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FLAG *F*-VECTORS OF POLYTOPES WITH FEW VERTICES

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FLAG F-VECTORS OF POLYTOPES WITH FEW VERTICES

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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Lexington, Kentucky

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ABSTRACT OF DISSERTATION

FLAG F-VECTORS OF POLYTOPES WITH FEW VERTICES

We may describe a polytope P as the convex hull of n points in space. Here we consider the numbers of chains of faces of P. The toric g-vector and CD-index of P are useful invariants for encoding this information. For a simplicial polytope P, Lee defined the winding number w_k in a Gale diagram corresponding to P. He showed that w_k in the Gale diagram equals g_k of the corresponding polytope. In this dissertation, we fully establish how to compute the g-vector for any polytope with few vertices from its Gale diagram. Further, we extend these results to polytopes with higher dimensional Gale diagrams in certain cases, including the case when all the points are in affinely general position. In the Generalized Lower Bound Conjecture, McMullen and Walkup predicted that if $g_k(P) = 0$ for some simplicial polytope P and some k, then P is $(k-1)$-stacked. Lee and Welzl independently use Gale transforms to prove the GLBC holds for any simplicial polytope with few vertices. In the context of Gale transforms, we will extend this result to nonpyramids with few vertices. We will also prove how to obtain the CD-index of polytopes dual to polytopes with few vertices in several cases. For instance, we show how to compute the CD-index of a polytope from the Gale diagram of its dual polytope when the Gale diagram is 2-dimensional and the origin is captured by a line segment.

KEYWORDS: polytopes, Gale transforms/diagrams, toric g-vector, triangulations, CD-index

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Date: April 27, 2016

FLAG F-VECTORS OF POLYTOPES WITH FEW VERTICES

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I dedicate this dissertation to Steve Kuhn and Ben Braun for encouraging me to embark on this journey and embrace the road less traveled.

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1.1 Introduction

People have long been interested in the number of faces of polytopes. Studying the faces of a given polytope allows us to better understand the polytope itself. Every convex d-polytope P can be defined as the convex hull of n vertices in $\mathbb{R}^d$. If P is d-dimensional, then P has f_i i-dimensional faces for $-1 \leq i \leq d-1$, where the empty set is the only face of dimension -1 and there are n 0-faces. For a simplicial d-polytope, the components of the h-vector satisfy $h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$ for $0 \leq k \leq d$. The symmetry of the h-vector, *i.e.*, $h_i = h_{d-i}$, is a consequence of the Dehn-Sommerville equations. The g-vector is defined by $g_0 = h_0 = 1$ and $g_i = h_i - h_{i-1}$, for $1 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$.

McMullen and Walkup stated the *generalized lower bound conjecture* (GLBC) in 1971 [23].

Conjecture 1.1.1 (McMullen-Walkup 1971)**.** *Let P be any simplicial d-polytope. Then*

1. $g_i(P) \geq 0$ *for* $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$.
2. *For* $1 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, *the following statements are equivalent.*
   - $g_k(P) = 0$;
   - P *is* $(k-1)$*-stacked; i.e., there exists a triangulation* T *of* P *for which there is no interior face of dimension less than* $d-k+1$.

In 1971 McMullen made a conjecture to characterize the sequences that are realizable as the f-vectors of simplicial polytopes [21]. There was not a complete proof of the g-conjecture until the 1980's when Billera and Lee proved the sufficiency of McMullen's conditions for f-vectors [5] and Stanley proved the necessity of these conditions [27]. With this complete proof, the statement is now called the g-theorem. The g-theorem establishes the non-negativity of the g_i for $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$, which is the first part of the GLBC. We are still interested in elementary proofs and interpretations. For a simplicial polytope P, Lee [18] interprets $g_k(P)$ as a winding number $w_k(o)$ in terms of k-splitters winding around the origin o in P's corresponding Gale diagram, which is a representation of the polytope itself. In Chapter 2, we generalize the simplicial case to some classes of convex polytopes Q. We show how to determine the g-vector of Q when all the points are in affinely general position and the origin lies in the relative interior of the convex hull of at least one k-splitter in the Gale diagram (see Theorem 2.1.1 and Corollary 2.1.5). We also show how to determine the g-vector of Q when the origin lies in the relative interior of the convex hull of at least one splitter spanned by at least $e+1$ points in affinely general position with respect to their span in the e-dimensional Gale diagram (see Theorem 2.2.9 and Corollary 2.2.12). Further, we argue how to determine the g-vector of Q when the origin coincides with at least one point in the Gale diagram (see Theorems 2.1.10 and 2.2.1). As a consequence, we establish how to determine the g-vector for any polytope with a two-dimensional Gale diagram.

When they made the GLBC, McMullen and Walkup verified that if P is $(k-1)$-stacked, then $g_k(P) = 0$ [23]. Later, Lee and Welzl independently showed that the forward implication holds for polytopes with few vertices, *i.e.*, the number of vertices is at most $d+3$ [18, 29]. Recent work by Murai and Nevo established the validity of this forward implication for all simplicial d-polytopes [25]. In Chapter 3, we extend the generalized lower bound theorem to general convex (*i.e.*, nonsimplicial) nonpyramidal polytopes Q with few vertices in the context of Gale transforms in Theorem 3.1.1 by proving that when $g_k = 0$ for such polytopes there exists a triangulation for which there is no interior face of dimension less than $d-k+1$. Our proof uses the fact that we can shift the points in the Gale transform in a certain way without changing the resulting triangulation of the polytope.

In Chapter 4, we prove how to interpret the CD-index of polytopes dual to polytopes with few vertices in several cases. For instance, we show how to compute the CD-index of a polytope from the 2-dimensional Gale diagram of its dual polytope when the origin is in the relative interior of the convex hull of a line segment (see Theorem 4.3.8). Then we conjecture how to extend this result to determine the CD-index of a polytope from the 2-dimensional Gale diagram of its dual polytope when the origin is in the relative interior of the convex hull of a line segment with multiple points on either end in Conjecture 4.3.20.

1.2 Convex Polytopes

First, we will introduce the notions of a polytope and its faces. For additional information on convex polytopes, see Grünbaum [14] and Ziegler [30].

Convex polyhedra are subsets of $\mathbb{R}^d$ that are intersections of finitely many closed halfspaces. Adding one more condition gives us the following definition of convex polytopes.

Definition 1.2.1 (cf. Ziegler 1995)**.** A *convex polytope* $P \subset \mathbb{R}^d$ is the bounded intersection of finitely many closed halfspaces.

There is an equivalent definition of convex polytopes.

Theorem 1.2.2 (cf. Ziegler 1995)**.** *Consider a subset $P \subset \mathbb{R}^d$. Then P is the convex hull of finitely many points exactly when P is the bounded intersection of finitely many closed halfspaces.*

This equivalence is the fundamental theorem for polytopes and proofs may be found in various sources, including [14] and [30]. The usefulness of both descriptions of polytopes cannot be overstated. There are some problems, as in combinatorial optimization, that are more approachable through the lens of halfspaces. There are other problems that more naturally encourage the use of the convex hull description of a polytope. Here we will use this latter definition. It is also important to note that throughout this dissertation, all polytopes are convex. Since we are always considering convex polytopes, we will often omit the term *convex*. Thus, anywhere the term *polytope* arises, *convex polytope* is understood. Throughout this dissertation, we also assume all polytopes are full dimensional; *i.e.*, a d-dimensional polytope is a subset of $\mathbb{R}^d$.

Studying the faces of polytopes helps us understand the structure of polytopes. For any polytope $P \subset \mathbb{R}^d$, the empty set and P itself are the *improper faces* of P. Every subset F of P is a *proper face* of P if $F = P \cap H$ for some supporting hyperplane H of P. Consider a 2-dimensional n-gon, P_n. Then P_n has $2n + 2$ faces, including the empty set, n vertices, n edges, and P_n itself. More generally, for any d-polytope P that has n vertices, there are n 0-dimensional faces and f_i i-dimensional faces for $1 \leq i \leq d - 1$. The empty set and P itself are also faces of dimension -1 and d, respectively. We say the 0-faces are *vertices*, the 1-faces are *edges*, the $(d-2)$-faces are *subfacets*, and the $(d-1)$-faces are *facets*.

Every d-polytope P has a dual d-polytope P^*, which is defined as follows.

Definition 1.2.3 (cf. Ziegler 1995)**.** Consider d-polytopes P and P^*. Then P and P^* are *dual polytopes* exactly when there is an inclusion inversing bijection between their face lattices.

For example, every facet of P corresponds to a vertex of P^*. Also every vertex of P corresponds to a facet of P^*. Notice that P_n is its own dual. Now consider the following 3-dimensional polytopes, which are dual to one another.

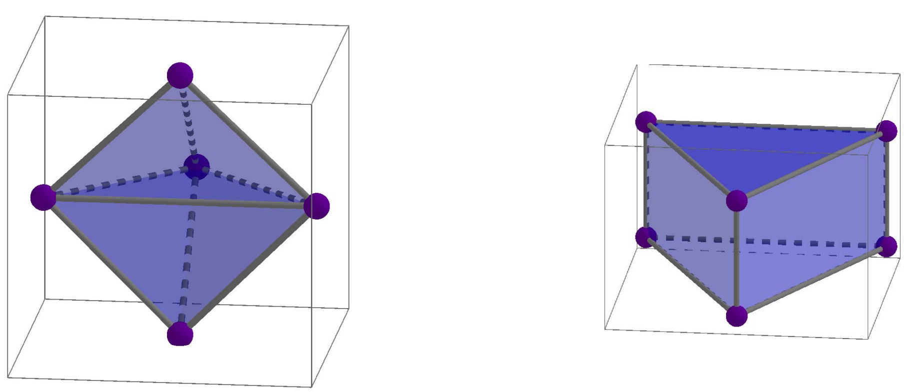

Figure 1.1: The triangular bipyramid and the triangular prism are dual 3-polytopes.

Example 1.2.4. In Figure 1.1, the triangular bipyramid is on the left and the triangular prism is on the right. It is obvious that the vertices of the bipyramid correspond to the facets of the triangular prism, and the facets of the bipyramid correspond to the vertices of the triangular prism.

We may use the above notions to introduce some important classes of polytopes. Simplicial polytopes form a well behaved class of convex polytopes. Every face of a d-dimensional *simplicial polytope* P is a simplex; in other words, for $0 \leq i \leq d-1$, each i-face of P consists of exactly $i+1$ vertices. On the other hand, every vertex of a *simple* d-dimensional polytope P is adjacent to exactly d facets. There is an important relationship between simplicial polytopes and simple polytopes, as stated in the following theorem.

Theorem 1.2.5 (cf. Brøndsted 1983)**.** *Consider dual d-polytopes P and P^*. Then P is simplicial exactly when P^* is simple.*

The proof of this result is straightforward and may be found in [7] and [30]. Let us now consider a simplicial 3-polytope, the octahedron, and its dual 3-polytope, the cube.

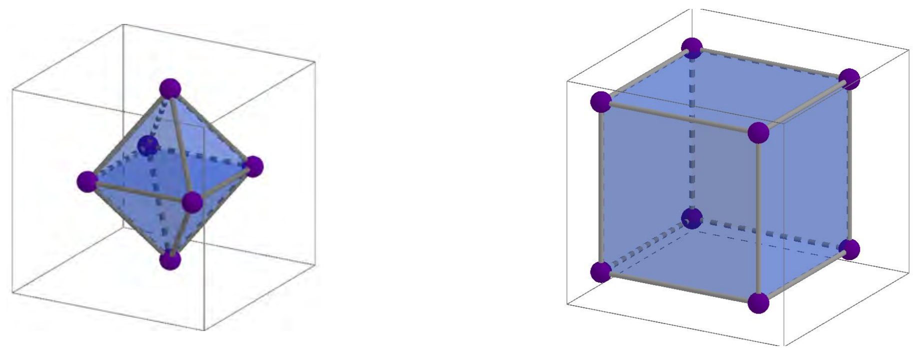

Figure 1.2: The octahedron and the cube are also dual polytopes.

Example 1.2.6. Every facet of the octahedron is a triangle, so the octahedron is a simplicial polytope. Every vertex in the cube is adjacent to three facets, so the cube is a simple polytope. Further, as Figure 1.2 illustrates, the octahedron and the cube are dual polytopes.

We are interested in studying the faces of polytopes. The f-vector is one useful tool for encoding the number of faces of a polytope in each dimension. For a d-polytope P, we have the following definition.

Definition 1.2.7 (cf. Ziegler 1995)**.** The f-vector of P, $f(P)$, is defined as $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where f_{-1} is always 1 and f_i denotes the number of i-dimensional faces of P, for $0 \leq i \leq d-1$.

We can determine the f-vectors for our examples above as follows.

Example 1.2.4 continued. The f-vector of the triangular bipyramid is $f(\text{triangular bipyramid}) = (1, 5, 9, 6)$, and the f-vector of the triangular prism is $f(\text{triangular prism}) = (1, 6, 9, 5)$.

Example 1.2.6 continued. The f-vector of the octahedron is $f(\text{octahedron}) = (1, 6, 12, 8)$, and the f-vector of the cube is $f(\text{cube}) = (1, 8, 12, 6)$.

As these examples show, the relationship between a polytope and its dual is reflected in their f-vectors. For any d-polytope P and its dual polytope P^*, $f_i(P) = f_{d-1-i}(P^*)$ for $0 \leq i \leq d-1$.

We will now explore further properties of polytopes.

1.3 The h- and g-vectors

Here we consider one way to encode some information about the chains of faces in a polytope.

In the previous section, we saw that the f-vector encodes information about a polytope's faces. The toric h- and g-vectors also encode information concerning the numbers of chains of faces of a general convex polytope. Stanley first introduced the following recursive definition of the toric h- and g-vectors in [26].

Definition 1.3.1 (Stanley 1987)**.** Let P be any d-polytope. Then we define the *toric h-vector* and the *toric g-vector* recursively using the following relations

$$h(\partial P, x) = \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)}$$

$$h(\partial P, x) = \sum_{i=0}^{d} h_i x^{d-i}$$

$$g(\partial P, x) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} g_i x^i,$$

where $g(\varnothing, x) = h(\varnothing, x) = 1$, $g_0 = g_0(\partial P) = h_0$, and $g_i = g_i(\partial P) = h_i - h_{i-1}$ for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Using this definition, we will compute the toric h- and g-vectors for some interesting examples. We will also highlight a few useful results. First, we will find the toric h- and g-vectors of 3-dimensional pyramids and bipyramids. Since these equations are defined recursively, we need the toric g- and h-vectors of all the lower dimensional polytopes which are faces of 3-dimensional pyramids and bipyramids. Let us start with the 0-polytope, which is the convex hull of a single point.

Example 1.3.2. When P is the 0-polytope, the only face of ∂P is the empty set of dimension -1. So

$$h(\partial P, x) = g(\partial P, x)(x-1)^{0-1-(-1)} = g(\varnothing, x) = h(\varnothing, x) = 1.$$

$$\begin{aligned} h(x) &= 1, \quad g(x) = 1 \\ h &= (1), \quad\ \ g = (1) \end{aligned}$$

Figure 1.3: Here is the line segment, a 1-dimensional polytope P.

Since we have the toric g-vector for the empty set and single points, then we may determine the toric h- and g-vectors of the 1-dimensional line segment (Figure 1.3).

Example 1.3.3. When P is the line segment, the faces of ∂P are one empty set and two points, of dimensions -1 and 0, respectively. So

$$\begin{aligned} h(\partial P, x) &= 1 \cdot g(\partial \varnothing)(x-1)^{1-1-(-1)} + 2 \cdot g(\partial \text{ point }, x)(x-1)^{1-1-0} \\ &= (x-1) + 2(1) \\ &= x+1. \end{aligned}$$

$$\begin{array}{ll} h(x) = x+1, & g(x) = 1 \\ h = (1,1), & g = (1) \end{array}$$

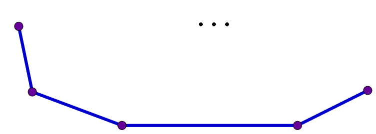

Figure 1.4: Here is the n-gon, a 2-dimensional polytope P.

Using these two examples, we may now compute the toric h- and g-vectors of a 2-dimensional n-gon (Figure 1.4).

Example 1.3.4. When P is an n-gon, the faces of ∂P are one empty set, n points, and n line segments, of dimensions -1, 0, and 1, respectively. So

$$\begin{aligned} h(\partial P, x) &= 1 \cdot g(\partial \varnothing)(x-1)^{2-1-(-1)} + n \cdot g(\partial \text{ point}, x)(x-1)^{2-1-0} \\ &\quad + n \cdot g(\partial \text{ line segment}, x)(x-1)^{2-1-1} \\ &= 1(1)(x-1)^2 + n(1)(x-1) + n(1)(1) \\ &= x^2 - 2x + 1 + nx - n + n \\ &= x^2 + (n-2)x + 1. \end{aligned}$$

$$\begin{array}{ll} h(x) = x^2 + (n-2)x + 1, & g(x) = 1 + (n-3)x \\ h = (1, n-2, 1), & g = (1, n-3) \end{array}$$

In the following two examples, we will find the toric h- and g-vectors for the 3-dimensional pyramid (Figure 1.5) and bipyramid (Figure 1.6) over a general n-gon.

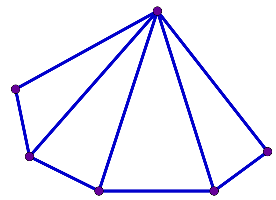

Figure 1.5: Here is the pyramid Q over an n-gon.

Example 1.3.5. Let Q be a pyramid over an n-gon. Then the faces of ∂Q are one empty set, $n+1$ points, $2n$ line segments, n triangles, and one n-gon, of dimensions -1, 0, 1, 2, and 2, respectively. So

$$\begin{aligned} h(\partial Q, x) &= 1((1)(x-1)^{3-1-(-1)}) + (n+1)((1)(x-1)^{3-1-0}) + 2n((1)(x-1)^{3-1-1}) \\ &\quad + n((1)(x-1)^{3-1-2}) + 1((1+(n-3)x)(x-1)^{3-1-2}) \\ &= 7(x-1)^3 + (n+1)(x-1)^2 + 2n(x-1) + n + 1 + (n-3)x \\ &= x^3 - 3x^2 + 3x - 1 + (n+1)(x^2 - 2x + 1) + 2nx - 2n + n + 1 + nx - 3x \\ &= x^3 + (n-2)x^2 + (n-2)x + 1. \end{aligned}$$

$$\begin{array}{ll} h(x) = x^3 + (n-2)x^2 + (n-2)x + 1, & g(x) = 1 + (n-3)x \\ h = (1, n-2, n-2, 1), & g = (1, n-3) \end{array}$$

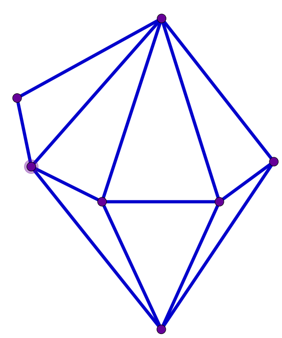

Figure 1.6: Here is the bipyramid R over an n-gon.

Example 1.3.6. Let R be a bipyramid over an n-gon. Then the faces of ∂R are one empty set, $n+2$ points, $3n$ line segments, and $2n$ triangles, of dimensions -1, 0, 1, and 2, respectively. So

$$\begin{aligned} h(\partial R, x) &= 1((1)(x-1)^{3-1-(-1)}) + (n+2)((1)(x-1)^{3-1-0}) \\ &\quad + 3n((1)(x-1)^{3-1-1}) + 2n((1)(x-1)^{3-1-2}) \\ &= (x-1)^3 + (n+2)(x-1)^2 + 3n(x-1) + 2n \\ &= x^3 - 3x^2 + 3x - 1 + (n+2)(x^2 - 2x + 1) + 3nx - 3n + 2n \\ &= x^3 + (n-1)x^2 + (n-1)x + 1. \end{aligned}$$

$$h(x) = x^3 + (n-1)x^2 + (n-1)x + 1, \quad g(x) = 1 + (n-3)x$$
$$h = (1, n-1, n-1, 1), \qquad g = (1, n-2)$$

As these examples illustrate, there is a relationship between the toric h- and g-vectors of a 2-dimensional n-gon and the toric h-vectors of the pyramid and bipyramid over that n-gon. In particular, $h(\partial Q, x) = xh(\partial P, x) + g(\partial P, x)$ and $h(\partial R, x) = (x+1)h(\partial P, x)$. As the next theorem states, these relationships hold for any d-polytope P and the pyramid Q and the bipyramid R over P. The proof involves easy computations and the results were observed, for instance, by Fine [13].

Theorem 1.3.7 (cf. Fine 2010)**.** *Let $P \in \mathbb{R}^d$ be any polytope. Let Q and R denote the pyramid and bipyramid, respectively, over P. Then $h(\partial Q, x) = xh(\partial P, x) + g(\partial P, x)$ and $h(\partial R, x) = (x+1)h(\partial P, x)$.*

First note that $h(\partial Q, x) = xh(\partial P, x) + g(\partial P, x)$ if and only if $g(\partial Q, x) = g(\partial P, x)$. Thus, we will prove the first statement by proving this equivalent statement $g(\partial Q, x) = g(\partial P, x)$ using an inductive argument on the dimension of faces.

In the following proof, pyr (F) denotes the pyramid over a face F on the boundary of P.

Proof. For any polytope P, we know

$$h(\partial P, x) = \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)}$$

$$= \sum_{i=0}^{d} h_i x^{d-i}$$

when $\dim(P) = d$ and $P \in \mathbb{R}^d$. Since Q and R denote the pyramid and bipyramid over P, respectively, then P and Q are both in $\mathbb{R}^{d+1}$.

Let us first consider the base case. From Example 1.3.2, we know that $g(\partial\text{point}, x) = 1$. The pyramid over a point is a line, and we found that $g(\partial\text{line segment}, x) = 1$. Thus, the g-vector of a point is equivalent to the g-vector of the pyramid over a point. Now suppose that the g-vector of $\partial\text{pyr}\,(F)$ is the same as the g-vector of ∂F for all F whose dimension is less than or equal to $d-1$.

In the case of Q, ∂Q contains ∂P, P itself, and pyr (F) for all F in ∂P. Then

$h(\partial Q, x)$

$$\begin{aligned}
&= \sum_{G \text{ face of } \partial Q} g(\partial G, x)(x-1)^{(d+1)-1-\dim(G)} \\
&= \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{(d-1-\dim(G))+1} + g(\partial P, x)(x-1)^{(d+1)-1-\dim(P)} \\
&\quad + \sum_{G=\text{pyr}(F): F \text{ face of } \partial P} g(\partial G, x)(x-1)^{(d+1)-1-\dim(G)} \\
&= (x-1) \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)} + g(\partial P, x)(x-1)^{d+1-1-d} \\
&\quad + \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{(d+1)-1-(\dim(G)+1)} \text{ by the induction assumption} \\
&= (x-1)h(\partial P, x) + g(P, x) + h(\partial P, x) \\
&= xh(\partial P, x) + g(\partial P, x).
\end{aligned}$$

In the case of R, ∂R contains ∂P and two sets of pyr (F) for all F in ∂P. Then $h(\partial R, x)$

$$\begin{aligned}
&= \sum_{G \text{ face of } \partial R} g(\partial G, x)(x-1)^{(d+1)-1-\dim(G)} \\
&= \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{(d-1-\dim(G))+1} \\
&\quad +2 \sum_{G=\text{pyr}(F): F \text{ face of } \partial P} g(\partial G, x)(x-1)^{(d+1)-1-\dim(G)} \\
&= (x-1) \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)} \\
&\quad +2 \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{(d+1)-1-(\dim(G)+1)} \text{ by the induction assumption} \\
&= (x-1)h(\partial P, x) + 2h(\partial P, x) \\
&= (x+1)h(\partial P, x).
\end{aligned}$$

■

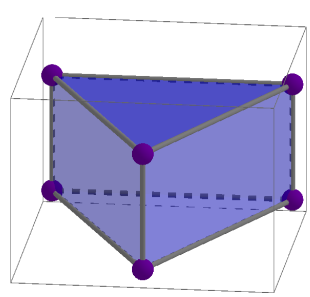

Figure 1.7: Here is the triangular prism.

Our earlier examples for low-dimensional polytopes provide us with enough information to compute the toric h- and g-vectors of the 3-dimensional triangular prism

(Figure 1.7). Since the triangular prism is a nonsimplicial polytope, we will discuss it in further detail throughout this dissertation.

Example 1.3.8. In Example 1.2.4, we determined the f-vector of the triangular prism. Now we will determine its toric h- and g-vectors. The h-vector is

$$\begin{aligned}
h(\partial P, x) &= 1((1)(x-1)^{3-1-(-1)}) + 6((1)(x-1)^{3-1-0}) + 9((1)(x-1)^{3-1-1}) \\
&\quad +3((1+x)(x-1)^{3-1-2}) + 2((1)(x-1)^{3-1-2}) \\
&= (x-1)^3 + 6(x-1)^2 + 9(x-1) + 3(1+x) + 2 \\
&= x^3 - 3x^2 + 3x - 1 + 6x^2 - 12x + 6 + 9x - 9 + 3 + 3x + 2 \\
&= x^3 + 3x^2 + 3x + 1.
\end{aligned}$$

$$\begin{array}{ll}
h(x) = x^3 + 3x^2 + 3x + 1, & g(x) = 1 + 2x \\
h = (1, 3, 3, 1), & g = (1, 2)
\end{array}$$

As we saw in Example 1.2.4, the bipyramid over a triangle is the dual of the triangular prism. Thus, knowing the toric g-vector of the triangular bipyramid is useful as well.

Example 1.3.9. From Example 1.3.6, we know that the g-vector of a n-gon bipyramid is $(1, n-2)$. Since the triangle is a 3-gon, the h- and g-vectors of the bipyramid over a triangle are $(1, 2, 2, 1)$ and $(1, 1)$, respectively.

Historically, h- and g-vectors were first defined in the simplicial (and dually simple) case. So we want to explore the relationship between the toric h- and g-vectors and the ordinary h- and g-vectors here.

For a simplicial d-polytope, the h-vector is defined by the following alternating sum.

Definition 1.3.10 (cf. Ziegler 1995)**.** For a simplicial d-polytope and $0 \le k \le d$, the k^{th} component of the h-vector is $h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$.

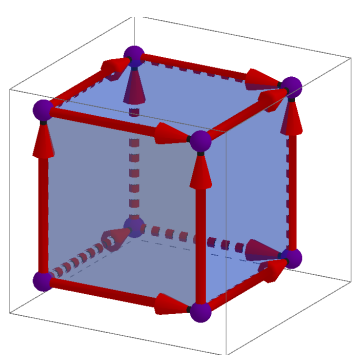

Figure 1.8: Orient the edges of the cube in the direction of the sweep.

The symmetry of the h-vector, *i.e.* $h_i = h_{d-i}$, is a consequence of the Dehn-Sommerville equations. From [7], for example, we know that we are able to determine the ordinary h-vector of a simplicial polytope by sweeping its dual. Here, we will compute the h-vector of the octahedron by sweeping the cube. To sweep a polytope, choose a hyperplane in general position so that it crosses each vertex at a different time as it moves parallel to itself across the polytope. Orient the edges in the direction of the sweep, as illustrated in Figure 1.8. Count the out-, or equivalently the in-, degree at each of its vertices as the plane 'sweeps' the polytope. Through this process, we see that one vertex has in-degree 0, three vertices have in-degree 1, three vertices have in-degree 2, and one vertex has in-degree 3. So the h-vector of the octahedron is $(1, 3, 3, 1)$ and, subsequently, the g-vector of the octahedron is $(1, 2)$. Now we will use the information we found in our earlier examples to compute the toric h- and g-vectors of the octahedron.

Example 1.3.11. We first considered the octahedron P in Example 1.2.6. Notice that P is a bipyramid over a 4-gon, the square. So we can use our results from Example 1.3.6 to find the following information.

$$\begin{array}{ll} h(x) = x^3 + 3x^2 + 3x + 1, & g(x) = 1 + 2x \\ h = (1, 3, 3, 1), & g = (1, 2) \end{array}$$

Notice that the simplicial h-vector of the octahedron matches its toric h-vector. This equality is not a coincidence. As Stanley proved in [26], the two h-vectors always match when P is a simplicial polytope. We will verify that this equality holds for a d-simplex and a simplicial d-polytope.

Example 1.3.12. Let P be a d-simplex. From sweeping, we know $h(x) = x^d + x^{d-1} + \ldots + x + 1 = \dfrac{x^{d+1} - 1}{x - 1}$. Then $h = (\underbrace{1, 1, \ldots, 1}_{d+1})$ and $g = (\underbrace{1, 0, \ldots, 0}_{\lfloor \frac{d}{2} \rfloor + 1})$. The toric h-vector should match.

$$\begin{aligned} h(\partial P, x) &= \sum_{j=-1}^{d-1} \binom{d+1}{j+1} (1)(x-1)^{d-1-j} \\ &= \textstyle\sum_{j=0}^{d} \displaystyle\binom{d+1}{j} (x-1)^{d-j} \\ &= (\textstyle\sum_{j=0}^{d+1} \displaystyle\binom{d+1}{j+1} (1)(x-1)^{d-j}) - \frac{1}{x-1} \\ &= \frac{(x-1+1)^{d+1}}{x-1} - \frac{1}{x-1} \\ &= \frac{x^{d+1} - 1}{x - 1}, \end{aligned}$$

which matches the h-vector obtained from sweeping.

Since it will prove to be very useful in future calculations, we will reiterate the following well known result.

Lemma 1.3.13. *For a simplex of arbitrary dimension, $g(\partial simplex, x) = 1$.*

Now we compute the h- and g-vectors for an arbitrary d-dimensional simplicial polytope P.

Example 1.3.14. Let P be a simplicial d-polytope. Then ∂P is a $(d-1)$-dimensional simplicial complex with f-vector $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where f_i denotes the number of i-dimensional faces. So

$$\begin{aligned} h(\partial P, x) &= \sum_{G \text{ face of } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)} \\ &= \sum_{i=-1}^{d-1} f_i (x-1)^{d-1-i} \\ &= \sum_{i=-1}^{d-1} f_i \sum_{j=0}^{d-1-i} \binom{d-1-i}{j} x^j (-1)^{d-1-i-j} \text{ by the Binomial Theorem.} \end{aligned}$$

Then the coefficient of x^{d-k} is:

$$\begin{aligned} &= \sum_{i=-1}^{d-1} f_i \binom{d-1-i}{d-k} (-1)^{d-1-i-(d-k)} \\ &= \sum_{i=0}^{d} f_{i-1} \binom{d-1-(i-1)}{d-k} (-1)^{-1-(i-1)+k} \\ &= \sum_{i=0}^{d} f_{i-1} \binom{d-i}{d-k} (-1)^{k-i}. \end{aligned}$$

We know $h_k = \sum_{j=0}^{k} f_{j-1} \binom{d-j}{d-k} (-1)^{k+j}$, which is equivalent to the previous line.

Consider two simplicial polytopes P and Q on disjoint vertex sets. Then (as sets of vertices) $\partial P * \partial Q = \{S \cup T | S \subset \partial(P), T \subset \partial(Q)\}$ denotes the simplicial join of P and Q. It is well known, and an easy computation to show, that the following holds.

Lemma 1.3.15. *For simplicial polytopes P and Q on disjoint vertex sets, $h(\partial P * \partial Q, x) = h(\partial P, x) \cdot h(\partial Q, x)$.*

A sequence that is nondecreasing to a certain term and then nonincreasing afterward is said to be *unimodal.* Stanley showed that the ordinary h-vector is unimodal [27]. Then Karu proved that the toric h-vector also behaves this way [15].

Theorem 1.3.16 (Karu 2004)**.** *The toric h-vector of any polytope Q is unimodal. Thus, the g-vector is nonnegative.*

We will use this fact when determining toric g-vectors from Gale diagrams in Chapter 2, thus extending some results of Lee [18] and Welzl [29].

1.4 The ab-, cd- and CD-indices

In addition to the f-vector and the toric h- and g-vectors, there are other ways to encode information about the numbers of faces of a polytope in each dimension. In this section, we will introduce three indices. It is common to use the ab-index and cd-index to encode information about numbers of chains of faces in polytopes. Alternatively, we may use the CD-index to encode the same information. Thus, we will focus on this last index. First, we will introduce and explore the ab- and cd-indices. For more information on the cd-index, see for example [28].

Let P be any d-polytope. Then the *flag f-vector* of P encodes the number of chains of faces of various types $S = \{S_1 < S_2 < \cdots < S_k\} \subseteq \{0, 1, \dots, d-1\}$. We define the flag f-vector more formally below.

Definition 1.4.1 (cf. Stanley 2012)**.** For a subset $S \subseteq \{0, 1, \dots, d-1\}$, an *$S$-flag* of P is a chain $\emptyset \subset F_1 \subset F_2 \subset \ldots \subset F_k \subset P$ with $\dim(F_i)=S_i$ for $i = 1, \dots, k$. Set $f(P)$ equal to the number of S-flags of P. Then $(f(P))_{S\subseteq\{0,1,\dots,d-1\}}$ is the *flag f-vector* of P.

The flag f-vector is an extension of the f-vector. In fact, all the single element subsets encode the f-vector itself. Thus, these entries of the flag f-vector are identical to the f-vector. Since we already found the f-vectors of the triangular bipyramid and the octahedron, we will now find their flag f-vectors.

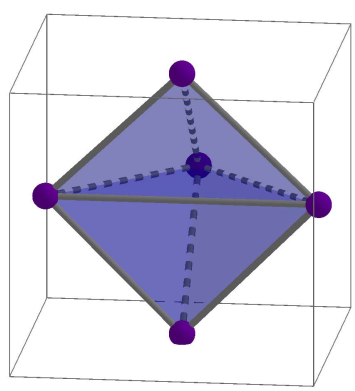

Figure 1.9: Triangular bipyramid

Example 1.3.9 continued. We are considering the triangular bipyramid P (Figure 1.9). From Example 1.2.4, $f(P) = (1, 5, 9, 6)$. As the picture illustrates, two vertices are incident to three edges and the other three vertices are incident to four edges. So the number of chains consisting of a 0-face and a 1-face is $f_{01} = 2(3) + 3(4) = 18$. Similarly, $f_{02} = 2(3) + 3(4) = 18$ and $f_{12} = 9(2) = 18$. It is also easy to see that $f_{012} = 2(3)(2) + 3(4)(2) = 36$. Therefore, $(f(P)) = (1, 5, 9, 6, 18, 18, 18, 36)$.

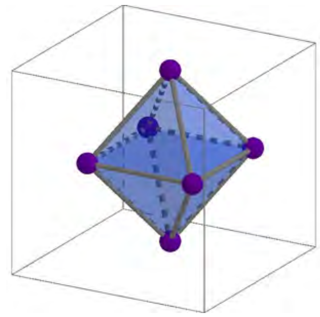

Figure 1.10: Octahedron

Example 1.3.11 continued. We are considering the octahedron P here (Figure 1.10). So $(f(P)) = (1, 6, 4 \cdot 3, 4 \cdot 2, 6 \cdot 4, 6 \cdot 4, 12 \cdot 2, 6 \cdot 4 \cdot 2) = (1, 6, 12, 8, 24, 24, 24, 48)$.

It is easy to see that every polytope has a flag f-vector. The information encoded by the flag f-vector is used to determine the ab-index for any polytope.

Definition 1.4.2 (Stanley 2012)**.** The *flag h-vector* of a polytope P is defined by the flag f-vector through the following relationship,

$$h(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P).$$

For $S \subseteq \{0, 1, \ldots, d-1\}$, let

$$w_i = \begin{cases} a & \text{if } i \notin S \\ b & \text{if } i \in S \end{cases}.$$

Then the *ab-index* of P is

$$\Psi(P) = \sum_{S \subseteq \{0,1,\ldots,d-1\}} h_S(P) w_0 w_1 \cdots w_{n-1},$$

where a and b are noncommuting variables.

Using an alternating sum based on set inclusion, we may determine the flag h-vector from the flag f-vector. Then we may determine the ab-index. This polynomial in the non-commutative variables a and b encapsulates information about the number of chains of faces. So let's find the flag h-vector and ab-index for both the triangular bipyramid and the octahedron.

Example 1.3.9 continued. For the triangular bipyramid P (see Figure 1.9), we have the following.

w_S	f_S	h_S	$(a+b)^3$	$(a+b)(ab+ba)$	$(ab+ba)(a+b)$
aaa	1	1	1		
baa	5	4	1		3
aba	9	8	1	4	3
aab	6	5	1	4	
bba	18	5	1	4	
bab	18	8	1	4	3
abb	18	4	1		3
bbb	36	1	1		

So $h(P) = (1, 4, 8, 5, 5, 8, 4, 1)$ and the ab-index of P is $\Psi(P) = 1a^3 + 4ba^2 + 8aba + 5a^2b + 5b^2a + 8bab + 4ab^2 + 1b^3$.

Example 1.3.11 continued. For the octahedron P (see Figure 1.10), we have the following.

w_S	f_S	h_S	$(a+b)^3$	$(a+b)(ab+ba)$	$(ab+ba)(a+b)$
aaa	1	1	1		
baa	6	5	1		4
aba	12	11	1	6	4
aab	8	7	1	6	
bba	24	7	1	6	
bab	24	11	1	6	4
abb	24	5	1		4
bbb	48	1	1		

So $h(P) = (1, 5, 11, 7, 7, 11, 5, 1)$ and the ab-index of P is $\Psi(P) = 1a^3 + 5ba^2 + 11aba + 7a^2b + 7b^2a + 11bab + 5ab^2 + 1b^3$.

As the tables in these two examples indicate, there is another index which encodes this information more concisely than the ab-index. Bayer and Klapper first introduced the cd-index in [3]. Our tables above suggest their definition.

Definition 1.4.3 (Bayer-Klapper 1991)**.** Define $c = a + b$ and $d = ab + ba$. For the non-commutative variables c and d, define the cd-index $\Phi(P)$ by rewriting the ab-index in terms of c and d, which can be done by the result of Bayer and Klapper.

Note that $c^3 = a^3 + ba^2 + aba + a^2b + b^2a + bab + ab^2 + b^3$, $cd = aba + a^2b + b^2a + bab$, and $dc = ba^2 + aba + bab + ab^2$. So we have the cd-index for both polytopes above.

Example 1.3.9 continued. For the triangular bipyramid P, the cd-index is $\Phi(P) = 1c^3 + 4cd + 3dc$.

Example 1.3.11 continued. For the octahedron P, the cd-index is $\Phi(P) = 1c^3 + 6cd + 4dc$.

The CD-index, which is equivalent to the cd-index, is yet another useful tool for encoding information about a polytope's faces introduced by Fine [12, 13]. Start by building up polytopes P from a point using pyramids, via applying C, and using prisms, via applying I.

Definition 1.4.4 (Fine 1995)**.** Let P be a polytope. We define CP as the pyramid, or cone, over P; and we define IP as the prism over P.

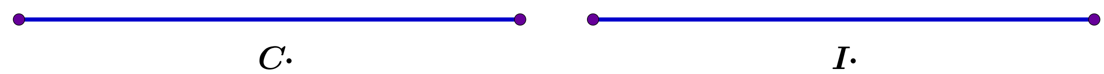

Figure 1.11: Here are equivalent ways to construct a line segment.

We make the following simple, but useful, observation.

Note 1.4.5. Figure 1.11 shows that a pyramid over a point is combinatorially equivalent to a prism over a point.

Thus, initially, we may apply either C or I. We may use different CI-sequences to build up certain polytopes from a point. In the following example, we will practice building up the tetrahedron.

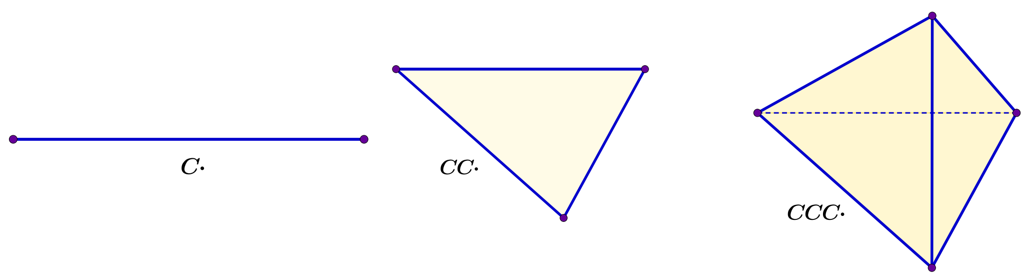

Figure 1.12: Here is one way to build the tetrahedron.

Example 1.4.6. We will build the tetrahedron from a point. For the first step, we may build a cone over a point. Then we build a cone over this edge. We end by building a cone over this triangle. As Figure 1.12 shows, we may build the tetrahedron using C^3.

Since building a prism over a point is equivalent to building a cone over a point, both C^3 and C^2I are valid IC-words used to describe the construction of the tetrahedron. In the third step, applying I rather than C builds a prism over a triangle rather than a pyramid over a triangle. Thus, we can also build the triangular prism from a point as shown below.

Example 1.3.8 continued. Consider the triangular prism P in Figure 1.7. As the picture shows, IC^2 is one IC-word which constructs P.

In Example 1.2.6, we noted that the cube is a simple polytope dual to the octahedron. We will determine an IC word for the cube below.

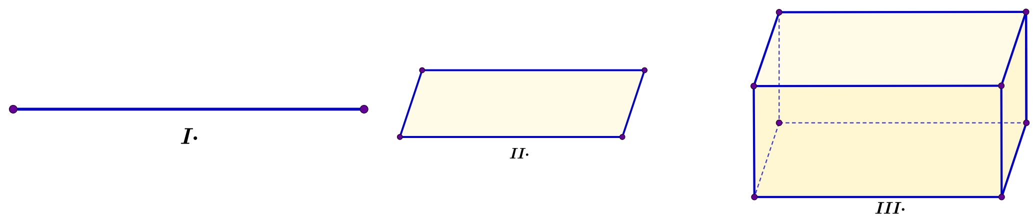

Figure 1.13: Here is one way to build the cube.

Example 1.4.7. We want to construct the cube Q as shown in Figure 1.13. Taking a prism over a point builds one edge. Taking a prism over the edge builds a square. Taking a prism over a square builds a cube. So I^3 is one IC-word of Q.

Unfortunately, it is not always possible to build up a polytope from a point using only the geometric operations C (pyramid) and I (prism) in a single word. For instance, neither the triangular bipyramid from Example 1.3.9 nor the octahedron from Example 1.3.11 may be constructed using only these operations. At the first step, applying C or I builds a line segment in $\mathbb{R}$. Applying the next operation builds a polytope in $\mathbb{R}^2$ and so on. So it is only possible to build 2^{d-1} polytopes of dimension d.

We may regard C and I as operations on flag vectors rather than regarding them as geometric operations. In other words, a CI-word w is equated to the flag f-vector of the polytope w(point). In [1, 2], Bayer and Billera proved that $F_d - 1$ is the dimension of the vector space spanned by flag f-vectors of d-polytopes, where F_d is the d^{th}-Fibonacci number ($F_0 = F_1 = 1$ and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$). Moreover, they showed that applying d-combinations of the pyramid and bipyramid operations to a point spans the flag vectors of d-polytopes [2]. Dually, CI-words of length d span the space of flag f-vectors [12]. But there are relations among these words. Through truncation, Fine shows the following result [12].

Theorem 1.4.8 (Fine 1995)**.** *For any polytope P, $IICP = ICCP + ICIP - CCIP$ (as a flag f-vector identity).*

Successively applying this relation and the fact that $I\cdot = C\cdot$, Fine defines a unique (C, IC)-polynomial for (the flag f-vector of) a polytope P. So let us revisit our example from earlier.

Example 1.4.7 continued. Above, we found that I^3 is an (I, C)-word for the cube Q. Use Note 1.4.5 to rewrite I^3 as I^2C. Now we may apply Fine's result to rewrite I^2C as $IC^2 + ICI - C^2I$. Then we apply Note 1.4.5 again to rewrite this polynomial as $2IC^2 - C^3$. So $2(IC)C - CCC$ is the unique (C, IC)-polynomial for Q.

Fine introduced the CD-index of a polytope with the following definition [13].

Definition 1.4.9 (Fine 1995)**.** Define $D = IC - CC$.

Note that C is regarded as having degree 1 and D as having degree 2. When determining the CD-index, the following theorem, which is equivalent to Theorem 1.4.8, is useful [12].

Theorem 1.4.10 (Fine 1995)**.** *For any polytope* P, $ID = DI$.

These relations allow us to compute the CD-index from (C, IC)-polynomials or IC-polynomials. By eliminating IC terms in the (C, IC) basis, the CD-index is a more concise way to encode information about a polytope. We quickly determine the CD-index for the cube and triangular prism.

Example 1.4.7 continued. Recall that the (C, IC)-polynomial for the cube Q is $2(IC)C - CCC$. So $2(D + CC)C - CCC = 2DC + CCC$ is the CD-index of Q.

Example 1.3.8 continued. For the triangular prism P, IC^2 is the (C, IC)-word. Thus, the DC-index of P is $DC + C^3$.

Recall that we were not able to determine the IC-word for the triangular bipyramid or octahedron through construction. So we need to regard C and I as operations on flag vectors. In [11], Ehrenborg and Readdy showed how the *cd*-index changes under the geometric operations C (pyramid) and I (prism). As a result, the CD-index of a polytope can be converted into its *cd*-index. Conversely, Lee [18] argued that setting $c = 2C - I$ and $d = IC - CI$ allows one to convert from the *cd*-index to the CD-index via an intermediate IC-polynomial.

Before finding the CD-index of the triangular bipyramid or octahedron, let's use these relationships to rewrite the *cd*-basis in terms of the CD-basis via the (C, IC)-basis.

We have

$$
\begin{aligned}
c^3 &= (2C - I)^3 \\
&= 8C^3 - 4C^2I - 4CIC + 2CI^2 - 4IC^2 + 2ICI + 2I^2C - I^3 \\
&= 4C^3 - 2CIC - 2IC^2 + I^2C \\
&= 4C^3 - 2C(D + C^2) - 2(D + C^2)C + I(D + C^2) \\
&= 4C^3 - 2CD - 2C^3 - 2DC - 2C^3 + ID + IC^2 \\
&= -2CD - 2DC + DI + (D + C^2)C \\
&= -2CD - 2DC + DC + DC + C^3 \\
&= -2CD + C^3.
\end{aligned}
$$

We also have

$$\begin{aligned}
cd &= 2CIC - 2C^2I - I^2C + ICI \\
&= 2CIC - 2C^3 - I^2C + IC^2 \\
&= 2C(D + C^2) - 2C^3 - I(D + C^2) + (D + C^2)C \\
&= 2CD + 2C^3 - 2C^3 - ID - IC^2 + DC + C^3 \\
&= 2CD - DI - (D + C^2)C + DC + C^3 \\
&= 2CD - DC - DC - C^3 + DC + C^3 \\
&= 2CD - DC.
\end{aligned}$$

Additionally,

$$\begin{aligned}
dc &= 2IC^2 - ICI - 2CIC + CI^2 \\
&= IC^2 - CIC \\
&= (D + C^2)C - C(D + C^2) \\
&= DC + C^3 - CD - C^3 \\
&= DC - CD.
\end{aligned}$$

Now we will quickly determine the CD-index for the following examples.

Example 1.3.9 continued. Recall that for the triangular bipyramid P, the cd-index is $\Phi(P) = 1c^3 + 4cd + 3dc$. Then the CD-index is

$$1(-2CD + C^3) + 4(2CD - DC) + 3(DC - CD)$$

$$= C^3 + 3CD - DC.$$

Example 1.3.11 continued. The cd-index of the octahedron P is $\Phi(P) = 1c^3+6cd+4dc$. So the CD-index is

$$1(-2CD + C^3) + 6(2CD - DC) + 4(DC - CD)$$

$$= C^3 + 6CD - 2DC.$$

Here we are going to highlight a couple of important results concerning the CD-index. The first result is proven by various people, including Fine and Lee [13, 16].

Theorem 1.4.11 (cf. Fine 2010, Lee 2016)**.** *For a simplicial d-polytope P, the CD-index of its (simple) dual P^* is of the form*

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} g_i(P) D^i C^{d-2i}.$$

The other theorem is a direct consequence of the geometric definition of C and the fact that $(CP)^* = CP^*$.

Theorem 1.4.12. *If Q is a k-fold pyramid over P, then the CD-index of Q^* is the CD-index of P^* premultiplied by C^k; i.e.,*

$$\psi(Q^*) = C^k \cdot \psi(P^*).$$

The following result by Fine allows us to determine the g-vector of a polytope Q from the CD-index of its dual Q^* [13].

Theorem 1.4.13 (Fine 2010)**.** *For a d-polytope Q, say $\psi(Q^*) = w(C, D)$ is the CD-index of its dual Q^*. Then $g(Q, x) = \psi(1, x)$.*

Therefore, it is possible to recover the g-vector of a polytope from the CD-index of its dual.

In Chapter 4, we will show how to determine a polytope's CD-index from one of its representations, the Gale diagram, in certain cases. Now that we know how to encode information about the polytope's faces using these different indices, we are going to explore a way to build up a polytope by shelling on its facets.

1.5 Shellings

There are many important interpretations of the components of the h-vectors of simplicial polytopes. As we saw when discussing the ordinary h-vector in Section 1.3, h_k of a simplicial polytope P counts the number of vertices of in-degree k when sweeping its dual P^* with a hyperplane. Alternatively, h_k counts the number of unique minimal new faces of P with exactly k vertices in an arbitrary shelling of the facets of P. These two interpretations are dual to one another in the simplicial case when the shelling is a line shelling. In particular, for any simplicial polytope P, $h_k(P)$ is found either by shelling P itself or sweeping its dual. Since they are counting something, the components of the h-vector are nonnegative. Thus, we will discuss shelling in detail here. For more general information on shelling, see [30].

For the simplicial case, Bayer and Lee gave the following definition of a shelling [4].

Definition 1.5.1 (Bayer-Lee 1993)**.** Let Δ be a simplical complex on V. Thus, Δ is closed under inclusion. Then Δ is *shellable* if it satisfies the following criteria.

1. For every i-face of Δ, there is a facet of Δ containing F (*i.e.*, Δ is *pure*).
2. The facets in Δ may be ordered $F_1, \ldots, F_n$ so that for $1 \leq k \leq n$, there is a unique face $G_k \in \mathcal{F}(F_k)$ such that $\mathcal{T}_k = \mathcal{F}(F_k) \bigcap \left[\bigcup_{j=1}^{k-1} \mathcal{F}(F_j) \right]$ is precisely the set of faces of F_k not containing G_k.

Here G_k is the unique minimal new face. It is also important to remember that any subset of facets of a simplex is shellable in any order. In this dissertation, we are extending results to general polytopes. So we will need the general definition for shelling the boundary complex of a $(k + 1)$-polytope [30].

Definition 1.5.2 (Ziegler 1995)**.** Let $\mathcal{C} = \partial P$ for a $(k + 1)$-polytope. A *shelling* of $\mathcal{C}$ is a linear ordering $F_1, \ldots, F_s$ of the facets of $\mathcal{C}$ such that either $\mathcal{C}$ is 0-dimensional (and, thus, facets are points) or it satisfies the following criteria.

1. The boundary complex $\mathcal{C}(\partial F_1)$ of the first facet F_1 has a shelling.

2. For $2 \leq j \leq s$, $F_j \bigcap \left[\bigcup_{i=1}^{j-1} F_i\right] = \bigcup_{i=1}^{r} G_i$ is nonempty and the beginning segment of a shelling $G_1, \ldots, G_r, \ldots, G_t$ of $\mathcal{C}(\partial F_j)$, the $(k-1)$-dimensional boundary complex of F_j.

We say that the complex of a polytope $\mathcal{C}(P)$ is *shellable* if the boundary complex $\mathcal{C}(\partial P)$ is shellable. So "shelling a polytope" means finding a shelling order for the facets of P. The following result allows us to reverse the ordering of any shelling [30].

Lemma 1.5.3 (Ziegler 1995)**.** *If $F_1, F_2, \ldots, F_s$ is a shelling order for the boundary of a polytope P, then $F_s, F_{s-1}, \ldots, F_1$ is also a shelling order of it.*

Bruggesser and Mani proved the following key result [8].

Theorem 1.5.4 (Bruggesser-Mani 1971)**.** *All polytopes are shellable.*

They used a "line" shelling argument to prove this result. In particular, the ordering of the facets of a polytope P corresponds to the ordering of the vertices of P^* for a particular dual of P (a polar dual) by sweeping with a hyperplane. Shellings that arise from Bruggesser and Mani's construction are called line shellings. Switching the orientation of the line reverses the ordering of a given line shelling and, subsequently, results in another line shelling. Various people, including Bruggesser and Mani, have shown that there is a shelling in which any specified facet comes first [8, 9, 20, 6]. Ziegler makes the following observation [30].

Corollary 1.5.5 (Ziegler 1995)**.** *Every regular subdivision of a polytope is shellable. In particular, every Schlegel diagram is shellable.*

When sweeping the d-polytope Q^* with a hyperplane H, each vertex has a corresponding $\mathbf{S}^*$ and $\mathbf{T}^*$ as H sweeps Q^* at v [19]. For a vertex v, $\mathbf{S}^*$ is the vertex figure of Q^* at v. For a vertex v, $\mathbf{T}^*$ is the intersection $\mathbf{S}^* \cap H$. So $\mathbf{S}^*$ is a polytope of dimension $d-1$ and $\mathbf{T}^*$ is a polytope of dimension $d-2$.

Sweeping the d-polytope Q^* corresponds to a shelling of Q induced by "lifting" the points in a Gale diagram, as described in Chapter 4. The vertex figure $\mathbf{S}^*$ is dual to the (co)facet $\mathbf{S}$ that we will shell on in the Gale diagram in Chapter 4. The intersection of the vertex figure with H is dual to the boundary of the intersection of the new facet with the old facets as we shell on a (co)facet in the Gale diagram in Chapter 4. Using the relationships from the previous section, Lee verifies how to calculate the CD-index of Q^* by shelling its dual Q [16].

Theorem 1.5.6 (Lee 2016)**.** *The CD-index of Q^* is $\psi(Q^*) = \sum_v \psi_v(Q_v^*)$, where*

$$\psi_v(Q_v^*) = \frac{1}{2}\Big((2C - I)\psi(\mathbf{S}_v^*) + (4D - I^2)\psi(\mathbf{T}_v^*)\Big).$$

Here, for a fixed j, $1 \leq j \leq s$, $\mathbf{S}_v = \partial F_j$ and $\mathbf{T}_v = \partial \bigcup_{i=1}^{r} G_i$. In the case that v is a simple vertex with i edges below the hyperplane and $d - i$ points above the hyperplane, then we have the following Corollary from Lee [16].

Corollary 1.5.7 (Lee 2016)**.** *Consider the d-polytope Q. Let v be a vertex of Q of in-degree i. Then*

$$\psi_v(Q^*) = -\frac{1}{2}\Delta_i^d = -\frac{1}{2}\Delta_{d-i}^d,$$

where $\Delta_i^d = D^i C^{d-2i} - D^{i+1}C^{d-2(i+1)}$ for $i \neq \lfloor \frac{d}{2} \rfloor$ and $\Delta^d_{\lfloor \frac{d}{2} \rfloor} = 2D^{\lfloor \frac{d}{2} \rfloor}$ when $2|d$ and $\Delta^d_{\lfloor \frac{d}{2} \rfloor} = D^{\lfloor \frac{d}{2} \rfloor}C$ otherwise.

Now that we have discussed various properties of polytopes, we are going to consider one useful representation of a polytope.

1.6 Gale Transforms and Gale Diagrams

In this section, we will discuss Gale transforms, Gale diagrams, and some useful properties. For more details and further information, see McMullen, Grünbaum, and Ziegler [22, 14, 30].

Before drawing attention to a few key results, we will define Gale transforms and diagrams and consider some examples.

Definition 1.6.1 (cf. Ziegler 1995)**.** Consider a set of points $V = \{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^d$. Suppose $P = \operatorname{conv} V \subseteq \mathbb{R}^d$. Then define a $(d+1) \times n$-matrix A as follows.

$$A = \begin{pmatrix} v_{1,1} & v_{2,1} & \cdots & v_{n,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,d} & v_{2,d} & \cdots & v_{n,d} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n-d-1)}$ be a basis for the $n-d-1$-dimensional nullspace of A. We define a $(n-d-1) \times n$-matrix $\overline{A}$ with rows $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n-d-1)}$ as follows.

$$\overline{A} = \begin{pmatrix} \alpha_{1,1} & \alpha_{2,1} & \cdots & \alpha_{n,1} \\ \alpha_{1,2} & \alpha_{2,2} & \cdots & \alpha_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1,n-d-1} & \alpha_{2,n-d-1} & \cdots & \alpha_{n,n-d-1} \end{pmatrix}$$

Then $\overline{V} = \{\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n}\}$ is the *Gale transform* of P, where $\overline{v}_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,n-d-1})$ for $1 \leq i \leq n$.

Note there is a natural correspondence between the points of V and the points of $\overline{V}$. Using this definition, we will find the Gale transform for the octahedron and cube below.

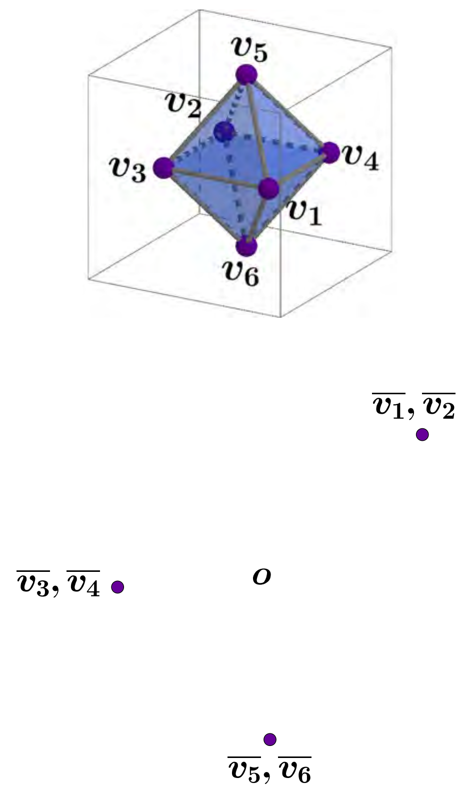

Figure 1.14: Here is the octahedron and its Gale transform.

Example 1.6.2. Consider the octahedron

$$P = \operatorname{conv}(\{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}).$$

Then A is

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and one possible $\overline{A}$ is

$$\overline{A} = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \end{pmatrix}$$

So the Gale transform is $\overline{V} = \{(1,1), (1,1), (-1,0), (-1,0), (0,-1), (0,-1)\}$. Figure 1.14 shows the octahedron and its Gale transform.

This example shows that there is a relationship between a d-dimensional polytope $P = \operatorname{conv}(\{v_1, v_2, \ldots, v_n\}) \subset \mathbb{R}^d$ and its Gale transform $\overline{V} \subset \mathbb{R}^{n-d-1}$. Before stating this well known result, we introduce the following definition.

Definition 1.6.3 (Lee-N)**.** We say that S *"captures"* a point q *if and only if* $q \in \operatorname{rel\,int}(\operatorname{conv}(S))$.

The following theorem gives the relationship between sets that "capture" the origin o in the Gale transform and faces of the corresponding polytope P.

Theorem 1.6.4 (cf. Grünbaum 2003)**.** *Consider any subset S of $\{1, 2, \ldots, n\}$. Then the set $\{v_i | i \in S\}$ is $V \cap F$ for some face F of P, $F \neq P$, if and only if the origin o is in the relative interior of conv$(\{\overline{v_i} | i \notin S\})$.*

In other words, the complement of every point set of a face of P must "capture" the origin o in the relative interior of its convex hull in P's corresponding Gale transform. So there is a one-to-one correspondence between the faces of a polytope P ($\neq P$) and the sets of points that capture o in the corresponding Gale transform. In our example above, notice that conv $(\{v_2, v_4\})$ is a face of P and $o \in$ rel int (conv $(\{\overline{v_1}, \overline{v_3}, \overline{v_5}, \overline{v_6}\})$). It is also true that $o \notin$ rel int (conv $(\{\overline{v_1}, \overline{v_2}, \overline{v_3}\})$) and conv $(\{v_4, v_5, v_6\})$ is not a face of P.

In Example 1.2.6, we noted that the cube is dual to the octahedron. So we will now find the Gale transform for the cube.

Example 1.6.5. Consider the cube $Q = \text{conv}\,(\{(1,1,1), (-1,1,1), (1,-1,1), (-1,-1,1), (1,1,-1), (-1,1,-1), (1,-1,-1), (-1,-1,-1)\})$.

Then A is

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and one possible $\overline{A}$ is

$$\overline{A} = \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

So the Gale transform is $\overline{V} = \{(1,1,1,1), (-1,-1,1,-1), (-1,1,-1,-1), (1,-1,-1,1), (1,-1,-1,-1), (-1,1,-1,1), (-1,-1,1,1), (1,1,1,-1)\}$. Since the Gale transform is 4-dimensional, we are not going to sketch it.

These two examples illustrate an important fact. When the number of points n and the dimension of the polytope d are close enough, then the dimension of the Gale transform is smaller than the dimension of the polytope itself. However, when n and d are not close enough, then the dimension of the Gale transform can be larger than the dimension of the polytope itself. Subsequently, Gale transforms are particularly useful representations of polytopes when $n - d$ is small. Furthermore, the Gale transform preserves many properties of the original polytope. So studying properties of the polytope through the Gale transform helps us better understand higher-dimensional polytopes. Therefore, in this dissertation, we will show how to determine information about polytopes with *few* vertices (*i.e.*, $n \leq d + 3$) from their Gale transforms. When interpreting the toric h- and g-vectors, we will actually study Gale diagrams, which are closely related to Gale transforms through the following well known relationship.

Definition 1.6.6 (cf. Grünbaum 2003)**.** Suppose that $\overline{V} \subset \mathbb{R}^{n-d-1}$ is the Gale transform of the d-polytope $P \subset \mathbb{R}^d$. Then $\overline{V'} \subset \mathbb{R}^d$ is a Gale diagram of P if and only if the origin $o \in \text{rel int}\,(\text{conv}\,(\{\overline{v^i}|i \in S\}))$ exactly when $o \in \text{rel int}\,(\text{conv}\,(\{\overline{v'^i}|i \in S\}))$ for every subset $S \subset \{1, 2, \ldots, n\}$.

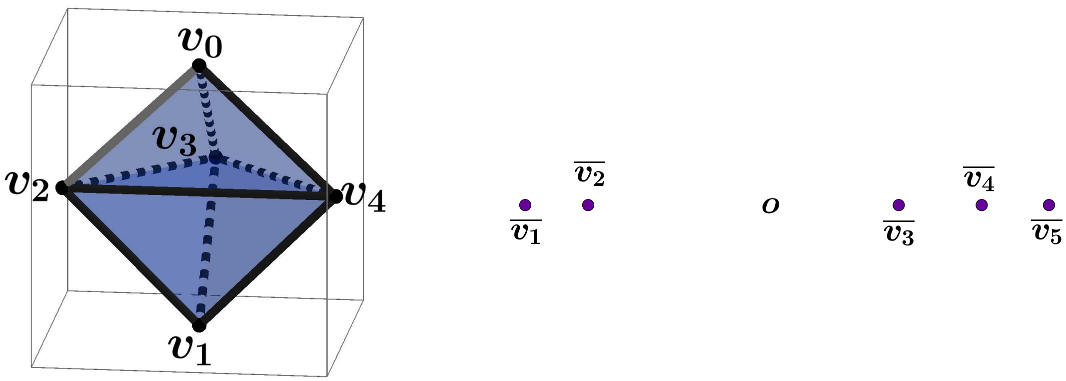

Figure 1.15: Here is the triangular bipyramid and its Gale diagram.

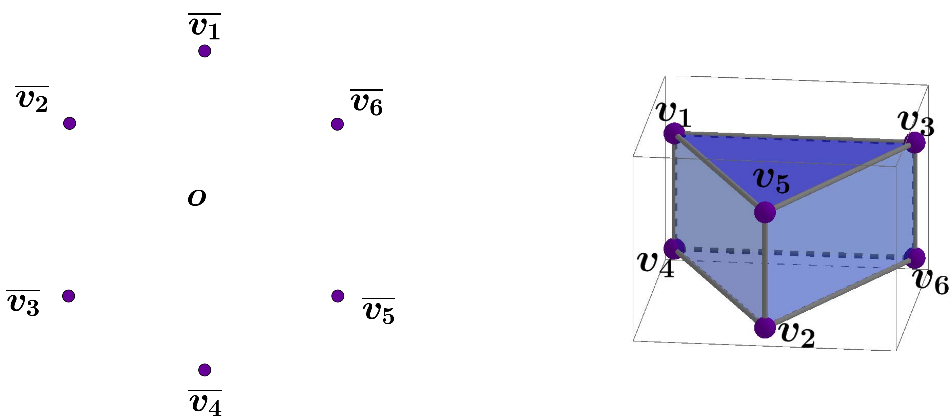

Figure 1.16: Here is the triangular prism and its Gale digram.

Every d-polytope $P \in \mathbb{R}^d$ corresponds to a unique Gale diagram in $\mathbb{R}^e$, up to combinatorial equivalence. Using this definition, we may construct Gale diagrams for the triangular bipyramid and the triangular prism without using matrices.

Example 1.2.4 continued. Figures 1.15 and 1.16 show the Gale diagrams for the triangular bipyramid and the triangular prism, respectively. Notice that the complement of every point set of a face of a given polytope does capture the origin o in its

Gale diagram. Further, the complement of every point set that does not capture the origin in the Gale diagram is not a face of that polytope.

The relationship between faces in P and the points in the Gale diagram leads to the following result, which is stated in [14].

Corollary 1.6.7 (cf. Grünbaum 2003). *The origin o in the Gale diagram coincides with i points, say $\{\overline{v_1}, \overline{v_2}, \dots \overline{v_i}\}$, from $\overline{V}$ if and only if the polytope P is an i-fold pyramid over the points $\{v_{i+1}, v_{i+2}, \dots, v_n\}$.*

A d-polytope P has a corresponding e-dimensional Gale diagram $\overline{V}$, where $e = n - d - 1$. Then the following result of McMullin shows how to determine the Gale diagram for any of the faces L of P from $\overline{V}$ [22].

Theorem 1.6.8 (McMullin 1979). *Regard L as a polytope itself. Then the Gale diagram of L is found by projecting onto the space orthogonal to $\overline{V \setminus L}$ in the Gale diagram corresponding to P and then deleting the points associated to $\overline{V \setminus L}$.*

When the number of vertices of a polytope is close enough to its dimension, then the Gale diagram is low-dimensional. Thus, this representation is particularly helpful when exploring properties of polytopes with few vertices.

1.7 Triangulations and Regular Triangulations

Here we briefly introduce triangulations and their significance in the upcoming chapters. There is some historical context in [14]. Additional information about triangulations may be found in [10].

McMullen and Walkup stated the *generalized lower bound conjecture* (GLBC) [23].

Conjecture 1.7.1 (McMullen-Walkup 1971). *Let P be any simplicial d-polytope. Then*

1. $g_i(P) \geq 0$ *for* $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$.
2. *For* $1 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, *the following statements are equivalent.*
   - $g_k(P) = 0$;
   - *P is $(k-1)$-stacked, i.e. P admits a subdivision into a simplicial complex whose $(d-k)$-faces are also faces of P.*

In 1971 McMullen made a conjecture to characterize the f-vectors of simplicial polytopes [21]. Billera and Lee proved the sufficiency of McMullen's conditions for f-vectors [5] and Stanley proved the necessity of these conditions [27]. With this complete proof, the statement is now called the g-theorem. The g-theorem establishes the non-negativity of the g_i for $0 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor$, which is Part 1 of the GLBC.

In this section, we focus our attention to Part 2 of the GLBC. An equivalent definition states that a d-polytope P is $(k-1)$-stacked exactly when P has a triangulation $\mathcal{T}$ for which there is no interior face of dimension less than $d - k + 1$. Consider any d-polytope $P = \operatorname{conv}(V)$ where $V = \{v_1, \dots, v_n\}$. Lee gives the following definition of a triangulation of P [17].

Definition 1.7.2 (Lee 1991)**.** A triangulation of P is a collection $\mathcal{T} = \{S_1, \ldots, S_m\}$ of subsets of V that satisfy the following conditions.

- $\dim(\text{conv}\,(S_i)) = d$ for $1 \leq i \leq m$;
- $|S_i| = d + 1$ for $1 \leq i \leq m$;
- $\displaystyle\bigcup_{i=1}^{m} \text{conv}\,(S_i) = P$;
- $\text{conv}\,(S_i) \cap \text{conv}\,(S_j) = \text{conv}\,(F)$ for some common proper face F of S_i and S_j where $1 \leq i < j \leq m$.

McMullen and Walkup proved that if P is $(k-1)$-stacked, then $g_k(P) = 0$ when they made the GLBC [23]. Then Lee and Welzl independently showed that the forward implication holds for polytopes with few vertices, *i.e.*, the number of vertices is at most $d+3$ [18, 29]. In his proof, Lee showed how to interpret the g_k as winding numbers in the Gale diagram. We will explore these concepts further in the next section.

In 2004, McMullen proved the following result [24].

Theorem 1.7.3 (McMullen 2004)**.** *If a simplicial polytope P has a triangulation $\mathcal{T}$ with no interior faces F such that dim$(F) \leq \left\lfloor \frac{d}{2} \right\rfloor$, then this triangulation $\mathcal{T}$ is unique.*

The validity of the forward implication for Part 2 of the GLBC is now fully verified due to recent work by Murai and Nevo [25]. So for all simplicial d-polytopes P, $g_k(P) = 0$ exactly when P is $(k-1)$-stacked. In fact, Murai and Nevo also showed that the forward implication holds in the more general context of homology spheres which satisfy the *weak Lefschetz property*. In Chapter 3, we extend this forward implication to general convex (*i.e.*, non-simplicial) nonpyramidal polytopes Q with few vertices in the context of Gale transforms.

1.8 Winding Numbers

Let $e \geq 1$, $n \geq e+1$, and $d = n-e-1$. Consider a set of points $\overline{V} = \{\overline{v}_1, ..., \overline{v}_n\} \subseteq \mathbb{R}^e$ in affinely general position.

Definition 1.8.1 (Lee 1991)**.** Let $\mathcal{X} = \{X \subset \overline{V} : |X| = e\}$. In $\mathbb{R}^e$, an X in $\mathcal{X}$ is a *k-splitter* if the hyperplane $H = \text{aff}\,(X)$ partitions the remaining $n - e$ points of $\overline{V}$ into two sets, at least one of which has cardinality k.

Example 1.8.2. Consider the 2-dimensional Gale diagram in Figure 1.17. The 7 points of $\overline{V}$ are evenly spaced around a circle. All the 2-splitters of these 7 points are shown.

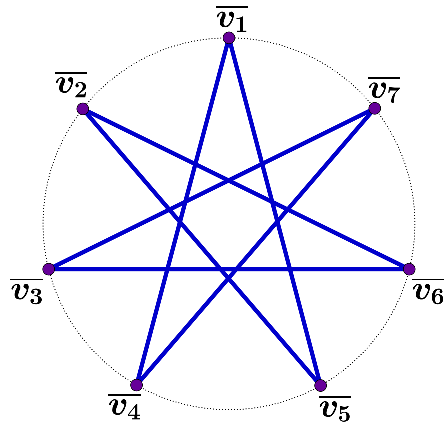

Figure 1.17: All the 2-splitters are marked in this 2-dimensional Gale diagram.

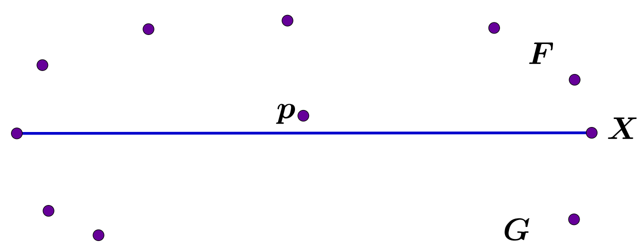

Figure 1.18: Here is the set up for computing the winding number.

Now we may discuss winding numbers. Choose any $p \in \mathbb{R}^e$ in affinely general position with respect to the points in $\overline{V}$. The hyperplane H spanned by a splitter X partitions $\overline{V} \setminus X$ into two sets F and G, where F is the set of points of $\overline{V}$ in the same open halfspace H^- as p and G is the set of points in the opposite open halfspace H^+. See Figure 1.18. Then Lee defined the winding number $w_k(p)$ of p as follows [18].

Definition 1.8.3 (Lee 1991)**.** The *sign of X with respect to p* is

$$sg(X) = \begin{cases} +1, & \text{if } \operatorname{card}(G) < \operatorname{card}(F) \\ -1, & \text{if } \operatorname{card}(G) > \operatorname{card}(F) \\ 0, & \text{if } \operatorname{card}(G) = \operatorname{card}(F) \end{cases}.$$

Let $\alpha(X)$ denote the *measure of the solid angle* of the cone which is determined by $\operatorname{conv}(X)$ and whose apex is p. (Total angle equals 1.)
Then the k^{th} *winding number* is $w_k = \sum_{\substack{k-\text{splitters} \\ X \in \mathcal{X}}} sg(X)\alpha(X)$ for a fixed k, $0 \leq k < \left\lfloor \frac{n-e}{2} \right\rfloor$.

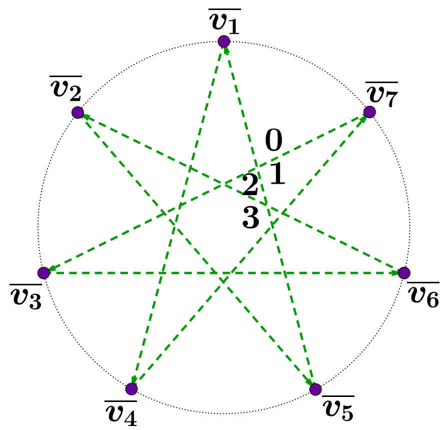

Figure 1.19: Here we want to determine w_2 in the Gale diagram.

Intuitively, the k^{th} winding number, $w_k(p)$, counts how many times k-splitters "wind around" p. So we may determine the k^{th} winding number of regions determined by the sets $\operatorname{conv}(X)$ for k-splitters X. We calculate w_2 in Figure 1.19.

Example 1.8.2 continued. We found all the 2-splitters of 7 points evenly spaced around a circle above. Here we determine the winding number $w_2(p)$ for different points p that fall in affinely general position with respect to the points of $\overline{V}$.

Then Lee proved the following result [18].

Theorem 1.8.4 (Lee 1991)**.** *Let $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$. Then the k^{th} winding number of p, $w_k(p)$, equals g_k for the corresponding simplicial polytope P, where the origin of the Gale diagram is chosen to be p.*

As stated in the previous section, it is known that $w_k(p) \ge 0$ for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$, as a consequence of the g-theorem. Using Gale diagrams, Lee and Welzl independently provided elementary proofs by providing interpretations of the g-vector for simplicial polytopes with few vertices (e.g., $e = 2$) [18, 29]. As noted in the last section, Lee and Welzl also independently proved that if $g_k(P) = 0$ for $1 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$, then P is $(k-1)$-stacked. In particular, they showed the following result holds.

Theorem 1.8.5 (Lee 1991, Welzl 2001)**.** *Suppose P is a simplicial d-polytope, with at most $d+3$ vertices, whose corresponding Gale transform has origin q. Let $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$. If $g_k(q) = 0$, then there exists a ray from q crossing no k-splitters. This ray determines the triangulation predicted by the GLBC.*

Example 1.8.2 continued. See Figure 1.20. Suppose the origin q in this Gale transform with 7 points corresponds to the 4-dimensional polytope Q. Since q falls in affinely general position with respect to the points in $\overline{V}$ of the Gale transform, then Q is a simplicial polytope. Further, since $g_2(Q) = g_2(q) = 0$, then there exists a ray

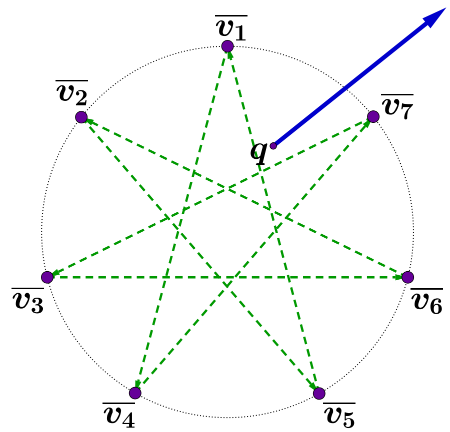

Figure 1.20: Here is a ray that does not cross any 2-splitters in the Gale transform.

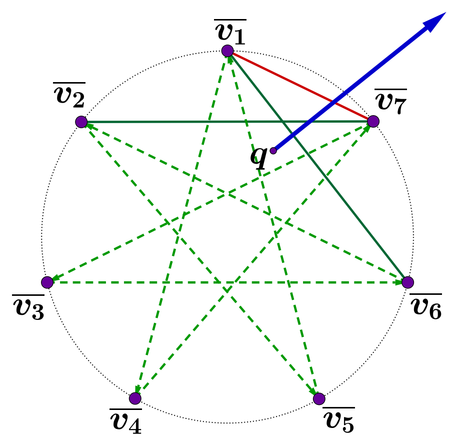

Figure 1.21: Here the ray crosses three splitters in the Gale transform.

that does not cross any 2-splitter. One such ray is indicated in Figure 1.20.

A *ray in general direction* crosses the relative interior of the convex hull of exactly one $X \in \mathcal{X}$ at any given point. The maximum simplices in the triangulation are the complements of the splitters crossed by the ray. Figure 1.21 verifies that the ray chosen above is in general direction. As it moves towards infinity, this ray crosses the 1-splitter $\{\overline{v}_1, \overline{v}_6\}$, the 1-splitter $\{\overline{v}_2, \overline{v}_7\}$, and the 0-splitter $\{\overline{v}_1, \overline{v}_7\}$. Thus, this ray determines a triangulation of Q for which the maximum simplices are $V \setminus \{v_1, v_6\}$, $V \setminus \{v_2, v_7\}$, and $V \setminus \{v_1, v_7\}$.

McMullen's earlier result [22] verifies that Theorem 1.8.5 determines a triangulation of the corresponding polytope.

Theorem 1.8.6 (McMullen 1979)**.** *Let $\overline{V}$ be the e-dimensional Gale transform with origin o corresponding to the d-polytope P. Adding a point z in affinely general position with respect to the points $\overline{V} \cup \{o\}$ determines a triangulation of P. The faces* *of this triangulation are the complements of the sets of points which together with z capture the origin o.*

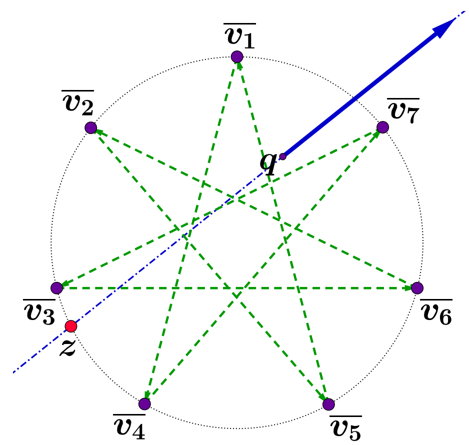

Figure 1.22: Here the point z is opposite the ray in the Gale transform.

Adding z in general position with respect to $\overline{V}$ determines a triangulation that is equivalent to the triangulation determined by a ray in general position. In fact, if the ray is pointing in the opposite direction of z, as in Figure 1.22 then these two methods determine the exact same triangulation. Triangulations obtained in this way are called *regular triangulations*.

By extending these results to certain non-simplicial polytopes in Chapter 2, we determine the g-vector for any polytope P whose Gale diagram is e-dimensional in various cases. In particular, we are able to determine the g-vector of every polytope with a 2-dimensional Gale diagram. Then we extend the triangulation result to general nonpyramidal polytopes with few vertices in Chapter 3.

Chapter 2 The Toric g-Vector and Gale diagrams

Using the winding number interpretation of Lee in [18], we know how to find the g-vector of any simplicial polytope from its Gale diagram. In particular, we compute the winding number of the origin, which falls in affinely general position with respect to the points $\overline{V}$ of the Gale diagram. Our goal in this chapter is to determine the g-vector of a general convex polytope from its Gale diagram. We will compute the g-vector for d-dimensional polytopes with e-dimensional Gale diagrams whose points are in various states of special position. In addition, we will provide winding interpretations in some cases. First, we will consider Gale diagrams $\overline{V}$ in which all the points of $\overline{V}$ are in affinely general position but the origin is not in affinely general position with respect to $\overline{V}$. Then we will consider Gale diagrams in which all the points $\overline{V}$ are in affinely general position except a subset of vertices which fall on a single hyperplane containing the origin. We will close this chapter with a complete analysis of polytopes with few vertices.

For notational ease, we will write V, instead of $\overline{V}$, for points in the Gale diagram throughout the rest of this dissertation. Whether V refers to the points of a polytope or points of its Gale diagram will be clear from the context.

2.1 Gale Diagrams in e-Dimensions whose Origin is not in Affinely General Position

We will first consider convex d-polytopes with e-dimensional Gale diagrams in which the points V of the diagram fall in affinely general position, except for the origin, which is not in affinely general position with respect to V. In the case when the origin q is in the relative interior of the convex hull of a single splitter, we will show that $g_k(q)$ equals the smaller of the winding numbers of the two regions immediately adjacent to q in Theorem 2.1.1. When the origin lands on multiple splitters, we argue that each splitter acts "independently" by Lemma 2.1.4. The g-vector for a polytope Q whose origin q lands on one of the vertices v in its Gale diagram is the same as the g-vector obtained when that vertex v is deleted in Theorem 2.1.10.

In this section, we start with a collection of n points, V, in affinely general position in $\mathbb{R}^e$. Consider a point q that is in affinely general position with respect to V except that q is in the relative interior of the convex hull of each of α splitters, say $X_1, X_2, \ldots, X_\alpha$, where X_i is a k_i-splitter for $1 \leq i \leq \alpha$, and $\operatorname{aff}(X_i) \neq \operatorname{aff}(X_j)$ for all $i \neq j$. Then there exists a p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitter except $X_1, X_2, \ldots, X_\alpha$.

Define $H_i = \operatorname{aff}(X_i)$ for all i. Let F_i be the points of V that are in the open halfspace of H_i containing p, and let G_i be points of V that are in the opposite open halfspace of H_i. See Figure 2.1. Then $V = F_i \cup G_i \cup X_i$ for each i. Note that placing the origin at p corresponds to a simplicial polyope P, and placing the origin at q corresponds to a nonsimplicial poytope Q.

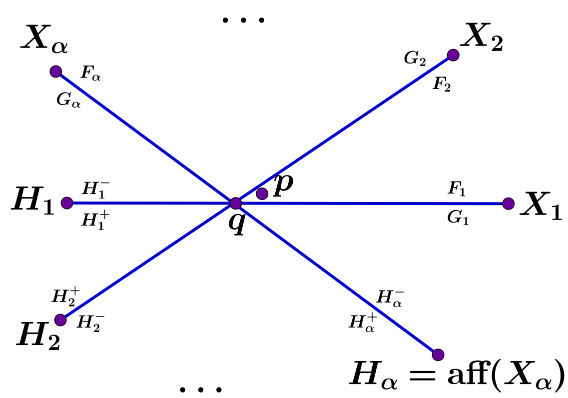

Figure 2.1: Multiple splitters capture the origin q here.

We know how to compute the winding number of p and, subsequently, we know how to determine the g-vector of P. How do we compute the g-vector of Q when q falls on the relative interior of the convex hull of some splitters in the corresponding Gale diagram? How do we interpret the winding number q in this case?

Single k-Splitter

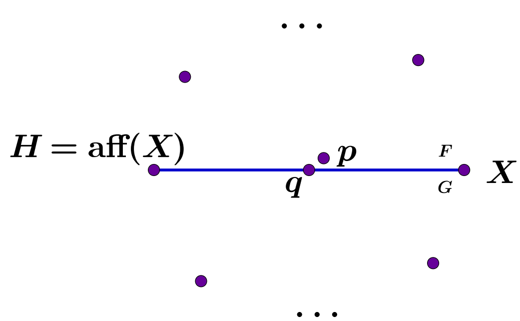

Figure 2.2: One k-splitter captures q here.

We will first determine the g-vector of Q when $\alpha = 1$. In other words, we assume that q falls in the relative interior of the convex hull of a single k-splitter X_1, or X. See Figure 2.2.

Theorem 2.1.1 (Lee-N)**.** *When the origin q in the Gale diagram falls in the relative interior of the convex hull of exactly one k-splitter, then $g_k(Q)$ equals the minimum g_k of its neighboring regions. Otherwise $g_i(Q) = g_i(P)$ for $i \neq k$, where P is the polytope* *corresponding to having the origin p in either of the regions immediately adjacent to q.*

Proof. We already know how to handle the simplicial case, so we want to take advantage of this fact when working on the nonsimplicial case. We will determine $g(\partial Q)$ in terms of $g(\partial P)$. We need to determine which faces are different for P and Q. Start by considering which sets capture p or q but not both.

Clearly, X captures only q. Since p is in affinely general position with respect to V, then any set that captures p in its interior must have full dimension e. Since $\overline{pq}$ does not meet the convex hull of any splitter except X, we also note that any set which captures p in its interior either captures q in its interior or on its boundary. Sets which capture both p and q in their relative interiors correspond to common faces of P and Q. Sets which capture p in their interiors and q on their boundaries are exactly the sets of the form $F' \cup X$ for F' a nonempty subset of F.

Since we know what is different between the sets that capture only p or q, then we also know exactly what is different about the faces of P and Q by considering complements by Definition 1.6.6. The nonsimplicial polytope Q has the face $V \setminus X = F \cup G$. The simplicial polytope P has all the faces of the form $V \setminus (F' \cup X) = (F \setminus F') \cup G$ for every nonempty subset F' of F.

Then we may compute the h-vector of ∂Q by subtracting terms for all the faces of ∂P not in ∂Q and adding the term for the face of ∂Q not in ∂P to the h-vector of ∂P.

$$\begin{aligned}
h(\partial Q, x) &= h(\partial P, x) \\
&- \sum_{G \text{ face of } \partial P \text{ not in } \partial Q} g(\partial G, x)(x-1)^{d-1-\dim(G)} \\
&+ \sum_{G \text{ face of } \partial Q \text{ not in } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)}
\end{aligned}$$

Since P is simplicial, all of its faces are simplices. For simplicial objects, we may use the ordinary h- and g-vectors [26]. Recall that $g(\partial \text{simplex}, x) = 1$ (Lemma 1.3.13). Also, for $1 \leq i \leq |F|$ and $|F'| = i$, there are $\binom{|F|}{i}$ sets of the form $(F \setminus F') \cup G$ of dimension $d-i$. Thus, we have the following:

$$\begin{aligned}
&\sum_{L \text{ face of } \partial P \text{ not in } \partial Q} g(\partial L, x)(x-1)^{d-1-\dim(L)} \\
=\ & \sum_{i=1}^{|F|} \binom{|F|}{i} (1) \cdot (x-1)^{d-1-(d-i)} \\
=\ & \sum_{i=1}^{|F|} \binom{|F|}{i} (x-1)^{i-1} \\
=\ & \frac{1}{x-1} \sum_{i=1}^{|F|} \binom{|F|}{i} (x-1)^{i} \\
=\ & \frac{1}{x-1} \left(x^{|F|}-1\right) \text{ by the Binomial Theorem} \\
=\ & \frac{x^{|F|}-1}{x-1} \\
=\ & x^{|F|-1} + x^{|F|-2} + \cdots + x^2 + x + 1.
\end{aligned}$$

Observe that the face $F \cup G$ is a simplicial $(d-1)$-polytope that attaches along the boundary of the complex of the faces of P after the above faces of P are removed. The facets of $\partial(F \cup G)$ are of the form $(F \setminus t) \cup (G \setminus s)$ where $t \in F$ and $s \in G$. They have cardinality $d-1$ and are contained in exactly one set of the form $(F \setminus t) \cup G$ where t is an element of F. Therefore, $\partial(F \cup G) = (2^{F \setminus t}) \cup (2^{G \setminus s}) = \partial 2^F * \partial 2^G$, where $*$ denotes the simplicial join. We need to determine $g(\partial(F \cup G), x)$. Since $\partial(F \cup G)$ is the join of the simplicial complexes $\partial 2^F$ and $\partial 2^G$, then we have that

$$\begin{aligned}
& h(\partial(F \cup G), x) \\
=\ & h(\partial \overline{F}, x) \cdot h(\partial \overline{G}, x) \text{ by Lemma 1.3.15} \\
=\ & (x^{|F|-1} + x^{|F|-2} + \cdots + x + 1) \cdot (x^{|G|-1} + x^{|G|-2} + \cdots + x + 1) \text{ as shown above.}
\end{aligned}$$

Since $|F| + |G| - 2 = d-1$, then $h(\partial(F \cup G), x)$ equals

$$x^{d-1} + 2x^{d-2} + \cdots + \ell x^{d-\ell} + \cdots + \ell x^{\ell-1} + \cdots + 2x + 1.$$

where $\ell = \min\{|F|, |G|\}$. Therefore, $g(\partial(F \cup G), x)$ equals

$$1 + x + \cdots + x^{\ell-1}.$$

Thus, we have the following:

$$\begin{aligned}
&\sum_{L \text{ face of } \partial Q \text{ not in } \partial P} g(\partial L, x)(x-1)^{d-1-\dim(L)} \\
=\ & (1 + x + \cdots + x^{\ell-1}) \cdot (x-1)^{d-1-(d-1)} \\
=\ & (1 + x + \cdots + x^{\ell-1}).
\end{aligned}$$

With both of these results, we are able to determine the change in the toric h-vector when we move the origin from p to q in the Gale diagram. The change in the toric h-vector is

$$-(1 + x + \cdots + x^{|F|-1}) + (1 + x + \cdots + x^{\ell-1}).$$

In the case that $\ell = |F| \leq |G|$, there is no change in the h-vector. In the case that $\ell = |G| < |F|$, then the change in the h-vector is

$$-x^G - x^{G+1} - \cdots - x^{|F|-1}.$$

By looking at the differences, we are able to determine the change in the toric g-vector when we move from p to q in the Gale diagram. In the former case, there is no change in the g-vector. So $g_\ell(Q) = g_\ell(P) = w_\ell(p)$. In the latter case, the change in the g-vector is $-x^{|G|}$. So $g_\ell(Q) = g_\ell(P) - 1 = w_\ell(p) - 1$. In either case, $g_\ell(Q)$ equals the minimum ℓ^{th} winding number of the two regions adjacent to q. ■

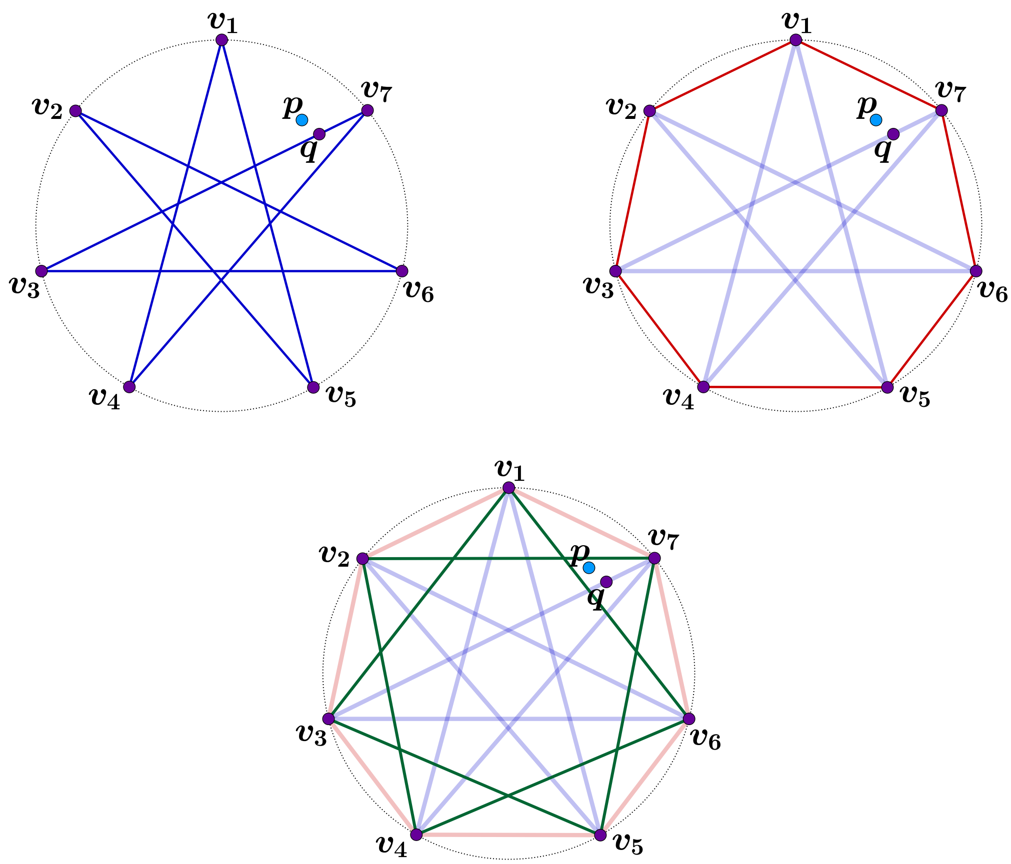

Figure 2.3: Use Theorem 2.1.1 to determine $g(Q)$.

Consider the following example.

Example 2.1.2. See Figure 2.3. In Example 1.8.2, we found $g_2(P) = w_2(p)$ for different points p that fall in affinely general position with respect to the points of V. The point q is in the relative interior of the convex hull of the 2-splitter $\{v_3, v_7\}$. As the figure here shows, p is in region adjacent to q for which g_2 is minimum. Let P be the simplicial 4-polytope corresponding to the Gale diagram in which the origin is p. Then by Theorem 2.1.1, $g(Q) = g(P) = (1, 1, 0)$.

This result leads to the following winding interpretation.

Corollary 2.1.3 (Lee-N)**.** *When the origin q in the Gale diagram falls on exactly one k-splitter, then that splitter is not winding around q.*

We know how to compute the g-vector of Q when q falls in the relative interior of the convex hull of exactly one k-splitter in the Gale diagram. How do we compute the g-vector of Q when q falls in the relative interiors of the convex hulls of multiple splitters simultaneously?

Multiple k-Splitters

Now we will determine the g-vector of Q when $\alpha > 1$; in other words, q falls in the relative interior of the convex hulls of multiple splitters $X_1, X_2, \ldots, X_\alpha$, where $\operatorname{aff}(X_i) \neq \operatorname{aff}(X_j)$ for all $i \neq j$. See Figure 2.1. We first need to consider the relationship between any pair of splitters that cross on their interiors.

For each i, let H_i^- and H_i^+ be the closed halfspaces of X_i on the same side as F_i and G_i, respectively.

Lemma 2.1.4 (Lee-N)**.** *If X_i and X_j contain a common point q in the relative interiors of their convex hulls for $1 \leq i < j \leq \alpha$, then there exists a point of X_j in each of G_i and F_i.*

Proof. Without loss of generality, we may consider $i = 1$ and $j = 2$. Since $X_1 \neq X_2$, then $H_1 \neq H_2$. We know that

$$q \in \operatorname{rel\,int}(\operatorname{conv}(X_1)) \cap \operatorname{rel\,int}(\operatorname{conv}(X_2)).$$

Then $q \in \operatorname{rel\,int}(\operatorname{conv}(X_1)) \subseteq H_1$, and, subsequently,

$$q \notin H_1^+ \setminus H_1. \tag{2.1}$$

Since $q \in \operatorname{rel\,int}(\operatorname{conv}(X_2)) \subseteq H_2$ and $H_1 \neq H_2$, then

$$q \in \operatorname{rel\,int}(\operatorname{conv}(X_2)) \nsubseteq H_1.$$

Now suppose there is no point of X_2 in F_1. Then $X_2 \subseteq H_1^+$. It follows that $\operatorname{rel\,int}(\operatorname{conv}(X_2)) \subseteq H_1^+$. Thus, we have

$$q \in \operatorname{rel\,int}(\operatorname{conv}(X_2)) \subseteq H_1^+ \setminus H_1,$$

which contradicts (2.1). Therefore, there must be at least one point of X_2 in F_1. Using the same argument with H_1^- and G_1 rather than H_1^+ and F_1, respectively, we conclude that there must be at least one point of X_2 in G_1. Relabeling also gives us that there is at least one point of X_1 in F_2 and at least one point of X_1 in G_2. Now, in general, we know that $X_j \cap (H_i^+ \setminus H_i) \neq \emptyset$ and $X_j \cap (H_i^- \setminus H_i) \neq \emptyset$. Therefore, $X_j \cap G_i \neq \emptyset$ and $X_j \cap F_i \neq \emptyset$. ∎

Corollary 2.1.5 (Lee-N)**.** *Let $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$. When the origin q in the Gale diagram falls on multiple k-splitters as described above, then we have to consider whether we are moving from the positive side (the closed halfspace of X_i with more points) towards the negative side (the closed halfspace X_i with fewer points) of each k-splitter X_i when moving the origin from p to q. We subtract one from $g_k(P)$ for every k-splitter which has p on its positive side.*

Proof. The sets of points capturing the origin q for each splitter are disjoint from Lemma 2.1.4. So the sets of faces that are changed when moving the origin from p to q for the different splitters are disjoint. The result then follows by applying Theorem 2.1.1 to each splitter. ■

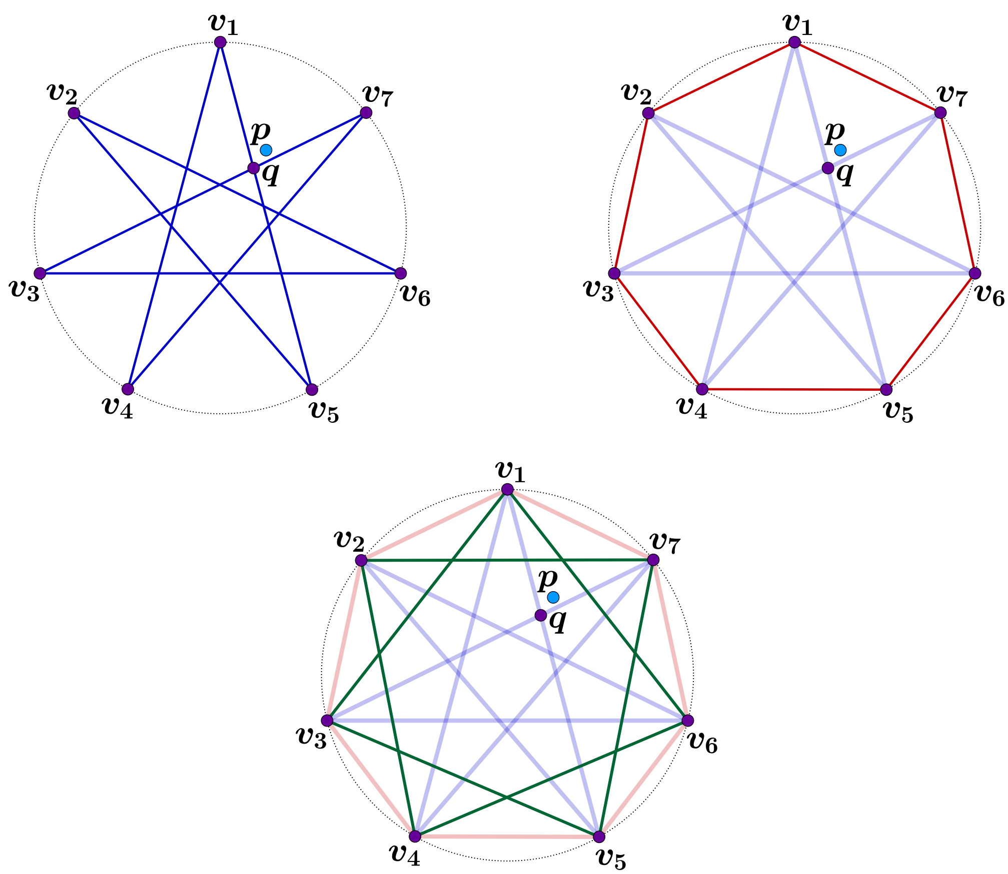

Figure 2.4: Use Corollary 2.1.5 to determine $g(Q)$.

Consider the following example.

Example 2.1.6. See Figure 2.4. In Example 1.8.2, we found $g_2(P) = w_2(p)$ for different points p that fall in affinely general position with respect to the points of V. The point q is in the relative interior of the convex hull of the 2-splitter $\{v_3, v_7\}$ and the 2-splitter $\{v_2, v_6\}$. Let P be the simplicial 4-polytope corresponding to the

Gale diagram in which the origin is p. Notice that p is on the negative side of both the 2-splitter $\{v_3, v_7\}$ and the 2-splitter $\{v_2, v_6\}$. So $g_2(Q) = g_2(P) - 0 = 0$ by Corollary 2.1.5. Then $g(Q) = g(P) = (1, 2, 0)$.

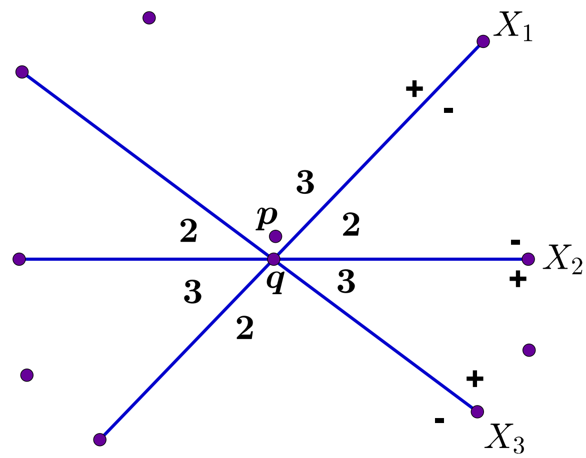

Figure 2.5: Determine $g_3(Q)$ using Corollary 2.1.5.

Note 2.1.7. *In general, $g_k(Q)$ does not equal the minimum w_k of its neighboring regions.*

The following example illustrates this important fact.

Example 2.1.8. See Figure 2.5. Consider this 2-dimensional Gale diagram with origin q corresponding to the non-simplicial 6-polytope Q. Then X_i is a 3-splitter for all $i = 1, 2, 3$. Further, these are the only 3-splitters which capture q. One may show that the numbers in each region indicate $w_3(o)$ for different points o that fall in affinely general position with respect to the points of V. Having the origin at p corresponds to the simplicial 6-polytope P, and $g_3(P) = 3$. Notice that p is on the positive side of X_1 and X_3. Then $g_3(Q) = g_3(P) - 2 = 3 - 2 = 1$ by Corollary 2.1.5. As indicated in the figure, the regions immediately adjacent to q correspond to $g_3 = 2$ or $g_3 = 3$. So $g_3(Q)$ is not the minimum g_3 of its neighboring regions.

Theorem 2.1.9 (Lee-N)**.** *When the origin q in the Gale diagram is in the relative interiors of the convex hulls of multiple splitters, then we only consider the k-splitters when determining $g_k(Q)$. In other words, we consider each set of k-splitters separately.*

With these results, we may determine the g-vector for any d-polytope whose e-dimensional Gale diagram consists of points falling in affinely general position except for the origin q falling in the relative interior of the convex hull of at least one splitter.

Origin Falls on a Point

Now we will determine the the g-vector of a d-polytope Q when the origin coincides with a point in the e-dimensional Gale diagram V. In other words, q falls on exactly one of the points v.

Consider the $(d-1)$-polytope Q' which corresponds to the e-dimensional Gale diagram consisting of the origin q and the $n-1$ points that do not coincide with q; *i.e.*, the set $V \ \setminus \ v$. Now we will argue that the g-vector of Q is the same as the g-vector of Q'.

Theorem 2.1.10 (Lee-N)**.** *When the origin q in the Gale diagram falls on exactly one point v, then we find $g(Q)$ by first deleting v from the Gale diagram.*

Proof. Since q coincides with v in the Gale diagram, we know that Q is a pyramid over Q' by Corollary 1.6.7. As noted in Section 1.3, building a pyramid over a polytope does not change its g-vector (Theorem 1.3.7). Therefore, the g-vectors of Q and Q' are equivalent. ■

If the points in $V \setminus v$ are in affinely general position with respect to each other and q, then Q' is a simplicial polytope. Thus, we may find the g-vector of Q' using the winding number interpretation provided by Lee [18].

Using Corollary 2.1.5 and Theorem 2.1.10, we may now determine the g-vector of any polytope whose Gale diagram consists of all points in affinely general position except for the origin, which lies in the relative interior of the convex hulls of some of the splitters and is itself in V. All of these results hold for e-dimensional Gale diagrams. In the next section, we will continue to study polytopes with e-dimensional Gale diagrams. We will now prove Theorems 2.2.9 and 2.2.9, which establish how to find the g-vector in certain cases of Gale diagrams whose points are in affinely general position except for a subset of points that falls on a single hyperplane containing the origin.

2.2 Considering Hyperplanes

We say the points V in $\mathbb{R}^e$ are in affinely general position if no more than $i+1$ points lie in any common i-dimensional affine space for $0 \leq i \leq e-1$.

First, we consider the case when more than one point lies on a 0-dimensional affine space. We will extend our previous result to argue that the g-vector for an e-dimensional Gale diagram in which multiple points coincide with the origin is the same as the g-vector for that Gale diagram without those stacked points at the origin. Then we will determine the g-vector for certain d-polytopes with e-dimensional Gale diagrams in which there exists a hyperplane H containing at least $e+1$ points.

Origin Falls on Multiple Points

We will determine the the g-vector of a d-polytope Q when the origin coincides with multiple points in the e-dimensional Gale diagram V. In other words, q coincides with the stacked points $v_1, v_2, \ldots, v_\sigma$, where $\sigma \geq 2$.

Consider the $(d-\sigma)$-polytope Q'' which corresponds to the e-dimensional Gale diagram consisting of the origin q and the $n-\sigma$ points that do not coincide with q; *i.e.*, the set $V \setminus \{v_1, v_2, \ldots, v_\sigma\}$. Now we will apply Theorem 2.1.10 to show that the g-vector of Q is the same as the g-vector of Q''.

Theorem 2.2.1 (Lee-N)**.** *Suppose the origin q in the Gale diagram falls on more than one point $\{v_1, v_2, \ldots, v_\sigma\}$ in V. Then we find $g(Q)$ by first deleting the points $v_1, v_2, \ldots, v_\sigma$ from the Gale diagram.*

Proof. Since q coincides with $\{v_1, v_2, \ldots, v_\sigma\}$ in the Gale diagram, we know that Q is a σ-fold pyramid over Q'' by Corollary 1.6.7. As noted in Section 1.3, building a pyramid over a polytope does not change its g-vector (Theorem 1.3.7). Then building a multipyramid over a polytope also does not change the g-vector. Thus, as in Theorem 2.1.10 the g-vectors of Q and Q'' are equivalent. ■

Further, suppose that the points in $V \setminus \{v_1, v_2, \ldots, v_\sigma\}$ are in affinely general position with respect to each other and q. Then Q'' is a simplicial polytope. Thus, we may find the g-vector of Q'' using the winding number interpretation provided by Lee [18].

We have now shown how to determine the g-vector of any d-polytope whose e-dimensional Gale diagram consists of all points in affinely general position except for multiple points coinciding with the origin. Next we will consider what happens when many points fall on a single hyperplane H in the Gale diagram.

For the remainder of this section, we will show how to compute the g-vector from e-dimensional Gale diagrams in which there exists a hyperplane H that satisfies the following conditions.

- H contains at least $e+1$ points from V and another point q;
- The set $Y = V \cap H$ affinely spans H;
- Any minimal affinely dependent subset of V of at most $e+1$ points lies in H;
- Every subset of e points in $Y \cup \{q\}$ is affinely independent.

Let $\gamma \geq e+1$ denote the cardinality of Y. Say $Y = \{v_1, v_2, \ldots, v_\gamma\}$, relabeling if necessary. Then q is in the relative interior of the convex hull of nonempty subsets of Y; denote them $\{Y_\beta\}$.

Here we generalize the notion of a splitter.

Definition 2.2.2 (Lee-N)**.** Suppose $e \geq 1$, $n \geq e+1$, and $d = n-e-1$. Consider a set of vertices $V = \{v_1, ..., v_n\}$ in affinely general position, where each $v_i \in \mathbb{R}^e$. In $\mathbb{R}^e$, a *splitter* is a maximal subset of V affinely spanning a hyperplane.

Thus, the nonempty subsets Y_β of Y are splitters. Then there exists a point p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitters except those Y_β of Y. There also exists a p' in affinely general position with respect to $V \cup q$ that satisfies the following conditions.

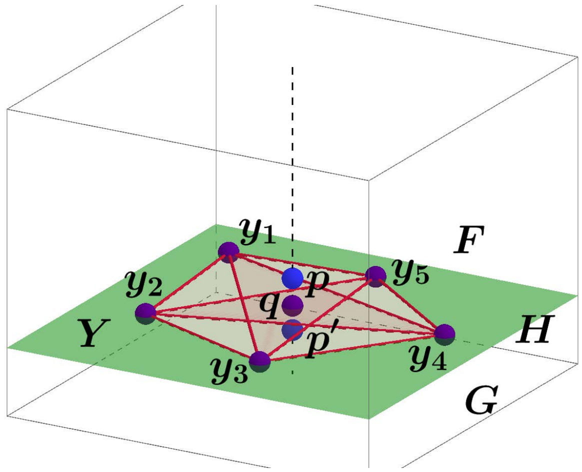

Figure 2.6: Here the splitter consisting of 5 points maximally spans H.

- The point p' is in the line spanned by p and q;
- The lines segment $\overline{pp'}$ does not meet the convex hull of any splitters except the same subsets Y_β of Y;
- The point p' is in the open halfspace of H not containing p.

Let F be the points of V that are in the open halfspace of H containing p, and let G be points of V that are in the opposite open halfspace of H containing p'. See Figure 2.6. Then $V = F \cup G \cup Y$. Note that placing the origin at p and p' corresponds to simplicial polytopes P and P', respectively, and placing the origin at q corresponds to a nonsimplicial polytope Q.

We know how to compute $g(P)$ and $g(P')$ [18]. How does the g-vector change when crossing a hyperplane H with more than e points? How do we compute $g_k(Q)$ when q is in H?

Crossing a Splitter

We will first determine the relationship between the g-vectors determined by placing the origin at p and p', respectively. In other words, we are considering what change corresponds to crossing the convex hull of a set of points on a single hyperplane H with at least $e+1$ points, which are in affinely general position with respect to H.

Proposition 2.2.3 (Lee-N)**.** *Suppose the origin in an e dimensional Gale diagram crosses H, as described above, from F to G. Then for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product*

$$(x-1)^{d-n+1}(x^{|F|} - x^{|G|}) \sum_{Y_\beta} (x-1)^{|Y_\beta|}.$$

Proof. Since we would like to compute the change in the toric g-vector, we are going to determine $g(\partial P')$ in terms of $g(\partial P)$. We need to determine which faces are different for P and P'. Let's start by considering which sets capture p or p' but not both.

Since p and p' are both in affinely general position with respect to V, then any set that captures p or p' in its interior must have full dimension e. Since $\overline{pq}$ does not meet the convex hull of any splitters except the subsets Y_β of Y, we also note that any set which captures p in its interior either captures q in its interior or on its boundary. Since $\overline{pp'}$ also does not meet the convex hull of any splitters except the subsets Y_β of Y, then any set which captures both p and q in its interior also captures p' in its interior.

Sets which capture both p and p' in their interiors correspond to common faces of P and P'. So we only need to consider sets which capture p in their interiors and q on their boundaries. Such sets do not capture p' in their interiors or on their boundaries. Sets which capture p in their interiors and q on their boundaries are exactly the sets of the form $F' \cup Y_\beta$ for every nonempty subset F' of F and every nonempty subset Y_β of Y that captures q.

Applying the same argument to p', we know that we only need to consider sets which capture p' in their interiors and q on their boundaries. Such sets do not capture p in their interiors or on their boundaries. Sets which capture p' in their interiors and q on their boundaries are exactly the sets of the form $G' \cup Y_\beta$ for every nonempty subset G' of G and every nonempty subset Y_β of Y that captures q.

Since we know what is different between the sets that capture only p or p', then we also know exactly what is different about the faces of P and P' by considering complements by Definition 1.6.6. P has all the faces of the form $V \setminus (F' \cup Y_\beta) = (F \setminus F') \cup G \cup (Y \setminus Y_\beta)$ for every nonempty subset F' of F and every nonempty subset Y_β of Y that captures q. P' has all the faces of the form $V \setminus (G' \cup Y_\beta) = F \cup (G \setminus G') \cup (Y \setminus Y_\beta)$ for every nonempty subset G' of G and every nonempty subset Y_β of Y that captures q.

Then we may compute the h-polynomial of $\partial P'$ by subtracting the terms of the h-polynomial for all the faces of ∂P not in $\partial P'$ and adding the terms of the h-polynomial for all the faces of $\partial P'$ not in ∂P to the h-polynomial of ∂P.

$$\begin{aligned} h(\partial P', x) &= h(\partial P, x) \\ &- \sum_{G \text{ face of } \partial P \text{ not in } \partial P'} g(\partial G, x)(x-1)^{d-1-\dim(G)} \\ &+ \sum_{G \text{ face of } \partial P' \text{ not in } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)} \end{aligned}$$

First, we will rewrite the second term. Since P is simplicial, all of its faces are simplices. Consider $L = (F \setminus F') \cup G \cup (Y \setminus Y_\beta)$. Since L is a simplex, then $g(\partial L, x) = 1$ by Lemma 1.3.13 and $\dim(L) = \operatorname{card}(L) - 1 = |F| - |F'| + |G| + |Y| - |Y_\beta| - 1 = n - |F'| - |Y_\beta| - 1$.

So we have the following:

$$\begin{aligned}
&\sum_{\text{L face of } \partial P \text{ not in } \partial P'} g(\partial L, x)(x-1)^{d-1-\dim(L)} \\
= &\sum_{\text{L face of } \partial P \text{ not in } \partial P'} (1)(x-1)^{d-1-(n-|F'|-|Y_\beta|-1)} \\
= &\sum_{\text{L face of } \partial P \text{ not in } \partial P'} (x-1)^{d-n+|F'|+|Y_\beta|} \\
= &\sum_{\emptyset \subsetneq T \subset F} \sum_{Y_\beta} (x-1)^{d-n+|F'|+|Y_\beta|} \\
= &(x-1)^{d-n} \sum_{\emptyset \subsetneq T \subset F} (x-1)^{|F'|} \sum_{Y_\beta} (x-1)^{|Y_\beta|} \\
= &(x-1)^{d-n} \sum_{i=1}^{|F|} \binom{|F|}{i} (x-1)^i \sum_{Y_\beta} (x-1)^{|Y_\beta|} \\
= &(x-1)^{d-n}(x^{|F|}-1) \sum_{Y_\beta} (x-1)^{|Y_\beta|} \text{ by the Binomial Theorem.}
\end{aligned}$$

Next, we will rewrite the third term. In this case, P' is simplicial and $L = F \cup (G \setminus G') \cup (Y \setminus Y_\beta)$. So $g(\partial L, x) = 1$ (Lemma 1.3.13) and $\dim(L) = \operatorname{card}(L) - 1 = |F| + |G| - |G'| + |Y| - |Y_\beta| - 1 = n - |G'| - |Y_\beta| - 1$.

Then we have the following:

$$\begin{aligned}
&\sum_{\text{L face of } \partial P \text{ not in } \partial P'} g(\partial L, x)(x-1)^{d-1-\dim(L)} \\
= &(x-1)^{d-n}(x^{|G|}-1) \sum_{Y_\beta} (x-1)^{|Y_\beta|} \text{ by a similar argument.}
\end{aligned}$$

With both of these results, we are able to determine the change in the toric h-vector when we move the origin in the Gale diagram from p to p'. The change in the toric h-vector is

$$\begin{aligned}
\Delta(x) &= (x-1)^{d-n}\left(-(x^{|F|}-1) + (x^{|G|}-1)\right) \textstyle\sum_{Y_\beta} (x-1)^{|Y_\beta|} \\
&= (x-1)^{d-n}(x^{|G|} - x^{|F|}) \textstyle\sum_{Y_\beta} (x-1)^{|Y_\beta|}.
\end{aligned}$$

Since the toric h-vector is unimodal by Theorem 1.3.16 [15], then $g(x)$ agrees with $(1-x) \cdot h(x)$ in the lower degree terms $\leq \left\lfloor \frac{d}{2} \right\rfloor$, *i.e.*, $h(P', x) - h(P, x) = \Delta(x)$. Both $h(P', x)$ and $h(P, x)$ are symmetric; hence, so is $\Delta(x)$. Thus, the change in $g(x)$ matches the lower order terms (x^j, $0 \leq j \leq \left\lfloor \frac{d}{2} \right\rfloor$) of $(1-x)\Delta(x)$. Therefore, the toric g-vector as we move the origin in the Gale diagram from p to p' is given by

$$\begin{aligned}
&(1-x)(x-1)^{d-n}(x^{|G|} - x^{|F|}) \sum_{Y_\beta} (x-1)^{|Y_\beta|} \\
&= -(x-1)^{d-n+1}(x^{|G|} - x^{|F|}) \sum_{Y_\beta} (x-1)^{|Y_\beta|} \\
&= (x-1)^{d-n+1}(x^{|F|} - x^{|G|}) \sum_{Y_\beta} (x-1)^{|Y_\beta|}
\end{aligned}$$

■

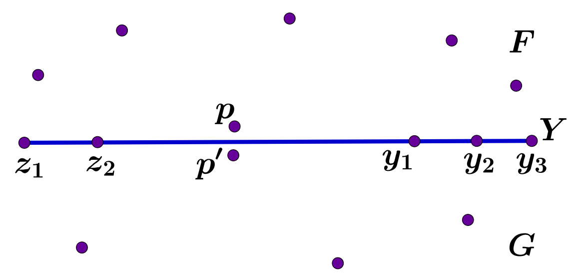

Figure 2.7: We will determine the change in the g-vector when crossing this splitter.

We will apply this result to the following example.

Example 2.2.4. Consider this 2-dimensional Gale diagram with 13 points as shown in Figure 2.7. Let the origin p correspond to the simplicial 10-polytope P and the origin p' correspond to the simplicial 10-polytope P'. Then the change in the k^{th} component of the toric g-vector when the origin moves from p to p' is given by the coefficient of x^k (for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor = 5$) in the product

$$(x-1)^{-2}(x^5 - x^3)\sum_{Y_\beta}(x-1)^{|Y_\beta|}$$

by Proposition 2.2.3.

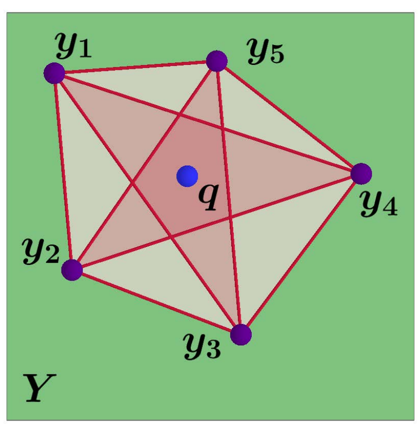

Figure 2.8: Here Y is a 2-dimensional Gale diagram corresponding to a simplicial 2-polytope P.

Now consider Y as a set of points in dimension $e-1$ within the ambient space H. In this case, q corresponds to a simplicial polytope $\widehat{Q}$ of dimension $|Y|-(e-1)-1 = \gamma-e$. Figure 2.8 shows the Gale diagram Y that corresponds to the hyperplane in Figure 2.6.

Proposition 2.2.5 (Lee-N). *Suppose the origin in H lies in the convex hull of points Y that affinely span H and are not in affinely general position. When the origin falls in affinely general position with respect to Y, then the k^{th} component of the toric h-vector of $\widehat{Q}$ is given by the coefficient of x^k in the product*

$$(x-1)^{d-n+1}\sum_{Y_\beta}(x-1)^{|Y_\beta|}.$$

Proof. Since $\widehat{Q}$ is simplicial, all of its faces are simplices. So we may use the ordinary h- and g-vectors and the fact that $g(\partial\text{simplex}, x) = 1$. Only the subsets Y_β of Y capture q. By considering complements, all the faces of $\widehat{Q}$ have the form $Y \setminus Y_\beta$. Since $Y \setminus Y_\beta$ is a simplex, then $\dim(Y \setminus Y_\beta) = \text{card}(Y \setminus Y_\beta) - 1 = |Y| - |Y_\beta| - 1 = \gamma - |Y_\beta| - 1$.

So we have the following:

$$\begin{aligned}
h(\partial\widehat{Q}, x) &= \sum_{\text{L face of } \partial\widehat{Q}} g(\partial L, x)(x-1)^{\dim(\widehat{Q})-1-\dim(L)} \\
&= \sum_{Y_\beta}(1)(x-1)^{(\gamma-e)-1-(\gamma-|Y_\beta|-1)} \\
&= \sum_{Y_\beta}(x-1)^{|Y_\beta|-e} \\
&= \sum_{Y_\beta}(x-1)^{|Y_\beta|+d-n+1} \\
&= (x-1)^{d-n+1}\sum_{Y_\beta}(x-1)^{|Y_\beta|}
\end{aligned}$$

■

Since q is in affinely general position with respect to Y, we may combine the previous two results in the following theorem.

Theorem 2.2.6 (Lee-N). *Suppose H and Y are defined as above. Then for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector corresponding to crossing H from p to p' in dimension e equals the k^{th} component of the product of the difference $x^{|F|} - x^{|G|}$ times the toric h-vector of $\widehat{Q}$.*

Proof. By Proposition 2.2.3, for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product

$$(x-1)^{d-n+1}(x^{|F|} - x^{|G|})\sum_{Y_\beta}(x-1)^{|Y_\beta|}$$

$$= (x^{|F|} - x^{|G|})(x-1)^{d-n+1}\sum_{Y_\beta}(x-1)^{|Y_\beta|}$$

$$= (x^{|F|} - x^{|G|})h(\partial\widehat{Q}, x) \text{ by Proposition 2.2.5.}$$

■

Figure 2.9: The 1-dimensional Gale diagram of Y.

Example 2.2.4 continued. Consider the 1-dimensional Gale diagram in Figure 2.9. Let the origin q in this Gale diagram correspond to the simplicial 3-polytope $\hat{Q}$. This Gale diagram is combinatorially equivalent to the Gale diagram of the triangular bipyramid in Figure 1.15. So $\hat{Q}$ is the triangular bipyramid from Example 1.2.4. Then $h(\partial\hat{Q}, x) = x^3 + 2x^2 + 2x + 1$ by Example 1.3.6. Since this is also the 1-dimensional Gale diagram of Y from Figure 2.7, then the change in the k^{th} component of the toric g-vector (for $0 \le k \le \lfloor \frac{d}{2} \rfloor = 5$) is given by the coefficient of x^k in the product

$$\begin{aligned} & (x^5 - x^3)(x^3 + 2x^2 + 2x + 1) \\ = & (x^8 + 2x^7 + 2x^6 + 1x^5) - (x^6 + 2x^5 + 2x^4 + 1x^3) \\ = & x^8 + 2x^7 + x^6 - x^5 - 2x^4 - 1x^3 \end{aligned}$$

by Theorem 2.2.6. Thus, $g(\partial P', x) = g(\partial P, x) - x^5 - 2x^4 - 1x^3$.

So we know how to compute the change in the toric g-vector when the origin moves across a hyperplane H whose points are not in affinely general position. How do we compute the toric g-vector of a polytope Q whose origin q lies on a hyperplane H with more than e points?

Landing on a Splitter

Now we will determine the toric g-vector for q when q falls on the hyperplane H with at least $e + 1$ points, as described after the proof of Theorem 2.2.1. See Figures 2.6 and 2.8.

Let $\ell = \min\{|F|, |G|\}$. Then the following result holds.

Proposition 2.2.7 (Lee-N)**.** *Suppose the origin in an e dimensional Gale diagram is in the convex hull of Y whose points span the hyperplane H, as described above. Then for $0 \le k \le \lfloor \frac{d}{2} \rfloor$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product*

$$(x-1)^{d-n+1}(x^{|F|} - x^{\ell}) \sum_{Y_\beta} (x-1)^{|Y_\beta|}.$$

Proof. Since we would like to compute the change in the toric g-vector, we are going to determine $g(\partial Q)$ in terms of $g(\partial P)$. We need to determine which faces are different for P and Q. Let's start by considering which sets capture p or q but not both.

Since p is in affinely general position with respect to V, then any set that captures p in its interior must have full dimension e. Further, $\overline{pq}$ does not meet the convex hull of any splitters except all the nonempty subsets Y_β of Y that capture q. We also note that any set which captures p in its interior either captures q in its interior or on its boundary.

Sets which capture both p and q in their interiors correspond to common faces of P and Q. So we only need to consider sets which capture p in their interiors and q on their boundaries. Sets which capture p in their interiors and q on their boundaries are exactly the sets of the form $F' \cup Y_\beta$ for every nonempty subset F' of F and every nonempty subset Y_β of Y that captures q.

The nonempty subsets Y_β of Y are the only sets which capture only q.

Since we know what is different between the sets that capture only p or q, then we also know exactly what is different about the faces of P and Q by considering complements (Definition 1.6.6). P has all the faces of the form $V \setminus (F' \cup Y_\beta) = (F \setminus F') \cup G \cup (Y \setminus Y_\beta)$ for every nonempty subset F' of F and every nonempty subset Y_β of Y that captures q. Q has all the faces of the form $V \setminus Y_\beta = F \cup G \cup (Y \setminus Y_\beta)$ for every nonempty subset Y_β of Y that captures q.

Then we may compute the h-polynomial of ∂Q by subtracting the terms of the h-polynomial for all the faces of ∂P not in ∂Q and adding the terms of the h-polynomial for all the faces of ∂Q not in ∂P to the h-polynomial of ∂P.

$$\begin{aligned} h(\partial Q, x) &= h(\partial P, x) \\ &- \sum_{G \text{ face of } \partial P \text{ not in } \partial Q} g(\partial G, x)(x-1)^{d-1-\dim(G)} \\ &+ \sum_{G \text{ face of } \partial Q \text{ not in } \partial P} g(\partial G, x)(x-1)^{d-1-\dim(G)} \end{aligned}$$

From the proof of Proposition 2.2.3 in the previous section, the second term may be rewritten as follows.

$$\begin{aligned} & \sum_{\text{L face of } \partial P \text{ not in } \partial Q} g(\partial L, x)(x-1)^{d-1-\dim(L)} \\ = \;& (x-1)^{d-n}(x^{|F|}-1) \sum_{Y_\beta} (x-1)^{|Y_\beta|} \end{aligned}$$

Next, we will rewrite the third term. In this case, $L = F \cup G \cup (Y \setminus Y_\beta)$. Recall that in the hyperplane H, no $e-1$ points fall on an $(e-2)$-dimensional affine space with q. So these sets Y_β that capture q must be full dimensional within H; *i.e.*, they are $(e-1)$-dimensional with at least e points. Note that any set of e points of Y spans the hyperplane H. In particular, each Y_β spans H.

McMullin proved that projecting orthogonally to Y_β and then deleting its points gives the Gale diagram of the face L, regarded as a polytope [22]. Since Y_β spans H, the Gale diagram of L is a one-dimensional line with the points F falling on one side of q, the points of G falling on the other side of q and $|Y| - |Y_\beta| = \gamma - |Y_\beta|$ points coinciding with q. Then $g(\partial L, x) = 1 + x + x^2 + \cdots + x^{\ell-1}$. If Y_β is a cofacet, then L is a facet of dimension $d-1 = n-e-2$ and cardinality $n-e$. The dimension and cardinality of L decreases by 1 for each additional point added to a cofacet. Thus, the $\dim(L) = n - |Y_\beta| - 2$ for the face L corresponding to any coface Y_β.

Then we have the following:

$$\begin{aligned}
& \sum_{L \text{ face of } \partial Q \text{ not in } \partial P} g(\partial L, x)(x-1)^{d-1-\dim(L)} \\
= & \sum_{Y_\beta} \frac{x^\ell - 1}{x-1}(x-1)^{d-1-(n-|Y_\beta|-2)} \\
= & \frac{x^\ell - 1}{x-1} \sum_{Y_\beta}(x-1)^{|Y_\beta|-e} \\
= & \frac{x^\ell - 1}{(x-1)^{e+1}} \sum_{Y_\beta}(x-1)^{|Y_\beta|} \\
= & (x-1)^{d-n}(x^\ell - 1) \sum_{Y_\beta}(x-1)^{|Y_\beta|}.
\end{aligned}$$

With both of these results, we are able to determine the change in the toric h-vector when we move the origin in the Gale diagram from p to q. The change in the toric h-vector is

$$\begin{aligned}
& (x-1)^{d-n} \left(-(x^{|F|} - 1) + (x^\ell - 1) \right) \sum_{Y_\beta}(x-1)^{|Y_\beta|} \\
= & (x-1)^{d-n}(x^\ell - x^{|F|}) \sum_{Y_\beta}(x-1)^{|Y_\beta|}.
\end{aligned}$$

As before, since the toric h-vector is symmetric by Theorem 1.3.16 [15], then $g(x)$ agrees with $(1-x) \cdot h(x)$ in the lower degree terms $\leq \left\lfloor \frac{d}{2} \right\rfloor$. Therefore, the change in the lower terms of the toric g-vector as we move the origin in the Gale diagram from p to q is given by

$$\begin{aligned}
& (1-x)(x-1)^{d-n}(x^\ell - x^{|F|}) \sum_{Y_\beta}(x-1)^{|Y_\beta|} \\
= & -(x-1)^{d-n+1}(x^\ell - x^{|F|}) \sum_{Y_\beta}(x-1)^{|Y_\beta|} \\
= & (x-1)^{d-n+1}(x^{|F|} - x^\ell) \sum_{Y_\beta}(x-1)^{|Y_\beta|}.
\end{aligned}$$

■

We will apply this result to the following example.

Example 2.2.8. Consider the 2-dimensional Gale diagram with 13 points as shown in Figure 2.10. Let the origin p correspond to the simplicial 10-polytope P and the origin q correspond to the nonsimplicial 10-polytope Q. Since $\ell = |G| < |F|$, the change in the k^{th} component of the toric g-vector when the origin moves from p to q is given by the coefficient of x^k (for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor = 5$) in the product

$$(x-1)^{-2}(x^5 - x^3) \sum_{Y_\beta}(x-1)^{|Y_\beta|}$$

by Proposition 2.2.7.

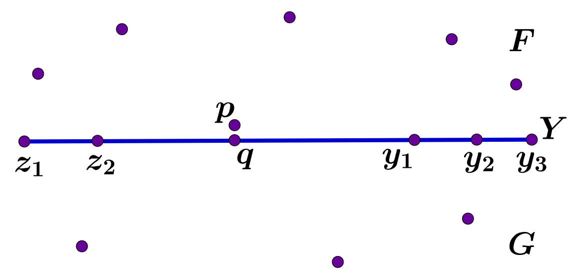

Figure 2.10: We will determine the change in the g-vector when landing on this splitter.

Recall that when considering Y as a set of points in dimension $e-1$ within the ambient space H, q corresponds to a simplicial polytope $\widehat{Q}$ of dimension $|Y|-(e-1)-1=\gamma-e$. Then we may use Proposition 2.2.5 to express $g(\partial Q)$ in terms of the $(\gamma-e)$-dimensional simplicial polytope $\widehat{Q}$.

Theorem 2.2.9 (Lee-N)**.** *Suppose H, Y, and ℓ are defined as above. Then for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector corresponding to moving the origin from p to q on H equals the k^{th} component of the product of the difference $x^{|F|}-x^{\ell}$ times the toric h-vector of $\widehat{Q}$.*

Proof. By Proposition 2.2.7, for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product

$$\begin{aligned}
& (x-1)^{d-n+1}(x^{|F|}-x^{\ell})\sum_{Y_\beta}(x-1)^{|Y_\beta|} \\
= \; & (x^{|F|}-x^{\ell})(x-1)^{d-n+1}\sum_{Y_\beta}(x-1)^{|Y_\beta|} \\
= \; & (x^{|F|}-x^{\ell})h(\partial\widehat{Q},x) \text{ by Proposition 2.2.5.}
\end{aligned}$$

∎

When $|F| \le |G|$, the following result is immediate.

Corollary 2.2.10 (Lee-N)**.** *Suppose H, Y, and ℓ are defined as above. Let $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$. If $|F| \le |G|$, then there is no change in the k^{th} component of the toric g-vector corresponding to moving the origin from p to q on H.*

Example 2.2.8 continued. Consider the 1-dimensional Gale diagram in Figure 2.9. Recall that the origin q in this Gale diagram correspond to the simplicial 3-polytope $\hat{Q}$. As noted in Example 2.2.4, $\hat{Q}$ is the triangular bipyramid from Example 1.2.4 and $h(\partial\hat{Q},x)=x^3+2x^2+2x+1$. Since this is the 1-dimensional Gale diagram of Y from Figure 2.10, then the change in the k^{th} component of the toric g-vector (for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor = 5$) is given by the coefficient of x^k in the product

$$(x^{|F|}-x^{\ell})h(\partial\widehat{Q},x)$$

by Theorem 2.2.9. Further, since $\ell = |G| < |F|$, this product is equivalent to the product

$$(x^{|F|} - x^{\ell})h(\partial\widehat{Q}, x).$$

Thus, $g(\partial Q, x) = g(\partial P, x) - x^5 - 2x^4 - 1x^3$ by the calculations in Example 2.2.4.

Multiple Splitters

We will discuss Lemma 2.1.4 in the context of splitters.

Consider an e-dimensional Gale diagram in which there exists α hyperplanes H_i, with $\alpha > 1$, satisfying the following conditions.

- Each H_i contains at least e points from V and the point $q \notin V$;
- The set $Y_i = V \cap H_i$ affinely spans H_i for each i;
- Any minimal affinely dependent subset of V of at most $e + 1$ points lies in one of the H_i's;
- Every subset of e points in any $Y_i \cup \{q\}$ is affinely independent;
- $H_i \neq H_j$ for all $i \neq j$.

Let $\gamma_i \geq e+1$ denote the cardinality of Y_i. Say $Y_i = \{v_{1,i}, v_{2,i}, \ldots, v_{\gamma,i}\}$, relabeling if necessary. Then q is in the relative interior of the convex hull of nonempty subsets of Y_i, denote them $\{Y_{\beta,i}\}$.

As we saw earlier in this section, nonempty subsets $Y_{\beta,i}$ of Y_i are splitters.

Then there exists a point p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitters except those $Y_{\beta,i}$ of Y. Let F_i be the points of V that are in the open halfspace of H_i containing p, and let G_i be points of V that are in the opposite open halfspace of H not containing p. Then $V = F_i \cup G_i \cup Y_i$ for each i. Note that placing the origin at p corresponds to simplicial polytopes P, and placing the origin at q corresponds to a nonsimplicial polytope Q.

For each i, let H_i^- and H_i^+ be the closed halfspaces of Y_i on the same side as F_i and G_i, respectively. Then we may extend Lemma 2.1.4 for k-splitters to the more general definition of splitters.

Lemma 2.2.11 (Lee-N)**.** *If Y_i and Y_j contain a common point q in their relative interiors for $1 \leq i < j \leq \alpha$, then there exists a point of Y_j in each of G_i and F_i.*

Proof. The argument in the proof of Lemma 2.1.4 relies on the hyperplanes and closed halfspaces determined by a given X_i. Replacing the X_i from the k-splitter setting with the Y_i from the current splitter setting does not change the argument. Thus, the proof of this result follows immediately from the proof of Lemma 2.1.4. ■

Therefore, the following results also hold in this more general setting.

Corollary 2.2.12 (Lee-N)**.** *Let $0 \le k \le \lfloor \frac{d}{2} \rfloor$. When the origin q in the Gale diagram falls on multiple splitters as described above, then each splitter acts independently. To determine the total change in the g-vector, we sum over the change for each splitter using Corollary 2.1.5 and Theorem 2.2.9.*

Proof. The sets of points capturing the origin q for each splitter are disjoint from Lemma 2.2.11. So the sets of faces that are changed when moving the origin from p to q for the different splitters are disjoint. The result then follows by applying Theorem 2.2.9 to each splitter. ∎

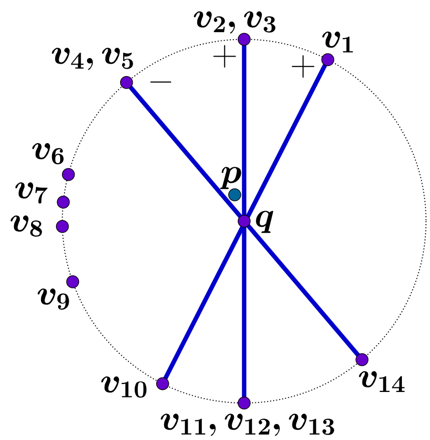

Figure 2.11: We will determine how the g-vector changes when the origin moves onto multiple splitters from p to q.

We will apply this result to the 2-dimensional Gale diagram in Figure 2.11.

Example 2.2.13. Consider the 2-dimensional Gale diagram with 14 points as shown in Figure 2.11. Let the origin p correspond to the simplicial 11-polytope P and the origin q correspond to the nonsimplicial 11-polytope Q. Note that q is captured by the 4-splitter $\{v_1, v_{10}\}$, the splitter $\{v_2, v_3, v_{11}, v_{12}, v_{13}\}$, and the splitter $\{v_4, v_5, v_{15}\}$. Since p is on the positive side of the 4-splitter $\{v_1, v_{10}\}$, then this splitter decreases $g_4(P)$ by one.

Notice that p is also on the positive side of the splitter $\{v_2, v_3, v_{11}, v_{12}, v_{13}\}$. The Gale diagram $\{v_2, v_3, v_{11}, v_{12}, v_{13}\}$ as a set of points in dimension 1 is combinatorially equivalent to the Gale diagram of the triangular bipyramid in Figure 1.15. As we calculated in Example 2.2.4, $h(\partial\text{triangular bipyramid}, x) = x^3+2x^2+2x+1$. Then the contribution of this splitter to the change in the k^{th} component of the toric g-vector (for $0 \le k \le \lfloor \frac{d}{2} \rfloor = 5$) is given by the coefficient of x^k in the product

$$\begin{aligned} & (x^7 - x^2)(x^3 + 2x^2 + 2x + 1) \\ = \; & (x^{10} + 2x^9 + 2x^8 + x^7) - (x^5 + 2x^4 + 2x^3 + x^2) \end{aligned}$$

by Theorem 2.2.9. So this splitter decreases $g_5(P)$ by one, $g_4(P)$ by two, $g_3(P)$ by two, and $g_2(P)$ by one.

Note that p is on the negative side of the splitter $\{v_4, v_5, v_{15}\}$. Then by Corollary 2.2.10, this splitter contributes no change to the g-vector as the origin moves from p to q.

Therefore, $g(\partial Q, x) = g(\partial P, x) - x^5 - 3x^4 - 2x^3 - 1x^2$.

2.3 Complete Analysis for Gale Diagrams of Polytopes with Few Vertices

Recall that a d-polytope with few vertices has at most $d+3$ vertices. In this section, we want to determine the g-vector of any polytope with few vertices from its Gale diagram. Polytopes with few vertices have Gale diagrams of dimension at most 2.

Gale Diagrams of Dimension 0 and 1

Theorem 2.3.1 (Lee-N)**.** *Let P be any d-dimensional polytope with a 0-dimensional Gale diagram. Then the g-polynomial of P is $g(P, x) = 1$.*

Proof. Any d-polytope P with 0-dimensional Gale diagram is a d-simplex. Then the result follows immediately from Lemma 1.3.13. ■

Now consider any d-polytope Q with a 1-dimensional Gale diagram. The $d+2$ points are collinear in the Gale diagram. So u points fall on one side of the origin q, σ points coincide with the origin q, and the remaining r points fall on the other side of the origin q where $u + \sigma + r = d + 2$, $u \geq r \geq 1$, and $\sigma \geq 0$.

Theorem 2.3.2 (Lee-N)**.** *Let Q be any d-dimensional polytope with a 1-dimensional Gale diagram. Suppose σ points coincide with the origin q. Let u points fall on one side of the origin q and r points fall on the other side of the origin q. If $r \leq u$, then $g(Q, x) = 1 + x + x^2 + \cdots + x^{r-1}$.*

Proof. By Corollary 1.6.7, Q is a σ-fold pyramid over the polytope $\hat{Q}$ whose 1-dimensional Gale diagram has u points on one side of the origin q and r points on the other side of the origin q. Since the origin q falls in affinely general position in the Gale diagram of $\hat{Q}$, then $\hat{Q}$ is a simplicial $(d-\sigma)$-polytope. Since $u \geq r$, the g-polynomial is $g(\hat{Q}, x) = 1 + x + x^2 + \cdots + x^{r-1}$ by Lee's result in [18].

Therefore, the result follows from Theorem 1.3.7. ■

Using these two results, we may determine the g-vector of any polytope whose Gale diagram has dimension 0 or 1. Now we turn out attention to polytopes with 2-dimensional Gale diagrams. Notice that polytopes with 2-dimensional Gale diagrams are the convex hulls of $d+3$ points.

Gale Diagrams of Dimension 2

When the $d+3$ points of V and the origin q all fall in affinely general position with respect to one another, then Q is a simplicial d-polytope. In this case, we determine $g(Q)$ using Lee's result [18].

For any polytope Q whose origin q coincides with σ point(s) (with $\sigma \geq 0$) in the Gale diagram V, the g-vector of Q is the same as the g-vector of the polytope $\hat{Q}$ whose Gale diagram consists of the points $\hat{V} = \{v \in V | v \neq q\}$ by Theorem 2.1.10 and Theorem 2.2.1. If all the points in $\hat{V} \cup \{q\}$ are in affinely general position with respect to each other, then the d-polytope Q is a σ-fold pyramid over the simplicial $(d-\sigma)$-polytope $\hat{Q}$. In this case, we determine $g(\hat{Q})$ using Lee's results [18] and, subsequently, know $g(Q)$.

In all other cases, the $d+4$ points of $V \cup \{q\}$ are not in affinely general position with respect to each another. Suppose that $Y_1, Y_2, \ldots, Y_\alpha$ are the only splitters that "capture" q. Then there exists a point p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitter except $Y_1, Y_2, \ldots, Y_\alpha$. To determine the total change in the g-vector, we sum over the change for each splitter using Corollary 2.1.5 and Theorem 2.2.9.

Using these results, we may determine the g-vector of any polytope whose Gale diagram is 2-dimensional. Now we will analyze splitters in 2-dimensional Gale diagrams further.

For the remainder of this section, we will show how to compute the g-vector from 2-dimensional Gale diagrams in which there exists a line H that satisfies the following conditions.

- The line H contains at least 2 points from V and the origin q;
- The set $Y = V \cap H$ affinely spans H;
- Any minimal affinely dependent subset of V of at most 3 points lies in H;
- Every subset of 2 points in $Y \cup \{q\}$ is affinely independent.

Without loss of generality, say u points $Y_1 = \{y_1, y_2, \ldots, y_u\}$ of Y fall on one side of q and the remaining r points $Y_2 = \{z_1, z_2, \ldots z_r\}$ of Y fall on the other side of q with $u \geq r \geq 1$. Note that all the points in $V \setminus Y$ are in affinely general position with respect to $V \cup \{q\}$. Then q is in the relative interior of the convex hull of nonempty subsets of Y, denote them Y_β. In this case, each Y_β is the union of any nonempty subset Y_1' of Y_1 with any nonempty subset Y_2' of Y_2.

Recall that the the nonempty subsets Y_β of Y are splitters. In particular, $Y_1' \cup Y_2'$ is a splitter for every nonempty subset Y_1' of Y_1 and every nonempty subset Y_2' of Y_2.

Then there exists a point p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitters except those Y_β of Y. There also exists a p' in affinely general position with respect to $V \cup q$ that satisfies the following conditions.

- The point p' is in the affine hull of $\{p, q\}$;

- The line segment $\overline{pp'}$ does not meet the convex hull of any splitters except the same subsets Y_β of Y;

- The point p' is in the open halfplane not containing p.

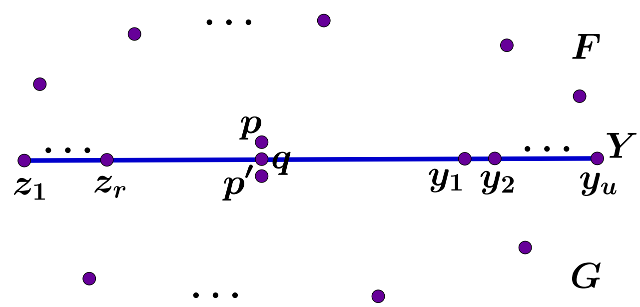

Figure 2.12: Here is a line in a 2-dimensional Gale Diagram with points in $V \cup \{p\} \cup \{p'\} \cup \{q\}$.

Let F be the points of V that are in the open halfplane of H containing p, and let G be points of V that are in the opposite open halfplane of H containing p'. See Figure 2.12. Then $V = F \cup G \cup Y = F \cup G \cup Y_1 \cup Y_2$. Note that placing the origin at p and p' corresponds to simplicial polytopes P and P', respectively, and placing the origin at q corresponds to a nonsimplicial polytope Q.

Since p and p' fall in affinely general position with respect to V, we know how to compute $g(P)$ and $g(P')$ [18]. How does the g-vector change when crossing a splitter with at least 2 points? How do we compute $g_k(Q)$ when q is in the relative interior of the convex hull of that splitter? In both of these cases, we show that various g_k's will change simultaneously and, thus, intuitively, there are multiple k-splitters winding around the point q.

Crossing a Splitter With Multiple Points on Each End

We will determine the change in the g-vector when the origin moves from p to p'. In other words, we show how the g-vector changes when the origin crosses the relative interior of the convex hull of a single splitter Y with $|Y_1| = u$ points on one end and $|Y_2| = r$ points on the other end.

Theorem 2.3.3 (Lee-N)**.** *When the origin in a 2-dimensional Gale diagram moves from p to p' crossing a single splitter Y from F to G with u points on one end and r points on the other end, then for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$ the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product*

$$(x^{r-1} + x^{r-2} + \cdots + x^2 + x + 1)(x^{u-1} + x^{u-2} + \cdots + x^2 + x + 1)\left(x^{|F|} - x^{|G|}\right).$$

Proof. By Theorem 2.2.6, for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product

$$(x^{|F|} - x^{|G|})h(\partial\widehat{Q}, x).$$

The sets Y_β which capture q are of the form $Y_1' \cup Y_2'$ for every nonempty subset Y_1' and Y_2' of Y_1 and Y_2, respectively. Since all the faces of $\partial\widehat{Q}$ are of the form $(Y_1 \backslash Y_1') \cup (Y_2 \backslash Y_2')$ for every nonempty subset Y_1' of Y_1 and every nonempty subset Y_2' of Y_2, then the $\partial\widehat{Q}$ is actually the simplicial join $\partial 2^{Y_1} * \partial 2^{Y_2}$. Therefore, $h(\partial\widehat{Q}, x) = h(\partial\overline{Y_1}, x) \cdot h(\partial\overline{Y_2}, x)$ by Lemma 1.3.15.

Then the change in the toric g-vector is encoded by

$$\frac{x^r - 1}{x - 1} \cdot \frac{x^u - 1}{x - 1}(x^{|F|} - x^{|G|})$$

and the result follows. ■

Then the following corollary is an immediate consequence.

Corollary 2.3.4 (Lee-N)**.** *When the same number of points are on both sides of the single splitter, i.e., $|F| = |G|$, then there is no change in the k^{th} component of the toric g-vector for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$.*

When there is exactly one point on each end of the splitter, then we know what change occurs by Theorem 2.1.1.

Corollary 2.3.5 (Lee-N)**.** *When $r = u = 1$, then the change in the toric g-vector is the same as the change in the toric g-vector when crossing a single ℓ-splitter, where $\ell = min\{|F|, |G|\}$ (Theorem 2.1.1).*

Proof. In the case that $r = u = 1$, our theorem states that the change in the g-polynomial is the restriction of the following product to terms of degree $\leq \left\lfloor \frac{d}{2} \right\rfloor$:

$$(1)(1)\left(x^{|F|} - x^{|G|}\right) = x^{|F|} - x^{|G|}.$$

When we also have $|F| < |G|$, then $g_{|F|}(P') = g_{|F|}(P) + 1$. So you increase by one when moving to the side with more points. When we also have $|F| > |G|$, then $g_{|G|}(P') = g_{|F|}(P) - 1$. So you decrease by one when moving to the side with fewer points. Thus, we get the change that we were expecting, and are generalizing, when crossing a *single* ℓ-splitter, where $\ell = \min\{|F|, |G|\}$. ■

Note that even when the points in $Y_1(Y_2)$ coincide, then these results would still hold. Whether the $u(r)$ points are distinct or coincide, the sets Y_β of Y do not change. Thus, the faces of P and P' are the same.

Example 2.3.6. Let $e = 2$. See Figure 2.13. Suppose $|Y_2| = 6$, $|F| = 3$, $|Y_1| = 4$, and $|G| = 5$. Then there are 18 points in V, $d = 18 - 2 - 1 = 15$, and $\left\lfloor \frac{d}{2} \right\rfloor = 7$. By

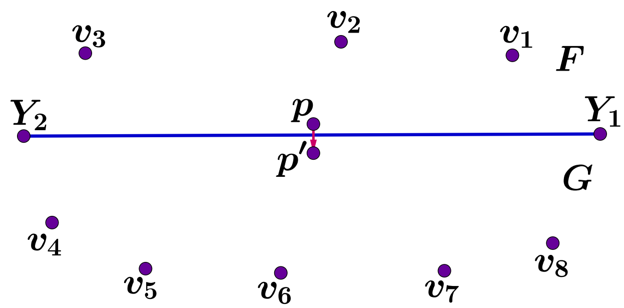

Figure 2.13: We will determine the change in the g-vector when crossing a line.

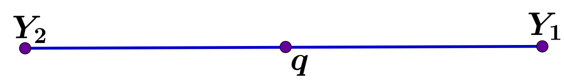

Figure 2.14: Here is the 1-dimensional Gale diagram of Y.

Theorem 2.3.3, for $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor = 7$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product

$$\begin{aligned} & (x^5 + x^4 + x^3 + x^2 + x + 1)(x^3 + x^2 + x + 1)\,(x^3 - x^5) \\ = \; & (x^8 + 2x^7 + 3x^6 + 4x^5 + 4x^4 + 4x^3 + 3x^2 + 2x + 1)\,(x^3 - x^5) \\ = \; & (1 + 2x + 3x^2 + 4x^3 + 4x^4 + 4x^5 + 3x^6 + 2x^7 + x^8)\,(x^3 - x^5)\,. \end{aligned}$$

After distributing, we have

$$\begin{array}{rrrrrrrrrrrr} & 1x^3 & +2x^4 & +3x^5 & +4x^6 & +4x^7 & +4x^8 & +3x^9 & +2x^{10} & +1x^{11} & & \\ & & & -1x^5 & -2x^6 & -3x^7 & -4x^8 & -4x^9 & -4x^{10} & -3x^{11} & -2x^{12} & -1x^{13} \\ = & 1x^3 & +2x^4 & +2x^5 & +2x^6 & +1x^7 & +0x^8 & -1x^9 & -2x^{10} & -2x^{11} & -2x^{12} & -1x^{13}. \end{array}$$

So the change in the toric g-vector is $(0, 0, 0, 1, 2, 2, 2, 1)$.

Now let $e = 1$. See Figure 2.14. Once more, $|Z| = 6$, and $|Y| = 4$. So there are 10 points in V, $d = 10 - 1 - 1 = 8$, and $\left\lfloor \frac{d}{2} \right\rfloor = 4$.

Then the g-vector corresponding to having the origin o is $(1, 1, 1, 1, 0)$. We may compute the g-vector by moving in along the splitter to the origin from either the left or the right. If we move in from the left, then we pass one positive 0-splitter, 1-splitter, 2-splitter, and 3-splitter. If we move in from the right, then we pass one positive 0-splitter, 1-splitter, 2-splitter, 3-splitter, and 4-splitter and one negative 4-splitter. In either case, the h-vector is $(1, 2, 3, 4, 4, 4, 3, 2, 1)$. Notice that multiplying this h-polynomial by $x^{|F|} - x^{|G|}$ matches the change in the g-vector found above.

Landing on a Splitter With Multiple Points on Each End

We now know how to compute the change in the toric g-vector when the origin moves from p to p' across the relative interior of the convex hull of a single splitter which has multiple points on either end in a 2-dimensional Gale diagram. Now we will determine the g-vector of q when q falls on the relative interior of the convex hull of a single splitter with multiple points on either end in a 2-dimensional Gale diagram.

Let $\ell = \min\{|F|, |G|\}$. Recall that when considering Y as a set of points in dimension 1 within the ambient space H, q corresponds to a simplicial polytope $\widehat{Q}$ of dimension $|Y| - 1 - 1 = u + r - 2$. Then the following result holds.

Theorem 2.3.7 (Lee-N)**.** *When the origin in a 2-dimensional Gale diagram moves from p to q in the relative interior of the convex hull of a single splitter Y with u points on one end and r points on the other end, then for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$ the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product*

$$(x^{r-1} + x^{r-2} + \cdots + x^2 + x + 1)(x^{u-1} + x^{u-2} + \cdots + x^2 + x + 1)\left(x^{|F|} - x^{\ell}\right).$$

Proof. By Theorem 2.2.9, for $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$, the change in the k^{th} component of the toric g-vector is given by the coefficient of x^k in the product

$$(x^{|F|} - x^{\ell})h(\partial\widehat{Q}, x).$$

Then the result follows immediately from the argument used in the proof of Theorem 2.3.3. ∎

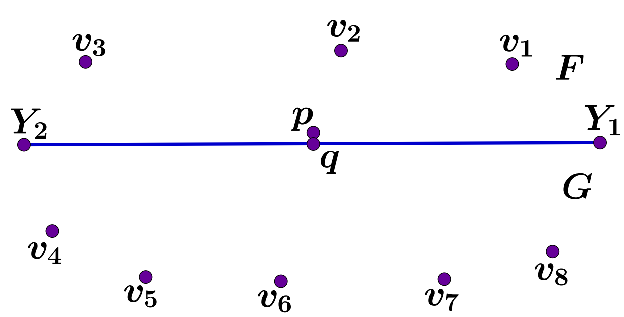

Figure 2.15: We will determine the change in the g-vector when landing on a line.

We will revisit the example introduced in the previous subsection.

Example 2.3.6 continued. See Figure 2.15. Consider this 2-dimensional Gale diagram with 18 points, where $|Y_1| = 4$ and $|Y_2| = 6$. Let the origin p correspond to the simplicial 15-polytope P, and let the origin q correspond to the nonsimplicial 15-polytope Q. Since $\ell = |F| < |G|$, then $g(Q) = g(P)$ by Theorem 2.3.7.

Let the origin p' correspond to the simplicial polytope P'. If the points in the Gale diagram were relabeled so that the points in F were in the same plane as p', then

$$g(Q) = g(P') - (x^5 - x^3)(x^3 + x^2 + x + 1)(x^5 + x^4 + x^3 + x^2 + x + 1)$$

by Theorem 2.3.7. The product is

$$\begin{array}{cccccccccccc} & -1x^3 & -2x^4 & -3x^5 & -4x^6 & -4x^7 & -4x^8 & -3x^9 & -2x^{10} & -1x^{11} & & \\ & & & +1x^5 & +2x^6 & +3x^7 & +4x^8 & +4x^9 & +4x^{10} & +3x^{11} & +2x^{12} & +1x^{13} \\ = & -1x^3 & -2x^4 & -2x^5 & -2x^6 & -1x^7 & +0x^8 & +1x^9 & +2x^{10} & +2x^{11} & +2x^{12} & +1x^{13}. \end{array}$$

So the change in the toric g-vector is $(0, 0, 0, -1, -2, -2, -2, -1)$. Notice that the change in the g-vector when moving the origin from p' to p is the negative of the change in the g-vector when moving the origin from p to p' (as computed at the end of the previous subsection).

Chapter 3 Triangulations and Gale Transforms

Part of McMullen and Walkup's GLBC states that for a simplicial d-polytope P, if $g_k(P) \geq 0$ for some $1 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, then P has a triangulation $\mathcal{T}$ in which there is no interior face of dimension less than $d-k+1$ [23]. Murai and Nevo proved that this statement holds for all simplicial polytopes [25].

3.1 Extension to Non-Pyramids

Recall that a polytope with few vertices has at most $d+3$ vertices. Lee and Welzl showed that this statement holds for polytopes with few vertices, *i.e.*, the number of vertices is at most $d+3$, and the triangulation is regular in this case [18, 29]. In his proof, Lee showed how to interpret the g_k as winding numbers in the Gale transform. We will extend the triangulation result in the Gale transform setting to general nonpyramidal polytopes with few vertices. Note that polytopes with few vertices have Gale transforms of dimension at most 2.

Suppose a d-polytope Q has a Gale transform in which σ points coincide with the origin q for $\sigma > 0$. Then by Corollary 1.6.7, Q is a σ-fold pyramid over the $(d-\sigma)$-polytope $\hat{Q}$. If there is a triangulation $\mathcal{T}$ of $\hat{Q}$ in which there is no interior face of dimension less than $d-\sigma-k+1$, then $\mathcal{T}$ of $\hat{Q}$ extends to a triangulation of Q where no interior face has dimension less than $d-k+1$. Therefore, we will restrict our attention to polytopes Q which are not pyramids.

Theorem 3.1.1 (Lee-N)**.** *Consider any d-polytope Q which is not a pyramid and has few vertices. If $g_k(Q) = 0$ for some k where $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$, then there exists a regular triangulation of Q for which there is no interior face of dimension less than $d-k+1$.*

Proof. Let Q be any d-polytope Q that is not a pyramid and has few vertices whose corresponding Gale transform is $V = \{v_1, v_2, \ldots, v_n\}$ with origin q. Further, suppose that $g_k(Q) = 0$ for some k where $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$.

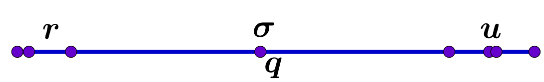

Figure 3.1: Here is a general 1-dimensional Gale transform.

Any d-polytope P with 0-dimensional Gale transform has all its points coincide with q and is a d-simplex and, thus, P is a pyramid.

For any d-polytope Q with a 1-dimensional Gale transform, the $d+2$ points are collinear in the Gale transform. Figure 3.1 depicts an arbitrary Gale transform of dimension 1. Since no points coincide with q (*i.e.*, $\sigma = 0$), then u points fall on one side of q and the remaining r points fall on the other side of q where $u + r = d + 2$. Without loss of generality, $u \geq r \geq 1$. Since q falls in affinely general position in the Gale transform of Q, then Q is a simplicial d-polytope. Then the result follows from Lee and Welzl [18, 29].

Therefore, previous work takes care of all d-dimensional polytopes that are not pyramids and have at most $d + 2$ points. Now we will prove that the above result holds for all d-dimensional polytopes that are not pyramids and have $d + 3$ vertices. In this case, Q's corresponding Gale transform is 2-dimensional.

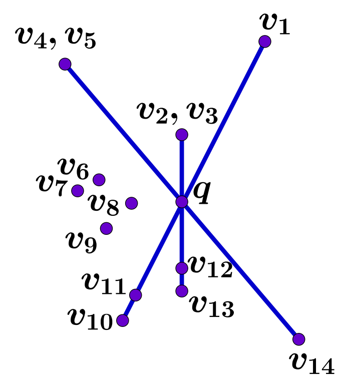

Figure 3.2: Three splitters capture q in this 2-dimensional Gale transform.

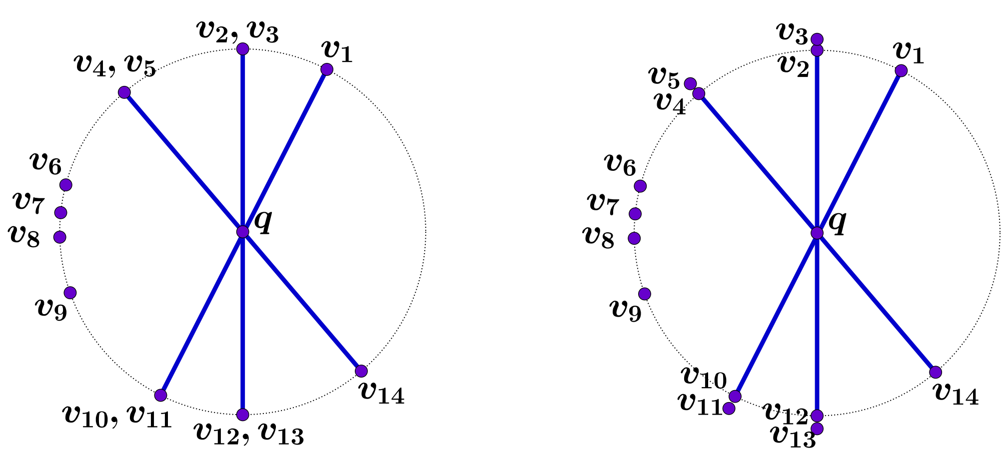

Figure 3.3: Positively scale the points of the Gale transform.

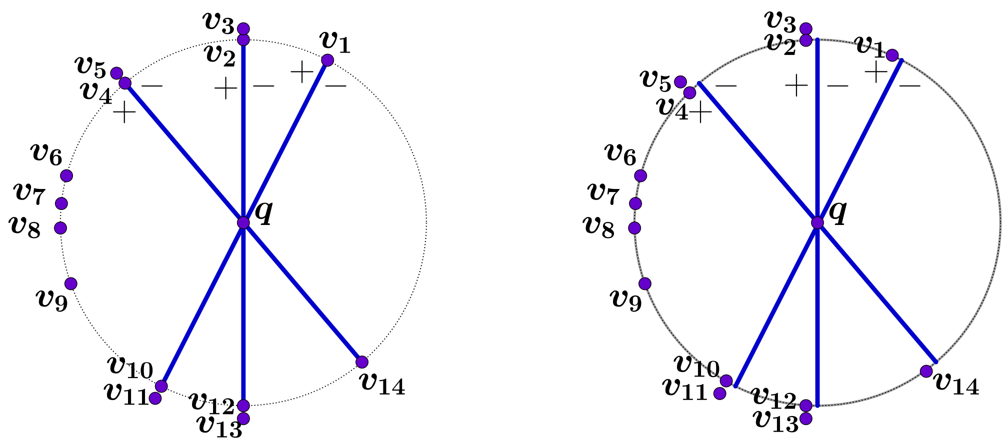

Figure 3.4: Shift the splitters capturing q in the Gale transform towards their larger sides.

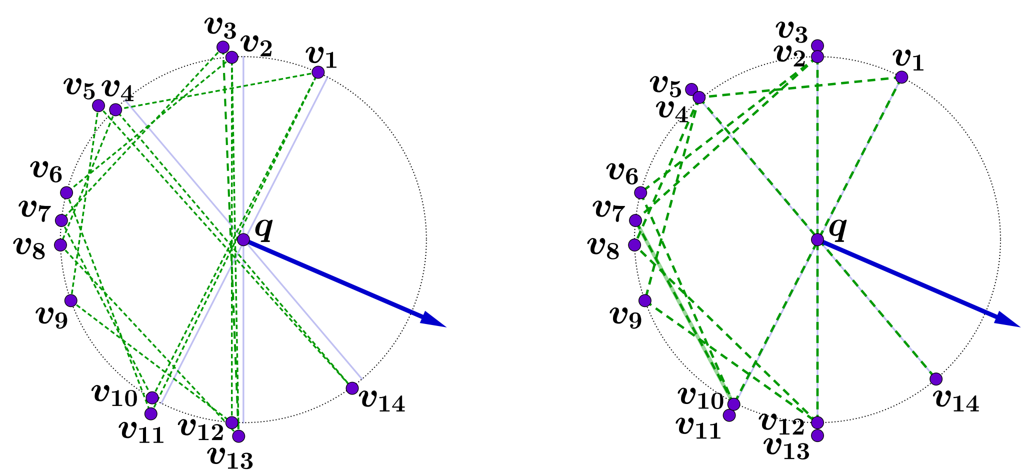

Figure 3.5: Find a ray in the Gale transform from origin q that does not cross any 3-splitters.

If the origin q falls in affinely general position with respect to the points V in the Gale transform, then its corresponding polytope Q is simplicial. Then the result follows from Lee and Welzl again [18, 29].

If q does not fall in affinely general position with respect to V, then the polytope Q is nonsimplicial. So q is in the relative interior of the convex hull of at least one splitter, as in Figure 3.2. Positively scale all the points in V independently with respect to q so that they fall on the boundary of a circle. Then positively scale any points in V that coincide on the boundary of the circle independently with respect to q by arbitrarily small positive amounts so that no points in the Gale transform coincide. See Figure 3.3. This positively scaled Gale transform also corresponds to the nonsimplicial polytope Q, since the sets of points capturing q and not capturing q are the same.

The line through each splitter divides the remaining points (which are not on that splitter) into two sets. Denote the side of the line with fewer points as the negative side, and denote the side of the line with more points as the positive side. Whenever the number of points on both sides is equivalent, arbitrarily choose one side of the line as the positive side and the other side as the negative side. Translate the line and all the points on it in an arbitrarily small parallel motion towards its positive side. See Figure 3.4. Once all the splitters are shifted by arbitrarily small amounts, then q falls in affinely general position with respect to all points.

Notice that the splitters no longer "capture" q. So the sets of points which do and do not capture q have now changed. Further, this shifted Gale transform corresponds to a simplicial polytope P of dimension d with $d+3$ vertices. Since q falls on the negative side of each splitter, then $g_k(P) = g_k(Q)$ for all k by Corollary 2.1.5 and Theorem 2.2.9. So P is a simplicial d-polytope and $g_k(P) = 0$.

Using Lee's Theorem 1.8.5, we find a ray r in general position from the origin q that does not cross any k-splitters. Scaling the Gale transform by arbitrarily small amounts and shifting guarantees that the ray crosses each splitter individually. Then we shift each splitter back towards its negative side so that it captures q once more. See Figure 3.5. Since $0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor$ and the splitter is moving from its negative side, this shift does not change what k-splitters r is crossing. So r still does not cross k-splitters in the Gale transform of Q. Therefore, there exists a regular triangulation of Q where no interior face has dimension less than $d-k+1$. ∎

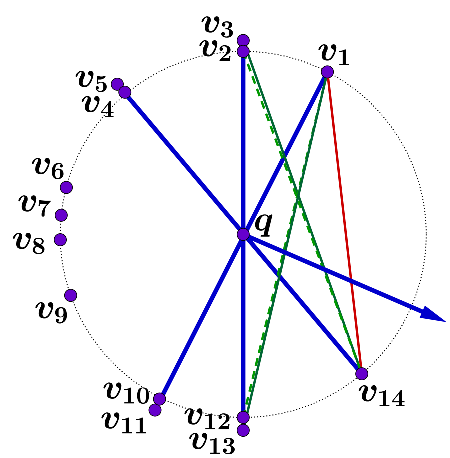

Figure 3.6: The ray in this Gale transform determines the regular triangulation of the polytope.

Consider the following example.

Example 3.1.2. Consider the 2-dimensional Gale transform with origin q and 14 points as shown in Figure 3.2. Let q correspond to the nonsimplicial 11-polytope Q. One may show that $g_3(Q) = 0$. As the figures throughout the proof illustrate, the chosen ray does not cross any 3-splitters. In fact, as it moves towards infinity, this ray crosses the 2-splitter $\{v_1, v_{12}\}$, the 1-splitter $\{v_1, v_{13}\}$, the 2-splitter $\{v_2, v_{14}\}$, the 1-splitter $\{v_3, v_{14}\}$, and the 0-splitter $\{v_1, v_{14}\}$. See Figure 3.6. Thus, this ray determines a triangulation of Q for which the maximum simplices are $\overline{V \setminus \{v_1, v_{12}\}}$, $\overline{V \setminus \{v_1, v_{13}\}}$, $\overline{V \setminus \{v_2, v_{14}\}}$, $\overline{V \setminus \{v_3, v_{14}\}}$, and $\overline{V \setminus \{v_1, v_{14}\}}$

3.2 Counter Example: Pyramid

Before our theorem, we discussed a d-polytope Q which is a σ-fold pyramid over the $(d-\sigma)$-polytope $\hat{Q}$. Since the dimension of $\hat{Q}$ is $(d-\sigma)$, we would have to restrict k so that $0 \leq k \leq \left\lfloor \frac{d-\sigma}{2} \right\rfloor$ in order for the above result to still hold. Otherwise, the required triangulation might not exist. Consider the following example.

Example 3.2.1. Suppose Q is a pyramid over P, where P is the regular octahedron. Note that Q is a 4-dimensional polytope and P is a 3-dimensional polytope. From Example 1.3.11, recall that $h(P) = (1, 3, 3, 1)$ and $g(P) = (1, 2)$. Then by Theorem 1.3.7, $h(Q) = (1, 3, 3, 3, 1)$. So $g(Q) = (1, 2, 0)$. In particular, $g_2(Q) = 0$ where $2 \leq \left\lfloor \frac{\dim(Q)}{2} \right\rfloor$. However, Q does not have any triangulation $\mathcal{T}$ for which there is no interior face of dimension less than $d-k+1 = 4-2+1 = 3$. If $\mathcal{T}$ was a triangulation of Q, the $\mathcal{T}$ would induce a triangulation $\mathcal{T}'$ of the base P where there is no interior face of dimension less than 2. Since every triangulation of the regular octahedron must contain an interior edge, then such a $\mathcal{T}$ can not exist.

Recall that a d-polytope with few vertices has at most $d+3$ vertices. In this chapter, we want to determine the CD-index of a polytope from the Gale diagram of its dual polytope, which has few vertices. Polytopes with few vertices have Gale diagrams of dimension at most 2.

4.1 Dimension 0 and 1

Recall that $C^d\cdot$ corresponds to taking a d-fold pyramid over a point. Further, $(CP)^* = CP^*$. Thus, for any d-dimensional polytope P with a 0-dimensional Gale diagram, $\psi(P^*) = C^d$.

Now consider any d-polytope Q with a 1-dimensional Gale diagram. Without loss of generality, say u points fall on one side of the origin q, σ points coincide with the origin q, and the remaining r points fall on the other side of the origin q where $u+\sigma+r = d+2$, $u \geq r \geq 1$, and $\sigma \geq 0$. Figure 3.1 depicts this arbitrary Gale diagram of dimension 1.

Theorem 4.1.1 (Lee-N)**.** *Let Q be any d-dimensional polytope with a 1-dimensional Gale diagram. Suppose σ points coincide with q where $0 \leq \sigma \leq d+2$. If u points fall on one side of q and the remaining r points fall on the other side of q, for $u \geq r \geq 1$, then*

$$\psi(Q^*) = \sum_{i=0}^{r-1} C^\sigma D^i C^{d-\sigma-2i}.$$

Proof. By Corollary 1.6.7, Q is a σ-fold pyramid over the polytope $\hat{Q}$ whose 1-dimensional Gale diagram has u points on one side of q and r points on the other side of q. Since q falls in affinely general position with respect to the points in the Gale diagram of $\hat{Q}$, then $\hat{Q}$ is a simplicial $(d-\sigma)$-polytope. So the CD-index of the polytope dual to $\hat{Q}$ is $\psi(\hat{Q}^*) = \sum_{i=0}^{\lfloor \frac{d-\sigma}{2} \rfloor} g_i D^i C^{d-\sigma-2i}$ by Theorem 1.4.11. Since $u \geq r$, then $g_i = 1$ for $0 \leq i \leq r-1$ and 0 otherwise. So $\psi(\hat{Q}^*) = \sum_{i=0}^{r-1} D^i C^{d-\sigma-2i}$.

Therefore, the result follow from Theorem 1.4.12. ■

Using these two results, we may compute the CD-index of a polytope whose dual polytope has a Gale diagram of dimension 0 or 1. We will consider a few important cases that hold for e-dimensional Gale diagrams before turning our attention to polytopes whose dual polytopes have 2-dimensional Gale diagrams.

4.2 General Dimension

Many different cases arise in e-dimensional Gale diagrams, as we saw in Chapter 2. Here we will discuss a few results that are straightforward in this general setting.

We will explain how to interpret the CD-index of a polytope from the e-dimensional Gale diagram of its dual polytope in the case that all the points in V are in affinely general position except for a subset which coincides with the origin.

Simplicial Polytopes

When the $d+e+1$ points of V and the origin p all fall in affinely general position with respect to one another, then P is a simplicial d-polytope. So computing the CD-index of P^* is straightforward.

Theorem 4.2.1 (cf. Fine 2010, Lee 2016)**.** *Let P be any d-dimensional simplicial polytope with an e-dimensional Gale diagram. Then*

$$\psi(P^*) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} g_i(P) D^i C^{d-2i}.$$

Proof. By Theorem 1.4.11, the CD-index of P^* is $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} g_i(P) D^i C^{d-2i}$. ■

Then we determine $g(P)$ using Lee's results [18].

Multipyramids over Polytopes

Now consider any polytope Q whose origin q coincides with σ point(s) in the Gale diagram V for $0 \leq \sigma \leq d+e+1$. Let $\hat{V} = \{v \in V | v \neq q\}$ be the Gale diagram of the $(d-\sigma)$-polytope $\hat{Q}$. Note that $\hat{Q} = Q$ and $\hat{V} = V$ when $\sigma = 0$. Recall the following relationship between $\psi(Q^*)$ and $\psi(\hat{Q}^*)$.

Theorem 1.4.12 *Let Q be any d-dimensional polytope with an e-dimensional Gale diagram V. Suppose σ points coincide with the origin q where $0 \leq \sigma \leq d+e+1$. If $\hat{V} = \{v \in V | v \neq q\}$ is the Gale diagram of the $(d-\sigma)$-polytope $\hat{Q}$, then*

$$\psi(Q^*) = C^\sigma \psi(\hat{Q}^*).$$

Proof. The d-polytope Q is a σ-fold pyramid over the $(d-\sigma)$-polytope $\hat{Q}$ by Corollary 1.6.7. Since $C^\sigma \hat{Q}$ corresponds to taking a σ-fold pyramid over the $(d-\sigma)$-polytope $\hat{Q}$ and $(C\hat{Q})^* = C\hat{Q}^*$, then the result follows. ■

This result allows us to focus our attention on the base $\hat{Q}$ of the pyramid Q. If all the points in $\hat{V} \cup \{q\}$ are in affinely general position with respect to each other, then computing the CD-index of Q^* is straightforward.

Corollary 4.2.2 (Lee-N). *Let Q be any d-dimensional polytope with an e-dimensional Gale diagram. Suppose σ points coincide with the origin q where $0 \leq \sigma \leq d+e+1$, and $\hat{V} \cup \{q\}$ are in affinely general position with respect to each other. Then*

$$\psi(Q^*) = \sum_{i=0}^{\lfloor \frac{d-\sigma}{2} \rfloor} g(\hat{Q}) C^\sigma D^i C^{d-\sigma-2i},$$

where $\hat{Q}$ is the $(d-\sigma)$-polytope corresponding to the e-dimensional Gale diagram $\hat{V}$ with origin q.

Proof. By Theorem 1.4.12, $\psi(Q^*) = C^\sigma \psi(\hat{Q}^*)$. So we need to understand $\psi(\hat{Q}^*)$.

Since all the points in $\hat{V} \cup \{q\}$ are in affinely general position with respect to each other, then $\hat{Q}$ is a simplicial $(d-\sigma)$-polytope. Therefore, $g(\hat{Q}) = \psi(\hat{Q}^*) = \sum_{i=0}^{\lfloor \frac{d-\sigma}{2} \rfloor} g_i(\hat{Q}) D^i C^{d-\sigma-2i}$ by Theorem 4.2.1. Recall that the g-vector of Q is the same as the g-vector of the polytope $\hat{Q}$ whose Gale diagram consists of the points $\hat{V} = \{v \in V | v \neq q\}$ by Theorem 2.1.10 and Theorem 2.2.1. Then then result follows. ∎

Since $\hat{Q}$ is simplicial, then we use Lee's result to determine $g(\hat{Q})$ [18].

In all other cases, the points of $\hat{V} \cup \{q\}$ are not in affinely general position with respect to each another. For the rest of this chapter, we are going to focus on the base $\hat{Q}$ of the pyramid Q in the case that $\hat{Q}$ has a 2-dimensional Gale diagram. (Since $\hat{Q}$ is the base of the pyramid, no points of $\hat{V}$ coincide with q.)

Recall Theorem 1.5.6, which states that the CD-index of Q^* is $\psi(Q^*) = \sum_v \psi_v(Q_v^*)$, where $\psi_v(Q_v^*) = \frac{1}{2}\Big((2C-I)\psi(\mathbf{S}_v^*) + (4D-I^2)\psi(\mathbf{T}_v^*)\Big)$. So we first show how to determine $\mathbf{S}$ and $\mathbf{T}$ for each (co)facet we shell on in an e-dimensional Gale diagram.

Shelling via Gale diagrams

Consider a set $\overline{V} = \{\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n}\}$ in $\mathbb{R}^d$, where $\overline{v_i} \in \mathbb{R}^d$ for $1 \leq i \leq d$. Let Q be the polytope that is the convex hull of this point set, and let $V \subset \mathbb{R}^e$ be a Gale diagram for this point set.

Lift the points of V by appending an $(e+1)^{\text{st}}$ coordinate to each point so that the new points are in affinely general position with respect to any nonvertical hyperplane. (Nonvertical means that the last coordinate of the normal vector is nonzero.) Call this new set $\hat{V} \subset \mathbb{R}^{e+1}$.

Consider the line $L = (q, t)$, $-\infty < t < \infty$, which we visualize as a vertical line.

A cofacet in V, which corresponds to a facet in Q ($(d-1)$-dimensional face), is a minimal set S such that S "captures" q. Therefore, it will be a k-simplex for some $0 \leq k \leq e$. The set $\hat{S}$ will also be a k-simplex, and will have exactly one point on L.

A cosubfacet in V, which corresponds to a subfacet in Q ($(d-2)$-dimensional face) is a minimal affinely dependent set S such that S captures q. Therefore, S will be a k-dimensional set of $k+2$ points for some $0 \leq k \leq e$. The set $\hat{S}$ will be a $(k+1)$-simplex, and will intersect L in a line segment through some points in rel int (conv $(\hat{S})$). Note that the "lower" endpoint of this segment will lie in rel int (conv (S_1)) for some unique face of $\hat{S}$, and likewise the "upper" endpoint of this segment will lie in rel int (conv (S_2)) for some unique face of $\hat{S}$. Then S_1 and S_2 will be the unique pair of cofacets containing the cosubfacet S.

In general, any coface S in V that is not a cofacet will lift to a subset $\hat{S}$ of $\hat{V}$ that will intersect L in a line segment through some points in rel int (conv $(\hat{S})$).

In particular, a coface in V which corresponds to a $(d-3)$-face in Q is an affinely dependent k-dimensional set S of $k+3$ points such that S captures q for some $0 \leq k \leq e$. The set $\hat{S}$ will be a $(k+1)$-dimensional set of $k+3$ points (hence minimally affinely dependent), and will also intersect L in a line segment through points in rel int (conv $(\hat{S})$).

Lemma 4.2.3 (Lee-N)**.** *Order the cofacets $\mathbf{S_1}, \ldots, \mathbf{S_m}$ of V in the order in which the $\hat{S}_i$ are intersected by L as t goes from $-\infty$ to ∞. This ordering corresponds to a shelling order of the facets $S'_1, \ldots, S'_m$ of Q, where $S'_i = \overline{V \setminus S_i}$ for all i.*

Proof. Claim: Fix $2 \leq i \leq m$. For every proper face F' that is in both S'_i and S'_j for some j, $1 \leq j < i$, there exists k, $1 \leq k < i$, such that $S'_i \cap S'_k$ is a subfacet containing F'. That is to say, as a polyhedral complex, $S'_i \cap (S'_1 \cup \cdots \cup S'_{i-1})$ is generated by a collection of subfacets of Q in S'_i.

Proof. In terms of V, let F, $\mathbf{S_i}$, and $\mathbf{S_j}$ be the corresponding cofaces, where $F \supset \mathbf{S_i}$ and $F \supset \mathbf{S_j}$. Then $p_i = L \cap \hat{\mathbf{S}}_\mathbf{i}$ is higher than $p_j = L \cap \hat{\mathbf{S}}_\mathbf{j}$. Note that the entire open line segment (p_j, p_i) from p_j to p_i lies in rel int (conv $(\hat{\mathbf{S}}_\mathbf{i} \cup \hat{\mathbf{S}}_\mathbf{j})$) $\subseteq$ conv $(\hat{F})$. Choose a point p on (p_j, p_i) sufficiently close to p_i such that there is a simplex $\hat{S}$, $\hat{\mathbf{S}}_\mathbf{i} \subset \hat{S} \subseteq (\hat{\mathbf{S}}_\mathbf{i} \cup \hat{\mathbf{S}}_\mathbf{j})$ that contains p in its relative interior. Consider the lower point q' of the intersection of L with conv $(\hat{S})$. There is a unique face $\hat{T}$ of $\hat{S}$ such that $q' \in$ rel int (conv $(\hat{T})$). This set $\hat{T}$ will correspond to the desired cofacet $\mathbf{S_k}$.

From the above we can identify the collection $S_1, \ldots, S_\ell$ of cosubfacets such that as a polyhedral complex, $S'_i \cap (S'_1 \cup \cdots \cup S'_{i-1})$ is generated by $S'_1, \ldots, S'_\ell$. These will be all the cosubfacets S for which the upper vertex of the line segment $L \cap \hat{S}$ lies in conv $\hat{\mathbf{S}}_\mathbf{i}$.

Claim: This collection of subfacets is the beginning of a shelling of the boundary of S'_i.

We can prove this by induction on dimension. Recall that a Gale diagram for S'_i can be obtained from that of Q by projecting all points of $V \setminus \mathbf{S_i}$ orthogonally onto the space W perpendicular to span $\mathbf{S_i}$. Now extend W to $\hat{W} = \text{span}\,(W \cup L)$. Project the points of $\hat{V}$ parallel to aff $\hat{S}_i$ onto $\hat{W}$. The result will be a lifting of the Gale diagram for S'_i in which $S_1, \ldots, S_\ell$ are now cofacets, and come first in the ordering of all cofacets of S'_i via the line L. (The base cases of $d = 0$ or $d = 1$ are easily handled.) ■

For CD-index calculations we are going to need the Gale diagram of the boundary $\mathbf{T_i}'$ of $S'_1 \cup \cdots \cup S'_\ell$.

Lemma 4.2.4 (Lee-N)**.** *If we place the origin at $L \cap \hat{\mathbf{S}}_{\mathbf{i}}$ and project within $\hat{V}$ as above, then this projection yields a Gale diagram for $\mathbf{T_i}$.*

Proof. The Gale diagram of the boundary T_i' of $S_1' \cup \cdots \cup S_\ell'$ is generated by all $(d-3)$-faces of S_i' that lie in at least one $(d-2)$-face of S_i' in $\{S_1' \cup \cdots \cup S_\ell'\}$ and in at least one $(d-2)$-face of S_i' not in $\{S_1' \cup \cdots \cup S_\ell'\}$. Thus we need minimal affinely dependent sets $\hat{R}$ in $\hat{V}$ that satisfy the following conditions simultaneously.

1. $\hat{T}$ contains a lifted cosubfacet containing $\hat{S}_i$ but lie below $\hat{S}_i$;

2. $\hat{T}$ contains a lifted cosubfacet containing $\hat{S}_i$ but lie above $\hat{S}_i$.

If we place the origin at $L \cap \hat{S}_i$ and project within $\hat{V}$ as above, such $\hat{T}$ projects to a simplex containing the origin in its relative interior. Thus this projection yields a Gale diagram for $\mathbf{T_i}$. ■

4.3 Dimension 2

As we saw in Chapter 2, many cases arise in 2-dimensional Gale diagrams. The results from Section 4.2 allow us to interpret the CD-index from 2-dimensional Gale diagrams whose points are in affinely general position except for a subset which coincides with the origin.

In all other cases, the points of $\hat{V} \cup \{q\}$ are not in affinely general position with respect to each another. For the rest of this chapter, we are going to focus on the base $\hat{Q}$ of the pyramid Q. So we are going to focus on determining the CD-index of a polytope Q^* from the Gale diagram of its dual Q in the case that Q has a 2-dimensional Gale diagram in which no points coincide with the origin q.

An ℓ-Splitter

Let V be the a collection of n points in affinely general position in $\mathbb{R}^2$. Consider a point q that is in affinely general position with respect to V except that q is in the relative interior of the convex hull of some l-splitter X. Define $H = \text{aff}(X)$. Let F be the points of V that are in one open halfplane of H, and let G be points of V that are in the opposite open halfplane of H. Then $V = F \cup G \cup X$. Say that $X = \{y_1, z_1\}$ and $|F| = \ell \leq |G| = m$, relabeling the points of V if necessary. Then there exists a point p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitter except X and p falls in the same halfplane as the points of F. See Figure 4.1. Note that placing the origin at p corresponds to a simplicial polyope P, and placing the origin at q corresponds to a nonsimplicial poytope Q. How does the CD-index of the dual polytope change when the origin in the Gale diagram moves from p to q onto the splitter with exactly 2 points?

Definition 4.3.1 (Lee-N)**.** Denote the *change in the CD-index from P^* to Q^** as the origin shifts from p to q in the relative interior of the convex hull of a line Y with at least 2 points as $\chi(u, r, \ell, m)$ if the $u + r + \ell + m$ points fall on a circle in such a way that there are u points and r points on either side of the splitter which "captures" q

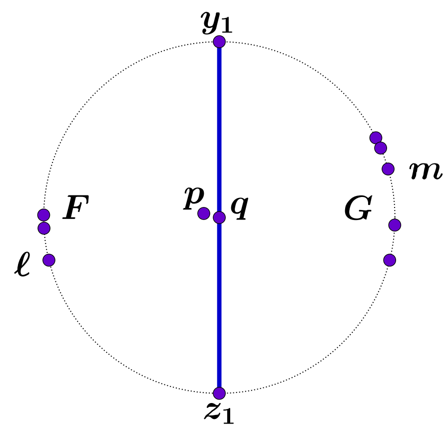

Figure 4.1: Here is a 2-dimensional Gale diagram with an ℓ-splitter.

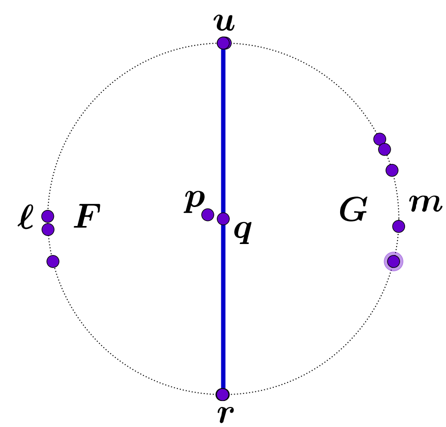

Figure 4.2: We want to determine $\chi(u, r, \ell, m)$ in this 2-dimensional Gale diagram.

and the points of F are on the left side of q and the points of G are on the right side of q. See Figure 4.2.

Here we will determine $\chi(1, 1, \ell, m)$, where $\ell \leq m$. Notice that we lose ℓ (co)facets and gain 1 (co)facet. To determine which (co)facets are lost, lift the 2-dimensional Gale diagram so that y_1 is higher than z_1 and the ℓ points in F and the m points in G are suitably low. Relabel the points of F $v_0, v_1, \ldots, v_\ell$ in such a way that v_i is higher than v_{i+1} for $0 \leq i \leq \ell - 2$. See Figure 4.3. Then all the points $\hat{V} \subset \mathbb{R}^3$ are in affinely general position with respect to any nonvertical plane. So we are able to use our results from the last section to find each $\mathbf{S}$ and $\mathbf{T}$ as we shell on the (co)facets. We are also able to determine which (co)facets are shelled off by considering how many points are split above each of the ℓ planes determined by the two endpoints

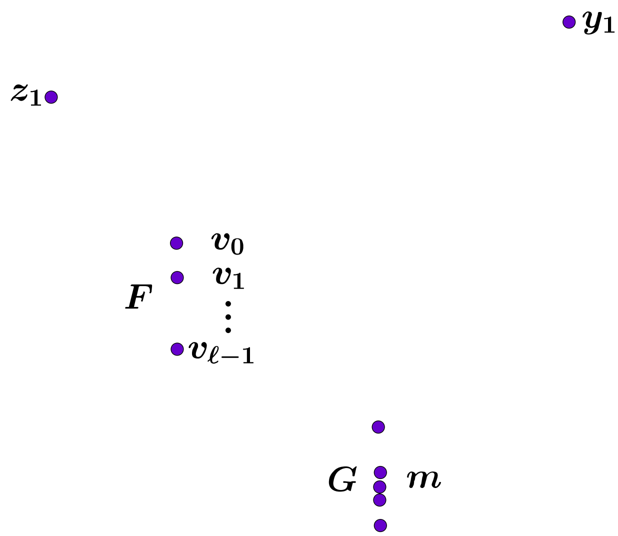

Figure 4.3: Lift the Gale diagram to find the (co)facets for shelling off facets lost.

and one point from F. We may calculate how many different ways i points are split above a plane as follows.

Lemma 4.3.2 (Lee-N)**.** *There is one i-splitter, for $0 \leq i \leq \ell - 1$.*

Proof. The number of points above the plane determined by y_1, z_1 and one point v_i from F is equivalent to the number of points in F above v_i for $0 \leq i \leq \ell - 1$. There are i points of F above v_i for $0 \leq i \leq \ell - 1$. So the result follows. ∎

This result tells us there is one (co)facet of each type $d - i$ for $0 \leq i \leq \ell - 1$ being shelled off. Due to the symmetry of $\Delta_i^d = \Delta_{d-i}^d$ as noted in Corollary 1.5.7, we may equivalently think of shelling off one (co)facet of each type i for $0 \leq i \leq \ell - 1$. So we may compute the total contribution of the (co)facets shelled off to the total change in the CD-index as we move onto the ℓ-splitter as follows.

Proposition 4.3.3 (Lee-N)**.** *The contribution from all the (co)facets shelled off to the change in the CD-index from P^* to Q^* as the origin moves from p to q onto an ℓ-splitter is*

$$-\frac{1}{2}\sum_{i=0}^{\ell-2}(D^iC^{2\ell-1-2i} - D^{i+1}C^{2\ell-1-2(i+1)}) - \frac{1}{2}D^{\frac{2\ell-2}{2}}C$$

if $\ell = m$ and

$$-\frac{1}{2}\sum_{i=0}^{\ell-1}(D^iC^{(\ell+m-1)-2i} - D^{i+1}C^{(\ell+m-1)-2(i+1)})$$

if $\ell < m$.

Proof. Shelling simplicial d-polytopes gives the CD-index of their dual polytopes. So shelling off the (co)facets lost allows us to determine the CD-index contribution of the vertices of their duals, which are simple vertices. Therefore, we may use Lee's Δ-formulas in [16]. When the simplicial (co)facet is d-dimensional, the contribution of the cofacets shelled off to the change in the CD-index from P^* to Q^* as the origin moves from p to q in the relative interior of the convex of hull of the ℓ-splitter X is given by

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor-1} B_i\cdot(D^iC^{d-2i}-D^{i+1}C^{d-2(i+1)})\\
&-\frac{1}{2}2B_i\cdot D^{\frac{d}{2}}\\
&-\frac{1}{2}\sum_{i=\lfloor \frac{d}{2}\rfloor+1}^{\ell-1} B_i\cdot(D^{d-i}C^{d-2(d-i)}-D^{d+1-i}C^{d-2(d+1-i)}).
\end{aligned}$$

when $2|d$, and

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor-1} B_i\cdot(D^iC^{d-2i}-D^{i+1}C^{d-2(i+1)})\\
&-\frac{1}{2}B_i\cdot D^{\frac{d-1}{2}}C\\
&-\frac{1}{2}\sum_{i=\lfloor \frac{d}{2}\rfloor+1}^{\ell-1} B_i\cdot(D^{d-i}C^{d-2(d-i)}-D^{d+1-i}C^{d-2(d+1-i)}).
\end{aligned}$$

otherwise; where B_i is the number of i-splitters crossed in the lifting above.

Here the dimension is $d=\ell+m-1$. If $\ell=m$, then $d=2\ell-1$ and $\lfloor \frac{d}{2}\rfloor=\ell-1$. So d is not divisible by d and the change is

$$-\frac{1}{2}\sum_{i=0}^{\ell-2}(D^iC^{(2\ell-1)-2i}-D^{i+1}C^{(2\ell-1)-2(i+1)})-\frac{1}{2}D^{\frac{(2\ell-1)-1}{2}}C.$$

When $\ell<m$, then $\ell\leq\lfloor \frac{d}{2}\rfloor$. So either $\ell=0$ and there are no cofacets lost; or $\ell>0$ and $0\leq\ell-1<\lfloor \frac{d}{2}\rfloor$. In both cases, there are no terms for $i\geq\lfloor \frac{d}{2}\rfloor$. ■

There is exactly one (co)facet shelled on to the polytope. We determine $\mathbf{S}$ as follows.

Lemma 4.3.4 (Lee-N)**.** *The CD-index of $\psi(\mathbf{S}^*)$ is* $\psi(\mathbf{S}^*)=\sum_{i=0}^{\ell-1}D^iC^{\ell+m-2(i+1)}$.

Proof. The (co)facet gained consists of y_1 and z_1. To obtain the Gale diagram for this (co)facet, project the 2-dimensional Gale diagram V onto the line orthogonal to $V\setminus\{y_1,z_1\}$ and then delete those two points [22]. So the Gale diagram $\hat{V}$ for the (co)facet is 1-dimensional with ℓ points on one side of q and m points on the other

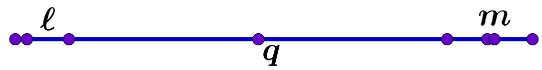

Figure 4.4: Here is the Gale diagram for $\mathbf{S}$.

side of q. See Figure 4.4. Since q is in affinely general position with respect to the points in $\hat{V}$, then $\mathbf{S}$ a simplicial polytope of dimension $l+m-2$. Since $\ell \leq m$, then $\psi(\mathbf{S}^*) = \sum_{i=0}^{\ell-1} D^i C^{\ell+m-2-2i}$ by Theorem 4.1.1. ■

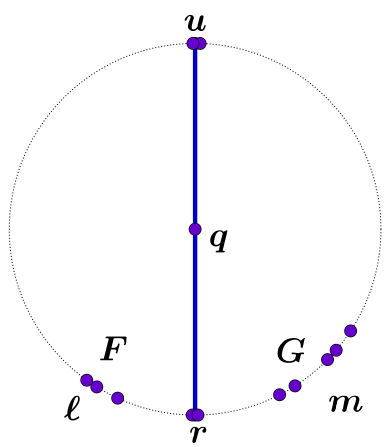

Figure 4.5: Here is the Gale diagram for $\mathbf{T}$ of *type* (u, r, ℓ, m).

Now, we need to determine $\mathbf{T}$ as follows. First, make note of the following definition.

Definition 4.3.5 (Lee-N)**.** We say $\mathbf{T}$ has *type* (u, r, ℓ, m) if the $u + r + \ell + m$ points fall on a circle in such a way that there are u points and r points on either side of the splitter which "captures" q and the points of F are on the lower side left of q and the points of G are on the lower side right of q. See Figure 4.5.

Lemma 4.3.6 (Lee-N)**.** *The CD-index* $\psi(\mathbf{T}^*)$ *is 0.*

Proof. To obtain the Gale diagram for $\mathbf{T}$, lift the 2-dimensional Gale diagram once more so that y_1 is higher than z_1 and the ℓ points in F and the m points in G are suitably low. Then the origin starts below the higher points y_1 and z_1 points and above the lower points in F and G points. As the origin moves up, it crosses exactly

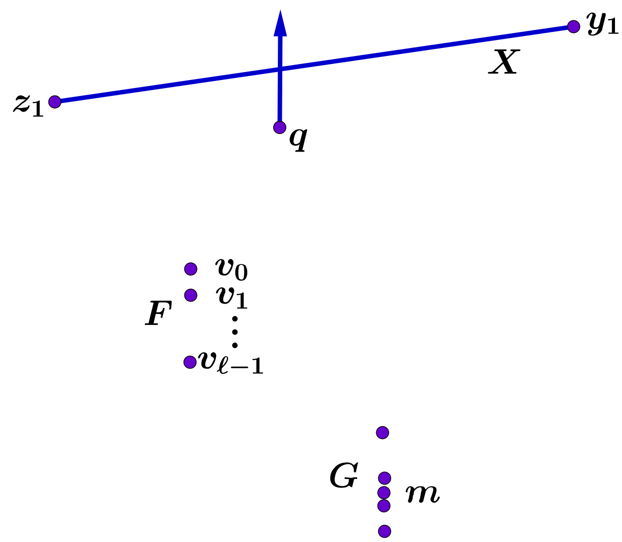

Figure 4.6: Lift the Gale diagram with an ℓ-splitter to determine a shelling order for the facets shelled on.

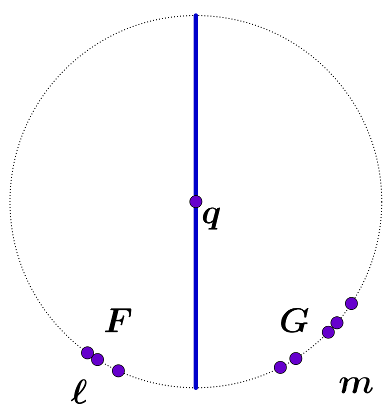

Figure 4.7: Project the lifting to obtain the Gale diagram of $\mathbf{T}$.

one splitter, namely X. See Figure 4.6. Project in the direction of the splitter X onto the plane perpendicular to the unlifted X and delete the two points. Then we have a 2-dimensional Gale diagram for $\mathbf{T}$ with no points on either side of the diagonal and the points of F on the low left side and the points of G on the low right side. See Figure 4.7. Therefore, this particular $\mathbf{T}$ is of type $(0, 0, \ell, m)$. Since no subsets "capture" q, then the result follows. ■

Let us determine $\chi(1, 1, 1, 1)$.

Example 4.3.7. When determining $\chi(1,1,1,1)$, we shell off one (co)facet and shell on one (co)facet. Notice that $n = 4$ and $d = 1$. So the facet shelled off has a CD-index of C. Projecting orthogonally shows that the CD-index of $\mathbf{S}^*$ is 1. Lifting and then projecting gives that the CD-index of $\mathbf{T}^*$ is 0. Thus,

$$\begin{aligned}
&\chi(1,1,1,1) = -\frac{1}{2}C + \frac{1}{2}\Big((2C - I)\cdot 1 + (4D - I^2)\cdot 0\Big)\\
&= -\frac{1}{2}C + C - \frac{1}{2}I\\
&= -\frac{1}{2}C + C - \frac{1}{2}C\\
&= -C + C\\
&= -D^0C^1 + D^0C^1D^0C^0\\
&= -D^0C^{1-0} + D^0C^{2-1}D^0C^{2-2}.
\end{aligned}$$

So there is no change in the CD-index from P^* to Q^* as the origin moves from p to q onto the 1-splitter.

Combining these results, we determine $\chi(1,1,\ell,m)$ as follows.

Theorem 4.3.8 (Lee-N)**.** *The change in the CD-index from P^* to Q^* as the origin moves from p to q onto X in the Gale diagram, as shown in Figure 4.1 is*

$$\chi(1,1,\ell,m) = -\sum_{i=0}^{\ell-1} D^iC^{\ell+m-1-2i} + \sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i}.$$

Proof. The previous example shows the computation holds when $\ell = m = 1$.

There are two cases that arise here. First, suppose that $\ell = m$. Then

$$\begin{aligned}
&\psi(Q^*) = \psi(P^*)\\
&-\frac{1}{2}\sum_{i=0}^{\ell-2}(D^iC^{2\ell-1-2i} - D^{i+1}C^{2\ell-1-2(i+1)}) - \frac{1}{2}D^{\frac{2\ell-2}{2}}C\\
&+\frac{1}{2}(2C - I)\sum_{i=0}^{\ell-1} D^iC^{2\ell-2(i+1)}\\
&+\frac{1}{2}(4D - I^2)0
\end{aligned}$$

So

$$\begin{aligned}
&\chi(1,1,\ell,\ell) = -\frac{1}{2}(D^0C^{2\ell-1} - D^{\ell-1}C^1) - \frac{1}{2}D^{\frac{2\ell-2}{2}}C\\
&+\sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i} - \frac{1}{2}\sum_{i=0}^{\ell-1} D^iIC^{2\ell-2-2i}
\end{aligned}$$

Since $2\ell - 2 - 2(\ell - 1) = 0$, then we peel off the last term of this last sum and rewrite the rest of the sum using the following relationship.

$$\begin{aligned}
&D^iIC^{2\ell-2-2i} = D^i(IC)C^{2\ell-3-2i}\\
&= D^i(D + C^2)C^{2\ell-3-2i}\\
&= D^{i+1}C^{2\ell-3-2i} + D^iC^{2\ell-1-2i}
\end{aligned}$$

Then

$$\chi(1,1,\ell,\ell) = -\frac{1}{2}D^0C^{2\ell-1} + \frac{1}{2}D^{\ell-1}C^1 - \frac{1}{2}D^{\frac{2\ell-2}{2}}C$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i} - \frac{1}{2}D^{\ell-1}I$$
$$-\frac{1}{2}\sum_{i=0}^{\ell-2}(D^{i+1}C^{2\ell-3-2i} + D^iC^{2\ell-1-2i})$$

So

$$\chi(1,1,\ell,\ell) = -\frac{1}{2}D^0C^{2\ell-1} + \frac{1}{2}D^{\ell-1}C^1 - \frac{1}{2}D^{\ell-1}C$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i} - \frac{1}{2}D^{\ell-1}C$$
$$-\frac{1}{2}\sum_{i=0}^{\ell-2} D^{i+1}C^{2\ell-3-2i} - \frac{1}{2}\sum_{i=0}^{\ell-2} D^iC^{2\ell-1-2i})$$

$$= -\frac{1}{2}D^0C^{2\ell-1} - \frac{1}{2}D^{\ell-1}C + \sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i}$$
$$-\frac{1}{2}\sum_{i=1}^{\ell-1} D^iC^{2\ell-1-2i} - \frac{1}{2}\sum_{i=0}^{\ell-2} D^iC^{2\ell-1-2i}$$

$$= -\frac{1}{2}D^0C^{2\ell-1} - \frac{1}{2}D^{\ell-1}C + \sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i}$$
$$-\frac{1}{2}D^{\ell-1}C^{2\ell-1-2(\ell-1)} - \sum_{i=1}^{\ell-2} D^iC^{2\ell-1-2i} - \frac{1}{2}D^0C^{2\ell-1}$$

$$= -D^0C^{2\ell-1} - \sum_{i=1}^{\ell-2} D^iC^{2\ell-1-2i} - D^{\ell-1}C^1$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i}$$

$$= -\sum_{i=0}^{\ell-1} D^iC^{2\ell-1-2i} + \sum_{i=0}^{\ell-1} CD^iC^{2\ell-2-2i}$$

And the result follows.

Now suppose that $\ell < m$ Then

$$\psi(Q^*) = \psi(P^*)$$
$$-\frac{1}{2}\sum_{i=0}^{\ell-1}(D^iC^{(\ell+m-1)-2i} - D^{i+1}C^{(\ell+m-1)-2(i+1)})$$
$$+\frac{1}{2}(2C-I)\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}$$
$$+\frac{1}{2}(4D-I^2)0$$

So

$$\chi(1,1,\ell,m) = -\frac{1}{2}(D^0C^{\ell+m-1} - D^\ell C^{m-\ell-1})$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2(i+1)} - \frac{1}{2}\sum_{i=0}^{\ell-1} D^iIC^{\ell+m-2(i+1)}$$

Since $\ell + m - 2(\ell - 1 + 1) = m - \ell \geq 1$, we rewrite the fourth term again using the relationship found above. Then

$$\chi(1,1,\ell,m) = -\frac{1}{2}D^0C^{\ell+m-1} + \frac{1}{2}D^\ell C^{m-\ell-1}$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i} - \frac{1}{2}\sum_{i=0}^{\ell-1}(D^{i+1}C^{\ell+m-3-2i} + D^iC^{\ell+m-1-2i}$$

So

$$\chi(1,1,\ell,m) = -\frac{1}{2}D^0C^{\ell+m-1} + \frac{1}{2}D^\ell C^{m-\ell-1}$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i}$$
$$-\frac{1}{2}\sum_{i=0}^{\ell-1} D^{i+1}C^{\ell+m-3-2i} - \frac{1}{2}\sum_{i=0}^{\ell-1} D^iC^{\ell+m-1-2i}$$
$$= -\frac{1}{2}D^0C^{\ell+m-1} + \frac{1}{2}D^\ell C^{m-\ell-1}$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i}$$
$$-\frac{1}{2}\sum_{i=1}^{\ell} D^iC^{\ell+m-1-2i} - \frac{1}{2}\sum_{i=0}^{\ell-1} D^iC^{\ell+m-1-2i}$$
$$= -\frac{1}{2}D^0C^{\ell+m-1} + \frac{1}{2}D^\ell C^{m-\ell-1}$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i} - \frac{1}{2}D^\ell C^{\ell+m-1-2(\ell)}$$
$$-\sum_{i=1}^{\ell-1} D^iC^{\ell+m-1-2i} - \frac{1}{2}D^0C^{\ell+m-1}$$
$$= -D^0C^{\ell+m-1} - \sum_{i=1}^{\ell-1} D^iC^{\ell+m-1-2i}$$
$$+\sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i}$$
$$= -\sum_{i=0}^{\ell-1} D^iC^{\ell+m-1-2i} + \sum_{i=0}^{\ell-1} CD^iC^{\ell+m-2-2i}$$

And the result follows. ■

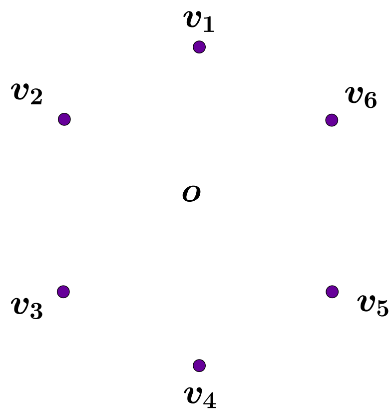

Figure 4.8: Here is the Gale diagram of the triangular prism P^*.

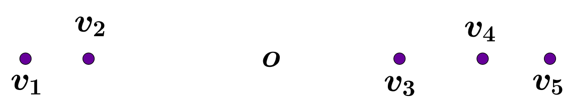

Figure 4.9: Here is the Gale diagram of the triangular bipyramid P.

We will use previous results to determine the CD-index of the triangular bipyramid and the triangular prism from the appropriate Gale diagram.

Example 4.3.9. To determine the CD-index of the triangular bipyramid P, consider the Gale diagram of its dual P^*, the triangular prism. See Figure 4.8. No matter which direction we move in to o, we cross one 0-splitter and two 1-splitters. When we move the origin onto o, then o is captured by three 2-splitters. By Theorem 1.4.11 and Lemma 2.1.4, the CD-index of P is $\psi(P) = C^3 + 2DC + 3 \cdot \chi(1,1,2,2)$. Using Theorem 4.3.8, we find that $\chi(1,1,2,2) = -C^3 - DC + C^3 + CD = CD - DC$. Therefore, $\psi(P) = C^3 + 2DC + 3CD - 3DC = C^3 - DC + 3CD$, which agrees with our earlier computations (see Example 1.3.9).

To determine the CD-index of the triangular prism P^*, consider the Gale diagram of its dual P, the triangualr bipyramid. See Figure 4.9. We cross one 0-splitter and one 1-splitter as we move to o from the left. By Theorem 1.4.11, $\psi(P^*) = C^3 + DC$, which also agrees with are earlier computations (see Example 1.3.8).

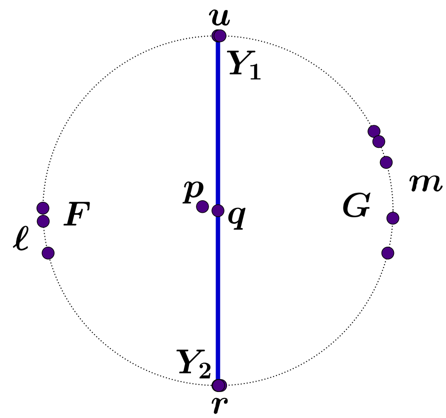

Figure 4.10: Here is a 2-dimensional Gale diagram with a splitter Y.

A Splitter

We would like to generalize the result for an ℓ-splitter to any splitter defined by at least 2 points. As in Section 2.3, we will consider 2-dimensional Gale diagrams in which there exists a line H that satisfies the following conditions.

- The line H contains at least 3 points from V and the origin q;
- The set $Y = V \cap H$ affinely spans H;
- Any minimal affinely dependent subset of V of at most 3 points lies in H;
- Every subset of 2 points in $Y \cup \{q\}$ is affinely independent.

Once more, the u points $Y_1 = \{y_1, y_2, \ldots, y_u\}$ of Y fall on one side of q and the remaining r points $Y_2 = \{z_1, z_2, \ldots z_r\}$ of Y fall on the other side of q with $u \geq r \geq 1$. Note that all the points in $V \setminus Y$ are in affinely general position with respect to $V \cup \{q\}$. Then q is in the relative interior of the convex hull of nonempty subsets of Y, denote them Y_β. In this case, each Y_β is the union of any nonempty subset Y_1' of Y_1 with any nonempty subset Y_2' of Y_2. Recall that the nonempty subsets Y_β of Y are splitters. In particular, $Y_1' \cup Y_2'$ is a splitter for every nonempty subset Y_1' of Y_1 and every nonempty subset Y_2' of Y_2.

Let F be the points of V that are in one open halfplane of H, and let G be points of V that are in the opposite open halfplane of H. Say that $|F| = \ell \leq |G| = m$, relabeling the points of V if necessary. So $l + m \geq 3$ and $V = F \cup G \cup Y = F \cup G \cup Y_1 \cup Y_2$. Then there exists a point p in affinely general position with respect to $V \cup \{q\}$ so that $\overline{pq}$ does not meet the convex hull of any splitters except those Y_β of Y and p falls in the same halfplane of H as the points of F. See Figure 4.10. Note that placing the origin at p corresponds to simplicial polytope P, and placing the origin at q corresponds to a nonsimplicial polytope Q. Since p

falls in affinely general position with respect to V, we know how to compute $\psi(P^*)$ by Theorem 1.4.11. How does the CD-index of the dual polytope change when the origin in the Gale diagram moves from p to q into the relative interior of the convex hull of the splitter with at least 3 points?

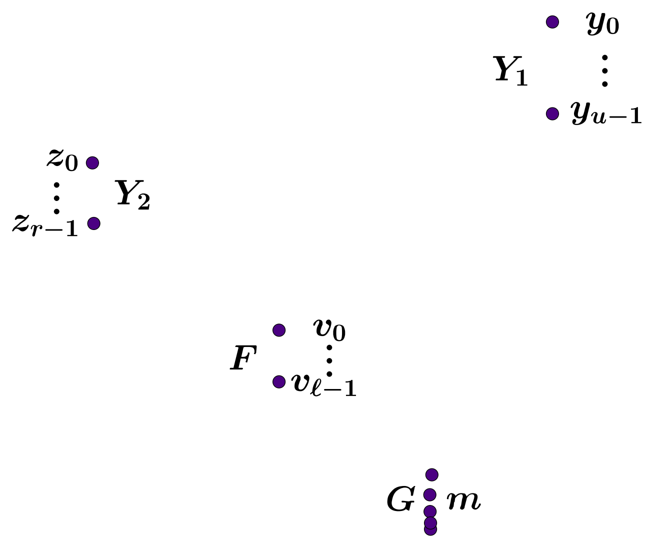

Figure 4.11: Lift the Gale diagram with a splitter Y to order the (co)facets for shelling.

Notice that we lose $ur\ell$ (co)facets and gain ur (co)facets. To determine which (co)facets are lost, lift the 2-dimensional Gale diagram so that the u points in Y_1 are higher than the r points in Y_2 and the ℓ points in F and the m points in G are suitably low. Relabel the points of F $v_0, v_1, \ldots, v_\ell$ in such a way that v_σ is higher than $v_{\sigma+1}$ for $0 \leq \sigma \leq \ell-2$. Also relabel the points of Y_1 $y_0, y_1, \ldots, y_{u-1}$ in such a way that y_j is higher than y_{j+1} for $0 \leq j \leq u-2$. Relabel the points of Y_2 $z_0, z_1, \ldots, z_{r-1}$ in such a way that z_k is higher than z_{k+1} for $0 \leq k \leq r-2$. Then all the points $\hat{V} \subset \mathbb{R}^3$ are in affinely general position with respect to any nonvertical plane. See Figure 4.11. So we are able to use our results from the last section to find each $\mathbf{S}$ and $\mathbf{T}$ as we shell on the (co)facets. We are also able to determine which (co)facets are shelled off by considering how many points are split above each of the $ur\ell$ planes determined by one point from Y_1, one point from Y_2, and one point from F. We may calculate how many different ways $i = \sigma + j + k$ points are split above a plane for $0 \leq i \leq u + r + \ell - 3$ as follows.

Lemma 4.3.10 (Lee-N)**.** *The number of i-splitters is given by the sum* $\binom{i+2}{2} - \binom{i+2-u}{2} - \binom{i+2-r}{2} - \binom{i+2-\ell}{2} + \binom{i+2-u-r}{2} + \binom{i+2-u-\ell}{2} + \binom{i+2-r-\ell}{2} + \binom{i+2-u-r-\ell}{2}$, *where* $\binom{p}{2} = 0$ *if* $p < 0$.

Proof. The number of points above the plane determined by one point y_j from Y_1, one point z_k from Y_2, and one point v_i from F is equivalent to the sum of the number of points in F above v_σ, the number of points in Y_1 above y_j, and the number of points in Y_2 above z_k for any given σ, j, and k. There are σ points of F above v_σ for $0 \leq \sigma \leq \ell - 1$. There are j points of Y_1 above y_j for $0 \leq j \leq u-1$. There are k points of Y_2 above z_k for $0 \leq k \leq r-1$.

Determining the number of i-splitters is equivalent to finding all possible combinations of $\sigma + j + k = i$ where $0 \leq \sigma \leq \ell - 1$, $0 \leq j \leq u-1$, and $0 \leq k \leq r-1$. The total number of such combinations is the coefficient of x^i in the product

$$\left(\sum_{j=0}^{u-1} x^j\right)\left(\sum_{k=0}^{r-1} x^k\right)\left(\sum_{\sigma=0}^{\ell-1} x^\sigma\right)$$

$$= \frac{1-x^u}{1-x}\frac{1-x^r}{1-x}\frac{1-x^\ell}{1-x}$$

$$= \frac{1-x^u-x^r-x^\ell+x^{u+r}+x^{u+\ell}+x^{r+\ell}-x^{u+r+\ell}}{(1-x)^3}$$

$$= (1-x^u-x^r-x^\ell+x^{u+r}+x^{u+\ell}+x^{r+\ell}-x^{u+r+\ell})\sum_{j=1}^{\infty}\binom{j+2}{2}x^j.$$

Then the result follows. ∎

This result tells us how to count how many (co)facets of each type are shelled off. Using this result, we may compute the total contribution of the (co)facets shelled off to the total change in the CD-index as we move onto the splitter Y.

Due to the symmetry of $\Delta_i^d = \Delta_{d-i}^d$ as noted in Corollary 1.5.7, we may compute the total contribution of the (co)facets shelled off to the total change in the CD-index as we move onto the splitter Y as follows.

Proposition 4.3.11 (Lee-N). *The contribution from all the (co)facets shelled off to the change in the CD-index from P^* to Q^* as the origin moves from p to q in the relative interior of the convex of hull of the splitter Y is given by the following sum.*

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor -1} B\cdot(D^iC^{d-2i}-D^{i+1}C^{d-2(i+1)})\\
&-\frac{1}{2}B\cdot 2D^{\frac{d}{2}}\\
&-\frac{1}{2}\sum_{i=\lfloor \frac{d}{2}\rfloor +1}^{u+r+\ell-3} B\cdot(D^{d-i}C^{d-2(d-i)}-D^{d+1-i}C^{d-2(d+1-i)}),
\end{aligned}$$

if $2|d$;

$$\begin{aligned}
&-\frac{1}{2}\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor -1} B\cdot(D^iC^{d-2i}-D^{i+1}C^{d-2(i+1)})\\
&-\frac{1}{2}B\cdot D^{\frac{d-1}{2}}C\\
&-\frac{1}{2}\sum_{i=\lfloor \frac{d}{2}\rfloor +1}^{u+r+\ell-3} B\cdot(D^{d-i}C^{d-2(d-i)}-D^{d+1-i}C^{d-2(d+1-i)}),
\end{aligned}$$

where $B = \binom{i+2}{2} - \binom{i+2-u}{2} - \binom{i+2-r}{2} - \binom{i+2-\ell}{2} + \binom{i+2-u-r}{2} + \binom{i+2-u-\ell}{2} + \binom{i+2-r-\ell}{2} + \binom{i+2-u-r-\ell}{2}$.

Proof. Shelling simplicial d-polytopes gives the CD-index of their dual polytopes. So shelling off the (co)facets lost allows us to determine the CD-index contribution of the vertices of their duals, which are simple vertices. Therefore, we may use Lee's Δ-formulas in [16] and the result follows immediately. ■

Now we need to explore the ur (co)facets which are shelled on by determining the **S** and **T** associated with each (co)facet shelled on. After finding the CD-index for **S**, we will find the CD-index for **T**.

Lemma 4.3.12 (Lee-N)**.** *The* CD*-index of a given* $\mathbf{S}^*$ *is*

$$C^{u+r-2}\left(\sum_{i=0}^{\ell-1} D^i C^{\ell+m-2(i+1)}\right).$$

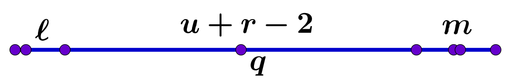

Figure 4.12: Here is the Gale diagram for **S** in the splitter case.

Proof. Each (co)facet consists of one point from either end of the splitter, *i.e.*, one point y_i from Y_1 and another point z_j from Y_2. To obtain the Gale diagram of one of these cofacets, project the 2-dimensional Gale diagram V onto the line orthogonal to $V \setminus \{y_i, z_j\}$ and then delete those two points [22]. So the Gale diagram for a (co)facet is 1-dimensional with two fewer points. See Figure 4.12. Notice that ℓ points fall on one side of q, $u+r-2$ points are stacked at q, and m points fall on the other side of q. Thus, the result follows from Theorem 4.1.1. ■

There are ur Q's and they are combinatorially equivalent. So they all have the same CD-index. Thus, we have the following result.

Corollary 4.3.13 (Lee-N)**.** *The sum of the CD-indices of all the* $\mathbf{S}^*$*'s is*

$$ur \cdot C^{u+r-2}\left(\sum_{i=0}^{\ell-1} D^i C^{\ell+m-2(i+1)}\right).$$

Using this result, we may compute the contribution of the ur Q's to the total change in the CD-index as we move onto the splitter.

Proposition 4.3.14 (Lee-N)**.** *The contribution from all the* $\mathbf{S}$*'s to the change in the* CD*-index from* P^* *to* Q^* *as the origin moves from* p *to* q *onto a splitter* Y *is*
$$\frac{ur}{2}\cdot C^{u+r-1}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)-\frac{ur}{2}\cdot DC^{u+r-3}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right).$$

Proof. Recall that $m\geq \ell$, $\ell+m\geq 2$, and $u+r\geq 3$. Then the following calculations hold. $\frac{1}{2}(2C-I)\cdot ur\cdot C^{u+r-2}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)$

$$= ur\cdot C^{u+r-1}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)-\frac{ur}{2}(IC)C^{u+r-3}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)$$
$$= ur\cdot C^{u+r-1}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)-\frac{ur}{2}(D+C^2)C^{u+r-3}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)$$
$$= ur\cdot C^{u+r-1}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)-\frac{ur}{2}\cdot DC^{u+r-3}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)-\frac{ur}{2}\cdot C^{u+r-1}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)$$
$$= \frac{ur}{2}\cdot C^{u+r-1}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)-\frac{ur}{2}\cdot DC^{u+r-3}\left(\sum_{i=0}^{\ell-1} D^iC^{\ell+m-2(i+1)}\right)$$ ■

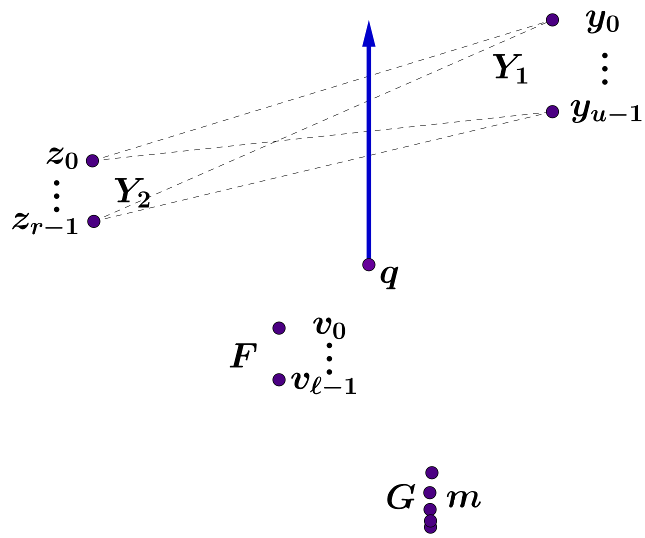

Figure 4.13: Lift the Gale diagram with a splitter Y to determine a shelling order for the facets shelled on.

Now that we understand the contribution of the $\mathbf{S}$'s to the change in the CD-index, we need to understand the contribution of the $\mathbf{T}$'s to the change in the CD-index.

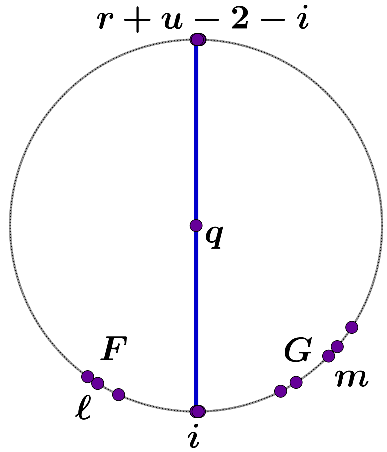

Figure 4.14: Project this lifting to obtain the Gale diagram for $\mathbf{T}$.

As we argued in the previous section, we obtain the Gale diagrams for the $\mathbf{T}$'s by lifting the 2-dimensional Gale diagram so that the u points are higher than the r points and the ℓ points and the m points are much lower. The origin starts below the higher u and r points and above the lower ℓ and m points. As the origin moves up, it crosses all possible splitters determined by one point from the r points and one point from the u points. See Figure 4.13. For a given splitter, project in the direction of the splitter onto the plane perpendicular to the unlifted splitter and delete the two points. Then we have a 2-dimensional Gale diagram for $\mathbf{T}$ with a total of $r+u-2$ points on either side of the diagonal and the ℓ points on the low left side and the m points on the low right side. See Figure 4.14. To compute the CD-contribution from all the $\mathbf{T}$'s, we will determine how many $\mathbf{T}$'s of each type arise.

Lemma 4.3.15 (Lee-N)**.** *All $\mathbf{T}$'s are of type $(r+u-2-i, i, \ell, m)$. There are $i+1$ $\mathbf{T}$'s for $0 \le i \le r-1$ and $r+u-1-i$ R's for $u \le i \le r+u-2$. When $r < u$, there are also r $\mathbf{T}$'s for $r \le i \le u-1$.*

Proof. Each $\mathbf{T}$ arises by projecting orthogonally when the origin crosses a splitter as it moves up. So the number of splitters of each type crossed is equivalent to the number of $\mathbf{T}$'s of each type arising during this process.

There is exactly one way to choose a point from the r (u) vertices to split off i points above that point for every i where $0 \le i \le r-1$ ($0 \le i \le u-1$). Then the number of i-splitters is given by the coefficient of x^i in the following product

$$\left(\sum_{i=0}^{r-1} x^i\right)\left(\sum_{i=0}^{u-1} x^i\right).$$

When $r \neq u$, this product may be written as follows.

$$(1+x+x^2+\ldots+x^{r-1})(1+x+x^2+\ldots+x^{u-1})$$

$$= 1+2x+3x^2+\ldots+rx^{r-1}+rx^r+\ldots+rx^{u-1}+(r-1)x^u+\ldots+2x^{u-1+r-2}+x^{u-1+r-1}$$

$$= \sum_{i=0}^{r-1}(i+1)x^i + \sum_{i=r}^{u-1} rx^i + \sum_{i=1}^{r-1}(r-i)x^{u-1+i}$$

$$= \sum_{i=0}^{r-1}(i+1)x^i + \sum_{i=r}^{u-1} rx^i + \sum_{i=u}^{r+u-2}(r+u-i-1)x^i$$

When $r = u$, the middle sum is empty.

The i runs over all possible i-splitters. So after projecting orthogonally, there will be i points below the origin and $r+u-2-i$ points above the origin. Since the coefficient of x^i records how many $\mathbf{T}$'s of each type arise as the origin moves up through all the possible splitters, then the result follows. ■

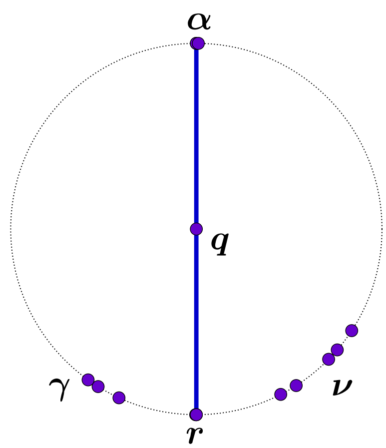

Figure 4.15: Here is the Gale diagram for $\mathbf{T}$ of *type* $(\alpha, \beta, \gamma, \nu)$.

Since we know how many $\mathbf{T}$'s arise of each type, it is now important to understand the CD-index of each type of $\mathbf{T}$. So we need to determine the CD-index of a general $\mathbf{T}$ of type $(\alpha, \beta, \gamma, \nu)$, as shown in Figure 4.15. First we need to calculate the number of w-splitters of each type crossed as the origin approaches the splitter in this general $\mathbf{T}$.

Lemma 4.3.16 (Lee-N)**.** *Consider a general* $\mathbf{T}$ *as described above. There are* $w+1$ *w-splitters for* $0 \le w \le min\{\alpha,\gamma\}-1$ *and* $\alpha+\gamma-w-1$ *w-splitters for* $max\{\alpha,\gamma\} \le w \le \alpha+\gamma-2$. *When* $\alpha \ne \gamma$*, the number of w-splitters for* $min\{\alpha,\gamma\} \le w \le max\{\alpha,\gamma\}-1$ *is* $min\{\alpha,\gamma\}$.

Proof. There is exactly one way to choose a point from the α (γ) vertices to split off w points above that point for every w where $0 \le w \le \alpha-1$ ($0 \le w \le \gamma-1$). Then the number of w-splitters is given by the coefficient of x^w in the following product

$$\left(\sum_{w=0}^{\alpha-1} x^w\right)\left(\sum_{w=0}^{\gamma-1} x^w\right).$$

When $\alpha \neq \gamma$, this product may be written as follows.

$$(1+x+x^2+\ldots+x^{\alpha-1})(1+x+x^2+\ldots+x^{\gamma-1})$$

$$= 1+2x+3x^2+\ldots+\min\{\alpha,\gamma\}x^{\min\{\alpha,\gamma\}-1}$$

$$+\min\{\alpha,\gamma\}x^{\min\{\alpha,\gamma\}}+\ldots+\min\{\alpha,\gamma\}x^{\max\{\alpha,\gamma\}-1}$$

$$+(\min\{\alpha,\gamma\}-1)x^{\max\{\alpha,\gamma\}}+\ldots+2x^{\max\{\alpha,\gamma\}-1+\min\{\alpha,\gamma\}-2}+x^{\max\{\alpha,\gamma\}-1+\min\{\alpha,\gamma\}-1}$$

$$= \sum_{w=0}^{\min\{\alpha,\gamma\}-1}(w+1)x^w + \sum_{w=\min\{\alpha,\gamma\}}^{\max\{\alpha,\gamma\}-1}\min\{\alpha,\gamma\}x^i$$

$$+\sum_{w=1}^{\min\{\alpha,\gamma\}-1}(\min\{\alpha,\gamma\}-w)x^{\max\{\alpha,\gamma\}-1+w}$$

$$= \sum_{w=0}^{\min\{\alpha,\gamma\}-1}(w+1)x^w + \sum_{w=\min\{\alpha,\gamma\}}^{\max\{\alpha,\gamma\}-1}\min\{\alpha,\gamma\}x^w + \sum_{w=\max\{\alpha,\gamma\}}^{\alpha+\gamma-2}(\alpha+\gamma-w-1)x^w$$

When $\alpha = \gamma$, the middle sum is empty and, thus, the above product may be written as follows.

$$(1+x+x^2+\ldots+x^{\alpha-1})(1+x+x^2+\ldots+x^{\alpha-1})$$

$$= 1+2x+3x^2+\ldots+\alpha x^{\alpha-1}+(\alpha-1)x^{\alpha}+\ldots+2x^{2\alpha-3}+x^{2\alpha-2}$$

$$= \sum_{w=0}^{\alpha-1}(w+1)x^w + \sum_{w=1}^{\alpha-1}(\alpha-w)x^{\alpha-1+w}$$

$$= \sum_{w=0}^{\alpha-1}(w+1)x^w + \sum_{w=\alpha}^{2\alpha-2}(2\alpha-w-1)x^w$$

The result follows from these two sums. ■

Summing over all possible w values and adding the change that occurs as we move the origin onto the splitter gives the entire CD-index of a general $\mathbf{T}^*$.

Corollary 4.3.17 (Lee-N)**.** *The CD-index of a general* $\mathbf{T}^*$ *is*

$$\sum_{w=0}^{min\{\alpha,\gamma\}-1}(w+1)D^wC^{\alpha+\beta+\gamma+\nu-3-2w} + \sum_{w=min\{\alpha,\gamma\}}^{max\{\alpha,\gamma\}-1} min\{\alpha,\gamma\}D^wC^{\alpha+\beta+\gamma+\nu-3-2w}$$

$$+\sum_{w=max\{\alpha,\gamma\}}^{\alpha+\gamma-2}(\alpha+\gamma-w-1)D^wC^{\alpha+\beta+\gamma+\nu-3-2w}+x(\alpha,\beta,\gamma,\nu),$$

where $x(\alpha,\beta,\gamma,\nu)$ *denotes the change in the* CD*-index as we move the origin onto the splitter from very close by.*

Simplifying algebraically, our previous results give the CD-index for the sum over all $\mathbf{T}$'s.

Corollary 4.3.18 (Lee-N)**.** *The sum of the CD-indices of all the $\mathbf{T}^*$'s that arise when starting with a Gale diagram of type (r, u, ℓ, m) is*

$$\sum_{i=0}^{r-1}(i+1)\Bigg(\sum_{w=0}^{min\{r+u-2-i,\ell\}-1}(w+1)D^w C^{r+u+\ell+m-5-2w}$$
$$+\sum_{w=min\{r+u-2-i,\ell\}}^{max\{r+u-2-i,\ell\}-1} min\{r+u-2-i,\ell\}D^w C^{r+u+\ell+m-5-2w}$$
$$+\sum_{w=max\{r+u-2-i,\ell\}}^{r+u+\ell-4-i}(r+u+\ell-3-i-w)D^w C^{r+u+\ell+m-5-2w}$$
$$+\chi(r+u-2-i,i,\ell,m)\Bigg)$$
$$+\sum_{i=r}^{u-1} r\Bigg(\sum_{w=0}^{min\{r+u-2-i,\ell\}-1}(w+1)D^w C^{r+u+\ell+m-5-2w}$$
$$+\sum_{w=min\{r+u-2-i,\ell\}}^{max\{r+u-2-i,\ell\}-1} min\{r+u-2-i,\ell\}D^w C^{r+u+\ell+m-5-2w}$$
$$+\sum_{w=max\{r+u-2-i,\ell\}}^{r+u+\ell-4-i}(r+u+\ell-3-i-w)D^w C^{r+u+\ell+m-5-2w}$$
$$+\chi(r+u-2-i,i,\ell,m)\Bigg)$$
$$+\sum_{i=u}^{r+u-2}(r+u-1-i)\Bigg(\sum_{w=0}^{min\{r+u-2-i,\ell\}-1}(w+1)D^w C^{(r+u+\ell+m-5-2w}$$
$$+\sum_{w=min\{r+u-2-i,\ell\}}^{max\{r+u-2-i,\ell\}-1} min\{r+u-2-i,\ell\}D^w C^{r+u+\ell+m-5-2w}$$
$$+\sum_{w=max\{r+u-2-i,\ell\}}^{r+u+\ell-4-i}(r+u+\ell-3-i-w)D^w C^{r+u+\ell+m-5-2w}$$
$$+\chi(r+u-2-i,i,\ell,m)\Bigg)$$

Using this result, we may compute the contribution of all ur $\mathbf{T}$'s to the total change in the CD-index as we move onto the splitter.

Proposition 4.3.19 (Lee-N)**.** *The contribution from all the $\mathbf{T}$'s to the change in the CD-index from P^* to Q^* as the origin moves from p to q is*

$$\frac{1}{2}(4D-I^2)\Bigg[\sum_{i=0}^{r-1}(i+1)\Bigg(\sum_{w=0}^{min\{r+u-2-i,\ell\}-1}(w+1)D^w C^{r+u+\ell+m-5-2w}$$

$$+\sum_{w=min\{r+u-2-i,\ell\}}^{max\{r+u-2-i,\ell\}-1} min\{r+u-2-i,\ell\} D^w C^{r+u+\ell+m-5-2w}$$

$$+\sum_{w=max\{r+u-2-i,\ell\}}^{r+u+\ell-4-i} (r+u+\ell-3-i-w) D^w C^{r+u+\ell+m-5-2w}$$

$$+\chi(r+u-2-i,i,\ell,m)\Bigg)$$

$$+\sum_{i=r}^{u-1} r\Bigg(\sum_{w=0}^{min\{r+u-2-i,\ell\}-1} (w+1) D^w C^{r+u+\ell+m-5-2w}$$

$$+\sum_{w=min\{r+u-2-i,\ell\}}^{max\{r+u-2-i,\ell\}-1} min\{r+u-2-i,\ell\} D^w C^{r+u+\ell+m-5-2w}$$

$$+\sum_{w=max\{r+u-2-i,\ell\}}^{r+u+\ell-4-i} (r+u+\ell-3-i-w) D^w C^{r+u+\ell+m-5-2w}$$

$$+\chi(r+u-2-i,i,\ell,m)\Bigg)$$

$$+\sum_{i=u}^{r+u-2} (r+u-1-i)\Bigg(\sum_{w=0}^{min\{r+u-2-i,\ell\}-1} (w+1) D^w C^{r+u+\ell+m-5-2w}$$

$$+\sum_{w=min\{r+u-2-i,\ell\}}^{max\{r+u-2-i,\ell\}-1} min\{r+u-2-i,\ell\} D^w C^{r+u+\ell+m-5-2w}$$

$$+\sum_{w=max\{r+u-2-i,\ell\}}^{r+u+\ell-4-i} (r+u+\ell-3-i-w) D^w C^{r+u+\ell+m-5-2w}$$

$$+\chi(r+u-2-i,i,\ell,m)\Bigg)\Bigg]$$

To rewrite an (IC)-polynomial as a CD-polynomial, the following equalities are useful.

Whenever the power of C is at least 2, then $I^2 D^a C^b$

$$\begin{aligned}
&= D^a I^2 C^b \\
&= D^a I(IC)C^{b-1} \\
&= D^a I(D+C^2)C^{b-1} \\
&= D^{a+1}(IC)C^{b-2} + D^a(IC)C^b \\
&= D^{a+1}(D+C^2)C^{b-2} + D^a(D+C^2)C^b \\
&= D^{a+2}C^{b-2} + D^{a+1}C^b + D^{a+1}C^b + D^aC^{b+2} \\
&= D^{a+2}C^{b-2} + 2D^{a+1}C^b + D^aC^{b+2}.
\end{aligned}$$

Whenever the power of the first C is exactly 1 and the power of the second C is at least 1, then $I^2D^aCD^bC^e$

$$\begin{aligned}
&= D^aI^2CD^bC^e \\
&= D^aI(IC)D^bC^e \\
&= D^aI(D+C^2)D^bC^e \\
&= D^aID^{b+1}C^e + D^a(IC)CD^bC^e \\
&= D^{a+b+1}(IC)C^{e-1} + D^a(IC)CD^bC^e \\
&= D^{a+b+1}(D+C^2)C^{e-1} + D^a(D+C^2)CD^bC^e \\
&= D^{a+b+2}C^{e-1} + D^{a+b+1}C^{e+1} + D^{a+1}CD^bC^e + D^aC^3D^bC^e.
\end{aligned}$$

We put together all of these pieces to determine the change in the CD-index from P^* to Q^* as the origin moves from p to q in the relative interior of the convex hull of the splitter Y.

Conjecture 4.3.20 (Lee-N)**.** *Denote the change in the CD-index from P^* to Q^* as the origin moves from p to q in the relative interior of the convex hull of the splitter Y as $\chi(u,r,\ell,m)$. Then $\chi(u,r,\ell,m) = \chi(r,u,\ell,m)$*

$$= -\sum_{\beta=0}^{\ell-1}\left(\sum_{\alpha=0}^{r-1} D^{\alpha+\beta}C^{d-2(\alpha+\beta)}\right) + \left(\sum_{\alpha=0}^{r-1} D^{\alpha}C^{u+r-1-2\alpha}\right)\left(\sum_{\beta=0}^{\ell-1} D^{\beta}C^{\ell+m-2(\beta+1)}\right)$$

Evidence We have computed the change in the CD-index for a few values of u, r, ℓ, and m. They are listed here to motivate our above conjecture.

$$\begin{array}{rcll}
\chi(1,1,2,2) & = & -D^0C^3 & -DC \\
 & & +D^0CD^0C^2 & +D^0CDC^0 \\
\chi(2,1,2,2) & = & \chi(1,2,2,2) & \\
 & = & -D^0C^4 & -DC^2 \\
 & & +D^0C^2D^0C^2 & +D^0C^2DC^0 \\
\chi(3,1,2,2) & = & \chi(1,3,2,2) & \\
 & = & -D^0C^5 & -DC^3 \\
 & & +D^0C^3D^0C^2 & +D^0C^3DC^0 \\
\chi(2,2,2,2) & = & -D^0C^5 & -DC^3 \\
 & & +D^0C^3D^0C^2 & +D^0C^3DC^0 \\
 & & -DC^3 & -D^2C \\
 & & +DCD^0C^2 & +DCDC^0 \\
\chi(3,2,2,2) & = & \chi(2,3,2,2) & \\
 & = & -D^0C^6 & -DC^4 \\
 & & +D^0C^4D^0C^2 & +D^0C^4DC^0 \\
 & & -DC^4 & -D^2C^2 \\
 & & +DC^2D^0C^2 & +DC^2DC^0
\end{array}$$

$$
\begin{aligned}
\chi(1,1,2,6) &= -D^0C^7 - DC^5 \\
&\quad +D^0CD^0C^6 + D^0CDC^4 \\
\chi(2,1,2,6) &= \chi(1,2,2,6) \\
&= -D^0C^8 - DC^6 \\
&\quad +D^0C^2D^0C^6 + D^0C^2DC^4 \\
\chi(3,1,2,6) &= \chi(1,3,2,6) \\
&= -D^0C^9 - DC^7 \\
&\quad +D^0C^3D^0C^6 + D^0C^3DC^4 \\
\chi(4,1,2,6) &= \chi(1,4,2,6) \\
&= -D^0C^10 - DC^8 \\
&\quad +D^0C^4D^0C^6 + D^0C^4DC^4 \\
\chi(2,2,2,6) &= -D^0C^9 - DC^7 - DC^7 - D^2C^5 \\
&\quad +D^0C^3D^0C^6 + D^0C^3DC^4 + DCD^0C^6 + DCDC^4 \\
\chi(3,2,2,6) &= \chi(2,3,2,6) \\
&= -D^0C^{10} - DC^8 - DC^8 - D^2C^6 \\
&\quad +D^0C^4D^0C^6 + D^0C^4DC^4 + DC^2D^0C^6 + DC^2DC^4 \\
\chi(3,3,2,6) &= -D^0C^{11} - DC^9 \\
&\quad +D^0C^5D^0C^6 + D^0C^5DC^4 \\
&\quad -DC^9 - D^2C^7 \\
&\quad +DC^3D^0C^6 + DC^3DC^4 \\
&\quad -D^2C^7 - D^3C^5 \\
&\quad +D^2CD^0C^6 + D^2CDC^4 \\
\chi(4,3,2,6) &= \chi(3,4,2,6) \\
&= -D^0C^{12} - DC^{10} \\
&\quad +D^0C^6D^0C^6 + D^0C^6DC^4 \\
&\quad -DC^{10} - D^2C^8 \\
&\quad +DC^4D^0C^6 + DC^4DC^4 \\
&\quad -D^2C^8 - D^3C^6 \\
&\quad +D^2C^2D^0C^6 + D^2C^2DC^4
\end{aligned}
$$

$$
\begin{aligned}
\chi(1,1,3,6) &= -D^0C^8 - DC^6 - D^2C^4 \\
&\quad +D^0CD^0C^7 + D^0CDC^5 + D^0CD^2C^3 \\
\chi(2,1,3,6) &= \chi(1,2,3,6) \\
&= -D^0C^9 - DC^7 - D^2C^5 \\
&\quad +D^0C^2D^0C^7 + D^0C^2DC^5 + D^0C^2D^2C^3 \\
\chi(3,1,3,6) &= \chi(1,3,3,6) \\
&= -D^0C^{10} - DC^8 - D^2C^6 \\
&\quad +D^0C^3D^0C^7 + D^0C^3DC^5 + D^0C^3D^2C^3 \\
\chi(2,2,3,6) &= -D^0C^{10} - DC^8 - D^2C^6 \\
&\quad +D^0C^3D^0C^7 + D^0C^3DC^5 + D^0C^3D^2C^3 \\
&\quad -DC^8 - D^2C^6 - D^3C^4 \\
&\quad +DCD^0C^7 + DCDC^5 + DCD^2C^3 \\
\chi(3,2,3,6) &= \chi(2,3,3,6) \\
&= -D^0C^{11} - DC^9 - D^2C^7 \\
&\quad +D^0C^4D^0C^7 + D^0C^4DC^5 + D^0C^4D^2C^3 \\
&\quad -DC^9 - D^2C^7 - D^3C^5 \\
&\quad +DC^2D^0C^7 + DC^2DC^5 + DC^2D^2C^3 \\
\chi(3,3,3,6) &= -D^0C^{12} - DC^{10} - D^2C^8 \\
&\quad +D^0C^5D^0C^7 + D^0C^5DC^5 + D^0C^5D^2C^3 \\
&\quad -DC^{10} - D^2C^8 - D^3C^6 \\
&\quad +DC^3D^0C^7 + DC^3DC^5 + DC^3D^2C^3 \\
&\quad -D^2C^8 - D^3C^6 - D^4C^4 \\
&\quad +D^2CD^0C^7 + D^2CDC^5 + D^2CD^2C^3
\end{aligned}
$$

So, it appears as though $\chi(j,k,\ell,m) = \chi(k,j,\ell,m)$ satisfies the following pattern.

$$
\begin{aligned}
&-D^0C^{j+k+\ell+m-3} - DC^{j+k+\ell+m-5} - \ldots - D^{\ell-1}C^{j+k+m-\ell-1} \\
&+D^0C^{j+k-1}D^0C^{\ell+m-2} + D^0C^{j+k-1}DC^{\ell+m-4} + \ldots + D^0C^{j+k-1}D^{\ell-1}C^{m-\ell} \\
&-DC^{j+k+\ell+m-5} - D^2C^{j+k+\ell+m-7} - \ldots - D^{\ell}C^{j+k+m-\ell-3} \\
&+DC^{j+k-3}D^0C^{\ell+m-2} + DC^{j+k-3}DC^{\ell+m-4} + \ldots + DC^{j+k-3}D^{\ell-1}C^{m-\ell} \\
&\ldots \\
&\ldots \\
&\ldots \\
&-D^{k-1}C^{j+\ell+m-k-1} - D^kC^{j+\ell+m-k-3} - \ldots - D^{k+\ell-2}C^{j+m-k-\ell+1} \\
&+D^{k-1}C^{j-k+1}D^0C^{\ell+m-2} + D^{k-1}C^{j-k+1}DC^{\ell+m-4} + \ldots + D^{k-1}C^{j-k+1}D^{\ell-1}C^{m-\ell}
\end{aligned}
$$

The next result follows immediately from Conjecture 4.3.20.

Corollary 4.3.21 (Lee-N)**.** *The g-vector of Q^* is the same as the g-vector of P^*.*

Proof. We may determine the change in the g-vector by plugging in x for D and 1 for C by Theorem 1.4.13 [13]. Then we obtain the following result.

$$
-\sum_{\beta=0}^{\ell-1}\left(\sum_{\alpha=0}^{r-1} x^{\alpha+\beta}\cdot 1^{d-2(\alpha+\beta)}\right) + \left(\sum_{\alpha=0}^{r-1} x^{\alpha}\cdot 1^{u+r-1-2\alpha}\right)\left(\sum_{\beta=0}^{\ell-1} x^{\beta}\cdot 1^{\ell+m-2(\beta+1)}\right)
$$

$$= -\sum_{\beta=0}^{\ell-1}\left(\sum_{\alpha=0}^{r-1} x^{\alpha}\cdot x^{\beta}\right) + \sum_{\alpha=0}^{r-1} x^{\alpha}\sum_{\beta=0}^{\ell-1} x^{\beta}$$
$$= -\sum_{\beta=0}^{\ell-1} x^{\beta}\sum_{\alpha=0}^{r-1} x^{\alpha} + \sum_{\alpha=0}^{r-1} x^{\alpha}\sum_{\beta=0}^{\ell-1} x^{\beta}$$
$$= 0$$

So there is no change in the g-vector. Since we are moving from the smaller side to the larger side, this result agrees with our g-vector calculations in Chapter 2 (Theorem 2.3.7). ■

This agreement with our earlier result further supports the validity of our conjecture.

Chapter 5 Future Work

Here are some unanswered questions worth pursuing.

- How do we read off the g-vector from any Gale diagram? How do we completely generalize the interpretation of the g-vector from the Gale diagram?
- To what extent does the triangulation part of the generalized lower bound theorem extend to non-simplicial polytopes?
- Prove the CD-Conjecture for all 2-dimensional Gale diagrams.
- How do we read off the CD-index from any Gale diagram? How do we completely generalize the interpretation of the CD-index from the Gale diagram?

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