{n,r}$, the derivatives $\partial_\tau^m D_x^\alpha\left(\mu(x,\,\cdot\,,\tau)\right)$ exist in $L^{\tilde{r}}(\mathbb{C})$, provided $(3m + |\alpha|) \leq n$. We are now ready to prove that the maps from $\mathbf{s}$ to μ, its derivatives, and q are continuous.

Lemma 4.2.3. *Given $r \in (1,2)$ and $n \geq 0$*

(i) For any m and α such that $3m + |\alpha| \leq n$, the maps

$$\begin{aligned}\mathcal{X}^{\epsilon}_{n,r} &\longrightarrow C(\mathbb{R}^2\times\mathbb{R}, L^{\tilde{r}}(\mathbb{C})) \\ \mathbf{s} &\mapsto \Big(\,(x,\tau) \mapsto \partial_\tau^m D_x^\alpha(\mu(x,\,\cdot\,,\tau) - 1)\Big)\end{aligned}$$

are continuous.

(ii) For any m and α such that $3m + |\alpha| + 2 \leq n$, the map

$$\begin{aligned}\mathcal{X}^{\epsilon}_{n,r} &\mapsto C_0(\mathbb{R}^2\times\mathbb{R}) \\ \mathbf{s} &\to \partial_\tau^m D_x^\alpha q(x,\tau)\end{aligned}$$

is continuous.

Proof. For any $n \geq 3m + |\alpha|$ we will show the map

$$\begin{aligned}\mathcal{X}^{\epsilon}_{n,r} &\longrightarrow C(\mathbb{R}^2\times\mathbb{R}, L^{\tilde{r}}(\mathbb{C})) \\ \mathbf{s} &\mapsto \Big(\,(x,\tau) \mapsto \partial_\tau^m D_x^\alpha[T_{x,\tau}(\mathbf{s})(1)]\Big)\end{aligned}$$

is continuous. From the computation

$$\partial_\tau^m D_x^\alpha T_{x,\tau}1 = \overline{\partial}^{-1}\left[e^{i\tau S}(-2ik_2)^{\alpha_1}(2ik_1)^{\alpha_2}[i(k^3+\bar{k}^3)]^m\mathbf{s}(\,\cdot\,)\right] \tag{4.14}$$

and Theorem 1.2.4, we see that it suffices to bound

$$\left\| | \cdot |^{\ell} (\mathbf{s}_1(\,\cdot\,) - \mathbf{s}_2(\,\cdot\,)) \right\|_{L^r(\mathbb{C})} \leq c \left\| \mathbf{s}_1 - \mathbf{s}_2 \right\|_{\mathcal{X}^{\epsilon}_{n,r}},$$

where $\ell = 3m + |\alpha| \leq n$. This is immediate from the definition.

We need to check that the expression (4.3) defines a continuous $C(\mathbb{R}^2 \times \mathbb{R}; L^{\tilde{r}}(\mathbb{C}))$-valued function of $\mathbf{s}$. By Lemma 4.2.2(ii), it suffices to show that $\mathbf{s} \mapsto \overline{\partial}^{-1} f$ (with f given by (4.4)) has the same property. A typical term of $\overline{\partial}^{-1} f$ takes the form

$$\overline{\partial}^{-1} \left[\mathbf{s}(\,\cdot\,) e^{i\tau S} k^{\beta} (i((\,\cdot\,)^3 + (\overline{\cdot})^3))^{\ell} \left(\partial_\tau^{\ell} \partial_x^{\beta} \mu \right) (x, \,\cdot\,, \tau) \right]$$

where $|\beta| + \ell \leq m - 1$. We have $|k|^{m-1} b f s \in L^2(\mathbb{C})$ and we assume inductively that the derivatives $\partial_\tau^{\ell} \partial_x^{\beta} \mu$ are continuous $C(\mathbb{R}^2 \times \mathbb{R}, L^{\tilde{r}}(\mathbb{C}))$-valued functions of $\mathbf{s}$ so that $f \in L^r(\mathbb{C})$. We can now use Theorem 1.2.4 to obtain the required continuity. Thus it remains to prove that $\mu(x, \,\cdot\,, k) - 1$ is a continuous $C(\mathbb{R}^2 \times \mathbb{R}; L^{\tilde{r}}(\mathbb{C}))$-valued function of $\mathbf{s}$. This is an immediate consequence of Lemma 4.2.2(ii) and equation (4.14).

Part (ii) follows directly from part (i). The proof is the same as in Lemma 4.1.5. The result follows from equation (1.24), part (i), and the fact that $\mathbf{s} \in \mathcal{X}^{\epsilon}_{n+1,r} \subset L^1_n$. □

4.3 Reality of Reconstructed q from the Symmetries of t(k)

We now use the symmetry from Section 3.4 to prove that q reconstructed from formula 1.24 is real. Our proof follows ideas of Grinevich and Manakov [13, Theorem 2] from the nonzero-energy Schrödinger equation adapted to zero-energy.

Lemma 4.3.1. *If $\mathbf{t} \in C_c^{\infty}(\mathbb{C})$ with $\mathbf{t}(k) = 0$ for $|k| < \epsilon$ for some $\epsilon > 0$ and $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$ then $q(x)$ defined by equation* (1.24) *is real. Moreover, the solution $\mu(x,k)$ of the $\overline{\partial}_k$-equation* (1.18) *satisfies*

$$\overline{\partial}_x (\partial_x + ik) \mu(x,k) = q(x) \mu(x,k)$$

in distribution sense.

Proof. Consider the real differential form

$$\omega = \frac{\mu(x,k)\mu(x,-k)}{k}\, dk + \frac{\overline{\mu(x,k)\mu(x,-k)}}{\bar{k}}\, d\bar{k}.$$

Using the symmetry $\mathbf{t}(k) = \overline{\mathbf{t}(-k)}$ and (1.18), it is not difficult to see that ω is a closed form. It now follows by Stokes' Theorem applied to the region $R^{-1} \leq |k| \leq R$, the large-k asymptotic behavior of μ and $\bar{\mu}$, and the identities

$$\oint_\gamma \frac{dk}{k} = - \oint_\gamma \frac{d\bar{k}}{\bar{k}} = 2\pi i,$$

true for any simple closed contour γ, that

$$\mu(x,0)^2 = \left[\overline{\mu(x,0)} \right]^2. \tag{4.15}$$

Thus, $\mu(x,0)$ is either purely real or purely imaginary. Next consider the differential operators

$$P_1\psi = -\overline{\partial}_x\left(\partial_x + ik\right)\psi + q\psi$$
$$P_2\psi = -\partial_x\left(\bar{\partial}_x - i\bar{k}\right)\psi + q\psi$$

where q is defined by the expansion (1.22) to be $q = i\bar{\partial}_x a_1$. Let $\chi_1 = P_1\mu$ and $\chi_2 = P_2\overline{\mu}$. We need to show $\chi_1 = \chi_2 = 0$. From the expansion (1.22), we have

$$\lim_{|k|\to\infty} \chi_1(x,k) = 0.$$

By (4.15), we have

$$\chi_1(x,0)^2 - \chi_2(x,0)^2 = 0, \tag{4.16}$$

and χ_1 and χ_2 satisfy

$$\left(\overline{\partial}_k\chi_1\right)(x,k) = e_{-x}(k)\mathbf{s}(k)\chi_2(x,k)$$
$$\left(\partial_k\chi_2\right)(x,k) = e_x(k)\overline{\mathbf{s}(k)}\chi_1(x,k).$$

where, for each fixed x,

$$\lim_{|k|\to\infty}\chi_2(x,k) = \lim_{|k|\to\infty} i\partial_x\bar{a}_1 + q + O\left(|k|^{-1}\right) = i\partial_x\bar{a}_1(x) + q(x).$$

It is not (yet) clear that $i\partial_x\bar{a}_1(x)+q(x) = 0$ but we will prove this using the condition (4.16).

To this end, consider the one-form

$$\eta = \frac{\chi_1(x,k)\chi_1(x,-k)}{k}\,dk + \frac{\chi_2(x,k)\chi_2(x,-k)}{\bar{k}}\,d\bar{k}.$$

A computation analogous to the one for ω together with (4.16) shows that η is a closed form and that

$$\chi_1(x,\infty)^2 - \chi_2(x,\infty)^2 = \chi_1(x,0)^2 - \chi_2(x,0)^2$$

where $\chi_i(x,\infty)$ means $\lim_{|k|\to\infty}\chi_i(x,k)$ for $i = 1,2$. By (4.16), the right-hand side is zero, and also $\chi_1(x,\infty) = 0$. We may conclude then that $\chi_2(x,\infty) = 0$, thus $q = \overline{i\bar{\partial}_x a_1} = \overline{q}$ as desired. □

Combining the above result with Lemma 4.2.3 we have:

Corollary 4.3.2. *If* $\mathbf{s} \in \mathcal{X}^{\epsilon}_{2,r}(\mathbb{C})$ *for* $r \in (1,2)$ *and* $\epsilon > 0$ *then* $q(x)$ *defined by equation* (1.24) *is real. Moreover, the solution* $\mu(x,k)$ *of the* $\overline{\partial}_k$*-problem* (1.18) *satisfies*

$$\overline{\partial}_x(\partial_x + ik)\mu(x,k) = q(x)\mu(x,k)$$

in the sense of distributions.

It is now simple to prove the interesting fact that all scattering transforms in $\mathcal{X}^{\epsilon}_{n,r}$ come from critical or subcritical potentials.

Proposition 4.3.3. *If* $\mathbf{s} \in \mathcal{X}^{\epsilon}_{2,r}$ *then* $q(x) = [\mathcal{Q}\mathbf{t}](x)$ *is critical or subcritical.*

Proof. This result follows easily using an approximation argument. Define $q_j(x) = \mathcal{Q}(\mathbf{t}_j)(x)$ where $\mathbf{t}_j(k) = \chi_j(k)\mathbf{t}(k)$ and χ_j is a smooth radial function which is 1 for $|k| > 1/j$ and 0 for $|k| < 1/(2j)$. Nachman [26, Theorem 4.1] proves that the solutions $\mu_j(x,k)$ to equation (1.18) satisfy $\inf_{x,k} |\mu_j(x,k)| > 0$ and from (4.15) $\mu_j(x,0)$ is either real or imaginary. By multiplying this by the appropriate constant and using Corollary 4.3.2, we have $c\mu_j(k,0)$ is a positive solution to the Schrödinger equation with potential q_j. By Lemma 4.2.3 $q_j \in C(\mathbb{R}^2)$, so by [8, Theorem 2.12] q_j is critical or subcritical. By Lemma 4.2.3, on any compact subset $\Omega \subset \mathbb{R}^2$ we have

$$\lim_{j\to\infty} |q_j(x) - q(x)| = 0.$$

By Definition 1.3.1, a potential q is either critical or subcritical if the associated quadratic form (1.8) is nonnegative. That is, for every q_j and any $\psi \in C^{\infty}_{c}(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \frac{1}{4}|\nabla\psi|^2 + q_{\epsilon}|\psi|^2 \, dm(x) \geq 0.$$

Taking limits preserves the non-negativity and the result follows. □

Now we pay close attention to identities in the large-k expansion of the reconstructed CGO solutions $\mu(x,k,\tau)$ in order to find that $q = i\overline{\partial}_x a_1$ solves the Novikov-Veselov equation. We make use of only three facts now: $\mu(x,k,\tau)$ has the large-k expansion (1.22), $\mu(x,k,\tau)$ and the terms in its expansion satisfies the Schrödinger equation (1.9), and μ and its expansion satisfies the $\overline{\partial}_k$-equation (1.18). This method is similar to that from [7], but they express the identities in terms of integrals involving $\mu(x,k,\tau)$. Here, we are using the coefficients in the large-k expansion of $\mu(x,k,\tau)$. The results in this chapter may be found in [23].

Corollary 5.0.4. *Assume* $\mathbf{s} \in \mathcal{X}^{\epsilon}_{n+1,r}$ *for* $n \geq 2$ *and define* $q = i\overline{\partial}_x a_1$*. The following identities hold:*

$$i\overline{\partial}_x a_n = -\overline{\partial}_x \partial_x a_{n-1} + (i\overline{\partial}_x a_1)a_{n-1}, \tag{5.1}$$

which for $n = 2$ *simplifies to*

$$i\overline{\partial}_x a_2 = \overline{\partial}_x \left(-\partial_x a_1 + i\frac{a_1^2}{2} \right). \tag{5.2}$$

Additionally we have

$$i\partial_x a_2 = \partial_x \left(-\partial_x a_1 + i\frac{a_1^2}{2} \right). \tag{5.3}$$

Proof. Identity (5.1) follows by plugging in the large-k expansion (1.22) into (1.18). Equation (5.2) can be rearranged to

$$\overline{\partial}_x \left(ia_2 + \partial_x a_1 - i\frac{a_1^2}{2} \right) = 0.$$

We then get identity (5.3) by applying Liouville's theorem to the analytic function $ia_2 + \partial_x a_1 - i\frac{a_1^2}{2}$ and noting that a_1, a_2, and $\partial_x a_1$ are bounded. □

Lemma 5.0.5. *If* $\mathbf{s} \in \mathcal{X}^{\epsilon}_{4,r}$ *for* $r \in (1,2)$ *and* $\epsilon > 0$ *then the function* $\mu(x,k,\tau)$ *defined by* (1.18) *satisfies the equation*

$$(\partial_\tau - ik^3)\mu = \left[\overline{\partial}_x^3 + (\partial_x + ik)^3 - 3u(\partial_x + ik) - 3\overline{u}\overline{\partial}_x\right]\mu(x,k) \tag{5.4}$$

where

$$u = i\partial_x a_1. \tag{5.5}$$

Proof. We have the two equations

$$\overline{\partial}_x(\partial_x + ik)\mu(x,k,\tau) = q(x,\tau)\mu(x,k,\tau)$$

and

$$\overline{\partial}_k \mu(x,k,\tau) = \mathbf{s}(k,\tau) e_{-x}(k) \overline{\mu(x,k,\tau)}.$$

Expanding $\mu(x,k,\tau)$ up to order k^3, we have

$$\mu(x,k,\tau) = 1 + \frac{a_1(x,\tau)}{k} + \frac{a_2(x,\tau)}{k^2} + \frac{a_3(x,\tau)}{k^3} + o(k^{-3}). \tag{5.6}$$

By commuting the following specially chosen differential operators through the $\bar{\partial}_k$ equation, we obtain the three identities:

$$\overline{\partial}_k (\partial_\tau - ik^3)\mu = \mathbf{s}(k,\tau) e_{-x}(k) \overline{(\partial_\tau - ik^3)\mu},$$

$$\overline{\partial}_k [\overline{\partial}_x^3 + (\partial_x + ik)^3]\mu = \mathbf{s}(k,\tau) e_{-x}(k) \overline{\left[\overline{\partial}_x^3 + (\partial_x + ik)^3\right] \mu},$$

and

$$\overline{\partial}_k [-3\bar{u}\overline{\partial}_x - 3u(\partial_x + ik)]\mu = \mathbf{s}(k,\tau) e_{-x}(k) \overline{\left[-3\bar{u}\overline{\partial}_x - 3u(\partial_x + ik)\right] \mu}.$$

We combine these in such a way that all nonnegative powers of k cancel out in order to obtain a formula for $\partial_\tau \mu$. Adding the three identities, we conclude that

$$\Psi = \left[\partial_\tau - (\overline{\partial}_x^3 + \partial_x^3 + 3ik\partial_x^2 - 3k^2\partial_x - 3u(\partial_x + ik) - 3\bar{u}\overline{\partial}_x)\right] \mu$$

satisfies

$$\overline{\partial}_k \Psi = \mathbf{s}(k,\tau) e_{-x}(k) \overline{\Psi}.$$

We will prove that $\Psi = O(k^{-1})$ so that Corollary 2.1.2 implies $\Psi \equiv 0$. Plugging (5.6) into the expression for Ψ, it is easy to see that we get no terms of order k^2 or higher. Collecting all terms of order k^1, we find

$$-3k\partial_x a_1 - 3iku = 0$$

which is zero by our choice of u in (5.5). Collecting terms of order k^0 gives us

$$3i\partial_x^2 a_1 - 3\partial_x a_1 - 3iua_1 = 3i\partial_x \left[\partial_x a_1 + ia_2 - \frac{i}{2} a_1^2\right] \tag{5.7}$$

which is zero by identity (5.3). Thus we have $\Psi = O(|k|^{-1})$, and by Theorem 1.2.6 $\Psi \equiv 0$. □

We expand (5.4) and take a $i\overline{\partial}_x$ derivative to get a formula for $\partial_\tau q$.

Corollary 5.0.6. *If* $\mathbf{s} \in \mathcal{X}_{5,r}^{\epsilon}$ *then the function* $q(x,\tau)$ *satisfies*

$$\partial_\tau q = \overline{\partial}_x^3 q + \partial_x^3 q - 3\partial_x(uq) - 3\overline{\partial}_x(\bar{u}q)$$

with

$$\partial_x q = \overline{\partial}_x u$$

Proof. We expand Ψ at order k^{-1} to get

$$\partial_\tau a_1 = \overline{\partial}_x^3 a_1 + \partial_x^3 a_1 + 3i\partial_x^2 a_2 - 3\partial_x a_3 - 3u\partial_x a_1 - 3iua_2 - 3\bar{u}\overline{\partial}_x a_1.$$

Applying the operator $i\overline{\partial}_x$ to both sides we get

$$\partial_\tau q = \overline{\partial}_x^3 q + \partial_x^3 q - 3\partial_x^2\overline{\partial}_x a_2 - 3i\partial_x\overline{\partial}_x a_3 - 3\overline{\partial}_x(u^2) + 3\overline{\partial}_x(ua_2) - 3\overline{\partial}_x(\bar{u}q).$$

Now we use equation (5.1) with $n = 2$ to get

$$\partial_\tau q = \overline{\partial}_x^3 q + \partial_x^3 q - 3\partial_x^2\overline{\partial}_x a_2 + 3\partial_x^2\overline{\partial}_x a_2 - 3\partial_x(qa_2) - 3\overline{\partial}_x(u^2) + 3\overline{\partial}_x(ua_2) - 3\overline{\partial}_x(\bar{u}q).$$

$$\partial_\tau q = \overline{\partial}_x^3 q + \partial_x^3 q - 3q\partial_x a_2 - 3\overline{\partial}_x(u^2) + 3u\overline{\partial}_x a_2 - 3\overline{\partial}_x(\bar{u}q).$$

We wish to show the evolution equation is equal to the NV evolution (1.6), so the final step is to prove the identity

$$-3q\partial_x a_2 + 3u\overline{\partial}_x a_2 - 3\overline{\partial}_x(u^2) = -3\partial_x(uq).$$

We use equations (5.2) and (5.3) to get

$$\begin{aligned}
3iq\partial_x\left(-\partial_x a_1 + i\frac{a_1^2}{2}\right) - 3iu\overline{\partial}_x\left(-\partial_x a_1 + i\frac{a_1^2}{2}\right) -3\overline{\partial}_x(u^2)& \\
&= -3q\partial_x u + 3u\partial_x q - 6u\overline{\partial}_x u \\
&= -3q\partial_x u - 3u\partial_x q \\
&= -3\partial_x(uq)
\end{aligned}$$

□

It now follows that the inverse scattering method yields solutions to the Novikov-Veselov equation.

Proof of Theorem 1.0.1. By Lemma 3.3.2, we have $\mathbf{s} \in \mathcal{X}_{5,r}^{\epsilon}$ when $q(x,0) \in W_\rho^{5,p}(\mathbb{R}^2)$ for $p \in (1,2)$, $\rho > 1$, $\epsilon > 0$, and $r \in (\tilde{p}', \infty)$. By Corollary 5.0.6, $q(x,\tau)$ solves the Novikov-Veselov equation. By Lemma 4.2.3 part (ii) $q(\,\cdot\,,\tau) \in C_0(\mathbb{R}^2 \times \mathbb{R})$ and hence $q(x,\tau) \to q(x,0)$, where $q(x,0)$, the inverse scattering transform of $\mathbf{s}(k,0)$, is the initial datum. By the same argument in Lemma 4.2.3 part (ii), we have $u(x,\tau) \in C_0(\mathbb{R}^2 \times \mathbb{R})$ making this the correct choice for u in (1.6). □

Chapter 6 Observations and Unsolved Problems

In a paper with Perry and Siltanen [24], we proved that certain supercritical perturbations of critical potentials have exceptional points. Suppose that $\psi_0 \in C^\infty(\mathbb{R}^2)$ is a real-valued, positive, radial function, and $\phi_0 - 1 \in C_c^\infty(B_1)$. Define a critical potential q by $q = (\overline{\partial}_x \partial_x \phi_0)/\phi_0$. Let $w \in C_c^\infty(B_1)$ be a nonnegative, radial function that is positive on a set of positive measure. We let $q_\lambda = q + \lambda w$ and $\mathbf{t}_\lambda = \mathcal{T}[q_\lambda]$. By Theorem 1.3.3 q_λ is supercritical for $\lambda < 0$ and subcritical for $\lambda > 0$.

Theorem 6.0.7. *[24, Theorem 1.2] Denote by $\mathbf{t}_\lambda$ the scattering transform of q_λ.*

1. *For $\lambda > 0$ sufficiently small, the exceptional set is empty and $\mathbf{t}_\lambda \in C^\infty$ away from $k = 0$.*
2. *For $\lambda < 0$ sufficiently small and a unique $r(\lambda) > 0$, the exceptional set is a circle C_λ of radius $r(\lambda)$ about the origin, and the function $\mathbf{t}_\lambda$ is C^∞ on $\mathbb{C} \setminus \{C_\lambda \cup 0\}$, while*
$$\lim_{|k| \to r(\lambda)} |\mathbf{t}_\lambda(k)| = \infty.$$
The radius $r(\lambda)$ obeys the formula
$$r(\lambda) = \exp\left(-\gamma + \frac{1 + O(|\lambda|)}{\mu(\lambda)}\right) \qquad \text{as } \lambda \uparrow 0$$
where γ is the Euler Mascheroni constant, and $\mu(\lambda)$ is the eigenvalue of the Dirichlet to Neumann operator q_λ corresponding to the constant function on S^1.

Part (1) follows from the work in this paper. What is more interesting is part (ii) where we have shown that supercritical potentials have a circle of exceptional points for small $\lambda < 0$. We proved the theorem by explicitly computing for which k the operator $I + S_k(\Lambda_q - \Lambda_0)$ has nontrivial kernel. Theorem 6.0.7 illustrates why the current incarnation of the inverse scattering method cannot be used to solve the Novikov-Veselov equation for supercritical potentials.

In the same paper [24], Siltanen computed the scattering transform for specific q_λ. We let $q_0 \equiv 0$ and define the perturbation $w(x)$ to be a radially symmetric function with profile
$$w(|x|) = \begin{cases} 1 & \text{for } 0 \le |x| \le R_1 \\ p(|x|) & \text{for } 0 \le |x| \le R_1 \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$
where $\tilde{p}(t) = 1 - 10t^3 + 15t^4 - 6t^5$ and
$$p(t) = \tilde{p}\left(\frac{t - R_1}{R_2 - R_1}\right).$$

We choose $R_1 = 0.8$ and $R_2 = 0.9$. $\lambda \in (-35, 35)$. If q_λ is radially symmetyric then so is $\mathbf{t}_\lambda(k)$. A plot of $\mathbf{t}_\lambda(|k|)$ is included in Figure 6.1. We see that an exceptional set appears for $\lambda < 0$. Additionally, you can see that around $\lambda = -8$ the exceptional

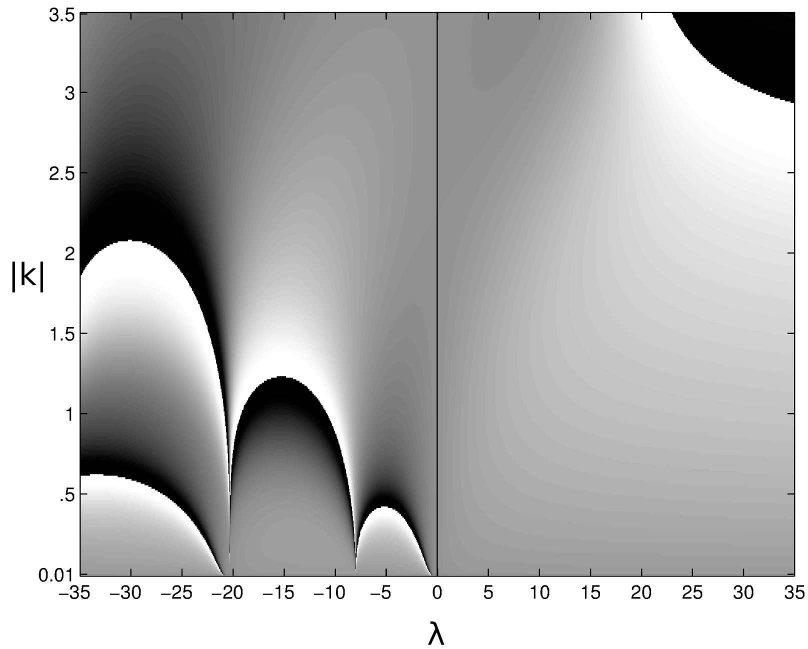


Figure 6.1: The function $\mathbf{t}_\lambda(|k|)$ for q_λ. The darker areas are where $\mathbf{t}_\lambda(|k|)$ is negative, and the lighter areas are $\mathbf{t}_\lambda(|k|)$ is positive.

set shrinks to a point. At $\lambda \approx -8.008$ there is a $L^r(\mathbb{R}^2)$ solution to the Schrödinger equation $(-\overline{\partial}_x \partial_x + q)\phi = 0$. We calculate the evolution of the Novikov-Veselov equation for potentials on both sides of this point. The evolutions were produced with code provided by Andreas Stahel which uses a Crank-Nicolson method [32]. We will focus on four potentials: q_8, q_{-7}, q_{-9}, and $q_{-8.008}$. The subcritical potential q_8 is expected to simply decay with time while we know nothing about the other functions should evolve given their singular scattering transforms. The initial data in Figure 6 for all the functions looks the same. For the negative functions we plot $-q_\lambda$ to make the evolutions easier to see. In Figure 6, we plot the evolutions at times $tau = 0.5$ and $\tau = 1.0$. We see that the subcritical potential q_8 decays as expected. The supercritical potential q_{-7} appears to be blowing up, $q_{-8.008}$ is stabilizing, and q_{-9} is splitting up into three spikes.

This leads to two conjectures: (1) when a potential has an eigenfunction at zero, the circle of singularities collapses, and (2) these potentials consist of a soliton part and a decaying part.

Finally, there is a theorem of Lassas, Mueller, and Siltanen [18] about the decay of certain reconstructed critical potentials, $q = \mathcal{Q}[\mathbf{t}]$, that we are unable to prove for subcritical potentials but likely still holds.

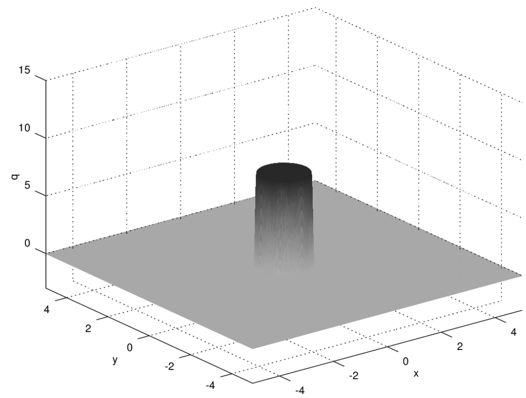


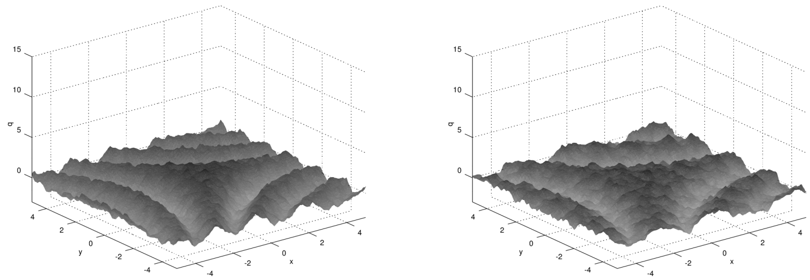
Figure 6.2: The initial value of the function q_8.

Theorem 6.0.8. *[18, Theorem 1.2] Let* $\mathbf{t} : \mathbb{C} \to \mathbb{C}$ *satisfy* $\mathbf{t}(k)/\bar{k}$ *and* $\mathbf{t}(k)/k$ *are Schwartz class. Then the function* $\mathcal{Q}\mathbf{t} : \mathbb{C} \to \mathbb{R}^2$ *is well-defined and continuous. Furthermore,*

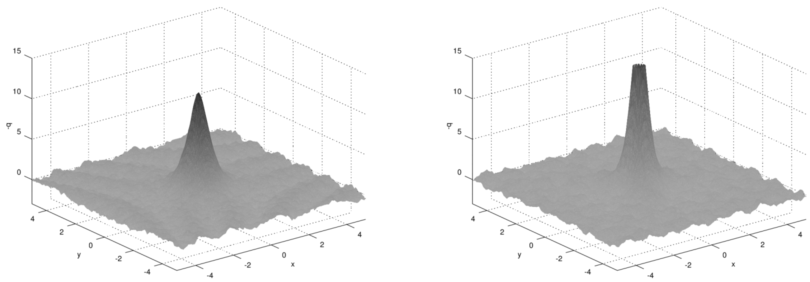
$$[\mathcal{Q}\mathbf{t}](x)| \leq C\langle x\rangle^{-2}$$

By Theorem 1.4.5, such scattering transforms must come from critical potentials. As of now, there is no similar proof that shows decay in $\mathcal{Q}[\mathbf{t}]$ when the scattering transform has the L^2 singularity that comes from subcritical potentials.

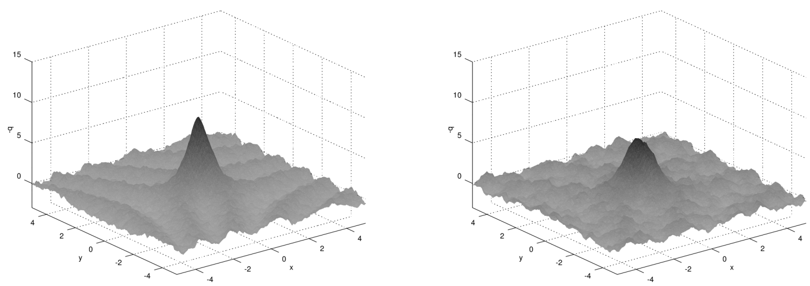
q_8



$-q_{-7}$



$-q_{-8.008}$



$-q_{-9}$

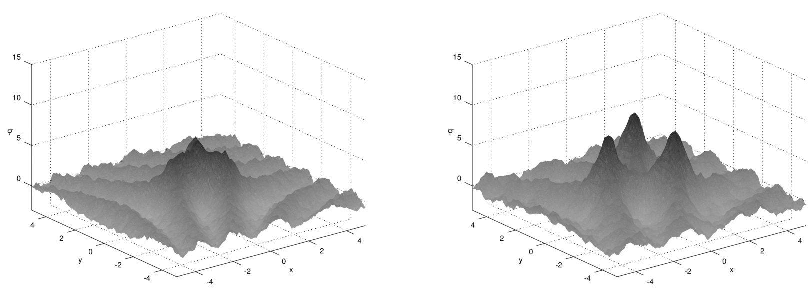


Figure 6.3: Solutions to the Novikov-Veselov equation at times $\tau = 0.5$ and $\tau = 1.0$. q_8 is a supercritical potential and is decaying. The other potentials are supercritical: q_{-7} is beginning to blow up, $q_{-8.008}$ is relatively stable, and q_{-9} is breaking up into three spikes.

Bibliography

[1] Angelopoulos, Y.: Well-posedness and ill-posedness results for the Novikov-Veselov equation. `ArXiV:1307.4110v1 [math.AP]`.

[2] Astala, K., Iwaniec, T., Martin, G.: Elliptic partial differential equations and quasiconformal mappings in the plane. Princeton Mathematical Series, **48**. Princeton University Press, Princeton, NJ, 2009.

[3] Boiti, M., Leon, J., Manna, M., Pempinelli, F.: On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane. Inverse Problems **3** (1987), no. 1, 25–36.

[4] Ben-Artzi, M., Koch, H., Saut, J.-C.: Dispersion estimates for third order equations in two dimensions. Comm. Partial Differential Equations **28** (2003), no. 11-12, 1943–1974.

[5] Brown, R. M., Uhlmann, G. A.: Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. Comm. Partial Differential Equations **22** (1997), no. 5-6, 1009–1027.

[6] Calderón, A.: On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), *Soc. Brasil Mat.*, (1980) 65–73, Rio de Janeiro.

[7] Croke, R., Mueller, J., Music, M., Perry, P., Siltanen, S., Stahel, A.: The Novikov-Veselov equation: theory and computation. Nonlinear wave equations: analytic and computational techniques, 25–70, Contemp. Math., **635** (2015)

[8] Cycon, H. L., Froese, R. G., Kirsh, W., Simon, B.: Schrödinger Operators with Application to Quantum Mechanics and Global Geometry. Springer-Verlag, Berlin (1987).

[9] Faddeev, L. D.: Increasing solutions of the Schrödinger equation, Dokl. Akad Nauk SSSR, **165** (1965), 514–517 (English translation: Sov. Phys. Dokl. **10** (1966), 1033–1035).

[10] Gardner, C., Greene, J., Kruskal, M., Miura, R.: Korteweg-deVries equation and generalization. VI. Methods for exact solution. *Comm. Pure Appl. Math.* **27** (1974), 97-133.

[11] Gesztesy, F., Zhao, Z.: On positive solutions of critical Schrödinger operators in two dimensions. *J. Funct. Anal.* **127** (1995), no. 1, 235–256.

[12] Grinevich, P. G.: The scattering transform for the two-dimensional Schrö-dinger operator with a potential that decreases at infinity at fixed nonzero energy. Uspekhi Mat. Nauk **55** (2000), no. 6(336), 3–70; translation in Russian Math. Surveys **55** (2000), no. 6, 1015–1083.

[13] Grinevich, P. G., Manakov, S. V.: Inverse problem of scattering theory for the two-dimensional Schrödinger operator, the $\overline{\partial}$-method and nonlinear equations. Funktsional. Anal. i Prilozhen. **20** (1986), no. 2, 14–24, 96. English translation: Functional Anal. Appl. **20** (1986), no. 2, 94–103.

[14] Grinevich, P. G., Novikov, R. G.: Faddeev eigenfunctions for point potentials in two dimensions. Phys. Lett. A **376** (2012), no. 12-13, 1102-1106.

[15] Grinevich, P. G., Novikov, S. P.: Inverse scattering problem for the two-dimensional Schrödinger operator at a fixed negative energy and generalized analytic functions. Plasma theory and nonlinear and turbulent processes in physics, Vol. 1, 2 (Kiev, 1987), 58–85, World Sci. Publishing, Singapore, 1988.

[16] Grinevich, P. G. , Novikov, S. P. : A two-dimensional "inverse scattering problem" for negative energies, and generalized-analytic functions. I. Energies lower than the ground state. (Russian) Funktsional. Anal. i Prilozhen. **22** (1988), no. 1, 23–33, 96; translation in Funct. Anal. Appl. **22** (1988), no. 1, 19–27.

[17] Kazeykina, A. V.: Solitons and large time asymptotics for solutions of the Novikov-Veselov equation. Thèse, École Polytechnique, 2012. Available at `http://hal.archives-ouvertes.fr/docs/00/76/26/62/PDF/these.pdf`

[18] Lassas, M., Mueller, J. L., Siltanen, S.: Mapping properties of the nonlinear Fourier transform in dimension two. Comm. Partial Differential Equations **32** (2007), no. 4-6, 591–610.

[19] Lassas, M., Mueller, J. L., Siltanen, S., Stahel, A.: The Novikov-Veselov equation and the inverse scattering method, Part I: Analysis. Phys. D **241** (2012), no. 16, 1322–1335.

[20] Manakov, S. V.: The method of the inverse scattering problem, and two-dimensional evolution equations. Uspehi Mat. Nauk **31** (1976), no. 5 (191), 245–246. English translation: Russian Math. Surveys **31** (1976), no. 5, 245246.

[21] Murata, M.: Structure of positive solutions to $(-\Delta + V)\psi = 0$ in $\mathbb{R}^n$. Duke Math. J. **53** (1986), no. 4, 869–943.

[22] Music, M.: The nonlinear Fourier transform for two-dimensional subcritical potentials. Inverse Probl. Imaging **8** (2014), no. 4, 1151-1167.

[23] Music, M., Perry, P.: Global Solutions for the Zero-Energy Novikov-Veselov Equation by Inverse Scattering *In preparation.*

[24] Music, M., Perry, P., Siltanen, S.: Exceptional Circles of Radial Potentials. Inverse Problems **29** (2013), no. 4, 045004.

[25] A. I. Nachman.: Global uniqueness for a two-dimensional inverse boundary value problem. (University of Rochester, Dept. of Mathematics Preprint Series), 19, 1993.

[26] Nachman, A. I.: Global uniqueness for a two-dimensional inverse boundary value problem. Ann. of Math. (2) **143** (1996), no. 1, 71–96.

[27] Novikov, R. G.: The inverse scattering problem on a fixed energy level for the two-dimensional Schrödinger operator *J. Funct. Anal.* **103** (1992) (2) 409–63.

[28] Novikov, S. P., Veselov, A. P.: Two-dimensional Schrödinger operator: inverse scattering transform and evolution equations. Solitons and coherent structures (Santa Barbara, Calif., 1985). *Phys. D* **18** (1986), no. 1-3, 267–273.

[29] Perry, P.: Miura maps and inverse scattering for the Novikov-Veselov equation. Anal. PDE **7** (2014), no. 2, 311–343.

[30] S. Siltanen. Electrical impedance tomography and Faddeev's Green functions. *Ann. Acad. Sci. Fenn. Mathematica Dissertationes*, 121, 1999.

[31] Simon, Barry.: Large time behavior of the L^p norm of Schrödinger semigroups. J. Funct. Anal. 40 (1981), no. 1, 66–83.

[32] Stahel, A.: Personal correspondence.

[33] Taimanov, I. A., Tsarev, S. P.: On the Moutard transformation and its applications to spectral theory and soliton equations. (Russian) Sovrem. Mat. Fundam. Napravl. **35** (2010), 101–117; translation in J. Math. Sci. (N. Y.) **170** (2010), no. 3, 371–387.

[34] Taimanov, I. A., Tsarev, S. P.: Faddeev eigenfunctions for two-dimensional Schrdinger operators via the Moutard transformation. Russian version appears in Teoret. Mat. Fiz. **176** (2013), no. 3, 408–416. Theoret. and Math. Phys. **176** (2013), no. 3, 1176–1183.

[35] Tsai, T.-Y.: The Schrödinger operator in the plane. *Inverse Problems* **9** (1993), no. 6, 763–787.

[36] Vekua, I. N.: *Generalized Analytic Functions.* Reading, Massachusetts: Addison-Wesley Publishing Co., 1962.

[37] Veselov, A. P., Novikov, S. P.: Finite-gap two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations. *Dokl. Akad. Nauk SSSR* **279** (1984), no. 1, 20–24. English translation: *Soviet Math. Dokl.* **30** (1984), no. 3, 588–591

Vita

Michael Estill Music

Education

University of Kentucky, Lexington, Kentucky, USA

- M.A., Mathematics, 2012

Georgia Institute of Technology, Atlanta, Georgia, USA

- B.S., Discrete Mathematics, 2010

Publications

- Music, M., Perry, P.: Global Solutions for the Zero-Energy Novikov-Veselov Equation by Inverse Scattering *In preparation.*
- Croke, R., Mueller, J., Music, M., Perry, P., Siltanen, S., Stahel, A.: The Novikov-Veselov equation: theory and computation. Nonlinear wave equations: analytic and computational techniques, 25–70, Contemp. Math., **635** (2015)
- Music, M.: The nonlinear Fourier transform for two-dimensional subcritical potentials. Inverse Probl. Imaging **8** (2014), no. 4, 1151-1167.
- Music, M., Perry, P., Siltanen, S.: Exceptional Circles of Radial Potentials. Inverse Problems **29** (2013), no. 4, 045004.

Awards

- *Max Steckler Fellowship*, University of Kentucky, 2012.