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LEFSCHETZ PROPERTIES AND ENUMERATIONS

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LEFSCHETZ PROPERTIES AND ENUMERATIONS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
David Cook II
Lexington, Kentucky

Director: Dr. Uwe Nagel, Professor of Mathematics
Lexington, Kentucky 2012

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ABSTRACT OF DISSERTATION

LEFSCHETZ PROPERTIES AND ENUMERATIONS

An artinian standard graded algebra has the weak Lefschetz property if the multiplication by a general linear form induces maps of maximal rank between consecutive degree components. It has the strong Lefschetz property if the multiplication by powers of a general linear form also induce maps of maximal rank between the appropriate degree components. These properties are mainly studied for the constraints they place, when present, on the Hilbert series of the algebra. While the majority of research on the Lefschetz properties has focused on characteristic zero, we primarily consider the presence of the properties in positive characteristic. We study the Lefschetz properties by considering the prime divisors of determinants of critical maps.

First, we consider monomial complete intersections in a finite number of variables. We provide two complements to a result of Stanley. We next consider monomial almost complete intersections in three variables. We connect the characteristics in which the weak Lefschetz property fails with the prime divisors of the signed enumeration of lozenge tilings of a punctured hexagon. Last, we study how perturbations of a family of monomial algebras can change or preserve the presence of the Lefschetz properties. In particular, we introduce a new strategy for perturbations rooted in techniques from algebraic geometry.

KEYWORDS: Commutative algebra, combinatorics, Lefschetz properties, monomial ideals, determinants

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Date: 11. April 2012

LEFSCHETZ PROPERTIES AND ENUMERATIONS

By
David Cook II

Director of Dissertation: Uwe Nagel

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Date: 11. April 2012

For Calico Womac, *requiescat in pace*
&
For Milli Rotkätzchen

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TABLE OF CONTENTS

Acknowledgments	iii
Table of Contents	iv
List of Figures	v
Chapter 1 Introduction	1
Chapter 2 The Lefschetz properties	5
2.1 Fundamental concepts	5
2.2 Algebraic tools	7
2.3 Stability of syzygy bundles	9
Chapter 3 Monomial complete intersections	11
3.1 Background	11
3.2 Bounding failure of the Lefschetz properties	13
3.3 Almost quadratically generated ideals	16
3.4 The presence of the Lefschetz properties for two variables	17
3.5 The presence of the Lefschetz properties for three variables	22
3.6 The presence of the Lefschetz properties in many variables	25
3.7 Conclusions	28
Chapter 4 Monomial almost complete intersections	30
4.1 Background	31
4.2 Punctured hexagons and friends	36
4.3 Interlude of signs	43
4.4 Determinants	47
4.5 Centralising the puncture	57
4.6 Interesting families and examples	62
4.7 Splitting type and regularity	66
Chapter 5 Subtlety to perturbation	72
5.1 Liftings and hyperplanes sections	72
5.2 The weak Lefschetz property	74
5.3 The strong Lefschetz property	79
Bibliography	81
Vita	85

LIST OF FIGURES

4.1	The interlaced basis region for $R = K[x, y, z]$	36
4.2	$H_{a,b,c,\alpha,\beta,\gamma}$, the interlaced basis region of $[R/I]_s$ and $[R/I]_{s+1}$	37
4.3	A lozenge tiling and its associated family of non-intersecting lattice paths	38
4.4	Converting lozenge tilings to families of non-intersecting lattice paths	39
4.5	A lozenge tiling and its associated perfect matching of edges	41
4.6	Converting lozenge tilings to perfect matchings of edges	42
4.7	<i>up</i> and <i>down</i> lozenge groups with lattice path pieces superimposed	44
4.8	An <i>up</i> lozenge group with labeling	44
4.9	An example of tilings T and U of $H_{9,8,9,4,3,3}$, for $k = 1$	45
4.10	An example of tilings T' and U' of $H_{9,8,9,4,3,3}$, for $k = 1$	46
4.11	Example of interpreting the sign	47
4.12	A $3 \times 6 \times 5$ plane partition and its associated lozenge tiling	49
4.13	A punctured hexagon with a trivial puncture	50
4.14	A punctured hexagon with a trivial side	51
4.15	A punctured hexagon with a maximal side	52
4.16	A punctured hexagon with $\gamma = 0$	54
4.17	The Mahonian terms in the evaluation of $\gamma = 0$	54
4.18	When $a = b$ and $\alpha = \beta$, then $H_{a,a,c,\alpha,\alpha,\gamma}$ is symmetric.	55
4.19	A punctured hexagon with an axis-central puncture.	58
4.20	A punctured hexagon with a gravity-central puncture.	60
4.21	When $A = B = \gamma = 0$, then $H_{a,b,c,\alpha,\beta,\gamma}$ has a unique tiling.	64

Chapter 1 Introduction

In this dissertation we will consider a pair of algebraic properties which were motivated by a topological theorem. In fact, the weak and strong Lefschetz properties are natural abstractions from the conclusion of the Hard Lefschetz Theorem. We study these properties by using techniques in combinatorics, number theory, and algebraic geometry.

Both Lefschetz properties have been studied extensively; the recent manuscript by Harima, Maeno, Morita, Numata, Wachi, and Watanabe [21] provides a wonderfully comprehensive exploration of the Lefschetz properties. In particular, the presence of the properties provides interesting constraints on the Hilbert functions of the algebras (see, e.g., [1], [22], [41], and [53]). For example, Harima, Migliore, Nagel, and Watanabe [22] completely classified the Hilbert functions of algebras with the Lefschetz properties.

The Lefschetz properties have been used in many contexts, e.g., Stanley [48] used, essentially, the strong Lefschetz property to prove the necessity of the conditions in the g -theorem, thus classifying the f -vectors of simplicial convex polytopes. Despite this utility much is still unknown about the presence of the Lefschetz properties, even in seemingly simple cases (see, e.g., [5] and [38]).

In 1980, Stanley [49] proved that every artinian monomial complete intersection over a polynomial ring has the strong Lefschetz property. See the survey [40] by Migliore and Nagel for a discussion of the amount and depths of research that this theorem has inspired. We emphasise that this theorem, and indeed most results about the Lefschetz properties, are specific to characteristic zero. We focus on establishing the presence or absence of the Lefschetz properties in positive characteristic. In particular, we consider artinian monomial complete intersections, artinian monomial *almost* complete intersections, and perturbations of the latter.

We now give an overview of the contents of the dissertation.

In Chapter 2 we introduce the fundamental concepts from commutative algebra that we will use throughout this thesis. Then we will provide an array of algebraic tools that are used extensively in our study of the Lefschetz properties.

In Chapter 3 we study artinian monomial complete intersection over a polynomial ring for the presence of the weak and strong Lefschetz properties in positive characteristic in contrast to the aforementioned theorem of Stanley. We first demonstrate that any possible failure must occur in small primes for both the weak Lefschetz property (Proposition 3.10) and the strong Lefschetz property (Theorem 3.11). We then provide a complement to Stanley's result in characteristic two:

Theorem. (Theorem 3.35) *Let $d_0 \geq \dots \geq d_n \geq 2$ be integers with $n \geq 1$, and let $I = (x_0^{d_0}, \dots, x_n^{d_n}) \subset R = K[x_0, \dots, x_n]$, where K is an infinite field of characteristic two. Then R/I has the strong Lefschetz property if and only if $n = 1$ and either (i) d_0 is odd and $d_1 = 2$ or (ii) $d_0 = 4k + 2$ for some $k \in \mathbb{N}$ and $d_1 = 3$.*

If we restrict to the case of generation by monomials all of the same degree, then

a more complete picture is possible. In particular, the weak Lefschetz property is classified by Brenner and Kaid [5] (for three variables) and Kustin and Vraciu [32] (for at least four variables), both in 2011. We provide a complement to this result with regard to the strong Lefschetz property:

Theorem. (Theorem 3.36) *Let $d \geq 2$, $n \geq 1$, and $I = (x_0^d, \dots, x_n^d) \subset R = K[x_0, \dots, x_n]$, where K is an infinite field of characteristic p . Then R/I has the strong Lefschetz property if and only if p is zero or p is a positive prime and either*

(i) $n = 1$ and $p^s > 2(d - 1)$, where s is the largest integer such that p^{s-1} divides $(2d - 1)(2d + 1)$, or

(ii) $n \geq 2$ and $p > (n + 1)(d - 1)$.

The above two results indicate that most of the interesting variation occurs in two variables. This lead us to ask Question 3.37: for which prime characteristics p does the algebra $K[x, y]/(x^a, y^b)$, where $a \geq b \geq 2$, fail to have the strong Lefschetz property? Similar behaviour holds for the weak Lefschetz property; in the case of generation in a single degree, the weak Lefschetz property always holds in two variables (Proposition 2.16) but requires more care in three or more variables (see [5, Theorem 2.6] and Theorem 3.31).

In Chapter 4 we examine artinian monomial *almost* complete intersections in three variables for the presence of the weak Lefschetz property, in both characteristic zero and positive characteristic. To the end, we introduce a natural correspondence between these algebras and punctured hexagons.

Let A be an artinian monomial almost complete intersection in three variables, that is, an artinian ideal with four monomial generators. In Proposition 4.10, we connect the prime characteristics where the weak Lefschetz property is absent for A to the prime divisors of the determinants of a binomial matrix N and a zero-one matrix Z . In Theorem 4.15 we prove that the determinant of N is the enumeration of signed lozenge tilings of the associated punctured hexagon H , and in Theorem 4.18 we prove that the determinant of Z is the enumeration of signed perfect matchings of an associated bipartite graph. Moreover, we show that the sign coming from the lozenge tilings agrees with the sign of the perfect matching. Then, we show in Theorem 4.23 that the determinant of N is the same, up to sign, as the determinant of Z .

Using the above connections, in Section 4.4 we provide a wide variety of results about both the presence of the weak Lefschetz property for artinian monomial almost complete intersections and the enumeration of signed lozenge tilings of punctured hexagons. One interesting example, which is surprising in its simplicity, is that when the puncture of the associated punctured hexagon has even side length, then the enumeration of signed lozenge tilings is positive:

Theorem. (Theorem 4.28) *Suppose $a > \alpha \geq 0, b > \beta \geq 0$, and $c > \gamma \geq 0$. Let $A = K[x, y, z]/I$, where $I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$, and suppose A has a semistable syzygy bundle. If $a + b + c$ is even, then the associated punctured hexagon has even side length. Moreover, A has the weak Lefschetz property in characteristic zero and when the characteristic is sufficiently large.*

This result is indeed applicable because we characterise the ideals that have a semistable syzygy bundle in Proposition 4.3.

We close Chapter 4 by considering the perturbations in the generic splitting type of the syzygy bundle of the algebra that come from the presence or absence of the weak Lefschetz property. Using this, we see that the weak Lefschetz property, at least in the case of artinian monomial almost complete intersection in three variables, does not just have algebraic and combinatorial interpretations, but it also has an interpretation in the terms of algebraic geometry:

Theorem. (Theorem 4.67) *Let $R = K[x, y, z]$, where K is a field of arbitrary characteristic. Let $I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$ be associated to a punctured hexagon; in particular, $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$ and $\text{syz } I$ is semistable (see Proposition 4.3). Set $s = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma) - 2$.*

Then the following conditions are equivalent:

- (i) *The algebra R/I has the weak Lefschetz property;*
- (ii) *the regularity of S/J is s , where $S/J = K[x, y]/(x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$.*
- (iii) *the determinant of the associated binomial matrix N (i.e., the enumeration of signed lozenge tilings of the punctured hexagon H) modulo the characteristic of K is non-zero; and*
- (iv) *the determinant of the associated zero-one matrix Z (i.e., the enumeration of signed perfect matchings of the bipartite graph associated to H) modulo the characteristic of K is non-zero.*

If the characteristic of K is zero, then there is one further equivalent condition:

- (v) *The generic splitting type of $\text{syz } I$ is $(s + 2, s + 2, s + 2)$.*

In Chapter 5 we consider the subtlety of the weak Lefschetz property under deformation. As a motivation, we note that in [38, Section 5] it was shown by example that a small *ad hoc* perturbation of a monomial ideal without the weak Lefschetz property may result in an ideal having the weak Lefschetz property for almost every field characteristic. Instead of an *ad hoc* approach, we propose a systematic way of deforming a monomial ideal that preserves the Hilbert function but possibly modifies the presence or absence of the weak Lefschetz property. In particular, we show that the general hyperplane section of a family of level artinian set of points has the weak Lefschetz property in almost every characteristic, whereas a special hyperplane section never has the weak Lefschetz property:

Theorem. (Corollary 5.8) *Let $t \geq 1$ be an integer and set $A = R/\overline{I}_t$, where*

$$\overline{I}_t := \left(\prod_{i=0}^t (x - iw), \prod_{i=0}^t (y - iw), \prod_{i=0}^t (z - iw), xyz \right) \subset R = K[w, x, y, z].$$

Then:

- (i) *If the characteristic of K is zero or greater than t , then the ideal \overline{I}_t defines a set of $3(t+1)t+1$ points in \mathbb{P}^3 that is level of type three.*
- (ii) *If $\ell \in [R]_1$ is a general linear form, then $A/\ell A$ has the weak Lefschetz property, regardless of the characteristic of K .*
- (iii) *If $\ell = w$, then the Artinian algebra $A/\ell A$ has the weak Lefschetz property if and only if t is odd and $\text{char } K \neq 2$.*

Chapter 2 The Lefschetz properties

This chapter provides a primer of the fundamental concepts and tools used throughout this dissertation. In particular, in Section 2.1 we give an overview of definitions and concepts relevant to the study of the Lefschetz properties. In Section 2.2 we describe new and old techniques for establishing the Lefschetz properties, particularly for monomial algebras.

2.1 Fundamental concepts

We assume the reader is familiar with basic ring and module theory. For further details on the fundamental concepts of commutative algebra, the reader may consult any graduate text on commutative algebra, such as [13] or [42].

The most basic object we will be working with is a graded ring.

Definition 2.1. A *graded ring* is a ring R with a decomposition of abelian groups

$$R = \bigoplus_{d \in \mathbb{Z}} [R]_d$$

such that

$$[R]_i \cdot [R]_j \subset [R]_{i+j}$$

for all integers i and j . The elements in $[R]_d$ are called the *homogeneous elements of degree d* , and an ideal of R generated by homogeneous elements is called a *homogeneous ideal*. Moreover, for an integer a , we denote the a^{th} *twist of R* as $R(-a)$, where $[R(-a)]_i := [R]_{i-a}$.

A graded ring we will be using for the remainder of the dissertation is the polynomial ring over a field.

Example 2.2. Consider the $(n + 1)$ -variate polynomial ring $R = K[x_0, \dots, x_n]$ over the field K . We will always consider R as a graded ring with the standard grading, that is, $\deg x_i = 1$. The d^{th} homogeneous component of R , $[R]_d$, is the set of degree d homogeneous polynomials in R . For example, if the degree of the variables x_i is set to one, then $[R]_0 = K$ and

$$[R]_1 = \{a_0x_0 + \dots + a_nx_n \mid a_i \in K\}.$$

Definition 2.3. A graded ring R is said to be *artinian* if there exists an integer e such that $[R]_d = 0$ for all $d > e$. If R/I is artinian, then we say that I is *artinian*.

That is, an artinian ring is one that is finite, when viewed as a vector space over the homogeneous component of degree zero.

For the remainder of this section let $R = K[x_0, \dots, x_n]$ be the $(n + 1)$ -variate polynomial ring over the field K , and let I be a homogeneous ideal of R .

Definition 2.4. Assume K is an infinite field. The algebra R/I is said to have the *strong Lefschetz property* if there exists a linear form $\ell \in R/I$ such that the map

$$\times \ell^k : [R/I]_d \rightarrow [R/I]_{d+k}$$

has maximal rank for all integers $d \geq 0$ and $k \geq 1$. In this case, ℓ is called a *strong Lefschetz element* of R/I .

If the property holds for $k = 1$, then R/I is said to have the *weak Lefschetz property*, and ℓ is called a *weak Lefschetz element* of R/I .

Clearly, the presence of the strong Lefschetz property implies the presence of the weak Lefschetz property.

Definition 2.5. The *Hilbert function* of R/I is the integer function $h_{R/I} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ defined by

$$h_{R/I}(d) := \dim_K [R/I]_d,$$

i.e., the dimension of $[R/I]_d$ as a vector space over K .

Further, if R/I is artinian with $[R/I]_d = 0$ for $d > e$, but $[R/I]_e \neq 0$, then we define the *h-vector* of R/I to be the finite sequence of positive integers

$$h(R/I) := (h_{R/I}(0), \dots, h_{R/I}(e)).$$

Thus, the Hilbert function encodes the size of each of the homogeneous components of a graded ring.

Example 2.6.

- (i) The Hilbert function in degree $d \geq 0$ of R is the number of monomials of degree d , that is, $h_R(d) = \binom{n+d}{n}$.
- (ii) Suppose $n = 2$, and consider the ideal $I = (x_0^3, x_1^3, x_2^2)$. Then R/I is artinian as every monomial of degree at least 6 is present in I , i.e., $[R/I]_6 = 0$. The *h-vector* of R/I is $(1, 3, 5, 5, 3, 1)$.

Definition 2.7. The *socle* of R/I is the set $\text{soc } R/I$ of all elements of R/I annihilated by every variable of R . If the socle is concentrated in a single degree, say t , then R/I is said to be *level of socle degree t* . Moreover, the K -dimension of the socle is called the *type of the socle*.

Example 2.8. (Example 2.6(ii) continued) Suppose $n = 2$, and consider the ideal $I = (x_0^3, x_1^3, x_2^2)$. Then $\text{soc } R/I = \{ax_0^2x_1^2x_2 \mid a \in K\}$ is concentrated in degree 5. Indeed, the degree 4 monomials in R/I are $x_0^2x_1x_2$, $x_0^2x_1^2$, and $x_0x_1^2x_2$, each of which divides $x_0^2x_1^2x_2$ hence is not in the socle. Thus R/I is a level algebra of socle degree 5.

2.2 Algebraic tools

Let $R = K[x_0, \dots, x_n]$ be the polynomial ring over the infinite field K .

Proposition 2.9. [38, Proposition 2.1] *Let I be a homogeneous artinian ideal of R . Set $A = R/I$, and let ℓ be a general linear form. Consider the maps $\varphi_d : [A]_d \rightarrow [A]_{d+1}$ given by multiplication by ℓ , for $d \geq 0$.*

- (i) *If φ_i is surjective, then φ_{i+1} is surjective.*
- (ii) *If R/I is level and φ_i is injective for some $i \geq 1$, then φ_{i-1} is injective.*
- (iii) *If R/I is level and $\dim_K[R/I]_i = \dim_K[R/I]_{i+1}$ for some i , then R/I has the weak Lefschetz property if and only if φ_i is injective (and hence is a bijection).*

Proposition 2.9(i) generalises to modules generated in degrees that are sufficiently small.

Lemma 2.10. *Let M be an R -module generated in degrees bounded by e , and let ℓ be a general linear form. If the map $\times \ell : [M]_d \rightarrow [M]_{d+1}$ is surjective, and $d \geq e$, then the map $\times \ell : [M]_{d+1} \rightarrow [M]_{d+2}$ is surjective.*

Proof. Consider the sequence

$$[M]_d \xrightarrow{\times \ell} [M]_{d+1} \rightarrow [M/\ell M]_{d+1} \rightarrow 0.$$

Notice the first map is surjective if and only if $[M/\ell M]_{d+1} = 0$. By assumption the map is surjective, so $[M/\ell M]_{d+1} = 0$. Hence $[M/\ell M]_{d+2}$ is zero as M has no generators with degree greater than d . \square

Thus, from this we get an analogous result to Proposition 2.9(ii) for non-level algebras.

Proposition 2.11. *Let I be a homogeneous artinian ideal of R . Set $A = R/I$, and let ℓ be a general linear form. If the map $\times \ell : [A]_{d-1} \rightarrow [A]_d$ is injective, and d is not greater than the smallest socle degree of A , then $\times \ell : [A]_{d-2} \rightarrow [A]_{d-1}$ is injective.*

Proof. The K -dual of A , M , is a shift of the canonical module of A and is generated in degrees that are a linear shift of the socle degrees of A . Consider now the map $\times \ell : [M]_i \rightarrow [M]_{i+1}$. Using Lemma 2.10 we see that once i is at least as large as the largest degree in which M is generated, and the map is surjective, then the map is surjective thereafter. The result then follows by duality. \square

Further recall that a monomial algebra has the weak (strong) Lefschetz property exactly when the sum of the variables is a weak (strong) Lefschetz element.

Proposition 2.12. [38, Proposition 2.2] *Let I be a homogeneous artinian ideal of R generated by monomials. Then R/I has the weak (strong) Lefschetz property if and only if $x_0 + \dots + x_n$ is a weak (strong) Lefschetz element of R/I .*

Hence, the weak Lefschetz property can be decided for monomial ideals, in a small number of cases, by simple invariants. The following lemma is a generalisation of [33, Proposition 3.7].

Lemma 2.13. *Let I be an artinian ideal of R generated by monomials. Suppose that a is the least positive integer such that $x_i^a \in I$, for $0 \leq i \leq n$, and suppose that the Hilbert function of R/I weakly increases to degree $s+1$. Then, for any positive prime p such that $a \leq p^m \leq s+1$ for some positive integer m , R/I fails to have the weak Lefschetz property in characteristic p .*

Proof. By Proposition 2.12, we need only consider $\ell = x_0 + \cdots + x_n$. Suppose the characteristic of K is p , then by the Frobenius endomorphism $\ell \cdot \ell^{p^m-1} = \ell^{p^m} = x_0^{p^m} + \cdots + x_n^{p^m}$. Moreover, as $a \leq p^m$, then $\ell^{p^m} = 0$ in A while $\ell \neq 0$ in A . Hence $\times \ell^{p^m-1} : [A]_1 \rightarrow [A]_{p^m}$ is not injective and thus A does not have the weak Lefschetz property. \square

Further, for monomial ideals, if the weak Lefschetz property holds in characteristic zero, then it holds for almost every characteristic.

Lemma 2.14. *Let I be an artinian ideal in R generated by monomials. If R/I has the weak Lefschetz property when $\text{char } K = 0$, then R/I has the weak Lefschetz property whenever $\text{char } K$ is sufficiently large.*

Proof. By Proposition 2.12, we need only consider if $\ell = x_0 + \cdots + x_n$ is a weak Lefschetz element. As R/I is artinian, then there are finitely many maps that need to be checked for the maximal rank property, and this in turn implies finitely many determinants that need to be computed. Further, because of the form of ℓ , the matrices in question are all zero-one matrices when the rows and columns are indexed by monomials. Thus, the maximum determinants are integers. Moreover, they are non-zero because R/I has the weak Lefschetz property in characteristic zero. Simply let p be the smallest prime larger than all prime divisors of the determinants, then the determinants are all non-zero modulo p and so R/I has the weak Lefschetz property if $\text{char } K \geq p$. \square

Conversely, again for monomial ideals, if the weak Lefschetz property holds in some positive characteristic, then it holds for characteristic zero.

Lemma 2.15. *Let I be an artinian ideal in R generated by monomials. If R/I has the weak Lefschetz property when $\text{char } K = p > 0$, then R/I has the weak Lefschetz property for $\text{char } K = 0$.*

Proof. The proof is the same as that of Lemma 2.14 except we notice that if an integer d is non-zero modulo a prime p , then d is not zero. \square

We note that any artinian ideal in two variables has the weak Lefschetz property. This was proven for characteristic zero in [22, Proposition 4.4] and then for arbitrary characteristic in [41, Corollary 7], though it was not specifically stated therein, as noted in [33, Remark 2.6]. We provide a brief, direct proof of this fact to illustrate the

weak Lefschetz property. The argument does not extend to three or more variables, even for monomial ideals.

Proposition 2.16. *Let $R = K[x, y]$, where K is an infinite field with arbitrary characteristic. Every artinian algebra R/I has the weak Lefschetz property.*

Proof. Assume I is minimally generated by f_1, \dots, f_m . Let $s = \min\{\deg f_i \mid 1 \leq i \leq m\}$, and let ℓ be a general linear form. As $[R]_i = [R/I]_i$ for $i < s$ and multiplication by ℓ in R is injective, we have that $[R/I]_{i-1} \rightarrow [R/I]_i$ is injective for $i < s$. Moreover, since $R/(I, \ell) \cong K[x]/(x^s)$ and $[K[x]/(x^s)]_i = 0$ for $i \geq s$, then the map $[R/I]_{i-1} \rightarrow [R/I]_i$ has a trivial cokernel for $i \geq s$, that is, the map is surjective for $i \geq s$. Hence R/I has the weak Lefschetz property with Lefschetz element ℓ . \square

2.3 Stability of syzygy bundles

The syzygy bundle of an ideal will be an important tool that we use in both Chapter 3 and Chapter 4. The syzygy bundle encodes the possible cancellations among the generators of an ideal.

Definition 2.17. Assume I is artinian. The *syzygy module* of $I = (f_1, \dots, f_m)$ is the module $\text{syz } I$ that fits into the exact sequence

$$0 \rightarrow \text{syz } I \rightarrow \bigoplus_{i=1}^m R(-\deg f_i) \rightarrow I \rightarrow 0.$$

The sheafification $\widetilde{\text{syz } I}$ is a vector bundle on \mathbb{P}^n , called the *syzygy bundle* of I .

The next two technical conditions essentially imply that the syzygy bundle is “very nice.”

Definition 2.18. A vector bundle E on projective space is said to be *semistable* if the inequality

$$\frac{c_1(F)}{\text{rk}(F)} \leq \frac{c_1(E)}{\text{rk}(E)}$$

holds for every coherent subsheaf $F \subset E$. If the inequality is always strict, then E is said to be *stable*.

Fortunately, Brenner gave a beautiful and simple classification of monomial ideals with (semi)stable syzygy bundles. Moreover, we only consider the (semi)stability of syzygy bundles of monomial ideals, so the following may be taken as the definition of (semi)stability.

Theorem 2.19. [3, Proposition 2.2 & Corollary 6.4] *Let $I = (f_1, \dots, f_m)$ be an artinian ideal in R generated by monomials. Then I has a semistable syzygy bundle if and only if, for every subset J of $\{1, \dots, m\}$ with at least two elements, the inequality*

$$\frac{d_J - \sum_{j \in J} \deg f_j}{|J| - 1} \leq \frac{-\sum_{i=1}^m \deg f_i}{m - 1}$$

holds, where d_J is the degree of the greatest common divisor of the f_j where $j \in J$. Further, I has a stable syzygy bundle if and only if the above inequality is always strict.

Example 2.20. [3, Corollary 7.2] Let $I = (x_0^{d_0}, \dots, x_n^{d_n})$ be an ideal of the ring $R = K[x_0, \dots, x_n]$, where $d_0 \geq \dots \geq d_n \geq 1$. Then the syzygy bundle of I is semistable if and only if $d_0 + \dots + d_n \geq nd_0$. Moreover, stability holds if and only if the inequality is strict.

(Example 2.6(ii) continued) More specifically, suppose $n = 2$, and consider the ideal $I = (x_0^3, x_1^3, x_2^2)$. Then $d_0 + d_1 + d_2 = 8$ and $nd_0 = 6$, so $\text{syz } I$ is stable. On the other hand, consider the ideal $J = (x_0^6, x_1^3, x_2^2)$. Then $d_0 + d_1 + d_2 = 11$ and $nd_0 = 12$, so $\text{syz } I$ is non-semistable.

Chapter 3 Monomial complete intersections

We start with a theorem that motivates the results in this chapter. It was first proven by Stanley [49] using algebraic topology, but has since been shown in many different ways. Most notably it has been proven by Watanabe [51] using representations of \mathfrak{gl}_2 , and later by Reid, Roberts, and Roitman [46] using purely algebraic techniques.

Theorem 3.1. (*[49, Theorem 2.4], [51, Corollary 3.5], [46, Theorem 10]*) *Every artinian monomial complete intersection over a polynomial ring has the strong Lefschetz property in characteristic zero.*

This result does *not* hold in positive characteristic. Moreover, there has been a great deal of recent interest in positive characteristic (see, e.g., [6, 11, 33]). Specifically, Brenner and Kaid [5] (for three variables) and Kustin and Vraciu [32] (for at least four variables) completely characterised the characteristics in which the weak Lefschetz property is present for monomial complete intersections generated by monomials all having the same degree.

The goal of this chapter is to provide complements to Theorem 3.1 in characteristic two (see Theorem 3.35) and further in the case of generation by monomials of the same degree (see Theorem 3.36). The remainder of the chapter is organised as follows: In Section 3.1 we describe a few old and new ways to establish the Lefschetz properties, specifically in the case of monomial complete intersections. In Section 3.2 we describe the characteristics in which the Lefschetz properties may fail, and prove they are bounded linearly in the degrees of the generating monomials. The proofs involve an analysis of the prime divisors of an associated determinant. In Section 3.3 we discuss, as an example, an interesting class of monomial complete intersections: those generated by quadrics, except possibly for one term.

As demonstrated in [32], when fewer variables are used, exploring the presence of the Lefschetz properties becomes more interesting. In Sections 3.4 and 3.5 we consider monomial complete intersections in two and three variables, respectively. In Section 3.6 we handle the case of at least four variables. Throughout these three sections, we use a variety of techniques to establish the presence and failure of the Lefschetz properties. These techniques include determining syzygy gaps, using basic number theory, and finding explicit syzygies of small degree. Finally, in Section 3.7 we close with the desired classifications and a few comments.

The contents of this chapter is taken from [8].

3.1 Background

Let K be an infinite field of arbitrary characteristic. All artinian monomial complete intersections over the polynomial ring $R = K[x_0, \dots, x_n]$ are of the form $R/I_{\underline{d}}$, where

$$I_{\underline{d}} = (x_0^{d_0}, x_1^{d_1}, \dots, x_n^{d_n}),$$

$\underline{d} = (d_0, d_1, \dots, d_n) \in \mathbb{N}^{n+1}$, and, without loss of generality, $d_0 \geq d_1 \geq \dots \geq d_n \geq 2$. Throughout the remainder of the chapter we use the above definition of $I_{\underline{d}}$.

The weak Lefschetz property

Notice that the socle degree of $R/I_{\underline{d}}$ is $t := d_0 + \dots + d_n - (n + 1)$. Moreover, if the largest generating degree is sufficiently large (relative to the socle degree), then the weak Lefschetz property always holds.

Proposition 3.2. [37, Proposition 5.2] *Let $\underline{d} \in \mathbb{N}^{n+1}$, $d_0 \geq d_1 \geq \dots \geq d_n \geq 2$, and $t = d_0 + \dots + d_n - (n + 1)$. If $d_0 > \lceil \frac{t}{2} \rceil$, then $R/I_{\underline{d}}$ has the weak Lefschetz property, regardless of the characteristic of K .*

In the case of an artinian monomial complete intersection, we have a series of conditions on the algebra that are equivalent to the algebra having the weak Lefschetz property.

Lemma 3.3. *Let $\ell = x_0 + \dots + x_n$. Suppose t is odd and set $s = \lfloor \frac{t}{2} \rfloor$. Then the following are equivalent (where the ordering on the d_i is ignored):*

- (i) *The algebra $R/I_{\underline{d}}$ has the weak Lefschetz property;*
- (ii) *the multiplication map $\times \ell : [R/I_{\underline{d}}]_s \rightarrow [R/I_{\underline{d}}]_{s+1}$ is an injection;*
- (iii) *the K -dimension of $[R/(I_{\underline{d}}, \ell)]_{s+1}$ is 0;*
- (iv) *the K -dimension of $[S/J_{\underline{d}}]_{s+1}$ is 0, where $S = K[x_1, \dots, x_n]$ and*

$$J_{\underline{d}} = ((x_1 + \dots + x_n)^{d_0}, x_1^{d_1}, \dots, x_n^{d_n}).$$

Proof. By Proposition 2.12, as $I_{\underline{d}}$ is a monomial ideal, it suffices to consider $\ell = x_0 + \dots + x_n$. Further note that as the Hilbert function of $R/I_{\underline{d}}$ is symmetric and t is odd, then $\dim_K [R/I_{\underline{d}}]_s = \dim_K [R/I_{\underline{d}}]_{s+1}$.

The equivalences follow as:

- (i) & (ii): use Proposition 2.9(ii) and duality;
- (ii) & (iii): $[R/(I_{\underline{d}}, \ell)]_{s+1}$ is the cokernel of the map in (ii); and
- (iii) & (iv): $[R/(I_{\underline{d}}, \ell)]_{s+1} \cong [S/J_{\underline{d}}]_{s+1}$. □

If the socle degree is even, then the weak Lefschetz property is sometimes inherited.

Corollary 3.4. *If t is even and $R[x_{n+1}]/I_{(\underline{d}, 2)}$ has the weak Lefschetz property, then $R/I_{\underline{d}}$ has the weak Lefschetz property.*

Proof. Notice that $s = \frac{t}{2} = \lfloor \frac{t+1}{2} \rfloor$ as t is even. Set $\ell := x_0 + \dots + x_n$.

By Lemma 3.3, if $R[x_{n+1}]/I_{(\underline{d}, 2)}$ has the weak Lefschetz property, then K -dimension of $[R/(I_{\underline{d}}, \ell^2)]_{s+1}$ is zero. Moreover, this is the cokernel of the map $\times \ell^2 : [R/I_{\underline{d}}]_{s-1} \rightarrow [R/I_{\underline{d}}]_{s+1}$; hence the map is a bijection. This implies $\times \ell : [R/I_{\underline{d}}]_{s-1} \rightarrow [R/I_{\underline{d}}]_s$ is an injection. Thus, using Proposition 2.9(iii) and duality we have that $R/I_{\underline{d}}$ has the weak Lefschetz property. □

The strong Lefschetz property

If the Hilbert function is symmetric, then demonstrating the strong Lefschetz property is equivalent to showing certain maps are bijections.

Remark 3.5. An artinian algebra A with socle degree t has the *strong Stanley property* if there exists a linear form $\ell \in A$ such that the map $\times \ell^{t-2k} : [A]_k \rightarrow [A]_{t-k}$ is a bijection for all $0 \leq k \leq \lfloor \frac{t}{2} \rfloor$. Clearly then, an artinian algebra has the strong Stanley property if and only if the algebra has the strong Lefschetz property and has a symmetric Hilbert function.

It follows that $R/I_{\underline{d}}$ has the strong Stanley property if and only if it has the strong Lefschetz property. Moreover, this provides a deeper connection between the strong and weak Lefschetz properties.

Proposition 3.6. $R/I_{\underline{d}}$ has the strong Lefschetz property if and only if each of the rings $R[x_{n+1}]/I_{(\underline{d}, t-2k)}$ has the weak Lefschetz property for $0 \leq k \leq \lfloor \frac{t}{2} \rfloor$.

Proof. Set $\ell := x_0 + \cdots + x_n$. The socle degree of $A_k := R[x_{n+1}]/I_{(\underline{d}, t-2k)}$ is $2(t-k) - 1$, which is odd, and further $\lfloor \frac{2(t-k)-1}{2} \rfloor = t-k-1$. By Lemma 3.3, A_k has the weak Lefschetz property if and only if the K -dimension of $[R/(I_{\underline{d}}, \ell^{t-2k})]_{t-k}$ is zero. The latter is equivalent to the map $\varphi_k := \times \ell^{t-2k} : [R/I_{\underline{d}}]_k \rightarrow [R/I_{\underline{d}}]_{t-k}$ being a bijection.

By Remark 3.5, $R/I_{\underline{d}}$ has the strong Lefschetz property if and only if it is the strong Stanley property, that is, if and only if φ_k is a bijection (i.e., A_k has the weak Lefschetz property) for $0 \leq k \leq \lfloor \frac{t}{2} \rfloor$. \square

3.2 Bounding failure of the Lefschetz properties

Let $R/I_{\underline{d}}$ be an artinian monomial complete intersection as defined in Section 3.1, and let $t = d_0 + \cdots + d_n - (n+1)$ be the socle degree of $R/I_{\underline{d}}$. Set $s := \lfloor \frac{t}{2} \rfloor$. If t is odd and $d_0 \leq \lceil \frac{t}{2} \rceil$, then Lemma 3.3(iv) holds if and only if the determinant of $M_{\underline{d}}$, the associated matrix defined by the map, is non-zero modulo the characteristic of K . We use this to describe the characteristics in which the Lefschetz properties may fail and to prove they are bounded linearly in the degrees of the generating monomials.

A connection to weak compositions

An ordered n -tuple $\underline{m} = (m_1, \dots, m_n) \in \mathbb{N}_0^n$ with $m_1 + \cdots + m_n = k$ is called a *weak composition* of k into n parts. Define the set $C(n, \underline{m}, k)$ to be the set of weak compositions \underline{a} of k into n parts such that \underline{a} is component-wise bounded by \underline{m} . For elements $\underline{a}, \underline{b} \in C(n, \underline{m}, k)$, define $\underline{a}! := a_1! \cdots a_n!$ and $\underline{b} - \underline{a} = (b_1 - a_1, \dots, b_n - a_n)$. Notice that if \underline{a} is component-wise bounded by \underline{b} , then $\underline{b} - \underline{a} \in C(n, \underline{m}, k)$.

Given an n -tuple $\underline{a} = (a_1, a_2, \dots, a_n)$, we define $x^{\underline{a}} = x_1^{a_1} \cdots x_n^{a_n}$. The matrix $M_{\underline{d}}$ has rows indexed by the monomials $x^{\underline{a}}$ of $[S/J_{\underline{d}}]_{s+1-d_0}$ and columns indexed by the monomials $x^{\underline{b}}$ of $[S/J_{\underline{d}}]_{s+1}$. The element in the $x^{\underline{a}}$ row and the $x^{\underline{b}}$ column is zero if \underline{a}

is larger than \underline{b} in at least one component, otherwise it is the multinomial coefficient

$$\binom{d_0}{b_1 - a_1, \dots, b_n - a_n} = \frac{d_0!}{(\underline{b} - \underline{a})!}.$$

Notice that the monomials in $[S/J_{\underline{d}}]_i$ are in bijection with the weak compositions in $C(n, \check{\underline{d}} - \underline{1}, i)$, where $\check{\underline{d}} = (d_1, \dots, d_n)$ and $\underline{1} = (1, \dots, 1)$. Hence the matrix $M_{\underline{d}}$ can be seen as a matrix with rows indexed by $\underline{a} \in C(n, \check{\underline{d}} - \underline{1}, s + 1 - d_0)$ and columns indexed by $\underline{b} \in C(n, \check{\underline{d}} - \underline{1}, s + 1)$ with entries given by zero if \underline{a} is larger than \underline{b} in at least one component and $\frac{d_0!}{(\underline{b} - \underline{a})!}$ otherwise.

Seeing $M_{\underline{d}}$ in this new light, a theorem of Proctor computes the determinant of $M_{\underline{d}}$ in terms of compositions.

Theorem 3.7. [44, Corollary 1] *Let $\underline{d} \in \mathbb{N}^{n+1}$, where $d_0 \geq d_1 \geq \dots \geq d_n \geq 2$, and suppose $d_0 + \dots + d_n - (n + 1)$ is odd. Set $s := \lfloor \frac{d_0 + d_1 + \dots + d_n - (n + 1)}{2} \rfloor$. Then*

$$|\det M_{\underline{d}}| = \frac{\prod_{\underline{a}} \underline{a}!}{\prod_{\underline{b}} \underline{b}!} \prod_{i=0}^{s+1-d_0} \langle i + 1 \rangle_{d_0}^{\delta_{s+1-d_0-i}},$$

where \underline{a} and \underline{b} run over $C(n, \check{\underline{d}} - \underline{1}, s + 1 - d_0)$ and $C(n, \check{\underline{d}} - \underline{1}, s + 1)$, respectively, $\langle x \rangle_m := x(x + 1) \cdots (x + m - 1)$, and $\delta_i = \#C(n, \check{\underline{d}} - \underline{1}, i) - \#C(n, \check{\underline{d}} - \underline{1}, i - 1)$.

Remark 3.8. By the work of Gessel and Viennot [17], we have that the determinant of $M_{\underline{d}}$ is the enumeration of signed non-intersecting lattice paths from the hyperplane $x_1 + \dots + x_n = s + 1 - d_0$ to the hyperplane $x_1 + \dots + x_n = s + 1$ in the parallelepiped of size $(d_1 - 1) \times \dots \times (d_n - 1)$.

If the top generating degree, d_0 , is as large as possible such that the preceding theorem is still applicable, then the matrix has one entry.

Lemma 3.9. *Let $n \geq 2$ and $d_1 \geq \dots \geq d_n \geq 2$; set $d_0 = d_1 + \dots + d_n - n$. Then the algebra $R/I_{(d_0, d_1, \dots, d_n)}$ has the weak Lefschetz property if and only if p does not divide $\binom{d_0}{d_1 - 1, \dots, d_n - 1}$.*

Proof. The socle degree is $t = 2(d_1 + \dots + d_n - n) - 1 = 2d_0 - 1$, and so the peak is $s = d_0 - 1$. Thus, $M_{\underline{d}}$ is the 1×1 matrix with entry $\binom{d_0}{d_1 - 1, \dots, d_n - 1}$, and so $\det M_{\underline{d}} = \binom{d_0}{d_1 - 1, \dots, d_n - 1}$. \square

Bounding failure

Using the above connection, and some algebraic considerations, we can bound the prime characteristics in which the weak Lefschetz property can fail.

Proposition 3.10. *Let $n \geq 2$ and $d_0 \geq \dots \geq d_n \geq 2$; set $t = d_0 + \dots + d_n - n$. Suppose K is a field of characteristic p , where p is a positive prime, and suppose $d_0 \leq \lfloor \frac{t}{2} \rfloor$. Then:*

(i) If $d_1 \leq p \leq d_0$ or $d_0 \leq p^m \leq \lceil \frac{t}{2} \rceil$, for some positive integer m , then $R/I_{\underline{d}}$ fails to have the weak Lefschetz property. In particular, injectivity fails in degree d_0 or p^m , respectively.

(ii) If $p > \lceil \frac{t+1}{2} \rceil$, then $R/I_{\underline{d}}$ has the weak Lefschetz property.

Proof. Set $\ell := x_0 + \cdots + x_n$ and $\varphi_k := \times \ell : [R/I_{\underline{d}}]_{k-1} \rightarrow [R/I_{\underline{d}}]_k$.

Assume $d_1 \leq p \leq d_0$. Then ℓ^{d_0} is zero in $R/I_{\underline{d}}$ as the coefficients $\binom{d_0}{i_1, \dots, i_n}$ are zero modulo p except for on $x_i^{d_0}$, $0 \leq i \leq n$, but these are in $I_{\underline{d}}$. Hence φ_{d_0} is not injective and so $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.

Next, assume $d_0 \leq p^m \leq \lceil \frac{t}{2} \rceil$, for some positive integer m . Then φ_{p^m} is not injective and $R/I_{\underline{d}}$ fails to have the weak Lefschetz property by Lemma 2.13. (Recall that $x_i^{d_0} \in I_{\underline{d}}$ for $0 \leq i \leq n$ and the Hilbert function of $R/I_{\underline{d}}$ weakly increases to $t - \lfloor \frac{t}{2} \rfloor = \lceil \frac{t}{2} \rceil$.)

Finally, assume $p > \lceil \frac{t+1}{2} \rceil$. We consider the two cases given by the parity of t .

Suppose t is odd; then $\lceil \frac{t+1}{2} \rceil = \lceil \frac{t}{2} \rceil$. Moreover, analysing Theorem 3.7 we see that the terms in the formula are bounded between 1 and $\lceil \frac{t+1}{2} \rceil$. Thus, $\det M_{\underline{d}}$ is not divisible by primes $p > \lceil \frac{t+1}{2} \rceil$, and $R/I_{\underline{d}}$ has the weak Lefschetz property if $p > \lceil \frac{t+1}{2} \rceil$.

Suppose t is even; then $\lceil \frac{t+1}{2} \rceil = \frac{t}{2} + 1 = \lceil \frac{t+2}{2} \rceil$. By the previous paragraph, $R[x_{n+1}]/I_{\underline{d},2}$ has the weak Lefschetz property for $p > \lceil \frac{t+2}{2} \rceil = \lceil \frac{t+1}{2} \rceil$. Hence, by Corollary 3.4 $R/I_{\underline{d}}$ has the weak Lefschetz property if $p > \lceil \frac{t+1}{2} \rceil$. \square

Notice that the algebras in Proposition 3.6, which we desire to show have the weak Lefschetz property, all have odd socle degree. We exploit this, along with the preceding proposition, to find a similar bound in the case of the strong Lefschetz property.

Theorem 3.11. *Suppose K is a field of characteristic p , where p is a positive prime. Then:*

(i) If $\max\{d_1, 2d_0 - t\} \leq p \leq d_0$ or $d_0 \leq p^m \leq t$, for some positive integer m , then $R/I_{\underline{d}}$ fails to have the strong Lefschetz property.

(ii) If $p > t$, then $R/I_{\underline{d}}$ has the strong Lefschetz property.

Proof. Set $\ell := x_0 + \cdots + x_n$ and $A_k := R[x_{n+1}]/I_{(\underline{d}, t-2k)}$. Recall that by Proposition 3.6, $R/I_{\underline{d}}$ has the strong Lefschetz property if and only if each A_k , for $0 \leq k \leq \lfloor \frac{t}{2} \rfloor$, has the weak Lefschetz property. Set $r := \min\{\lfloor \frac{t}{2} \rfloor, t - d_0\}$ and notice that the largest generating degree of A_k is $\max\{d_0, t - 2k\}$. Thus A_k satisfies the hypotheses of Proposition 3.10 if and only if $0 \leq k \leq r$.

Suppose $d_0 \leq \lceil \frac{t}{2} \rceil$ and $\max\{d_1, 2d_0 - t\} \leq p \leq d_0$ or $d_0 \leq p^m \leq t$, for some positive integer m . Then $\max\{d_1, 2d_0 - t\} = d_1$, and by Proposition 3.10(i) $R/I_{\underline{d}}$ fails to have the weak Lefschetz property, hence fails to have the strong Lefschetz property.

Suppose $d_0 > \lceil \frac{t}{2} \rceil$ and $\max\{d_1, 2d_0 - t\} \leq p \leq d_0$. We then have that $0 < t - d_0 < \lfloor \frac{t}{2} \rfloor$ and A_{t-d_0} fails to have the weak Lefschetz property by Proposition 3.10(i).

Let $0 \leq k \leq r$. Then by Proposition 3.10(i), A_k fails to have the weak Lefschetz property if $\max\{t - 2k, d_0\} \leq p^m \leq t - k$, for some positive integer m . Hence

ranging k from 0 to r we get that $R/I_{\underline{d}}$ fails to have the weak Lefschetz property for $d_0 \leq p^m \leq t$.

On the other hand, by Proposition 3.10(ii), A_k has the weak Lefschetz property for $p > t - k$. Hence if $p > t$, then each A_k has the weak Lefschetz property and $R/I_{\underline{d}}$ has the strong Lefschetz property. \square

Case (ii) of the preceding theorem can be recovered with some work from results of Lindsey [34, Lemma 5.2 and Corollary 5.3], or Hara and Watanabe's proof of [20, Proposition 8].

3.3 Almost quadratically generated ideals

In this section we consider monomial complete intersections that are quadratically generated, except possibly for one generator. In particular, we consider $d_0 \geq d_1 = \dots = d_n = 2$.

Using Proposition 3.10 and Theorem 3.11, we can completely classify the weak Lefschetz property and largely classify the strong Lefschetz property in this case.

Proposition 3.12. *Let $\underline{d} = (d, 2, \dots, 2)$ be an $(n + 1)$ -tuple, where $d \geq 2$. Suppose p is the characteristic of K . Then:*

- (i) *If $d \leq n$, then $R/I_{\underline{d}}$ has the weak Lefschetz property if and only if $p = 0$ or $p > \lceil \frac{d+n-1}{2} \rceil$.*
- (ii) *If $d > n$, then $R/I_{\underline{d}}$ has the weak Lefschetz property, regardless of the field characteristic.*
- (iii) *If $d \leq n + 1$, then $R/I_{\underline{d}}$ has the strong Lefschetz property if and only if $p = 0$ or $p > d + n - 1$.*
- (iv) *If $d > n + 1$, then $R/I_{\underline{d}}$ has the strong Lefschetz property if $p = 0$ or $p > d + n - 1$.*

Proof. Note that the socle degree of $R/I_{\underline{d}}$ is $t = d + n - 1$.

Part (i) follows from Proposition 3.10, as $d_1 = 2$. Part (ii) follows directly from Proposition 3.2, as $d > \lceil \frac{d+n-1}{2} \rceil$ if and only if $d > n$.

Parts (iii) and (iv) follow directly from Theorem 3.11, where we need only notice $\max(2, 2d - t) \leq 2$ if and only if $2d - t \leq 2$ if and only if $d \leq n + 1$. \square

This, however, leaves open a question.

Question 3.13. Let $d > n + 1$. For which primes p does $R/I_{\underline{d}}$ have the strong Lefschetz property?

Remark 3.14. Suppose $2 \leq d \leq n$ and $n - d$ is even, i.e., d and n have the same parity. The matrix $M_{\underline{d}}$ was studied by Wilson [52] and Krämer [26] as it is the incidence matrix of $\frac{n-d}{2}$ -subsets vs. $\frac{n+d}{2}$ -subsets of the n -set $\{1, \dots, n\}$. It was further studied by Hara and Watanabe [20] as it is related to the presence of the strong Lefschetz property for the algebra $S/I_{\underline{e}}$, where $e_0 = \dots = e_{n-1} = 2$. We note that in the

latter, the authors only explicitly prove the presence of the strong Lefschetz property in characteristic zero. However, their proof easily extends to show that the strong Lefschetz property exists in positive characteristic if and only if the characteristic is greater than n .

We note that the matrix $M_{\underline{d}}$ has its determinant calculated in each of the papers listed in the preceding remark, and further it is a specialisation of Proctor's evaluation given in Theorem 3.7. We give yet another equivalent form of the determinant evaluation as it provides (more easily) a few interesting examples. As in the remark, suppose $2 \leq d \leq n$ and $n - d$ is even. Then

$$\begin{aligned} |\det M_{\underline{d}}| &= d! \binom{\frac{d}{2}}{\frac{n-d}{2}} \prod_{j=0}^{\frac{n-d}{2}-1} \binom{\frac{n+d}{2}-j}{\frac{n-d}{2}-j} \binom{n}{j} \\ &= \prod_{j=1}^{\frac{n+d}{2}} j^{\binom{\min\{\frac{n-d}{2}, \frac{n+d}{2}-j\}}{\frac{n-d}{2}-j}}, \end{aligned}$$

where the first comes from [20, Proposition 6] and the second is a simple alternate representation of the first

If $d = n$, then $|\det M_{\underline{d}}| = n!$. If $d = n - 2$, then $|\det M_{\underline{d}}| = (n - 2)!n(n - 1)$. But the most interesting case is $d = 2$ and $n = 2m$, for some positive integer m . In this case,

$$|\det M_{\underline{d}}| = \prod_{j=1}^{m+1} j^{C_{m+j, m+1-j}},$$

where $C_{m+j, m+1-j} = \frac{j}{m+1} \binom{2m+2}{m+1-j}$ comes from a generalisation of the Catalan numbers called Catalan's triangle [47, A009766]. In particular, the exponents on the integers $1, \dots, m + 1$ are exactly the $(2m + 1)$ st diagonal of Catalan's triangle.

3.4 The presence of the Lefschetz properties for two variables

Recall that any homogeneous artinian ideal in two variables has the weak Lefschetz property by Proposition 2.16. On the other hand, the strong Lefschetz property is much more subtle. By Proposition 3.6, $R/I_{(a,b)}$ has the strong Lefschetz property if and only if $B_k = R[x_2]/I_{(a,b, a+b-2-2k)}$ has the weak Lefschetz property for $0 \leq k \leq \lfloor \frac{a+b-2}{2} \rfloor$. In this case, if $k > b - 2$, then B_k always has the weak Lefschetz property, by Proposition 3.2; hence we need only to consider $0 \leq k \leq b - 2$.

A few particular cases stand out. Using Lemma 3.9, we have that the algebra B_0 has the weak Lefschetz property in characteristic p if and only if p does not divide $\binom{a+b-2}{b-1}$. Similarly, B_{b-2} has the weak Lefschetz property in characteristic p if and only if p does not divide $\binom{a}{b-1}$.

For $2 \leq b \leq 3$, we characterise the strong Lefschetz property with the above. We single out these cases because they play a special role in the classification of the strong Lefschetz property in characteristic two for arbitrary $R/I_{\underline{d}}$ given in Section 3.7.

Lemma 3.15. *Let $R = K[x, y]$ and p be the characteristic of K . Then:*

- (i) $R/I_{(a,2)}$, for $a \geq 2$, has the strong Lefschetz property if and only if p does not divide a .
- (ii) $R/I_{(a,3)}$, for $a \geq 3$, has the strong Lefschetz property if and only if $p = 2$ and $a \equiv 2 \pmod{4}$ or $p \neq 2$ and a is not equivalent to $-1, 0$, or 1 modulo p

Proof. By the comments above, $R/I_{(a,2)}$ has the strong Lefschetz property if and only if p does not divide $\binom{a}{1} = a$.

Similarly, $R/I_{(a,3)}$ has the strong Lefschetz property if and only if p does not divide either $\binom{a+1}{2}$ or $\binom{a}{2}$. That is, p does not divide $\binom{a+1}{2}\binom{a}{2} = \frac{1}{4}(a-1)a^2(a+1)$. Analysing this, we see that this is equivalent to the claim. \square

Syzygy gaps

Let $a \geq b$, and let $B_k = R[x_2]/I_{(a,b,a+b-2-2k)}$ for $0 \leq k \leq b-2$. Notice that $a+b > a+b-2-2k$ as $k \geq 0$, and $b+(a+b-2-2k) > a$ as $b-2 \geq k$. Thus, by [3, Corollary 3.2], B_k has a stable syzygy bundle, and so by [4, Theorem 2.2], B_k has the weak Lefschetz property if and only if the syzygy bundle of B_k splits on the line $x+y+z$ with twists, say, $s_0 \leq s_1$, such that $s_1 - s_0 \leq 1$, i.e., the *syzygy gap* (introduced by Monsky [43]) of $(x^a, y^b, (x+y)^{a+b-2-2k})$ in R is at most one. Moreover, it is easy to see that $s_0 + s_1 = -2(a+b-1-k)$, and hence the parity of the syzygy gap $s_1 - s_0$ is even.

Han [19] provides a way to compute the syzygy gap via a continuation of the syzygy gap function. Define $\delta : \mathbb{N}^3 \rightarrow \mathbb{N}_0$ to be the syzygy gap of $(x^a, y^b, (x+y)^c)$ in $K[x, y]$ (notice δ depends on the characteristic of K). Let $\delta^* : [0, \infty)^3 \rightarrow [0, \infty)$ be the continuous continuation of δ . Define $\mathbb{Z}_{\text{odd}}^3$ to be the integer triples (u, v, w) such that $u+v+w$ is odd. Further, define μ , the *Manhattan distance* on \mathbb{R}^3 , to be $\mu((a, b, c), (u, v, w)) = |u-a| + |v-b| + |w-c|$.

Theorem 3.16. [19, Theorems 2.25 and 2.29] *Let K be an algebraically closed field of characteristic $p > 0$, and assume the entries $(a, b, c) \in [0, \infty)^3$ satisfy $a \leq b \leq c < a+b$. If there exists a negative integer s and a triple $(u, v, w) \in \mathbb{Z}_{\text{odd}}^3$ such that $\mu(p^s(a, b, c), (u, v, w)) < 1$, then $\delta^*(a, b, c) > 0$. Otherwise, if no such s and (u, v, w) exist, then $\delta^*(a, b, c) = 0$.*

This approach was used by Brenner and Kaid in [5] to classify the characteristics in which $K[x, y, z]/(x^d, y^d, z^d)$ has the weak Lefschetz property.

Characteristic two

Let n be a positive integer, and $n = b_s 2^s + \dots + b_0 2^0$ be its binary representation, i.e., $b_i \in \{0, 1\}$. Define the *bit-positions* of n to be the set $\mathcal{B}(n)$ of indices i such that $b_i = 1$ in the binary representation of n . For example, $\mathcal{B}(42) = \{1, 3, 5\}$ and $\mathcal{B}(2^m) = \{m\}$, for $m \geq 0$.

The following theorem is due to Kummer [29]. (We thank Fabrizio Zanello for pointing us to this reference.)

Theorem 3.17. *If $n \geq k \geq 0$ and p is a prime, then the largest power of p dividing $\binom{n}{k}$ is the number of carries that occur in the addition of k and $n - k$ in base- p arithmetic.*

An immediate (and simple) corollary of this theorem is a classification for when binomial coefficients are odd.

Corollary 3.18. *If $n \geq k \geq 0$, then $\binom{n}{k}$ is odd if and only if $\mathcal{B}(k)$ and $\mathcal{B}(n - k)$ are disjoint.*

Using the above classification, we get a useful intermediate result.

Lemma 3.19. *If $a \geq b \geq 2$, then $\binom{a+b-2}{b-1}$ is odd and $\binom{a}{b-1}$ is odd if and only if $a = 2^m \ell$ and $b = 2^m + 1$, where $m \geq 0$ and $\ell \geq 3$ odd.*

Proof. Suppose $\binom{a+b-2}{b-1}$ and $\binom{a}{b-1}$ are odd. Then by Corollary 3.18, $\mathcal{B}(a - 1)$ and $\mathcal{B}(b - 1)$ are disjoint as are $\mathcal{B}(b - 1)$ and $\mathcal{B}(a - b + 1)$. Thus, $\mathcal{B}(a) = \mathcal{B}(b - 1) \cup \mathcal{B}(a - b + 1)$ has at least two elements, as $\mathcal{B}(b - 1)$ and $\mathcal{B}(a - b + 1)$ each have at least one element.

Suppose $\mathcal{B}(a - 1)$ contains $0, \dots, m - 1$ but not m . Then $\mathcal{B}(a)$ contains m but not $0, \dots, m - 1$. Further, for $i > m$, $i \in \mathcal{B}(a - 1)$ if and only if $i \in \mathcal{B}(a)$. As $\mathcal{B}(b - 1) \subset \mathcal{B}(a)$, and $\mathcal{B}(b - 1)$ has at least one element, then $\mathcal{B}(b - 1)$ and $\mathcal{B}(a - 1)$ being disjoint implies $\mathcal{B}(b - 1) = \{m\}$. That is, $b - 1 = 2^m$ and so $b = 2^m + 1$. Moreover, as $\mathcal{B}(a)$ contains m but not $0, \dots, m - 1$, $a = 2^m \ell$, where ℓ is odd. As $a \geq b$, then $a \geq 3$.

On the other hand, suppose $a = 2^m \ell$ and $b = 2^m + 1$. Then $\mathcal{B}(a - 1) = \{0, \dots, m - 1\} \cup \{m + 1 + i \mid i \in \mathcal{B}(\ell)\}$, $\mathcal{B}(b - 1) = \{m\}$, and $\mathcal{B}(a - b + 1) = \mathcal{B}(2^{m+1} \frac{\ell - 1}{2})$ only contains values at least $m + 1$. Thus $\mathcal{B}(a - 1)$ and $\mathcal{B}(b - 1)$ are disjoint as are $\mathcal{B}(b - 1)$ and $\mathcal{B}(a - b + 1)$, and so by Corollary 3.18 we have that $\binom{a+b-2}{b-1}$ and $\binom{a}{b-1}$ are odd. \square

Using the syzygy gap method described in Subsection 3.4, we complete the classification in the exceptional cases.

Lemma 3.20. *If $a = 2^m \ell$ and $b = 2^m + 1$, where $m \geq 0$ and $\ell \geq 3$ odd, then $R/I_{(a,b,a+b-2(k+1))}$ fails to have the weak Lefschetz property in characteristic two if and only if $1 \leq k \leq b - 3$.*

Proof. If $k = 0$ or $k = b - 2$, then $R/I_{(a,b,a+b-2)}$ or $R/I_{(a,b,a-b+2)}$ has the weak Lefschetz property by Lemmas 3.9 and 3.19.

Suppose $1 \leq k \leq b - 3 = 2^m - 2$. We have the following:

$$\begin{aligned} \mu \left(\frac{1}{2^m}(a, b, a + b - 2 - 2k), (\ell, 1, \ell) \right) &= \left| \frac{2^m \ell}{2^m} - \ell \right| + \left| \frac{2^m + 1}{2^m} - 1 \right| \\ &\quad + \left| \frac{2^m(\ell + 1) - 1 - 2k}{2^m} - \ell \right| \\ &= \frac{1}{2^m} + \left| \frac{2k + 1}{2^m} - 1 \right| \\ &= \begin{cases} 1 - \frac{k}{2^{m-1}} & \text{if } k \leq 2^{m-1}, \\ \frac{k+1}{2^{m-1}} - 1 & \text{if } k > 2^{m-1}. \end{cases} \end{aligned}$$

Notice that $1 - \frac{k}{2^{m-1}} < 1$ if and only if $\frac{k}{2^{m-1}} > 0$ if and only if $k > 0$. Further, $\frac{k+1}{2^{m-1}} - 1 < 1$ if and only if $k < 2^m - 1$ if and only if $k \leq 2^m - 2 = b - 3$.

Thus, $\mu\left(\frac{1}{2^m}(a, b, a + b - 2 - 2k), (\ell, 1, \ell)\right) < 1$ for $1 \leq k \leq b - 3$. Notice $2\ell + 1$ is odd. Hence, by Theorem 3.16, $R/I_{(a,b,a+b-2(1+k))}$ fails to have the weak Lefschetz property in characteristic two. \square

Combining the above two lemmas, we classify the strong Lefschetz property in characteristic two for the two-variable case.

Corollary 3.21. *Let $a \geq b \geq 2$. Then $R/I_{(a,b)}$ fails to have the strong Lefschetz property in characteristic two if and only if one of the following hold:*

- (i) $b = 2$ and a is even,
- (ii) $b = 3$ and $a \not\equiv 2 \pmod{4}$, or
- (iii) $b \geq 4$.

Proof. Parts (i) and (ii) follow from Lemma 3.15 (alternatively, see Lemma 3.19 and Lemma 3.20, after considering each case).

Recall that by Proposition 3.6, $R/I_{(a,b)}$ has the strong Lefschetz property if and only if each $B_k := S/I_{(a,b,a+b-2-2k)}$ has the weak Lefschetz property, for $0 \leq k \leq b-2$.

Suppose that $b \geq 4$. If $a \neq 2^m \ell$ for some $m \geq 0$ or $b \neq 2^m + 1$ for some $l \geq 3$ odd, then by Lemma 3.19, $\binom{a+b-2}{b-1}$ is even or $\binom{a}{b-1}$ is even. That is, B_0 or B_{b-2} , respectively, fails to have the weak Lefschetz property in characteristic two.

On the other hand, if $a = 2^m \ell$ and $b = 2^m + 1$, where $m \geq 0$ and $l \geq 3$ odd, then for $0 < k < b - 2$, B_k fails to have the weak Lefschetz property in characteristic two, by Proposition 3.20. Note that $b \geq 4$ implies $b - 2 \geq 2$. \square

Generation in a single degree

Using the syzygy gap method in Subsection 3.4, we get the following classification of the strong Lefschetz property for $R/I_{(d,d)}$.

Theorem 3.22. *Let $R = K[x, y]$, where p is the characteristic of K , and $I_d = (x^d, y^d)$, where $d \geq 2$. Then R/I_d has the strong Lefschetz property if and only if $p = 0$ or $2d - 2 < p^s$, where s is the largest integer such that p^{s-1} divides $(2d - 1)(2d + 1)$.*

Proof. By Theorem 3.11, if $d \leq p \leq 2d - 2$, then the strong Lefschetz property fails and if $p > 2d - 2$ then the strong Lefschetz property holds. By Corollary 3.21, if $p = 2$, then the strong Lefschetz property fails. Hence, we need only to consider $2 < p < d$.

Next, notice that if such a triple $(u, v, w) \in \mathbb{Z}_{\text{odd}}^3$ exists, then $u = v$ and so w is odd. Otherwise, if $u \neq v$, then $|m - u| + |m - v| \geq |u - v| \geq 1$ for all $m \in \mathbb{R}$ by the triangle inequality; in particular, this holds for $m = \frac{d}{p^s}$.

Set s to be the largest integer such that p^{s-1} divides $(2d - 1)(2d + 1)$. Further, set $e = 2d + 1$, if p divides $2d + 1$, otherwise set $e = 2d - 1$.

As the sum $d + d + 2(d - 1 - k)$ is even, if $r = 0$, then $p^r(d, d, 2(d - 1 - k))$ is at least one from every point in $\mathbb{Z}_{\text{odd}}^3$, under the Manhattan distance. Suppose $0 < r < s$, then p divides e ; set $e = p^r n$ for some odd integer n (recall e is odd). If $e = 2d - 1$, then $d = \frac{p^r n + 1}{2}$. The minimal value of $\left| \frac{d}{p^r} - u \right|$ is $\frac{p^r - 1}{2p^r}$ at $u = \frac{n+1}{2}$. The minimal value of $\left| \frac{2(d-1-k)}{p^r} - w \right|$ is $\frac{1+2k}{p^r}$ at $w = n$. However, $2\frac{p^r-1}{2p^r} + \frac{1+2k}{p^r} = \frac{p^r+2k}{p^r}$ is at least one for all k . Similarly, if $e = 2d + 1$, then the Manhattan distance to any point in $\mathbb{Z}_{\text{odd}}^3$ is at least one.

Suppose $2d - 2 < p^s$. Let $r \geq s$; then $2d - 2 < p^r$, and so $\frac{d}{p^r} \leq \frac{1}{2}$. Hence, we may set $u = v = 0$, and thus $w \geq 1$ as w must be odd. As $2(d - 1 - k) \leq 2d - 2 < p^r$, we must choose $w = 1$. However, for all $k \geq 0$,

$$\mu \left(\frac{1}{p^r}(d, d, 2(d - 1 - k)), (0, 0, 1) \right) = \frac{p^r + 2 + 2k}{p^r} > 1.$$

Suppose $2d - 2 \geq p^s$; then $e > p^s$, and we can write $e = p^s n + j$, where p does not divide n and $0 < j < p^s$ ($j > 0$ as p^s does not divide e). Notice that $n > 0$ as $e > p^s$. We consider two cases, given by the parity of n .

Suppose n is even, then j is odd as e is odd. As p^s is odd and j is odd, then $j \neq p^s - 1$ and $j \neq p^s - 3$. Assume $e = 2d - 1$, that is, p does not divide $2d + 1$. Notice, $j \neq p^s - 2$, otherwise, $2d - 1 = p^s(n + 1) - 2$, and so $2d + 1 = p^s(n + 1)$, which contradicts our choice of e . Thus, $j \leq p^s - 4$. Set $u = v = \frac{n}{2}$, $w = n - 1$, and $k = j + 1$. As $n \geq 2$, $2p^s < e$ and so $p^s \leq d$. This in turn implies $k = j + 1 < p^s - 2 \leq d - 2$; thus, k is applicable. As $e = 2d - 1$, then $d = \frac{p^s n + j + 1}{2}$. Further,

$$\begin{aligned} \mu \left(\frac{1}{p^s}(d, d, 2(d - 1 - k)), (u, v, w) \right) &= 2 \left| \frac{n}{2} + \frac{j + 1}{2p^s} - \frac{n}{2} \right| + \left| n + \frac{-j - 3}{p^s} - (n - 1) \right| \\ &= \frac{j + 1}{p^s} + \frac{p^s - j - 3}{p^s} \\ &= \frac{p^s - 2}{p^s} \\ &< 1. \end{aligned}$$

Assume $e = 2d + 1$, that is, p does divide $2d + 1$. In this case, set $u = v = \frac{n}{2}$, $w = n - 1$, and $k = j$. Notice, $k \leq p^s - 2$. As $n \geq 2$, $2p^s < e$ and so $p^s \leq d$. This in turn implies $k = j \leq p^s - 2 \leq d - 2$; thus, k is applicable. As $e = 2d + 1$, then $d = \frac{p^s n + j - 1}{2}$. Further,

$$\begin{aligned} \mu \left(\frac{1}{p^s}(d, d, 2(d - 1 - k)), (u, v, w) \right) &= 2 \left| \frac{n}{2} + \frac{j - 1}{2p^s} - \frac{n}{2} \right| + \left| n + \frac{-j - 3}{p^s} - (n - 1) \right| \\ &= \frac{j - 1}{p^s} + \left| \frac{p^s - j - 3}{p^s} \right| \tag{*} \\ &= \frac{p^s - 2}{p^s} \\ &< 1. \end{aligned}$$

Note for (\star) : If $j \leq p^s - 3$, then the absolute value disappears; on the other hand, if $j = p^s - 2$, then the latter term is $\frac{1}{p^s}$.

Suppose n is odd, then j is even as e is odd. Set $u = v = \frac{n+1}{2}$, $w = n$, and $k = \frac{j}{2} - 1$. Notice that $n \geq 1$ and $j \geq 2$. As $j < p^s < 2d - 1 \leq e$, then $j \leq 2d - 3$. So $k = \frac{j}{2} - 1 \leq d - 2$; thus, k is applicable. Suppose $e = 2d - 1$, then $d = \frac{p^s n + j + 1}{2}$.

$$\begin{aligned} \mu \left(\frac{1}{p^s} (d, d, 2(d-1-k)), (u, v, w) \right) &= 2 \left| \frac{n}{2} + \frac{j+1}{2p^s} - \frac{n+1}{2} \right| + \left| n + \frac{1}{p^s} - n \right| \\ &= \frac{p^s - j - 1}{p^s} + \frac{1}{p^s} \\ &= \frac{p^s - j}{p^s} \\ &< 1. \end{aligned}$$

When $e = 2d + 1$, the result follows similarly with the finally fraction being $\frac{p^s - j + 2}{p^s}$. We notice that if $j = 2$, then $e = 2d + 1 = p^s n - 2$ and so $2d - 1 = p^s n$. That is, p divides $2d - 1$, contradicting our choice of e . Thus, $j \geq 4$. \square

3.5 The presence of the Lefschetz properties for three variables

In this section, we focus entirely on the strong Lefschetz property for $R/I_{(d,d,d)}$, where $d \geq 2$. We use the method of Kustin and Vraciu [32] that is based on finding syzygies of low enough degree which we recall next.

Minimal degree syzygies

Let $S = K[x_1, \dots, x_n]$ and $\underline{d} = (d_0, d_1, \dots, d_n) \in \mathbb{N}^{n+1}$. Define $\phi_{\underline{d}} : \bigoplus_{i=0}^n S(-d_i) \rightarrow S$ by the matrix $[(x_1 + \dots + x_n)^{d_0}, x_1^{d_1}, \dots, x_n^{d_n}]$, and let $\text{syz}(\underline{d}) := \ker \phi_{\underline{d}}$. Next, define $\text{Kos}(\underline{d})$ to be the S -submodule of $\text{syz}(\underline{d})$ generated by the Koszul relations on the entries of the matrix defining $\phi_{\underline{d}}$, and define $\overline{\text{syz}}(\underline{d})$ to be the quotient $\text{syz}(\underline{d}) / \text{Kos}(\underline{d})$. Last, for a non-zero graded module M , the *minimal generator degree* of M is the smallest d such that M_d is non-zero; we denote this by $\text{mgd } M$.

Proposition 3.23. [32, Corollary 2.2(4 & 6)] *Let $\underline{d} = (d_0, d_1, \dots, d_n) \in \mathbb{N}^{n+1}$, and set $t = d_0 + \dots + d_n - (n+1)$. Then $R/I_{\underline{d}}$ has the weak Lefschetz property if and only if $\lfloor \frac{t+3}{2} \rfloor \leq \text{mgd } \overline{\text{syz}}(\underline{d})$.*

Thus, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property if we can demonstrate that there exists a non-Koszul syzygy of small enough degree.

Finding syzygies

First, we describe an explicit non-Koszul syzygy of $S/I_{(k,k+j,k+j,k)}$ that will be used repeatedly in the proceeding proof. This is a generalisation of the syzygy described in the proof of [32, Lemma 4.2].

Lemma 3.24. *Let $j \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then $(-f_{k+j}, g_k, (-1)^{k+j+1}g_k, f_{k+j})$ is a non-Koszul syzygy in $\overline{\text{syz}}(k, k+j, k+j, k)$, where*

$$f_k := \frac{y^k - (-z)^k}{y+z} = \sum_{i=0}^{k-1} y^i (-z)^{k-i-1}$$

and

$$g_k := \frac{x^k - (x+y+z)^k}{y+z} = - \sum_{i=0}^{k-1} \binom{k}{i} x^i (y+z)^{k-i-1}.$$

Proof. Notice that $(-f_{k+j}, g_k, (-1)^{k+j+1}g_k, f_{k+j}) \in \text{syz}(k, k+j, k+j, k)$ as

$$\begin{aligned} & -f_{k+j}x^k + g_k y^{k+j} + (-1)^{k+j+1}g_k z^{k+j} + f_{k+j}(x+y+z)^k \\ &= f_{k+j}((x+y+z)^k - x^k) + g_k(y^{k+j} - (-z)^{k+j}) \\ &= \frac{y^{k+j} - (-z)^{k+j}}{y+z}((x+y+z)^k - x^k) + \frac{x^k - (x+y+z)^k}{y+z}(y^{k+j} - (-z)^{k+j}) \\ &= 0. \end{aligned}$$

Furthermore, it is clear that $f_{k+j} \notin (x^k, y^{k+j}, z^{k+j})$ since f_{k+j} is a polynomial in y and z of degree $k+j-1$. Thus, the described syzygy is non-Koszul. \square

In order to demonstrate that the algebra $R/I_{(d,d,d)}$ does not have the strong Lefschetz property, we classify the weak Lefschetz property for $S/I_{(d,d,d,d-3)}$.

Proposition 3.25. *Let $d \geq 6$, and set $\underline{d} = (d, d, d, d-3)$. Then $R/I_{\underline{d}}$ has the weak Lefschetz property in characteristic p if and only if $p = 0$ or $p > 2d-3$.*

Proof. Set β to be the quadruple $(x^d, y^d, z^d, (x+y+z)^{d-3})$. The proof follows from several cases.

(i) *Characteristic two:* Let $p = 2$. If $d \neq 2^m + 1$ for some $m \in \mathbb{N}$, then $d = 2^m - k$ for some $0 \leq k \leq 2^{m-1} - 2$, and $2d-3 = 2^{m+1} - 2k - 3 \geq 2^m + 1$. Thus, $d \leq 2^m \leq 2d-3$ and so $R/I_{\underline{d}}$ fails to have the weak Lefschetz property by Proposition 3.10.

Suppose $d = 2^m + 1$ for some $m \in \mathbb{N}$. Then $\alpha = (yz, xz, xy, xyz(x+y+z)^2)$ is a syzygy in $\text{syz}(\underline{d})$ as

$$\begin{aligned} \alpha \cdot \beta &= x^{2^m+1}yz + xy^{2^m+1}z + xyz^{2^m+1} + xyz(x+y+z)^{2^m} \\ &= xyz(x^{2^m} + y^{2^m} + z^{2^m} + (x+y+z)^{2^m}) \\ &= xyz(x+y+z + (x+y+z))^{2^m} \\ &= 0. \end{aligned}$$

Further, $xyz(x+y+z)^2 \notin (x^d, y^d, z^d)$ and $\deg \alpha = d+2 \leq 2d-3$, as $d \geq 6$. Hence by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.

Note that many of the following cases are proven almost identically to the case in the preceding paragraph. In each of the forward cases, we provide only the syzygy, as the rest is straightforward to check.

(ii) *Characteristic three:* Let $p = 3$, and write $2d = 3q + r$ with unique $q, r \in \mathbb{N}$ such that $0 \leq r \leq 2$. Suppose $q = 3^m$ and $r = 1$, then $d = 3^m + \frac{3^m+1}{2} = 3j - 1$, where $j = \frac{3^m+1}{2}$. Let α be

$$(x^{j-1}y^j(x+y+z)^{j-3}, (x-z)^{3^m}(x+y+z)^{j-3}, -y^jz^{j-1}(x+y+z)^{j-3}, -(x-z)^{3^m}y^j).$$

Then α is in $\overline{\text{syz}}(\underline{d})$, and $\deg \alpha = 2d - 3$. Thus, by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.

Suppose $3^m < q \leq 2 \cdot 3^m - 1$ for some m , then $d \leq \frac{6 \cdot 3^m - 3 + r}{2} \leq 3^{m+1}$ and $2d - 3 = 3q + (r - 3) \geq 3(3^m + 1) - 3 = 3^{m+1}$. Thus, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property by Proposition 3.10.

Suppose $2 \cdot 3^m \leq q < 3^{m+1}$ for some m . Set $k = d - 3^{m+1}$, so $0 \leq k \leq \frac{3^{m+1}-1}{2}$. Let α be

$$(y^kz^k(x+y+z)^j, x^kz^k(x+y+z)^j, x^ky^k(x+y+z)^j, -x^ky^kz^k(x+y+z)^{\max\{0, 3-k\}}),$$

where $j = \max\{0, k - 3\}$. Then α is in $\overline{\text{syz}}(\underline{d})$, and $\deg \alpha \leq 2d - 3$. Thus, by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.

(iii) *Characteristic at least five:* Let $p \geq 5$ be prime, and let f_k and g_k be defined as in Lemma 3.24. Write $2d = qp + r$ with unique $q, r \in \mathbb{N}$ such that $0 \leq r < p$. Notice that q and r must have the same parity as p is odd. We distinguish two sub-cases based on the parity of q and r .

(a) *The quotient is even:* Suppose q and r are even. Set $j = \max\{0, \frac{r}{2} - 3\}$, and α to be

$$\left(y^{\frac{r}{2}}z^{\frac{r}{2}}(x+y+z)^j(-f_{\frac{q}{2}}^p), x^{\frac{r}{2}}z^{\frac{r}{2}}(x+y+z)^jg_{\frac{q}{2}}^p, \right. \\ \left. x^{\frac{r}{2}}y^{\frac{r}{2}}(x+y+z)^j((-1)^{\frac{q}{2}+1}g_{\frac{q}{2}}^p), x^{\frac{r}{2}}y^{\frac{r}{2}}z^{\frac{r}{2}}(x+y+z)^{\max\{0, 3-\frac{r}{2}\}}f_{\frac{q}{2}}^p \right).$$

Then α is in $\overline{\text{syz}}(\underline{d})$, and $\deg \alpha \leq 2d - 3$. Thus, by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.

(b) *The quotient is odd:* Suppose q and r are odd. First, suppose $r = 1$. Then set $j = d - \frac{q-1}{2}p$, and α to be

$$\left((x+y+z)^{j-3}(-f_{\frac{q+1}{2}}^p), x^jy^{j-1}(x+y+z)^{j-3}g_{\frac{q-1}{2}}^p, \right. \\ \left. x^jz^{j-1}(x+y+z)^{j-3}((-1)^{\frac{q-1}{2}+2}g_{\frac{q-1}{2}}^p), x^jf_{\frac{q+1}{2}}^p \right).$$

Notice that $d + j - 1 = \frac{q+1}{2}p$. Then α is in $\overline{\text{syz}}(\underline{d})$, and $\deg \alpha = 2d - 3$. Thus, by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.

Last, suppose $r \geq 3$. Then set $j = d - r - \frac{q-1}{2}p$, and α to be

$$\left(x^j(-f_{\frac{q+1}{2}}^p), y^jg_{\frac{q+1}{2}}^p, z^j((-1)^{\frac{q+1}{2}+1}g_{\frac{q+1}{2}}^p), (x+y+z)^{j+3}f_{\frac{q+1}{2}}^p \right).$$

Then α is in $\overline{\text{syz}}(\underline{d})$, and $\deg \alpha \leq 2d - 3$. Thus, by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property. \square

Remark 3.26. Each of the syzygies described in the preceding proof are modifications of extant syzygies by means of the Frobenius homomorphism and multiplying by an appropriate ring element. This is similar to the approach used by Kustin and Vraciu [32].

Further, to discuss the cases left out in Proposition 3.25, we notice that the determinants associated to $(4, 4, 4, 1)$, and $(5, 5, 5, 2)$ are $20 = 2^2 \cdot 5$ and $-43750 = -2 \cdot 5^5 \cdot 7$, respectively. Thus, $R/I_{(4,4,4,1)}$ fails to have the weak Lefschetz property in exactly characteristics 2 and 5. Similarly, $R/I_{(5,5,5,2)}$ fails to have the weak Lefschetz property in exactly characteristics 2, 5, and 7.

Theorem 3.27. *Let $d \geq 2$, and set $\underline{d} = (d, d, d)$. Then $R/I_{\underline{d}}$ has the strong Lefschetz property in characteristic p if and only if $p = 0$ or $p > 3(d - 1)$.*

Proof. By Theorem 3.11, if $d \leq p \leq 3(d - 1)$, then $R/I_{\underline{d}}$ fails to have the strong Lefschetz property, and if $p > 3(d - 1)$, then $R/I_{\underline{d}}$ has the strong Lefschetz property.

If $d \geq 6$, then by Proposition 3.25, for $2 \leq p < d$, $S/I_{(d,d,d,d-3)}$ fails to have the weak Lefschetz property. Thus by Proposition 3.6, $R/I_{\underline{d}}$ fails to have the strong Lefschetz property for $2 \leq p < d$ as $d - 3 = t - 2k$, where $t = 3d - 3$ and $k = d$.

For the remaining four cases, we consider $k = 0$ and use Lemma 3.9. In particular, notice that $\binom{3}{1,1,1} = 2 \cdot 3$, $\binom{6}{2,2,2} = 2 \cdot 3^2 \cdot 5$, $\binom{9}{3,3,3} = 2^4 \cdot 3 \cdot 5 \cdot 7$, and $\binom{12}{4,4,4} = 2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$. Hence, for $2 \leq d \leq 5$, $S/I_{(d,d,d,t)}$ fails to have the weak Lefschetz property for $2 \leq p < d$, and so $R/I_{(d,d,d)}$ fails to have the strong Lefschetz property. \square

3.6 The presence of the Lefschetz properties in many variables

We first consider the strong Lefschetz property in characteristic two when $n \geq 2$, that is, when R has at least three variables. Then we consider the strong Lefschetz property for $I_{\underline{d}}$ having generators of the same degree $d_0 = \dots = d_n$ in at least four variables.

Characteristic two

We expand Corollary 3.18 to classify when multinomial coefficients are odd.

Lemma 3.28. *Let $a_0 \geq \dots \geq a_n \geq 1$. Then the following are equivalent:*

- (i) $\binom{a_0 + \dots + a_n}{a_0, \dots, a_n}$ is odd,
- (ii) $\mathcal{B}(a_i)$ and $\mathcal{B}(a_j)$ are disjoint for all $0 \leq i < j \leq n$, and
- (iii) $\mathcal{B}(a_{i_1} + \dots + a_{i_m})$ and $\mathcal{B}(a_j)$ are disjoint for any $1 \leq m < n$ and $j \notin \{i_1, \dots, i_m\} \subsetneq [n]$.

Proof. Set $M = \binom{a_0 + \dots + a_n}{a_0, \dots, a_n}$.

(i) \Rightarrow (ii): Notice $\binom{a_i + a_j}{a_i}$ divides M for all $0 \leq i < j \leq n$. Thus, if M is odd, then so is $\binom{a_i + a_j}{a_i}$. Hence, by Corollary 3.18, $\mathcal{B}(a_i)$ and $\mathcal{B}(a_j)$ are disjoint.

(ii) \Rightarrow (iii): Let $1 \leq m < n$ and $j \notin \{i_1, \dots, i_m\} \subsetneq [n]$. As $\mathcal{B}(a_0), \dots, \mathcal{B}(a_n)$ are disjoint, $\mathcal{B}(a_{i_1} + \dots + a_{i_m}) = \cup \mathcal{B}_{i_k}$ is disjoint from $\mathcal{B}(a_j)$.

(iii) \Rightarrow (i): Recall that

$$M = \binom{a_0 + \dots + a_n}{a_0, \dots, a_n} = \prod_{i=2}^n \binom{a_1 + \dots + a_i}{a_i}.$$

As $\mathcal{B}(a_1 + \dots + a_{i-1})$ and $\mathcal{B}(a_i)$ are disjoint, by Corollary 3.18, $\binom{a_1 + \dots + a_i}{a_i}$ is odd. Hence M is a product of odd integers, that is, M is odd. \square

By the preceding lemma, for certain pairs of multinomial coefficients, one must be even.

Lemma 3.29. *Let $n \geq 2$, $a_0 \geq \dots \geq a_n \geq 1$, and suppose $a_0 \geq a_1 + \dots + a_n$. Then $\binom{a_0 + \dots + a_n}{a_0, \dots, a_n}$ is even or $\binom{a_0 + 1}{a_0 + 1 - (a_1 + \dots + a_n), a_1, \dots, a_n}$ is even.*

Proof. Suppose $\binom{a_0 + \dots + a_n}{a_0, \dots, a_n}$ and $\binom{a_0 + 1}{a_0 + 1 - (a_1 + \dots + a_n), a_1, \dots, a_n}$ are odd. By Lemma 3.28, the $\mathcal{B}(a_i)$ are disjoint for all $0 \leq i \leq n$, $\mathcal{B}(a_0)$ and $\mathcal{B}(a_1 + \dots + a_n)$ are disjoint, and $\mathcal{B}(a_0 + 1 - (a_1 + \dots + a_n))$ and $\mathcal{B}(a_1 + \dots + a_n)$ are disjoint. Thus we have that $\mathcal{B}(a_0 + 1) = \mathcal{B}(a_0 + 1 - (a_1 + \dots + a_n)) \cup \mathcal{B}(a_1 + \dots + a_n)$. Notice that since $a_n \geq 1$, each $\mathcal{B}(a_i)$ has at least one element, and so $\mathcal{B}(a_1 + \dots + a_n)$ has at least n elements.

Suppose $\mathcal{B}(a_0)$ contains $0, \dots, m-1$ but not m . Then for $k > m$, $k \in \mathcal{B}(a_0 + 1)$ if and only if $k \in \mathcal{B}(a_0)$. Moreover, $\mathcal{B}(a_0 + 1)$ contains m but not $0, \dots, m-1$. As $\mathcal{B}(a_1 + \dots + a_n) \subset \mathcal{B}(a_0 + 1)$, and the former has at least $n \geq 2$ elements, there exists a $k \in \mathcal{B}(a_1 + \dots + a_n) \subset \mathcal{B}(a_0 + 1)$ with $k > m$. Thus $\mathcal{B}(a_1 + \dots + a_n)$ and $\mathcal{B}(a_0)$ have k in common, contradicting $\mathcal{B}(a_0)$ and $\mathcal{B}(a_1 + \dots + a_n)$ being disjoint. This in turn contradicts $\binom{a_0 + \dots + a_n}{a_0, \dots, a_n}$ being odd. \square

As a corollary, we classify the strong Lefschetz property in characteristic two for all monomial complete intersections in at least three variables.

Corollary 3.30. *Let $d_0 \geq \dots \geq d_n \geq 2$ with $n \geq 2$. Then $R/I_{\underline{d}}$ fails to have the strong Lefschetz property in characteristic two.*

Proof. If $d_0 \leq \lceil \frac{t}{2} \rceil$, then $\frac{t+1}{d_0} \geq 2$ and so $d_0 \leq 2^m \leq t$, for some $m \in \mathbb{N}$. Thus, by Theorem 3.11 $R/I_{\underline{d}}$ fails to have the strong Lefschetz property in characteristic two.

Set $\ell := x_0 + \dots + x_n$ and $B_k := R[x_{n+1}]/I_{(\underline{d}, t-2k)}$. Recall that by Proposition 3.6, $R/I_{\underline{d}}$ has the strong Lefschetz property if and only if each B_k has the weak Lefschetz property, for $0 \leq k \leq \lfloor \frac{t}{2} \rfloor$.

Suppose $d_0 > \lceil \frac{t}{2} \rceil$, that is, $d_0 \geq d_0 + \dots + d_n - n$. Notice B_0 has the weak Lefschetz property in characteristic 2 if and only if $\binom{t}{d_0-1, \dots, d_n-1}$ is odd. Further, $t - d_0 \leq \lfloor \frac{t}{2} \rfloor$ as $d_0 > \lceil \frac{t}{2} \rceil$, and B_{t-d_0} has the weak Lefschetz property in characteristic 2 if and only if $\binom{d_0}{d_1-1, \dots, d_n-1, d_0-(d_1+\dots+d_n-n)}$ is odd (notice that $t - 2(t - d_0) = 2d_0 - t = d_0 + 1 - (d_1 + \dots + d_n - n)$).

By Lemma 3.29, $\binom{t}{d_0-1, \dots, d_n-1}$ is even or $\binom{d_0}{d_1-1, \dots, d_n-1, d_0-(d_1+\dots+d_n-n)}$ is even, thus B_0 or B_{t-d_0} fails to have the weak Lefschetz property in characteristic 2. Hence, $R/I_{\underline{d}}$ fails to have the strong Lefschetz property in characteristic two. \square

Generation in a single degree

In this subsection, we consider monomial complete intersections generated by monomials of the same degree, that is, $d_0 = \cdots = d_n = d \geq 2$. Notice that the socle degree is $(n+1)(d-1)$.

The case when $n = 1$ is handled in Section 3.4. In particular, the weak Lefschetz property is classified in Proposition 2.16, and the strong Lefschetz property is classified in Theorem 3.22. Brenner and Kaid [5, Theorem 2.6] classify the weak Lefschetz property when $n = 2$. We note that Kustin, Rahmati, and Vraciu [31] relate this result to the projective dimension of $K[x, y, z]/(x^d, y^d, z^d): (x^n + y^n + z^n)$. Kustin and Vraciu [32, Theorem 4.3] classify the weak Lefschetz property when $n = 3$. Further still, Kustin and Vraciu [32] prove the surprising classification of the weak Lefschetz property when $n \geq 4$. We recall the last here, as we will use it.

Theorem 3.31. [32, Theorem 6.4] *Let $d \geq 2$ and $n \geq 4$. Then $R/I_{(d, \dots, d)}$ has the weak Lefschetz property if and only if the characteristic of K is 0 or greater than $\left\lfloor \frac{(n+1)(d-1)}{2} \right\rfloor$.*

As a corollary of the above theorem, we get a classification of the strong Lefschetz property when $n \geq 4$.

Corollary 3.32. *Let $d \geq 2$ and $n \geq 4$. Then $R/I_{(d, \dots, d)}$ has the strong Lefschetz property if and only if the characteristic of K is 0 or greater than $(n+1)(d-1)$.*

Proof. As the strong Lefschetz property implies the weak Lefschetz property, we combine Theorems 3.11 and 3.31 to verify the claim. \square

Before classifying the strong Lefschetz property for $n = 3$, we prove a more general lemma regarding the weak Lefschetz property and monomial complete intersections generated by monomials all having the same degree, except one.

Lemma 3.33. *Let $\underline{d} \in \mathbb{N}^{n+1}$, where $d \geq 3$, $n \geq 4$, $d_0 = \cdots = d_{n-1} = d$ and $d_n = d-1$. If d or n is odd, and $2 \leq p < d$, then $R/I_{\underline{d}}$ fails to have the weak Lefschetz property.*

Proof. Set $\underline{e} = (d, \dots, d) \in \mathbb{N}^{n+1}$.

By Theorem 3.31, $R/I_{\underline{e}}$ fails the weak Lefschetz property for $2 \leq p < d$. Thus, by Proposition 3.23 and as $(n+1)(d-1) + 3$ is odd, $\text{mgd } \overline{\text{syz}} \underline{d} < \left\lfloor \frac{(n+1)(d-1)+3}{2} \right\rfloor = \frac{(n+1)(d-1)+2}{2}$.

Let $\alpha = (z_0, \dots, z_n)$ be a homogeneous representative of a nonzero syzygy in $\overline{\text{syz}} \underline{e}$ of degree $\text{mgd } \overline{\text{syz}} \underline{e}$ such that $z_0 \notin (x_1^d, \dots, x_n^d)$. Without loss of generality we may further assume x_n^{d-1} does not divide z_0 (otherwise, the degree of α would be at least $(n+1)(d-1)$, which is larger than $\frac{(n+1)(d-1)+2}{2}$), that is, $z_0 \notin (x_1^d, \dots, x_{n-1}^d, x_n^{d-1})$.

Then $\alpha' = (z_0, \dots, x_n z_n)$ is a homogeneous nonzero syzygy in $\text{syz } \underline{d}$. Further, as z_0 is not a member of $(x_1^d, \dots, x_{n-1}^d, x_n^{d-1})$, and all relations in $\text{Kos } \underline{d}$ must have a S -linear combination of $x_1^d, \dots, x_{n-1}^d, x_n^{d-1}$ in the first entries, then α' is not in $\text{Kos } \underline{d}$. Thus, α' is a homogeneous representative of a nonzero syzygy in $\overline{\text{syz}} \underline{d}$.

Notice, the degree of α' is the degree of α , and is strictly bounded above by $\frac{(n+1)(d-1)+2}{2}$. Hence, $\text{mgd}\overline{\text{syz}}\underline{d} < \frac{(n+1)(d-1)+2}{2}$. Notice though, $\left\lceil \frac{n(d-1)+(d-2)+3}{2} \right\rceil = \frac{(n+1)(d-1)+2}{2}$. Therefore, by Proposition 3.23, $R/I_{\underline{d}}$ fails to have the weak Lefschetz property for $2 \leq p < d$. \square

From this we get a classification of the strong Lefschetz property when $n = 3$.

Proposition 3.34. *Let $d \geq 2$. Then $R/I_{(d,d,d,d)}$ has the strong Lefschetz property if and only if the characteristic of K is 0 or greater than $4(d-1)$.*

Proof. By Theorem 3.11, we need only to consider $2 \leq p < d$.

By Proposition 3.6, $R/I_{(d,d,d,d)}$ fails the strong Lefschetz property if the ring $R[z]/I_{(d,d,d,d,4d-4-2k)}$ fails to have the weak Lefschetz property for some $0 \leq k \leq 2d-2$.

Suppose d is even, then $4d-4-2k = d$ when $k = \frac{3d-4}{2} < 2d-2$. Thus, using Theorem 3.31 we see that $R/I_{(d,d,d,d)}$ fails to have the strong Lefschetz property for $p \leq \left\lceil \frac{5(d-1)}{2} \right\rceil$. As $d < \left\lceil \frac{5(d-1)}{2} \right\rceil$ for all d , then the claim holds.

Suppose d is odd, then $4d-4-2k = d-1$ when $k = \frac{3d-3}{2} < 2d-2$. By Lemma 3.33 we see that $R/I_{(d,d,d,d)}$ fails to have the strong Lefschetz property for $2 \leq p < d$. \square

3.7 Conclusions

We combine Corollaries 3.21 and 3.30 to get the following theorem classifying the strong Lefschetz property for monomial complete intersections in characteristic two.

Theorem 3.35. *Let $d_0 \geq \dots \geq d_n \geq 2$ with $n \geq 1$, and let $I = (x_0^{d_0}, \dots, x_n^{d_n}) \subset R = K[x_0, \dots, x_n]$, where K is an infinite field of characteristic two. Then R/I has the strong Lefschetz property if and only if $n = 1$ and either (i) d_0 is odd and $d_1 = 2$ or (ii) $d_0 = 4k + 2$ for some $k \in \mathbb{N}$ and $d_1 = 3$.*

Moreover, combining Theorem 3.22 ($n = 1$), Theorem 3.27 ($n = 2$), Proposition 3.34 ($n = 3$), and Corollary 3.32 ($n \geq 4$), we completely classify the strong Lefschetz property for monomial complete intersections generated by monomials all having the same degree.

Theorem 3.36. *Let $d \geq 2$, $n \geq 1$, and $I = (x_0^d, \dots, x_n^d) \subset R = K[x_0, \dots, x_n]$, where K is an infinite field of characteristic p . Then R/I has the strong Lefschetz property if and only if p is zero or p is a positive prime and either*

- (i) $n = 1$ and $p^s > 2(d-1)$, where s is the largest integer such that p^{s-1} divides $(2d-1)(2d+1)$, or
- (ii) $n \geq 2$ and $p > (n+1)(d-1)$.

By Theorem 3.11, for a monomial complete intersection generated in degrees $d_0 \geq \dots \geq d_n \geq 2$, the presence of the strong Lefschetz property is uniform for primes at least d_0 . However, for small primes (those less than d_0), the strong Lefschetz property appears to behave chaotically when arbitrary degree sequences $\underline{d} = (d_0, \dots, d_n)$

are considered. However, some restrictions, such as characteristic two or a fixed generating degree, can limit this apparent chaos to only the case of two variables. This suggests that perhaps more focus should be given to two variables.

Question 3.37. For which prime characteristics p does the algebra $K[x, y]/(x^a, y^b)$, where $a \geq b \geq 2$, fail to have the strong Lefschetz property?

Unfortunately, Proposition 3.10 has a gap when the socle degree t is even and $p = \frac{t}{2} + 1$. Note, Corollary 3.4 cannot be used in this specific case. As an example, consider $A = K[w, x, y]/(w^5, x^5, y^5)$ and $B = K[w, x, y, z]/(w^5, x^5, y^5, z^2)$. In this case, A has the weak Lefschetz property in characteristic 7, but B does not.

Experimentally, the weak Lefschetz property always holds. We have checked all non-degenerate complete intersections for $2 \leq p \leq 7$. Further, we have checked all non-degenerate complete intersections in up to six variables for $11 \leq p \leq 19$. We formalise the above experimental results.

Conjecture 3.38. *Let t be the socle degree of $R/I_{\underline{d}}$. If t is even and the characteristic of K is $p = \frac{t}{2} + 1$, then the algebra $R/I_{\underline{d}}$ has the weak Lefschetz property.*

Conjecture 3.38 is true when $n = 2$.

Remark 3.39. Let $a \geq b \geq c \geq 2$ such that $t = a + b + c - 3 = 2(p - 1)$ for some prime p , and suppose that $a \leq \lceil \frac{t}{2} \rceil = p - 1$. Set $A = K[x, y, z]/(x^a, y^b, z^c)$. Let $\alpha = p - b$ and $\beta = b - 1$, and notice that $0 < \alpha < a$. Consider the following commutative diagram, where $B = K[x, y, z]/(x^a, y^b, z^c, x^\alpha y^\beta)$ and $\ell = x + y + z$.

$$\begin{array}{ccc} [A]_{p-2} & \xrightarrow{\times \ell} & [A]_{p-1} \\ \downarrow \cong & & \uparrow \\ [B]_{p-2} & \xrightarrow{\times \ell} & [B]_{p-1} \end{array}$$

Thus, the top map is injective if the bottom map is. Using Proposition 4.37, we see that B has the weak Lefschetz property, and thus the bottom map is injective, if the characteristic of K is at least $\frac{a+b+c+\alpha+\beta}{3} = p$. Hence the top map is injective and A has the weak Lefschetz property in characteristic p .

Moreover, we conjecture that when d_0 is “small” (i.e., when the weak Lefschetz property is *not* guaranteed to hold by Proposition 3.2), then the strong Lefschetz property only holds when guaranteed by Theorem 3.11. Notice that Theorems 3.35 and 3.36 provide evidence for this conjecture.

Conjecture 3.40. *Suppose $d_0 \leq \lceil \frac{t}{2} \rceil$, where t is the socle degree of $R/I_{\underline{d}}$. Then $R/I_{\underline{d}}$ has the strong Lefschetz property if and only if the characteristic of K is either 0 or greater than t .*

Chapter 4 Monomial almost complete intersections

The starting point of this chapter was an intriguing conjecture in [38] on the weak Lefschetz property of certain algebras. In this chapter we make progress on the above conjecture and illustrate the depth of the problem by considering a larger class of algebras and relating the problem to *a priori* seemingly unrelated questions in combinatorics and algebraic geometry. This builds on the work of many authors.

In [9], Nagel and the author found a connection between certain families of level artinian monomial almost complete intersections and lozenge tilings of hexagons; independently, Li and Zanello [33] found a similar connection for artinian monomial complete intersections (see also Corollary 4.30). However, both were without combinatorial bijection until one was found by Chen, Guo, Jin, and Liu [6]; Boyle, Migliore, and Zanello [2] have pushed this connection further. Brenner and Kaid [5] also consider artinian monomial complete intersections in three variables with generators all of the same degree. We also note that in their study of pure O -sequences Boij, Migliore, Miró-Roig, Nagel, and Zanello [1] have explored the relation between the weak Lefschetz property and pure O -sequences.

In this chapter we extend the connection found by Chen, Guo, Jin, and Liu to a connection between artinian monomial almost complete intersections in three variables and lozenge tilings of more general regions that we call punctured hexagons. In Section 4.1 we introduce the algebras we are interested in: artinian monomial almost complete intersections in three variables. These are the ideals discussed in [4, Corollary 7.3] and [38, Section 6]. We also demonstrate that the prime characteristics in which the weak Lefschetz property fails for a given algebra are exactly the prime divisors of the determinants of two different matrices. In Section 4.2 we organise the monomials generating the peak homogeneous components of such an algebra in a plane. It turns out that the monomials fill a punctured hexagon. We argue that the determinant of one of the associated matrices is the enumeration of the signed lozenge tilings of the punctured hexagon, up to sign. In Section 4.3 we show that the determinants of the two matrices are the same, up to sign.

In Section 4.4 we find closed formulae for the determinant of the above matrices, up to sign, in several cases. In Section 4.5 we explore two different ways to centralise the puncture. In particular, one of the central conditions is equivalent to the associated algebra being level, that is, its socle is concentrated in one degree. In Section 4.6 we describe several interesting families of algebras whose associated matrices have specified determinants. In Section 4.7 we explicitly determine (in one case, depending on the presence of the weak Lefschetz property) the splitting type of all artinian monomial almost complete intersections. Moreover, in the case of ideals associated to punctured hexagons, we relate the weak Lefschetz property to a number of other problems in algebra, combinatorics, and algebraic geometry.

The contents of this chapter come from [11], a joint work with Uwe Nagel.

4.1 Background

Let K be an infinite field, and consider the ideal

$$I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$$

in $R = K[x, y, z]$, where $0 \leq \alpha < a$, $0 \leq \beta < b$, and $0 \leq \gamma < c$. If $\alpha = \beta = \gamma = 0$, then we define $I_{a,b,c,0,0,0}$ to be (x^a, y^b, z^c) which is a complete intersection and is studied extensively in [33] and [6]. Assume at most one of α, β , and γ is zero.

Proposition 4.1. [38, Proposition 6.1] *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be defined as above. Assume, without loss of generality, that $0 \leq \alpha \leq \beta \leq \gamma$.*

- (i) *If $\alpha = 0$, then R/I has socle type 2 with socle degrees $a+\beta+c-3$ and $a+b+\gamma-3$; thus R/I is level if and only if $b - \beta = c - \gamma$.*
- (ii) *If $\alpha > 0$, then R/I has socle type 3 with socle degrees $\alpha+b+c-3$, $a+\beta+c-3$, and $a+b+\gamma-3$; thus R/I is level if and only if $a - \alpha = b - \beta = c - \gamma$.*
- (iii) *Moreover, the minimal free resolution of R/I has the form*

$$\begin{array}{ccccccc}
& & & R(-a - \beta - \gamma) & & & \\
& & & \oplus & & & \\
& & & R(-\alpha - b - \gamma) & & R(-\alpha - \beta - \gamma) & \\
& & & \oplus & & \oplus & \\
& & R(-a - b - \gamma) & & & & \\
& & \oplus & R(-\alpha - \beta - c) & & R(-a) & \\
0 \rightarrow & R(-a - \beta - c) & \rightarrow & \oplus & \rightarrow & \oplus & \rightarrow R \rightarrow R/I \rightarrow 0 \\
& \oplus & & R(-a - b) & & R(-b) & \\
& R^n(-\alpha - b - c) & & \oplus & & \oplus & \\
& & & R(-a - c) & & R(-c) & \\
& & & \oplus & & & \\
& & & R^n(-b - c) & & &
\end{array} \tag{4.1}$$

where $n = 1$ if $\alpha > 0$ and $n = 0$ if $\alpha = 0$.

Moreover, we see that in characteristic zero the weak Lefschetz property follows for certain choices of the parameters.

Proposition 4.2. [38, Theorem 6.2] *Let K be an algebraically closed field of characteristic zero. Then $R/I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if $a+b+c+\alpha+\beta+\gamma \not\equiv 0 \pmod{3}$.*

Semi-stability

The syzygy module $\text{syz } I$ of $I = I_{a,b,c,\alpha,\beta,\gamma}$ fits into the exact sequence

$$0 \longrightarrow \text{syz } I \longrightarrow R(-\alpha - \beta - \gamma) \oplus R(-a) \oplus R(-b) \oplus R(-c) \longrightarrow I_{a,b,c,\alpha,\beta,\gamma} \longrightarrow 0.$$

The sheafification $\widetilde{\text{syz } I}$ is a rank 3 bundle on \mathbb{P}^2 , and it is called the *syzygy bundle* of I . Recall that a vector bundle E on projective space is said to be *semistable* if, for every coherent subsheaf $F \subset E$, the inequality $\frac{c_1(F)}{\text{rk}(F)} \leq \frac{c_1(E)}{\text{rk}(E)}$ holds.

We analyse when $I_{a,b,c,\alpha,\beta,\gamma}$ has a semistable syzygy bundle. (Note, the slightly awkward definition of s in the following is kept for consistency with [38, Section 7], the starting point of this work.)

Proposition 4.3. *Let K be an algebraically closed field of characteristic zero. Further, let $I = I_{a,b,c,\alpha,\beta,\gamma}$, and define the following rational numbers*

$$\begin{aligned} s &:= \frac{1}{3}(a + b + c + \alpha + \beta + \gamma) - 2, \\ A &:= s + 2 - a, \\ B &:= s + 2 - b, \\ C &:= s + 2 - c, \text{ and} \\ M &:= s + 2 - (\alpha + \beta + \gamma). \end{aligned}$$

Then I has a semistable syzygy bundle if and only if the following conditions all hold:

- (i) $0 \leq M$,
- (ii) $0 \leq A \leq \beta + \gamma$,
- (iii) $0 \leq B \leq \alpha + \gamma$, and
- (iv) $0 \leq C \leq \alpha + \beta$.

Proof. Using [3, Corollary 7.3] we have that I has a semistable syzygy bundle if and only if

- (a) $\max\{a, b, c, \alpha + \beta + \gamma\} \leq s + 2$,
- (b) $\min\{\alpha + \beta + c, \alpha + b + \gamma, a + \beta + \gamma\} \geq s + 2$, and
- (c) $\min\{a + b, a + c, b + c\} \geq s + 2$.

Notice that condition (a) is equivalent to A, B, C , and M being non-negative. Moreover, condition (b) is equivalent to the upper bounds on A, B , and C . We claim that condition (c) follows directly from condition (a).

Indeed, by condition (a) we have that $C + M \geq 0$ and so $A + B + C + M = s + 2 \geq A + B = 2(s + 2) - a - b$, thus $a + b \geq s + 2$. Similarly, we have $a + c \geq s + 2$ and $b + c \geq s + 2$. Thus condition (c) holds if condition (a) holds. \square

This gives further conditions on the parameters that force the weak Lefschetz property in characteristic zero (see [4, Theorem 3.3]). This extends [38, Lemma 6.7].

Corollary 4.4. *Let K be an algebraically closed field of characteristic zero, and let $I = I_{a,b,c,\alpha,\beta,\gamma}$. If any of the conditions (i)-(iv) in Proposition 4.3 fail, then R/I has the weak Lefschetz property.*

The above definitions of $s, A, B, C,$ and M are not without purpose. Before going further, we make a few comments about the given parameters.

Remark 4.5. Suppose $s, A, B, C,$ and M are defined as in Proposition 4.3. Then clearly s is an integer if and only if $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$; if s is an integer, then so are $A, B, C,$ and M . Further, $A + B + C + M = s + 2$ and $A + B + C = \alpha + \beta + \gamma$.

Associated matrices

Given the minimal free resolution of R/I (see (4.1)), we can easily compute the h -vector of R/I as a weighted sum of binomial coefficients dependent only on the parameters $a, b, c, \alpha, \beta,$ and γ .

We say $h(A)$ has *twin peaks* if there exists an integer s such that $h_s = h_{s+1}$. When $I_{a,b,c,\alpha,\beta,\gamma}$ has parameters as in Proposition 4.3 and s is an integer, then the algebras $R/I_{a,b,c,\alpha,\beta,\gamma}$ always have twin peaks and the peaks are bounded by the socle degrees. This extends the results in [38, Lemma 7.1] wherein the level algebras $R/I_{a,b,c,\alpha,\beta,\gamma}$ with twin peaks are identified.

Lemma 4.6. *Assume the parameters of $I = I_{a,b,c,\alpha,\beta,\gamma}$ satisfy the conditions in Proposition 4.3 and suppose $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$. Then R/I has twin peaks in degrees s and $s + 1$. Moreover, $s + 1$ is bounded above by the socle degrees of R/I .*

Proof. The upper bounds on $A, B,$ and C are exactly those required to force the ultimate and penultimate terms in the minimal free resolution of R/I , see Proposition 4.1(iii), to not contribute to the h -vector for degrees up to $s + 1$. Moreover, as $A, B, C,$ and M are non-negative, and using $\binom{n+1}{2} - \binom{n}{2} = n$ for $n \geq 0$, then

$$\begin{aligned} h_{s+1} - h_s &= \left(\binom{s+3}{2} - \binom{A+1}{2} - \binom{B+1}{2} - \binom{C+1}{2} - \binom{M+1}{2} \right) \\ &\quad - \left(\binom{s+2}{2} - \binom{A}{2} - \binom{B}{2} - \binom{C}{2} - \binom{M}{2} \right) \\ &= s + 2 - (A + B + C + M) \\ &= 0. \end{aligned}$$

Suppose, without loss of generality, that $\alpha \leq \beta \leq \gamma$. The socle degrees of R/I are $\alpha + b + c - 3$, $a + \beta + c - 3$, and $a + b + \gamma - 3$, with the first removed if $\alpha = 0$. The following argument shows that $\alpha + b + c - 3$ is at least $s + 1$, however, up to a change of labels it shows that each of the socle degrees is at least $s + 1$.

As we are considering the socle degree $\alpha + b + c - 3$, we may assume $\alpha \geq 1$. Notice that $\alpha + b + c - 3 = 2A + B + C + 2M + \alpha - 3$, which is at least $s + 1 = A + B + C + M - 1$ exactly when $A + M + \alpha \geq 2$. If $A + M \geq 1$, then we are done. Suppose $A + M = 0$, then $A = M = 0$ and $b + c = \alpha + \beta + \gamma$. Moreover, since $b > \beta$ and $c > \gamma$, then $\alpha + \beta + \gamma = b + c \geq \beta + \gamma + 2$. Thus $\alpha \geq 2$. \square

A consequence of the previous lemma is that exactly one map need be considered for each algebra in order to determine the presence of the weak Lefschetz property.

Corollary 4.7. *Assume the parameters of $I = I_{a,b,c,\alpha,\beta,\gamma}$ satisfy the conditions in Proposition 4.3 and suppose $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$. Then R/I has the weak Lefschetz property if and only if the map $\times(x + y + z) : [R/I]_s \rightarrow [R/I]_{s+1}$ is injective (or surjective).*

Proof. This follows immediately from Lemma 4.6 by using Propositions 2.9(iii), 2.11, and 2.12. \square

This leads to the definition of two matrices with determinants that determine the weak Lefschetz property. The first is a zero-one matrix and the second is a matrix of binomial coefficients.

Proposition 4.8. *Assume the parameters of $I = I_{a,b,c,\alpha,\beta,\gamma}$ satisfy the conditions in Proposition 4.3 and suppose $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$.*

Then there exists a matrix $Z = Z_{a,b,c,\alpha,\beta,\gamma}$ such that

- (i) Z is a square integer matrix of size h_s ,
- (ii) R/I has the weak Lefschetz property if and only if $\det Z \not\equiv 0 \pmod{\text{char } K}$, and
- (iii) the entries of Z are given by

$$(Z)_{i,j} = \begin{cases} 1 & n_j \text{ is a multiple of } m_i, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{m_1, \dots, m_{h_s}\}$ and $\{n_1, \dots, n_{h_s}\}$ are the monomial bases of $[R/I]_s$ and $[R/I]_{s+1}$, respectively, and are given in lexicographic order.

Proof. We notice that the map $\times(x + y + z) : [R/I]_s \rightarrow [R/I]_{s+1}$ can be represented as a matrix Z with rows and columns indexed by fixed monomial bases of $[R/I]_s$ and $[R/I]_{s+1}$, respectively. This follows immediately from viewing $[R/I]_d$ as a vector space over K .

Claim (i) follows from Lemma 4.6 wherein it is shown that $h_s = h_{s+1}$. Since Z is square, then the injectivity of $\times(x + y + z) : [R/I]_s \rightarrow [R/I]_{s+1}$ is equivalent to Z being invertible, that is, equivalent to $\det Z$ being non-zero in K . Thus, claim (ii) follows from Corollary 4.7 wherein it is shown that the injectivity of the map $\times(x + y + z) : [R/I]_s \rightarrow [R/I]_{s+1}$ exactly determines the presence of the weak Lefschetz property for R/I . Claim (iii) follows immediately from the construction of the map. \square

The following generalises the results in [38, Theorem 7.2 and Corollary 7.3].

Proposition 4.9. *Assume the parameters of $I = I_{a,b,c,\alpha,\beta,\gamma}$ satisfy the conditions in Proposition 4.3, and suppose $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$.*

Then there exists a matrix $N = N_{a,b,c,\alpha,\beta,\gamma}$ such that

- (i) N is a square integer matrix of size $C + M$,

(ii) R/I has the weak Lefschetz property if and only if $\det N \not\equiv 0 \pmod{\text{char } K}$,
and

(iii) the entries of N are given by

$$(N)_{i,j} = \begin{cases} \binom{c}{A+j-i} & \text{if } 1 \leq i \leq C, \\ \binom{\gamma}{A+C-\beta+j-i} & \text{if } C+1 \leq i \leq C+M. \end{cases}$$

Proof. Notice that $R/(I, x+y+z) \cong S/J$, where $S = K[x, y]$ and

$$J = (x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma).$$

Thus the sequence

$$[R/I]_d \xrightarrow{\times(x+y+z)} [R/I]_{d+1} \rightarrow [R/(I, x+y+z)]_{d+1} \rightarrow 0$$

implies that $\times(x+y+z) : [R/I]_s \rightarrow [R/I]_{s+1}$ is injective exactly when $[S/J]_{s+1} = 0$. Hence it suffices to show that all $s+2$ monomials of the form $x^i y^j$ where $i+j = s+1$ are in J .

Clearly if $i \geq a$ or $j \geq b$, then $x^i y^j$ is in J . This leaves $s+2 - (s+2-a) - (s+2-b) = s+2 - A - B = C+M$ monomials that are not trivially in J . Thus there are $C+M$ equations and unknowns, all of which only involve the non-monomial terms (after reduction by the monomial terms). Associated to this system of equations is a square integer matrix of size $C+M$, call it N . Then N is invertible if and only if $\det N$ is non-zero in K . Thus, claims (i) and (ii) hold.

There are $s+2-c = C$ ways to scale $(x+y)^c$ and $s+2-(\alpha+\beta+\gamma) = M$ ways to scale $x^\alpha y^\beta (x+y)^\gamma$ to be degree $s+1$. In both cases consider the binomial coefficient indexed by the degree of y . Then $(N)_{i,j}$ is the coefficient on $x^{a-j} y^{A+j-1}$ in the scaling $x^{C-i} y^{i-1} (x+y)^c$ for $1 \leq i \leq C$, i.e., $\binom{c}{A+j-i}$, and in the scaling $x^{C+M-i} y^{i-C-1} x^\alpha y^\beta (x+y)^\gamma$ for $C+1 \leq i \leq C+M$, i.e., $\binom{\gamma}{A+C-\beta+j-i}$. Thus claim (iii) holds. \square

Clearly $\det Z_{a,b,c,\alpha,\beta,\gamma}$ and $\det N_{a,b,c,\alpha,\beta,\gamma}$ must both be either zero or have the same set of prime divisors. We can determine a few of the prime divisors from the known failure of the weak Lefschetz property.

Proposition 4.10. *Assume the parameters of $I = I_{a,b,c,\alpha,\beta,\gamma}$ satisfy the conditions in Proposition 4.3, and suppose $a+b+c+\alpha+\beta+\gamma \equiv 0 \pmod{3}$. If K has positive characteristic p and there exists a positive integer m such that*

$$\max\{a, b, c\} \leq p^m \leq s+1 = \frac{1}{3}(a+b+c+\alpha+\beta+\gamma) - 1,$$

then

(i) R/I fails to have the weak Lefschetz property,

(ii) p is a prime divisor of the determinant of $Z_{a,b,c,\alpha,\beta,\gamma}$, and

(iii) p is a prime divisor of the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$.

Proof. By Lemma 4.6, the Hilbert function of R/I weakly increases to degree $s + 1$, hence part (i) follows by Lemma 2.13. Parts (ii) and (iii) then follow from Propositions 4.8 and 4.9, respectively. \square

In the next section we will see a nice combinatorial interpretation for both matrices as well as the defined values s, A, B, C , and M .

4.2 Punctured hexagons and friends

Recall the definition of s, A, B, C , and M , and the conditions thereon, from Proposition 4.3. In this section we assume, without exception, that $I = I_{a,b,c,\alpha,\beta,\gamma}$ has parameters matching these conditions and further that $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$.

Punctured hexagons

Notice that every monomial in $[R]_d$ is of the form $x^i y^j z^k$ where i, j , and k are non-negative integers such that $i + j + k = d$. Hence we can organise the monomials in $[R]_d$ into a triangle of side-length $d + 1$ with x^d at the lower-center, y^d at the upper-right, and z^d at the upper-left. (See Figure 4.1(i).)



(i) The monomial triangle for $[R]_3$ (ii) The interlaced region for $[R]_2$ and $[R]_3$

Figure 4.1: The interlaced basis region for $R = K[x, y, z]$

Notice that we can interlace the monomials of $[R]_{d-1}$ within the monomials of $[R]_d$. If we stay consistent with our orientation (i.e., largest power of x at the lower-center, largest power of y at the upper-right, and largest power of z at the upper-left), then two monomials are adjacent if and only if one divides the other. (See Figure 4.1(ii).) We call such a figure the *interlaced basis region* of $[R]_{d-1}$ and $[R]_d$.

If we compute the interlaced basis region of $[R/I_{a,b,c,\alpha,\beta,\gamma}]_s$ and $[R/I_{a,b,c,\alpha,\beta,\gamma}]_{s+1}$, then we get a punctured hexagonal region.

Theorem 4.11. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ satisfy the conditions in Proposition 4.3, and suppose $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$. Then the interlaced basis region $H_{a,b,c,\alpha,\beta,\gamma}$ of $[R/I]_s$ and $[R/I]_{s+1}$ is in the shape of a hexagon with side-lengths (in clockwise cyclic order, starting at the bottom) $(A, B + M, C, A + M, B, C + M)$*

$$(A, B + M, C, A + M, B, C + M)$$

and with a puncture in the shape of an equilateral triangle of side-length M . The puncture has sides parallel to the sides of the hexagon of lengths $A + M, B + M,$ and $C + M$. Moreover, the puncture is located $\alpha, \beta,$ and γ units from the sides of length $A + M, B + M,$ and $C + M,$ respectively. (See Figure 4.2.)

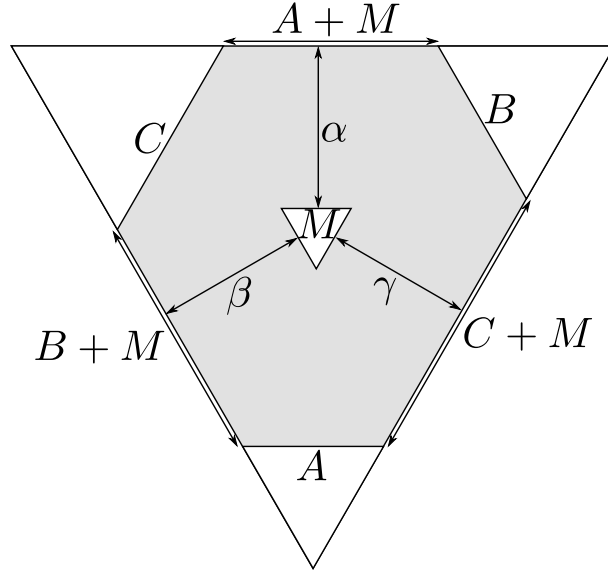


Figure 4.2: $H_{a,b,c,\alpha,\beta,\gamma}$, the interlaced basis region of $[R/I]_s$ and $[R/I]_{s+1}$

Proof. The interlaced basis region of $[R/I]_s$ and $[R/I]_{s+1}$ corresponds to a spatial placement of the monomials of the associated components of R/I . As I is a monomial ideal, we can easily get restrictions on the monomials $x^i y^j z^k$ in the region:

- (i) The generator x^a forces $0 \leq i < a$; this corresponds to the lower-center missing triangle which has side-length $s + 2 - a = A$.
- (ii) The generator y^b forces $0 \leq j < b$; this corresponds to the upper-right missing triangle which has side-length $s + 2 - b = B$.
- (iii) The generator z^c forces $0 \leq k < c$; this corresponds to the upper-left missing triangle which has side-length $s + 2 - c = C$.
- (iv) The generator $x^\alpha y^\beta z^\gamma$ forces one of $i < \alpha, j < \beta,$ or $k < \gamma$ to also hold; this corresponds to the center missing triangle, which has side-length $s + 2 - \alpha - \beta - \gamma = M$. This further forces the particular placement of the puncture.

Moreover, the conditions in Proposition 4.3 force the regions to have non-negative side-lengths and to not overlap. \square

Remark 4.12. The ideals $I = I_{a,b,c,\alpha,\beta,\gamma}$ in Theorem 4.11 are in bijection with their hexagonal regions (assuming a fixed orientation and assuming a puncture of side-length zero is still considered to be in a particular position). Suppose we have a punctured hexagonal region, as in Figure 4.2, with parameters A, B, C, M, α , and β . Then $a = B + C + M$, $b = A + C + M$, $c = A + B + M$, and $\gamma = A + B + C - (\alpha + \beta)$.

Moreover, we notice that, in characteristic zero, these ideals are exactly the artinian monomial almost complete intersections which do not immediately have the weak Lefschetz property from Proposition 4.2 or Proposition 4.3.

Notice that by Lemma 4.6 we have $h_s = h_{s+1}$, so the region $H_{a,b,c,\alpha,\beta,\gamma}$ has the same number of upward pointing triangles as it has downward pointing triangles. In particular, it may then be possible to tile the region by lozenges (i.e., rhombi with unit side-lengths and angles of 60° and 120° ; we also note a pair of alternate names used in the literature: calissons and diamonds).

Non-intersecting lattice paths

We follow [7, Section 5] (similarly, [15, Section 2]) to translate lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}$ to families of non-intersecting lattice paths. An example of a lozenge tiling and its associated family of non-intersecting lattice paths is given in Figure 4.3.

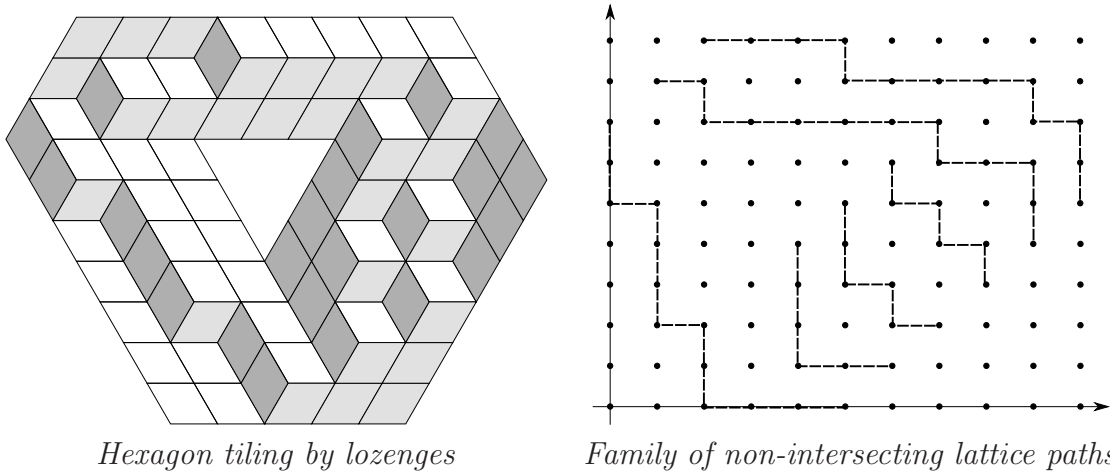
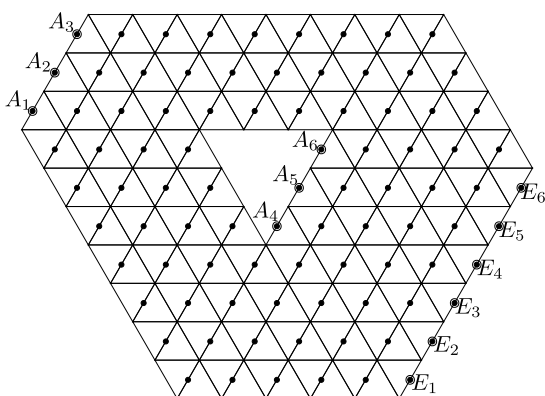


Figure 4.3: A lozenge tiling and its associated family of non-intersecting lattice paths

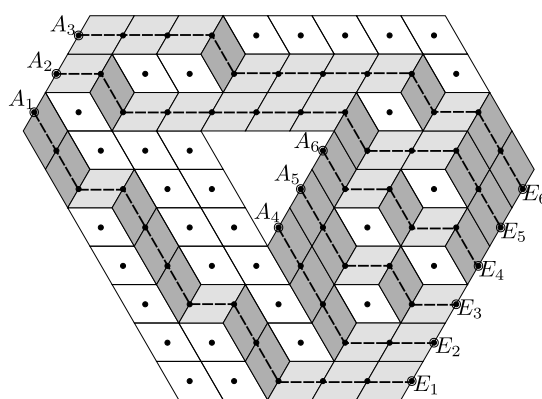
In order to transform a lozenge tiling of a punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$ into a family of non-intersecting lattice paths, we follow three simple steps (see Figure 4.4):

- (i) Mark the midpoints of the triangle edges parallel to the sides of length C and $C + M$ with vertices. Further, label the midpoints, always moving lower-left to upper-right,

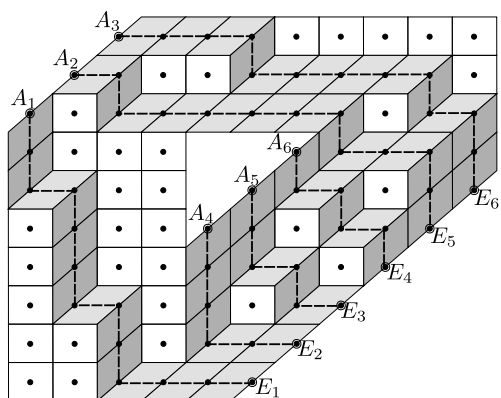
- a) along the hexagon side of length C as A_1, \dots, A_C ,
 - b) along the puncture as A_{C+1}, \dots, A_{C+M} , and
 - c) along the hexagon side of length $C + M$ as E_1, \dots, E_{C+M} .
- (ii) Using the lozenges as a guide, we connect any pair of vertices that occur on a single lozenge.
- (iii) Thinking of motion parallel to the side of length A as horizontal and motion parallel to the side of length B as vertical, we orthogonalise the lattice (and paths) and consider the lower-left vertex as the origin.



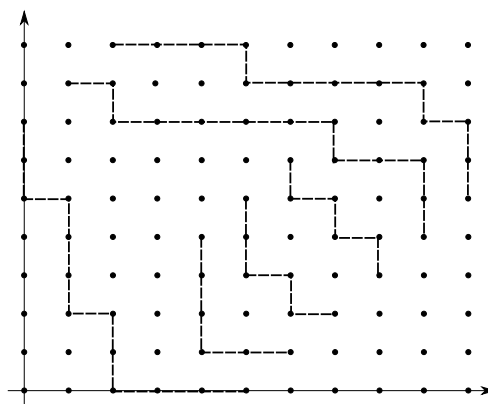
(i) Mark midpoints with vertices and label particular vertices



(ii) Connect vertices using the tiling



(iii) Orthogonalise the path family



The family by itself

Figure 4.4: Converting lozenge tilings to families of non-intersecting lattice paths

Given the above transformation of $H_{a,b,c,\alpha,\beta,\gamma}$ to the integer lattice, we see that A_i and E_j have easy to compute coordinates:

$$A_i = \begin{cases} (i - 1, B + M + i - 1) & \text{if } 1 \leq i \leq C, \\ (\beta + i - C - 1, B - \alpha + i - 1) & \text{if } C + 1 \leq i \leq C + M, \end{cases}$$

and

$$E_j = (A + j - 1, j - 1) \text{ for } 1 \leq j \leq C + M.$$

Now we associate to each family of non-intersecting lattice paths a permutation and use it to assign a sign to the family of paths.

Definition 4.13. Let L be a family of non-intersecting lattice paths as above, and let $\lambda \in \mathfrak{S}_{C+M}$ be the permutation so that A_i is connected to $E_{\lambda(i)}$. We define the *sign* of L to be the signature (or sign) of the permutation λ . That is, $\text{sgn } L := \text{sgn } \lambda$.

Now we are ready to use a beautiful theorem relating (signed) enumerations of families of non-intersecting lattice paths with determinants. In particular, we use a theorem first given by Lindström in [35, Lemma 1] and stated independently in [17, Theorem 1] by Gessel and Viennot. Stanley gives a very nice exposition of the topic in [50, Section 2.7].

Here we give a specialisation of the theorem to the case when all edges have the same weight—one. In particular, this result is given in [7, Lemma 14].

Theorem 4.14. Let $A_1, \dots, A_n, E_1, \dots, E_n$ be distinct lattice points on \mathbb{N}_0^2 . Then

$$\det_{1 \leq i, j \leq n} (P(A_i \rightarrow E_j)) = \sum_{\lambda \in \mathfrak{S}_n} \text{sgn}(\lambda) P_\lambda^+(A \rightarrow E),$$

where $P(A_i \rightarrow E_j)$ is the number of lattice paths from A_i to E_j and, for each permutation $\lambda \in \mathfrak{S}_n$, $P_\lambda^+(A \rightarrow E)$ is the number of families of non-intersecting lattice paths with paths going from A_i to $E_{\lambda(i)}$.

Thus, we have an enumeration of the signed lozenge tilings of a punctured hexagon with signs given by the non-intersecting lattice paths.

Theorem 4.15. The enumeration of signed lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}$, with signs given by the signs of the associated families of non-intersecting lattice paths (Definition 4.13), is given by $\det N_{a,b,c,\alpha,\beta,\gamma}$, where the matrix $N_{a,b,c,\alpha,\beta,\gamma}$ is defined in Proposition 4.9.

Proof. Notice that the number of lattice paths from (u, v) to (x, y) , where $u \leq x$ and $v \geq y$, is given by $\binom{x-u+v-y}{x-u}$ as there are $x - u + v - y$ steps and $x - u$ must be horizontal steps (equivalently, $v - y$ must be vertical steps). Thus the claim follows immediately from the steps above. \square

However, we need not consider all $(C + M)!$ permutations $\lambda \in \mathfrak{S}_{C+M}$ as the vast majority will always have $P_\lambda^+(A \rightarrow E) = 0$. Given our choice of A_i and E_j the only possible choices of λ are given by

$$\lambda_k = \left(\begin{array}{ccc|ccc|ccc} 1 & \cdots & k & k+1 & \cdots & C & C+1 & \cdots & C+M \\ 1 & \cdots & k & M+k+1 & \cdots & C+M & k+1 & \cdots & M+k \end{array} \right),$$

where $0 \leq k \leq C$ and k corresponds to the number of lattice paths that go below the puncture. In particular, the three parts of λ_k correspond to the paths going below,

above, and starting from the puncture. We call these permutations the *admissible permutations* of $H_{a,b,c,\alpha,\beta,\gamma}$.

We will use this connection to compute determinants in Section 4.4, but first we look at an alternate combinatorial connection.

Perfect matchings

Lozenge tilings of a punctured hexagon can be associated to perfect matchings on a bipartite graph. This connection was first used by Kuperberg in [30] to study symmetries on plane partitions. An example of a lozenge tiling and its associated perfect matching of edges is given in Figure 4.5.

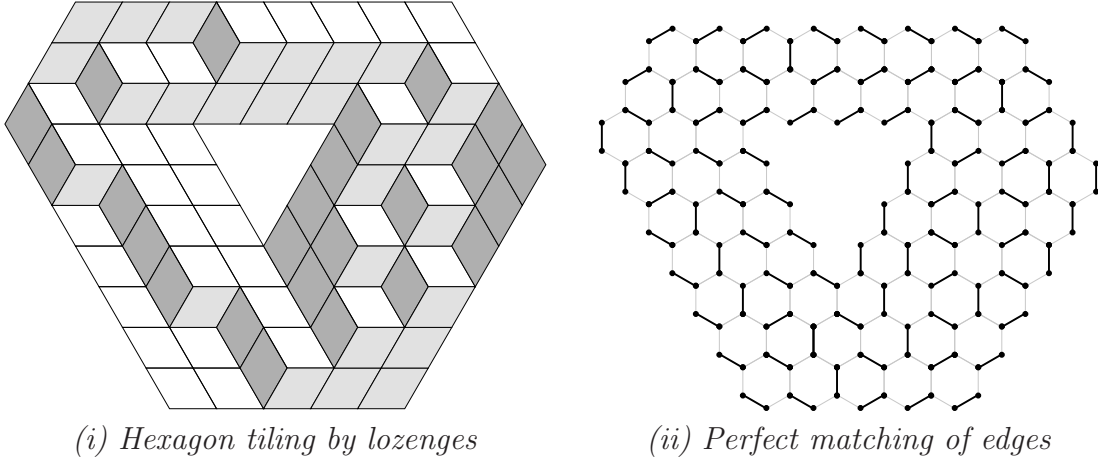


Figure 4.5: A lozenge tiling and its associated perfect matching of edges

In order to transform a lozenge tiling of a punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$ into a perfect matching of edges, we follow three simple steps (see Figure 4.6):

- (i) Put a vertex at the center of each triangle.
- (ii) Connect the vertices whose triangles are adjacent.
- (iii) Select the edges which the lozenges cover—this set is the perfect matching.

Notice that the graph associated to the punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$ is a bipartite graph with colour classes given by monomials in $[R/I]_s$ and $[R/I]_{s+1}$. Thus we can represent this bipartite graph by a bi-adjacency matrix with rows enumerated by the monomials in $[R/I]_s$ and columns enumerated by the monomials in $[R/I]_{s+1}$. We fix the order on the monomials to be the lexicographic order. Clearly then the matrix $Z_{a,b,c,\alpha,\beta,\gamma}$ from Proposition 4.8 is the bi-adjacency matrix described here.

Consider the permanent of $Z = Z_{a,b,c,\alpha,\beta,\gamma}$, that is,

$$\text{perm } Z = \sum_{\pi \in \mathfrak{S}_{h_s}} \prod_{i=1}^{h_s} (Z)_{i,\pi(i)}.$$

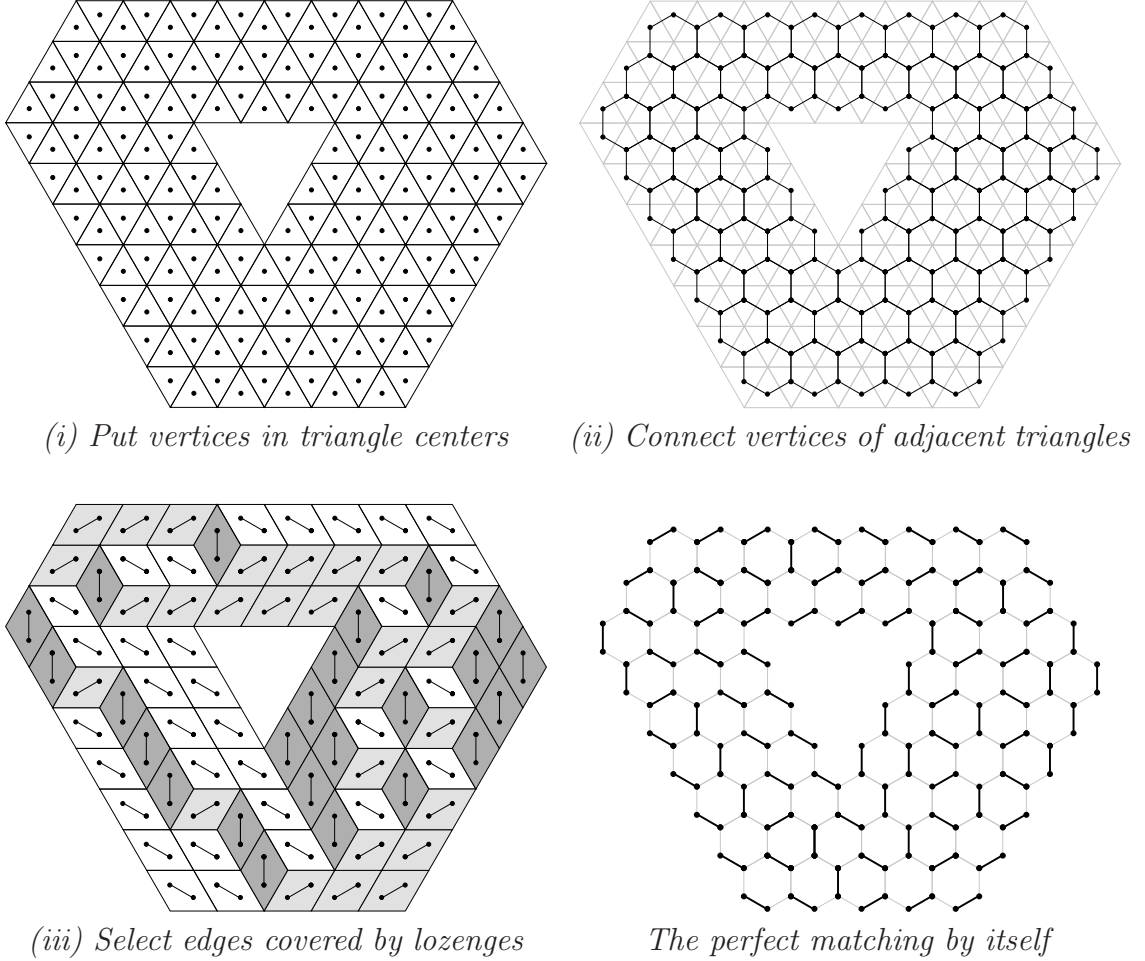


Figure 4.6: Converting lozenge tilings to perfect matchings of edges

As Z has entries which are either zero or one, we see that all summands in $\text{perm } Z$ are either zero or one. Moreover, each non-zero summand corresponds to a perfect matching, as it corresponds to an isomorphism between the two colour classes of the bipartite graph, namely, the monomials in $[R/I]_s$ and $[R/I]_{s+1}$. Thus, $\text{perm } Z$ enumerates the perfect matchings of the bipartite graph associated to $H_{a,b,c,\alpha,\beta,\gamma}$, and hence $\text{perm } Z$ also enumerates the lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}$.

Proposition 4.16. *The number of lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}$ is $\text{perm } Z_{a,b,c,\alpha,\beta,\gamma}$.*

Since each perfect matching is an isomorphism between the two colour classes, it can be seen as a permutation $\pi \in \mathfrak{S}_{h_s}$. As with Definition 4.13, it is thus natural to assign a sign to a given perfect matching.

Definition 4.17. Let P be a perfect matching of the bipartite graph associated to $H_{a,b,c,\alpha,\beta,\gamma}$, and let $\pi \in \mathfrak{S}_{h_s}$ be the associated permutation (as described above). We define the *sign* of P to be the signature of the permutation π . That is, $\text{sgn } P := \text{sgn } \pi$.

Since the sign is the sign that is used in computing the determinant of the matrix $Z_{a,b,c,\alpha,\beta,\gamma}$, we get an enumeration of the signed lozenge tilings of a punctured hexagon with signs given by the perfect matchings.

Theorem 4.18. *The enumeration of signed perfect matchings of the bipartite graph associated to $H_{a,b,c,\alpha,\beta,\gamma}$, with signs given by the signs of the related perfect matchings, is given by $\det Z_{a,b,c,\alpha,\beta,\gamma}$, where the matrix $Z_{a,b,c,\alpha,\beta,\gamma}$ is defined in Proposition 4.8.*

Remark 4.19. Kasteleyn [24] provided, in 1967, a general method for computing the number of perfect matchings of a planar graph as a determinant. Moreover, he provided a classical review of methods and applications of enumerating perfect matchings. Planar graphs, such as the “honeycomb graphs” described here, are studied for their connections to physics; in particular, honeycomb graphs model the bonds in dimers (polymers with only two structural units) and perfect matchings correspond to so-called *dimer coverings*. Kenyon [25] gave a modern recount of explorations on dimer models, including random dimer coverings and their limiting shapes.

Remark 4.20. Recall that Proposition 4.10 provides a numerical constraint that determines some of the prime divisors of the determinants of the matrices $Z_{a,b,c,\alpha,\beta,\gamma}$ and $N_{a,b,c,\alpha,\beta,\gamma}$ by means of some algebra deciding the weak Lefschetz property for the algebra $R/I_{a,b,c,\alpha,\beta,\gamma}$. Hence, by Theorems 4.15 and 4.18, we see that information from algebra can indeed be used to determine some of the prime divisors of the enumerations of signed lozenge tilings and of signed perfect matchings.

Finally, we note that in [45], Propp gives a history of the connections between lozenge tilings (of non-punctured hexagons), perfect matchings, plane partitions, and non-intersecting lattice paths.

4.3 Interlude of signs

In the preceding section we discussed three related combinatorial structures from which we can extract the primes p for which the algebras $R/I_{a,b,c,\alpha,\beta,\gamma}$ fail to have the weak Lefschetz property. Therein we discussed two different ways to assign a sign to a lozenge tiling: by the associated family of non-intersecting lattice paths (Definition 4.13) and by the associated perfect matching (Definition 4.17). We now show that the two signs indeed agree.

Fix a hexagonal region $H = H_{a,b,c,\alpha,\beta,\gamma}$, and fix a lozenge tiling T of H . As discussed in Section 4.2, we associate to the tiling T a family of non-intersecting lattice paths L_T and a perfect matching P_T . Moreover, we introduced a permutation $\lambda_T \in \mathfrak{S}_{C+M}$ associated to L_T (see Definition 4.13) and a permutation $\pi_T \in \mathfrak{S}_{h_s}$ associated to P_T via $Z_{a,b,c,\alpha,\beta,\gamma}$ (see Definition 4.17).

We first notice that “rotating” particular lozenge groups of T do not change the permutation associated to the non-intersecting lattice paths.

Lemma 4.21. *Let T be a lozenge tiling of $H_{a,b,c,\alpha,\beta,\gamma}$. Pick any triplet of lozenges in T which is either an up or a down grouping, as in Figure 4.7, and let U be T with*

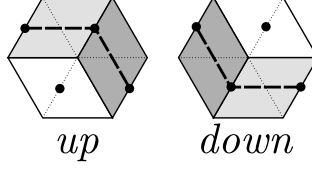


Figure 4.7: *up* and *down* lozenge groups with lattice path pieces superimposed

the triplet exchanged for the other possibility (i.e., rotated 180°). Then U is a lozenge tiling of $H_{a,b,c,\alpha,\beta,\gamma}$ and $\lambda_T = \lambda_U$. Moreover, $\pi_U = \tau\pi_T$, for some three-cycle $\tau \in \mathfrak{S}_{h_s}$.

Proof. First, we note that if T is a lozenge tiling of $H_{a,b,c,\alpha,\beta,\gamma}$ then clearly so is U as the change does not modify any tiles besides the three in the triplet.

Next, notice that exchanging the triplet in T for its rotation only modifies the associated family of non-intersecting lattice paths in one path. Moreover, it does not change the starting or ending points of the path, merely the order in which it gets there, that is, either right then down or down then right. Thus, $\lambda_T = \lambda_U$.

Last, suppose, without loss of generality, that our chosen triplet is an *up* lozenge group. Label the three upward pointing triangles in the triplet i, j, k as in Figure 4.8. Thus we see that $\pi_U(i) = \pi_T(k)$, $\pi_U(j) = \pi_T(i)$, $\pi_U(k) = \pi_T(j)$, and $\pi_U(m) = \pi_T(m)$

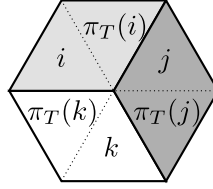


Figure 4.8: An *up* lozenge group with labeling

for m not i, j , or k . Hence $\pi_U = \tau\pi_T$ where τ is the three-cycle $(\pi_T(k), \pi_T(j), \pi_T(i))$. \square

It follows that two lozenge tilings that have the same λ permutation have π permutations with the same sign.

Proposition 4.22. *For each $H_{a,b,c,\alpha,\beta,\gamma}$ there exists a constant $i \in \{1, -1\}$ such that for all lozenge tilings T of $H_{a,b,c,\alpha,\beta,\gamma}$ the expression $\text{sgn } L_T = i \cdot \text{sgn } P_T$ holds, where L_T is the family of non-intersecting lattice paths associated to T and P_T is the family of perfect matchings associated to T .*

Proof. Step 1:

Let T and U be two lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}$ with $\lambda_T = \lambda_U$. As $\lambda_T = \lambda_U$, then the families of non-intersecting lattice paths associated to T and U start and end at the same places. Hence T can be modified by a series of, say n , rotations, as in Lemma 4.21, to U . Thus

$$\pi_U = \tau_n \tau_{n-1} \cdots \tau_1 \pi_T,$$

where $\tau_1, \dots, \tau_n \in \mathfrak{S}_{h_s}$ are three cycles by Lemma 4.21. As $\text{sgn } \tau_i = 1$ for $1 \leq i \leq n$, and sgn is a group homomorphism, we see that $\text{sgn } \pi_T = \text{sgn } \pi_U$.

Step 2:

By the comments following Theorem 4.15 we only need to consider the admissible permutations $\lambda_0, \dots, \lambda_C$. Moreover, $\text{sgn } \lambda_k = (-1)^{M(C-k)}$ so $\text{sgn } \lambda_k = (-1)^M \text{sgn } \lambda_{k+1}$.

Let T and U be two lozenge tilings of $H = H_{a,b,c,\alpha,\beta,\gamma}$ with $\lambda_T = \lambda_k$ and $\lambda_U = \lambda_{k+1}$. That is, $\text{sgn } \lambda_T = (-1)^M \text{sgn } \lambda_U$. First, $\alpha \geq C - k$ by the existence of T as $C - k$ paths go above the puncture and so must go through a gap of size α , and similarly $\beta \geq k + 1$ by the existence of U .

By *Step 1*, we may pick T and U however we wish, as long as $\lambda_T = \lambda_k$ and $\lambda_U = \lambda_{k+1}$. In particular, let T , and similarly U , be defined as follows (see Figure 4.9):

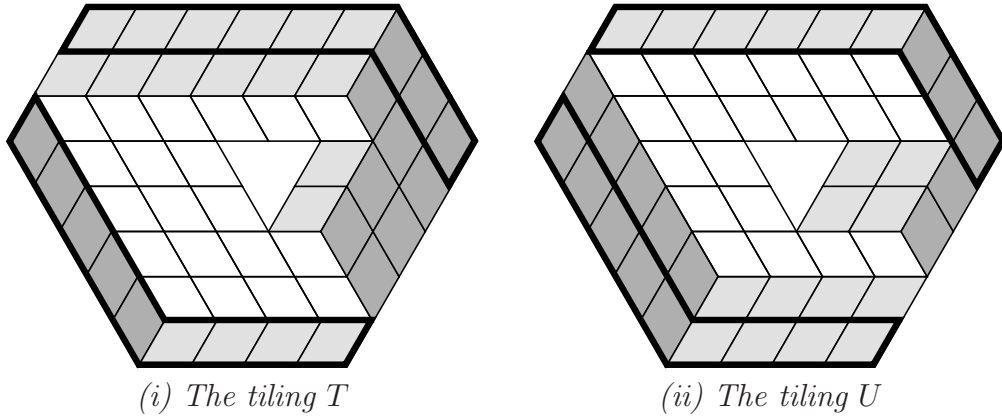


Figure 4.9: An example of tilings T and U of $H_{9,8,9,4,3,3}$, for $k = 1$, which are “minimal” below the puncture and “maximal” everywhere else; both tilings have the regions of similarity highlighted.

- (i) The $C - k$ paths above the puncture ($C - k - 1$ for U) always move right before moving down.
- (ii) The M paths from the puncture always move right before moving down.
- (iii) The k paths below the puncture ($k + 1$ for U) always move down before moving right.

With the idea of *up* and *down* triplets from Lemma 4.21, we can say a path is “minimal” if it contains no *up* triplets and a path is “maximal” if it contains no *down* triplets. Thus, T and U are “minimal” below the puncture and “maximal” everywhere else.

Given this choice, T and U have exactly the same paths for the top $C - k - 1$ paths above the puncture and the bottom k paths below the puncture. Hence we can trim off these paths to make two new tilings, T' and U' , of the new punctured hexagon $H' = H_{B+M+1, A+M+1, c, \alpha-(C-k-1), \beta-k, \gamma}$. Notice that H and H' have the same

$A, B, M,$ and $\gamma,$ only $C, \alpha,$ and β have changed; in particular, $C' = 1.$ See Figure 4.10 parts (i) and (ii) for an example of the tilings T' and U' with their region-of-difference highlighted in bold.

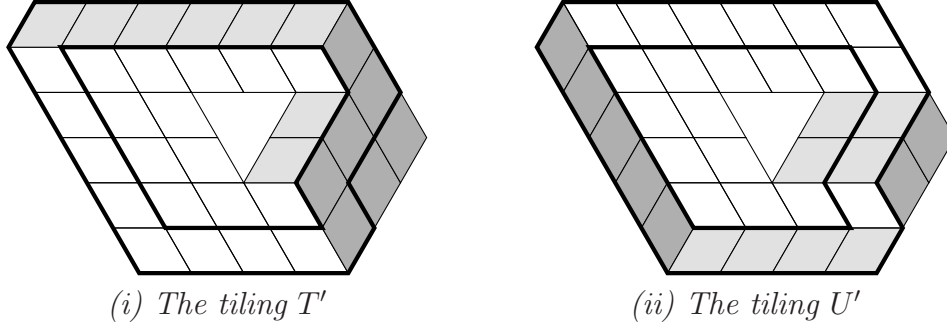


Figure 4.10: The punctured hexagon $H_{7,6,9,3,2,3};$ both tilings have the region-of-difference highlighted.

Clearly then T' and U' differ in four ways: (i) the upper path in T' except the small overlap near the end, (ii) the lower path in $U',$ (iii) the position of the bend in the puncture-paths, and (iv) the part past the bend of the bottom puncture-path in $T'.$ The difference between T' and U' is exactly $2(A + B + M) + M - 1$ tiles; moreover the region-of-difference forms a cycle so that there exists a $(2(A + B + M) + M - 1)$ -cycle, $\sigma,$ such that $\pi_{T'} = \sigma\pi_{U'}.$ We then have

$$\operatorname{sgn} \pi_{T'} = (-1)^{2(A+B+M)+M-1-1} \operatorname{sgn} \pi_{U'} = (-1)^M \operatorname{sgn} \pi_{U'}.$$

That is, $\operatorname{sgn} \pi_T = (-1)^M \operatorname{sgn} \pi_U.$ Since $\operatorname{sgn} \lambda_T = (-1)^M \operatorname{sgn} \lambda_U,$ the claim follows. \square

We conclude that $Z_{a,b,c,\alpha,\beta,\gamma}$ and $N_{a,b,c,\alpha,\beta,\gamma}$ have the same determinant, up to sign.

Theorem 4.23. *Consider the punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}.$ Then*

$$|\det Z_{a,b,c,\alpha,\beta,\gamma}| = |\det N_{a,b,c,\alpha,\beta,\gamma}|.$$

Proof. Combine Theorems 4.15 and 4.18 via Proposition 4.22. \square

Moreover, when the puncture is of even length, the determinant and permanent of $Z_{a,b,c,\alpha,\beta,\gamma}$ are the same.

Corollary 4.24. *Consider the punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}.$ If M is even, then*

$$\operatorname{perm} Z_{a,b,c,\alpha,\beta,\gamma} = |\det Z_{a,b,c,\alpha,\beta,\gamma}|.$$

Proof. A simple analysis of the proof of Proposition 4.22 implies that when M is even then $\operatorname{sgn} \pi_T = \operatorname{sgn} \pi_U$ for all tilings T and U of $H_{a,b,c,\alpha,\beta,\gamma}.$ Thus, the enumeration of signed lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}$ is, up to sign, the enumeration of (unsigned) lozenge tilings of $H_{a,b,c,\alpha,\beta,\gamma}.$ Thus, the claim follows from Proposition 4.16 and Theorem 4.18. \square

Remark 4.25. We make a pair of remarks regarding the preceding corollary.

- (i) The corollary can be viewed as a special case of Kasteleyn’s theorem on enumerating perfect matchings [24]. To see this, notice that when M is even, then all “faces” of the bipartite graph have size congruent to 2 (mod 4).
- (ii) The corollary extends [6, Theorem 1.2], where punctured hexagons with trivial puncture (i.e., $M = 0$) are considered. We further note that [25, Section 3.4] provides, independently, essentially the same proof as [6], and the proof of Lemma 4.21 builds on this technique.

We conclude this section with some observations on the signs introduced here.

Let T be a lozenge tiling of $H_{a,b,c,\alpha,\beta,\gamma}$, and let L_T and P_T be the associated family of non-intersecting lattice paths and perfect matching, respectively. By Proposition 4.22, we may assume that $\text{sgn } L_T = \text{sgn } P_T$. Thus we may assign to T the sign $\text{sgn } T = \text{sgn } L_T$.

Recall that there are C admissible permutations $\lambda_0, \dots, \lambda_C$ (see the discussion after Theorem 4.15) associated to $H_{a,b,c,\alpha,\beta,\gamma}$. Further, $\text{sgn } \lambda_k = (-1)^{M(C-k)}$ and so if M is even then $\text{sgn } \lambda_k = 1$ for all k . Hence, we need only consider M odd. In this case, $\text{sgn } \lambda_k = 1$ if and only if $C - k$ is even. Thus, the sign of T is $(-1)^{C-k}$.

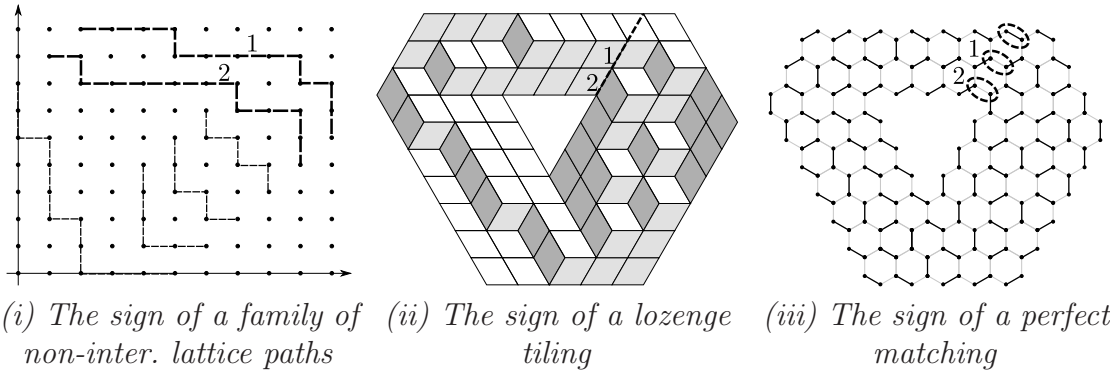


Figure 4.11: Example of interpreting the sign

By definition of λ_k , $C - k$ is the number of lattice paths in the family that go above the puncture; see Figure 4.11(i). For the lozenge tiling T , $C - k$ is the number of edges of lozenges of T that touch the line formed by extending the edge of the puncture parallel to the side of length C to the side of length $A + M$; see Figure 4.11(ii). Note that this interpretation is in line with the definition of the statistic $n(\cdot)$ in [7, Section 2]. Last, for the perfect matching, $C - k$ is the number of *non*-selected edges that correspond to those on the edge described for lozenge tilings; see Figure 4.11(iii).

4.4 Determinants

We continue to use the notation introduced in Proposition 4.3 and Theorem 4.11. Throughout this section we assume that A, B, C , and M meet conditions (i)-(iv) of Proposition 4.3 and $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$.

We will discuss properties of the determinant of the matrix $N_{a,b,c,\alpha,\beta,\gamma}$ given in Proposition 4.9 using Theorem 4.15. In particular, we are chiefly interested in whether the determinant is zero and if we can compute an upper bound on the prime divisors. In some cases we can explicitly compute the determinant.

A few properties

First, a brief remark about the polynomial nature of the determinants.

Remark 4.26. The argument in [7, Section 6] demonstrates that for fixed A, B , and C and α, β , and γ satisfying certain restraints, then the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$ is a polynomial in M , the side-length of the puncture of $H_{a,b,c,\alpha,\beta,\gamma}$, with integer coefficients for M of a fixed parity. This argument centers around an alternate bijection between the lozenge tilings and non-intersecting lattice paths.

We note that the argument is completely independent of the restrictions on α, β , and γ . Thus, their argument can be easily seen to generalise to show that, for fixed A, B, C, α, β , and γ , the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$ is polynomial in M , for M of a fixed parity.

We demonstrate that every punctured hexagonal region $H_{a,b,c,\alpha,\beta,\gamma}$ has at least one tiling.

Lemma 4.27. *Every region $H_{a,b,c,\alpha,\beta,\gamma}$ has at least one lozenge tiling.*

Proof. In this case, it is easier to show there exists a family L of non-intersecting lattice paths. In particular, it is sufficient to show that the sum of the maximum numbers of paths that can go above and below the puncture is at least C . By analysis of $H_{a,b,c,\alpha,\beta,\gamma}$, we see that at most $\min\{C, \beta, B + C - \alpha\}$ paths can go below the puncture and up to $\min\{C, \alpha, A + C - \beta\}$ paths can go above the puncture. However, as $0 \leq A, B, C$ and $C \leq \alpha + \beta$, then $\min\{C, \beta, B + C - \alpha\} + \min\{C, \alpha, A + C - \beta\} \geq C$. \square

Thus when M is even, the determinant is always positive.

Theorem 4.28. *If $a + b + c$ is even, then M is even and $\det N_{a,b,c,\alpha,\beta,\gamma} > 0$. Thus*

$$I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$$

has the weak Lefschetz property in characteristic zero and when the characteristic is sufficiently large.

Proof. Recall the definition of the admissible permutations λ_k , for $0 \leq k \leq C$ (see the discussion following Theorem 4.15). Since M is even, then $\text{sgn } \lambda_k = 1$ for $0 \leq k \leq C$ and hence $\det N_{a,b,c,\alpha,\beta,\gamma}$ is the number of tilings of $H_{a,b,c,\alpha,\beta,\gamma}$. Thus, by Lemma 4.27, $\det N_{a,b,c,\alpha,\beta,\gamma} > 0$. \square

Mahonian determinants

MacMahon computed the number of plane partitions (finite two-dimensional arrays that weakly decrease in all columns and rows) in an $A \times B \times C$ box as (see, e.g., [45, Page 261])

$$\text{Mac}(A, B, C) := \frac{\mathcal{H}(A)\mathcal{H}(B)\mathcal{H}(C)\mathcal{H}(A+B+C)}{\mathcal{H}(A+B)\mathcal{H}(A+C)\mathcal{H}(B+C)},$$

where A, B , and C are non-negative integers and $\mathcal{H}(n) := \prod_{i=0}^{n-1} i!$ is the *hyperfactorial* of n . David and Tomei proved in [12] that plane partitions in an $A \times B \times C$ box are in bijection with lozenge tilings in a non-punctured hexagon of side-lengths (A, B, C, A, B, C) . We note that Propp states on [45, Page 258] that Klarner was likely the first to have observed this. See Figure 4.12 for an illustration of the connection.

5 5 5 4 4 1
 4 3 3 3 2 1
 4 3 3 2 0 0

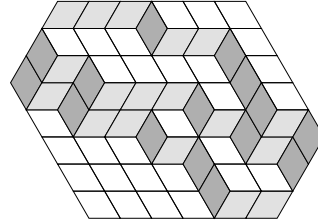


Figure 4.12: A $3 \times 6 \times 5$ plane partition and its associated lozenge tiling (with light grey as the top faces of the boxes)

We can use MacMahon’s formula to compute the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$ in many cases. Also, note that the prime divisors of $\text{Mac}(A, B, C)$ are sharply bounded above by $A + B + C - 1$. A first case is when the puncture is trivial. This extends [9, Theorem 4.5] where the level algebras of this family are considered.

Proposition 4.29. *If $a + b + c = 2(\alpha + \beta + \gamma)$, then $M = 0$ and $\det N_{a,b,c,\alpha,\beta,\gamma}$ is*

$$\text{Mac}(A, B, C).$$

Thus, $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if the characteristic of K is zero or at least $A + B + C = \alpha + \beta + \gamma = \frac{1}{2}(a + b + c)$.

Proof. When $M = 0$ then there is no puncture in the region $H_{a,b,c,\alpha,\beta,\gamma}$. Hence the region is a simple hexagon with side-lengths (A, B, C, A, B, C) , exactly the region to which MacMahon’s formula applies. \square

This result allows us to recover some earlier results about complete intersections.

Corollary 4.30. *If $a + b + c$ is even, then the complete intersection $J = (x^a, y^b, z^c)$ has the weak Lefschetz property if and only if the characteristic of K is not a prime divisor of $\text{Mac}(A, B, C)$. That is, the algebra R/J has the weak Lefschetz property if the characteristic of K is zero or at least $A + B + C = \alpha + \beta + \gamma = \frac{1}{2}(a + b + c)$.*

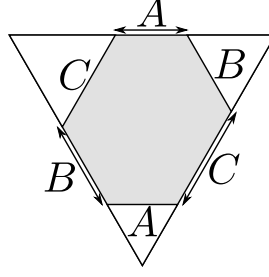


Figure 4.13: When the puncture has side-length zero, the region is a simple hexagon.

Proof. Set $\alpha = \frac{1}{2}(-a + b + c)$, $\beta = \frac{1}{2}(a - b + c)$, $\gamma = \frac{1}{2}(a + b - c)$, and consider $I = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$. Then Proposition 4.29 applies to I and the mixed term, $x^\alpha y^\beta z^\gamma$, has total degree $s + 2$. Thus we have that $[R/I]_i \cong [R/J]_i$ for $i \leq s + 1$. That is, the twin peaks of R/I are isomorphic to the twin peaks of the complete intersection R/J . Hence R/J has the weak Lefschetz property if and only if R/I has the weak Lefschetz property, and Proposition 4.29 gives the claim. \square

In particular, the corollary recovers [33, Theorem 3.2(1)] when combined with Proposition 4.9 and [6, Theorem 1.2] when combined with Corollary 4.24. Further, the special case in [33, Theorem 4.2] can be recovered if we set $a = \beta + \gamma$, $b = \alpha + \gamma$, and $c = \alpha + \beta$.

MacMahon's formula can be used again in another special case, when $C = 0$. (Notice if A or B is zero, then we can simply relabel the sides to ensure C is zero.) We notice this extends [9, Theorem 4.3] where the level algebras of this family are considered.

Proposition 4.31. *If $c = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$, then $C = 0$ and $\det N_{a,b,c,\alpha,\beta,\gamma}$ is*

$$\text{Mac}(M, A - \beta, B - \alpha).$$

Thus, $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if the characteristic of K is zero or at least $A + B + M - \alpha - \beta = c - \alpha - \beta$.

Proof. In this case, it is easier to consider families of non-intersecting lattice paths. In particular, since $C = 0$, then the only starting points, the A_i , are those on the puncture. Further, since lattice paths must move only right and down, then we can focus on the isolated region between the puncture and the bottom-right edge. If we convert this region back into a punctured hexagon, then it is just a hexagon without a puncture and with side-lengths $(M, A + C - \beta, B + C - \alpha, M, A + C - \beta, B + C - \alpha)$. \square

Remark 4.32. Notice that in the preceding proof, we show that the only possible lattice paths come from the puncture to the opposite edge. Converting this back to the language of lozenge tilings, we see this means that a large region of the figure has fixed tiles leaving only a small region in which variation can occur. See Figure 4.14 for an illustration of this.

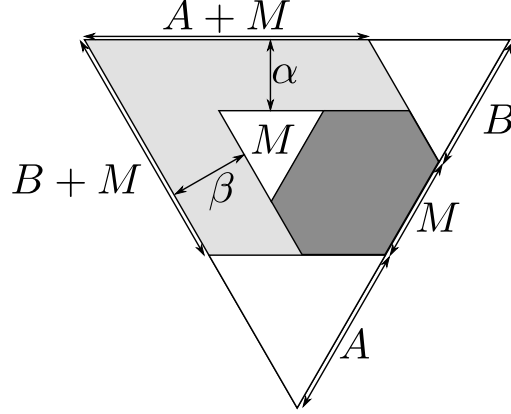


Figure 4.14: When C is zero, the lightly shaded region has tiles that are fixed, leaving the only variation in the darkly shaded region.

Further, given the condition in Proposition 4.31, we see that the pure power of z , z^c , has total degree $c = s + 2$. Thus, if we let $I = I_{a,b,c,\alpha,\beta,\gamma}$, then we have that $[R/I]_i \cong [R/J]_i$ for $i \leq s + 1$, where $J = (x^a, y^b, x^\alpha y^\beta z^\gamma)$. Thus, the twin peaks of R/I are isomorphic to the twin peaks of the non-artinian algebra R/J .

Corollary 4.33. *Let $J = (x^a, y^b, x^\alpha y^\beta z^\gamma)$ and $c = \frac{1}{2}(a+b+\alpha+\beta+\gamma)$, with parameters still suitably restricted. Then the map*

$$[R/J]_i \xrightarrow{\times(x+y+z)} [R/J]_{i+1}$$

is injective for $i \leq c$.

Further, MacMahon's formula can be used when C is maximal, that is, $C = \alpha + \beta$.

Proposition 4.34. *If $c = \frac{1}{2}(a + b + \gamma) - \alpha - \beta$, then $C = \alpha + \beta$ and $\det N_{a,b,c,\alpha,\beta,\gamma}$ is*

$$\text{Mac}(A, B, C + M).$$

Thus, $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if the characteristic of K is zero or at least $A + B + C + M = s + 2 = c + \alpha + \beta$.

Proof. In this case, it is easier to consider families of non-intersecting lattice paths. In particular, since $C = \alpha + \beta$, then $\gamma = A + B$ and so the puncture has a point touching the side labeled C ; see Figure 4.15. Thus the lattice paths starting from A_1, \dots, A_β have the first M moves being down and the lattice paths starting from $A_{\beta+1}, \dots, A_C$ have the first M moves being right. However, we then see that each A_i "starts" on the same line, the line running through the lower-right side of the puncture. If we convert the region-of-interest back into a punctured hexagon, then it is a simple hexagon with side-lengths $(A, B, C + M, A, B, C + M)$. \square

The next case considered, when the mixed term is in two variables, needs a special determinant calculation which may be of independent interest.

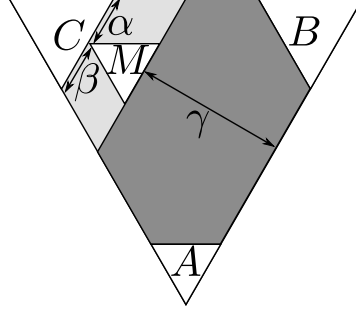


Figure 4.15: When C is maximal, the lightly shaded region has tiles which are fixed, leaving the only variation in the darkly shaded region.

Lemma 4.35. *Let T be an n -by- n matrix defined as follows*

$$(T)_{i,j} = \begin{cases} \binom{p}{q+j-i} & \text{if } 1 \leq j \leq m, \\ \binom{p}{q+r+j-i} & \text{if } m+1 \leq j \leq n, \end{cases}$$

where p, q, r , and m are non-negative integers and $1 \leq m \leq n$. Then

$$\det T = \text{Mac}(m, q, r) \text{Mac}(n-m, p-q-r, r) \frac{\mathcal{H}(q+r)\mathcal{H}(p-q)\mathcal{H}(n+r)\mathcal{H}(n+p)}{\mathcal{H}(n+p-q)\mathcal{H}(n+q+r)\mathcal{H}(p)\mathcal{H}(r)}.$$

Proof. In this case, we can use [7, Equation (12.5)] to evaluate $\det T$ to be

$$\prod_{1 \leq i < j \leq n} (L_j - L_i) \prod_{i=1}^n \frac{(p+i-1)!}{(n+p-L_i)!(L_i-1)!},$$

where $L_j = q+j$ if $1 \leq j \leq m$ and $L_j = q+r+j$ if $m+1 \leq j \leq n$. If we split the products in the previously displayed equation relative to the split in L_j , then we get the following equations:

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (L_j - L_i) &= \left(\prod_{1 \leq i < j \leq m} (j-i) \right) \left(\prod_{m < i < j \leq n} (j-i) \right) \left(\prod_{1 \leq i \leq m < j \leq n} (r+j-i) \right) \\ &= (\mathcal{H}(m)) (\mathcal{H}(n-m)) \left(\frac{\mathcal{H}(n+r)\mathcal{H}(r)}{\mathcal{H}(n+r-m)\mathcal{H}(m+r)} \right) \end{aligned}$$

and

$$\begin{aligned}
\prod_{i=1}^n \frac{(p+i-1)!}{(n+p-L_i)!(L_i-1)!} &= \left(\prod_{i=1}^n (p+i-1)! \right) \left(\prod_{i=1}^m \frac{1}{(n+p-q-i)!(q+i-1)!} \right) \\
&\quad \left(\prod_{i=m+1}^n \frac{1}{(n+p-q-r-i)!(q+r+i-1)!} \right) \\
&= \left(\frac{\mathcal{H}(n+p)}{\mathcal{H}(p)} \right) \left(\frac{\mathcal{H}(n+p-m-q)\mathcal{H}(q)}{\mathcal{H}(n+p-q)\mathcal{H}(m+q)} \right) \\
&\quad \left(\frac{\mathcal{H}(p-q-r)\mathcal{H}(m+q+r)}{\mathcal{H}(n+p-m-q-r)\mathcal{H}(n+q+r)} \right).
\end{aligned}$$

Bringing these equations together we have that $\det T$ is

$$\begin{aligned}
\det T &= \frac{\mathcal{H}(m)\mathcal{H}(q)\mathcal{H}(r)\mathcal{H}(m+q+r)}{\mathcal{H}(m+r)\mathcal{H}(m+q)} \times \frac{\mathcal{H}(n-m)\mathcal{H}(p-q-r)\mathcal{H}(n+p-m-q)}{\mathcal{H}(n+r-m)\mathcal{H}(n+p-m-q-r)} \\
&\quad \times \frac{\mathcal{H}(n+r)\mathcal{H}(n+p)}{\mathcal{H}(p)\mathcal{H}(n+p-q)\mathcal{H}(n+q+r)}.
\end{aligned}$$

After minor manipulation, this yields the claimed result. \square

Remark 4.36. Lemma 4.35 generalises the result of [33, Lemma 2.2] where the case $r = 1$ is discussed. Further, when $r = 0$, then $\det T = \text{Mac}(n, p - q, q)$, as expected (see the running example, $\det \binom{a+b}{a-i+j}$, in [27]).

The case when the mixed term has only two variables follows immediately.

Proposition 4.37. *If $\gamma = 0$, then $|\det N_{a,b,c,\alpha,\beta,\gamma}|$ is*

$$\begin{aligned}
&\text{Mac}(\beta - A, A, M) \text{Mac}(\alpha - B, B, M) \\
&\quad \times \frac{\mathcal{H}(A+M)\mathcal{H}(B+M)\mathcal{H}(C+M)\mathcal{H}(A+B+C+M)}{\mathcal{H}(a)\mathcal{H}(b)\mathcal{H}(c)\mathcal{H}(M)}.
\end{aligned}$$

Thus, the type 2 ideal

$$I_{a,b,c,\alpha,\beta,0} = (x^a, y^b, z^c, x^\alpha y^\beta)$$

has the weak Lefschetz property if the characteristic of K is zero or at least $A + B + C + M$.

Proof. As $\gamma = 0$, $N = N_{a,b,c,\alpha,\beta,\gamma}$ has entries given by

$$(N)_{i,j} = \begin{cases} \binom{c}{A+j-i} & \text{if } 1 \leq i \leq C, \\ \left\{ \begin{array}{l} 1 \text{ if } j = i + \beta - A - C \\ 0 \text{ if } j \neq i + \beta - A - C \end{array} \right\} & \text{if } C + 1 \leq i \leq C + M. \end{cases}$$

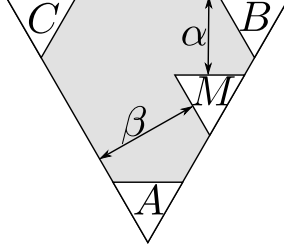


Figure 4.16: When γ is zero, the starting points A_{C+1}, \dots, A_{C+M} coincide with the M consecutive ending points $E_{A-\beta+1}, \dots, E_{A-\beta+M}$.

Further, if we define the matrix T by

$$(T)_{i,j} = \begin{cases} \binom{c}{A+j-i} & \text{if } 1 \leq j \leq \beta - A, \\ \binom{c}{A+M+j-i} & \text{if } \beta - A + 1 \leq j \leq C \end{cases},$$

then $|\det N| = |\det T|$ due to the structure of the lower-part of N . Thus, if we let $p = c, q = A, r = M, m = \beta - A$, and $n = C$, then by Lemma 4.35 we have the desired determinant evaluation.

Moreover, $\alpha + M$ and $\beta + M$ are smaller than $A + B + C + M$, so the prime divisors of $\det N$ are strictly bounded above by $A + B + C + M$. \square

Remark 4.38. Proposition 4.37 deserves a pair of comments:

- (i) The evaluation of the determinant includes two Mahonian terms and a third non-Mahonian term. It should be noted that both hexagons associated to the Mahonian terms actually show up in the punctured hexagon. See Figure 4.17

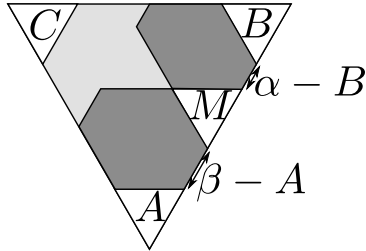


Figure 4.17: The darkly shaded hexagons correspond to the two Mahonian terms in the determinantal evaluation.

where the darkly shaded hexagons correspond to the Mahonian terms. It is not clear (to us) where the third term comes from, though it may be of interest to note that if one subtracts M from each hyperfactorial, before the evaluation, then what remains is $\text{Mac}(A, B, C)$.

- (ii) We notice the proposition also extends [38, Lemma 6.6] where it was shown that the associated almost complete intersection always has the weak Lefschetz property in characteristic zero (i.e., the determinant is non-zero). That is, all

level type 2 artinian monomial almost complete intersections in R have the weak Lefschetz property in characteristic zero.

Exploring symmetry

When $a = b$ (equivalently, $A = B$) and $\alpha = \beta$, then $H_{a,a,c,\alpha,\alpha,\gamma}$ is symmetric; see Figure 4.18. In this case, c is even exactly when $M = \frac{1}{3}(2a + c - 4\alpha - 2\gamma)$ is even; similarly, γ is even exactly when $C = \frac{1}{3}(2a - 2c + 2\alpha + \gamma)$ is even. Moreover, $\alpha = A + \frac{1}{2}(C - \gamma)$.

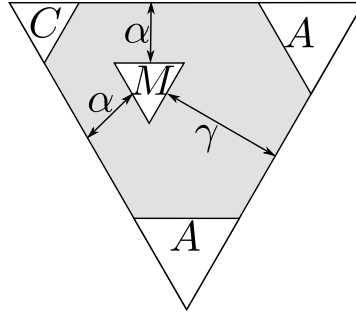


Figure 4.18: When $a = b$ and $\alpha = \beta$, then $H_{a,a,c,\alpha,\alpha,\gamma}$ is symmetric.

When C and M are odd, we can exploit symmetry to show $\det N_{a,a,c,\alpha,\alpha,\gamma}$ is 0. This result extends the evaluation in [38, Corollary 7.4] and offers a (more) direct combinatorial proof, rather than one based on linear algebra.

Proposition 4.39. *If c and γ are odd, $a = b$, and $\alpha = \beta$, then $H_{a,a,c,\alpha,\alpha,\gamma}$ is symmetric with an odd puncture (i.e., M odd; see Figure 4.18) and $\det N_{a,a,c,\alpha,\alpha,\gamma}$ is 0. Thus,*

$$I_{a,a,c,\alpha,\alpha,\gamma} = (x^a, y^a, z^c, x^\alpha y^\alpha z^\gamma)$$

never has the weak Lefschetz property, regardless of characteristic.

Proof. Recall the admissible partitions of $H_{a,a,c,\alpha,\alpha,\gamma}$ are $\lambda_0, \dots, \lambda_C$. For $0 \leq i \leq \frac{C-1}{2}$ we see that $P_{\lambda_i}^+(A \rightarrow E) = P_{\lambda_{C-i}}^+(A \rightarrow E)$ by symmetry, and further that $\text{sgn } \lambda_i = -\text{sgn } \lambda_{C-i}$, as $\text{sgn } \lambda_k = (-1)^{M(C-k)}$ and C is odd. Hence, $\det N_{a,b,c,\alpha,\beta,\gamma} = \sum_{i=0}^C \text{sgn } \lambda_i P_{\lambda_i}^+(A \rightarrow E) = 0$. \square

From the preceding proof we see that if we consider c even instead of c odd (i.e., M even instead of M odd), then $\det N_{a,a,c,\alpha,\alpha,\gamma}$ is even, when γ is odd (i.e., C is odd).

Recall the definitions of A, B, C , and M from Proposition 4.3, $H_{a,a,c,\alpha,\alpha,\gamma}$ from Theorem 4.11, and $N_{a,b,c,\alpha,\beta,\gamma}$ from Proposition 4.9. If C or M is even, then the region $H_{a,a,c,\alpha,\alpha,\gamma}$ is symmetric and we offer the following conjecture for a closed form for $\det N_{a,a,c,\alpha,\alpha,\gamma}$. Notice that in this case $\alpha = A + \frac{1}{2}(C - \gamma)$.

Conjecture 4.40. Suppose $a = b$ and $\alpha = \beta$ so $H_{a,a,c,\alpha,\alpha,\gamma}$ is symmetric. If c or γ is even, then $\det N_{a,b,c,\alpha,\beta,\gamma}$ is

$$\begin{aligned} & (-1)^{M \lceil \frac{C}{2} \rceil} \times \frac{\mathcal{H}(\lfloor \frac{M}{2} \rfloor) \mathcal{H}(\lfloor \frac{M}{2} \rfloor + A) \mathcal{H}(\lfloor \frac{M}{2} \rfloor + \frac{C+\gamma}{2}) \mathcal{H}(\lfloor \frac{M}{2} \rfloor + A + \frac{C-\gamma}{2})}{\mathcal{H}(\lfloor \frac{M+C}{2} \rfloor) \mathcal{H}(\lfloor \frac{M+\gamma}{2} \rfloor) \mathcal{H}(\lfloor \frac{M+C}{2} \rfloor + A) \mathcal{H}(\lfloor \frac{M-\gamma}{2} \rfloor + A)} \\ & \times \frac{\mathcal{H}(\lceil \frac{M}{2} \rceil) \mathcal{H}(\lceil \frac{M}{2} \rceil + A) \mathcal{H}(\lceil \frac{M}{2} \rceil + \frac{C+\gamma}{2}) \mathcal{H}(\lceil \frac{M}{2} \rceil + A + \frac{C-\gamma}{2})}{\mathcal{H}(\lceil \frac{M+C}{2} \rceil) \mathcal{H}(\lceil \frac{M+\gamma}{2} \rceil) \mathcal{H}(\lceil \frac{M+C}{2} \rceil + A) \mathcal{H}(\lceil \frac{M-\gamma}{2} \rceil + A)} \\ & \times \frac{\mathcal{H}(A - \lfloor \frac{\gamma}{2} \rfloor) \mathcal{H}(\lfloor \frac{C}{2} \rfloor) \mathcal{H}(\lfloor \frac{\gamma}{2} \rfloor) \mathcal{H}(A - \lceil \frac{\gamma}{2} \rceil) \mathcal{H}(\lceil \frac{C}{2} \rceil) \mathcal{H}(\lceil \frac{\gamma}{2} \rceil)}{\mathcal{H}(\gamma) \mathcal{H}^2(A + \frac{C-\gamma}{2})} \\ & \times \frac{\mathcal{H}(M+C) \mathcal{H}(M+\gamma) \mathcal{H}(M+A + \lfloor \frac{C}{2} \rfloor) \mathcal{H}(M+A + \lceil \frac{C}{2} \rceil) \mathcal{H}(M+2A+C)}{\mathcal{H}(M+2A) \mathcal{H}^2(M+A+C) \mathcal{H}^2(M + \frac{C+\gamma}{2})}. \end{aligned}$$

Further, the ideal

$$I_{a,a,c,\alpha,\alpha,\gamma} = (x^a, y^a, z^c, x^\alpha y^\alpha z^\gamma)$$

has the weak Lefschetz property when the characteristic of K is zero or at least $2A + C + M$.

Remark 4.41. The above symmetry conjecture deserves a few remarks.

- (i) Note that by Remark 4.26, $\det N_{a,b,c,\alpha,\beta,\gamma}$ is polynomial in M . Further, the conjectured form of the determinant would imply that the polynomial factors completely into linear terms and has degree $AC + \lfloor \frac{\gamma}{2}(C - \frac{\gamma}{2}) \rfloor$.
- (ii) If Conjecture 4.40 were shown to hold, then it would complete the (-1) -enumeration of symmetric punctured hexagons when combined with Proposition 4.39,
- (iii) As expected, the conjecture corresponds to Proposition 4.31 when $C = 0$, to Proposition 4.34 when $A = \frac{1}{2}\gamma$ (this implies $\alpha = \frac{1}{2}C$ and so $C = 2\alpha$, which is maximal), and to Proposition 4.37 when $\gamma = 0$. Moreover, when $A = C = \gamma$, then $H_{a,a,c,\alpha,\alpha,\gamma}$ has an *axis-central* puncture (see Section 4.5) and the conjecture corresponds to Corollary 4.46.

We give an example of using the symmetry conjecture.

Example 4.42. Consider $A = B = 8$, $C = 6$, $\gamma = 2$, and M even. Then $\alpha = \beta = 10$, $a = b = 14 + M$, and $c = 16 + M$. Moreover, the region $H_{14+M,14+M,16+M,10,10,2}$ is symmetric and does not fall into the case of Remark 4.41(iii).

Supposing Conjecture 4.40 holds, then the (-1) -enumeration of the punctured hexagon $H_{14+M,14+M,16+M,10,10,2}$ is

$$\begin{aligned} & \frac{1}{-2^{34} 3^{16} 5^{67} 6} \times (M+1)(M+3)^3(M+4)^2(M+5)^3(M+7)(M+2)^2(M+13)^4 \\ & \times (M+14)^6(M+15)^5(M+16)^6(M+17)^3(M+18)^4(M+19)(M+20)^2. \end{aligned}$$

Thus, $I_{14+M,14+M,16+M,10,10,2} = (x^{14+M}, y^{14+M}, z^{16+M}, x^{10}y^{10}z^2)$ has the weak Lefschetz property when the characteristic of the ground field is 0 or at least $M + 21$.

So far, in every case where we can bound the prime divisors of $\det N_{a,b,c,\alpha,\beta,\gamma}$ from above, we can do so linearly in the parameters (actually, always by at most $s + 2$). This may, however, not always be the case. We provide the following example to demonstrate that this is true, but also as a contrast to the symmetry conjecture, where some restrictions lead to a (conjectured) closed form.

Example 4.43. Consider the level and type 3 algebra given by R/I , where

$$I_{1+t,4+t,7+t,1,4,7} = (x^{1+t}, y^{4+t}, z^{7+t}, xy^4z^7)$$

and $t \geq 4$. By Remark 4.26, we have that $\det N = \det N_{1+t,4+t,7+t,1,4,7}$ is a polynomial in t . Hence we can use interpolation to determine the polynomial in terms of t ; in particular, $\det N_{1+t,4+t,7+t,1,4,7}$ is

$$\frac{4}{\mathcal{H}(7)}(t-3)(t-2)(t-1)^3t^3(t+1)^2(t+2)(t+4)(t+6)(t^2+6t-1)$$

if t is odd and otherwise is

$$\frac{4}{\mathcal{H}(7)}(t-2)^2(t-1)^2t^4(t+1)^2(t+2)(t+5)(t+7)(t^2+2t-9).$$

In 1857, Bouniakowsky conjectured that for every irreducible polynomial $f \in \mathbb{Z}[t]$ of degree at least 2 with common divisor $d = \gcd\{f(i) \mid i \in \mathbb{Z}\}$, there exists infinitely many integers t such that $\frac{1}{d}f(t)$ is prime. We note that the weaker Fifth Hardy-Littlewood conjecture, which states that $t^2 + 1$ is prime for infinitely many positive integers t , is a special case of the Bouniakowsky conjecture.

When t is odd, the determinant has the quadratic factor $t^2 + 6t - 1$. If we let $t = 2k + 1$, then this factor becomes $2(2k^2 + 8k + 3)$, which is an irreducible polynomial over $\mathbb{Z}[k]$ with common divisor 2 (when $k = 4$ then the polynomial evaluates to $134 = 2 \cdot 67$). Hence the quadratic factor of the determinant is prime for infinitely many odd integers t , assuming the Bouniakowsky conjecture. Similarly the quadratic factor of the determinant for t even is prime for infinitely many even integers t , again assuming the Bouniakowsky conjecture.

Hence, assuming the Bouniakowsky conjecture, for large enough t , the upper bound on the prime divisors of the determinant grows quadratically in t .

The above example falls in to the case of Proposition 4.50(ii)(a) or the second open case immediately following the proposition, depending on the parity of t .

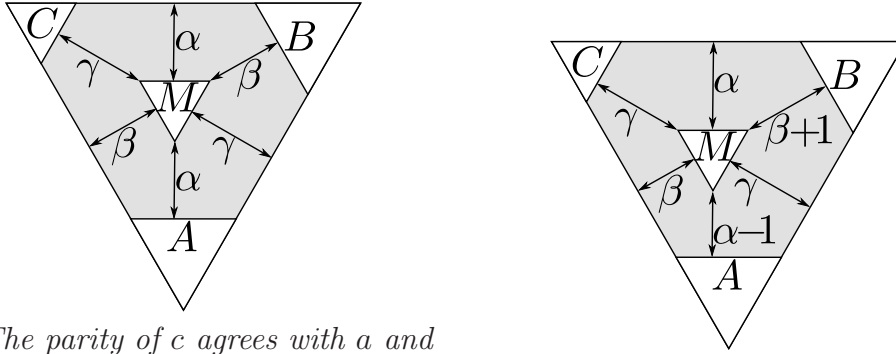
4.5 Centralising the puncture

In this section we consider two subtly different ways to centralise the puncture of a punctured hexagon. The first, *axis-central*, forces the puncture to be centered along each axis, individually. The second, *gravity-central*, forces the puncture to be the same distance, simultaneously, from the three sides of the hexagon that are parallel to the puncture-sides.

Throughout this section we assume, in addition to the conditions in Proposition 4.3 and $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$, that $I_{a,b,c,\alpha,\beta,\gamma}$ has type 3, that is, α , β , and γ are non-zero.

Axis-central

We define a punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$ to have an *axis-central* puncture if the puncture is “central” as defined in [7, Section 1]. Specifically, for each side of the puncture, the puncture-side should be the same distance from the parallel hexagon-side as the puncture-vertex opposite the puncture-side is from the other parallel hexagon-side; see Figure 4.19(i). However, when c has a different parity than both a and b , then an adjustment has to be made; in particular, translate the puncture parallel to the hexagon-side of length C one-half unit toward the side of length A ; see Figure 4.19(b).



(i) The parity of c agrees with a and b .

(ii) The parity of c differs from a and b .

Figure 4.19: A punctured hexagon with an axis-central puncture.

When $H_{a,b,c,\alpha,\beta,\gamma}$ has an axis-central puncture, then the ideal has a nice form. Suppose first that a , b , and c have the same parity. Then $\alpha = a - M - \alpha$ so $a = 2\alpha + M$; similarly, $b = 2\beta + M$ and $c = 2\gamma + M$. Thus, if we set $t = M$, then

$$I_{2\alpha+t, 2\beta+t, 2\gamma+t, \alpha, \beta, \gamma} = (x^{2\alpha+t}, y^{2\beta+t}, z^{2\gamma+t}, x^\alpha y^\beta z^\gamma).$$

The conditions in Proposition 4.3 simplify to $\alpha \leq \beta + \gamma$, $\beta \leq \alpha + \gamma$, $\gamma \leq \alpha + \beta$, and $t \geq 0$.

Now, suppose the parity of c differs from that of both a and b . Then $\alpha = a - M - \alpha + 1$, $\beta = b - M - \beta - 1$, and $\gamma = c - M - \gamma$, so $a = 2\alpha + M - 1$, $b = 2\beta + M + 1$, and $c = 2\gamma + M$. Thus, if we set $t = M$, then

$$I_{2\alpha+t-1, 2\beta+t+1, 2\gamma+t, \alpha, \beta, \gamma} = (x^{2\alpha+t-1}, y^{2\beta+t+1}, z^{2\gamma+t}, x^\alpha y^\beta z^\gamma).$$

The conditions in Proposition 4.3 simplify to $\alpha \leq \beta + \gamma + 1$, $\beta \leq \alpha + \gamma - 1$, $\gamma \leq \alpha + \beta$, and $t \geq 0$.

Much to our fortune, the determinants of $N_{a,b,c,\alpha,\beta,\gamma}$ have been calculated for punctured hexagons with axis-central punctures. We recall the four theorems here, although we forgo the exact statements of the determinant evaluations; the explicit evaluations can be found in [7]. Note, the abbreviations CEKZ comes from the four authors of [7]: Ciucu, Eisenkölbl, Krattenthaler, and Zare.

Theorem 4.44. [7, Theorems 1, 2, 4, & 5] *Let A, B, C , and M be non-negative integers and let H be the associated hexagon with an axis-central puncture. Then*

- (1) *The number of lozenge tilings of H is $\text{CEKZ}_1(A, B, C, M)$ if A, B , and C share a common parity.*
- (2) *The number of lozenge tilings of H is $\text{CEKZ}_2(A, B, C, M)$ if A, B , and C do not share a common parity.*
- (4) *The number of signed lozenge tilings of H is*
 - (i) $\text{CEKZ}_4(A, B, C, M)$ *if A, B , and C are all even, and*
 - (ii) 0 *if A, B , and C are all odd.*
- (5) *The number of signed lozenge tilings of H is $\text{CEKZ}_5(A, B, C, M)$ if A, B , and C do not share a common parity.*

Moreover, the four functions CEKZ_i are polynomials in M with integer coefficients which factor completely into linear terms. Further, each can be expressed as a quotient of products of hyperfactorials and, in each case, the largest hyperfactorial term is $\mathcal{H}(A + B + C + M)$.

Thus, we calculate the permanent of $Z_{a,b,c,\alpha,\beta,\gamma}$.

Corollary 4.45. *Let $H_{a,b,c,\alpha,\beta,\gamma}$ be a hexagon with an axis-central puncture. Then*

$$\text{perm } Z_{a,b,c,\alpha,\beta,\gamma} = \begin{cases} \text{CEKZ}_1(A, B, C, M) & \text{if } a, b, \text{ and } c \text{ share a common parity;} \\ \text{CEKZ}_2(A, B, C, M) & \text{otherwise.} \end{cases}$$

Proof. This follows from Proposition 4.16 and Theorem 4.44. □

Moreover, we calculate the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$, and thus can completely classify when the algebra $R/I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property.

Corollary 4.46. *Let $H_{a,b,c,\alpha,\beta,\gamma}$ be a hexagon with an axis-central puncture. If M is even, then*

$$\det N_{a,b,c,\alpha,\beta,\gamma} = \begin{cases} \text{CEKZ}_1(A, B, C, M) & \text{if } a, b, \text{ and } c \text{ have the same parity;} \\ \text{CEKZ}_2(A, B, C, M) & \text{otherwise.} \end{cases}$$

If M is odd, then

$$\det N_{a,b,c,\alpha,\beta,\gamma} = \begin{cases} \text{CEKZ}_4(A, B, C, M) & \text{if } a, b, c, \text{ and } s + 2 \text{ have the same parity;} \\ 0 & \text{if } a, b, \text{ and } c \text{ have the same parity} \\ & \text{different from the parity of } s + 2; \\ \text{CEKZ}_5(A, B, C, M) & \text{if } a, b, \text{ and } c \text{ do not have the same parity.} \end{cases}$$

Thus, $R/I_{a,b,c,\alpha,\beta,\gamma}$ always fails to have the weak Lefschetz property if a, b, c , and M are odd, regardless of the field characteristic. Otherwise, $R/I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if the field characteristic is zero or at least $A + B + C + M$.

Proof. This follows from Theorem 4.15 and Theorem 4.44. \square

As we will see in the following subsection, having a gravity-central puncture is equivalent to the associated algebra being level.

Question 4.47. Consider the punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$. Is there an algebraic property P of algebras such that $H_{a,b,c,\alpha,\beta,\gamma}$ has an axis-central puncture if and only if $R/I_{a,b,c,\alpha,\beta,\gamma}$ has property P ?

Gravity-central

We define a punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$ to have a *gravity-central* puncture if the vertices of the puncture are each the same distance from the perpendicular side of the hexagon; see Figure 4.20. That is, we have that $B + C - \alpha = A + C - \beta = A + B - \gamma$,

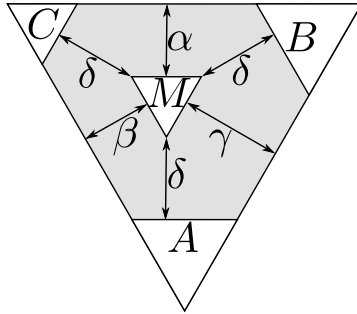


Figure 4.20: A punctured hexagon with a gravity-central puncture.

which simplifies to the relation $a - \alpha = b - \beta = c - \gamma$, and this is exactly the condition in Proposition 4.1(ii) for $R/I_{a,b,c,\alpha,\beta,\gamma}$ to be level and type 3. Thus, if we let t be this common difference, then we can rewrite $I_{a,b,c,\alpha,\beta,\gamma}$ as

$$I_{\alpha+t,\beta+t,\gamma+t,\alpha,\beta,\gamma} = (x^{\alpha+t}, y^{\beta+t}, z^{\gamma+t}, x^\alpha y^\beta z^\gamma).$$

Without loss of generality, assume $0 < \alpha \leq \beta \leq \gamma$. Then the conditions in Proposition 4.3 simplify to $t \geq \frac{1}{3}(\alpha + \beta + \gamma)$ and $\gamma \leq 2(\alpha + \beta)$.

The ideals $I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ are studied extensively in [38, Sections 6 & 7]. In particular, [38, Conjecture 6.8] makes a guess as to when $R/I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ has the weak Lefschetz property in characteristic zero. We recall the conjecture here, though we present it in a different but equivalent form.

Conjecture 4.48. *Consider the ideal $I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ in R where K has characteristic zero, $0 < \alpha \leq \beta \leq \gamma \leq 2(\alpha + \beta)$, $t \geq \frac{1}{3}(\alpha + \beta + \gamma)$, and $\alpha + \beta + \gamma$ is divisible by three.*

If $(\alpha, \beta, \gamma, t)$ is not $(2, 9, 13, 9)$ or $(3, 7, 14, 9)$, then $R/I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ fails to have the weak Lefschetz property if and only if t is even, $\alpha + \beta + \gamma$ is odd, and $\alpha = \beta$ or $\beta = \gamma$.

Remark 4.49. [38, Conjecture 6.8] is presented in a format that does not elucidate the reasoning behind it. We present the conjecture differently so it says that the weak Lefschetz property fails in two exceptional cases and also when a pair of parity conditions and a symmetry condition hold.

We add further support to the conjecture.

Proposition 4.50. *Let $I = I_{\alpha+t, \beta+t, \gamma+t, \alpha, \beta, \gamma}$ be as in Conjecture 4.48. Then*

- (i) *R/I fails to have the weak Lefschetz property when t is even, $\alpha + \beta + \gamma$ is odd, and $\alpha = \beta$ or $\beta = \gamma$;*
- (ii) *R/I has the weak Lefschetz property when*
 - a) *t and $\alpha + \beta + \gamma$ have the same parity, or*
 - b) *t is odd and $\alpha = \beta = \gamma$ is even.*

Proof. Part (i) follows from Proposition 4.39 (also by [38, Corollary 7.4]). Part (ii)(a) implies M is even and so follows by Theorem 4.28. Part (ii)(b) follows from [7, Theorem 4], which is recalled here in Theorem 4.44(4)(i). \square

We note that Conjecture 4.48 remains open in two cases, both of which are conjectured to have the weak Lefschetz property:

- (i) t even, $\alpha + \beta + \gamma$ odd, and $\alpha < \beta < \gamma$;
- (ii) t odd, $\alpha + \beta + \gamma$ even, and $\alpha < \beta$ or $\beta < \gamma$.

Remark 4.51. Notice that the second open case in the above statement is solved for the cases when $\alpha = \beta$ or $\beta = \gamma$ if Conjecture 4.40 is true.

Axis- and gravity-central

Suppose a, b , and c have the same parity. Then the punctured hexagons that are both axis- and gravity-central are precisely those with $a = b = c = \alpha + t$ and $\alpha = \beta = \gamma$. In this case, we strengthen [38, Corollary 7.6].

Corollary 4.52. *Consider the level, type 3 algebra A given by*

$$R/I_{\alpha+t, \alpha+t, \alpha+t, \alpha, \alpha, \alpha} = R/(x^{\alpha+t}, y^{\alpha+t}, z^{\alpha+t}, x^\alpha y^\alpha z^\alpha),$$

where $t \geq \alpha$. Then A fails to have the weak Lefschetz property in characteristic zero if and only if α is odd and t is even.

In [28], Krattenthaler described a bijection between cyclically symmetric lozenge tilings of the punctured hexagon considered in the previous corollary and descending plane partitions with specified conditions.

If c has a different parity than a and b , then $\alpha - 1 = \beta + 1 = \gamma$. Thus for $\alpha \geq 3$ and M non-negative we have that the ideals of the form

$$I_{2\alpha+M, 2\alpha+M-2, 2\alpha+M-1, \alpha, \alpha, \alpha} = (x^{2\alpha+M}, y^{2\alpha+M-2}, z^{2\alpha+M-1}, x^\alpha y^{\alpha-2} z^{\alpha-1}),$$

are precisely those that are both axis- and gravity-central.

4.6 Interesting families and examples

In this section, we give several interesting families and examples.

Large prime divisors

Throughout the two preceding sections, when we could bound the prime divisors of $\det N$ above, we bounded them above by (at most) $s + 2$. However, this need not always be the case, as demonstrated in Example 4.43. We provide here a few exceptional-looking though surprisingly common cases.

Example 4.53. Recall that $s + 2 = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$. In each case, we specify the parameter set by a sextuple $(a, b, c, \alpha, \beta, \gamma)$.

- (i) Consider the parameter set $(4, 6, 6, 1, 1, 3)$. Then $s + 2 = 7$ and $\det N = 11$. This is the smallest $s + 2$ so that $\det N$ has a prime divisor greater than $s + 2$.
- (ii) For the parameter set $(20, 20, 20, 3, 8, 13)$, we get $s + 2 = 28$ and

$$\det N = 2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 17^2 \cdot 19^6 \cdot 23^5 \cdot 20554657.$$

Hence $\det N$ is divisible by a prime that is over 700000 times large than $s + 2$. Moreover, 20554657 is greater than the multiplicity of the associated algebra.

- (iii) Consider the parameter set $(7, 12, 13, 1, 7, 2)$. Then $s + 2 = 14$ and $\det N = 13 \cdot 17 \cdot 23$. This is the smallest $s + 2$ so that $\det N$ has more than one prime divisor greater than $s + 2$.
- (iv) Last, for the parameter set $(8, 12, 15, 2, 8, 5)$, we get $s + 2 = 17$ and $\det N = 2 \cdot 11 \cdot 13^2 \cdot 179 \cdot 197$. In this case, notice that $\det N$ has two prime divisors both greater than $a + b + c + \alpha + \beta + \gamma$, the sum of the generating degrees of $R/I_{a,b,c,\alpha,\beta,\gamma}$.

Given the previous example and Example 4.43, it seems unlikely that there is a reasonably simple closed formula for the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$ in general, as opposed to the case of a symmetric region (see Conjecture 4.40).

Fixed determinants

For any positive integer n , there is an infinite family of punctured hexagons with exactly n tilings. Note the algebras are type 2 if β is zero or $c = n + \beta + 1$ and type 3 otherwise.

Proposition 4.54. *Let n be a positive integer. If $\beta \geq 0$ and $c \geq n + \beta + 1$, then*

$$\det N_{c-\beta-1,\beta+2,c,c-n-\beta-1,\beta,n} = n.$$

Hence the ideal

$$I_{c-\beta-1,\beta+2,c,c-n-\beta-1,\beta,n} = (x^{c-\beta-1}, y^{\beta+2}, z^c, x^{c-n-\beta-1}y^\beta z^n)$$

has the weak Lefschetz property when the characteristic of K is either zero or not a prime divisor of n .

Proof. In this case, $s = c - 2$, $A = \beta + 1$, $B = c - \beta - 2$, $C = 0$, and $M = 1$.

Using Proposition 4.31 we have that

$$\det N_{c-\beta-1,\beta+2,c,c-n-\beta-1,\beta,n} = \text{Mac}(M, A - \beta, B - \alpha) = \text{Mac}(1, 1, n - 1) = n.$$

Alternatively, from Proposition 4.9 we have that

$$N_{c-\beta-1,\beta+2,c,c-n-\beta-1,\beta,n} = \left(\binom{\gamma}{A + C - \beta} \right) = \left(\binom{n}{1} \right) = (n).$$

Clearly then the determinant is n . □

Thus for any prime p , Proposition 4.54 provides infinitely many monomial almost complete intersections that fail to have the weak Lefschetz property exactly when the field characteristic is p .

A result of Proposition 4.54 is an infinite (in fact, two dimensional) family whose members have a unique tiling. Note that the algebras are type 2 if β is zero or $c = \beta + 2$ and type 3 otherwise.

Corollary 4.55. *If $\beta \geq 0$ and $c \geq \beta + 2$, then $\det N_{c-\beta-1, \beta+2, c, c-\beta-2, \beta, 1}$ is 1. That is,*

$$I_{a,b,c,\alpha,\beta,\gamma} = (x^{c-\beta-1}, y^{\beta+2}, z^c, x^{c-\beta-2}y^\beta z)$$

has the weak Lefschetz property independent of the field characteristic.

Another family whose members have a unique tiling comes from Proposition 4.37. Note that it is a three dimensional family but also that all of the associated algebras are type 2.

Proposition 4.56. *If $a = b = \alpha + \beta + c$ and $\gamma = 0$, then $A = B = 0$ (see Figure 4.21) and $\det N_{a,b,c,\alpha,\beta,\gamma}$ is 1. That is,*

$$I_{a,b,c,\alpha,\beta,\gamma} = (x^{\alpha+\beta+c}, y^{\alpha+\beta+c}, z^c, x^\alpha y^\beta)$$

has the weak Lefschetz property independent of the field characteristic.

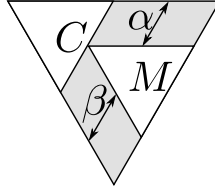


Figure 4.21: When $A = B = \gamma = 0$, then $H_{a,b,c,\alpha,\beta,\gamma}$ has a unique tiling.

Proof. This follows from Proposition 4.37. □

Several questions were asked in [38], two of which we can answer in the affirmative.

Remark 4.57. Question 8.2(2c) asked if there exist non-level almost complete intersections which never have the weak Lefschetz property. The almost complete intersection

$$R/I_{5,5,3,2,2,1} = R/(x^5, y^5, z^3, x^2y^2z)$$

is non-level and never has the weak Lefschetz property, regardless of field characteristic, as $\det N_{5,5,3,2,2,1} = 0$ by Proposition 4.39.

Further, we notice here that Question 8.2(2b) in [38] is answered in the affirmative by the comments following Question 7.12 in [38]. In particular, $I_{11,18,22,2,9,13}$ is a level almost complete intersection which has odd socle degree (39) and never has the weak Lefschetz property, as $\det N_{11,18,22,2,9,13} = 0$.

Minimal multiplicity

The Huneke-Srinivasan Multiplicity Conjecture, which was proven by Eisenbud and Schreyer [14, Corollary 0.3], shows that the multiplicity of a Cohen-Macaulay module gives nice bounds on the possible shifts of the Betti numbers. Moreover, as the algebras A can be viewed as finite dimensional vector spaces, then the multiplicity *is* the dimension of A as a vector space. Thus, algebras that have minimal multiplicity while retaining a particular property are the smallest, in the above sense, examples one can generate.

Example 4.58. Possibly of interest are a few cases of minimal multiplicity with regard to the weak Lefschetz property.

The following examples never have the weak Lefschetz property, that is, the determinant of their associated matrix $N_{a,b,c,\alpha,\beta,\gamma}$ is 0. Note that both examples are type 3.

- (i) The unique level ideal with minimal multiplicity is

$$I_{3,3,3,1,1,1} = (x^3, y^3, z^3, xyz).$$

Its Hilbert function is $(1, 3, 6, 6, 3)$ and so it has multiplicity 19. It is worth noting that this ideal is extensively studied in [4, Example 3.1] and is the basis for an exploration of the subtlety of the Lefschetz properties in Chapter 5.

- (ii) The unique non-level ideal with minimal multiplicity is

$$I_{5,5,3,2,2,1} = (x^5, y^5, z^3, x^2y^2z).$$

Its Hilbert function is $(1, 3, 6, 9, 12, 12, 9, 4, 1)$ and so it has multiplicity 57. Further, this ideal is the example given in Remark 4.57.

Moreover, the following examples always have the weak Lefschetz property, regardless of the base field characteristic. That is to say, the determinant of their associated matrix $N_{a,b,c,\alpha,\beta,\gamma}$ is 1.

- (i) The two level ideals with minimal multiplicity are

$$I_{1,2,3,0,1,2} = (x, y^2, z^3, yz^2) \text{ and } I_{1,3,3,0,1,1} = (x, y^3, z^3, yz).$$

Both ideals have Hilbert function $(1, 2, 2)$ and thus multiplicity 5. However, both ideals are isomorphic to ideals in $K[y, z]$.

- (ii) The unique level, type 2 ideal without x as a generator and with minimal multiplicity is

$$I_{2,2,3,1,1,0} = (x^2, y^2, z^3, xy).$$

Its Hilbert function is $(1, 3, 3, 2)$ and so it has multiplicity 9.

(iii) The unique level, type 3 ideal with minimal multiplicity is

$$I_{3,3,6,1,1,4} = (x^3, y^3, z^6, xyz^4).$$

Its Hilbert function is $(1, 3, 6, 8, 9, 9, 7, 3)$ and so it has multiplicity 46.

(iv) The unique non-level, type 2 ideal with minimal multiplicity is

$$I_{2,2,3,0,1,1} = (x^2, y^2, z^3, yz).$$

Its Hilbert function is $(1, 3, 3, 1)$ and so it has multiplicity 8.

(v) The unique non-level, type 3 ideal with minimal multiplicity is

$$I_{2,2,4,1,1,2} = (x^2, y^2, z^4, xyz^2).$$

Its Hilbert function is $(1, 3, 4, 4, 2)$ and so it has multiplicity 14.

Notice that example (ii) and (iv) in the above enumeration differ only slightly in the mixed term yet one is level and the other is not. It should also be noted that all of the above examples were found via an exhaustive search in the finite space of possible ideals using Macaulay2 [18].

4.7 Splitting type and regularity

Throughout this section we assume K is an algebraically closed field of characteristic zero.

Recall the definition of the ideals given in Section 4.1; consider

$$I = I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma),$$

where $0 \leq \alpha < a, 0 \leq \beta < b, 0 \leq \gamma < c$, and at most one of α, β , and γ is zero. In this section we consider the splitting type of the syzygy bundles of the artinian algebras R/I , regardless of any extra conditions on the parameters.

Recall, also from Section 4.1, that the syzygy module $\text{syz } I$ of I is defined by the exact sequence

$$0 \longrightarrow \text{syz } I \longrightarrow R(-\alpha - \beta - \gamma) \oplus R(-a) \oplus R(-b) \oplus R(-c) \longrightarrow I_{a,b,c,\alpha,\beta,\gamma} \longrightarrow 0$$

and the syzygy bundle $\widetilde{\text{syz } I}$ on \mathbb{P}^2 of I is the sheafification of $\text{syz } I$. Its restriction to the line $H \cong \mathbb{P}^1$ defined by $\ell = x + y + z$ splits as $\mathcal{O}_H(-p) \oplus \mathcal{O}_H(-q) \oplus \mathcal{O}_H(-r)$. The arguments in [38, Proposition 2.2] (recalled here in Proposition 2.12) imply that (p, q, r) is the splitting type of the restriction of $\widetilde{\text{syz } I}$ to a general line. Thus, we call (p, q, r) the *generic splitting type* of $\text{syz } I$.

In order to compute the generic splitting type of $\text{syz } I$, we use the observation that $R/(I, \ell) \cong S/J$, where $S = K[x, y]$, and $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$. Define $\text{syz } J$ by the exact sequence

$$0 \longrightarrow \text{syz } J \longrightarrow S(-\alpha - \beta - \gamma) \oplus S(-a) \oplus S(-b) \oplus S(-c) \longrightarrow J \longrightarrow 0 \quad (4.2)$$

using the possibly non-minimal set of generators $\{x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma\}$ of J . Then $\text{syz } J \cong S(-p) \oplus S(-q) \oplus S(-r)$. The Castelnuovo-Mumford regularity of a homogeneous ideal I is denoted by $\text{reg } I$.

Remark 4.59. For later use, we record the following facts on the generic splitting type (p, q, r) of $\text{syz } I_{a,b,c,\alpha,\beta,\gamma}$.

- (i) As the sequence in (4.2) is exact, we see that $p + q + r = a + b + c + \alpha + \beta + \gamma$.
- (ii) Further, if any of the generators of J are extraneous, then the degree of that generator is one of p, q , and r .
- (iii) As regularity can be read from the Betti numbers of R/J , we get that $\text{reg } J+1 = \max\{p, q, r\}$.

Before moving on, we prove a useful lemma.

Lemma 4.60. *Let $S = K[x, y]$, where K is a field of characteristic zero, and let a, b, α, β , and γ be non-negative integers with $\alpha + \beta + \gamma < a + b$. Without loss of generality, assume that $0 < a - \alpha \leq b - \beta$. Then $\text{reg}(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma)$ is*

$$\begin{cases} a + \beta + \gamma - 1 & \text{if } \alpha = 0 \text{ and } 0 < \gamma \leq b - \beta - a; \\ \alpha + b - 1 & \text{if } 0 < \alpha, \gamma \leq b - \beta + \alpha - a, \text{ and } 0 < \beta \text{ or } 0 < \gamma; \\ \lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \rceil - 1 & \text{if } \gamma > b - \beta + \alpha - a. \end{cases}$$

Further still, we always have $\text{reg}(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma) \leq \lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \rceil - 1$.

Proof. We proceed in three steps.

First, consider $\gamma = 0$, $0 < \alpha$, and $0 < \beta$. Then by the form of the minimal free resolution of the quotient algebra $S/(x^a, y^b, x^\alpha y^\beta)$ we have that $\text{reg}(x^a, y^b, x^\alpha y^\beta) = \alpha + b - 1$.

Second, consider $\gamma > 0$ and $\alpha = \beta = 0$. By [22, Proposition 4.4], the algebra $S/(x^a, y^b)$ has the strong Lefschetz property in characteristic zero. Thus the Hilbert function of $S/(x^a, y^b, (x+y)^\gamma)$ is

$$\dim_K [S/(x^a, y^b, (x+y)^\gamma)]_j = \max\{0, \dim_K [S/(x^a, y^b)]_j - \dim_K [S/(x^a, y^b)]_{j-\gamma}\}.$$

By analysing when the difference becomes non-positive, we get that the regularity is $a + \gamma - 1$ if $\gamma \leq b - a$ and $\lceil \frac{1}{2}(a + b + \gamma) \rceil - 1$ if $\gamma > b - a$.

Third, consider $\gamma > 0$ and $0 < \alpha$ or $0 < \beta$. Notice that

$$(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma) : x^\alpha y^\beta = (x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma).$$

Considering the short exact sequence (with end terms truncated)

$$[S/(x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma)](-\alpha - \beta) \xrightarrow{\times x^\alpha y^\beta} S/(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma) \rightarrow S/(x^a, y^b, x^\alpha y^\beta),$$

where the first map is multiplication by $x^\alpha y^\beta$, we obtain

$$\text{reg}(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma) = \max\{\alpha + \beta + \text{reg}(x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma), \text{reg}(x^a, y^b, x^\alpha y^\beta)\}.$$

The claims then follows by simple case analysis. \square

Recall that the semistability of $\text{syz } I_{a,b,c,\alpha,\beta,\gamma}$ is completely determined by the parameters $a, b, c, \alpha, \beta, \gamma$ in Proposition 4.3.

Non-semistable syzygy bundle

We first consider the case when the syzygy bundle is not semistable. We distinguish three cases. It turns out that in two cases, at least one of the generators of J is extraneous.

Proposition 4.61. *Let K be a field of characteristic zero and suppose $I = I_{a,b,c,\alpha,\beta,\gamma}$ is an ideal of R . Let $J = (x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma)$ be an ideal of S . We assume, without loss of generality, that $a \leq b \leq c$ so that $C \leq B \leq A$.*

- (i) *If $M < 0$, then the generator $x^\alpha y^\beta (x+y)^\gamma$ of J is extraneous. The generic splitting type of $\text{syz } I$ is $(a+c, b, \alpha+\beta+\gamma)$ if $c \leq b-a$; otherwise the generic splitting type of $\text{syz } I$ is $(\lfloor \frac{1}{2}(a+b+c) \rfloor, \lceil \frac{1}{2}(a+b+c) \rceil, \alpha+\beta+\gamma)$.*
- (ii) *If $M \geq 0$ and $C < 0$, then the generator $(x+y)^c$ of J is extraneous. The generic splitting type of $\text{syz } I$ is $(a+b+\alpha+\beta+\gamma-r-1, r+1, c)$, where $r = \text{reg}(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma)$ (which is given in Lemma 4.60).*
- (iii) *If $M \geq 0, C \geq 0$, and $A > \beta+\gamma$, then the only destabilising sub-bundle of $\text{syz } I$ is $\text{syz}(x^a, x^\alpha y^\beta z^\gamma)$ and so the generic splitting type of $\text{syz } I$ is $(\lfloor \frac{1}{2}(\alpha+b+c) \rfloor, \lceil \frac{1}{2}(\alpha+b+c) \rceil, a+\beta+\gamma)$.*

Proof. Assume $M < 0$, then $\frac{1}{2}(a+b+c) < \alpha+\beta+\gamma$ and when $c \geq a+b$ then

$$a+b-1 \leq \frac{1}{2}(a+b+c) - 1 < \alpha+\beta+\gamma.$$

By Lemma 4.60 the regularity of $(x^a, y^b, (x+y)^c)$ is $a+b-1$ when $c \geq a+b$ and $\lfloor \frac{1}{2}(a+b+c) \rfloor - 1$ otherwise; hence we have that $x^\alpha y^\beta (x+y)^\gamma$ is contained in $(x^a, y^b, (x+y)^c)$ and the first claim follows.

Assume $M \geq 0$ and $C < 0$, then $2(\alpha+\beta+\gamma) \leq a+b+c$, $c \geq \frac{1}{2}(a+b+\alpha+\beta+\gamma)$, and when $\alpha+\beta+\gamma \geq a+b$ then $2(\alpha+\beta+\gamma) \leq a+b+c$ implies $c \geq a+b$. By Lemma 4.60, the regularity of $(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma)$ is $a+b-1$ if $\alpha+\beta+\gamma \geq a+b$ and at most $\lfloor \frac{1}{2}(a+b+\alpha+\beta+\gamma) \rfloor - 1$ otherwise; hence we have that $(x+y)^c$ is contained in $(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma)$ and the second claim follows.

Last, assume $M \geq 0, C \geq 0$, and $A > \beta+\gamma$. Note that since $A+B+C = \alpha+\beta+\gamma$ we then have that $B+C < \alpha$ and, in particular, $B < \alpha+\gamma$ and $C < \alpha+\beta$. Using Brenner's combinatorial criterion for the semi-stability of syzygy bundles of monomial ideals (see [3, Corollary 6.4]), we see that that $\mathcal{S} = \text{syz}(x^a, x^\alpha y^\beta z^\gamma) \cong R(-r)$, where $r = a+\beta+\gamma$, is the only destabilising sub-bundle of $\text{syz } I$. Further, $(\text{syz } I)/\mathcal{S}$ is a semistable rank two vector bundle, so by Grauert-Müllich theorem, the quotient has

generic splitting type (p, q) where $0 \leq q - p \leq 1$. Thus, if we consider the short exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \text{syz } I \longrightarrow (\text{syz } I)/\mathcal{S} \longrightarrow 0,$$

then the third claim follows after restricting to ℓ . \square

In the third case, when $A > \beta + \gamma$, the associated ideal $J \subset S$ may be minimally generated by four polynomials, unlike in the other two cases.

Example 4.62. Consider the ideals

$$I_{4,5,5,3,1,1} = (x^4, y^5, z^5, x^3yz) \text{ and } J = (x^4, y^5, (x+y)^5, x^3y(x+y))$$

in R and S , respectively. Notice that in this case, $0 \leq C \leq B \leq A$, $0 \leq M$, and $A > \beta + \gamma$ so the syzygy bundle of $R/I_{4,5,5,3,1,1}$ is non-semistable and its generic splitting type is determined in Proposition 4.61(iii). Further, J is minimally generated by the four polynomials x^4 , y^5 , $xy^3(2x+y)$, and x^3y^2 .

Semistable syzygy bundle

Order the entries of the generic splitting type (p, q, r) of the semistable syzygy bundle $\widetilde{\text{syz } I}$ such that $p \leq q \leq r$. Then by Grauert-Mülich theorem we have that $r - q$ and $q - p$ are both non-negative and at most 1. Moreover, [4, Theorem 2.2] specialises in our case.

Theorem 4.63. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$. If R/I has the weak Lefschetz property, then $p = q$ or $q = r$ and $r - p \leq 1$; otherwise $q = p + 1$ and $r = p + 2$.*

When $a + b + c + \alpha + \beta + \gamma \not\equiv 0 \pmod{3}$, then the generic splitting type of $\text{syz } I$ and regularity of J can be computed easily.

Proposition 4.64. *Let $R = K[x, y, z]$ where K is a field of characteristic zero. Suppose $I = I_{a,b,c,\alpha,\beta,\gamma}$ is an ideal of R with a semistable syzygy bundle and let $J = (x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma)$ be an ideal of S . Let $k = \lfloor \frac{1}{3}(a + b + c + \alpha + \beta + \gamma) \rfloor$. Then $\text{reg } J = k$ and the generic splitting type of $\text{syz } I$ is*

$$\begin{cases} (k, k, k+1) & \text{if } a + b + c + \alpha + \beta + \gamma = 3k + 1, \text{ and} \\ (k, k+1, k+1) & \text{if } a + b + c + \alpha + \beta + \gamma = 3k + 2. \end{cases}$$

Proof. Let (p, q, r) be the generic splitting type of $\text{syz } I$, so $a + b + c + \alpha + \beta + \gamma = 3(s+2) = p + q + r$. By Proposition 4.2, R/I has the weak Lefschetz property so $p = q$, $q = r$, and $r - p \leq 1$. Clearly if $p = q = r$ then $p + q + r = 3p$ is 0 modulo 3 so cannot be $a + b + c + \alpha + \beta + \gamma$.

If $p = q < r$, then $r = p + 1$ and $p + q + r = 3p + 1$. This matches the case when $a + b + c + \alpha + \beta + \gamma = 3k + 1$, so $p = k$ and the splitting type of $\text{syz } I$ is $(k, k, k + 1)$.

Similarly, if $p < q = r$, then $q = r = p + 1$ and $p + q + r = 3p + 2$. This matches the case when $a + b + c + \alpha + \beta + \gamma = 3k + 2$, so $p = k$ and the splitting type of $\text{syz } I$ is $(k, k + 1, k + 1)$.

In both cases, we have that $k - 1 \leq \text{reg } J \leq k$ by Remark 4.59(iii). However, we see that $\dim_K [R/I]_{k-2} < \dim_K [R/I]_{k-1}$ so $\dim_K [R/(I, x + y + z)]_{k-1} = \dim_K [S/J]_{k-1} > 0$ and thus $\text{reg } J > k - 1$. Hence $\text{reg } J = k$. \square

The generic splitting type of $I_{a,b,c,\alpha,\beta,\gamma}$, when the ideal is associated to a punctured hexagon, depends on the ideal having the weak Lefschetz property.

Proposition 4.65. *Let $R = K[x, y, z]$ where K is a field of characteristic zero. Suppose $I = I_{a,b,c,\alpha,\beta,\gamma}$ is an ideal of R with a semistable syzygy bundle (see Proposition 4.3) and $a + b + c + \alpha + \beta + \gamma \equiv 0 \pmod{3}$. Let $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ be an ideal of S and let $s + 2 = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$. Then*

- (i) *If R/I has the weak Lefschetz property, then the generic splitting type of $\text{syz } I$ is $(s + 2, s + 2, s + 2)$ and $\text{reg } J = s + 1$.*
- (ii) *If R/I does not have the weak Lefschetz property, then the generic splitting type of $\text{syz } I$ is $(s + 1, s + 2, s + 3)$ and $\text{reg } J = s + 2$.*

Proof. Let (p, q, r) be the generic splitting type of $\text{syz } I$, so $a + b + c + \alpha + \beta + \gamma = 3(s + 2) = p + q + r$.

Assume that R/I has the weak Lefschetz property. Suppose $p \neq q$, then $q = r = p + 1$ and $p + q + r = 3p + 2$, similarly, if $q \neq r$, then $p = q$ and $r = p + 1$ so $p + q + r = 3p + 1$; neither case is 0 modulo 3, hence cannot be $3(s + 2)$. Thus $p = q = r = s + 2$. Further we then see that $\text{reg } J = s + 1$ by Remark 4.59(iii).

Now assume R/I fails to have the weak Lefschetz property. Then $p + q + r = 3p + 3 = 3(s + 2)$ so $p + 1 = s + 2$ and $p = s + 1$. Thus, the generic splitting type of $\text{syz } I$ must be $(s + 1, s + 2, s + 3)$. As R/I has twin-peaks at $s + 1$ and $s + 2$ by Corollary 4.7, we see that $\text{reg } J \leq s + 1$ if and only if R/I has the weak Lefschetz property; so $\text{reg } J \geq s + 2$. However, by Remark 4.59(iii) we have that $\text{reg } J + 1 \leq s + 3$ so $\text{reg } J \leq s + 2$, hence $\text{reg } J = s + 2$. \square

This proposition can be combined with the results in the previous sections to compute the generic splitting type of many of syzygy bundles of the artinian algebras $R/I_{a,b,c,\alpha,\beta,\gamma}$.

Example 4.66. Consider the ideal $I_{7,7,7,3,3,3} = (x^7, y^7, z^7, x^3 y^3 z^3)$ which never has the weak Lefschetz property, by Proposition 4.39. The generic splitting type of $\text{syz } I_{7,7,7,3,3,3}$ is $(9, 10, 11)$. Notice that the similar ideal $I_{6,7,8,3,3,3} = (x^6, y^7, z^8, x^3 y^3 z^3)$ has the weak Lefschetz property in characteristic zero as $\det N_{6,7,8,3,3,3} = -1764$ and the generic splitting type of $\text{syz } I_{6,7,8,3,3,3}$ is $(10, 10, 10)$.

If $I = I_{a,b,c,\alpha,\beta,\gamma}$ is not associated to a punctured hexagon, then we have seen in Proposition 4.2 and Corollary 4.4 that R/I has the weak Lefschetz property in characteristic zero. We summarise part of our results by pointing out that in the case

when I is associated to a punctured hexagon then deciding the presence of the weak Lefschetz property is equivalent to determining other invariants of the algebra.

Theorem 4.67. *Let $R = K[x, y, z]$ where K is a field of arbitrary characteristic. Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be associated to a punctured hexagon; in particular, $a+b+c+\alpha+\beta+\gamma \equiv 0 \pmod{3}$ and $\text{syz } I$ is semistable (see Proposition 4.3). Set $s = \frac{1}{3}(a+b+c+\alpha+\beta+\gamma) - 2$.*

Then the following conditions are equivalent:

- (i) *The algebra R/I has the weak Lefschetz property;*
- (ii) *the regularity of S/J is s ;*
- (iii) *the determinant of $N_{a,b,c,\alpha,\beta,\gamma}$ (i.e., the enumeration of signed lozenge tilings of the punctured hexagon $H_{a,b,c,\alpha,\beta,\gamma}$) modulo the characteristic of K is non-zero; and*
- (iv) *the determinant of $Z_{a,b,c,\alpha,\beta,\gamma}$ (i.e., the enumeration of signed perfect matchings of the bipartite graph associated to $H_{a,b,c,\alpha,\beta,\gamma}$) modulo the characteristic of K is non-zero.*

Moreover, if the characteristic of K is zero, then there is one further equivalent condition:

- (v) *The generic splitting type of $\text{syz } I$ is $(s+2, s+2, s+2)$.*

Proof. Combine Corollary 4.7, Propositions 4.8 and 4.9, Theorems 4.15 and 4.18, and Proposition 4.65. □

This relates the weak Lefschetz property to a number of other problems in algebra, combinatorics, and algebraic geometry.

Jumping lines

Recall that a *jumping line* is a line, $L = 0$, over which the syzygy bundle splits differently than in the generic case, $x + y + z = 0$. Since $I = I_{a,b,c,\alpha,\beta,\gamma}$ is a monomial ideal it is sufficient to consider the two cases $z = 0$ and $y + z = 0$.

Proposition 4.68. *Let $R = K[x, y, z]$ where K is a field of characteristic zero and let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be an ideal of R . The splitting type of $\text{syz } I$ on the line $z = 0$ is $(c, \alpha+b, a+\beta)$ if $\gamma = 0$ and $(c, \alpha+\beta+\gamma, a+b)$ if $\gamma > 0$. And the splitting type of $\text{syz } I$ on the line $y + z = 0$ is $(c, a+\beta+\gamma, \alpha+b)$ if $\beta + \gamma < b \leq c$ and $(c, \alpha + \beta + \gamma, a + b)$ if $b \leq \min\{c, \beta + \gamma\}$.*

Proof. All four cases follow immediately by analysing the monomial algebra S/J isomorphic to $R/(I, L)$, where $L = 0$ is the splitting line, and using Lemma 4.60 to compute the regularities. □

Chapter 5 Subtlety to perturbation

Herein we propose a systematic way of deforming a monomial ideal without the weak Lefschetz property to an ideal with the weak Lefschetz property (in almost every characteristic) and the same Hilbert function as the original ideal. This could potentially be useful, for example, if one expects an ideal to have a unimodal Hilbert function. Indeed, showing that the deformed ideal has the weak Lefschetz property would then imply the desired unimodality.

The basic idea is to lift the monomial ideal to a finite set of points. We then expect the general hyperplane section of this set of points to have the Lefschetz properties. We test this idea in the case of level monomial ideals in three variables of low type that do not have the weak Lefschetz property. If the type is one, then such an ideal is a complete intersection, so it has the weak Lefschetz property in characteristic zero. The latter is also true if the type is two by [1, Theorem 7.17]. Thus, we focus on a family of almost complete intersections of type three that do not have the weak Lefschetz property. Lifting such an ideal to a finite set of points we get a level set of points in 3-space of type three; recall that a subscheme of \mathbb{P}^n is called *level* if any, hence every, Artinian reduction is level (as an algebra). We show that the general hyperplane section of the level set of points has the weak Lefschetz property in almost every characteristic, whereas a special hyperplane section never has the weak Lefschetz property (see Corollary 5.8). Notice that examples of level sets of points in \mathbb{P}^3 of type three such that *every* Artinian reduction fails the weak Lefschetz property have been constructed in [36, Section 3].

This chapter is organised as follows. In Section 5.1 we recall the method of lifting an Artinian monomial ideal to a set of points and we introduce the family of Artinian monomial ideals that we focus on. In Section 5.2, we use this family to explore the subtlety of the weak Lefschetz property under various hyperplane sections and in arbitrary characteristic. Remark 5.6 and Theorem 5.7 show that general hyperplane sections of our liftings have the weak Lefschetz property but that there exists a special hyperplane section where it fails. Finally, in Section 5.3 we comment on the strong Lefschetz property in characteristic zero.

The contents of this chapter comes from the published work [10], a joint work with Uwe Nagel.

5.1 Liftings and hyperplanes sections

Let $R = K[x_0, \dots, x_n]$ and $S = K[x_1, \dots, x_n]$ be standard graded polynomial rings over an infinite field K . Let $J \subset S$ be a homogeneous ideal and $I \subset R$ be a homogeneous radical ideal. Then we say that J *lifts* to I if x_0 is a non-zero-divisor of R/I and $(I, x_0)/(x_0) \cong J$. If such an I exists, then J is called a *liftable* ideal.

If $\alpha = (a_1, \dots, a_n) \in \mathbb{N}_0^n$, then define $x^\alpha := x_1^{a_1} \cdots x_n^{a_n}$; the degree of x^α is $|\alpha| = \sum_{i=1}^n a_i$. For each $1 \leq j \leq n$, choose an infinite set $\{p_{j0}, p_{j1}, \dots\} \subset K$ of distinct elements. Then to $\alpha \in \mathbb{N}_0^n$ we associate the point $\bar{\alpha} := [1 : p_{1a_1} : \cdots : p_{na_n}] \in \mathbb{P}_K^n$ and

to x^α we associate the homogeneous polynomial

$$\overline{x^\alpha} := \prod_{j=1}^n \prod_{i=0}^{a_j-1} (x_j - p_{ji}x_0) \in R.$$

Using this construction, it was shown in [23, Theorem 4.9] and [16, Theorem 2.2] that monomial ideals are liftable given that the field K is infinite.

Theorem 5.1. *Let $I \subset S$ be a monomial ideal with minimal generators $x^{\alpha_1}, \dots, x^{\alpha_m}$, and assume K is infinite. Then I lifts to the ideal $\overline{I} := (\overline{x^{\alpha_1}}, \dots, \overline{x^{\alpha_m}}) \subset R$.*

Thus, given an Artinian monomial ideal $I \subset S$, the lifted ideal \overline{I} is a saturated ideal of a reduced set of points. Moreover, using [39, Proposition 2.6] we get that \overline{I} is level if and only if I is level. Hence, starting with a level Artinian monomial ideal, the action of lifting yields a (Krull) dimension one saturated ideal of a reduced level set of points.

Let Z be a subscheme of \mathbb{P}_K^n and let H be a hyperplane (i.e., $\text{codim } H = 1$) in \mathbb{P}_K^n . Then $Z \cap H$ is a *hyperplane section* of Z . Given a linear form $h \in R$, we abuse notation and call (\overline{I}, h) a *hyperplane section* of \overline{I} . If I is Artinian, then (\overline{I}, h) is an Artinian reduction of \overline{I} if and only if $\dim R/(\overline{I}, h) = 0$. Specifically, $R/(\overline{I}, x_0) \cong S/I$ is an Artinian reduction of \overline{I} , hence all Artinian reductions of \overline{I} have the same Hilbert function as I .

A family of almost complete intersections

Let $S = K[x, y, z]$ be the standard graded polynomial ring in three variables over an infinite field K . Then for $t \geq 1$, define I_t to be the ideal

$$I_t := (x^{t+1}, y^{t+1}, z^{t+1}, xyz) \subset S.$$

Proposition 5.2. *For $t \geq 1$, the ideal $I_t \subset S$ defined above has the following properties:*

- (i) S/I_t is level and Artinian,
- (ii) The minimal free resolution of S/I_t has the form

$$0 \longrightarrow S^3(-3-2t) \longrightarrow \begin{array}{c} S^3(-3-t) \\ \oplus \\ S^3(-2-2t) \end{array} \longrightarrow \begin{array}{c} S(-3) \\ \oplus \\ S^3(-1-t) \end{array} \longrightarrow S \longrightarrow S/I_t \longrightarrow 0,$$

in particular, S/I_t has socle type 3, and

- (iii) The Hilbert function of S/I_t is given by

$$h_{S/I_t}(d) = \begin{cases} 1 & \text{if } d = 0; \\ 3d & \text{if } 1 \leq d \leq t; \\ 3(2t+1-d) & \text{if } t < d \leq 2t; \\ 0 & \text{if } t > 2t. \end{cases}$$

Proof. The first two statements follow immediately from [38, Proposition 6.1] and the third statement follows from (ii). \square

A member of this family, I_2 , was discussed in [4, Example 3.1] where it was used to answer negatively the question of whether every almost complete intersection in S has the weak Lefschetz property. Motivated by this, a larger family containing the I_t is discussed in Chapter 4, [3, Corollary 7.3], [38, Sections 6 and 7], and [9]. We continue this exploration by considering hyperplane sections of a lift of I_t .

We consider the particular lift of I_t in $R = K[w, x, y, z]$ given by $p_{xi} = p_{yi} = p_{zi} = i$ for $0 \leq i \leq t$, that is, the homogeneous ideal

$$\overline{I}_t := \left(\prod_{i=0}^t (x - iw), \prod_{i=0}^t (y - iw), \prod_{i=0}^t (z - iw), xyz \right) \subset R.$$

If $2 \leq \text{char } K \leq t$, then \overline{I}_t is not a *true* lifting of I_t , but we will consider it nonetheless. When \overline{I}_t is a true lifting, i.e., $\text{char } K = 0$ or $\text{char } K > t$, then \overline{I}_t is the ideal of the level set of points

$$\{[1 : a : b : c] \mid 0 \leq a, b, c \leq t \text{ and at least one of } a, b, c \text{ is zero}\} \subset \mathbb{P}_K^3,$$

which is in bijection with the standard monomials of S/I_t ; recall that a subscheme of \mathbb{P}^n is called level if any, hence every, Artinian reduction is level (as an algebra).

Given the lift \overline{I}_t of I_t , we consider the hyperplanes in R of the form $w + ax$ for $a \in K$. If $a \in N := \{y \mid yi = -1 \text{ for some } i \in \{1, 2, \dots, t\}\}$, then $w + ax$ is a zero-divisor of R/\overline{I}_t and so $(\overline{I}_t, w + ax)$ is non-Artinian. Suppose $a \notin N$, then $w + ax$ is a non-zero-divisor of R/\overline{I}_t and so the hyperplane section $(\overline{I}_t, w + ax)$ is Artinian. Further still, $R/(\overline{I}_t, w + ax) \cong S/J_{t,a}$ where

$$J_{t,a} := \left(x^{t+1}, \prod_{i=0}^t (iax + y), \prod_{i=0}^t (iax + z), xyz \right) \subset S. \quad (5.1)$$

Specifically, $J_{t,0} = I_t$.

We will next analyse the ideals $J_{t,a}$ for the presence of the Lefschetz properties.

5.2 The weak Lefschetz property

In Proposition 2.12, it was shown that $x + y + z$ (and through a similar argument, $x + y - z$) is a weak Lefschetz element of an Artinian monomial algebra if and only if the algebra has the weak Lefschetz property. However, $S/J_{t,a}$ is not a monomial algebra unless $a = 0$. We investigate whether the linear form $\ell := bx + cy - z$ is a weak Lefschetz element of $S/J_{t,a}$.

Notice that $S/(J_{t,a}, \ell) \cong T/L_{t,a}$ where $T = K[x, y]$ and

$$L_{t,a} := \left(x^{t+1}, \prod_{i=0}^t (iax + y), \prod_{i=0}^t ((ia + b)x + cy), xy(bx + cy) \right) \subset T. \quad (5.2)$$

The second and third generators of $L_{t,a}$ are products of linearly-consecutive binomial terms and can be described using the unsigned Stirling numbers of the first kind.

The *unsigned Stirling numbers of the first kind*, denoted $s_{n,k}$, are defined recursively, for $1 \leq k \leq n$, as $s_{n+1,k} = s_{n,k-1} + ns_{n,k}$ with the initial conditions $s_{1,1} = 1$ and $s_{n,0} = 0$, for $n \geq 1$. In particular, $\prod_{i=0}^{n-1} (x+i) = \sum_{k=1}^n s_{n,k} x^k$ (see, e.g., [50, Theorem 1.3.4]).

We take the convention that both the empty product and 0^0 are one.

Lemma 5.3. *Let $0 \neq a$ and b be in K and let k and n be integers with $0 \leq k \leq n$. Define V to be the set of n elements of K of the form $ia + b$, for $0 \leq i < n$. Then the sum of all products of the elements of subsets in V with k elements, denoted $d_{n,n-k}(a, b)$, is*

$$\sum_{i=0}^k s_{n,n-i} \binom{n-i}{k-i} a^i b^{k-i}.$$

Proof. By way of preparation we make two remarks. First, for all $n \geq 1$, if $k = 0$, then

$$d_{n,n}(a, b) = 1, \quad (5.3)$$

as \emptyset is the unique subset of size zero. Second, for all $n \geq 1$, if $k = n$, then

$$d_{n,0}(a, b) = \prod_{i=0}^{n-1} (ia + b) = \sum_{k=1}^n s_{n,k} a^{n-k} b^k = \sum_{i=0}^n s_{n,n-i} \binom{n-i}{n-i} a^i b^{n-i}. \quad (5.4)$$

Now use induction on $n \geq 1$. If $n = 1$, then we are done by (5.3) and (5.4). Suppose that $n \geq 1$ and $1 \leq k \leq n$. Let V be the set of n elements of K of the form $ia + b$ for $0 \leq i < n$ and $W = V \cup \{na + b\}$; that is, W is the set of $n + 1$ elements of K of the form $ia + b$ for $0 \leq i \leq n$. Then the sum of all products of the elements of subsets of W with k elements is the sum of all products of the elements of subsets of V with k elements plus the sum of all products of the elements of subsets of V with $k - 1$ elements, each scaled by $na + b$. That is,

$$d_{n+1,(n+1)-k}(a, b) = d_{n,n-k}(a, b) + (na + b)d_{n,n-(k-1)}(a, b). \quad (5.5)$$

If $k = 0$ or $k = n + 1$, then we are done by (5.3) and by (5.4), respectively. If $1 \leq k \leq n$, then by (5.5) and the induction hypothesis we get

$$\begin{aligned} d_{n+1,(n+1)-k}(a, b) &= d_{n,n-k}(a, b) + (na + b)d_{n,n-(k-1)}(a, b) \\ &= \sum_{i=0}^k s_{n,n-i} \binom{n-i}{k-i} a^i b^{k-i} + (na + b) \sum_{i=0}^{k-1} s_{n,n-i} \binom{n-i}{k-1-i} a^i b^{k-1-i} \\ &= \sum_{i=0}^k s_{n,n-i} \binom{n+1-i}{k-i} a^i b^{k-i} + \sum_{i=1}^k n s_{n,n-(i-1)} \binom{n+1-i}{k-i} a^i b^{k-i} \\ &= \sum_{i=0}^k (s_{n,n-i} + n s_{n,n-(i-1)}) \binom{n+1-i}{k-i} a^i b^{k-i} \\ &= \sum_{i=0}^k s_{n+1,(n+1)-i} \binom{n+1-i}{k-i} a^i b^{k-i}, \end{aligned}$$

where we use the properties of binomial coefficients and the unsigned Stirling numbers of the first kind, as needed. \square

It should be noted that $d_{n,n-k}(1, 0) = s_{n,n-k}$ as the set V is then $\{0, 1, \dots, n-1\}$.

Using Lemma 5.3, the second generator of $L_{t,a}$ can be described using the unsigned Stirling numbers of the first kind as

$$\prod_{i=0}^t (iax + y) = \sum_{i=0}^t s_{t+1,t+1-i} a^i x^i y^{t+1-i} = \sum_{i=0}^t d_{t+1,t+1-i}(a, 0) x^i y^{t+1-i}$$

and the third generator of $L_{t,a}$ can be described using Lemma 5.3 as

$$\prod_{i=0}^t ((ia + b)x + cy) = \sum_{i=0}^t d_{t+1,t+1-i}(a, b) c^{t+1-i} x^i y^{t+1-i}.$$

Now we return to studying the weak Lefschetz property. We have explicitly described the coefficients of the generators of $L_{t,a}$, hence we can use linear algebra to determine whether $S/J_{t,a}$ has $\ell = bx + cy - z$ as a weak Lefschetz element. Define $N = \{y \mid yi = -1 \text{ for some } i \in \{1, 2, \dots, t\}\}$, as above.

Proposition 5.4. *Consider the algebra $A = S/J_{t,a}$ as in Equation (5.1) and $a \notin N$. Let $M_{t,a,b,c}$ be the $(t+1) \times (t+1)$ K -matrix given by*

$$\begin{bmatrix} s_{t+1,1}a^t & s_{t+1,2}a^{t-1} & s_{t+1,3}a^{t-2} & \cdots & s_{t+1,t}a & 1 \\ d_{t+1,1}(a,b)c & d_{t+1,2}(a,b)c^2 & d_{t+1,3}(a,b)c^3 & \cdots & d_{t+1,t}(a,b)c^t & c^{t+1} \\ b & c & 0 & \cdots & 0 & 0 \\ 0 & b & c & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & c & 0 \end{bmatrix}.$$

Then the algebra $A = S/J_{t,a}$, from Equation (5.1), has $\ell = bx + cy - z$ as a weak Lefschetz element if and only if $\det M_{t,a,b,c}$ is nonzero in K .

Thus, $\det M_{t,a,b,c} \neq 0 \in K$ for some $b, c \in K$ if and only if A has the weak Lefschetz property.

Proof. Since $J_{t,a}$ is an Artinian reduction of the lift of I_t , then their Hilbert functions are equal. Thus using Proposition 5.2(iii), we have then that the Hilbert function of $S/J_{t,a}$ is strictly unimodal from 0 to $2t$ and has a twin-peak at t and $t+1$; that is, $h_{S/J_{t,a}}(t) = 3t = h_{S/J_{t,a}}(t+1)$. Hence by [38, Proposition 2.1], $S/J_{t,a}$ has the weak Lefschetz property if and only if the map $[S/J_{t,a}]_t \xrightarrow{bx+cy-z} [S/J_{t,a}]_{t+1}$ is an isomorphism for some $b, c \in K$ and thus it suffices to check whether $[T]_{t+1} \subset L_{t,a}$.

Thus, the matrix $M_{t,a,b,c}$ corresponds to the system of equations which needs to be solved to determine if a polynomial in $[T]_{t+1}$ with no x^{t+1} term is in $L_{t,a}$. Hence, $\det M_{t,a,b,c} \neq 0 \in K$ if and only if $M_{t,a,b,c}$ is invertible in K , i.e., $[T]_{t+1} \subset L_{t,a}$. \square

Furthermore, we give a closed form for the determinant of $M_{t,a,b,c}$.

Proposition 5.5. *Assuming b, c both nonzero, then the determinant of $M_{t,a,b,c}$ is*

$$(-1)^t c^t \left(\prod_{i=1}^t (ai + b) - \prod_{i=1}^t (aci - b) \right).$$

Proof. By straightforward Gaussian elimination, we get

$$\begin{aligned} \det M_{t,a,b,c} &= b^{t-1} c^t \sum_{j=1}^t \left(\left(-\frac{c}{b} \right)^{t-j} ca^{t+1-j} s_{t+1,j} - \left(-\frac{1}{b} \right)^{t-j} d_{t+1,j}(a, b) \right) \\ &= b^{t-1} c^t \left(\sum_{j=1}^t \left(-\frac{c}{b} \right)^{t-j} ca^{t+1-j} s_{t+1,j} \right. \\ &\quad \left. - \sum_{k=1}^{t+1} a^{t+1-k} s_{t+1,k} \sum_{j=1}^t \left(-\frac{1}{b} \right)^{t-j} b^{k-j} \binom{k}{k-j} \right) \\ &= (-1)^t c^t \sum_{j=1}^{t+1} a^{t+1-j} b^{j-1} s_{t+1,j} ((-1)^j c^{t+1-j} + 1) \\ &= (-1)^t c^t \left(\sum_{j=1}^{t+1} a^{t+1-j} b^{j-1} s_{t+1,j} + \sum_{j=1}^{t+1} (-1)^j a^{t+1-j} b^{j-1} c^{t+1-j} s_{t+1,j} \right) \\ &= (-1)^t c^t \left(\prod_{i=1}^t (ai + b) - \prod_{i=1}^t (aci - b) \right). \end{aligned}$$

□

Given the above determinant calculation, we make the following observations.

Remark 5.6. If we specialise the parameters a, b , and c suitably, then we get three nice determinant evaluations.

- (i) *When $a = 0$:* Then $J_{t,0} = I_t$ and $\det M_{t,0,b,c} = b^t c^t ((-1)^t - 1)$. Hence $S/J_{t,0}$ has the weak Lefschetz property if and only if t is odd and $\text{char } K \neq 2$, recovering [9, Proposition 3.1].
- (ii) *When $b = c = 1$:* Then

$$\det M_{t,a,1,1} = \begin{cases} 2 \sum_{i=1}^{t/2} a^{2i-1} s_{t+1,t+1-(2i-1)} & \text{if } t \text{ is even;} \\ -2(1 + \sum_{i=1}^{\lfloor t/2 \rfloor} a^{2i} s_{t+1,t+1-2i}) & \text{if } t \text{ is odd.} \end{cases}$$

Thus in characteristic zero, $x + y - z$ is a weak Lefschetz element of $S/J_{t,a}$ if $a \neq 0$ and $a \notin N$.

- (iii) *When $a = b = c = 1$:* Then $\det M_{t,1,1,1} = (-1)^t (t+1)!$. Hence $x + y - z$ is a weak Lefschetz element of $S/J_{t,1}$ if and only if $\text{char } K = 0$ or $\text{char } K > t + 1$, i.e., \overline{I}_t is a true lifting of I_t .

Theorem 5.7. *Let K be any infinite field and $S = K[x, y, z]$. Then for all a in K such that $a \neq 0$ and $a \notin N$ and for all positive integers t , the algebra $A = S/J_{t,a}$ has the weak Lefschetz property.*

Proof. Let a, c be nonzero elements of K . Then $b = ac$ is nonzero and moreover

$$\det M_{t,a,b,c} = (-1)^t c^t \prod_{i=1}^t (ai + b) = (-1)^t a^t c^t \prod_{i=1}^t (i + c) \in K[x, y, z].$$

As K is infinite, there exists a nonzero c in K such that $i + c \neq 0$ for $1 \leq i \leq t$. Hence, $\det M_{t,a,b,c}$ is nonzero in K . Therefore, if $a \notin N$, then A has the weak Lefschetz property. \square

We partially summarise our results as follows:

Corollary 5.8. *Let $t \geq 1$ be an integer and set $A = R/\overline{I}_t$, where*

$$\overline{I}_t := \left(\prod_{i=0}^t (x - iw), \prod_{i=0}^t (y - iw), \prod_{i=0}^t (z - iw), xyz \right) \subset R = K[w, x, y, z].$$

Then:

- (i) *If the characteristic of K is zero or greater than t , then the ideal \overline{I}_t defines a set of $3(t+1)t+1$ points in \mathbb{P}^3 that is level of type three.*
- (ii) *If $\ell \in [R]_1$ is a general linear form, then $A/\ell A$ has the weak Lefschetz property.*
- (iii) *If $\ell = w$, then the Artinian algebra $A/\ell A$ has the weak Lefschetz property if and only if t is odd and $\text{char } K \neq 2$.*

Proof. Claim (i) follows by Theorem 5.1 and Proposition 5.2.

In order to prove (ii), notice that Theorem 5.7 shows that $A/\ell' A$ has the weak Lefschetz property, where $\ell' = w + ax$ and $a \neq 0$ is any element in $K \setminus N$. Let $L \in R$ be another general linear form. Then, for all $j \in \mathbb{Z}$, one has

$$\dim_K[A/(\ell, L)A]_j \leq \dim_K[A/(\ell', L)A]_j$$

Since $A/\ell' A$ has the weak Lefschetz property, this must be an equality, hence $A/\ell A$ also has the weak Lefschetz property.

Part (iii) has been shown in Remark 5.6(i). \square

Specialising to $t = 2$, we get an example reminiscent of [4, Example 3.1], which showed that, in characteristic zero, for any degree three form f , the ideal (x^3, y^3, z^3, f) has the weak Lefschetz property if and only if $f \neq xyz$ modulo x^3, y^3, z^3 .

Example 5.9. Let $t = 2$, $\text{char } K \neq 2$, and $a \in K \setminus \{-1, -\frac{1}{2}\}$, then

$$J_{t,a} = (x^3, y(ax + y)(2ax + y), z(ax + z)(2ax + z), xyz)$$

is Artinian. Moreover, $S/J_{t,a}$ has the weak Lefschetz property if and only if $a \neq 0$, that is, $J_{t,a}$ is non-monomial.

5.3 The strong Lefschetz property

In case, $a = 0$, where $S/J_{t,a}$ is a monomial algebra, the strong Lefschetz property fails spectacularly.

Proposition 5.10. *The algebra $S/J_{t,0} = S/I_t$ has the strong Lefschetz property if and only if $t = 1$ and $\text{char } K \neq 2$.*

Proof. Let $\ell = x + y - z$, then by [38, Proposition 2.2], ℓ is a strong Lefschetz element of $S/J_{t,0}$ if and only if $S/J_{t,0}$ has the strong Lefschetz property.

If t is even or $\text{char } K = 2$ then, by Proposition 5.4, $S/J_{t,0}$ fails to have the weak Lefschetz property hence fails to have the strong Lefschetz property.

Suppose then t is odd and $\text{char } K \neq 2$. If $t = 1$, then, by Proposition 5.4, $S/J_{1,0}$ has the weak Lefschetz property. As the regularity of $S/J_{1,0}$ is two and the map $[S/J_{1,0}]_0 \xrightarrow{\ell^2} [S/J_{1,0}]_2$ is injective since $\ell^2 \notin J_{1,0}$, then $S/J_{1,0}$ has the strong Lefschetz property.

Suppose $t \geq 3$ and let $A = S/J_{t,0}$. As $\dim_K [A]_2 = 6 = \dim_K [A]_{2t-1}$ by Proposition 5.2(iii), then it suffices to show the map $\varphi : [A]_2 \xrightarrow{\ell^{2t-3}} [A]_{2t-1}$ is not injective. Let $p, q \in K$, not both zero, such that $p(2t-2) + qt = 0$; such a non-trivial solution exists in K regardless of characteristic. Consider then $f = p(x^2 + y^2 + z^2) + q(xy + xz + yz)$ which is a nonzero element of $[A]_2$. As $\ell^{2t-3}f$ is equivalent to $(p(2t-2) + qt)(x^t y^{t-1} + x^{t-1} y^t + x^t z^{t-1} - x^{t-1} z^t + x^t z^{t-1} - x^{t-1} z^t)$ modulo I_t , then $\ell^{2t-3}f$ is in I_t . Hence, φ is not injective and thus A fails to have the strong Lefschetz property. \square

Now we consider the case when the Artinian algebra $S/J_{t,a}$ is *not* a monomial algebra.

Remark 5.11. Suppose K is a field of characteristic zero. Let $a \notin N$ be a nonzero element of K and let $A = S/J_{t,a}$. As the Hilbert function of A is symmetric from 1 to $2t$ with peak $t, t+1$ and A is level by Proposition 5.2, then using [38, Proposition 2.1] it suffices to show for $1 \leq i \leq t$ the following hold:

- (i) $[A]_{t-i+1} \xrightarrow{\ell^{2i-1}} [A]_{t+i}$ is an isomorphism,
- (ii) $[A]_{t-i} \xrightarrow{\ell^{2i}} [A]_{t+i}$ is an injection, and
- (iii) $[A]_{t-i+1} \xrightarrow{\ell^{2i}} [A]_{t+i+1}$ is a surjection.

As A has the weak Lefschetz property, if part (i) is shown for all i , then parts (ii) and (iii) follow immediately:

- (ii) $[A]_{t-i} \xrightarrow{\ell^{2i}} [A]_{t+i} = [A]_{t-i} \xrightarrow{\ell} [A]_{t-i+1} \xrightarrow{\ell^{2i-1}} [A]_{t+i}$ is a composition of injective maps, hence injective.
- (iii) $[A]_{t-i+1} \xrightarrow{\ell^{2i}} [A]_{t+i+1} = [A]_{t-i+1} \xrightarrow{\ell^{2i-1}} [A]_{t+i} \xrightarrow{\ell} [A]_{t+i+1}$ is a composition of surjective maps, hence surjective.

Thus, in order to show A has the strong Lefschetz property it is sufficient to show part (i) holds for $2 \leq i \leq t$.

Using Macaulay2 [18], we have verified that in characteristic zero, if $t \leq 30$, then there exists some $a \in K$ such that ℓ is a strong Lefschetz element of $S/J_{t,a}$. We suspect that this is always true.

Conjecture 5.12. *Suppose K is a field of characteristic zero. Let $a \notin N$ be a nonzero element of K and let $A = S/J_{t,a}$. Then A has the strong Lefschetz property with strong Lefschetz element $\ell = x + y - z$.*

Thus, if the conjecture holds, then, at least in characteristic zero, there is only one “bad” choice for the strong Lefschetz property and it is, interesting in its own right, the only monomial case.

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