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Deletion-Induced Triangulations

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DISSERTATION

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ABSTRACT OF DISSERTATION

Deletion-Induced Triangulations

Let $d > 0$ be a fixed integer and let $\mathbf{A} \subseteq \mathbb{R}^d$ be a collection of $n \geq d+2$ points which we lift into $\mathbb{R}^{d+1}$. Further let k be an integer satisfying $0 \leq k \leq n-(d+2)$ and assign to each k-subset of the points of $\mathbf{A}$ a (regular) triangulation obtained by deleting the specified k-subset and projecting down the lower hull of the convex hull of the resulting lifting. Next, for each triangulation we form the characteristic vector defined by Gelfand, Kapranov, and Zelevinsky by assigning to each vertex the sum of the volumes of all adjacent simplices. We then form a vector for the lifting, which we call the k-compound GKZ-vector, by summing all the characteristic vectors. Lastly, we construct a polytope $\Sigma_k(\mathbf{A}) \subseteq \mathbb{R}^{|\mathbf{A}|}$ by taking the convex hull of all obtainable k-compound GKZ-vectors by various liftings of $\mathbf{A}$, and note that $\Sigma_0(\mathbf{A})$ is the well-studied secondary polytope corresponding to $\mathbf{A}$. We will see that by varying k, we obtain a family of polytopes with interesting properties relating to Minkowski sums, Gale transforms, and Lawrence constructions, with the member of the family with maximal k corresponding to a zonotope studied by Billera, Fillamen, and Sturmfels. We will also discuss the case $k = d = 1$, in which we can provide a combinatorial description of the vertices allowing us to better understand the graph of the polytope and to obtain formulas for the numbers of vertices and edges present.

KEYWORDS: Discrete Geometry, Polytopes, Secondary Polytopes, Lawrence Polytopes

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Dedicated to Mrs. Sheri Rhine,
my surrogate graduate school mom who
saved my bacon on countless occasions.

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1.1 Introduction

In this dissertation, we will discuss the construction and properties of a family of polytopes which, in a sense, refines the secondary polytope corresponding to a point set [13] and which reaches maximum refinement in the form of a zonotope corresponding to the point set studied in [1]. The secondary polytope corresponding to a point set $\mathbf{A}$ is one which has a vertex corresponding to every regular triangulation of $\mathbf{A}$. We can define regular triangulations as those which are obtained by taking general liftings of the point set $\mathbf{A}$ into one higher dimension and projecting down the simplices of the lower hull of the resulting polytope onto the convex $\mathbf{A}$. The zonotope, on the other hand, has a vertex corresponding to every set of compatible orientations which can be placed on each of the simplices of a general liftings of the points of $\mathbf{A}$ into one higher dimension.

To obtain polytopes which interpolate between the secondary and the zonotope, we think of fixing a size k which will determine the cardinality of the sets which we will delete. We will then consider the set of triangulations that we can obtain from a particular lifting of $\mathbf{A}$ by deleting every k-subset of lifted points in $\mathbf{A}$ in turn and projecting the lower hull triangulations of the convex hull of the remaining $|\mathbf{A}| - k$ points. By adapting the method outlined by Gelfand, Kapranov, and Zelevinsky in [6] to construct the secondary, we obtain a method to construct a polytope corresponding to the different k deletion-induced triangulations of $\mathbf{A}$, which we will call the k deletion-induced polytope. We can see that by choosing $k = 0$ we obtain the secondary, and by choosing k large enough to leave only simplices in the lifting, we essentially record all orientations of every simplex in the lifting, and hence recover the zonotope.

In the first chapter, we will outline the specifics of the construction of the various k deletion-induced polytopes and follow up with some structural results regarding this family of polytopes. In particular, we will see that for each $k > 0$, we can obtain the k deletion-induced polytope as the Minkowski sum of various secondary polytopes corresponding to appropriate subsets of our initial point set $\mathbf{A}$. This relation to secondary polytopes will allow us to determine the dimension of the family of polytopes. We will also examine alternate constructions of the k deletion-induced polytope which will yield polytopes with equivalent combinatorial structure. We then examine the construction of the secondary polytope using the Gale transform in [1] and adapt this construction to study the k deletion-induced polytope. Further, we uncover a connection between our family of polytopes and the Lawrence polytope, which will yield yet another construction for the k deletion-induced polytopes.

In the second chapter, we restrict our view to collections $\mathbf{A}$ of points in $\mathbb{R}^1$ and consider the 1 deletion-induced polytope. For this particular restriction, we are able to describe a combinatorial structure which underlies the vertices. This allows us to efficiently determine the various types of 1 deletion-induced triangulations and to define a combinatorial condition for when two vertices will form an edge in the polytope. We also provide counting arguments which allow us to determine the number of vertices and edges present in the 1 deletion-induced polytope coming from the set of the first n integers in $\mathbb{R}^1$.

For good general references on polytopes, see [13] and [8].

1.2 Definitions and Notation

Let $n \geq d+2$ and $k \leq n-(d+2)$ be positive integers and let $\mathbf{A}$ denote a collection of points in general position in $\mathbb{R}^d$ with $|\mathbf{A}| = n$. By general position, we mean that no set of $d+1$ points is affinely dependent. Further, let $\omega \in \mathbb{R}^n$ be thought of as a "lifting vector" with coordinate ω_i being appended to the point $\mathbf{v}_i \in \mathbf{A}$, $1 \leq i \leq n$, to lift each point off of the natural embedding of $\mathbb{R}^d \subset \mathbb{R}^{d+1}$ and into $\mathbb{R}^{d+1}$. We will refer to the set of lifted points as $\omega(\mathbf{A})$ and will initially only consider vectors ω for which the lifted points $\omega(\mathbf{A})$ are general position. We will defer the description of what we mean by "general" position for a lifting vector until after the first definition. To each ω, we assign a set of triangulations which we will call the deletion-induced triangulations.

Definition 1.2.1 Fix the set of points $\mathbf{A}$ and assign to each general lifting vector ω a set of $\binom{n}{k}$ triangulations via the following method: List the k-subsets of n, $K_1, K_2, \ldots K_{\binom{n}{k}}$ and for i from 1 to $\binom{n}{k}$ disregard the points in the set $K_i \subset \mathbf{A}$ and consider only the lifting of the remaining points $\omega(\mathbf{A} \setminus K_i)$. Project the lower hull of $\operatorname{conv}(\omega(\mathbf{A} \setminus K_i))$ onto the natural embedding of $\mathbb{R}^d \subset \mathbb{R}^{d+1}$ to obtain a (regular) triangulation $\mathcal{T}_i$ of $\operatorname{conv}(\mathbf{A} \setminus K_i)$. We refer to the set of $\binom{n}{k}$ triangulations assigned to ω in this manner as the *deletion-induced triangulations* corresponding to ω.

Note, a lifting which is in general position with respect to the deletion size k will be defined as one where *no* lower hull after a k deletion will have $d+2$ points in affinely dependent position. That is, a lifting ω corresponding to $\mathbf{A}$ may have collinearities/coplanarities/etc. and still be considered in general position as long as these collinearities/coplanarities/etc. are not revealed in the lower hull by any k deletion. If $k = n-(d+2)$, however, this forces us to require that no set of $d+2$ points in $\omega(\mathbf{A})$ are affinely dependent.

Example 1.2.2 Suppose we start with the set $\mathbf{A} = \{(2,-3),(1,-9),(8,7),(-7,-7),(7,-1),(-1,5),(-8,9)\} \subset \mathbb{R}^2$. If we apply the lifting vector $\omega = \langle 8,5,2,2,4,8,3\rangle$ to our set $\mathbf{A}$, we obtain the diagram shown in Figure 1.1, where the red lines indicate the lower hull and the green lines indicate the upper hull.

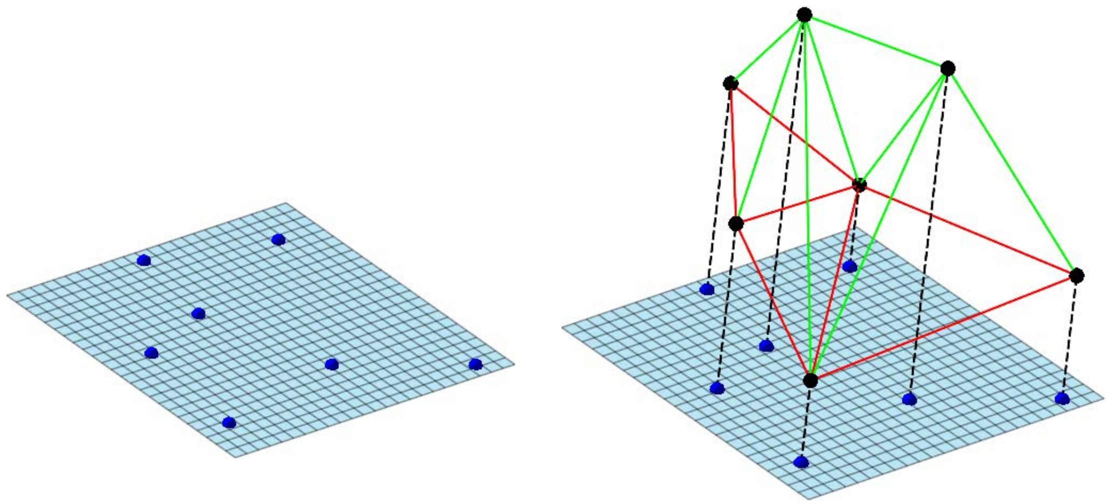
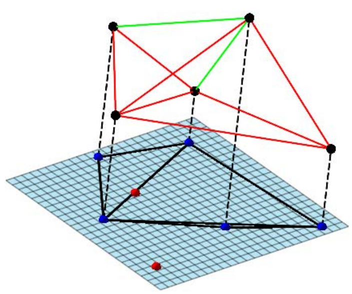
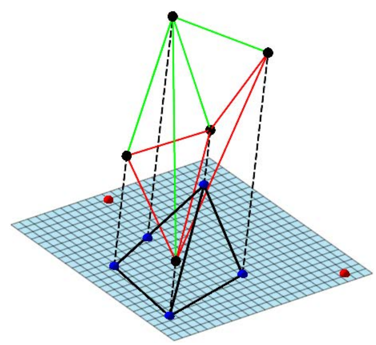


Figure 1.1: Point lifting

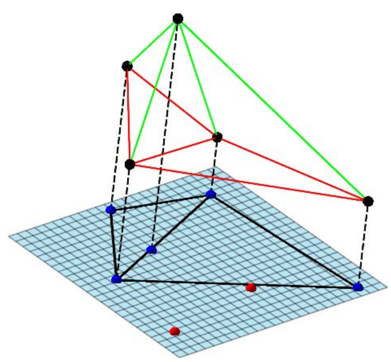
If we were to take 2 deletions, i.e., set $k = 2$, we would be considering deletions such as those shown in Figure 1.2. Notice that there would be a total of $\binom{7}{2} = 21$ triangulations in this particular set of 2 deletion-induced triangulations, but only 3 are shown. In particular, the deletion of the points i and j is indicated by $K_{i,j}$. $\quad\diamondsuit$



(i) $K_{1,3}$



(ii) $K_{2,7}$



(iii) $K_{3,6}$

Figure 1.2: 2 deletions of a point set

Definition 1.2.3 Each triangulation $\mathcal{T}_i$ is assigned a *characteristic vector*, denoted $\phi(\mathcal{T}_i)$, according to the procedure outlined in [6]. Note, this is also referred to as the GKZ-vector in [3]. Specifically, $\phi(\mathcal{T}_i) \in \mathbb{R}^n$ where coordinate j records the sum of the d-dimensional volumes of all simplices in the triangulation $\mathcal{T}_i$ incident to the point indexed by j in $\mathbf{A}$. If $j \in K_i$, then j is a deleted point and coordinate j of $\phi(\mathcal{T}_i)$ is assigned a 0.

Example 1.2.4 Suppose we focus on the particular 2 deletion induced triangulation $\mathcal{T}_{1,3}$ corresponding $K_{1,3}$ (Figure 1.2i) of $\mathbf{A}$ in Example 1.2.2. This triangulation, along with the areas of its triangles, is shown in Figure 1.3.

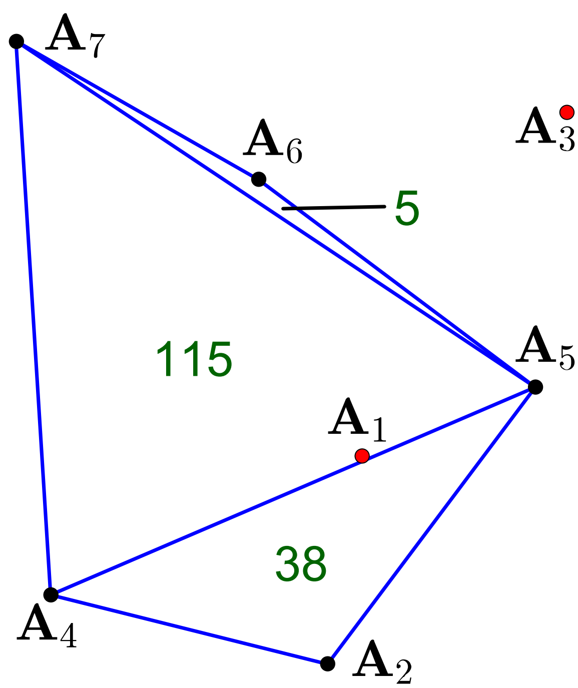


Figure 1.3: Triangulation $\mathcal{T}_{1,3}$

The characteristic vector corresponding to this triangulation is given by:

$$\phi(\mathcal{T}_{1,3}) \;=\; \langle 0, 38, 0, 38 + 115, 5 + 115 + 38, 5, 115 + 5\rangle \;=\; \langle 0, 38, 0, 153, 158, 5, 120\rangle.$$

$\diamondsuit$

Definition 1.2.5 We collect the information for the entire set of deletion-induced triangulations corresponding to the lifting ω and deletion size k into a single vector which we call the *k-compound GKZ-vector associated to ω*, denoted by $\overline{\phi}_k(\omega)$. If $\mathcal{T}_1, \ldots, \mathcal{T}_{\binom{n}{k}}$ denote the set of deletion-induced triangulations we obtain under k-deletion, we set

$$\overline{\phi}_k(\omega) \; := \; \sum_{i=1}^{\binom{n}{k}} \phi(\mathcal{T}_i).$$

If a vector $\mathbf{v} = \overline{\phi}_k(\omega)$ for some lifting ω, we will say that the lifting ω *realizes* the k compound GKZ-vector $\mathbf{v}$, or equivalently, ω realizes the corresponding set of k deletion-induced triangulations $\mathcal{T}_1, \ldots, \mathcal{T}_{\binom{n}{k}}$ which compose $\mathbf{v}$.

Proposition 1.2.6 *Two different general liftings ω and ω' of $\mathbf{A}$ produce the same k-compound* GKZ*-vector iff the corresponding deletion-induced triangulations $\mathcal{T}_i$ and $\mathcal{T}_i'$ match for deletions $i = 1, \ldots, \binom{n}{k}$.*

Proof. In [6] (and [3]), it is shown that if ω is a lifting vector in general position corresponding to some set of points $\mathbf{C}$, then for the triangulation $\mathcal{T}'$ corresponding to the lower hull of $\operatorname{conv}(\omega(\mathbf{C}))$ and *any* other regular triangulation $\mathcal{T} \neq \mathcal{T}'$ of $\mathbf{C}$, we have that $\omega \cdot \phi(\mathcal{T}') < \omega \cdot \phi(\mathcal{T})$. Extending to k-compound-GKZ vectors, we see that a set of deletion-induced triangulations will uniquely minimize the inner product with a general lifting ω exactly when the triangulation corresponding to each individual deletion K_i for $1 \leq i \leq \binom{n}{k}$ is coming from a projection of the lower hull of $\operatorname{conv}(\mathbf{A} \setminus K_i)$. ■

For a given regular triangulation $\mathcal{T}$ of the underlying point set $\mathbf{A}$ and lifting ω, the inner product $\omega{\cdot}\phi(\mathcal{T})$ can be interpreted as $(d{+}1)$ times the sum of $(d{+}1)$-dimensional volumes of simplicial prisms whose bases are determined by the triangulation $\mathcal{T}$ and heights at each vertex are determined by ω. It should be clear that such a sum would be minimized (for fixed lifting ω and varying triangulations $\mathcal{T}$) if $\mathcal{T}$ corresponds to the lower hull of $\operatorname{conv}(\omega(\mathbf{A}))$, and that such a $\mathcal{T}$ would be the *unique* minimizer if ω is a general lifting (only simplices appear as lower hull facets).

Definition 1.2.7 We construct the *k-deletion-induced polytope corresponding to* $\mathbf{A}$, $\Sigma_k(\mathbf{A})$, by first forming the set

$$\mathcal{B} \; := \; \left\{ \, \boldsymbol{v} \mid \boldsymbol{v} \; = \; \overline{\phi}_k(\omega) \text{ for some lifting } \omega \text{ of } \mathbf{A} \right\},$$

and taking

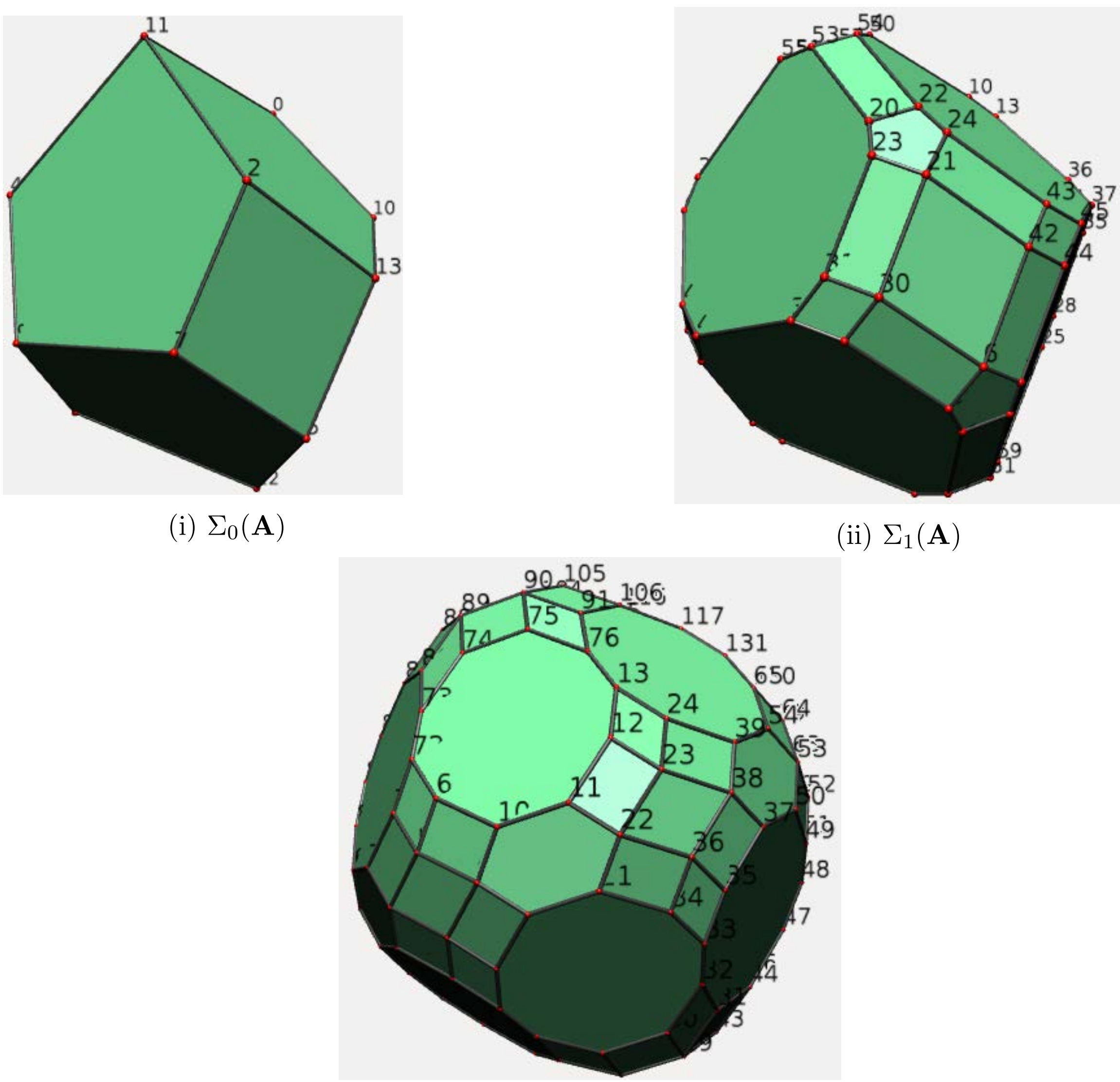
$$\Sigma_k(\mathbf{A}) \; := \; \operatorname{conv}(\mathcal{B}) \,.$$

That $\mathcal{B}$ is a finite set follows from the fact that $\overline{\phi}_k(\omega)$ can be uniquely determined if one knows the orientations of every set of $d+2$ points in $\omega(\mathbf{A})$, which gives the rough upper bound $|\mathcal{B}| \leq 2^{\binom{n}{d+2}}$. This bound is an overestimate, however, since not every set of orientations is realizable by a lifting and because two liftings with slightly differing sets of orientations may still potentially produce the same compound GKZ-vector for non-maximal k.

Corollary 1.2.8 *Each k-compound GKZ-vector is a vertex of $\Sigma_k(\mathbf{A})$. In other words, no k-compound GKZ-vector lies in the interior of the convex hull of the other k-compound GKZ-vectors.*

Proof. In Proposition 1.2.6, we saw that if ω were a lifting which produced the k-compound GKZ-vector $\mathbf{v}$, then $\omega \cdot \mathbf{v} < \omega \cdot \mathbf{v}'$ for any k-compound GKZ-vector $\mathbf{v}' \neq \mathbf{v}$. Thus $\omega \cdot \mathbf{x} = \omega \cdot \mathbf{v}$ is a vertex supporting hyperplane for the k-compound GKZ-vector $\mathbf{v}$. ■

Figure 1.4 shows the family $\Sigma_k(\mathbf{A})$ for $\mathbf{A}$ corresponding to a (regular) hexagon, with $k = 0$ (the associahedron), $k = 1$ and $k = 2$.



(i) $\Sigma_0(\mathbf{A})$

(ii) $\Sigma_1(\mathbf{A})$

(iii) $\Sigma_2(\mathbf{A})$

Figure 1.4: $\Sigma_k(\mathbf{A})$ for a hexagon - $k = 0, 1, 2$

1.3 Structural Results

If we let $\mathbf{v}_1, \mathbf{v}_2, \ldots$, denote the vertices of $\Sigma_k(\mathbf{A})$ (i.e., the distinct k-compound-GKZ vectors), then each of the vertices $\mathbf{v}_i$ will satisfy the following $d+1$ linearly distinct relations:

- $\langle 1, 1, \ldots, 1\rangle \cdot \mathbf{v}_i \;=\; \sum_{j=1}^{\binom{n}{k}} (d+1)\mathrm{vol}\left(\mathrm{conv}\left(\mathbf{A}\backslash K_j\right)\right),$
- $\langle x_{s,1}, x_{s,2}, \ldots, x_{s,n}\rangle \cdot \mathbf{v}_i \;=\; \sum_{j=1}^{\binom{n}{k}} (d+1)\mathrm{vol}\left(\mathrm{conv}\left(\mathbf{A}\backslash K_j\right)\right) c_{s,(\mathbf{A}\backslash K_j)},\; s = 1, \ldots, d,$

where $x_{s,t}$ denotes the s^{th} coordinate of the t^{th} point in $\mathbf{A}$ and $c_{s,(\mathbf{A}\backslash K_j)}$ denotes the s^{th} coordinate of the centroid of $\mathrm{conv}(\mathbf{A}\backslash K_j)$. This is a natural consequence of the relations found to be satisfied by "ordinary" GKZ-vectors in [3] and [6]. This tells us that $\Sigma_k(\mathbf{A})$ is not full-dimensional and has at *most* dimension $n - (d+1)$.

Theorem 1.3.1 *The k' deletion-induced triangulations are uniquely determined by the k deletion-induced triangulations for any $k' < k$ and fixed general lifting ω of a set of points $\mathbf{A}$.*

Proof. We induct on the size of k and begin with the basis case $k = 1$. The claim is that if we know the d-simplices that appear on the lower hull after each 1-deletion, then we can determine which d-simplices appear on the lower hull of $\mathrm{conv}(\mathbf{A})$. Clearly any d-simplex that appears on the lower hull of $\mathrm{conv}(\mathbf{A})$ must appear in the lower hull in all single deletions but those of its vertices. This condition is sufficient to identify the d-simplices of the original lower hull, as any simplex that is not in the original lower hull will fail to appear in the lower hull of at least one other deletion than those of its vertices. Specifically, if $k = 1$, then $d + 3 \leq n$. If a simplex is not on the original lower hull, then there is a point p in $\mathbf{A}$ (not one of the vertices of the simplex) which lies "below" this simplex, hiding it from view. Since $n \geq d + 3$ there is at least one point which we can delete that is neither p, nor one of the vertices of our simplex, thus we are guaranteed that this simplex does not appear on the lower hull under the deletion of this particular point.

Next assume that knowledge of the m deletions up to some $m \geq 1$ uniquely determines the m' deletions for all $0 \leq m' \leq m$ and consider the k deletions for $k = m+1$ (supposing that this k still satisfies $k \leq n-(d+2)$). If we collect all of the k deletions which involve the deletion of some fixed point $v_i \in \mathbf{A}$, then we see that this set is exactly the m deletions of the point set $\mathbf{A} \setminus v_i$. By the inductive hypothesis, we can reconstruct all of the m' deletions of $\mathbf{A}\setminus v_i$ for all $0 \leq m' \leq m$. In the original point set $\mathbf{A}$, this means we have determined the deletion-induced triangulations corresponding to any deletion set which included the point v_i. Repeating this process for all v_i as

i ranges from 1 to n gives us knowledge of any non-trivial deletion-induced triangulation. Since this includes all of the 1 deletion-induced triangulations, our base case gives us that the original lower hull of $\mathbf{A}$ may also be determined. ■

The following definition and theorem will allow us to extend our definition of k-compound GKZ-vectors while maintaining the combinatorial structure of the the polytope that is obtained. This will be useful in new constructions which we will introduce later in the section.

Definition 1.3.2 Let $N_k := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$ and let $S_1, S_2, \ldots, S_{N_k}$ enumerate the subsets of $\{1, 2, \ldots, n\}$ of cardinality $\leq k$. Further, let χ be a characteristic vector of length N_k with multiplicity. That is, let $\chi \in [0, 1, 2, \ldots]^{N_k}$ such that a y in position z of χ, $\chi(z) = y$, indicates that the subset S_z appears y times in the collection of subsets indicated by χ.

If χ is such that $\chi(z) \geq 1$ for all z where S_z indicates a subset of $\{1, 2, \ldots, n\}$ of cardinality equal to k, then we define *augmented k-compound GKZ-vectors* by

$$\widetilde{\phi}_k(\omega) \ := \ \sum_{i=1}^{N_k} \chi(i) \phi(\mathcal{T}_i),$$

where $\mathcal{T}_i$ is the regular triangulation given by projecting the lower hull of $\operatorname{conv}(\omega(\mathbf{A} \setminus S_i))$ onto $\mathbb{R}^d$. We then form the *augmented k deletion-induced polytope* corresponding to $\mathbf{A}$ and χ by

$$\widetilde{\Sigma}_{\chi,k}(\mathbf{A}) \ := \ \operatorname{conv}\left(\left\{\mathbf{v} \mid \mathbf{v} = \widetilde{\phi}_k(\omega) \text{ for some general lifting } \omega \text{ of } \mathbf{A}\right\}\right)$$

Theorem 1.3.3 *If χ is selected according to the conditions in Definition 1.3.2, then the polytopes $\Sigma_k(\mathbf{A})$ and $\widetilde{\Sigma}_{\chi,k}(\mathbf{A})$ have identical normal fans, and hence have identical combinatorial structure.*

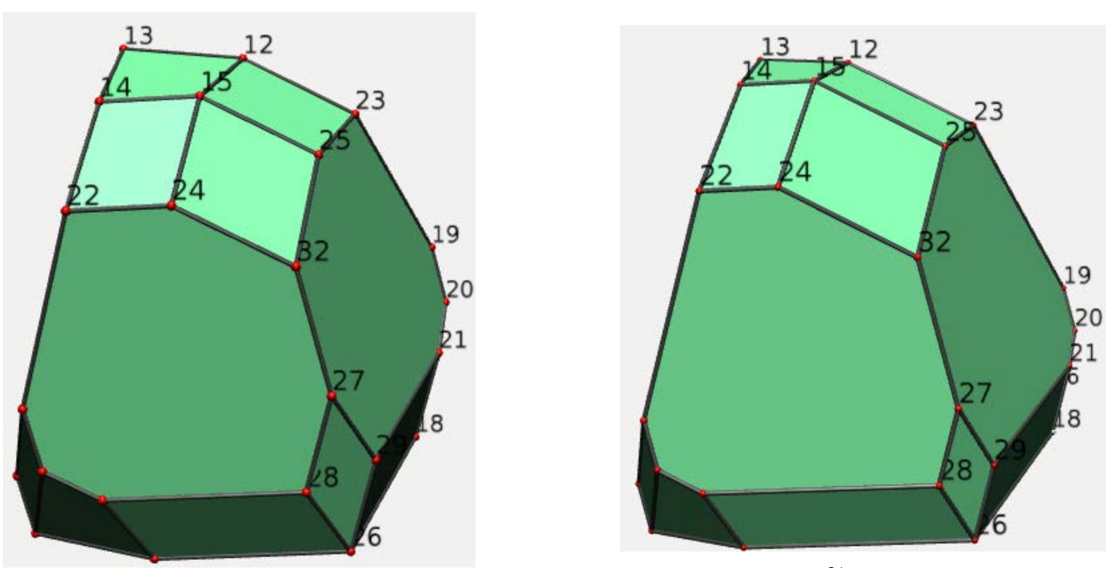
Proof.

It is sufficient to demonstrate that the vertices of the two polytopes can be put into bijective correspondence and to show that the normal cones for paired vertices are identical. We will make use of the fact that any normal vector for a vertex supporting hyperplane to a vertex $\widetilde{\mathbf{v}}$ of $\widetilde{\Sigma}_{\chi,k}(\mathbf{A})$ or a vertex $\mathbf{v}$ of $\Sigma_k(\mathbf{A})$ can be re-interpreted as a lifting vector which realizes that vertex. In other words, the normal vector must correspond to a general lifting with respect to k deletions, and hence a general lifting with respect to all lower order deletions, which has lower hulls matching the set of triangulations corresponding to $\widetilde{\mathbf{v}}$ or $\mathbf{v}$ under the appropriate deletions.

For the bijection, we will map a vertex $\widetilde{\mathbf{v}}$ of $\widetilde{\Sigma}_{\chi,k}(\mathbf{A})$ to the unique vertex $\mathbf{v}$ of $\Sigma_k(\mathbf{A})$ which has the same triangulations appearing after corresponding deletions of size k. Such a vertex is always available to map to as any normal vector for a vertex supporting hyperplane for $\widetilde{\mathbf{v}}$ is also a lifting which realizes $\widetilde{\mathbf{v}}$, and hence which realizes a corresponding $\mathbf{v}$ (it produces the appropriate triangulations under k deletions, we simply ignore the lower order deletions). The fact that the lifting is one-to-one follows from Theorem 1.3.1, and the fact that it is surjective is clear as the same lifting ω which realizes a vector $\mathbf{v}$ of $\Sigma_k(\mathbf{A})$ will realize an augmented k-compound GKZ-vector with the same k deletion induced triangulations and determined lower order deletions.

It is clear that for a matched pair of vertices $\mathbf{v}$ and $\widetilde{\mathbf{v}}$ any normal vector for a vertex supporting hyperplane of $\mathbf{v}$ can be considered as a lifting which realizes $\mathbf{v}$, which must also be a lifting which realizes $\widetilde{\mathbf{v}}$, which then serves as a normal for a vertex supporting hyperplane for $\widetilde{\mathbf{v}}$ and vice versa. ■

Example 1.3.4 If we construct the 1 deletion-induced polytope for a point set $\mathbf{A}$, $\Sigma_1(\mathbf{A})$, and then take the 1 compound-GKZ vectors and append the characteristic vector for the original lower hull of the corresponding lifting to obtain $\widetilde{\Sigma}_{\chi,k}(\mathbf{A})$, then the combinatorial structure of the two polytopes will be identical by Theorem 1.3.3. If we take $\mathbf{A} = \{1, 2, 3, 4, 5\}$, $k = 1$, and the previously described χ, the two polytopes are shown in Figure 1.5. We can see in the pictures that the normal vectors defining the facets do not appear to change, only the sizes of facets seem to change. In particular, the rhombus at top front consisting of vertices $\{15, 25, 32, 24\}$ appears to grow in size in this particular augmented k deletion-induced polytope. ♢



(i) $\Sigma_1(\mathbf{A})$ (ii) $\widetilde{\Sigma}_{\chi,1}(\mathbf{A})$

Figure 1.5: Comparing $\Sigma_k(\mathbf{A})$ and $\widetilde{\Sigma}_{\chi,k}(\mathbf{A})$

Definition 1.3.5 If P_A and P_B are two polytopes in $\mathbb{R}^d$, then the *vector sum* (or *Minkowski Sum*) of these two polytopes is defined to be

$$P_A + P_B \ := \ \{\mathbf{v}_A + \mathbf{v}_B \ : \ \mathbf{v}_A \in P_A, \ \mathbf{v}_B \in P_B\}\,.$$

The Minkowski sum $P_A + P_B$ of two polytopes is a polytope, and any vertex of $P_A + P_B$ must arise as the sum of two vertices from P_A and P_B, though not all such sums are guaranteed to be vertices of $P_A + P_B$. For more references on Minkowski sums, see [11], [13], [12].

Example 1.3.6 For an example of a Minkowski sum of two polygons, consider the pentagon P_A and triangle P_B in Figure 1.6.

(i) P_A (ii) P_B

Figure 1.6: Minkowski summands

The construction of the Minkowski sum and the final result $P_A + P_B$ are shown in Figure 1.7. $\diamondsuit$

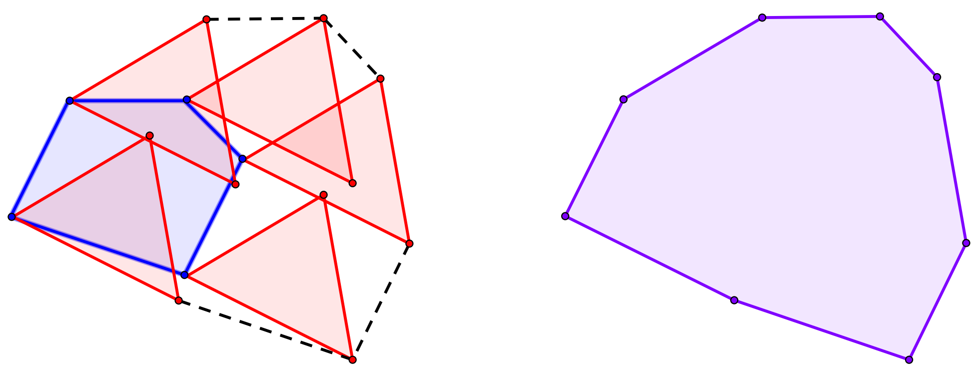


Figure 1.7: Minkowski sum $P_A + P_B$

Theorem 1.3.7 *The polytope $\Sigma_k(\mathbf{A})$ is equal to the Minkowski sum of the polytopes $\Sigma_0(\mathbf{A} \setminus K_1), \Sigma_0(\mathbf{A} \setminus K_2), \ldots, \Sigma_0(\mathbf{A} \setminus K_{\binom{n}{k}})$, where $K_1, \ldots, K_{\binom{n}{k}}$ denoted the k-subsets of the points in $\mathbf{A}$. In other words, the k deletion-induced polytope is given by the Minkowski sum of the secondary polytopes corresponding to the various k-deletions of the point set $\mathbf{A}$.*

Proof.

We first note that the vertices of the polytope obtained by the Minkowski sum correspond to some sum of $\binom{n}{k}$ vertices, one from each individual secondary polytope. In other words, if M denotes the polytope that we obtain from the Minkowski sum of the secondaries, then every vertex $\mathbf{v}$ of M is of the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_{\binom{n}{k}}$, where $\mathbf{v}_i$ is a vertex of $\Sigma_0(\mathbf{A} \setminus K_i)$ for $1 \leq i \leq \binom{n}{k}$. Note, however, that not every point of this form is guaranteed to be a vertex of M. If we consider the set of all possible combinations of secondary vertices, we see that we obtain all possible sums of characteristic vectors $\phi_k(\mathcal{T}_1) + \cdots + \phi_k(\mathcal{T}_{\binom{n}{k}})$, where each $\mathcal{T}_i$ can be independently chosen to be *any* regular triangulation of the corresponding point set $\mathbf{A} \setminus K_i$ for $1 \leq i \leq \binom{n}{k}$. Thus we see that all possible k-compound GKZ-vectors occur as sums of appropriate vertices of these secondaries. Further, we know that any k-compound GKZ-vector $\mathbf{u}$ will be extremal in M, due to the fact that if ω is a lifting which realizes $\mathbf{u}$, then the set of deletion-induced triangulations which $\mathbf{u}$ encodes, $\mathcal{T}_{\mathbf{u},1}, \ldots, \mathcal{T}_{\mathbf{u},\binom{n}{k}}$, will uniquely minimize the inner products $\omega \cdot \phi_k(\mathcal{T}_{\mathbf{u},i})$ for $1 \leq i \leq \binom{n}{k}$ simultaneously. Thus if we let

$$\mathcal{V} = \left\{ \mathbf{v} \;\middle|\; \begin{array}{c} \mathbf{v}=\mathbf{v}_1+\mathbf{v}_2+\cdots+\mathbf{v}_{\binom{n}{k}}, \text{ where } \mathbf{v}_i \\ \text{is a vertex of } \Sigma_0(\mathbf{A}\setminus K_i) \text{ for } 1\leq i\leq\binom{n}{k} \end{array} \right\},$$

then we see that the linear function $\omega \cdot \mathbf{x} - \omega \cdot \mathbf{u}$ is uniquely minimized by $\mathbf{u}$ if $\mathbf{x}$ ranges over $\mathcal{V}$, and hence $\omega \cdot \mathbf{x} = \omega \cdot \mathbf{u}$ defines a vertex supporting hyperplane for $\mathbf{u}$ in M. Lastly, we note that if $\mathbf{v} \in \mathcal{V}$ does *not* correspond to a k-compound GKZ-vector, then $\mathbf{v}$ will *not* be extremal in M. To see why, suppose that $\mathbf{v}$ was extremal, then there would exist a vertex supporting hyperplane with normal ω' such that $\omega' \cdot \mathbf{x} > \omega' \cdot \mathbf{v}$ for all $\mathbf{x} \in \mathcal{V} \setminus \mathbf{v}$. This implies that the set of deletion-induced triangulations encoded by $\mathbf{v}$, $\mathcal{T}_{\mathbf{v},1}, \ldots, \mathcal{T}_{\mathbf{v},\binom{n}{k}}$, is the *only* set of triangulations which minimize each inner product $\omega' \cdot \phi_k(\mathcal{T}_{\mathbf{v},i})$ for all $1 \leq i \leq \binom{n}{k}$ simultaneously. Then we would have that ω' is a lifting which has $\mathbf{v}$ as a k-compound GKZ-vector. This demonstrates that the polytopes M and $\Sigma_k(\mathbf{A})$ are (geometrically) identical. ■

Since the dimension of each secondary polytope $\Sigma_0(\mathbf{A} \setminus K)$, for $0 \leq |K| = k \leq n - (d+2)$, is guaranteed to be $n - (d+1)$ for a point set $\mathbf{A}$ (and hence $\mathbf{A} \setminus K$) in general position, we obtain that the k deletion-induced polytope $\Sigma_k(\mathbf{A})$ must also have dimension $n-(d+1)$. In other words, every relation satisfied by the k-compound GKZ-vectors arises as a consequence of the $d+1$ relations mentioned at the beginning of the section.

Corollary 1.3.8 *The polytope $\Sigma_k(\mathbf{A})$ is a zonotope if k is assigned the maximal possible value for the point set $\mathbf{A}$, i.e. if $k = n - (d+2)$.*

Proof. By Theorem 1.3.7, we are given that $\Sigma_k(\mathbf{A})$ is given by the Minkowski sum of the polytopes $\Sigma_0(\mathbf{A} \setminus K_1), \Sigma_0(\mathbf{A} \setminus K_2), \ldots, \Sigma_0(\mathbf{A} \setminus K_{\binom{n}{k}})$. Since $k = n - (d+2)$, we have that $|\mathbf{A} \setminus K_i| = d+2$ for each $1 \leq i \leq \binom{n}{k}$, and hence $\omega(\mathbf{A} \setminus K_i)$ forms a simplex in $\mathbb{R}^{d+1}$. Since a simplex can have only two orientations, we see that there are only two possible lower hulls of $\text{conv}(\omega(\mathbf{A} \setminus K_i))$ and hence only two possible k deletion-induced triangulations of $\mathbf{A} \setminus K_i$ for this size k. This tells us that each secondary polytope $\Sigma_0(\mathbf{A} \setminus K_i)$ is the convex hull of two points and hence forms a line segment in $\mathbb{R}^d$, which implies that $\Sigma_k(\mathbf{A})$ is a zonotope. ■

The zonotopes arising in Corollary 1.3.8 are well studied in [1]. We will see a second way to construct these zonotopes later in the section.

Next we will discuss how to utilize Gale diagrams to study k deletion-induced triangulations and the corresponding polytope $\Sigma_k(\mathbf{A})$, but first we will give a brief review how a Gale diagram is constructed and how it can help us study the set of regular triangulations, and hence the secondary polytope $\Sigma_0(\mathbf{A})$, corresponding to a set of points $\mathbf{A}$ in $\mathbb{R}^d$. What follows is a summary of Gale transforms based on [14], [1], and [8].

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^d$ denote column vectors corresponding to the points in $\mathbf{A}$ and form a matrix out of the homogenized coordinates

$$M \;:=\; \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{(d+1)\times n}$$

Since $|A| = n > d+2$ and no set of $d+1$ points in $\mathbf{A}$ is affinely dependent, we have that M has full row rank, and hence the dimension of the nullspace of M, $\mathcal{N}(M)$, is $n-(d+1)$. We can then form a new matrix $\overline{M}$ where the rows correspond to any set of basis elements of $\mathcal{N}(M)$ and label the columns by

$$\overline{M} \;=\; \begin{bmatrix} \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & \ldots & \overline{\mathbf{v}}_n \end{bmatrix}_{n-(d+1)\times n}.$$

The set of n points given by the columns of $\overline{M}$: $\overline{\mathbf{v}}_1, \ldots, \overline{\mathbf{v}}_n \in \mathbb{R}^{n-(d+1)}$ is referred to as a *Gale transform* corresponding to the point set $\mathbf{A}$. (Note that some points may occur with multiplicity in this collection.) We will use $\overline{M}$ to refer to refer to both the matrix as well as the set of columns of the matrix. We consider the corresponding points $\mathbf{v}_i$ and $\overline{\mathbf{v}}_i$ as being paired for $1 \leq i \leq n$ and note that since the row $[1, \ldots, 1]$ is guaranteed to be orthogonal to every row of $\overline{M}$, we have that $\overline{\mathbf{v}}_1 + \ldots + \overline{\mathbf{v}}_n = \mathbf{0}$. The Gale transform can be utilized to study the faces of the polytope $\operatorname{conv}(\mathbf{A})$ in the following manner: if some collection of points $\mathcal{F} = \{v_{i_1}, \ldots, v_{i_s}\} \subseteq \mathbf{A}$ forms a face of $\operatorname{conv}(\mathbf{A})$ of dimension $< d$, then there is some hyperplane with normal $\mathbf{n}$ such that

$$\mathbf{n}\cdot\mathbf{v}_{i_1} \;=\; \cdots \;=\; \mathbf{n}\cdot\mathbf{v}_{i_s} \quad \text{and} \quad \mathbf{n}\cdot\mathbf{v} > \mathbf{n}\cdot\mathbf{v}_{i_1}\ \forall\ \mathbf{v} \notin \mathcal{F}.$$

If we let $\mathbf{r} = [\mathbf{n},\ -(\mathbf{n}\cdot\mathbf{v}_{i_1})] \in \mathbb{R}^{d+1}$ be a row vector, then we see that we can interpret the row vector $\mathbf{r}M$ as a vector in the set $[0,+]^n$. Specifically, coordinate i of $\mathbf{r}M$ is a 0 precisely if $\mathbf{v}_i \in \mathcal{F}$ and is positive $(+)$ precisely when $\mathbf{v}_i \notin \mathcal{F}$ for $1 \leq i \leq n$. We also note that this new row vector $\mathbf{rM}$ is in the row span of M, and is hence perpendicular to each row of $\overline{M}$. Thus we have

$$\begin{bmatrix} \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & \dots & \overline{\mathbf{v}}_n \end{bmatrix} \cdot (\mathbf{r}M)^\top = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{1\times n}$$

If we let $\overline{\mathcal{F}} \subset \overline{M}$ be the corresponding collection to $\mathcal{F} \subset M$, we see that the above is simply a positive linear combination of the points in $\overline{M} \setminus \overline{\mathcal{F}}$ which produces the origin. With some positive rescaling, we see this implies that collections of points $\overline{F} \subset \overline{M}$ which correspond to complements $M \setminus \mathcal{F}$ where $\mathcal{F}$ forms a face of conv($\mathbf{A}$) will capture, i.e., contain, the origin in the relative interior of their convex hull, relint(conv($\overline{\mathcal{F}}$)). It is not hard to reverse the argument and show that if $\mathbf{0} \in$ relint(conv($\overline{\mathcal{F}}$)), then the collection $\mathcal{F} \subset M$ corresponding to $\overline{M} \setminus \overline{\mathcal{F}}$ forms a face of conv($\mathbf{A}$). Thus we note that the facets of conv($\mathbf{A}$) are given as the complements corresponding to *minimal* collections $\overline{\mathcal{F}}$ such that $\mathbf{0} \in$ relint(conv($\overline{\mathcal{F}}$)).

Further, we note that independently scaling the points in the Gale transform by some *positive* scalar factors does not change which collections capture the origin in the relative interior of their convex hulls. Thus the nonzero points of the Gale transform are traditionally scaled so that they lie on the unit sphere in $\mathbb{R}^{n-(d+1)}$ centered at the origin. Any new such placement of the n points of $\overline{M}$ in $\mathbb{R}^{n-(d+1)}$ which preserves which sets of points capture the origin in the relative interior of their convex hulls is referred to as a *Gale diagram* corresponding to the point set $\mathbf{A}$.

Example 1.3.9 Consider the point set $\mathbf{A} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{0, 1, 2, 3\} \subset \mathbb{R}$. Constructing the Gale transform, we have

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \overline{M} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

If we form a Gale diagram by scaling the points onto the unit circle centered at the origin in $\mathbb{R}^2$, we obtain Figure 1.8.

We see that the minimal sets which capture the origin in the relative interior of their convex hull are $\{\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \overline{\mathbf{v}}_3\}$ and $\{\overline{\mathbf{v}}_2, \overline{\mathbf{v}}_3, \overline{\mathbf{v}}_4\}$, which correctly indicates that the only facets of conv($\mathbf{A}$) are $\mathbf{v}_1$ and $\mathbf{v}_4$. $\Diamond$

To study the regular triangulations of the point set $\mathbf{A}$, we first form the Gale diagram $\overline{M}$ associated to $\mathbf{A}$ by positively scaling the nonzero points of the Gale transform so that they lie on the unit sphere. We then insert a new point $\overline{\mathbf{z}}$ on the unit sphere in the Gale diagram so that $\overline{\mathbf{z}}$ lies on no hyperplane spanned by $n-d-2$ elements from $\{\overline{\mathbf{v}}_1, \dots, \overline{\mathbf{v}}_n\}$, i.e., $\overline{\mathbf{z}}$ does not lie on the boundary of any cone formed by elements of $\overline{M}$. We see that $\overline{M} \cup \{\overline{\mathbf{z}}\}$ is no longer a Gale diagram which corresponds to $\mathbf{A} \subset \mathbb{R}^d$, but instead corresponds to a new point set $\mathbf{A}' \cup \{\mathbf{z}\} \subset \mathbb{R}^{d+1}$.

It can be shown that this new set of points $\mathbf{A}'$ is of the form $\{(\mathbf{v}_1, \lambda_1), \dots, (\mathbf{v}_n, \lambda_n)\}$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, and that $\mathbf{z}$ corresponds to the point at infinity (see [14] and

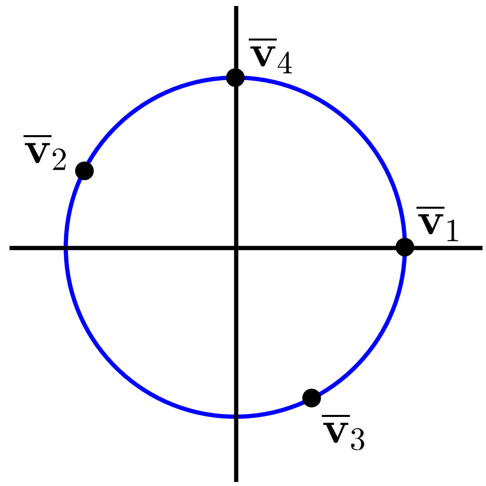


Figure 1.8: Gale diagram for 4 points on a line

[10]). In other words, the point set with Gale diagram $\overline{M} \cup \{\overline{\mathbf{z}}\}$ can be thought of as $\omega(\mathbf{A})$ for a general lifting vector $\omega = \langle \lambda_1, \ldots, \lambda_n \rangle$ together with a point $\mathbf{z}$ at infinity (the fact that the lifting is in general position follows from the placement of $\overline{\mathbf{z}}$ in affinely general position). We will think of the points $\mathbf{v}_i$ in $\mathbf{A}$ as being in bijective correspondence with the points $(\mathbf{v}_i, \lambda_i)$ in $\mathbf{A}'$ and will identify collections $\mathcal{F} \subset \mathbf{A}$ with collections $\mathcal{F}' \subset \mathbf{A}'$ in the natural manner. We see that any facet given by $\mathcal{F}$ in $\text{conv}(\mathbf{A})$ will have a corresponding facet $\mathcal{F}' \cup \{\mathbf{z}\}$ in $\text{conv}(\mathbf{A}')$, since $\mathbf{0} \in \text{relint}(\text{conv}((\overline{M} \setminus \overline{\mathcal{F}}))$ before and after the addition of the point $\overline{\mathbf{z}}$ to the Gale diagram. These facets of $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}\})$ which contain the point at infinity $\mathbf{z}$ can be thought of as the *lateral facets*. The only "new" facets that we obtain correspond to sets $M \setminus \mathcal{S}$ where $\mathbf{0} \in \text{relint}(\text{conv}((\overline{\mathcal{S}} \cup \{\overline{\mathbf{z}}\}))$. Since $\overline{\mathbf{z}}$ is placed in affinely general position, all such new faces $M \setminus \mathcal{S}$ will be simplices, and will in fact correspond to the simplices in the lower hull of $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}\})$.

We see that a particular d-dimensional simplex $\mathcal{S}$ appears in in the lower hull of $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}\})$ exactly when $\mathbf{0} \in \text{relint}(\ \text{conv}((\overline{M} \setminus \overline{\mathcal{S}}) \cup \{\overline{\mathbf{z}}\}))$. This is equivalent to the condition that the opposite point $-\overline{\mathbf{z}}$ lies in the interior of the cone formed by the vectors of the set $\overline{M} \setminus \overline{\mathcal{S}}$. Thus if we think about removing the boundaries for all such cones formed by $n - d + 1$ elements of $\overline{M}$ from $\mathbb{R}^{n-(d+1)}$, we see that the sphere is divided into different regions where any two $-\overline{\mathbf{z}}$, $-\overline{\mathbf{z}}'$ placed in the same region would give identical lower hulls for the corresponding sets $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}\})$ and $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}'\})$, whereas two $-\overline{\mathbf{z}}$, $-\overline{\mathbf{z}}'$ placed in *differing* regions would give distinct lower hulls for the corresponding sets $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}\})$ and $\text{conv}(\mathbf{A}' \cup \{\mathbf{z}'\})$.

We finally note that there will be a region in this spherical complex for *every* possible lower hull of $\text{conv}(\omega(\mathbf{A}))$ for arbitrary general liftings ω, as it is possible to reverse-engineer the lifting ω to determine an affinely general placement of $\overline{\mathbf{z}}$ in the Gale diagram to make the corresponding $\mathbf{A}'$ match $\omega(\mathbf{A})$. Thus the regions in this spherical complex are in bijective correspondence with the set of regular triangulations of **A**.

Example 1.3.10 If we return to the Gale diagram discussed in Example 1.3.9, we see that by removing the boundaries of all possible cones spanned by two elements of $\overline{M}$ we divide the surface of the circle into 4 distinct regions depicted in Figure 1.9. If we think of placing the opposite of $\overline{\mathbf{z}}$, $-\overline{\mathbf{z}}$, into each of these regions, we obtain the 4 corresponding representative liftings, $\omega(\mathbf{A})$, pictured in Figure 1.10. Notice that the lower hulls of these liftings give all possible distinct regular triangulations of **A** after projection. $\diamondsuit$

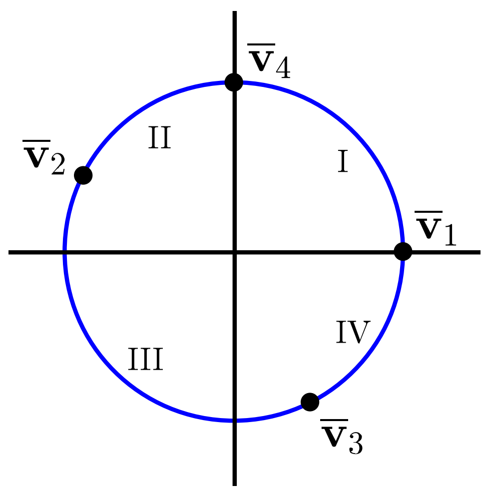


Figure 1.9: Gale diagram regions - line example

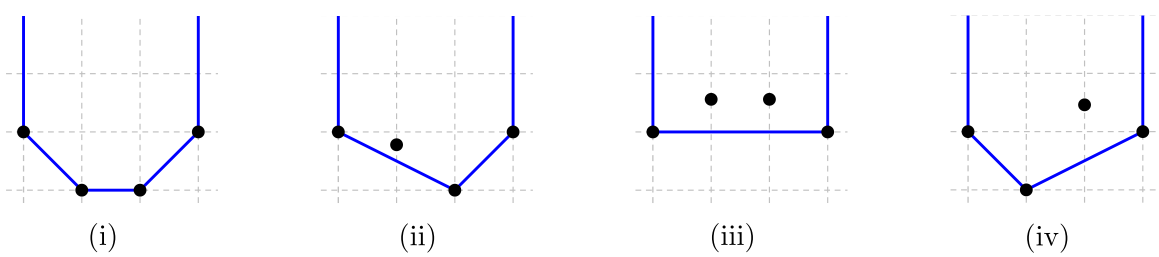


Figure 1.10: Representative liftings - line example, $k = 0$

Example 1.3.11 If we consider the hexagon given by

$$\mathbf{A} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{(-1, 1.5), (-2, 0), (-1, -1.5), (1, -1.5), (2, 0), (1, 1.5)\} \subset \mathbb{R}^2,$$

we find that the 2-dimensional Gale transform is given by

$$M = \begin{bmatrix} -1 & -2 & -1 & 1 & 2 & 1 \\ 1.5 & 0 & -1.5 & -1.5 & 0 & 1.5 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \overline{M} = \begin{bmatrix} 1 & 0 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{bmatrix}.$$

If we form a Gale diagram by scaling the points onto the unit sphere centered at the origin in $\mathbb{R}^3$, we obtain Figure 1.11.

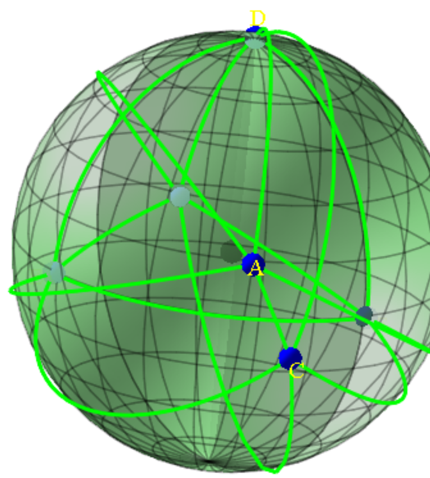


Figure 1.11: Gale diagram regions - hexagon example, $k = 0$

It is not easy to see from the static picture of the 3-dimensional sphere, but by "removing" the boundaries of all cones of 3 elements of $\overline{M}$, we obtain 14 distinct regions on the sphere which correspond to the 14 possible regular triangulations of the hexagon. This correctly reflects the number of vertices of the corresponding secondary polytope, which is also known as the associahedron corresponding to the hexagon, see [9], [3], and [13]. This polytope was pictured in Figure 1.4i. $\diamondsuit$

In [1], Billera, Filliman, and Sturmfels demonstrated another method to construct the secondary polytope $\Sigma_0(\mathbf{A})$ corresponding to the point set $\mathbf{A}$. In this construction, the point set $\mathbf{A}$ is translated so that $\sum_{\mathbf{v}\in\mathbf{A}} \mathbf{v} = \mathbf{0}$ prior to constructing the Gale diagram $\overline{M}$. This way, we can consider the original set of points as the Gale diagram of $\overline{M}$ and vice versa. We then think about moving the origin about in the interior of conv($\mathbf{A}$) such that it never lies on an affine hyperplane through d elements of $\mathbf{A}$. Each new placement of the origin in $\mathbf{A}$ will correspond to a scaling of the points of $\overline{M}$, thus we can form polytopes $\mathcal{P}_i = \text{conv}(\overline{M})$ for i ranging from 1 to $R = \#$(regions of conv($\mathbf{A}$) formed by all hyperplanes through d elements). By taking the Minkowski sum of the polars of these polytopes, we obtain a polytope $\mathcal{P}$ whose normal fan is a common refinement of the normal fans of the polars of $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_R$, and which was demonstrated to be combinatorially equivalent to the secondary polytope $\Sigma_0(\mathbf{A})$.

The following theorem describes how to alter the Gale diagram construction to describe the set of k deletion-induced triangulations associated to various general liftings ω of our point set $\mathbf{A}$.

Theorem 1.3.12 *Let $\overline{M}$ be a Gale diagram corresponding to the point set $\mathbf{A}$, represented by the matrix M, where the nonzero points have been scaled to lie on the unit sphere centered at the origin in $\mathbb{R}^{n-(d+1)}$. Denote the columns of $\overline{M}$, $\overline{\mathbf{v}}_1, \ldots, \overline{\mathbf{v}}_n$, and their opposites, $-\overline{\mathbf{v}}_1, \ldots, -\overline{\mathbf{v}}_n$, as the positive and negative elements of $\overline{M}$, respectively, and let $\overline{\mathbf{z}}$ be placed so that it lies on no hyperplane spanned by $n-(d+1)$ elements from $\{\pm\overline{\mathbf{v}}_1, \ldots, \pm\overline{\mathbf{v}}_n\}$. Then a collection $\mathcal{F} \subset M$ of points forms a lower hull facet of the lifting $\omega(\mathbf{A}) \cup \{\mathbf{z}\}$ (for some placement of $\overline{\mathbf{z}}$ in the Gale diagram) after the deletion of a cardinality k set $\widetilde{K}$ of M iff there exists a (possibly empty) subset $K \subseteq \widetilde{K}$ such that $\mathbf{0} \in \operatorname{relint}(\operatorname{conv}((\overline{M} \setminus (\overline{F} \cup \overline{K})) \cup -\overline{K}))$.*

Proof. We have previously seen that if we focus on the Gale diagram consisting of $\overline{\mathbf{v}}_1, \ldots, \overline{\mathbf{v}}_n$ and $\overline{\mathbf{z}}$ in general position, then the polytope P which possesses this Gale diagram can be thought of as the convex hull of a general lifting ω (determined by the placement of $\overline{\mathbf{z}}$) of the points of $\mathbf{A}$ and a point $\mathbf{z}$ at infinity. If $\mathcal{F} = \{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_s}\} \subset M$ denotes a lower hull facet of P after the k-deletion of some points $\widetilde{K} = \mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_k} \subset M$ ($\mathcal{F} \cap \widetilde{K} = \emptyset$), then there is some hyperplane with normal $\mathbf{n}$ and subset $K \subseteq \widetilde{K}$ (possibly empty) such that

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{v}' \quad \forall\, \mathbf{v}, \mathbf{v}' \in \mathcal{F}, \qquad \mathbf{n} \cdot \mathbf{v} > \mathbf{n} \cdot \mathbf{v}' \quad \forall\, \mathbf{v} \in \mathcal{F},\ \mathbf{v}' \in K,$$
$$\text{and} \quad \mathbf{n} \cdot \mathbf{v} < \mathbf{n} \cdot \mathbf{v}' \quad \forall\, \mathbf{v} \in \mathcal{F},\ \mathbf{v}' \in M \setminus (\mathcal{F} \cup K).$$

Thus if we let $\mathbf{r} = [\mathbf{n},\ -(\mathbf{n} \cdot \mathbf{v}_{i_1})] \in \mathbb{R}^{d+1}$ be a row vector, then we see that we can interpret the row vector $\mathbf{r}M$ as a vector in the set $[0, +, -]^n$. Specifically, coordinate i of $\mathbf{r}M$ is a 0 precisely if $\mathbf{v}_i \in \mathcal{F}$, is positive (+) precisely when $\mathbf{v}_i \notin \mathcal{F} \cup K$, and is negative (−) when $\mathbf{v}_i \in K$ for $1 \leq i \leq n$. This new row vector $\mathbf{rM}$ is in the row span of M, and is hence perpendicular to each row of $\overline{M}$. Thus we have

$$\begin{bmatrix} \overline{\mathbf{v}}_1 & \overline{\mathbf{v}}_2 & \ldots & \overline{\mathbf{v}}_n \end{bmatrix} \cdot (\mathbf{r}M)^\top = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{1\times n}$$

which implies that $\mathcal{F}$ is a facet of the lower hull of $\operatorname{conv}(\mathbf{A} \setminus K)$ precisely when $\mathbf{0} \in \operatorname{relint}(\operatorname{conv}(\overline{\mathcal{F}}' \cup (-\overline{K})))$, where $\overline{\mathcal{F}}' = \overline{M} \setminus (\overline{\mathcal{F}} \cup \overline{K})$ and $-\overline{K}$ denotes the negatives of the points in $\overline{K}$.

As is the case when studying facets of the lower hull of liftings of $\mathbf{A}$ with the Gale diagram, it is easy to see that if there are some subsets $\overline{K}, \overline{\mathcal{F}}' \subset \overline{M}$, $\overline{K} \cap \overline{\mathcal{F}}' = \emptyset$, $|\overline{K}| \leq k$, such that $\mathbf{0} \in \operatorname{relint}(\operatorname{conv}(\overline{\mathcal{F}}' \cup (-\overline{K})))$, then the previous argument may be reversed to show that the set $\mathcal{F} \subset M$ corresponding to $\overline{M} \setminus (\overline{K} \cup \overline{\mathcal{F}}')$ will form a face of the lower hull of P after the deletion of any k point collection $\widetilde{K}$ such that $K \subseteq \widetilde{K}$ and $\mathcal{F} \cap \widetilde{K} = \emptyset$. Similarly, if some set of k deletion-induced triangulations is realizable by some lifting ω of $\mathbf{A}$, then a placement of $\overline{\mathbf{z}}$ can be determined in the Gale diagram which will correspond to the set of points $\omega(\mathbf{A}) \cup \{\mathbf{z}\}$ where $\mathbf{z}$ is the point at infinity, which will hence produce the correct lower hulls after k-deletions. ■

Example 1.3.13 If we alter the Gale diagram for the set of 4 points on a line discussed in Example 1.3.9 by adding in the negatives of every point, we obtain the diagram in Figure 1.12 which can be seen to split the circle into eight regions if we remove the boundaries of all cones formed by two distinct positive points $\overline{\mathbf{v}}_i$ and $\overline{\mathbf{v}}_j$ (which tell us which simplices are in the original lower hull) and cones formed by a positive and a negative point $\overline{\mathbf{v}}_i$ and $-\overline{\mathbf{v}}_j$ (which tell us which simplices only appear in the lower hull after deletion of the point $\mathbf{v}_j \in \mathbf{A} \cup \{\mathbf{z}\}$). Note we do not consider cones involving more negative elements of $\overline{M}$ as we only go so far as to take $k = 1$ deletions for a set of this size in this dimension.

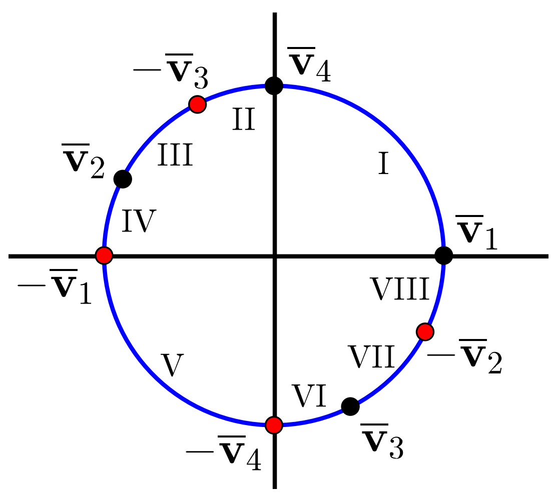


Figure 1.12: Gale diagram regions, line example, $k = 1$

If we place *negative* $\overline{\mathbf{z}}$ in each region, then Theorem 1.3.12 guarantees that the collection of representative polytopes $\text{conv}(\omega(\mathbf{A}) \cup \{\mathbf{z}\})$ corresponding to each placement of $-\overline{\mathbf{z}}$ will contain liftings which realize *every* possible 1-compound GKZ-vector corresponding to the point set $\mathbf{A}$. Figure shows 8 representative liftings obtained by the placement of $-\overline{\mathbf{z}}$ in the corresponding region of Figure 1.12.

For an example of how to interpret what sort of lifting is produced from each region, suppose that $-\overline{\mathbf{z}}$ is placed in region I. Then we see that $-\overline{\mathbf{z}}$ is in the cones formed by $(\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_4)$, $(\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2)$, and $(\overline{\mathbf{v}}_3, \overline{\mathbf{v}}_4)$, which implies that $(\mathbf{v}_2, \mathbf{v}_3)$, $(\mathbf{v}_3, \mathbf{v}_4)$, and $(\mathbf{v}_1, \mathbf{v}_2)$ are original lower hull facets, respectively. Next we want to consider what new minimal sets can capture the origin with $\overline{\mathbf{z}}$ when we consider cones with one

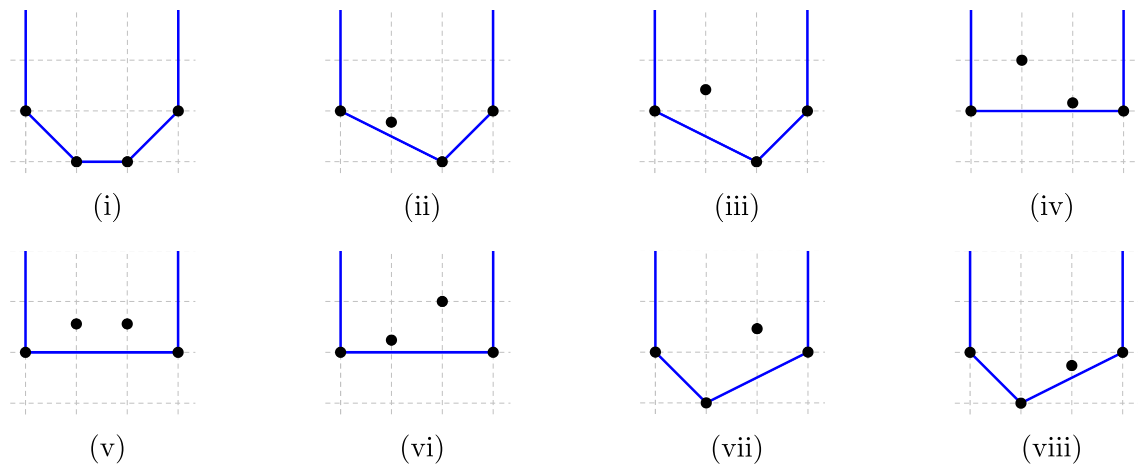


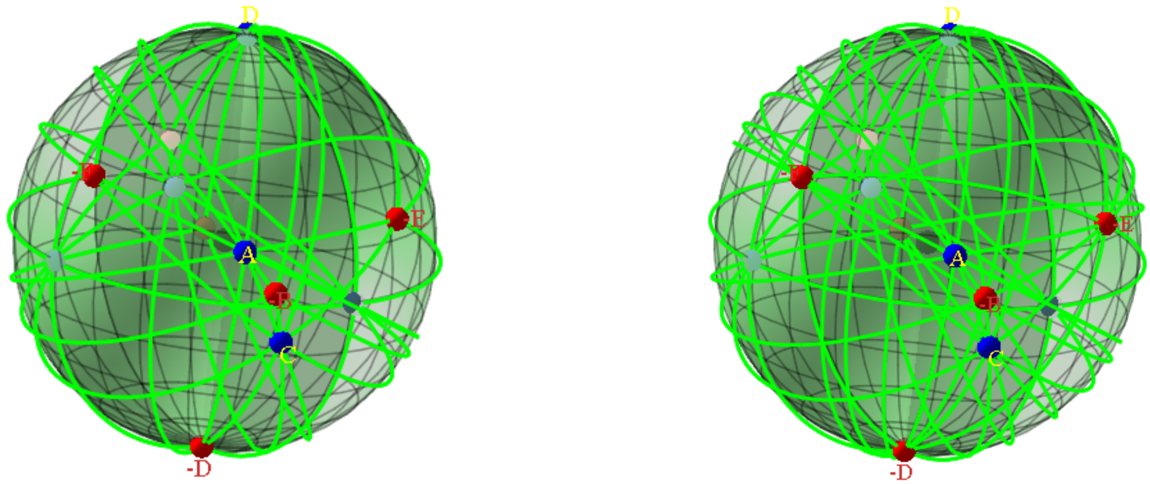
Figure 1.13: Representative liftings - line example, $k = 1$

negative element. Since $-\overline{\mathbf{z}}$ in this region does not lie in a cone formed by $-\overline{\mathbf{v}}_1$ and any positive $\overline{\mathbf{v}}_i$, we see no new lower hull facets are introduced after the deletion of $\mathbf{v}_1$, similarly with the deletion of $\mathbf{v}_4$. Since $-\overline{\mathbf{z}}$ is in the cones $(\overline{\mathbf{v}}_1, -\overline{\mathbf{v}}_3)$ and $(\overline{\mathbf{v}}_4, -\overline{\mathbf{v}}_2)$, we gather that $(\mathbf{v}_2, \mathbf{v}_4)$ forms a new lower hull facet after the deletion of $\mathbf{v}_3$ and $(\mathbf{v}_1, \mathbf{v}_3)$ forms a new lower hull after the deletion of $\mathbf{v}_2$.

To see the particular effects of the further subdivision of the sphere by the addition of the negative elements of $\overline{M}$, focus on regions IV, V, and VI in Figure 1.12. We see that these regions formed the single region III in Figure 1.9 of Example 1.3.10. This reflects the fact that all representative liftings of regions IV, V, and VI, have the same original lower hull, but have differing 1 deletion-induced triangulations. $\Diamond$

Example 1.3.14 We can also alter the Gale diagram for the hexagon discussed in Example 1.3.11 by adding in the negatives of every point. In this example, however, we have the option to choose k as high as 2.

Figure 1.14i is the diagram that we would use to study the 1-deletions of the hexagon. In this diagram, we have removed all boundaries of cones formed by 3 positive elements of $\overline{M}$ (these corresponded to the original lower hull) and the boundaries of cones formed by 2 positive elements of $\overline{M}$ and 1 negative element (these corresponded to new lower hull facets which appear after the 1-deletion of the point corresponding to the negative element used in the cone). We see that some new arcs (removed boundaries of cones) appear, specifically between positive and negative elements of $\overline{M}$. This subdivides the sphere into more regions than were present in Figure 1.11, and reflects the fact that two liftings ω and ω' which had the same original lower hulls, i.e., represented the same region in Figure 1.11, may have had distinct 1-deletion induced triangulations, which would require them to lie in distinct regions of Figure 1.14i. The deletion-induced polytope corresponding to this subdivision of the sphere was shown in Figure 1.4ii.



(i) (ii)

Figure 1.14: Gale diagram regions - hexagon example, $k = 1$, $k = 2$

Figure 1.14ii is the diagram that we would use to study the 2-deletions of the hexagon. We remove the same cones as we did for $k = 1$, but now we also remove cones which are formed by two negative elements and 1 positive element of $\overline{M}$. These cones allowed us to detect the new lower hull simplices which would appear after the deletion of the two points corresponding to the negative elements of the cone. We see that the surface of the sphere is subdivided further by arcs connecting two negatives of elements of $\overline{M}$. The polytope corresponding to this maximal subdivision was shown in Figure 1.4iii. $\diamondsuit$

We have seen that if we choose a maximal value for k, then by removing the boundaries of all cones formed by k' up to $n-(d+2)$ negative elements and $n-(d+1)-k'$ positive elements of $\overline{M}$, we obtain the finest possible subdivision of the sphere obtained by deleting the boundaries of cones consisting of any set of $n-(d+1)$ positive/negative elements of $\overline{M}$. In particular, this finest possible subdivision of the surface of the sphere can be thought of as being induced by deleting a particular central hyperplane arrangement from $\mathbb{R}^{n-(d+1)}$, which leads us to a second construction of the zonotope $\Sigma_{n-(d+2)}(\mathbf{A})$.

The creation of this particular zonotope via a hyperplane argument is discussed in [1]. In particular, we saw that knowledge of the orientation of every set of $d+2$ points in a general lifting $\omega(\mathbf{A})$ of our point set will determine the k deletion-induced triangulations for any k; in particular $k = n-(d+2)$. The set of sets of compatible orientations that can be placed on the $(d+2)$-tuples of points in $\mathbf{A}$ can be put into a natural bijection with the cells of the hyperplane arrangement given by the zero sets of the polynomials arising from the $(d+2)\times(d+2)$ minors of the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \dots & \mathbf{v}_n \\ x_1 & x_2 & x_3 & \dots & x_n \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}_{(d+2)\times n},$$

where the $\mathbf{v}_i$ correspond to the points of $\mathbf{A}$ in column vector form and the variables x_i give an arbitrary lifting vector $\omega = \langle x_1, x_2, \dots, x_n \rangle$. Thus if we let $\mathbf{n}_1, \dots, \mathbf{n}_{\binom{n}{d+2}}$ denote the normal vectors to the various hyperplanes in our arrangement, then defining l_i to be the line segment from $\mathbf{0}$ to the tip of the vector $\mathbf{n}_i$ in $\mathbb{R}^n$ for $1 \leq i \leq \binom{n}{d+2}$, we have that our zonotope is combinatorially equivalent to the Minkowski sum $\sum_{i=1}^{\binom{n}{d+2}} l_i \subset \mathbb{R}^n$ ([13]).

Lastly, we outline a construction for deletion-induced polytopes similar to the method used to construct the secondary in [1], which was described earlier. To this end, we note that we had considered the positions of the negatives of points $-\overline{\mathbf{v}}_1, \dots, -\overline{\mathbf{v}}_n$ in the Gale diagram when studying the lower hull facets after particular deletions, but we hadn't thought of these points as proper members of the Gale diagram. We see that by considering these points as part of the collection $\overline{M}$, then the corresponding point set $\mathbf{A}$ must increase by n points and increase in dimension by n. In particular, adding the negatives of points in the Gale diagram corresponds to applying Lawrence extensions to the point set $\mathbf{A}$ ([13], [7]).

Definition 1.3.15 If V is a point set in $\mathbb{R}^d$, adding the negative of an element $-\overline{\mathbf{v}}_i$ into $\overline{M}$, for some $\overline{\mathbf{v}}_i \in \overline{M}$, produces a new Gale diagram corresponding to a Lawrence extension on V. A *Lawrence extension* $\sigma_i(V)$, $1 \leq i \leq |V|$, is the point set we obtain from V by replacing the point $\mathbf{v}_i \in V$ with $\mathbf{v}_i^+ := \mathbf{v}_i + c_1 \cdot \mathbf{e}_{d+1}$, and $\mathbf{v}_i^- := \mathbf{v}_i + c_2 \cdot \mathbf{e}_{d+1}$, where $0 < c_1 < c_2 \in \mathbb{R}$, and $\mathbf{e}_{d+1}$ is the $(d+1)^{\text{st}}$ standard basis vector. Note that the "lower" point $\mathbf{v}_i^+$ is assigned a "+" because it corresponds to the positive version of $\overline{\mathbf{v}}_i$ in the Gale diagram, and $\mathbf{v}_i^-$ corresponds to $-\overline{\mathbf{v}}_i$. Traditionally, we choose $c_1 = 1$ and $c_2 = 2$, but by varying the value of these weights, we end up with a construction which is combinatorially equivalent (In the sense that the convex hull of $\sigma_i(V)$ has the same combinatorial structure).

For a point set $V \subset \mathbb{R}^d$ with no coloops (i.e., the deletion of any single vector from the set of vectors corresponding to V still leaves a set which spans $\mathbb{R}^d$), the *Lawrence Polytope* corresponding to the point set V, $\Lambda(V)$, is defined by

$$\Lambda(V) \ := \ \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_{|V|}(V) \subset \mathbb{R}^{d+|V|}.$$

Example 1.3.16 The Lawrence polytope $\Lambda(V)$ corresponding to $V = \{v_1, v_2, v_3\}$ consisting of three copies of the origin in $\mathbb{R}^0$ is depicted in Figure 1.15. $\diamondsuit$

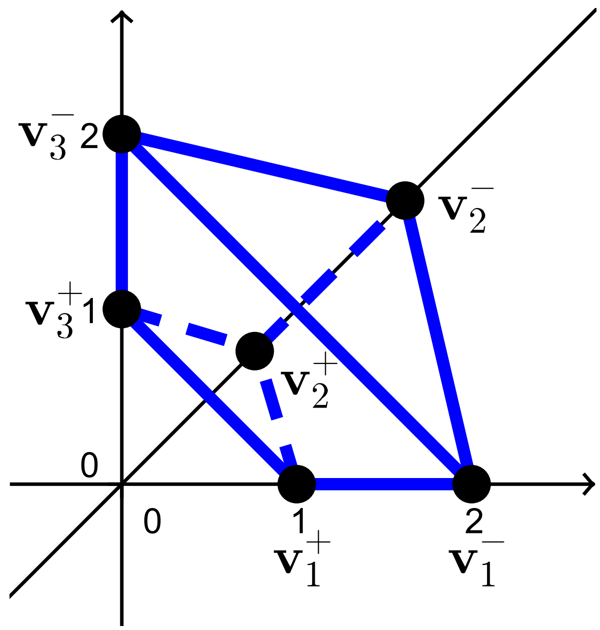


Figure 1.15: Lawrence polytope $\Lambda(V)$

This means that for a particular k-subset, $K \subset \mathbf{A}$, the different possible deletion-induced triangulations of the set $\mathbf{A} \setminus K$ should be, in some way, connected to the secondary polytope corresponding to the "k-Lawrence extension" L_K, where

$$L_K \ := \ \sigma_{\mathbf{v}_{i_1}} \circ \cdots \circ \sigma_{\mathbf{v}_{i_k}}(\mathbf{A}), \qquad \text{where} \ \ K = \{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\} \subset \mathbf{A}.$$

This follows from the fact that *both* of these combinatorial structures rely on varying the general placement of $\overline{\mathbf{z}}$ into the Gale diagram given by the points $\{\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \ldots, \overline{\mathbf{v}}_n\} \cup \{-\overline{\mathbf{v}}_i \mid \overline{\mathbf{v}}_i \in K\}$.

Example 1.3.17 Suppose we wish to study the deletion-induced triangulations of the point set studied in Example 1.8, specifically where we are deleting the element $\mathbf{v}_2 = 1$ from $\mathbf{A}$. This means we only consider adding $-\overline{\mathbf{v}}_2$ into the Gale diagram, producing Figure 1.16.

If we think of adding the *opposite* of $\overline{\mathbf{z}}$, $-\overline{\mathbf{z}}$, into the various regions depicted in Figure 1.16, we determine 5 representative liftings which model the only possible triangulations induced after the deletion of $\mathbf{v}_2 = 1 \in \mathbf{A}$. Five possible representative liftings for the corresponding regions in Figure 1.16 are shown in Figure 1.17 (read left to right, top to bottom), along with the triangulations induced by their original lower hulls $\mathcal{T}_\emptyset$ (top) and lower hulls after the deletion of $\mathbf{v}_2 = 1$, $\mathcal{T}_2$ (bottom).

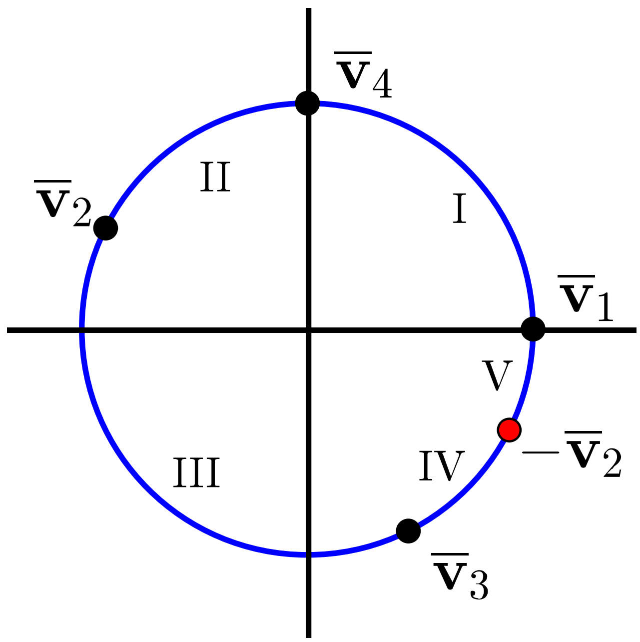


Figure 1.16: $\mathbf{v}_2$-deletion Gale diagram

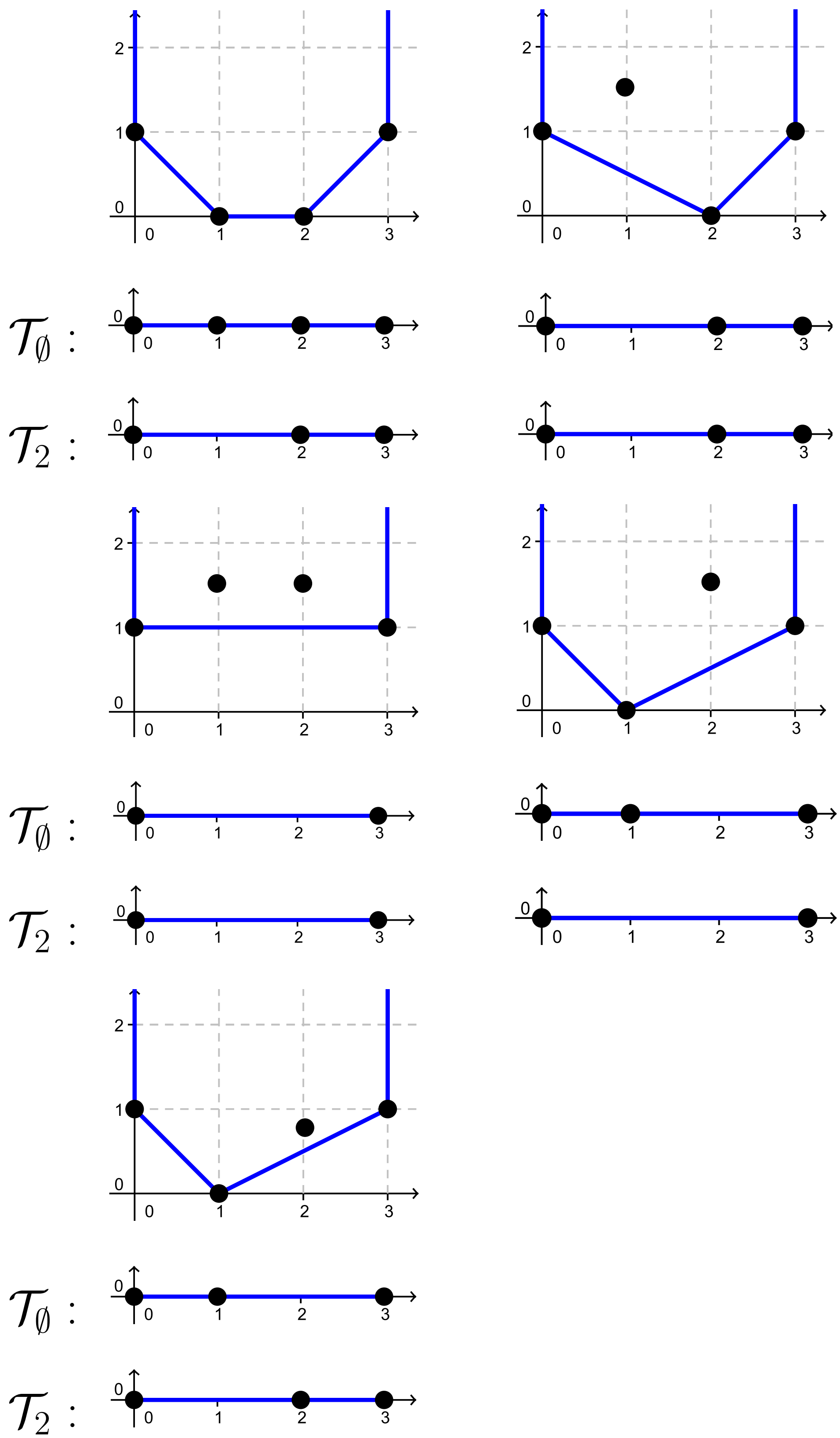


Figure 1.17: $\mathbf{v}_2$-deletion-induced triangulations

The compound GKZ-vectors obtained by adding the characteristic vectors corresponding to the triangulations $\mathcal{T}_\emptyset$ and $\mathcal{T}_2$ for the representative liftings of each region in Figure 1.16 are shown below:

Region I:	$\langle 1,2,2,1\rangle + \langle 2,0,3,1\rangle$	$=$	$\langle 3,2,5,2\rangle$,
Region II:	$\langle 2,0,3,1\rangle + \langle 2,0,3,1\rangle$	$=$	$\langle 4,0,6,2\rangle$,
Region III:	$\langle 3,0,0,3\rangle + \langle 3,0,0,3\rangle$	$=$	$\langle 6,0,0,6\rangle$,
Region IV:	$\langle 1,3,0,2\rangle + \langle 3,0,0,3\rangle$	$=$	$\langle 4,3,0,5\rangle$,
Region V:	$\langle 1,3,0,2\rangle + \langle 2,0,3,1\rangle$	$=$	$\langle 3,3,3,3\rangle$.

Next, we will utilize the same diagram in Figure 1.16, but we will instead use it to study the regular triangulations corresponding to the Lawrence extension $\sigma_{\mathbf{v}_2}(\mathbf{A})$ (using $c_1 = 2$, $c_2 = 4$), pictured in Figure 1.18.

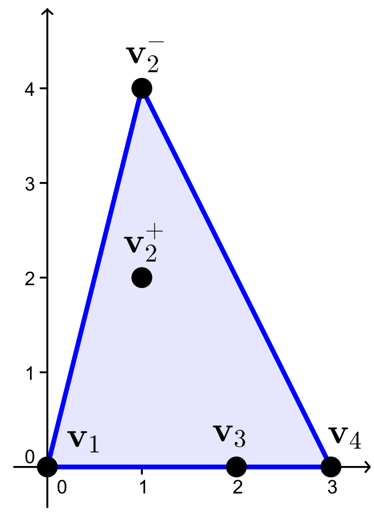


Figure 1.18: Lawrence extension $\sigma_{\mathbf{v}_2}(\mathbf{A})$

The 5 regular triangulations of this Lawrence extension corresponding to the placement of $-\overline{\mathbf{z}}$ in the 5 regions of Figure 1.16 are shown in Figure 1.19.

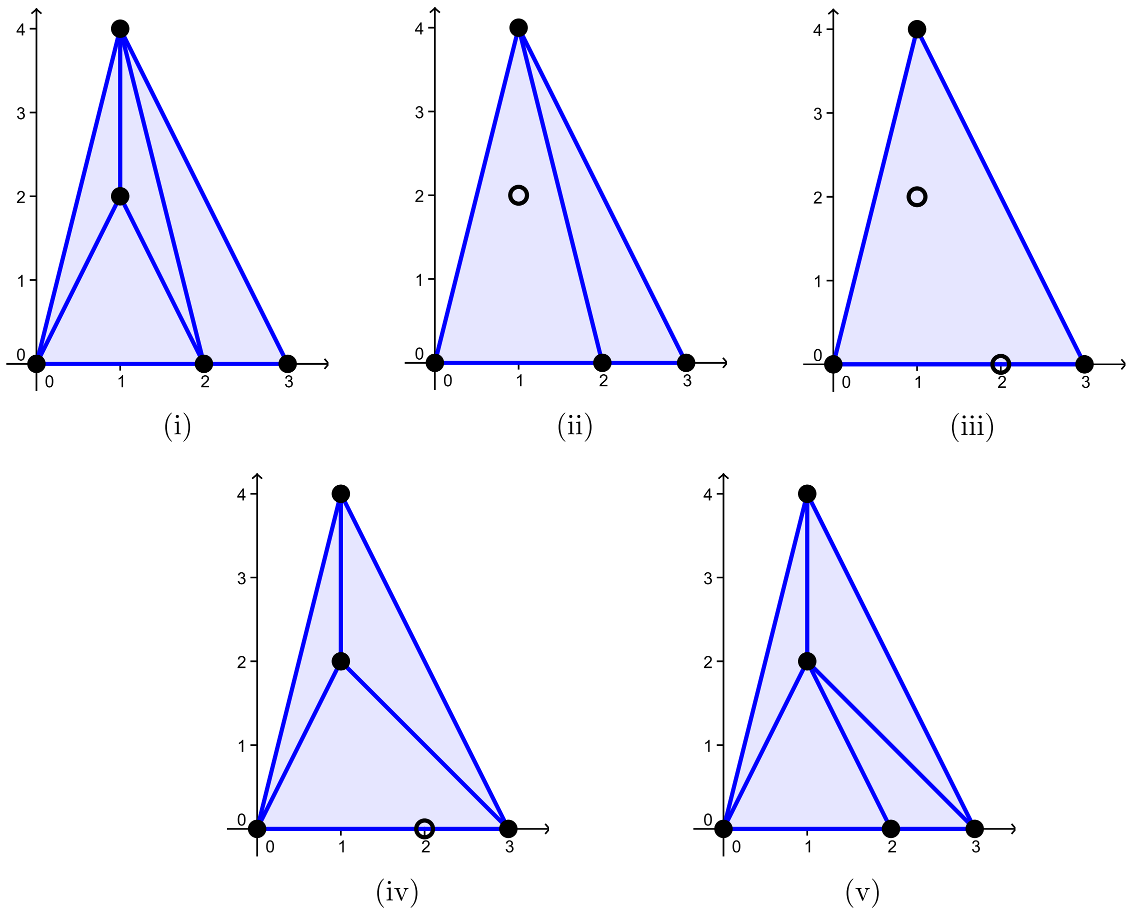


Figure 1.19: Regular triangulations of the Lawrence extension

If we order the points in the Lawrence extension by $\mathbf{v}_1, \mathbf{v}_2^-, \mathbf{v}_2^+, \mathbf{v}_3, \mathbf{v}_4$, then the corresponding characteristic vectors for the triangulations given by the regions of Figure 1.16 are shown below:

Region I: $\langle 3, 4, 4, 5, 2\rangle$,
Region II: $\langle 4, 6, 0, 6, 2\rangle$,
Region III: $\langle 6, 6, 0, 0, 6\rangle$,
Region IV: $\langle 4, 3, 6, 0, 5\rangle$,
Region V: $\langle 3, 3, 6, 3, 3\rangle$.

$\Diamond$

Definition 1.3.18 Let $\mathbf{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a collection of points in $\mathbb{R}^d$ of cardinality $n \geq d + 2$ such that no $(d + 1)$ points are affinely dependent and let $K = \{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\} \subset \mathbf{A}$ be such that $|K| = k$ for some integer $0 \leq k \leq n - (d + 2)$. Further, let L_K denote the k-Lawrence extension:

$$L_k \ := \ \sigma_{\mathbf{v}_{i_1}} \circ \ldots \circ \sigma_{\mathbf{v}_{i_k}}(\mathbf{A}) \subset \mathbb{R}^{d+k}.$$

For any simplex $\mathcal{S}$ in a triangulation of L_K, we see that the simplex must contain *at least* one version, $\mathbf{v}_{i_j}^-$ or $\mathbf{v}_{i_j}^+$ corresponding to each point $\mathbf{v}_{i_j} \in K$, otherwise the simplex would not have the appropriate dimension. For a particular simplex, we will define the following sets:

$$\begin{aligned} K^+(\mathcal{S}) &:= \{\mathbf{v}_{i_j} \in K \mid \mathbf{v}_{i_j}^+ \in \mathcal{S},\ \mathbf{v}_{i_j}^- \notin \mathcal{S}\}, \\ K^-(\mathcal{S}) &:= \{\mathbf{v}_{i_j} \in K \mid \mathbf{v}_{i_j}^- \in \mathcal{S},\ \mathbf{v}_{i_j}^+ \notin \mathcal{S}\}, \\ K^\pm(\mathcal{S}) &:= \{\mathbf{v}_{i_j} \in K \mid \mathbf{v}_{i_j}^+ \in \mathcal{S},\ \mathbf{v}_{i_j}^- \in \mathcal{S}\}. \end{aligned}$$

It is easy to see that these sets will partition K. We will also define the *support of $\mathcal{S}$ in $\mathbf{A}$*, $\text{supp}_{\mathbf{A}}(\mathcal{S})$, by

$$\text{supp}_{\mathbf{A}}(\mathcal{S}) := (\mathcal{S} \cap \mathbf{A}) \cup K^\pm(\mathcal{S}).$$

Lemma 1.3.19 *Let $\mathbf{A}$ and K be defined as in Definition 1.3.18 and construct the k-Lawrence extension so that if $\sigma_{\mathbf{v}_{i_j}}$ is the extension from $\mathbb{R}^{d+j-1}$ to $\mathbb{R}^{d+j}$, then the corresponding extension heights c_1 and c_2 are chosen so that $\mathbf{v}_{i_j}^- = \mathbf{v}_{i_j} + 2(d+j)\mathbf{e}_{d+j}$ and $\mathbf{v}_{i_j}^+ = \mathbf{v}_{i_j} + (d+j)\mathbf{e}_{d+j}$. If $\text{vol}_r(\mathcal{S})$ denotes the r-dimensional volume of an r-dimensional simplex $\mathcal{S}$, then we have that for any $(d+k)$-dimensional simplex $\mathcal{S}$ in a triangulation of L_K,*

$$\text{vol}_{d+k}(\mathcal{S}) = 2^{|K^-(\mathcal{S})|} \cdot \text{vol}_d(\text{supp}_{\mathbf{A}}(\mathcal{S})),$$

where $K^-(\mathcal{S})$ is as defined in Definition 1.3.18.

Proof. Note that Figure 1.20 may be helpful to look at when visualizing this proof. Fix an arbitrary simplex $\mathcal{S}$ in L_K and proceed by induction on k, noting that the base case for $k = 0$ is trivially satisfied. Suppose the assumption holds up to some $k = r$ and consider the next case $k = r + 1$. We break into three cases depending on which set $K^+(\mathcal{S}), K^-(\mathcal{S})$, or $K^\pm(\mathcal{S})$ the "last" element $v_{i_k} \in K$ falls into for the simplex $\mathcal{S}$:

<u>Case 1</u>: $\mathbf{v}_{i_k} \in K^+(\mathcal{S})$

In this case, $\mathcal{S}$ can be thought of as a pyramid over a $(d+k-1)$-dimensional simplex $\mathcal{S}'$ with apex $\mathbf{v}_{i_k}^+$ at height $d+k$. Since $\text{supp}_{\mathbf{A}}(\mathcal{S}) = \text{supp}_{\mathbf{A}}(\mathcal{S}')$ and $K^-(\mathcal{S}) = K^-(\mathcal{S}')$, we see by induction that the claim is satisfied.

<u>Case 2</u>: $\mathbf{v}_{i_k} \in K^-(\mathcal{S})$

In this case, $\mathcal{S}$ can be thought of as a pyramid over a $(d + k - 1)$-dimensional simplex $\mathcal{S}'$ with apex $\mathbf{v}_{i_k}^+$ at height $2(d + k)$. Since $\text{supp}_{\mathbf{A}}(\mathcal{S}) = \text{supp}_{\mathbf{A}}(\mathcal{S}')$ and

$|K^-(\mathcal{S})| - 1 = |K^-(\mathcal{S}')|$, we see by induction that the claim is satisfied.

Case 3: $\mathbf{v}_{i_k} \in K^{\pm}(\mathcal{S})$

In this case, we note that our simplex $\mathcal{S}$ may be expressed as the difference of two pyramids with common base $\mathcal{S}' = (\mathcal{S} \setminus \{\mathbf{v}_{i_k}^+, \mathbf{v}_{i_k}^-\}) \cup \{\mathbf{v}_{i_k}\}$ and with apexes $\mathbf{v}_{i_k}^-$ and $\mathbf{v}_{i_k}^+$, with heights $2(d+k)$ and $d+k$, respectively. This difference has the same volume as the pyramid P with base $\mathcal{S}'$ and apex $\mathbf{v}_{i_k}^+$, which satisfies $\mathrm{supp}_{\mathbf{A}}(P) = \mathrm{supp}_{\mathbf{A}}(\mathcal{S})$ and $|K^-(P)| = |K^-(\mathcal{S})|$. Since this reduces to Case I, we see by induction that our claim is satisfied. ■

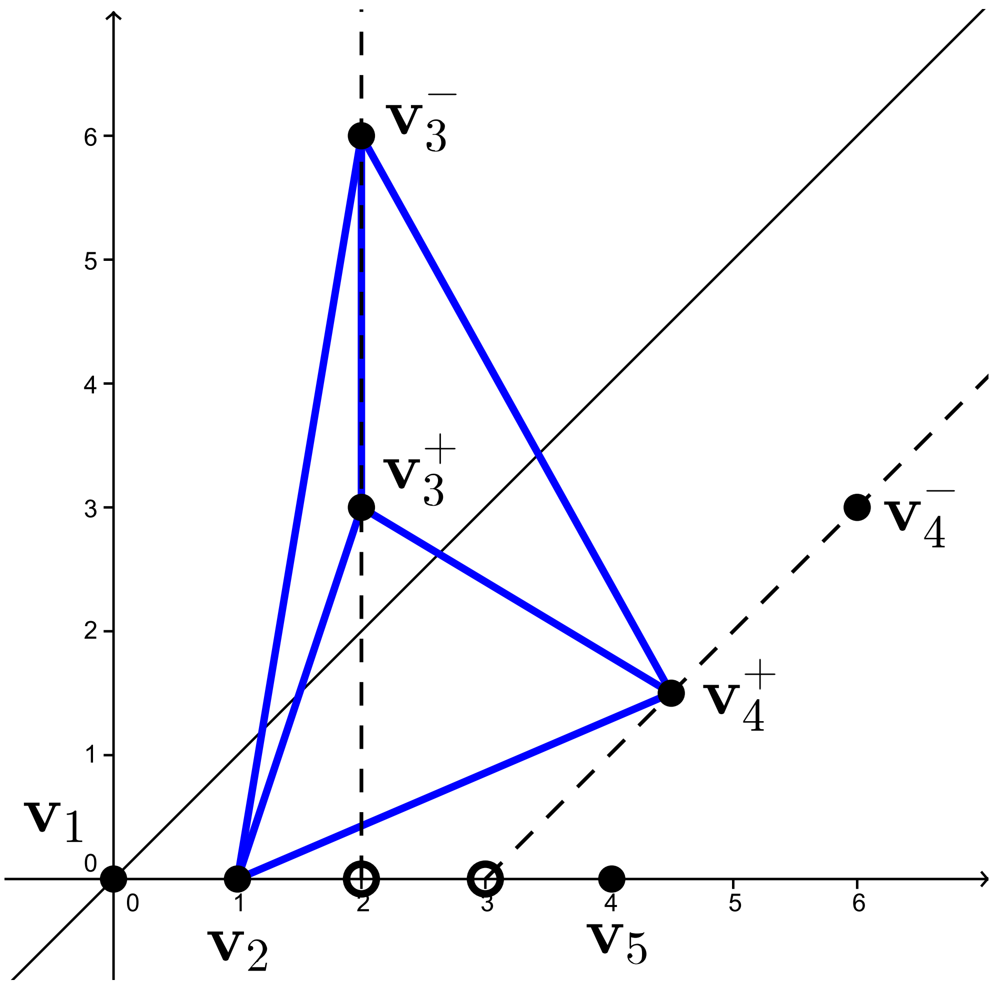


Figure 1.20: Double Lawrence extension in $\mathbb{R}^3$

Theorem 1.3.20 *If $L_{K_1}, \ldots, L_{K_{\binom{n}{k}}}$ denote the $\binom{n}{k}$ k-Lawrence extension polytopes of $\mathbf{A}$, then the k deletion-induced polytope $\Sigma_k(\mathbf{A})$ is combinatorially equivalent to a Minkowski sum of certain projections of the secondary polytopes corresponding to $L_{K_1}, \ldots, L_{K_{\binom{n}{k}}}$.*

Proof. Let $\mathbf{A} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be given and focus on a particular subset $K = \{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\} \subset \mathbf{A}$ of cardinality k. Let $\overline{M}$ be a Gale diagram corresponding to $\mathbf{A}$ where the negatives of the points corresponding to $\mathbf{v}_{i_j} \in K$ have been added. We have seen previously that this diagram is not only the Gale diagram corresponding to the k-Lawrence extension L_K (defined in Definition 1.3.18), but also could be used to study the possible deletion induced triangulations obtainable by removing K (or any of its subsets) from a lifting ω of $\mathbf{A}$. If we place a new point $-\overline{\mathbf{z}}$ in general position somewhere into the diagram, we saw that this induced a representative lifting $\omega(\mathbf{A})$ which corresponded to a list of deletion-induced triangulations, and also produced a particular regular triangulation of L_K.

If we let

$$\mathbf{u} := \sum_{K' \subset K} \phi(\mathcal{T}_{K'}) \qquad \text{where } \mathcal{T}_{K'} \text{ corresponds to the lower hull of } \omega(\mathbf{A} \setminus K'),$$

and let $\mathbf{u}'$ denote the characteristic vector corresponding to our regular triangulation of L_K, our aim is to first produce a projection $P : \mathbb{R}^{d+k} \longrightarrow \mathbb{R}^d$ which will send $\mathbf{u}' \mapsto \mathbf{u}$ for $-\overline{\mathbf{z}}$ placed in any general region. First label the set $\mathbf{A} \setminus K$ by $\{\mathbf{v}_{i_{k+1}}, \ldots, \mathbf{v}_{i_{n-k}}\}$ and assume, without loss of generality, that coordinates of the characteristic vectors $\mathbf{u}$ and $\mathbf{u}'$ are ordered by:

$$\begin{aligned} \mathbf{u} &:= (\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}, \mathbf{v}_{i_{k+1}}, \ldots, \mathbf{v}_{i_{n-k}}), \\ \mathbf{u}' &:= (\mathbf{v}^+_{i_1}, \mathbf{v}^-_{i_1}, \ldots, \mathbf{v}^+_{i_k}, \mathbf{v}^-_{i_k}, \mathbf{v}_{i_{k+1}}, \ldots, \mathbf{v}_{i_{n-k}}). \end{aligned}$$

The proposed projection P that will map $\mathbf{u}'$ to $\mathbf{u}$ is defined by

$$\begin{aligned} P &:= (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}, x_{2k+2} \ldots, x_{n+k}) \\ &\mapsto (x_1 + x_2 - \mathrm{vol}(L_K), \ldots, x_{2k-1} + x_{2k} - \mathrm{vol}(L_K), x_{2k+1}, x_{2k+2}, \ldots, x_n) \end{aligned}$$

To verify that this is the correct projection, first note that the presence of a particular $(d+k)$-dimensional simplex $\mathcal{S}$ in L_K implies that $\mathrm{supp}_{\mathbf{A}}(\mathcal{S})$ forms a d-dimensional simplex in $\omega(\mathbf{A})$ where only the elements in $K^+(\mathcal{S})$ appear "below" the simplex, and all other elements $K^-(\mathcal{S})$ appear "above" the simplex (recall that in the Gale diagram we are looking at complements of these sets, this is why elements $\mathbf{v}_{i_j} \in K^+(\mathcal{S})$ appear below, because these correspond to $-\overline{\mathbf{v}}_{i_j}$ being used to capture the origin).

Now, if $\mathbf{v} \in \mathbf{A} \setminus K$, and $\mathcal{S}$ is a simplex which contains $\mathbf{v}$ in our triangulation of L_K, then Lemma 1.3.19 implied that $\mathcal{S}$'s contribution to the coordinate corresponding to $\mathbf{v}$ in $\mathbf{u}'$ is equal to $2^{|K^-|} \cdot \mathrm{vol}_d(\mathrm{supp}_{\mathbf{A}}(\mathcal{S}))$. When we consider the corresponding simplex $\mathrm{supp}_{\mathbf{A}}(\mathcal{S})$ in $\omega(\mathbf{A})$, we see that it only contributes its volume to the coordinate corresponding to $\mathbf{v}$ in $\mathbf{u}$ in deletions of sets $K' \subseteq K$ where $K^+ \subseteq K'$. In other words, in order for this simplex to be picked up in a lower hull after a deletion, we must at least delete all of the vertices that appear below this simplex in $\omega(\mathbf{A})$. There are $2^{|K^-|}$ such sets K', hence the coordinates corresponding to $\mathbf{v}$ in $\mathbf{u}$ and $\mathbf{u}'$ will be equal if $\mathbf{v} \notin K$.

Next, suppose that $\mathbf{v}_{i_j} \in K$. The only simplices $\mathcal{S}$ of L_K which will correspond to simplices $\text{supp}_{\mathbf{A}}(\mathcal{S})$ in $\omega(\mathbf{A})$ which contain $\mathbf{v}_{i_j}$ are those which have $\mathbf{v}_{i_j} \in K^{\pm}(\mathcal{S})$. If we can restrict our view to the contribution of these simplices, Lemma 1.3.19 would guarantee that the sum of their volumes would appropriately match the coordinate corresponding to $\mathbf{v}$ in $\mathbf{u}$. In order to obtain the sum of volumes of this restricted set of simplices, we note that every simplex $\mathcal{S}$ must be adjacent to *at least one* of $\mathbf{v}_{i_j}^+$ or $\mathbf{v}_{i_j}^-$ in L_K. Thus if we sum the volumes of all simplices adjacent to $\mathbf{v}_{i_j}^+$ and the volumes of all simplices adjacent to $\mathbf{v}_{i_j}^-$ and subtract the total volume of L_K, we obtain the sum of volumes of all simplices $\mathcal{S}$ adjacent to *both* $\mathbf{v}_{i_j}^+$ and $\mathbf{v}_{i_j}^-$. This is equivalent to summing the coordinates corresponding to $\mathbf{v}_{i_j}^+$ and $\mathbf{v}_{i_j}^-$ in $\mathbf{u}'$ and subtracting the total volume of L_K. This establishes that P is indeed the correct projection.

Finally, we note that by projecting the secondary of L_K for a particular K, we are obtaining polytopes whose vertices are of the form $\mathbf{u}$ described previously. By taking the Minkowski sum of these polytopes projected from $L_{K_1}, \ldots, L_{K_{\binom{n}{k}}}$, we will obtain the polytope $\widetilde{\Sigma}_{\chi,k}(\mathbf{A})$ where χ is defined as follows:

$$\chi(x) = \binom{n}{k - \#(x)},$$

where $\#(x)$ denotes the cardinality of the subset corresponding to coordinate x. In other words, the characteristic vectors corresponding to deletions of k-subsets are each counted once, characteristic vectors corresponding to deletions of $(k-1)$-subsets are each counted n times, and so on down to the characteristic vector corresponding to the original lower hull (no deletions) which is counted $\binom{n}{k}$ times. This polytope was proven to be combinatorially equivalent to $\Sigma_k(\mathbf{A})$ in Theorem 1.3.3. ■

Example 1.3.21 Consider the secondary polytope corresponding to the Lawrence extension $\sigma_{\mathbf{v}_2}(\mathbf{A})$ constructed as the convex hull of the characteristic vectors provided in Example 1.3.17. Theorem 1.3.20 implies that we can construct the following projection to map the characteristic vectors of the secondary onto the compound GKZ-vectors also provided in Example 1.3.17*:*

$$P \; : \; \mathbb{R}^5 \longrightarrow \mathbb{R}^4, \qquad (x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2 + x_3 - \text{Area}(\sigma_{\mathbf{v}_2}(\mathbf{A})), x_4, x_5),$$

where coordinates x_2 and x_3 in the characteristic vector recorder the area of triangles incident to $\mathbf{v}_2^-$ and $\mathbf{v}_2^+$, respectively, and Area($\sigma_{\mathbf{v}_2}(\mathbf{A})$)=6. We verify the projection for each region in Figure 1.16:

$$\begin{array}{llcl} \textit{Region I:} & P(\langle 3,4,4,5,2\rangle) & = & \langle 3,2,5,2\rangle, \\ \textit{Region II:} & P(\langle 4,6,0,6,2\rangle) & = & \langle 4,0,6,2\rangle, \\ \textit{Region III:} & P(\langle 6,6,0,0,6\rangle) & = & \langle 6,0,0,6\rangle, \\ \textit{Region IV:} & P(\langle 4,3,6,0,5\rangle) & = & \langle 4,3,0,5\rangle, \\ \textit{Region V:} & P(\langle 3,3,6,3,3\rangle) & = & \langle 3,3,3,3\rangle, \end{array}$$

If we were to repeat this process for elements $\mathbf{v}_1, \mathbf{v}_3$, and $\mathbf{v}_4$, then take the Minkowski sum of each of the projected polytopes, Theorem 1.3.20 would imply that the result would be combinatorially equivalent to $\Sigma_1(\mathbf{A})$, where $\mathbf{A} = \{0, 1, 2, 3\} \subset \mathbb{R}$.

$\Diamond$

1.4 Further Questions

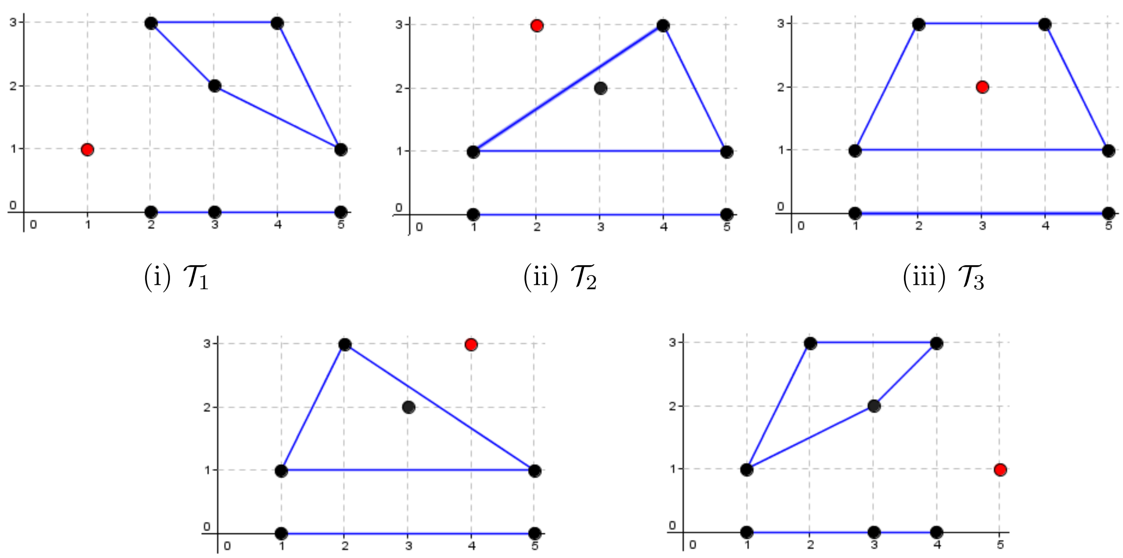
The deletion construct discussed so far gives a method to construct a family of polytopes corresponding to a point set $\mathbf{A}$ with the secondary polytope corresponding to $\mathbf{A}$ as the "base" member. We have seen that the Gale diagrams of each successive member in the family correspond to finer and finer subdivisions of the surface of a hypersphere, and so, in a sense, each polytope in the family is a refinement of the previous member.

The secondary polytope introduced in [5] was later described as a special case of fiber polytope arising from the projection of the $(n-1)$-simplex onto a polytope with n vertices. It would be interesting to see what connections, if any, could be made using this deletion process to study more general fiber polytopes than the secondary. Of particular interest would be the study of zonotopes, which arise as the projections of high dimensional cubes.

2.1 Definitions and Notation

In this chapter, we will explicitly focus on point sets $\mathbf{A} \subset \mathbb{R}$, i.e. $d = 1$. Specifically, let $n > 4$ be an integer and let $\mathbf{A}$ denote the set of integers $\{1, \ldots, n\}$. As in the previous chapter, we will let $\omega \in \mathbb{R}^n$ be thought of as a "lifting vector" with coordinate ω_i being appended to the point $\mathbf{v}_i \in \mathbf{A}$, $1 \leq i \leq n$, to lift each point off of the x-axis and into $\mathbb{R}^2$. We will refer to the set of lifted points as $\omega(\mathbf{A})$ and will initially only consider ω such that no set of 3 points in $\omega(\mathbf{A})$ are collinear. In this chapter, we will also fix $k = 1$ and so only consider 1 deletion-induced triangulations. To each ω, we assign the set of 1 deletion-induced triangulations as outlined previously.

Example 2.1.1 Let $\mathbf{A} = \{1, 2, 3, 4, 5\}$ and consider the lifting $\omega = \langle 1, 3, 2, 3, 1 \rangle$. The set of 1 deletion-induced triangulations $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$, and $\mathcal{T}_5$ corresponding to this ω is shown in Figure 2.1. $\diamondsuit$



(i) $\mathcal{T}_1$ (ii) $\mathcal{T}_2$ (iii) $\mathcal{T}_3$

(iv) $\mathcal{T}_4$ (v) $\mathcal{T}_5$

Figure 2.1: 1 deletion-induced triangulations

Example 2.1.2 The following are the characteristic vectors corresponding to the triangulations given in Figure 2.1:

$$\phi(\mathcal{T}_1) = \langle 0,1,3,0,2\rangle, \qquad \phi(\mathcal{T}_2) = \langle 4,0,0,0,4\rangle, \qquad \phi(\mathcal{T}_3) = \langle 4,0,0,0,4\rangle$$
$$\phi(\mathcal{T}_4) = \langle 4,0,0,0,4\rangle, \qquad \phi(\mathcal{T}_5) = \langle 2,0,3,1,0\rangle.$$

$\Diamond$

For the lifting ω given in Example 2.1.1, we find $\overline{\phi}_1(\omega) = \langle 14,1,6,1,14\rangle$. For the rest of this chapter, we will omit referring to the "1-compound" GKZ-vectors since k will always be fixed at 1. Instead, we will just call them compound GKZ-vectors. To further investigate the properties of the compound GKZ-vectors when $k = d = 1$, we introduce some machinery.

Definition 2.1.3 We define the *orientation function* $\mathcal{O}$ on the set of lifted points $\omega(\mathbf{A})$, as follows:

$$\mathcal{O}_\omega \ : \ \mathbf{A}^3 \longrightarrow \mathbb{R}, \quad (i,\ j,\ k) \ \mapsto \ \det \begin{vmatrix} i & j & k \\ \omega(i) & \omega(j) & \omega(k) \\ 1 & 1 & 1 \end{vmatrix}$$

where $i < j < k$. We will omit the subscript if it is clear from which lifting we are considering orientations.

Definition 2.1.4 Utilizing the orientation function described in Definition 2.1.3, we classify the points $(j,\ \omega(j))$ corresponding to a particular lifting ω of $\mathbf{A}$ via the following. The point $(j,\ \omega(j))$ is

- *Visible (from below) in* ω if $\mathcal{O}(i,j,k) < 0$ for all pairs (i,k) with $i < j$ and $k > j$,
- *Not Visible (from below) in* ω if there exists i,k such that $i < j < k$ and $\mathcal{O}(i,j,k) > 0$,
- *Semi-Visible (from below) in* ω if i and k are the positions of the nearest visible neighbors to the left and right of j, respectively, and $\mathcal{O}(i,j,k) = 0$.

We will extend these descriptions to particular deletions of ω in the natural manner and will usually refer to a particular point $(j,\ \omega(j))$ in a lifting simply as "j". Note that initially we only consider liftings which produce points in general position, thus no point would be semi-visible. We will require this distinction later in the paper.

Example 2.1.5 Consider the lifting $\omega = \langle 4,2,1,2,4,4,2\rangle$ where we delete the point 3, illustrated in Figure 2.2. The points are classified as follows: points 1, 2, and 7 are visible, point 4 is semi-visible, and points 5 and 6 are not visible in this particular deletion of ω. $\Diamond$

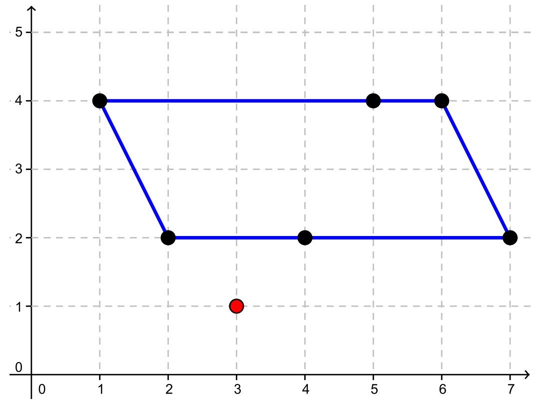


Figure 2.2: Visibility example

Definition 2.1.6 We will define the *support* of a characteristic vector $\phi(\mathcal{T})$ as the set of indices of nonzero positions in $\phi(\mathcal{T})$, and denote it by $\text{supp}(\phi(\mathcal{T}))$. Note that we can determine the entries in the characteristic vector with knowledge of the support and vice versa, which in turn allows us to reconstruct the triangulation $\mathcal{T}$.

Proposition 2.1.7 *Two different liftings ω and ω' of* $\mathbf{A}$ *produce the same compound* GKZ-*vector iff the corresponding deletion-induced triangulations $\mathcal{T}_i$ and $\mathcal{T}_i'$ match for deletions $i = 1, \ldots, n$.*

Proof. This result for characteristic vectors of regular triangulations is given in [3] and [6], but we will reproduce the spirit of the argument here and extend to sets of deletion-induced triangulations as it will be useful to reference later in the chapter. The fact that two liftings with identical sets of deletion-induced triangulations will produce the same compound GKZ-vectors is clear, as the characteristic vectors will match at each deletion. Next, for a fixed lifting ω and *any* regular triangulation $\mathcal{T}$ of the set of points $\mathbf{A}$, consider the inner product $\omega \cdot \phi(\mathcal{T})$. For liftings into $\mathbb{R}^2$, this inner product records the sum of 2 times the areas of the trapezoids whose bases are determined by $\mathcal{T}$ and whose heights are determined by the lifting ω. It is an easy algebraic exercise to show the following:

- If $\mathcal{T}$ is a triangulation of $\mathbf{A}$ satisfying $i \notin \text{supp}(\phi(\mathcal{T}))$ for some position i which is visible in ω, then $\mathcal{T}'$ satisfying $\text{supp}(\phi(\mathcal{T}')) = \text{supp}(\phi(\mathcal{T})) \cup \{i\}$ gives the relation $\omega \cdot \phi(\mathcal{T}') < \omega \cdot \phi(\mathcal{T})$.
- If $\mathcal{T}$ is a triangulation of $\mathbf{A}$ satisfying $i \in \text{supp}(\phi(\mathcal{T}))$ for some position i which is not visible in ω, then $\mathcal{T}'$ satisfying $\text{supp}(\phi(\mathcal{T}')) = \text{supp}(\phi(\mathcal{T})) \setminus \{i\}$ gives the relation $\omega \cdot \phi(\mathcal{T}') < \omega \cdot \phi(\mathcal{T})$.

- If $\mathcal{T}$ is a triangulation of $\mathbf{A}$ satisfying $i \in \mathrm{supp}(\phi(\mathcal{T}))$ for all positions i which are visible in ω, then $\mathcal{T}'$ satisfying $\mathrm{supp}(\phi(\mathcal{T}')) \triangle \mathrm{supp}(\phi(\mathcal{T})) = \{j\}$, for some position j which is semi-visible in ω gives the relation $\omega \cdot \phi(\mathcal{T}') \; = \; \omega \cdot \phi(\mathcal{T})$.

This allows us to conclude that a characteristic vector $\phi(\mathcal{T})$ minimizes $\omega \cdot \phi(\mathcal{T})$ exactly when $\mathrm{supp}(\phi(\mathcal{T}))$ contains all visible positions of ω, no non-visible positions, and some (possibly empty) subset of semi-visible positions of ω. Since we initially only consider general liftings, we do not have semi-visible points, and hence the support of characteristic vectors must be exactly the set of visible points of ω, which in turn corresponds to selecting the triangulation obtained by projecting the lower hull of $\mathrm{conv}(\omega(\mathbf{A}))$ in order to minimize this inner product. Since $\overline{\phi}(\omega)$ is constructed precisely as the sum of these lower hull induced triangulations, we have that $\omega \cdot \overline{\phi}(\omega) \; \leq \omega \cdot \overline{\phi}(\widetilde{\omega})$ for any lifting $\widetilde{\omega}$. Further, we are given that $\omega \cdot \overline{\phi}(\omega) = \omega \cdot \overline{\phi}(\omega')$, so it must be the case that the set of deletion-induced triangulations corresponding to ω', $\mathcal{T}_1', \ldots, \mathcal{T}_n'$ minimize the individual inner products $\omega \cdot \phi(\mathcal{T}_i')$ which in turn implies that the triangulations $\mathcal{T}_i$ and $\mathcal{T}_i'$ must match for $i = 1, \ldots, n$, as both must be given by projections of the lower hulls of $\mathrm{conv}(\omega(\mathbf{A}) \setminus \{i\})$ for $i = 1, \ldots, n$. ■

Definition 2.1.8 We construct the *deletion-induced polytope corresponding to* $\mathbf{A}$, $\Sigma_1(n)$, by first forming the set

$$\mathcal{B} \; := \; \left\{ \, \boldsymbol{v} \mid \boldsymbol{v} \; = \; \overline{\phi}(\omega) \text{ for some lifting } \omega \text{ of } \mathbf{A} \right\},$$

and taking

$$\Sigma_1(n) \; := \; \mathrm{conv}\,(\mathcal{B})\,.$$

In this particular chapter, we denote the polytope by $\Sigma_1(n)$ and not $\Sigma_1(\mathbf{A})$, since $\mathbf{A}$ will always correspond to n equally spaced points on the line.

That $\mathcal{B}$ is a finite set follows from the fact that $\overline{\phi}(\omega)$ can be uniquely determined if one knows the orientations of every set of 3 points in $\omega(\mathbf{A})$, which gives the rough upper bound $|\mathcal{B}| \leq 2^{\binom{n}{3}}$. This bound is a large overestimate, however, since not every set of orientations is realizable by a lifting and because two liftings with slightly differing sets of orientations may still potentially produce the same compound GKZ-vector.

Corollary 2.1.9 *Each compound GKZ-vector is a vertex of* $\Sigma_1(n)$*. In other words, no compound GKZ-vector lies in the interior of the convex hull of the other compound GKZ-vectors.*

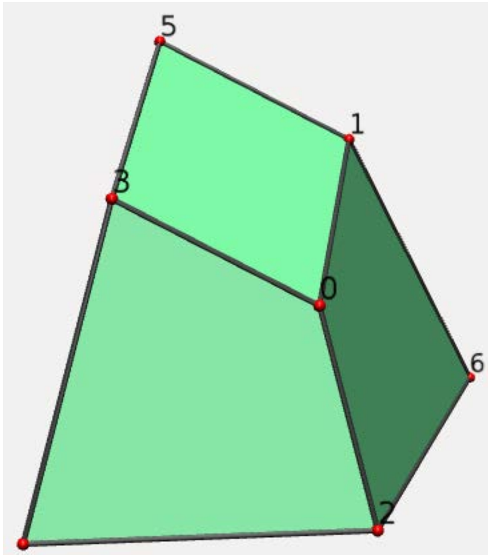
Proof. In Proposition 2.1.7, we saw that if ω was a lifting which produced the compound GKZ-vector $\mathbf{v}$, then $\omega \cdot \mathbf{v} < \omega \cdot \mathbf{v}'$ for any compound GKZ-vector $\mathbf{v}' \neq \mathbf{v}$. Thus $\omega \cdot \mathbf{x} = \omega \cdot \mathbf{v}$ is a vertex supporting hyperplane for the compound GKZ-vector $\mathbf{v}$. ■

If we let $\mathbf{v}_1, \mathbf{v}_2, \ldots$ denote the vertices of $\Sigma_1(n)$ (i.e. the distinct compound-GKZ vectors), then each of the vertices $\mathbf{v}_i$ will satisfy the following two linearly distinct relations:

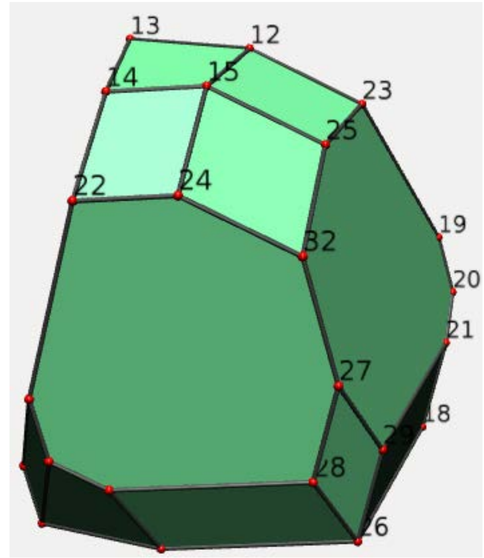
- $\langle 1, 1, \ldots, 1 \rangle \cdot \mathbf{v}_i = \sum_{j=1}^{n} (d+1)\mathrm{vol}\,(\mathrm{conv}\,(\mathbf{A} \backslash \{j\})),$
- $\langle 1, 2, \ldots, n \rangle \cdot \mathbf{v}_i = \sum_{j=1}^{n} (d+1)\mathrm{vol}\,(\mathrm{conv}\,(\mathbf{A} \backslash \{j\}))\, c_{(\mathbf{A} \backslash j)},$

where $c_{(\mathbf{A} \backslash j)}$ denotes the position of the centroid of $\mathrm{conv}(\mathbf{A} \backslash j)$. This is a natural consequence of the relations found to be satisfied by "ordinary" GKZ-vectors in [3] and [6]. We have established indirectly in Chapter 1 that $\Sigma_1(n)$ has dimension $n-2$, and later in Section 2.2, after we describe and utilize a combinatorial structure underlying the vertices, we will be able to give a different argument for this fact.

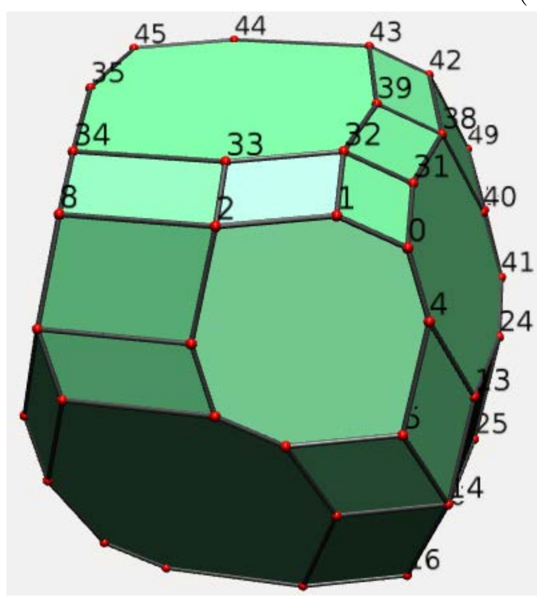
Figure 2.3 shows the family $\Sigma_k(\mathbf{A})$ for $\mathbf{A}$ corresponding to 5 evenly spaced points on a line, with $k = 0, 1$ and 2.



(i) $\Sigma_0(\mathbf{A})$



(ii) $\Sigma_1(\mathbf{A})$



(iii) $\Sigma_2(\mathbf{A})$

Figure 2.3: $\Sigma_k(\mathbf{A})$ for 5 points on a line - $k = 0, 1, 2$

2.2 Combinatorial Structure

In order to further investigate the properties of $\Sigma_1(n)$, we will outline a combinatorial structure underlying the compound GKZ-vectors. In particular, we will describe a map which sends compound GKZ-vectors / representative liftings to words in a special alphabet.

Definition 2.2.1 Let $\mathcal{C}$ denote the set of equivalence classes on liftings of $\mathbf{A}$ where we declare two liftings to be equivalent if they are associated with the same compound

GKZ-vectors. Consider the map

$$f: \ \mathcal{C} \ \rightarrow [M, L, R, LR, E]^n, \quad f(\omega) \mapsto P(\omega),$$

which assigns to each lifting ω an n-tuple of letters $P(\omega)$. Each position i from 1 to n in a particular lifting ω will be assigned a letter via the following method:

- $i \mapsto M$ if and only if the point $(i, \omega(i))$ is a part of the lower hull of $\text{conv}(\omega(\mathbf{A}))$. Note, this means positions 1 and n will always be assigned an M.
- $i \mapsto LR$ if and only if the point $(i, \omega(i))$ is not visible in the original lower hull, but becomes visible after the single deletion of *each* its nearest "M neighbors" to the left and right.
- $i \mapsto L$ if and only if the point $(i, \omega(i))$ only becomes visible after the deletion of its nearest left M neighbor, but it is *not* visible after the deletion of its right M neighbor.
- $i \mapsto R$ if and only if the point $(i, \omega(i))$ only becomes visible after the deletion of its nearest right M neighbor, but it is *not* visible after the deletion of its left M neighbor.
- $i \mapsto E$ if and only if the point $(i, \omega(i))$ is *not* visible in the deletion of its left M neighbor *nor* its right M neighbor, and hence not visible in *any* single deletion of $\mathbf{A}$.

It is not difficult to check that this map is well defined on $\mathcal{C}$. We will call the n-tuple $P(\omega)$ the *deletion pattern* associated to ω and will interchangeably refer to a vertex by the compound-GKZ vector which formed it or by its associated deletion pattern. This is reasonable since two distinct compound-GKZ vectors cannot share the same pattern.

The letters were chosen as an extension of the combinatorial structure that can be placed on the vertices of the secondary polytope ([13], [1]) corresponding to points on a line. The secondary polytope is formed by taking the convex hull of all characteristic vectors corresponding to regular triangulations formed by projecting just the lower hull of $\text{conv}(\omega(\mathbf{A}))$ and taking no deletions. In the case of the secondary, the only characteristic that we need keep track of for a particular lifting is whether a point is a "+" (not visible from below in the lifting), or a "−" (visible from below in the lifting). This led to the choice of "M" to represent minus. As for the points that are not visible, we see that the only possible deletions which could reveal them were the nearest visible neighbors to the left and right which should make the meaning behind L, R, and LR is self-explanatory. The "E" corresponds to "empty" reflecting the fact that no single deletion can bring this point into visibility.

Example 2.2.2 The lifting $\omega = \langle 1, 3, 4, 2, 3, 5, 1\rangle$ in Figure 2.4 will have the associated deletion pattern $P = (M, L, E, LR, R, R, M)$. $\diamondsuit$

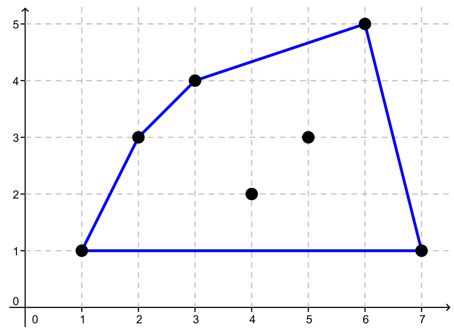


Figure 2.4: Deletion pattern example

For the proof of the following theorem, when we are considering a particular lifting ω, we will write $l(i, j)$ for $1 \leq i, j \leq n$ with $i \neq j$, to refer to the line through the lifted points $(i, \omega(i))$ and $(j, \omega(j))$.

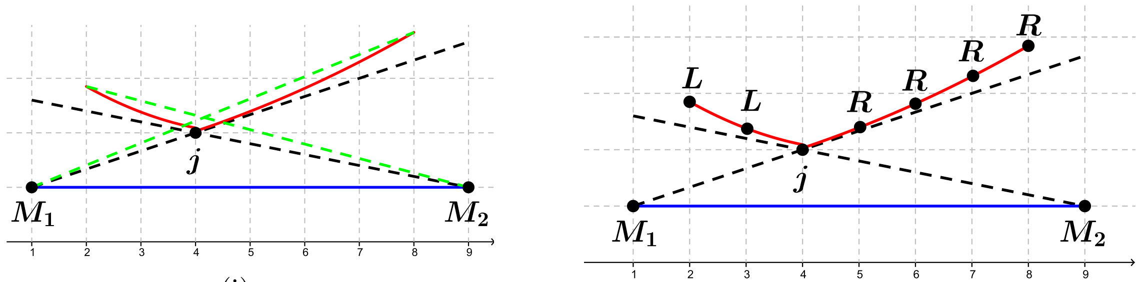
Theorem 2.2.3 *Let P be an arbitrary deletion pattern of length n and let $P[i]$ denote the i^{th} position in the pattern P. The following conditions are necessary and sufficient for the pattern P to be realized by a lifting, i.e. $P = P(\omega)$ for some lift ω:*

i) *$P[1]$ and $P[n]$ must both be M's.*

ii) *$P[2]$ must be in the set $\{M, L, LR\}$ and $P[n-1]$ must be in the set $\{M, R, LR\}$.*

iii) *Let $i < k$ with $P[j] \neq M$ for all $i < j < k$. Then it cannot be the case that both $P[i]$ belongs to the set $\{R, LR\}$ and $P[k]$ belongs to the set $\{L, LR\}$. In other words, a term containing an R cannot appear before a term containing an L if they are not separated by an M.*

Proof. It is easy to see that the conditions are necessary for a pattern P to be realized by a lifting ω. In order to show that they are also sufficient, first consider a pattern P which has only two M's, $M_1 < M_2$. It is clear that if $|P| = 2$ or 3 then a lifting ω which supports P is constructible. If $|P| \geq 4$, then we are guaranteed that 2 is assigned an L or LR and $n-1$ is assigned an R or LR (though not both an LR). If pattern P has a position assigned an LR, let j denote the index of this position. Otherwise, let j denote the position exactly midway between the last L and the first R appearing in P (guaranteed to exist if $|P| \geq 4$ and doesn't have an LR). Set $\omega(M_1) = \omega(M_2) = 0$ and set $\omega(j) = 1$. If we let $f(x) = (x - j)^2 + 1$, then for positions i assigned an L, we set $\omega(i) = l(j, M_2)(x) + \epsilon_L f(x)$ and for positions i assigned an R we set $\omega(i) = l(M_1, j)(x) + \epsilon_R f(x)$ for some small $\epsilon_L, \epsilon_R > 0$. We see that we can shrink ϵ_L small enough so that the x coordinate of the intersection of lines $l(2, M_2)$ and $l(M_1, j)$ occurs strictly between j and $j+1$. Similarly we can shrink ϵ_R small enough so that the x-coordinate of the intersection of lines $l(M_1, n-1)$ and $l(j, M_2)$ occurs strictly between $j-1$ and j. This process is depicted in Figure 2.5.



(i) (ii)

Figure 2.5: Single interval construction example

To finish describing the lifting vector ω, consider the positions i which are assigned an E. We set $\omega(i)$ large enough so that $(i, \omega(i))$ lies above both lines $l(1, n-1)$ and $l(2, n)$. This guarantees that these positions will not be revealed by single deletion, and we can clearly place the lifted E's to avoid forming any collinearities with the previously placed points. It is clear that this lifting ω is in general position and will correspond to the pattern P.

For patterns P with strictly more than 2 M's, we can build up the corresponding lifting by an inductive process. We will label the positions of the M's appearing in the pattern P with $M_1, M_2, M_3, \ldots$, and begin by placing $\omega(M_1) = \omega(M_2) = 0$ as before. This time, however, we will not necessarily be guaranteed positions with both L- and R-visibility which would be used to "hide" the positions that lack L- and R-visibility. We get around this by structuring the M's to hide visibility for the appropriate points. We notice that if $M_2 - M_1 \leq 3$, it is apparent how to construct the first interval, so we will focus on cases where $M_2 - M_1 \geq 4$.

If the interval $[M_1, M_2]$ has an LR appearing, we will denote this position with j. If no LR appears, we note that we *are* guaranteed positions with L-visibility, so let i_L denote the last L position in this interval. If there is a first R position in this interval, let j denote the position exactly halfway between the two, otherwise, set $j = i_L + \frac{1}{2}$. Regardless of how j was picked, we set $\omega(j) = 1$. If we let $g(x)$ denote the equation of line $l(j, M_2)$ and $f(x) = (x-j)^2 + 1$, then for all positions i assigned an L with $M_1 < i < j$, we can set $\omega(i) = g(i) + \epsilon f(i)$ for some small $\epsilon > 0$. We see that we can choose ϵ small enough so that line $l(2, M_2)$ will intersect $l(M_1, j)$ at $x < j + \frac{1}{4}$, which is depicted in Figure 2.6i. Since all positions assigned an R will be placed above line $l(M_1, j)$ and have x-coordinate greater than or equal to $j+1$, this guarantees that $l(2, M_2)$ hides all of these points from L-visibility. Also, notice that the point $(j, 1)$ is visible after the deletion of M_1 since all points with L visibility lie above the line $l(j, M_2)$.

Next, we set $\omega(M_3)$ high enough so that this point lies strictly above line $l(M_1, j)$ (to ensure that the point $(j, 1)$ would be visible under deletion of M_2 in case this position corresponded to an LR). If there were positions assigned an L between M_1 and j, let i_L denote the rightmost of these positions. We will also want to guarantee that $\omega(M_3)$ is strictly below $l(M_1, i_L)$, so that $l(M_1, M_3)$ hides all L positions from visibility under the deletion of M_2. Note that $\omega(i_L) > \omega(j)$ in the case that i_L exists, so it is always possible for $\omega(M_3)$ to fit appropriately between these lines. In practice we will want to place $\omega(M_3)$ very close to $l(M_1, j)$ for reasons which we will outline shortly. This process is depicted in Figure 2.6ii. This extra restriction is not always necessary to hide the L's (many times they will be appropriately hidden by other points), but it will be helpful when we are discussing liftings which support edges of $\Sigma_1(n)$.

To place the R's, let (x_m, y_m) denote the midpoint of the line segment from $(j, 1)$ to $(M_3, \omega(M_3))$. We will consider a new point $q = (x_m, y_m - \epsilon)$ for some small $\epsilon > 0$ and let $P(x)$ denote the equation of the unique (necessarily upward opening) parabola between $(j, 1)$, $(x_m, y_m - \epsilon)$, and $(M_3, \omega(M_3))$. Since M_3 lies above $l(M_1, j)$, we see that we can choose ϵ small enough so that $P(x) > l(M_1, j)(x)$ for all $x \in [j + 1, M_3]$. We then set $\omega(i) = P(i)$ for any position i assigned an R with $j < i < M_2$, which is depicted in Figure 2.6iii. Since the slope of $l(M_1, i)$ must be steeper than the slope of $l(M_1, i)$ for any such i, we see that point $(j, 1)$ and all points assigned an R between j and M_2 must be visible under the deletion of M_2. Further note that since we could place $\omega(M_3)$ arbitrarily close to the line $l(M_1, j)$ and we are free to make the drop ϵ of point q from line $l(j, M_3)$ as slight as we like, we can orchestrate the placement of the R's such that if there is a last R position, call it k, we can make $|l(M_1, M_3)(x) - l(k, M_3)(x)|$ as small as we like for any fixed position $x > M_3$. We will utilize this fact when we append more lifted intervals to this one.

For any positions assigned an E in the entire pattern, we will wish to guarantee that they lie above both the line $l(M_1, n-1)$ and the line $l(2, M_n)$. Since we do not yet know the lifting heights of positions $n - 1$ and M_n, we will assume that the E's have been placed suitably high. To recap the construction so far, we note all L positions and $(j, 1)$ are visible after the removal of M_1, all R positions and $(j, 1)$ are visible after the removal of M_2, the line $l(M_1, M_3)$ hides the L positions from R-visibility, and the line $l(2, M_2)$ hides all the R positions from L-visibility, as seen in Figure 2.6iv.

Now suppose that we have finished determining the height of our lifting up to some M_m for $2 \leq m < n$ and have already reached forward and placed M_{m+1} to ensure the appropriate visibility of any L-positions occurring between M_{m-1} and M_m after the deletion of M_m. Further, if there is a last R position in $[M_{m-1}, M_m]$, which we will call k, we will make the assumption that $|l(k, M_{m+1})(x) - l(M_{m-1}, M_{m+1})(x)|$ can be made arbitrarily small over all $1 \leq x \leq n$ without affecting the visibility of any points in $[M_1, M_m]$ (Note that our base case satisfied this condition). We will

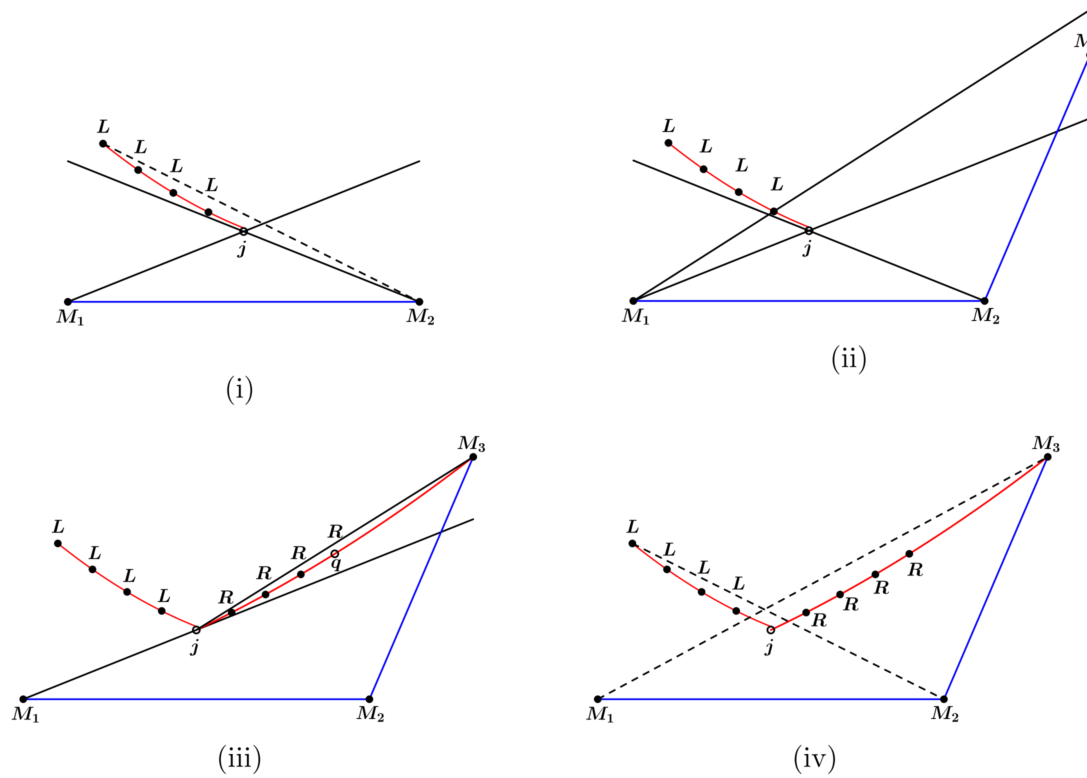


Figure 2.6: First interval construction example

start by assuming that $M_{m+1} < n$, so we know that there are additional M's to place. Before determining the lifting heights of the non-M positions between M_m and M_{m+1}, we first locate a position j to separate the L's from the R's as we did in $[M_1, M_2]$. We are no longer guaranteed either an LR or last L in this interval, so we must add a few steps. If the interval $[M_m, M_{m+1}]$ has an LR, let j denote its position. If not, then check for the existence of a position i_L denoting the position of the last L in the interval, and a position i_R denoting the first R. If both exist, set j equal to $(i_L + i_R)/2$. If only i_L exists, set $j = i_l + \frac{1}{2}$, and similarly, if only i_R exists, set $j = i_R - \frac{1}{2}$. Lastly, note that if neither an LR, i_L, nor i_R exists in this interval, then either $M_{m+1} - M_m = 1$, or all positions in (M_m, M_{m+1}) are assigned an E. In either case it is clear how the interval should be constructed, and we only need place M_{m+2} so that it lies strictly above $l(M_m, M_{m+1})$.

Supposing that we *are* able to find a position j in this interval, then we lift M_{m+2} high enough so that the intersection of lines $l(M_m, M_{m+2})$ and $l(k, M_{m+1})$ occurs on the line $x = j$, where k denotes the rightmost position with R-visibility (assigned an LR or R) in $[M_{m-1}, M_m]$. Notice that we may utilize our previous condition on the lines $l(k, M_{m+1})$ and $l(M_{m-1}, M_{m+1})$ to assure that the intersection of $l(M_{m-1}, M_{m+1})$ and $l(M_m, M_{m+2})$ occurs within some arbitrarily small interval around j. If no such position k exists, then lift M_{m+2} to the appropriate height so that the intersection of lines $l(M_m, M_{m+2})$ and $l(M_{m-1}, M_{m+1})$ occurs on the line $x = j$. Note this is always possible and guarantees that M_{m+2} will lie strictly above the line $l(M_m, M_{m+1})$. This step is illustrated in Figure 2.8i.

Within the interval $[M_m, M_{m+1}]$ our goal is to place any position assigned an L inside the triangle bounded by lines $l(M_m, M_{m+2})$, $l(M_{m-1}, M_{m+1})$, and $x = M_m$, any R's in the triangle formed by $l(M_m, M_{m+2})$, $l(M_{m-1}, M_{m+1})$, and $x = M_{m+1}$, any LR's in the triangle formed by lines $l(M_m, M_{m+2})$, $l(M_{m-1}, M_{m+1})$, and $l(M_m, M_{m+1})$, and lastly, any E's above both lines $l(M_m, M_{m+2})$ and $l(M_{m-1}, M_{m+1})$. These regions are outlined in Figure 2.7, and we note that the placement of M_{m+2} dependent on the position j guarantees that a point *can* be lifted into the appropriate region when necessary (not so much of an issue if we failed to assign a position j). Although it is possible to have a lifting where an R can lie below line $l(M_{m-1}, M_{m+1})$, for instance, as long as there are nearby L positions to hide it from visibility when M_m is deleted, we are avoiding such pathological examples, and will only rely on the positions of nearby M's: M_{m-1}, M_m, M_{m+1} and M_{m+2}, to restrict visibility to the points in (M_m, M_{m+1}) appropriately. This is another condition which will become very handy when we are constructing liftings which support edges of $\Sigma_1(n)$.

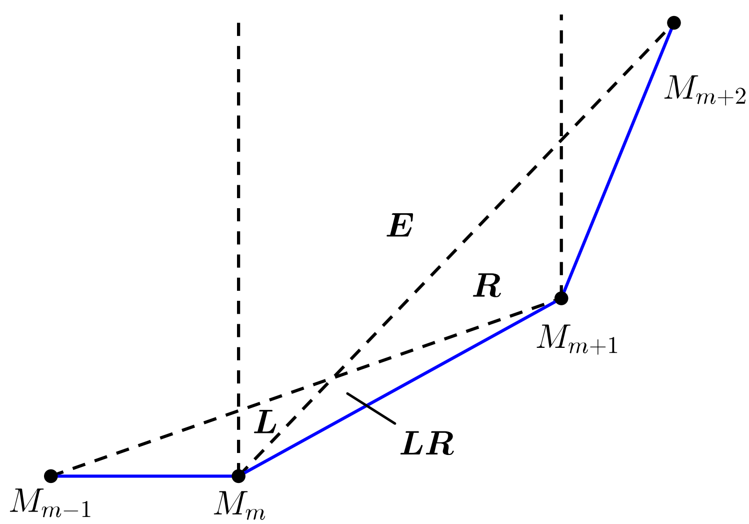


Figure 2.7: Deletion regions

We first focus on lifting the L's in interval $[M_m, M_{m+1}]$ so that not only do they have the appropriate visibility, but also they do not adversely affect the visibility of any position with R-visibility in $[M_{m-1}, M_m]$. If interval $[M_{m-1}, M_m]$ had a point with R-visibility (assigned an LR or an R), let k denote the rightmost such position. If y denotes the y-coordinate of the intersection of $l(k, M_{m+1})$ and $l(M_m, M_{m+2})$, we will set $\omega(j) = y - \epsilon$ for some small $\epsilon > 0$ (recall that the intersection of these lines was orchestrated to occur on $x = j$). If we let $P_L(x)$ denote the (necessarily upward opening) parabola through $(k, \omega(k))$, $(j, y - \epsilon)$, and $(M_{m+1}, \omega(M_{m+1}))$, then we note that ϵ can be chosen small enough so that any lifted position with R-visibility in $[M_{m-1}, M_m]$ lies above $P_L(x)$. Further, a sufficiently small ϵ will also guarantee that $P_L(x) > l(M_m, M_{m+2})(x)$ for $M_m \leq x \leq j - \frac{1}{4}$ (this requires the condition that restricts the x-coordinate of the intersection of $l(M_{m-1}, M_{m+1})$ and $l(M_m, M_{m+2})$ to occur within some arbitrarily small interval around j) . With such an ϵ chosen, we set $\omega(i) = P_L(i)$ for any position i assigned an L with $M_m < i < j$. We notice that neither position j nor one of these positions i is capable of hiding an R in $[M_{m-1}, M_m]$ from visibility under the deletion of M_m. Further, each of these positions i and position j is visible under the deletion of M_m, and all of these i are shielded from R-visibility by the line $l(M_m, M_{m+2})$. If the previous interval $[M_{m-1}, M_m]$ had no position k with R-visibility, we repeat this argument with $k = M_{m-1}$ (No need to worry about affecting R-visibility in the previous interval). This step is illustrated in Figure 2.8ii.

We invoke a similar process to lift the R's in $[M_m, M_{m+1}]$ and consider the (necessarily upward opening parabola) $P_R(x)$ through the points $(M_m, \omega(M_m))$, $(j, y - \epsilon)$, and $(M_{m+2}, \omega(M_{m+2}))$. We can shrink ϵ even further if necessary to guarantee that $P_R(x) > l(M_{m-1}, M_{m+1})(x)$ for all $j + \frac{1}{4} < x \leq M_{m+2}$ (again relying on the condition which forces the x-coordinate of the intersection of $l(M_{m-1}, M_{m+1})$ and $l(M_m, M_{m+2})$ to occur within some arbitrarily small interval around j). Notice that shrinking ϵ further will not adversely affect the visibility of the L's that we had placed previously, though it will perturb them a bit. We can then set $\omega(i) = P_R(i)$ for any i assigned an R with $j < i < M_{m+1}$. This step is illustrated in Figure 2.8iii. Notice that when we delete M_{m+1} we are guaranteed that position j and all of these i are visible. Further, all of these positions i are shielded from L-visibility by the line $l(M_{m-1}, M_{m+1})$. Figure 2.8iv depicts all of the newly lifted points in our interval. We note that if there is a last R in the interval $[M_m, M_{m+1}]$, say at position k', we can choose ϵ sufficiently small enough to guarantee that $|l(M_m, M_{m+2})(x) - l(k', M_{m+2})(x)|$ is as small as we like for any fixed position $x > M_{m+2}$, which will satisfy one of our inductive hypotheses.

Our last case to consider is if $M_{m+1} = n$, that is, if there was no need to reach forward to place M_{m+2}. In this case, we can put an artificial M_{m+2} so that $M_{m+2} = M_{m+1} + 1$ and proceed with the construction of the lifting as we did in previous intervals. Although we no longer really have the line $l(M_m, M_{m+2})$ to block any L's in $[M_m, M_{m+1}]$ from R-visibility, we *are* guaranteed to have some points with R-visibility (that is, as long as $M_{m+1} \neq M_m + 1$, in which case this argument would be unnecessary). We note that for any position i with R-visibility in this interval, the line $l(M_m, i)$ will serve to hide our L's from R-visibility.

We see that if we string enough of these constructions together, we will be able to complete the lifting interval by interval. As was mentioned earlier, we will want to double check that all E's are lifted sufficiently high enough so that they lie above both the line $l(M_1, n-1)$ and the line $l(2, M_n)$ (which are now determined). The last thing to note is that we may have coincidentally introduced some collinearities while constructing this lifting. We know, however, that all of the important orientations on triples of points which determine visibility are strictly positive or negative. Thus we

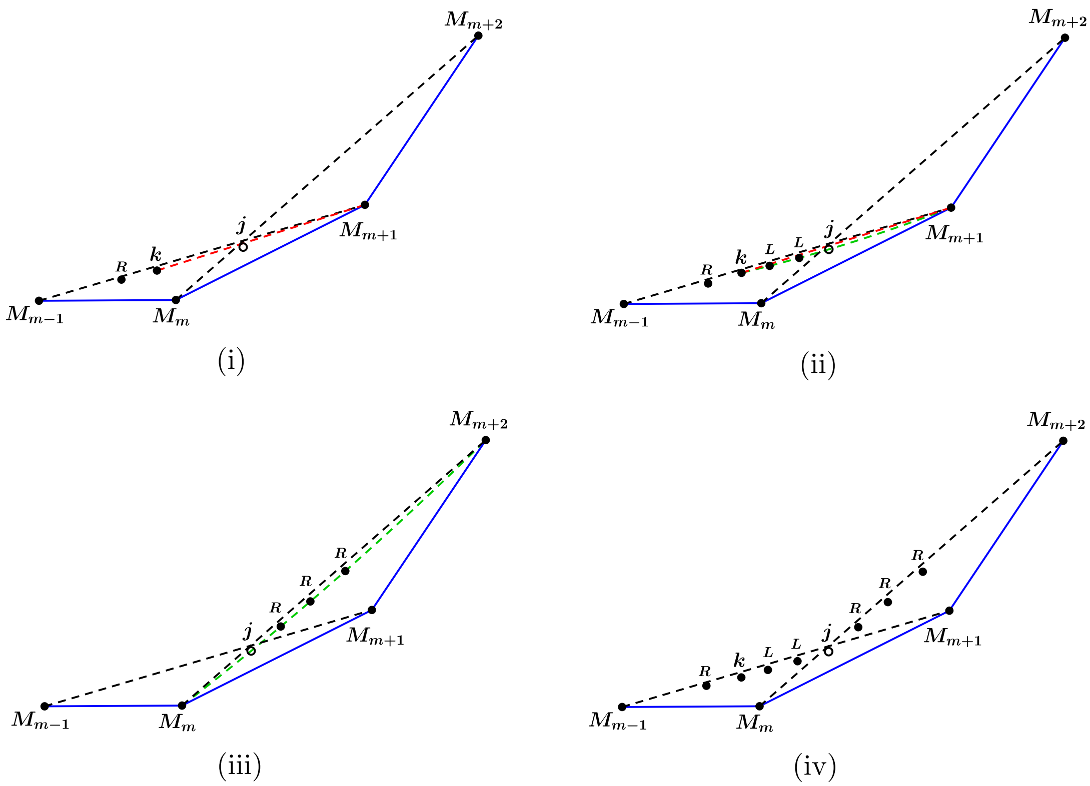


Figure 2.8: Middle interval construction

may make small enough perturbations on the points to break any collinearities while maintaining the appropriate visibility of the points. This will give us a lifting ω in general position which corresponds to the pattern P. ■

Although we have established the dimension of these polytopes in the previous chapter, we can utilize our combinatorial structure to produce an explicit set of vertices of $\Sigma_1(n)$ whose affine span is $n-2$ dimensional. To this end, let P_1 denote the length n pattern consisting entirely of M's and for i from 2 to $n-1$ let P_i denote the length n pattern consisting entirely of M's except for an LR at position i. Clearly these patterns are all valid, and so correspond to vertices of $\Sigma_1(n)$. Working out the compound GKZ-vectors $\overline{\phi}(v_j)$ from the given deletion patterns P_j, we find:

$$
\begin{aligned}
\overline{\phi}(v_1) &= \langle n,\ 2n-2,\ 2n,\ 2n,\ \ldots,\ 2n,\ 2n-2,\ n\rangle. \\
\overline{\phi}(v_2) &= \langle 2n-3,\ 4,\ 3n-3,\ 2n,\ 2n,\ \ldots\ ,2n,\ 2n-2,\ n\rangle, \\
\overline{\phi}(v_3) &= \langle n,\ 3n-5,\ 6,\ 3n-3,\ 2n,\ 2n,\ \ldots,\ 2n,\ 2n-2,\ n\rangle, \\
\overline{\phi}(v_{n-2}) &= \langle n,\ 2n-2,\ 2n,\ 2n,\ \ldots,\ 2n,\ 3n-3,\ 6,\ 3n-5,\ n\rangle, \quad \text{(the reverse of } v_3) \\
\overline{\phi}(v_{n-1}) &= \langle n,\ 2n-2,\ 2n,\ 2n,\ \ldots,\ 2n,\ 3n-3,\ 4,\ 2n-3\rangle, \quad \text{(the reverse of } v_2)
\end{aligned}
$$

For $4 \leq i \leq n-3$, we have

$$\overline{\phi}(v_i) = \langle n,\ 2n-2,\ 2n,\ 2n,\ \ldots,\ 2n,\ \underbrace{3n-3}_{i-1},\ \underbrace{6}_{i},\ \underbrace{3n-3}_{i+1},\ 2n,\ \ldots,\ 2n,\ 2n-2,\ n\rangle.$$

Next we form a matrix whose rows correspond to the homogenized coordinates of the compound GKZ-vectors corresponding to our chosen set of vertices:

$$\begin{pmatrix}
1 & n & 2n-2 & 2n & 2n & 2n & 2n & \cdots & 2n & 2n & 2n-2 & n \\
1 & 2n-3 & 4 & 3n-3 & 2n & 2n & 2n & \cdots & 2n & 2n & 2n-2 & n \\
1 & n & 3n-5 & 6 & 3n-3 & 2n & 2n & \cdots & 2n & 2n & 2n-2 & n \\
1 & n & 2n-2 & 3n-3 & 6 & 3n-3 & 2n & \cdots & 2n & 2n & 2n-2 & n \\
1 & n & 2n-2 & 2n & 3n-3 & 6 & 3n-3 & \cdots & 2n & 2n & 2n-2 & n \\
1 & n & 2n-2 & 2n & 2n & 3n-3 & 6 & \cdots & 2n & 2n & 2n-2 & n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
1 & n & 2n-2 & 2n & 2n & 2n & 2n & \cdots & 6 & 3n-3 & 2n-2 & n \\
1 & n & 2n-2 & 2n & 2n & 2n & 2n & \cdots & 3n-3 & 6 & 3n-5 & n \\
1 & n & 2n-2 & 2n & 2n & 2n & 2n & \cdots & 2n & 3n-3 & 4 & 2n-3
\end{pmatrix}.$$

This matrix is row equivalent to the following full row rank matrix:

$$\begin{pmatrix}
1 & n & 2n-2 & 2n & 2n & 2n & 2n & \cdots & 2n & 2n & 2n-2 & n \\
0 & n-3 & \star & \star & \star & \star & \star & \cdots & \star & \star & \star & \star \\
0 & 0 & n-3 & \star & \star & \star & \star & \cdots & \star & \star & \star & \star \\
0 & 0 & 0 & n-3 & \star & \star & \star & \cdots & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & n-3 & \star & \star & \cdots & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & n-3 & \star & \cdots & \star & \star & \star & \star \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \star & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & n-3 & \star & \star & \star \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & n-3 & \star & \star
\end{pmatrix},$$

which implies that the affine span of our chosen vertices, and hence the polytope $\Sigma_1(n)$, is $n-2$ dimensional for $n \geq 4$.

2.3 Vertex Count

In order to enumerate the number of distinct possible compound GKZ-vectors, and hence the number of vertices of $\Sigma_1(n)$, we will utilize the combinatorial structure of the compound GKZ-vectors outlined in the previous section. To begin, we introduce the concept of an interval of a deletion pattern and classify 3 different types.

Definition 2.3.1 An *interval* in a deletion pattern is a maximal block of consecutive coordinates where the first (leftmost) coordinate is an M and which contains no other M's. Intervals can be partitioned into 3 disjoint types:

i) A *type* A interval contains either the first M of the deletion pattern or the entry in position $n-1$, but not both.

ii) A *type* $\hat{A}$ interval contains every entry of the deletion pattern except the rightmost M.

iii) A *type* B interval contains neither the first nor the penultimate M.

The *length* of an interval will refer to the number of coordinates that it contains.

Example 2.3.2 To illustrate the different types of intervals, consider the following two patterns:

$$(\underbrace{M, L, E, R}_{\textit{Type-A}}, \underbrace{M, L, E, LR, E}_{\textit{Type-B}}, \underbrace{M, L, E, E, R}_{\textit{Type-B}}, \underbrace{M}_{\textit{Type-A}}, M)$$

$$(\underbrace{M, L, E, E, L, L, E, LR, E, E, R, R, R, E, E}_{\textit{Type-}\hat{A}}, M)$$

Note that every pattern is always terminated by a single "M", which we will refer to as a trivial interval. ♢

In order to count the total number of valid deletion patterns of length n, we note that every such pattern is either: a composition of a type A interval followed by some number of type B intervals (possibly 0) followed by a type A interval, or a single type $\hat{A}$ interval of length $n-1$ (each case terminated by a trivial interval). In order to find the o.g.f. (ordinary generating function) for valid deletion patterns, we will first find the o.g.f.'s for the intervals of each type.

We will start with type B intervals as they have the least number of restrictions on their entries. Let b_n denote the number of type B intervals of length n and let $B(x) := \sum_{n\geq 1} b_n x^n$ be the corresponding o.g.f. To determine an expression for b_n, we consider two separate cases:

Case 1: The interval has no LR term present.
In this case, there are $n-1$ places for the last (rightmost) L to appear in the interval. After this, we are free to choose whether the remaining $n-2$ entries are E or not, with the non-empty entries being determined by their position relative to the last L. If there is no last L, then we can freely choose which of the $n-1$ entries are E, with the rest being forced as R. Thus we have $(n-1)2^{n-2} + 2^{n-1}$ type B intervals of length n with no LR term. Note that this formula correctly gives one trivial type B

interval of length 1.

<u>Case 2</u>: The interval has an LR term present.
We have $n-1$ positions to place the LR, and we can freely choose which of the remaining $n-2$ entries are E, with the non-empty positions being determined by their position relative to the LR. This leaves us with $(n-1)2^{n-1}$ type B intervals of length n with an LR appearing.

Combining the two cases, we can obtain the closed form of $B(x)$:

$$B(x) \;=\; \sum_{n\geq 1} \left((n-1)2^{n-2} + 2^{n-1} + (n-1)2^{n-1}\right) x^n \;=\; \frac{x}{(1-2x)^2} \qquad (2.1)$$

Next we will consider type A intervals and let a_n denote the number of type A intervals of length n with $A(x) := \sum_{n\geq 1} a_n x^n$ as the corresponding o.g.f. We will solve for a_n using a count similar to that of b_n, but we must exercise caution, as the 2^{nd} and $(n-1)^{\text{st}}$ entries of valid deletion patterns have some extra restrictions. Without loss of generality, we will assume that we are counting type A intervals which contain the first M of the deletion pattern as the other case is symmetric. We will again break into two cases depending on the entry in the "special" second position, with the trivial length 1 case being considered afterwards.

<u>Case 1</u>: The second position is an L.
Filling the remaining $n-2$ positions is analogous to enumerating the number of type B intervals of length $n-1$, thus we find there are $(n-1)2^{n-2}$ type A intervals of this form.

<u>Case 2</u>: The second position is an LR.
We may freely choose which of the remaining $n-2$ positions are E, with all non-empty positions being forced as R's. This gives us 2^{n-2} type A intervals of this form.

Combining the two cases and noting that we have a single trivial type A interval of length 1 we obtain the following formula for $A(x)$:

$$A(x) \;=\; x + \sum_{n\geq 2} \left((n-1)2^{n-2} + 2^{n-2}\right) x^n \;=\; \frac{x(1-2x+2x^2)}{(1-2x)^2}.$$

Lastly, we consider the type $\hat{A}$ intervals of length n which have restrictions on both the 2^{nd} and $(n-1)^{\text{st}}$ entries. We will let $\hat{a}_n$ denote the number of type $\hat{A}$ intervals of length n and $\hat{A}(x) := \hat{a}_n x^n$ be the corresponding o.g.f. We will consider the degenerate cases $n = 1, 2$ separately and assume for now that $n \geq 3$. This means that we truly have two separate restricted positions, and we break into cases depending on the entries in these locations:

Case 1: The first position is an L and the $(n-1)^{\text{st}}$ is an R.

Filling the remaining $n-3$ positions is analogous to enumerating the number of type B intervals of length $n-2$, thus we find there are $(n-2)2^{n-3}$ type $\hat{A}$ intervals of this form.

Case 2: The first position is an LR and the $(n-1)^{\text{st}}$ is an R.
In this case, we are free to choose which of the remaining $n-3$ positions are filled with an E and the rest must be R's. This gives us 2^{n-3} type $\hat{A}$ intervals of this form.

Case 3: The first position is an L and the $(n-1)^{\text{st}}$ is an LR.
As in the previous case, we have 2^{n-3} type $\hat{A}$ intervals of this form.

Noting that there is a single type $\hat{A}$ interval of length 1 and a single of length 2, we find that $\hat{A}(x)$ is given by the following:

$$\hat{A}(x) \;=\; x + x^2 + \sum_{n\geq 3}\left((n-2)2^{n-3} + 2^{n-3} + 2^{n-3}\right)x^n \;=\; \frac{x(1-3x+3x^2)}{(1-2x)^2}.$$

Lastly, we let f_n denote the number of valid deletion patterns of length n and $F(x) := \sum_{n\geq 1} f_n x^n$ be the corresponding o.g.f. Recalling that deletion patterns are built up as particular compositions of the 3 types of intervals, we find that

$$\begin{aligned} F(x) &= \Big(\hat{A}(x) + A(x)\left(B^0(x) + B^1(x) + B^2(x) + \cdots\right)A(x)\Big)\,x \\ &= \frac{(x^3+2x^2-3x+1)x^2}{1-5x+4x^2}, \end{aligned}$$

where the middle expression is multiplied by x to account for the trivial length 1 interval that ends all deletion patterns. Our last task is to unpack this o.g.f. to solve for a formula giving the number of vertices of $\Sigma_1(n)$:

$$\begin{aligned} F(x) &= \frac{(x^3+2x^2-3x+1)x^2}{1-5x+4x^2}, \\ &= \frac{77}{256} + \frac{13}{64}x + \frac{13}{16}x^2 + \frac{1}{4}x^3 - \frac{1}{3}\left(\frac{1}{1-x}\right) + \frac{25}{768}\left(\frac{1}{1-4x}\right), \\ &= x^2 + 2x^3 + \sum_{n\geq 4}\left(\frac{-1+25\cdot 4^{n-4}}{3}\right)x^n. \end{aligned}$$

This establishes the following theorem.

Theorem 2.3.3 *For* $|\mathbf{A}| = n \geq 4$, $\Sigma_1(n)$ *has* $\dfrac{-1+25\cdot 4^{n-4}}{3}$ *vertices.*

2.4 Edges

In order to derive a formula for the number of edges of $\Sigma_1(n)$ we must first outline a combinatorial condition to determine exactly when a pair of vertices form an edge in the polytope. In other words, we must look for liftings ω which "support" exactly two vertices. This means that if we think of the lifting vector as the normal vector to a hyperplane, we must have $\omega \cdot \mathbf{v} = \omega \cdot \mathbf{v}' = k$, for two compound GKZ-vectors $\mathbf{v}$ and $\mathbf{v}'$ and some integer k. Further, for all other compound GKZ-vectors $\mathbf{u} \neq \mathbf{v}, \mathbf{v}'$, we require $\omega \cdot \mathbf{u} > k$.

Recalling Proposition 2.1.7, we note that a lifting in general position (i.e. one which possesses no semi-visible points in any of its deletions) will support only a single vertex, namely the vertex given by the compound GKZ-vector corresponding to this lifting. If our lifting had some collinearities, however, we saw that two distinct triangulations could be supported by the same lifting so long as both had supports containing all visible points, no non-visible points, and which differed only on the semi-visible points. Example 2.4.1 outlines a case where a single lifting can support two distinct lifting patterns.

Example 2.4.1 Consider the lifting $\omega = \langle 1, 3, 2, 1 \rangle$ and the two distinct deletion patterns $P = (M, L, LR, M)$ and $P' = (M, L, R, M)$. Figure 2.9 depicts the four deletions of the lifting, and the four deletion-induced triangulations corresponding to each P and P'. We see that the deletions of 2, 3, and 4 have no semi-visible points, hence when breaking the inner product up into its components $\omega \cdot \overline{\phi}(\omega) = \omega \cdot \phi(\mathcal{T}_1) + \cdots + \omega \cdot \phi(\mathcal{T}_n)$, there is a single unique triangulation $\mathcal{T}_i$ of the appropriate point set which minimizes $\omega \cdot \phi(\mathcal{T}_i)$ for $i = 2, 3, 4$. We see that both P and P' have these appropriate triangulations, and so both minimize the inner products corresponding to these deletions. In deletion 1, however, we have a single point which is semi-visible. This means that there are exactly two possible triangulations which could minimize the inner product with ω for this deletion, one that includes the semi-visible point, and another which doesn't. We see that P produces one of the possible triangulations and P' the other.

The end result is that if $\mathbf{v}$ and $\mathbf{v}'$ are the compound GKZ-vectors corresponding to patterns P and P', respectively, then $\omega \cdot \mathbf{v} = \omega \cdot \mathbf{v}' \leq \omega \cdot \mathbf{u}$, for any other compound GKZ-vector $\mathbf{u}$. Further, we see that patterns P and P' are the *only* two patterns which minimize the inner product as there are no other sets of triangulations which would minimize the inner product of each deletion. This means that this lifting ω provides the normal vector for an edge supporting hyperplane of $\Sigma_1(4)$. $\Diamond$

The following definition allows us to quantify how many vertices can be supported by a particular non-general lifting.

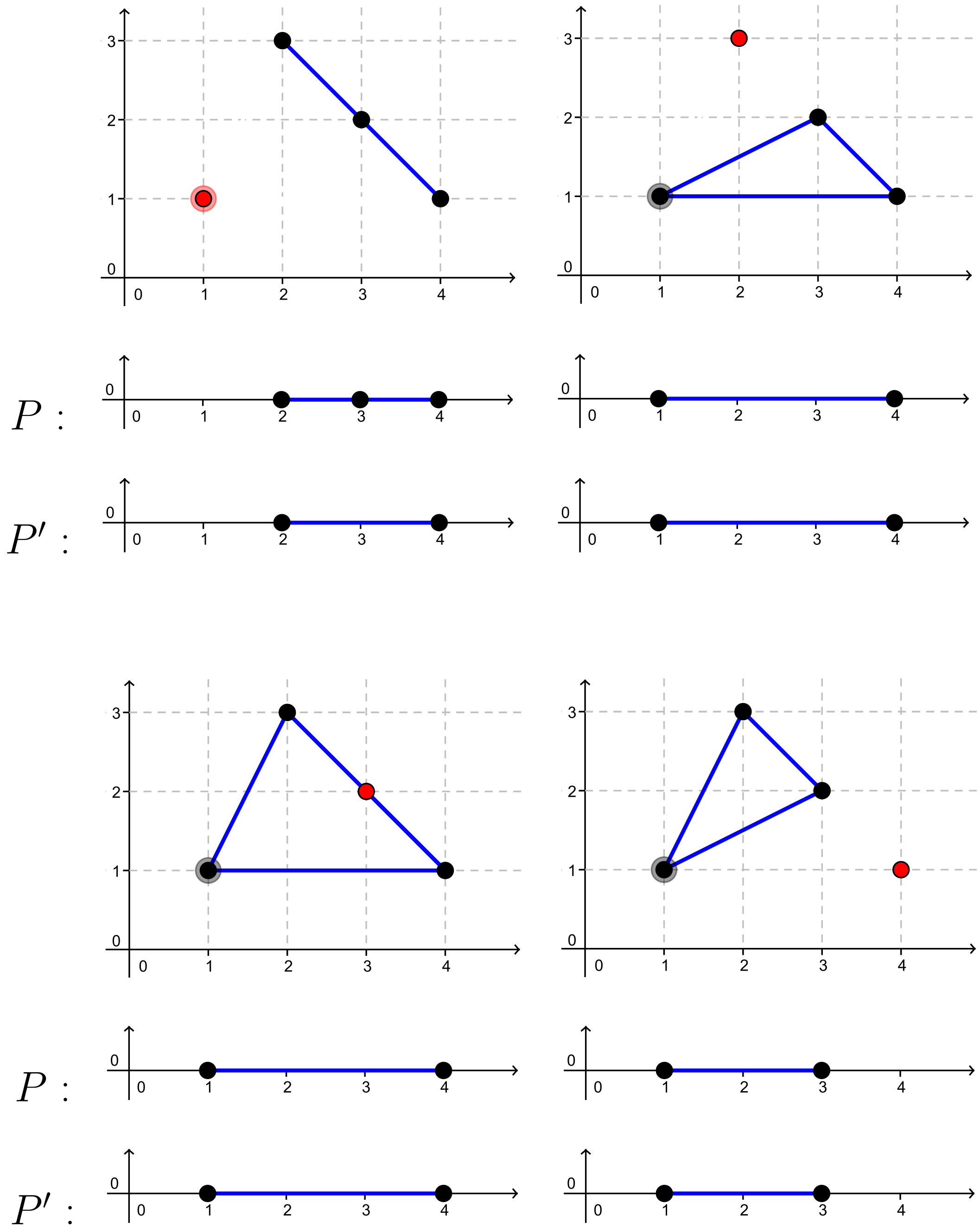
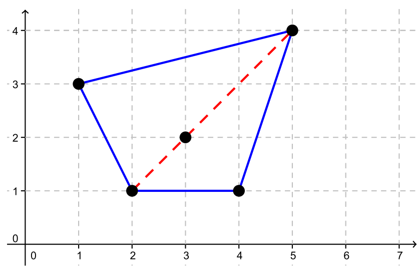


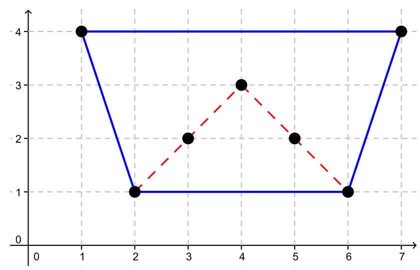
Figure 2.9: Edge-supporting lifting

Definition 2.4.2 An *essential collinearity* is a triple of collinear points $(i, \omega(i))$, $(j, \omega(j))$, $(k, \omega(k))$, $i < j < k$, in $\omega(\mathbf{A})$ such that there is at least one single deletion of $\mathbf{A}$ in which the outer two points are visible and the middle point is semi-visible. Note for $|\mathbf{A}| = n \geq 4$, this means that collinearities where $(i, \omega(i))$ and $(k, \omega(k))$ are already visible on the lower hull of $\mathrm{conv}(\omega(\mathbf{A}))$ qualify as essential. We will continue to refer to a point $(i, \omega(i))$ simply by "i".

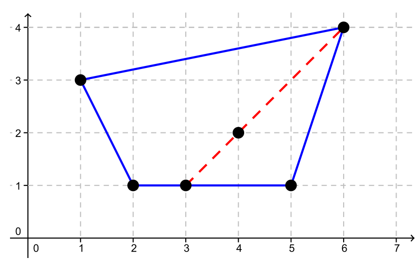
Example 2.4.3 Consider liftings $\omega_1, \omega_2, \omega_3$, and ω_4 depicted in Figure 2.10. We see in ω_1 we have the single essential collinearity $(2,3,5)$ revealed by the deletion of point 4. Although ω_2 has a couple of collinearities: (2,3,4) and (4,5,6), neither are essential because they do not become visible under single deletion. Lifting ω_3 has two essential collinearities, namely (2,3,5) and (3,4,6). Lifting ω_4 has the four essential collinearities (2,3,5), (2,4,5), (2,3,4), and (3,4,5). $\diamondsuit$



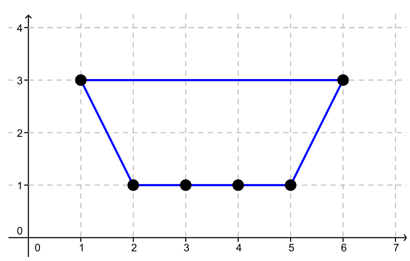
(i) $\omega_1 = \langle 3,1,2,1,4 \rangle$



(ii) $\omega_2 = \langle 4,1,2,3,2,1,4 \rangle$



(iii) $\omega_3 = \langle 3,1,1,2,1,4 \rangle$



(iv) $\omega_4 = \langle 3,1,1,1,1,3 \rangle$

Figure 2.10: Essential collinearities

Lemma 2.4.4 *A lifting can support exactly two compound GKZ-vectors, and hence an edge in* $\Sigma_1(n)$, *if and only if it has a single essential collinearity. Further, if two compound GKZ-vectors do form an edge, their corresponding deletion patterns differ from each other in exactly one spot, and the differing labels must form one of the pairs* $(M, LR), (LR, L), (LR, R), (L, E)$ *or* (R, E).

Proof. If a lifting has no essential collinearities, then that lifting has no semi-visible points in any of its deletions. This means that the underlying triangulations $\mathcal{T}_i$ which will minimize the inner product $\omega \cdot \phi(\mathcal{T}_i)$ in each deletion i are uniquely determined, which implies that this lifting can only support a single vertex.

Next suppose that a lifting has at least two essential collinearities. This means that we must have two distinct positions i and j with $1 \leq i, j \leq n$ which are semi-visible in at least one deletion-induced triangulation. We can form at least four distinct deletion patterns which will be supported by this lifting by independently choosing to make the two semi-visible points either visible or not (if we incidentally have more semi-visible positions, these can also be set to visible or non-visible status as necessary). The liftings which produce these four sets of triangulations can be constructed by slight perturbations of the semi-visible points of the original lifting.

Now that we have established that edge supporting liftings must have a single collinearity, we pick an arbitrary lifting with a single collinearity and show that it will support exactly two vertices of the form mentioned above. We break into cases depending on *how* the single essential collinearity becomes visible. That is, either the middle point j is already semi-visible in the lower hull, or it must be the case that it becomes semi-visible in the single deletion one of its nearest M neighbors to the left or to the right (if j is not semi-visible in the original lower hull, then it is impossible for it to be revealed by *both* the deletion of its left and right nearest M neighbors). Notice that in each case, position j is the only semi-visible position, thus the presence or absence of all other positions in each deletion-induced triangulation is determined, and hence their letter descriptor in the deletion pattern is fixed.

<u>Case 1</u>: The middle point j is semi-visible in the original lower hull.

Being visible in the original lower hull forces the letters assigned to i and k to be M's, and hence the single deletions of both i and k must make j visible. This rules out the letters E, L, and R for position j. We see that a pattern with j set to either M or LR would be supported. Setting j to an M would correspond to perturbing j downward so that it becomes visible in all deletions but its own. Setting j to an LR would make it visible only in the deletions that it is required to be, namely i and k, and setting it to be non-visible in the other deletions.

<u>Case 2</u>: The middle point j becomes semi-visible after the deletion of its left M neighbor and is not visible in the deletion of k.

Here there is only a single triangulation where we have the opportunity to choose whether to make j visible or not. If we perturb up so that it is not visible in the deletion of its left neighbor, then it is not visible in any deletions and hence must be an E. If we perturb down to make it visible in this deletion, then it must be an L.

<u>Case 3</u>: The middle point j becomes semi-visible after the deletion of its left M neighbor and is visible in the deletion of k.

In this case, j is not visible in the lower hull which rules out the possibility of supporting a pattern with j assigned an M. Since it does become visible after a deletion of k (which incidentally forces k to be an M), then we also rule out assigning j the letters E and L. Assigning j to either LR or R would be supported, however, and corresponds to perturbing j down or up, respectively.

Case 4: The middle point j becomes semi-visible after the deletion of its right M neighbor and is not visible under the deletion of i.

This is analogous to Case 2, and shows that this lifting supports patterns with j assigned an E or an L.

Case 5: The middle point j becomes semi-visible after the deletion of its right M neighbor and is visible under the deletion of i.

This is analogous to Case 3, and shows that this lifting supports patterns with j assigned an LR or an R.

In each case, a lifting with a single essential collinearity supports exactly two vertices which differ in a single position where the differences are exactly of the form described. ■

The previous lemma allows us to observe that for a lifting with a single essential collinearity, we may still assign letters to describe the deletion pattern on all positions but the one in the middle of the essential collinearity, call it position j, as the visibility or non-visibility in each deletion is exactly determined. We can also see that the inner product of a compound GKZ-vector and this lifting will be minimized exactly when the deletion patterns match on all positions but j, and the letter assigned to j in the GKZ vector is one of the two possibilities determined by how the essential collinearity in the lifting becomes visible.

Definition 2.4.5 The following functions will aid in the proof of Theorem 2.4.6 and will describe the positions of various neighbors for a given position i and lifting ω. They are defined as follows:

$$M_L, M_R, M_L^2, M_R^2, M_L', M_R' \ : \ \{1, 2, \ldots, n\} \times \mathbb{R}^n \ \longrightarrow \{1, 2, \ldots, n\} \cup \{\emptyset\},$$

$$
\begin{aligned}
M_L(i,\omega) &\mapsto j, && \text{where } j \text{ is the position of the nearest visible neighbor } (M) \text{ to the left of } i \text{ on the original lower hull of } \omega. \text{ If no such } j \text{ exists, we return } \emptyset,\\
M_R(i,\omega) &\mapsto j, && \text{where } j \text{ is the position of the nearest visible neighbor } (M) \text{ to the right of } i \text{ on the original lower hull of } \omega. \text{ If no such } j \text{ exists, we return } \emptyset,\\
M_L^2(i,\omega) &\mapsto j, && \text{where } j \text{ is the position of the second nearest visible neighbor } (M) \text{ to the left of } i \text{ on the original lower hull of } \omega. \text{ If no such } j \text{ exists, we return } \emptyset,\\
M_R^2(i,\omega) &\mapsto j, && \text{where } j \text{ is the position of the second nearest visible neighbor } (M) \text{ to the right of } i \text{ on the original lower hull of } \omega. \text{ If no such } j \text{ exists, we return } \emptyset,\\
M_L'(i,\omega) &\mapsto j, && \text{where } j \text{ is the position of the nearest visible neighbor to the left of } i \text{ after the deletion of } M_L(i,\omega) \text{ in } \omega. \text{ If } M_L(i,\omega)=\emptyset \text{ or this new nearest neighbor does not exist, we return } \emptyset,\\
M_R'(i,\omega) &\mapsto j, && \text{where } j \text{ is the position of the nearest visible neighbor to the right of } i \text{ after the deletion of } M_R(i,\omega) \text{ in } \omega. \text{ If } M_R(i,\omega)=\emptyset \text{ or this new nearest neighbor does not exist, we return } \emptyset.
\end{aligned}
$$

Note that the label assigned to a position $M_L'(i,\omega), M_R'(i,\omega)$ (assuming it is not $\emptyset$) does not have to be an M. It could be brought into the lower hull after the deletion. If it is clear from context, we will omit writing which lifting ω we are analyzing.

Theorem 2.4.6 *There is an edge between two vertices if and only if the two corresponding deletion patterns differ in exactly one coordinate, and the differing entries are connected by an edge in the following graph:*

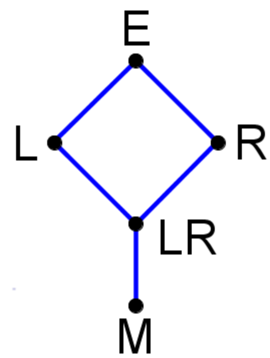


Figure 2.11: Diamond swap graph

Note: throughout the rest of this section, we will refer to the replacing of a single element in a deletion pattern by an adjacent element in the graph depicted in Figure 2.11 as a "diamond swap". We will refer to the position being changed in the diamond swap as the "swapped position".

Proof. Lemma 2.4.4 provided the necessity of this statement, what remains is to prove the sufficiency. So suppose that two deletion patterns P_A and P_B are given which match the conditions claimed to produce an edge. Our goal is to produce a lifting ω with a single essential collinearity that supports these two vertices (the fact that it *can* only support two vertices was also established in Lemma 2.4.4).

To produce this lifting, take the patterns corresponding to the two given vertices, P_A and P_B, and perform the construction outlined in Theorem 2.2.3 to obtain two liftings ω_A and ω_B which support the vertices. For certain types of diamond swaps, we will require the liftings to satisfy additional properties, so let $\overline{\omega_A}$ and $\overline{\omega_B}$ denote liftings which correspond to the *reverse* of patterns A and B, respectively, which are then reflected across the line $x = (n+1)/2$. We will refer to the swapped position in the two corresponding patterns as position j. Note that in any lifting, $M'_L(j)$ and $M'_R(j)$ will not be $\emptyset$ for positions j with $2 < j < n-1$, but $M'_L(j) = \emptyset$ if $j = 2$ and $M'_R(j) = \emptyset$ if $j = n-1$. We will not need the "existence" of $M'_L(j)$ and $M'_R(j)$ in the liftings ω_A and ω_B in these restricted circumstances due to the fact that the types of swaps which are allowable at these positions are suitably limited. Lastly, we see that we need not distinguish between consideration of ω_A or ω_B as $M_L(j, \omega_A) = M_L(j, \omega_B)$, $M_R(j, \omega_A) = M_R(j, \omega_B)$, $M'_L(j, \omega_A) = M'_L(j, \omega_B)$, and $M'_R(j, \omega_A) = M'_R(j, \omega_B)$.

For the various types of diamond swaps, construct ω as follows:

$$LR-M: \quad \omega = \frac{\omega_A}{\left|\det\begin{pmatrix} \omega_A(M_L(j)) & \omega_A(j) & \omega_A(M_R(j)) \\ M_L(j) & j & M_R(j) \\ 1 & 1 & 1 \end{pmatrix}\right|} + \frac{\omega_B}{\left|\det\begin{pmatrix} \omega_B(M_L(j)) & \omega_B(j) & \omega_B(M_R(j)) \\ M_L(j) & j & M_R(j) \\ 1 & 1 & 1 \end{pmatrix}\right|},$$

$$\begin{matrix} LR-L \\ \text{or } R-E \end{matrix}: \quad \omega = \frac{\overline{\omega_A}}{\left|\det\begin{pmatrix} \overline{\omega_A}(M_L(j)) & \overline{\omega_A}(j) & \overline{\omega_A}(M'_R(j)) \\ M_L(j) & j & M'_R(j) \\ 1 & 1 & 1 \end{pmatrix}\right|} + \frac{\overline{\omega_B}}{\left|\det\begin{pmatrix} \overline{\omega_B}(M_L(j)) & \overline{\omega_B}(j) & \overline{\omega_B}(M'_R(j)) \\ M_L(j) & j & M'_R(j) \\ 1 & 1 & 1 \end{pmatrix}\right|},$$

$$\begin{matrix} LR-R \\ \text{or } L-E \end{matrix}: \quad \omega = \frac{\omega_A}{\left|\det\begin{pmatrix} \omega_A(M'_L(j)) & \omega_A(j) & \omega_A(M_R(j)) \\ M'_L(j) & j & M_R(j) \\ 1 & 1 & 1 \end{pmatrix}\right|} + \frac{\omega_B}{\left|\det\begin{pmatrix} \omega_B(M'_L(j)) & \omega_B(j) & \omega_B(M_R(j)) \\ M'_L(j) & j & M_R(j) \\ 1 & 1 & 1 \end{pmatrix}\right|}.$$

We see that this new lifting ω does indeed create at least one collinearity between precisely the points required. Since we have taken a positive scaling of two liftings, we still need to verify that the non-swapped positions in the new lifting still have the appropriate visibility (recall the letters in the deletion patterns match in every position but j). In other words, if a position was an R in both of the original liftings, we need to make sure that it remains an R in the new lifting. For ease of notation, we will simply call the lifting which we use to represent P_A and P_B by ω_A and ω_B, i.e. we will drop the bar notation which indicates that the liftings corresponding to $LR-L$ and $R-E$ swaps came from a reflection process.

Due to the nature of ω's construction, we see that the orientation of any three points in ω is given as the sum of positive scalings of those points' orientations in the original two liftings:

$$\mathcal{O}_\omega(a,b,c) \;=\; c_1 \cdot \mathcal{O}_{\omega_A}(a,b,c) + c_2 \cdot \mathcal{O}_{\omega_B}(a,b,c), \quad \text{where } 1 \le a,b,c \le n, \text{ and } c_1, c_2 > 0.$$

This will immediately give us that (non-swapped) M's in the original patterns will be "preserved" (also assigned the letter M) in the new lifting. To see this, suppose i is the position of a non-swapped M in the original liftings for which there exists $h < i < k$ such that $\mathcal{O}_\omega(h,i,k) \ge 0$. This implies that at least one of $\mathcal{O}_{\omega_A}(h,i,k)$ or $\mathcal{O}_{\omega_B}(h,i,k)$ is positive, which is impossible.

A similar argument will show that E's in the original liftings are also preserved. In Theorem 2.2.3, we constructed the liftings so that any position i assigned an E satisfied $\mathcal{O}(1,i,n-1), \mathcal{O}(2,i,n) > 0$. This gives $\mathcal{O}_\omega(1,i,n-1), \mathcal{O}_\omega(2,i,n) > 0$, which forces position i to be assigned an E in the new lifting.

To finish the rest of the argument, we will need to split into two cases depending on what type of swap is occurring:

Case 1: The swap is *not* between an LR and an M.

We next wish to establish that the *only* M's appearing in the new lifting correspond to M's in the original liftings. Suppose that position $i \neq j$ in the original liftings does not correspond to an M, and consider $M_L(i)$ and $M_R(i)$. With these types of swaps, we are guaranteed that the positions recorded by M_L (and M_R) match in both ω_A and ω_B, so there is no need to specify the lifting. We have that $\mathcal{O}_{\omega_A}(M_L(i), i, M_R(i)), \mathcal{O}_{\omega_B}(M_L(i), i, M_R(i)) > 0$ and hence $\mathcal{O}_\omega(M_L(i), i, M_R(i)) > 0$, which guarantees that i cannot be assigned an M in the new lifting.

Next, we wish to show that non-swapped, non-M positions $i \neq j$ cannot *lose* L- or R-visibility. That is, L's and R's in the original liftings cannot become E's in the new lifting, nor could an LR in the originals become an L or R in the new. Since the arguments are symmetric, we focus on losing L-visibility and assume that position i was visible under left deletion in the original liftings, but is no longer visible under left deletion in ω. This implies there exist positions $h < i < k$ with $h \neq M_L(i)$ such that $\mathcal{O}_\omega(h, i, k) \geq 0$, which in turn requires that either $\mathcal{O}_{\omega_A}(h, i, k)$ or $\mathcal{O}_{\omega_B}(h, i, k)$ is non-negative. This contradicts the fact that i had L-visibility in the original liftings.

The last requirement to check is that no non-swapped, non-E position could *gain* L- or R-visibility. We will first focus on L-visibility and assume that position $i \neq j$ does not have L-visibility in the original liftings (assigned an R). To show that i cannot gain L-visibility, we need to provide fixed positions $h < i < k$ with $h \neq M_L(i)$ such that both $\mathcal{O}_{\omega_A}(h, i, k)$ and $\mathcal{O}_{\omega_B}(h, i, k)$ are strictly positive, which guarantees that $\mathcal{O}_\omega(h, i, k) > 0$ and hence i cannot gain L-visibility in the new lifting.

If $M_L^2(i, \omega_A), M_L^2(i, \omega_B) \neq \emptyset$ (we know if they do exist, they match in this case), then according to the construction outlined in Theorem 2.2.3, we have $\mathcal{O}(M_L^2(i), i, M_R(i)) > 0$ in both original liftings, which prevents the gain of L-visibility in ω. If this is not the case, then i lies between the first two M's in the patters P_A and P_B, and we also note that $M_L(i) = 1$ and $i \neq 2$ as position 2 cannot have the label R. We next check whether $M_L'(i, \omega_A) = M_L'(i, \omega_B)$ (we know neither can be $\emptyset$ for these i's). If this position is fixed in both liftings, then $M_L'(i), M_R(i)$ will serve as viable candidates for h and k. The only way that the position $M_L'(i)$ is not fixed in the original liftings is if we are making an $L-E$ or $LR-R$ swap. In this case, we reference Theorem 2.2.3 again and note that if position 2 is not an M, it is placed so that the line through 2 and the second M in the pattern hide any R positions between them. Thus we see $\mathcal{O}_\omega(2, i, M_R(i)) > 0$ in both liftings by construction.

The argument that position i cannot gain R-visibility runs similarly, with the difference being that we may need to utilize the fact that we can construct the liftings so that if position $n-1$ is not an M, we can guarantee that the line between the penultimate M and position $n-1$ hides any L positions between them. This is accomplished by reflecting the initial pattern, performing the construction, then reflecting that lifting back across the line $x = (n+1)/2$. Notice that we only need to perform this reflection process in the event that the swap is from $R-E$ or $LR-R$, which avoids any problems involving an overlap between which side of the pattern is going to require this special restriction.

Case 2: The swap is between an LR to an M. Again we will establish that the only M's appearing in the new lifting ω come from M's in the original lifting. The difference in this case is that for some non-M, non-swapped position $i \neq j$ in the original liftings, we may have $M_L(i, \omega_A) \neq M_L(i, \omega_B)$ or $M_R(i, \omega_A) \neq M_R(i, \omega_B)$ (we do know that both exist for the i's that we are considering, however). If they do both match for each lifting, then the argument in Case I shows that position i cannot possibly become an M in the new lifting. The only way that they do not match is if i is in the same interval as the swapped LR, or equivalently has the swapped M appearing at the edges of its interval. We have already removed the possibility of i being assigned an E in the original liftings, so the only other possibilities are that $i < j$, so $M_L(i, \omega_A) = M_L(i, \omega_B)$, and is assigned an L or $i > j$, so $M_R(i, \omega_A) = M_R(i, \omega_B)$, and is assigned an R. Any other labeling of i would render one of the patterns P_A or P_B invalid. We see, however, that if $i < j$, then $\mathcal{O}(M_L(i), i, j) > 0$ regardless of whether j is an M or an LR, and similarly if $i > j$ that $\mathcal{O}(j, i, M_R(i)) > 0$ regardless of the labeling on j. This establishes that the M's appearing in the new lifting ω correspond exactly to the M's in the original liftings, with the exception of the swapped M.

Next we will argue that non-swapped, non-M positions $i \neq j$ cannot lose L- or R-visibility. We will start off by assuming that position i has L-visibility in the original liftings, but not in the new lifting. If $M_L(i, \omega_A) = M_L(i, \omega_B)$ (recall they both must exist), then the previous argument shows that the loss of L-visibility will produce a contradiction. There is the possibility that the position of $M_L(i)$ may not be fixed for a position i if $j = M_L(i)$ in one of the liftings, but in this case, we are guaranteed that the label of i must be an R or an E, otherwise one of the original patterns is invalid; thus in either case, it is impossible for i to lose L-visibility. The argument to show that these type of positions cannot lose R-visibility is similar.

Lastly we verify that no non-swapped, non-E position could gain L- or R-visibility. We start with a position $i \neq j$ that does not have L-visibility in the original liftings, which forces i to be assigned an R. Our aim here is the same as it was in the previous case, to find positions $h < i < k$ with $h \neq M_L(i, \omega)$ such that $\mathcal{O}(h, i, k) > 0$ in both ω_A and ω_B. If $M_L^2(i, \omega_A) = M_L^2(i, \omega_B) \neq \emptyset$, then $M_L^2(i, \omega_A), M_R(i, \omega)$ will serve as viable candidates for h and k. If $M_L^2(i) \neq \emptyset$ in at least one of the original liftings, then we can separate into two separate cases. If swapped position $M_L(i) = j$ in at least one lifting, then $j \neq M_L(i, \omega)$ and M_R serve as candidates for h and k as $\mathcal{O}(j, i, M_R(i, \omega)) > 0$ for i assigned an R whether j is an LR or an M. If $M_L^2(i) = j$ in one of the liftings, then $j \neq M_L(i, \omega)$ and $M_R(i, \omega)$ will still work as candidates for h and k. We see that $\mathcal{O}(j, i, M_R(i, \omega)) > 0$ when j is an M by our particular construction; further, $\mathcal{O}(j, i, M_R(i, \omega)) > 0$ when j is an LR, because we would have $\mathcal{O}(M_L(j), j, M_R(i, \omega)) < 0$ by virtue of j being an LR, and this would force $\mathcal{O}(j, i, M_R(i, \omega)) > 0$ for any i assigned an R with

$j < i < M_R(i, \omega)$. This follows from the fact that $M_L(j, \omega_A)) = M_L(j, \omega_B)) = M_L^2(i)$ in this scenario. Since the previous two cases cover all possibilities where $M_L^2(i)$ exists in at least one lifting, the last case to consider is if $M_L^2(i) = \emptyset$ in either lifting. This will force i to occur in a left Type-A interval, but we are guaranteed that $M_L'(i, \omega_A) = M_L'(i, \omega_B) \neq \emptyset$ as we must have $j > i$. The argument in Case I shows that this type of i cannot gain L-visibility. The case for not gaining R-visibility is similar. Notice that when we need to use the existence of $M_L'(i)$ or $M_R'(i)$ in the Type-A intervals, we are guaranteed that the position will match in both liftings, otherwise we would be able to find candidates for h and k with a different argument. This is why we do not require the reflection tactic that was used in Case I.

The previous cases demonstrate that letters can be assigned to all positions $i \neq j$ in the new lifting ω as their visibility in each deletion is exactly determined. Further, we see that the letters applied to position $i \neq j$ must exactly correspond to the matching letters assigned to i in P_A and P_B, which in turn demonstrates that the collinearity that we have constructed in ω must indeed be essential. Although ω may coincidentally have other collinearities, the previous argument shows that they *cannot* be essential (we may perturb these points slightly to break all other collinearities if we wish, without affecting the visibility of ω). By construction, we see that $\omega \cdot \mathbf{v}$ is minimized over the set of compound-GKZ vectors exactly when $\mathbf{v}$ corresponds to the compound GKZ-vector corresponding to either P_A or P_B. $\blacksquare$

Example 2.4.7 In this example, we will demonstrate why it was useful to have the liftings in Theorem 2.4.6 be constructed via the method outlined in Theorem 2.2.3. Consider the liftings $\omega_A = \langle 2.4, 0, 0.7, 0.5, 0, 2 \rangle$ and $\omega_B = \langle 2.4, 0, 3.7, 0.8, 0, 2 \rangle$, pictured in Figure 2.12 which have the corresponding deletion patterns $P_A = (M, M, L, R, M, M)$ and $P_B = (M, M, E, R, M, M)$.

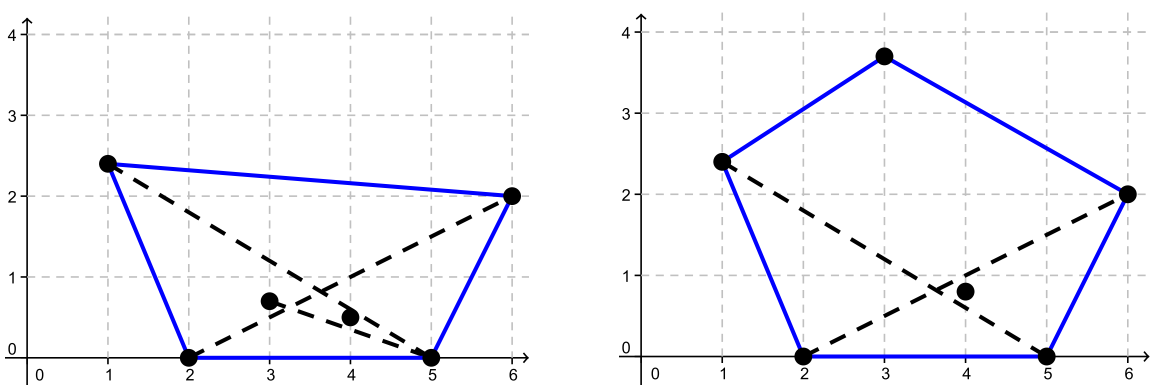


Figure 2.12: Vertex-supporting liftings

Note that if we were to construct a lifting ω_A' to support P_A via the method outline in Theorem 2.2.3, we would have that position 4 (which is assigned an R) has $M_L^2(4) = 1$ and $M_R(4) = 5$, and would satisfy $\mathcal{O}_{\omega_A'}(1, 4, 5) > 0$. In the given lifting which supports pattern P_A, however, we have that $\mathcal{O}_{\omega_A}(1, 4, 5) < 0$ and instead the orientation $\mathcal{O}_{\omega_A}(3,4,5) > 0$ denies L-visibility to position 4.

We see that we can perform an $L - E$ diamond swap on position 4 to get from pattern P_A to P_B, so they should form an edge in $\Sigma_1(6)$. If we utilize the sum of positive scalings of ω_A and ω_B (as described in Theorem 2.4.6) to create an edge supporting lifting ω with the appropriate essential collinearity between positions 1, 3, and 5, we obtain

$$\begin{aligned}\omega &= \frac{\langle 2.4, 0, 0.7, 0.5, 0, 2\rangle}{\left| det \begin{pmatrix} 1 & 3 & 5 \\ 2.4 & 0.7 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right|} + \frac{\langle 2.4, 0, 3.7, 0.8, 0, 2\rangle}{\left| det \begin{pmatrix} 1 & 3 & 5 \\ 2.4 & 3.7 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right|} \\ &= \langle 1.44, 0, 0.72, 0.33, 0, 1.2\rangle.\end{aligned}$$

Figure 2.13 shows the constructed lifting ω which contains the appropriate collinearity, but the visibility of position 4 has changes from an R in the original liftings to an LR in the new lifting. This prevents the collinearity that we created from being classified as essential; it is not revealed by a single deletion. Thus the lifting that we have created is a vertex supporting lifting, corresponding to the pattern $P_C = (M, M, E, LR, M, M)$.

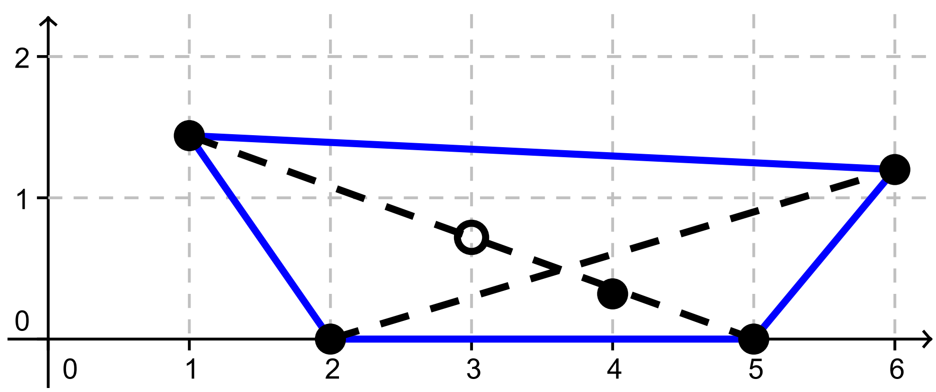
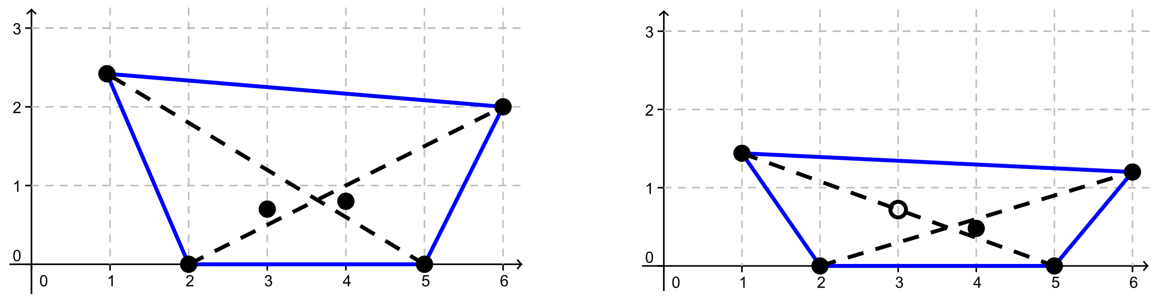


Figure 2.13: Edge-supporting lifting counterexample

The first image in Figure 2.14 illustrates an altered lifting ω'_A corresponding to pattern P_A satisfying $\mathcal{O}_{\omega'_A}(M_L^2(4), 4, M_R(4)) = \mathcal{O}_{\omega'_A}(1, 4, 5) > 0$ (Notice the construction in Theorem 2.2.3 would guarantee a lifting with this property). If we repeat the edge-supporting lifting construction with ω'_A and ω_B, we see that we obtain the second lifting, ω' in Figure 2.14, which indeed would be uniquely minimized by the compound GKZ-vectors corresponding to patterns P_A and P_B. $\Diamond$

Corollary 2.4.8 *The polytope $\Sigma_1(n)$ is neither simplicial, nor simple for $n \geq 5$.*



(i) ω'_A (ii) ω'

Figure 2.14: Rectified edge-supporting lifting

Proof. To see that the polytope $\Sigma_1(n)$ is not simplicial, pick an arbitrary face defining hyperplane and consider its normal vector as a lifting ω. If the lifting has one or fewer essential collinearities, it must support either an edge or a vertex. If the lifting has two or more essential collinearities, we saw in Lemma 2.4.4 that it must support at least 4 vertices. Since $\Sigma_1(n)$ can have no triangles as faces, it cannot be simplicial for $n \geq 5$.

Having established that $\Sigma_1(n)$ is $n-2$ dimensional in Section 2.2, we need to find a vertex of $\Sigma_1(n)$ with more than $n-2$ incident edges to establish that $\Sigma_1(n)$ is not simple. To this end, assume n is even and consider the deletion pattern $P = (M, LR, M, LR, M, \ldots, LR, M, M)$ which corresponds to a vertex. We see that this vertex has $(n-2)/2$ LR's, with only one in special position 2. Any LR not in position 2 may be replaced with either an M, L, or an R to produce a valid deletion pattern, and hence produce 3 edges incident to this vertex, whereas the LR in position 2 produces only 2 edges. Thus we see that this vertex has at least $3(n-2)/2-1$ edges, which is strictly greater than the required $n-2$ for $n \geq 5$. The argument if n is odd runs similarly. ■

2.5 Edge Count

In order to enumerate the number of edges of $\Sigma_1(n)$, we wish to count all possible diamond swaps between valid deletion patterns of length n. We must take care, however, as performing a diamond swap on a valid deletion pattern may produce a deletion pattern that no longer satisfies the conditions of Theorem 2.2.3, as illustrated in Example 2.5.1.

Example 2.5.1 The following is a diamond swap on a valid pattern which produces an invalid pattern:

$$\begin{array}{cccccccc} (& M, & L, & R, & M, & R, & M &) \\ & & & & \downarrow & & & \\ (& M, & L, & {\color{red}R}, & LR, & R, & M &) \end{array}$$

◊

To get around this problem, we will only count an edge when the diamond swap is made in a particular direction. Specifically, we will take the graph in Theorem 2.4.6 and turn it into the digraph depicted in Figure 2.15:

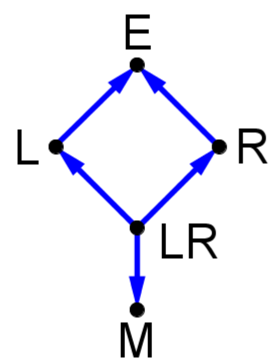


Figure 2.15: Diamond swap digraph

We now consider a particular diamond swap as forming an edge only if the swap conforms to the direction of the corresponding edge in the digraph. For example, a swap *from* an LR *to* an M will be counted as an edge, whereas a swap *from* an M *to* an LR would not. This serves the dual purpose of avoiding a double-count of the edges, and preventing us from performing a swap which would result in a non-valid deletion pattern. For example, any LR can be swapped to an M without violating the conditions outlined in Theorem 2.2.3, yet we cannot swap an M to an LR if there is already an LR present in the immediate intervals surrounding the M. The only case we have yet to be careful of is that we cannot swap an L in position 2 or an R in position $n-1$ to an E, nor can we swap an LR in position 2 or $n-1$ to an R or L, respectively, as this would violate the conditions in Theorem 2.2.3. With these conditions in mind, we see that LR's in position 2 or $n-1$ generate 2 edges each, while LR's in any other position generate 3 edges each. L's and R's in positions 2 and $n-1$, respectively, do not generate any edges, whereas L's and R's in any other position will generate 1 edge each. We note that E's and M's are not considered to generate edges under our conditions.

Our plan of attack will proceed as follows: We will first go through and count the total number of LR's appearing in every deletion pattern of length n. For the time being, we will assume that each of these LR's contribute 3 edges to the total edge count, one each for a swap with an M, an L, and an R. After the count is finished, we will need to correct for the fact that LR's in the special positions 2 and $n-1$ actually only contribute 2 edges. We will perform a separate count of these LR's in special positions 2 and $n-1$ and subtract off the appropriate number of edges. Lastly, we will perform a count of the total number of L's and R's appearing in any length n valid deletion pattern and note that each occurrence contributes exactly one edge for a swap with an E. We will again run into the issue of L's and R's appearing in positions 2 and $n-2$ being unable to swap, but we can ignore the contributions from these L's and R's directly in the count instead of having to deal with the erroneous edges after the fact.

The count of all LR's appearing in a valid deletion pattern of length n will be the easiest as most of the work is done for us already in Section 2.3. Let $b_{n,k}^{LR}$, $a_{n,k}^{LR}$, and $\hat{a}_{n,k}^{LR}$ denote the number of length n intervals of type B, A, and $\hat{A}$, respectively, which have exactly k LR's appearing, and further, let $B_{LR}(x,y) := \sum_{n\geq 1, k\geq 0} b_{n,k}^{LR} x^n y^k$, $A_{LR}(x,y) := \sum_{n\geq 1,k\geq 0} a_{n,k}^{LR} x^n y^k$, and $\hat{A}_{LR}(x,y) := \sum_{n\geq 1,k\geq 0} \hat{a}_{n,k}^{LR} x^n y^k$ denote the corresponding o.g.f.'s. To determine these o.g.f.'s, we simply need to return to the arguments counting the length n intervals of each type and tag the contributions that contain LR's with a y. We obtain:

$$
\begin{aligned}
B_{LR}(x,y) &= \sum_{n\geq 1} 2^{n-2}(n+1)x^n + \sum_{n\geq 2}(n-1)2^{n-2}yx^n = \frac{x(1-x+xy)}{(2x-1)^2}, \\
A_{LR}(x,y) &= x + \sum_{n\geq 1}\left(2^{n-3}n + (n-2)y2^{n-3} + 2^{n-2}y\right) \\
&= \frac{x(1-3x+3x^2+xy-x^2y)}{(2x-1)^2}, \\
\hat{A}_{LR}(x,y) &= x + yx^2 + \sum_{n\geq 3,k\geq 0}\left(\left(2^{n-3}+2^{n-3}+(n-3)2^{n-4}\right)y + 2^{n-4}(n-1)\right)x^n \\
&= y\frac{x^2(1-2x+x^2)}{(2x-1)^2} + \frac{x(1-4x+5x^2-x^3))}{(2x-1)^2}.
\end{aligned}
$$

Now we let $f_{n,k}^{LR}$ denote the number of length n deletion patterns which have exactly k LR's appearing and let $F_{LR}(x,y) := \sum_{n\geq 1,k\geq 0} f_{n,k}^{LR} x^n y^k$ denote the corresponding o.g.f. As in the previous section, we will utilize the fact that deletion patterns are built by very specific compositions of the three types of intervals to find that

$$F_{LR}(x,y) = \Big(A_{LR}(x,y)\cdot\big(B^0_{LR}(x,y)+B^1_{LR}(x,y)+\cdots\big)\cdot A_{LR}(x,y)+\hat{A}_{LR}(x,y)\Big)\,x,$$
$$= \frac{x^2(x^3-2x^2y+4x^2-4x+xy+1)}{1-5x+5x^2-x^2y}.$$

Now we will use $F_{LR}(x,y)$ to determine a new O.G.F $F_{\#LR}(x)$ with the property that $[x^n]F_{\#LR}(x)$ records the total number of all LR's occurring in all valid deletion patterns of length n. To obtain this new o.g.f. we first consider the function

$$y\frac{\partial}{\partial y}F_{LR}(x,y) = \sum_{n\geq 1,k\geq 0} kf_{n,k}x^ny^k,$$

which has the property that $[x^ny^k]y\dfrac{\partial}{\partial y}F_{LR}(x,y)$ records the total number of LR's appearing in all length n valid deletion patterns that each contain exactly k LR's. This, in turn, will help us count all the edges obtained by performing diamond swaps on LR's. Specifically, we find that

$$y\frac{\partial}{\partial y}\frac{x^2(x^3-2x^2y+4x^2-4x+xy+1)}{1-5x+5x^2-x^2y} = \frac{yx^3(1-6x+11x^2-6x^3+x^4)}{(-1+5x-5x^2+x^2y)^2}. \quad (2.2)$$

Lastly, we note that by setting $y=1$ in equation (2.2), we obtain $F_{\#LR}(x)$. We compute

$$F_{\#LR}(x) = \frac{x^3(1-6x-6x^3+x^4+11x^2)}{(1-5x+4x^2)^2}$$

At this stage we have almost obtained the total number of edges of $\Sigma_1(n)$ arising from diamond swaps originating with an LR. All that remains is to count the number of LR's in the special positions 2 and $n-1$ as these only contribute 2 edges, as opposed to the 3 edges contributed by LR's in non-special positions. Let $a^{S:LR}_{n,k}$ and $\hat{a}^{S:LR}_{n,k}$ denote the number of length n valid deletion patterns with exactly k LR's appearing in special position (i.e. 2 or $n-1$) of type A and $\hat{A}$ respectively, and let $A_{S:LR}(x,y) := \sum_{n\geq 1,k\geq 0} a^{S:LR}_{n,k}x^ny^k$, and $\hat{A}_{S:LR}(x,y) := \sum_{n\geq 1,k\geq 0}\hat{a}^{S:LR}_{n,k}x^ny^k$ denote the corresponding o.g.f.'s. Again we return to the arguments in Section which enumerate the type A and type $\hat{A}$ valid deletion patterns of length n and tag the pieces which contribute an LR in special positions 2 or $n-1$ with a y. We obtain:

$$
\begin{aligned}
A_{S:LR}(x,y) &= x + \sum_{n\geq 2} \left(2^{n-3}n + (n-2)2^{n-3} + 2^{n-2}y\right) x^n \\
&= \frac{x(1-3x+xy+4x^2-2x^2y)}{(1-2x)^2}, \\
\hat{A}_{S:LR}(x,y) &= x + x^2y^2 + \sum_{n\geq 3} \left(2^{n-3}y + 2^{n-3}y + (n-2)2^{n-3}\right) x^n \\
&= \frac{x^2y(2x+y-2xy)}{1-2x} + \frac{x(1-4x+5x^2)}{(1-2x)^2}.
\end{aligned}
$$

Notice that in $\hat{A}_{S:LR}(x,y)$, we have the term x^2y^2, since the deletion pattern M, LR, M is considered to have 2 LR's, one for each special position; this is an isolated special case that only occurs in deletion patterns of length 3. As for the type B intervals, we see they never contribute an LR in special position, so we may use equation (2.1) for the following argument.

Let $f_{n,k}^{S:LR}$ denote the number of length n deletion patterns which have exactly k LR's appearing in positions 2 and $n-1$ and let $F_{S:LR}(x,y) := \sum_{n\geq 1, k\geq 0} f_{n,k}^{S:LR} x^n y^k$ denote the corresponding o.g.f. We find:

$$
\begin{aligned}
&F_{S:LR}(x,y) \\
&\quad= \Big(A_{S:LR}(x,y)\left(B^0(x)+B^1(x)+B^2(x)+\cdots\right)A_{S:LR}(x,y) + \hat{A}_{S:LR}(x,y)\Big)\, x, \\
&\quad= \frac{x^2(-8x^3y+5x^3y^2+4x^3+4x^2y-5x^2y^2+3x^2-4x+xy^2+1)}{4x^2-5x+1}.
\end{aligned}
$$

Again we will take a partial derivative with respect to y and multiply by y to obtain an o.g.f. whose x^ny^k coordinate counts the total number of LR's appearing in special positions 2 and $n-1$ in all length n valid deletion patterns with exactly k LR's in special position:

$$
y\frac{y}{\partial y}F_{S:LR}(x,y) = \frac{x^2y(-8x^3+10x^3y+4x^2-10x^2y+2xy)}{4x^2-5x+1} \tag{2.3}
$$

Lastly, we set $y=1$ in Equation (2.3) to obtain $F_{\#S:LR}(x)$, the o.g.f. whose x^n coordinate counts the total number of LR's in special positions 2 and $n-1$ in all length n valid deletion patterns. We compute

$$
F_{\#S:LR}(x) := y\frac{\partial}{\partial y}F_{S:LR}(x,y)\Big|_{y=1} = \frac{x^2(2x^3-6x^2+2x)}{4x^2-5x+1}.
$$

Now that we know how many LR's appear in the special positions 2 and $n-1$ and how many *total* LR's appear in all length n valid deletion patterns, we can compute exactly how many edges in $\Sigma_1(n)$ occur as a result of diamond swapping an LR. We let e_n^{LR} denote the total number of edges occurring as a result of diamond swaps of LR's appearing in length n valid deletion patterns and let $E_{LR}(x)$ be the corresponding o.g.f. Since each LR in non-special position contributes 3 edges from diamond swaps, where each LR in special positions contribute only two edges, we find:

$$E_{LR}(x) \; := \; \sum_{n\geq 1} e_n^{LR} x^n \; = \; 3F_{\#LR}(x) - F_{\#S:LR}(x) \; = \; \frac{x^3(1-2x+16x^3-5x^4-7x^2)}{(1-5x+4x^2)^2}. \tag{2.4}$$

Now that we have finished counting the edges obtained from diamond swaps on LR's, we switch focus to counting the edges coming from diamond swaps on L's and R's. We recall that each L and R in a non-special position generates exactly one edge when we swap to an E, and L's and R's in special position generate none. With this in mind, we make the following definitions: let $b_{n,k}^{L,R}$, $a_{n,k}^{L,R}$, and $\hat{a}_{n,k}^{L,R}$ denote the number of length n intervals with exactly k L's and R's appearing in non-special positions in type B, A, and $\hat{A}$ intervals, respectively. Further let $B_{L,R}(x,y) := \sum_{n\geq 1, k\geq 0} b_{n,k}^{L,R} x^n y^k$, $A_{L,R}(x,y) := \sum_{n\geq 1, k\geq 0} a_{n,k}^{L,R} x^n y^k$, and $\hat{a}_{L,R}(x,y) := \sum_{n\geq 1, k\geq 0} \hat{a}_{n,k}^{L,R} x^n y^k$ denote the corresponding o.g.f.'s.

We first consider type B and focus on the coefficient of x^n and note that in this type of interval we do not have to worry about L's and R's in special position. We break into two cases depending on whether or not an LR appears in the interval:

Case 1: The interval has no LR term present.

In this case, we break into two subcases based on whether or not there is a rightmost L term present in the interval. If the rightmost L is present, we have a y for that L term, we have $n-1$ choices to place the rightmost L, and we have the possibility of 0 to $n-2$ of the remaining positions being E or non-E, where each non-E position will be tagged with a y. Notice that the value of the non-E positions is determined by their relative position to the rightmost L. Thus this subcase contributes a $y(n-1)\sum_{i=0}^{n-2}\binom{n-2}{i}y^i$ to the coefficient of x^n.

Next, we consider the case where there is no rightmost L present in the interval. In this case, we need only choose the location of the non-E terms which must all be R's. Notice that we may have anywhere from 0 to $n-1$ non-E terms. This subcase contributes $\sum_{i=0}^{n-1}\binom{n-1}{i}y^i$ to the coefficient of x^n.

Case 2: The interval has an LR term present.

In this case, we have $n-1$ locations to place the LR and $n-2$ remaining positions which may be E or non-E. Notice that the location of the term with respect to the LR will determine whether the non-E term must be an L or an R. Thus if we sum over all possible choices of non-E vs. E locations, we obtain a contribution of $(n-1)\sum_{i=0}^{n-2}\binom{n-2}{i}y^i$ to the coefficient of x^n.

Summing together all contributions to $[x^n]B_{L,R}(x,y)$, we find

$$\begin{aligned} B_{L,R}(x,y) &= \\ &\sum_{n\geq 1}\left(y(n-1)\sum_{i=0}^{n-2}\binom{n-2}{i}y^i+\sum_{i=0}^{n-1}\binom{n-1}{i}y^i+(n-1)\sum_{i=0}^{n-2}\binom{n-2}{i}y^i\right)x^n \\ &= \frac{x}{(1-xy-x)^2}. \end{aligned}$$

Next we consider type A intervals and focus on the coefficient of x^n. Recall again that we do not append a y for L's and R's occurring in the special positions 2 and $n-1$ since these will not contribute any edges. As in the vertex count, we will only consider type A intervals which contain the first M since the other case is symmetric. We focus on intervals with $n\geq 2$ and break into two cases depending on the value in special position 2:

Case 1: Position 2 is an LR.

In this case we may freely choose which of the $n-2$ remaining positions are non-E, with the rest being forced to be R's. Notice that each non-E position chosen will be tagged with a y. The contribution to the coefficient of x^n from this case is $\sum_{i=0}^{n-2}\binom{n-2}{i}y^i$.

Case 2: Position 2 is an L.

We will consider two subcases based on whether or not an LR appears after position 2 in the interval:

Case 2.a: No LR appears in the interval.

We further break into subcases based on whether or not a rightmost L appears (ignoring the L in position 2). If such a rightmost L appears, we append a y for that L, choose from the $n-2$ available positions to place that L, and choose which of the remaining $n-3$ positions will be non-E. This particular situation contributes $y(n-2)\sum_{i=0}^{n-3}\binom{n-3}{i}y^i$ to the coefficient of x^n.

If no rightmost L appears after the L in position 2, this implies that all non-E positions must be R's. We may freely choose which of the remaining $(n-2)$ positions are non-E, obtaining a contribution of $\sum_{i=0}^{n-2}\binom{n-2}{i}y^i$ to the coefficient of x^n.

<u>Case 2.b</u>: An LR appears in the interval.

In this case, we have $n-2$ positions to place the LR. After this, we may freely choose which of the $n-3$ remaining positions are non-E. Notice that the relative placement with respect to the LR will determine the value, L or R, of any non-E term. We obtain a contribution of $(n-2)\sum_{i=0}^{n-3}\binom{n-3}{i}y^i$ to the coefficient of x^n from this case.

Summing together all contributions to $[x^n]A_{L,R}(x,y)$ and noting the special case $n=1$, we find

$$\begin{aligned} A_{L,R}(x,y) &= x+\sum_{n\geq 2}\left(\sum_{i=0}^{n-2}\binom{n-2}{i}y^i+(n-2)\sum_{i=0}^{n-3}\binom{n-3}{i}y^i\right.\\ &\quad\left.+\sum_{i=0}^{n-2}\binom{n-2}{i}y^i+(n-2)\sum_{i=0}^{n-3}\binom{n-3}{i}y^i\right)x^n\\ &= \frac{x(x^2y^2+x^2y-2xy+1)}{(xy+x-1)^2}. \end{aligned}$$

Lastly, we consider type $\hat{A}$ intervals and focus on the coefficient of x^n. Again we recall that L's and R's in the special positions 2 and $n-1$ will not be tagged with a y and focus on the cases $n\geq 3$. We break into 3 cases depending on the values in positions 2 and $n-1$:

<u>Case 1</u>: Position 2 is an LR, position $n-1$ is an R.

In this case, all non-E positions are forced to be R's. We get to freely choose which of the $n-3$ positions are non-E and tag with y's accordingly. The contribution from this case is $\sum_{i=0}^{n-3}\binom{n-3}{i}y^i$.

<u>Case 2</u>: Position 2 is an L, position $n-1$ is an LR.

This is the same as Case 1 by symmetry, so we get another contribution of $\sum_{i=0}^{n-3}\binom{n-3}{i}y^i$ to the coefficient of x^n.

<u>Case 3</u>: Position 2 is an L, position $n-1$ is an R.

We break into subcases based on whether or not an LR appears in the interval:

Case 3.a: The interval contains an LR.

In this case, we have $n-3$ positions to place the LR. Once placed, we may freely choose which of the remaining $n-4$ positions are non-E, with their value, L or R, being determined by their placement with respect to the LR. For this case we get a contribution of $(n-3)\sum_{i=0}^{n-4}\binom{n-4}{i}y^i$.

Case 3.b: The interval does not contain an LR.

If we ignore the L in position 2 and find that there is still a rightmost L occurring, we will note that there are $n-3$ possible locations for this rightmost L. After we place this L and tag it with a y, we may freely choose which of the remaining $n-4$ positions are non-E, with their specific value being determined by their placement with respect to the last L. This subcase contributes $(n-3)y\sum_{i=0}^{n-4}\binom{n-4}{i}y^i$.

If there is no rightmost L occurring after the L in position 2, this means that all non-E positions must be filled by R's. We may freely choose which of the remaining $n-3$ positions are non-E to obtain a contribution of $\sum_{i=0}^{n-3}\binom{n-3}{i}y^i$.

Summing together all contributions to $[x^n]\hat{A}_{L,R}(x,y)$ and noting the special cases $n=1,2$, we find

$$\begin{aligned}\hat{A}_{L,R}(x,y) &= x+x^2\sum_{n\geq 3}\left(2\sum_{i=0}^{n-3}\binom{n-3}{i}y^i+(n-3)\sum_{i=0}^{n-4}\binom{n-4}{i}y^i\right.\\ &\quad\left.+(n-3)y\sum_{i=0}^{n-4}\binom{n-4}{i}y^i+\sum_{i=0}^{n-3}\binom{n-3}{i}y^i\right)x^n\\ &= \frac{x(x^3y^2+x^2y^2-x^3+2x^2-2xy-x+1)}{(1-xy-x)^2}.\end{aligned}$$

We are now in the position to begin to compute the number of edges resulting from diamond swaps on L's and R's. Let $f_{n,k}^{L,R}$ denote the number of length n deletion patterns with exactly k L's and R's appearing, all of which are in non-special positions. Further, let $F_{n,k}^{L,R}(x,y):=\sum_{n\geq 1,k\geq 0}f_{n,k}^{L,R}x^ny^k$ denote the corresponding o.g.f. Utilizing the compositional nature of valid deletion patterns, we find

$$
\begin{aligned}
F^{L,R}_{n,k}(x,y) & \\
&= \Big(A_{L,R}(x,y)\cdot\big(B^0_{L,R}(x,y)+B^1_{L,R}(x,y)+\cdots\big)\cdot A_{L,R}(x,y)+\hat{A}_{L,R}(x,y)\Big)\,x,\\
&= \frac{x^2(2x^3y^2+x^2y^2-x^3-2x^2y+3x^2-2xy-x+1)}{x^2y^2+2x^2y+x^2-2xy-3x+1}.
\end{aligned}
$$

Again we will take a partial derivative with respect to y and multiply by y to obtain an o.g.f. whose $x^n y^k$ coordinate counts the total number of L's and R's appearing in non-special positions in all length n valid deletion patterns with exactly k L's and R's in non-special position.

$$
y\frac{y}{\partial y}F_{L,R}(x,y) \;=\; \frac{2yx^5(2x^2y^2+3x^2y+x^2-8xy-5x+6)}{(x^2y^2+2x^2y+x^2-2xy-3x+1)^2}. \qquad (2.5)
$$

Lastly, we set $y=1$ in Equation (2.5) to obtain $E_{L,R}(x)$, the o.g.f. whose x^n coefficient counts the total number of L's and R's in non-special positions in all length n valid deletion patterns, and hence all the edges arising as diamond swaps on L's and R's. We compute

$$
E_{L,R}(x) \;:=\; y\frac{\partial}{\partial y}F_{L,R}(x,y)\Big|_{y=1} \;=\; \frac{2x^5(6x^2-13x+6)}{(4x^2-5x+1)^2}. \qquad (2.6)
$$

If we let e_n denote the total number of edges of $\Sigma_1(n)$ and let $E(x) := \sum_{n\geq 1} e_n x^n$ denote the corresponding o.g.f., we may put together Equations (2.4) and (2.6) and find that

$$
\begin{aligned}
E(x) &= E_{LR}(x)+E_{L,R}(x)\\
&= \frac{x^3(1-2x+16x^3-5x^4-7x^2)}{(1-5x+4x^2)^2}+\frac{2x^5(6x^2-13x+6)}{(4x^2-5x+1)^2}\\
&= \frac{x^3(1-2x-10x^3+7x^4+5x^2)}{(1-5x+4x^2)^2}.
\end{aligned}
$$

Lastly, we unpack $E(x)$ to determine a closed formula for the number of edges of $\Sigma_1(n)$. We find:

$$
\begin{aligned}
E(x) &= \frac{163}{256} + \frac{149}{256}x + \frac{15}{32}x^2 + \frac{7}{16}x^3 - \frac{19}{27(1-x)} - \frac{1745}{27648(1-4x)} + \frac{1}{9(1-x)^2} \\
&\quad + \frac{175}{9216(1-4x)^2}, \\
&= x^3 + \sum_{n\geq 4} \frac{525 \cdot 4^n n - 1220 \cdot 4^n + 3072n - 16384}{27648} x^n.
\end{aligned}
$$

This established the following theorem.

Theorem 2.5.2 *For* $|\mathbf{A}| = n \geq 4$, $\Sigma_1(n)$ *has* $\dfrac{525 \cdot 4^n n - 1220 \cdot 4^n + 3072n - 16384}{27648}$ *edges.*

2.6 Further Questions

Up to this point, we have been able to utilize the combinatorial structure to understand the vertices and edges of $\Sigma_1(n)$, but we have not yet been able to determine how to describe combinatorial conditions which indicate the structure of the higher dimensional faces. We strongly suspect that the number of essential collinearities present in a lifting determines the dimension of the face that it will support in $\Sigma_1(n)$. In particular, we have seen that a lifting supports a vertex iff it has no essential collinearities, and that it supports an edge iff it has exactly one essential collinearity. This naturally leads us to suspect that the facets of $\Sigma_1(n)$ are supported by liftings which are nontrivially maximally collinear. This means that not all points are collinear, but the addition of one more collinearity in the lifting *would* force all points to be collinear.

Example 2.6.1 If we consider $n = 6$, then the lifting $\omega = \left\langle 2, 1, 0, 1, 1, \frac{3}{2} \right\rangle$ can be seen to be maximally nontrivially collinear in Figure 2.16.

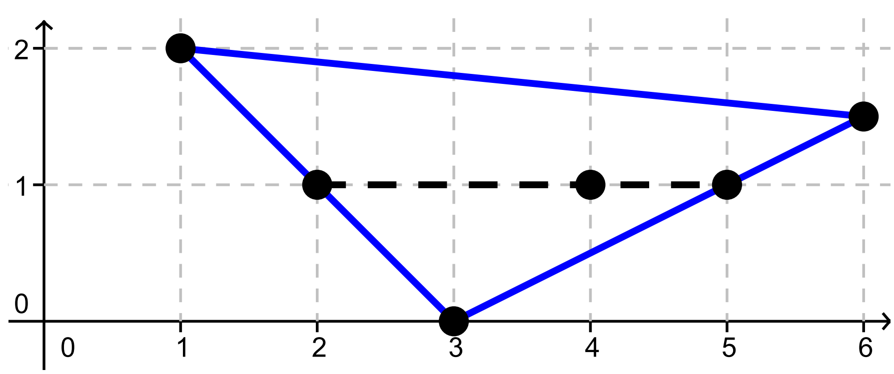


Figure 2.16: Maximally nontrivially collinear lifting

It was verified in Polymake [4] that this lifting supports a facet of $\Sigma_1(6)$ consisting of vertices with patterns

$$\begin{array}{l}(M, LR, M, E, LR, M),\\(M, LR, M, L, LR, M),\\(M, M, M, E, LR, M),\\(M, M, M, L, LR, M),\\(M, M, M, E, M, M),\\(M, M, M, L, M, M),\\(M, LR, M, E, M, M),\\(M, LR, M, L, M, M).\end{array}$$

$\diamondsuit$

When considering facets of the secondary polytope corresponding to n points on a line (combinatorially equivalent to a hypercube), it can be shown that each facet can be supported by a lifting where all but one of the points are set at height 0 and the remaining point is lifted to height 1. Example 2.6.1 shows that maximally non-trivially collinear liftings may be more convoluted in the case of 1 deletion-induced polytopes.

Another possible avenue of inquiry would be trying to determine a way to adapt the combinatorial structure assigned to 1-deletions of a line to study higher order deletions, like 2-deletions.

Additionally, it would be interesting to determine if a combinatorial structure could be placed on the 1-deletions of n-gons. The previous chapter outlined a method to construct these particular deletion-induced polytopes, but we have yet to determine a formula for the number of vertices they would contain. Unlike the combinatorial structure used for points on a line, this new structure could not be based on the "visibility" of the points of $\mathbf{A}$ in the lifting, as *all* of the points of an n-gon would be visible in *every* lifting.

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