2013

ON A PALEY-WIENER THEOREM FOR THE ZS-AKNS SCATTERING TRANSFORM

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ON A PALEY-WIENER THEOREM FOR THE ZS-AKNS SCATTERING TRANSFORM

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Ryan D. Walker
Lexington, Kentucky

Director: Dr. Peter A. Perry, Professor of Mathematics
Lexington, Kentucky 2013

ABSTRACT OF DISSERTATION

ON A PALEY-WIENER THEOREM FOR THE ZS-AKNS SCATTERING TRANSFORM

In this thesis, we establish an analog of the Paley-Wiener Theorem for the ZS-AKNS scattering transform on the set of real potentials in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. We first prove that if the real-valued potential $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is supported on $[\alpha, +\infty)$, then the left-hand ZS-AKNS reflection coefficient has the form

$$r_-(z) = \int_\alpha^\infty e^{2izx} C(x)\, dx$$

where $C \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Using the Riemann-Hilbert approach to inverse scattering, we make a shift of contours and obtain a converse to this result: if $r_-(z)$ is the Fourier transform of a function $C \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ supported on $[\alpha, +\infty)$, then $w(x)$ has support on $[\alpha, +\infty)$. We then show that the function C in this representation determines the potential $w(x)$ locally in the sense that only the values of C on $[\alpha, x]$ are required to recover $w(x)$. We also demonstrate one application of our techniques to the study of an inverse spectral problem for a half-line Miura potential Schrödinger equation.

KEYWORDS: Scattering transform, nonlinear Paley-Wiener Theorem, Riemann-Hilbert problems, singular Schrödinger operators, Miura potentials

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Date: April 29, 2013

ON A PALEY-WIENER THEOREM FOR THE ZS-AKNS SCATTERING TRANSFORM

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To my parents

ACKNOWLEDGMENTS

I could not have completed this thesis without the guidance, kindness, and camaraderie of a number of wonderful individuals at the University of Kentucky and elsewhere.

I am deeply thankful to my advisor Peter Perry for his training, patience, and tireless draft revision. I am immensely fortunate to have had such an inspiring mentor as my tour guide through the beautiful world of 1-dimensional inverse problems.

I am also very grateful to Kenneth McLaughlin at the University of Arizona who suggested the idea for the shift of contours argument in the Riemann-Hilbert problem and introduced me to the elegance of the Riemann-Hilbert technique. This thesis would not be what it is without his inspiration. I also thank Rostyslav Hryniv for suggesting the analogue of the m-function, and for giving valuable feedback on a draft.

The dedication of faculty and staff in the University of Kentucky Department of Mathematics is exceptional. I am grateful for countless interesting conversations, brilliant lectures, and fond memories.

I thank my colleagues at Blue Shield of California, and particularly my manager Aaron Lai, who have been tremendously supportive of me during the final year of this project.

I thank my friends and family for their companionship through the many trials and triumphs of my PhD program. My partner, Keara Harman, has stood by me through the most stressful periods, and I could not have finished without her love and support.

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Chapter 1 Introduction

1.1 Overview

This thesis studies the direct and inverse scattering of a 2×2 Zakharov-Shabat-Ablowitz-Kaup-Newell-Segur (ZS-AKNS) system [1, 48]

$$\begin{cases} \dfrac{d}{dx}\Psi = iz\sigma_3\Psi + Q(x)\Psi, \qquad z \in \mathbb{C}, x \in \mathbb{R} \\ \\ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & w(x) \\ w(x) & 0 \end{bmatrix} \end{cases} \tag{1.1}$$

where w is a real-valued potential drawn from the set

$$X = L^1(\mathbb{R}) \cap L^2(\mathbb{R}).$$

We may also write the ZS-AKNS equation in the form

$$P\Psi = z\Psi$$

where

$$P = -i \begin{bmatrix} d/dx & -w \\ w & -d/dx \end{bmatrix}$$

is the the self-adjoint Dirac operator on $L^2(\mathbb{R}) \otimes M_{2\times 2}(\mathbb{C})$, the space of square 2×2 matrix-valued functions with entries in $L^2(\mathbb{R})$.[1]

It is not hard to see that any two fundamental solutions Ψ_1, Ψ_2 to the ZS-AKNS equation are related by

$$\Psi_1(x, z) = \Psi_2(x, z)M(z)$$

for some matrix $M(z)$. The key to the direct scattering theory of the ZS-AKNS equation is to consider the solutions $\Psi_\pm$ to (1.1) which satisfy the respective asymptotic conditions

$$\lim_{x\to\pm\infty} |\Psi_\pm(x, z) - \exp(ixz\sigma_3)| = 0.$$

The solutions $\Psi_\pm$ are called *Jost solutions* and these solutions turn out to be related in a very special way. For $z \in \mathbb{R}$, we have that

$$\Psi_+(x, z) = \Psi_-(x, z)R(z),$$

where

$$R(z) = \begin{bmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix}$$

[1] See [28] for a thorough discussion of the self-adjointness of the operator P.

with $|a(z)|^2 - |b(z)|^2 = 1$. For each potential $w \in X$ and every $z \in \mathbb{R}$, we use this relation to define the quantities

$$r_-(z) = \frac{b(z)}{a(z)} \qquad r_+(z) = -\frac{\bar{b}(z)}{a(z)}$$

called respectively the *left and right reflection coefficients*. We will explain in Chapter 2 the physical meaning of these coefficients.

The idea behind the scattering of (1.1) is to show that the ZS-AKNS potential w uniquely determines the reflection coefficients $r_\pm(z)$ and that $r_\pm(z)$ uniquely determine $w(x)$. We define the *direct scattering transforms* on X to be the nonlinear mappings

$$\mathcal{D}_P^\pm : w \mapsto r_\pm.$$

The *inverse scattering transforms* are the nonlinear mappings

$$\mathcal{I}_P^\pm : r_\pm \mapsto w.$$

The direct scattering problem for ZS-AKNS is to completely characterize $\mathcal{D}_P^\pm$; the inverse scattering problem is to characterize $\mathcal{I}_P^\pm$. The ZS-AKNS scattering problem has been well-studied and there are now several general treatments. These include the classical Gelfand-Levitan-Marchenko integral equation approach developed in [1, 48] and the Riemann-Hilbert approach pioneered by Beals and Coifman in [2].

Throughout this work, we will use the Fourier transform convention

$$\hat{f}(z) = \mathcal{F}f(z) = \int_{\mathbb{R}} f(x)e^{2izx}\,dx$$
$$\check{f}(x) = \mathcal{F}^{-1}f(x) = \frac{1}{\pi}\int_{\mathbb{R}} f(z)e^{-2ixz}\,dz.$$

Though the direct maps $\mathcal{D}_P^\pm$ are nonlinear, they share many properties with the 1-dimensional Fourier transform $\mathcal{F}$. Our analysis will develop some new analogs of classical Fourier results for the scattering transforms. The most important of these analogs is that the support of the potential $w \in X$ determines the analyticity and decay of the reflection coefficients and vice versa.

The major result of this thesis is the following Paley-Wiener type theorem for the scattering transforms.

Theorem 1.1.1 (Nonlinear Paley-Wiener Theorem for ZS-AKNS)**.** *Let $r_\pm \in \widehat{X}$ be the left and right reflection coefficient for the real valued w-potential ZS-AKNS equation where $w \in X$. Then*

1. supp $w(x) \subseteq [\alpha, +\infty)$ *if and only if r_- extends to an analytic function on $\mathbb{C}^+$ satisfying*
$$|r_-(R + iT)| \leq Ce^{-2\alpha T} \qquad T \geq 0.$$

2. $\operatorname{supp} w(x) \subseteq (-\infty, \beta]$ *if and only if* r_+ *extends to an analytic function on* $\mathbb{C}^+$ *satisfying*

$$|r_+(R+iT)| \leq Ce^{2\beta T} \qquad T \geq 0.$$

Note that in the statement of this theorem and throughout the thesis we make no notational distinction between a function and any analytic extension it possesses. The theorem is a nonlinear analog of the following Paley-Wiener Theorem for the Fourier transform.

Theorem 1.1.2 (Two-Sided Paley-Wiener Theorem for the Fourier Transform)**.** *Suppose that* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*. Then*

1. $\operatorname{supp} f \subseteq [\alpha, +\infty)$ *if and only if* $\hat{f}$ *extends to an analytic function on* $\mathbb{C}^+$ *satisfying*

$$|\hat{f}(R+iT)| \leq Ce^{-2\alpha T} \qquad T \geq 0.$$

2. $\operatorname{supp} f \subseteq (-\infty, \beta]$ *if and only if* $\hat{f}$ *extends to an analytic function on* $\mathbb{C}^-$ *satisfying*

$$|\hat{f}(R+iT)| \leq Ce^{-2\beta T} \qquad T \leq 0.$$

A proof of this theorem may be found in Appendix B, where we also collect some other useful facts about the 1-dimensional Fourier transform.

The main contribution of this thesis is a compelling example of how simple Fourier analogies can reveal new results in inverse spectral problems, and make known results in this area more intuitive. The analogy between the Fourier transform and the scattering transform has long been recognized (see, for example, [1]). There are many instances in the literature of how Fourier analogies have been profitably exploited in the study of 1-dimensional inverse theory. A Paley-Wiener analogy inspired Christer Bennewitz's single-page proof [5] of the Local Borg-Marchenko theorem for the 1-dimensional Schrödinger equation with classical potentials. Bennewitz extends this analysis in [6] to the study of more general Sturm-Liouville inverse problems. The series of papers [45, 47, 46] take an approach quite similar to ours in developing a Fourier analogy and nonlinear Paley-Wiener theorem to resolve an inverse scattering problem for the 1-dimensional Helmholtz equation. A nonlinear Paley-Wiener theorem also plays an important role in [18], where the authors apply the inverse theory of the ZS-AKNS system to a pulse design problem in magnetic resonance imaging.

1.2 Plan for the Thesis

Chapters 2 and 3 of this thesis are devoted to proving Theorem 1.1.1. To outline the approach taken in these chapters, we must introduce the *real-valued* function spaces:

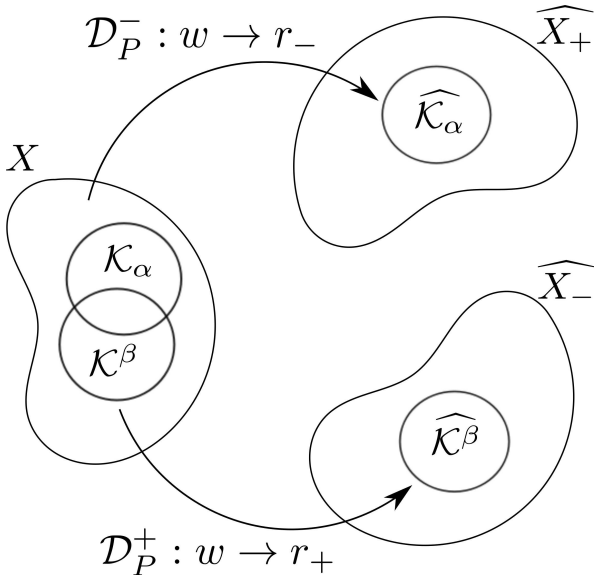

Figure 1.1: $\mathcal{D}_P^+$ is a bijection between X and $\widehat{X_+}$, and between $\mathcal{K}_\alpha$ and $\widehat{\mathcal{K}_\alpha}$. $\mathcal{D}_P^-$ is a bijection between X and $\widehat{X_-}$, and between $\mathcal{K}^\beta$ and $\widehat{\mathcal{K}^\beta}$.

$$
\begin{aligned}
X &= L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \\
\mathcal{K}_\alpha &= \{w \in X : \operatorname{supp} w \subseteq [\alpha, +\infty)\} \\
\mathcal{K}^\beta &= \left\{w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : \operatorname{supp} w \subseteq (-\infty, \beta]\right\} \\
\mathcal{K}_\alpha^\beta &= \left\{w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : \operatorname{supp} w \subseteq [\alpha, \beta]\right\} \\
\widehat{X_\pm} &= \left\{r(z) = \int_{-\infty}^{\infty} e^{\pm 2iz\xi} C(\xi)\, d\xi : C \in X, \|r\|_{L^\infty(\mathbb{R})} < 1\right\} \\
\widehat{\mathcal{K}_\alpha} &= \left\{r(z) = \int_{\alpha}^{+\infty} C(x) e^{2izx}\, dx : C \in X, \|r\|_{L^\infty(\mathbb{R})} < 1\right\} \\
\widehat{\mathcal{K}^\beta} &= \left\{r(z) = \int_{-\infty}^{\beta} C(x) e^{-2izx}\, dx : C \in X, \|r\|_{L^\infty(\mathbb{R})} < 1\right\}.
\end{aligned}
$$

The diagram in Figure 1.1 may prove useful for recalling the relationship between the scattering maps and the various function spaces required for our results.

In [21], the authors prove that $\mathcal{D}_P^-$ is a bijection from X to $\widehat{X_+}$ and that $\mathcal{D}_P^+$ is a bijection from X to $\widehat{X_-}$. We will prove the following refinement of this result, which is really a more convenient formulation of Theorem 1.1.1.

Theorem 1.2.1. *Let $\alpha, \beta \in \mathbb{R}$.*

1. *The mapping $\mathcal{D}_P^-$ is a bijection from $\mathcal{K}_\alpha$ to $\widehat{\mathcal{K}_\alpha}$.*
2. *The mapping $\mathcal{D}_P^+$ is a bijection from $\mathcal{K}^\beta$ to $\widehat{\mathcal{K}^\beta}$.*

The proof of Theorem 1.2.1 naturally divides into two parts. In the forward part of the proof, we show that the reflection coefficients for half-line supported potentials have the Fourier representation specified by Theorem 1.2.1. In the inverse part, the Fourier representation of the reflection coefficient is given and we must deduce that the potential w has the required support properties.

In Chapter 2, we review the direct scattering theory of the ZS-AKNS system. Following [21], we develop some new representation formulas for the solutions $\Psi_\pm$ to (1.1). Working with these formulas, we will show that $\mathcal{D}_P^+$ maps $\mathcal{K}_\alpha$ to $\widehat{K_\alpha}$ and $\mathcal{D}_P^-$ maps $\mathcal{K}^\beta$ to $\widehat{\mathcal{K}^\beta}$.

In Chapter 3, we use the approach developed in [2] to formulate the inverse scattering problem for the ZS-AKNS equation as a Riemann-Hilbert problem. We then introduce a factorization of the Riemann-Hilbert problem that will allow us to read off the support of the potential given the data $r_\pm$. Our approach in this chapter is quite reminiscent of a standard shift of contours argument from linear Fourier theory.

Chapter 4 of this thesis presents a localization result for the direct scattering map on the space $\mathcal{K}_\alpha$. If $w \in \mathcal{K}_\alpha$ then we know that

$$r_-(z) = \int_\alpha^\infty C(x) e^{2izx}\, dx$$

for $C \in X$. What is the relation between C and w? We use the Riemann-Hilbert reconstruction formula for the potential to show that only the values of C on $[\alpha, x]$ are needed to recover the potential w on $[\alpha, x]$.

Theorem 1.2.2. *Let $r_-(z) = \mathcal{D}_-^+ w(z)$ where $w \in \mathcal{K}_\alpha$. Then*

$$r_-(z) = \int_\alpha^\infty e^{2iz\xi} C(\xi)\, d\xi$$

where $C \in X$ and for a.e. x $w(x)$ is completely determined by from the values of $C(\xi)$ with $\xi \in [\alpha, x]$.

As an immediate consequence, we have

Corollary 1.2.1. *Let $w, \tilde{w} \in \mathcal{K}_\alpha$ and let*

$$r_-(z) = D_P^- w(z) = \int_\alpha^\infty e^{2iz\xi} C(\xi)\, d\xi$$
$$\tilde{r}_-(x) = D_P^- \tilde{w}(z) = \int_\alpha^\infty e^{2iz\xi} \tilde{C}(\xi)\, d\xi$$

where $C, \tilde{C} \in X$. If $C(\xi) = \tilde{C}(\xi)$ for a.e. $\xi \in [\alpha, \beta]$, then $w(x) = \tilde{w}(x)$ on $[\alpha, \beta]$.

In Chapter 5, we present an application of the techniques developed for the ZS-AKNS equation to the of study an inverse spectral problem for Schrödinger operators with singular Miura-type potentials. To sketch the results of this chapter, we need to introduce additional notation and recall a few ideas from the Weyl-Titchmarsh theory of the 1-dimensional Schrödinger equation.

Let

$$H^1(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \right\}$$

and let $H^{-1}(\mathbb{R})$ be the topological dual to $H^1(\mathbb{R})$.

Consider the formal Schrödinger operator

$$L_q = -\frac{d^2}{dx^2} + q(x)$$

where $q(x)$ is a distribution in $H^{-1}(\mathbb{R})$. Following [43], for $q \in H^{-1}(\mathbb{R})$, we consider the quadratic form

$$\mathfrak{t}_{\mathfrak{q}}(\phi) = \int_{-\infty}^{\infty} |\phi'(x)|^2 \, dx + \left\langle q, |\phi|^2 \right\rangle, \tag{1.2}$$

associated to the formally defined operator L_q. When the form $\mathfrak{t}_q$ is semi-bounded and closed on $L^2(\mathbb{R})$ then it corresponds to a unique self-adjoint operator on $L^2(\mathbb{R})$ (see [29], VI §2, part 1), and it is this operator that we identify with L_q.[2]

A potential $q \in H^{-1}(\mathbb{R})$ having the representation

$$q(x) = w'(x) + w(x)^2$$

for $w \in L^2_{\text{loc}}(\mathbb{R})$ is called a *Miura potential*. Kappeler, Perry, Shubin, and Topolav characterize these potentials in [26]. Following [21], we consider a special subclass of distributions in $H^{-1}(\mathbb{R})$ which may be represented as

$$q(x) = w'(x) + w(x)^2$$

for some $w \in X$, supported on $[\alpha, +\infty)$. Denote this subclass by $\mathcal{M}_\alpha$. From [26], the representation $q = w' + w^2$ for $q \in \mathcal{M}_\alpha$ is unique and the Schrödinger form is positive and closed. Thus, the Schrödinger operator L_q for $q \in \mathcal{M}_\alpha$ is a well-defined, self-adjoint operator on $L^2(\mathbb{R})$. We may proceed to study its spectral theory.

In the classical spectral theory for L_q with $q \in L^1_{\text{loc}}[0, +\infty)$, an important role is played by the *Weyl-Titchmarsh m-function.* To briefly introduce this object, consider the Schrödinger equation

$$L_q u = z^2 u \tag{1.3}$$

for $q \in L^1_{\text{loc}}[0, +\infty)$ and self-adjoint boundary conditions imposed at $x = 0$.[3]

For any $z \in \mathbb{C}^+$, we may solve this equation to obtain the solution $u(x, z)$ satisfying the asymptotic conditions

$$\lim_{x \to +\infty} \left| u(x, z) - e^{ixz} \right| = 0$$
$$\lim_{x \to +\infty} \left| u'(x, z) - ize^{ixz} \right| = 0.$$

[2] See [25] for a complete treatment of the self-adjointness of L_q for $q \in H^1(\mathbb{R})$.

[3] To keep our summary completely consistent with Simon's presentation in [42], we consider only the case of L_q on the half-line $[0, +\infty)$. It is straightforward to translate the results to any half-line $[\alpha, +\infty)$, as we describe in Chapter 5.

The solution $u(x,z)$ is unique up to a multiplicative constant and the Weyl-Titchmarsh m-function may be defined as[4]

$$m(z^2) = \left.\frac{u'(x,z)}{u(x,z)}\right|_{x=\alpha}. \tag{1.4}$$

Here $(')$ denotes the derivative in the x variable.

The classical direct spectral problem for L_q is to provide a complete characterization of the mapping

$$\mathcal{D}_L : q \mapsto m$$

and the inverse spectral problem is to characterize the mapping

$$\mathcal{I}_L : m \mapsto q.$$

The characterization of $\mathcal{D}_L$ is essentially due to Hermann Weyl in the early 1900s. See, for example, the thorough treatment provided by Chapter 9 of [10]. The classical approach to the inverse theory is to derive the Gelfand-Levitan-Marchenko integral equations for L_q. The solution to the inverse problem by this approach are detailed, for example, in [30]. More recent techniques include Barry Simon's A-function approach [42] and the de Branges space method of Christian Remling in [39].

Classical uniqueness results for the mapping D_L were obtained by Borg in [7] and Marchenko in [32]. Borg, for example, proves:

Theorem 1.2.3 (Borg [7])**.** *Let q_1, q_2 be real-valued, continuous potentials, and let m_1, m_2 be the corresponding m-functions for the operators L_{q_1}, L_{q_2}. Then $m_1 = m_2$ if and only if $q_1 = q_2$.*

Simon in [42] considerably enhances this result. He proves that, in fact, the asymptotics of the m-function determine the behavior of the potential locally.

Theorem 1.2.4 (Simon [42])**.** *Let $q_1, q_2 \in L^1_{\text{loc}}[0,+\infty)$. Let L_{q_1}, L_{q_2} have the respective m-functions m_1, m_2. Then*

$$m_1(-k^2) - m_2(-k^2) = \tilde{o}(e^{-2k\beta}),$$

as $k \to +\infty$ if and only if $q_1 = q_2$ a.e. on $[0,\beta]$.

The asymptotic notation $\widetilde{o}$ is from [42] and defined as follows. If $g(x) \to 0$ as $|x| \to +\infty$, then we write $f(x) = \widetilde{o}\,(g(x))$ when

$$\forall \epsilon > 0, \quad \lim_{x\to+\infty} \frac{|f(x)|}{|g(x)|^{1-\epsilon}} = 0.$$

As shown in Appendix 2 of [42], if

$$f(k) = \int_0^\infty e^{-2k\xi} C(\xi)\, d\xi = \tilde{o}\left(e^{-2k\beta}\right)$$

[4]Here, we assume that L_q is in the limit-point case at $+\infty$. See the discussion in Chapter 5.

then $C(\xi)$ is supported on $[0, \beta]$, which is a Paley-Wiener type result for the Laplace transform.

The key to Simon's method is to prove the existence of a function $A \in L^1(\mathbb{R})$ such that

$$m(-k^2) = -k - \int_0^\infty A(\xi) e^{-2k\xi} \, d\xi.$$

Among other remarkable properties of the A-function, is the fact that the values of A on $[0, x]$ uniquely determine $q(x)$. In light of our local determination result (Theorem 1.2.2), our expansion of

$$r_-(z) = \int_\alpha^\infty C(\xi) e^{2iz\xi} \, d\xi$$

for $w \in \mathcal{K}_\alpha$ is a ZS-AKNS analog of Simon's A-function expansion of the m-function.

Our work with the ZS-AKNS equation on $\mathcal{K}_\alpha$ also provides a means to extend the direct and inverse spectral theory of L_q to the set of potentials $\mathcal{M}_0$.

Let $q \in \mathcal{M}_0$. In analogy to (1.4), we associate to the operator L_q the *modified m-function*

$$m(z^2) = \left. \frac{u^{[1]}(x, z)}{u(x, z)} \right|_{x=0} \tag{1.5}$$

where u is an L^2-solution to the q-potential Schrödinger equation and $u^{[1]}$ is the regularization

$$u^{[1]} = u' - wu.$$

Now if $\Psi^{(i)}(x, z)$ is a column of the solution Ψ to the ZS-AKNS system (1.1) then

$$\chi(x, z) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \cdot \Psi^{(i)}(x, z) \tag{1.6}$$

is a solution to the Schrödinger equation (1.3). The same correspondence allows the authors of [21, 24] to study the scattering theory on the line for the Schrödinger equation with Miura potential via the Gelfand-Levitan-Marchenko approach to the ZS-AKNS system. We therefore study the modified m-function through the machinery we have developed for the ZS-AKNS system with potentials in $\mathcal{K}_0$. Using the ZS-AKNS results for $w \in \mathcal{K}_0$, we will characterize the mappings

$$\begin{aligned} \mathcal{D}_L &: (w' + w^2) \mapsto m \\ \mathcal{I}_L &: m \mapsto (w' + w^2). \end{aligned}$$

where m is the modified m-function. We then deploy our local determination result (1.2.1) for the ZS-AKNS equation to prove the following analog of Simon's Theorem 1.2.4.

Theorem 1.2.5. *Let $q_1, q_2 \in \mathcal{M}_0$ and let L_{q_1}, L_{q_2} have the respective modified m-functions m_1, m_2. If*

$$m_1(-k^2) - m_2(-k^2) = \tilde{o}(e^{-2k\beta}),$$

as $k \to +\infty$ then $q_1 = q_2$ on $[0, \beta]$.

The recent preprint [17] establishes a local Borg-Marchenko Theorem (Theorem 4.8) that is considerably more general than our Theorem 1.2.5. In the language of these authors, the correspondence between a column solution of the ZS-AKNS equation and the Schrödinger equation expressed by (1.6) is an example of a *supersymmetry*, and it is not difficult to generalize this relationship to the case of matrix valued $w(x)$. The supersymmetry can then be exploited to connect the Weyl-Titchmarsh theory of Dirac systems developed by Clark and Gesztesy [8, 9] to the spectral theory of the Miura (matrix) potential Schrödinger equation. We believe our result (1.2.5) and its simple proof complements the more technical development presented by these authors, as in our work the Dirac operator Weyl-Titchmarsh technology is replaced by standard inverse scattering and complex analysis techniques.

Chapter 2 Direct Scattering for the ZS-AKNS Equation

2.1 Overview

In this chapter, we prove the forward part of Theorem 1.2.1. The main result is:

Theorem 2.1.1. *The map $\mathcal{D}_P^-$ on the set $\mathcal{K}_\alpha$ takes its image in $\widehat{\mathcal{K}_\alpha}$.*

Explicitly, the reflection coefficient for $w \in \mathcal{K}_\alpha$ can be written as

$$r_-(z) = \int_\alpha^\infty e^{2iz\xi} C(\xi)\, d\xi, \tag{2.1}$$

where $C \in X$. Using a symmetry of the real potential ZS-AKNS equation (Proposition 2.3.5 below), the theorem immediately implies:

Corollary 2.1.1. *The map $\mathcal{D}_P^+$ on the set $\mathcal{K}^\beta$ takes its image in $\widehat{\mathcal{K}^\beta}$.*

Let us also explain how Theorem 2.1.1 and Corollary 2.1.1 imply the forward direction of the Nonlinear Paley-Wiener Theorem 1.1.1. To see why, suppose that (2.1) holds. A priori, the reflection coefficient $r_-(z)$ for $w \in X$ is defined only for $z \in \mathbb{R}$. Consider the extension of $r_-(z)$ obtained by allowing $z \in \overline{\mathbb{C}^+}$ in equation (2.1). If $\gamma \subset \mathbb{C}^+$ is any simple closed curve, it is easy to see that

$$\begin{aligned}\int_\gamma |r_-(z)|\ dz &\leq \int_\gamma e^{-2\alpha \Im z} \|C\|_1\, dz \\ &\leq \left(\max_{z \in \gamma}\ e^{-2\alpha \Im z} \right) \text{length}(\gamma) \|C\|_1\end{aligned}$$

We may then apply Fubini's Theorem to obtain

$$\int_\gamma r_-(z)\, dz = 0$$

for any closed curve $\gamma \subset \mathbb{C}^+$. By Morera's Theorem, $r_-(z)$ is analytic in $\mathbb{C}^+$. Moreover, the continuation of $r_-(z)$ satisfies

$$|r_-(R+iT)| \leq Ce^{-2\alpha T} \qquad T \geq 0.$$

By a completely analogous computation, if

$$r_+(z) = \int_{-\infty}^\beta C(x) e^{-2izx}\, dx$$

then $r_+(z)$ continues analytically to $\mathbb{C}^+$ and the continuation of $r_+(z)$ satisfies

$$|r_+(R+iT)| \leq Ce^{2\beta T} \qquad T \geq 0.$$

Therefore, the forward part of the Nonlinear Paley-Wiener Theorem 1.1.1 is a simple consequence of Theorem 2.1.1 and Corollary 2.1.1. Before proceeding to the proofs of our main results, we must first make rigorous the definitions of the direct scattering maps $\mathcal{D}_P^\pm$.

2.2 Direct Scattering Theory on the Line

In this section, we present a standard derivation of the direct scattering theory for the ZS-AKNS system over the class of real potentials X. Our presentation follows closely the treatments that can be found, for example, in the references [1, 12, 21, 36, 50].

Notation and Preliminaries

Let $M_2(\mathbb{C})$ be the space of 2×2 matrices. We will sometimes consider $M_2(\mathbb{C})$ equipped with the matrix 2-norm:

$$|A|_2 = \max_{\substack{|x|=1 \\ x\in\mathbb{C}^2}} \frac{|Ax|}{|x|},$$

where $|\cdot|$ denotes the standard Euclidean length on $\mathbb{C}^2$. We recall that $|A|_2$ is equal to the largest singular value of A, i.e. the square root of the largest eigenvalue of A^*A.[1]

Define the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

By $\mathbf{1}$ we denote the identity matrix in $M_2(\mathbb{C})$; by I we denote an identity operator on a given function space.

Recall that a function $f : \mathbb{R} \to \mathbb{C}$ is *absolutely continuous on* $\mathbb{R}$ if and only if there exists a function $g \in L^1(\mathbb{R})$ and a constant $c \in \mathbb{C}$ such that for all $x \in \mathbb{R}$

$$f(x) = c + \int_{-\infty}^{x} g(y)\, dy.$$

By the Lebesgue Differentiation Theorem, if f is absolutely continuous on $\mathbb{R}$ then f is differentiable almost everywhere with derivative $f'(x) = g(x)$.

The 2×2 ZS-AKNS system is the first order system

$$\frac{d}{dx}\Psi = iz\sigma_3\Psi + Q(x)\Psi \tag{2.2}$$

with

$$Q(x) = \begin{bmatrix} 0 & w(x) \\ w(x) & 0 \end{bmatrix}$$

and $w \in X$.[2]

[1] See, for example, Lemma 1.7 of [13] where the author collects a number of useful properties of matrix norms.

[2] Actually, this is only a special case of the ZS-AKNS system. In more general treatments, w is allowed to be complex-valued and

$$Q(x) = \begin{bmatrix} 0 & w(x) \\ \overline{w(x)} & 0 \end{bmatrix}.$$

A function $\Psi : \mathbb{R} \to M_2(\mathbb{C})$ is a solution to the ZS-AKNS equation if Ψ is absolutely continuous and satisfies (2.2) almost everywhere. To make explicit the dependence of Ψ on the spectral parameter $z \in \mathbb{C}$, we will often write $\Psi(x, z)$.

The entries of the matrix $\Psi(x, z)$ are the scalars $\psi_{ij}(x, z)$. When each of the component functions $\psi_{ij}(\cdot, z)$ of $\Psi(\cdot, z)$ belong to X, we will say that $\Psi(\cdot, z)$ belongs to the space

$$X \otimes M_2(\mathbb{C}) = \left\{ \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix} : \psi_{ij} \in X \right\}.$$

We now recall the definition of the Hardy spaces $H^2(\mathbb{C}^{\pm})$, which play an important role in the direct and inverse scattering theory of the ZS-AKNS equation. A function f analytic in $\mathbb{C}^+$ belongs to $H^2(\mathbb{C}^+)$ if

$$\sup_{T>0} \left(\int_{-\infty}^{\infty} |f(R + iT)|^2 \, dR \right)^{1/2} < +\infty.$$

The definition for $H^2(\mathbb{C}^-)$ is analogous. These spaces have the useful characterizations

$$H^2(\mathbb{C}^+) = \left\{ f(z) = \int_0^{\infty} \tilde{f}(x) e^{2ixz} \, dx : \tilde{f} \in L^2(\mathbb{R}) \right\}$$

and

$$H^2(\mathbb{C}^-) = \left\{ f(z) = \int_{-\infty}^{0} \tilde{f}(x) e^{2ixz} \, dx : \tilde{f} \in L^2(\mathbb{R}) \right\};$$

see for example Chapter 2.3 of [16]. We will also require the Hardy spaces $H^{\infty}(\mathbb{C}^{\pm})$: $f \in H^{\infty}(\mathbb{C}^{\pm})$ if

$$\sup_{T>0} \left(\sup_{R \in \mathbb{R}} |f(R \pm iT)| \right) < +\infty.$$

For a detailed treatment of these important function spaces, we suggest the standard references [15, 16].

Symmetries of the ZS-AKNS Equation

The ZS-AKNS equation possesses a number of symmetries which are useful in developing its direct and inverse scattering theory. In this section, we highlight some of the structural features of (2.2) that will prove useful for establishing a number of results in this thesis.

Proposition 2.2.1. *Let $\Psi(x, z)$ be a solution to the ZS-AKNS equation (2.2) with potential $w \in X$. Then:*

1. $\det \Psi(x, z)$ *is independent of* x.
2. *If*

$$\Psi_1(x, z) = \sigma_1 \overline{\Psi(x, \bar{z})} \sigma_1$$

then $\Psi_1(x, z)$ is also a solution to (2.2) with potential w.

3. *Let $\Psi_1(x,z), \Psi_2(x,z)$ be solutions to (2.2) with potential w. If* $\det\ \Psi_1(x_0,z) \neq 0$ *for some x_0 then $\Psi_1(x,z)^{-1}\Psi_2(x,z)$ is independent of x.*

Proof. Since $\Psi(x,z)$ solves (2.2), for a.e. $x \in \mathbb{R}$, we easily compute

$$\begin{aligned}\frac{d}{dx}\det\Psi(x,z) &= \frac{d}{dx}\left(\psi_{11}\psi_{22} - \psi_{21}\psi_{12}\right)\\ &= \psi_{11}'\psi_{22} + \psi_{11}\psi_{22}' - \psi_{21}'\psi_{12} - \psi_{21}\psi_{12}'\\ &= (iz\psi_{11} + w(x)\psi_{21})\psi_{22} + \psi_{11}(-iz\psi_{22} + w(x)\psi_{12})\\ &\quad - (-iz\psi_{21} + w(x)\psi_{11})\psi_{12} - \psi_{21}(iz\psi_{12} + w(x)\psi_{22})\\ &= 0.\end{aligned}$$

Therefore, $\det\Psi(x,z) = c$ for a.e. x with $c \in \mathbb{C}$ a constant. But since $\Psi(x,z)$ is continuous, $\det\Psi(x,z) = c$ holds for every x. Conclude that $\det\Psi(x,z)$ is independent of x.

For part (2), note that

$$\sigma_1\sigma_3 = -\sigma_3\sigma_1$$

and that $\sigma_1 Q(x) = Q(x)\sigma_1$. Then

$$\begin{aligned}\frac{d}{dx}\Psi_1(x,z) &= \frac{d}{dx}\left[\sigma_1\overline{\Psi(x,\bar{z})}\sigma_1\right]\\ &= -iz\sigma_1\sigma_3\overline{\Psi(x,\bar{z})}\sigma_1 + \sigma_1 Q(x)\overline{\Psi(x,\bar{z})}\sigma_1\\ &= iz\sigma_3\Psi_1(x,z) + Q(x)\Psi_1(x,z).\end{aligned}$$

Conclude that $\Psi_1(x,z)$ is a solution to the ZS-AKNS equation whenever $\Psi(x,z)$ is.

For part (3), $\Psi_1(x,z)$ is invertible for all x since $\det\Psi_1(x_0,z) \neq 0$ and $\det\Psi_1(x,z) \neq 0$ is independent of x. Compute

$$\begin{aligned}\frac{d}{dx}\left(\Psi_1^{-1}\Psi_2\right) &= -\Psi_1^{-1}\Psi_1'\Psi_1^{-1}\Psi_2 + \Psi_1^{-1}\Psi_2'\\ &= -\Psi_1^{-1}\left(iz\sigma_3\Psi_1 + Q\cdot\Psi_1\right)\Psi_1^{-1}\Psi_2\\ &\qquad + \Psi_1^{-1}\left(iz\sigma_3\Psi_2 + Q\cdot\Psi_2\right)\\ &= 0,\end{aligned}$$

holding for a.e. x. □

The following useful result is a simple consequence of the facts collected in the previous proposition.

Proposition 2.2.2. *Let $\Psi_1(x,z), \Psi_2(x,z)$ be two solutions to the ZS-AKNS equation (2.2) for a fixed potential $w \in X$. Suppose that* $\det\ \Psi_1(x_0,z) \neq 0$ *for some $x_0 \in \mathbb{R}$. Then there exists a matrix $M(z)$ so that*

$$\Psi_2(x,z) = \Psi_1(x,z)M(z).$$

Proof. Set

$$M(z) = \Psi_1(x,z)^{-1}\Psi_2(x,z)$$

By Proposition 2.2.1, $M(z)$ is independent of x and clearly

$$\Psi_1(x,z) = \Psi_2(x,z)M(z).$$

□

The reality of $w(x)$ introduces further symmetry to the ZS-AKNS equation. The following proposition shows how to translate results specialized to potentials supported on the right half-line $[\alpha, +\infty)$ to potentials supported on the left half-line $(-\infty, -\alpha]$.

Proposition 2.2.3. *Suppose $\Psi(x,z)$ is a solution to the ZS-AKNS equation (2.2) with $w \in X$. Then*

$$\tilde{\Psi}(x,z) = \Psi(-x,-z).$$

satisfies (2.2) with w replaced by $\widetilde{w}(x) = -w(-x)$.

Proof. We easily check that

$$\begin{aligned}\frac{d}{dx}\tilde{\Psi}(x,z) &= -\Psi_x(-x,-z)\\ &= -\left[-iz\sigma_3\Psi(-x,-z) + Q(-x)\Psi(-x,-z)\right]\\ &= iz\sigma_3\tilde{\Psi}(x,z) + \tilde{Q}(x)\tilde{\Psi}(x,z)\end{aligned}$$

holds for a.e. x. □

Jost Solutions and the Direct Scattering Map

Taking $w(x) = 0$ in (2.2), we obtain the *free ZS-AKNS equation*. It is easy to see that the free ZS-AKNS equation has the *free solution*

$$\exp(ixz\sigma_3) = \begin{bmatrix} e^{ixz} & 0 \\ 0 & e^{-ixz}\end{bmatrix}.$$

For matrix potential $Q(x) \neq 0$, the ZS-AKNS equation is a perturbation of the free equation. If $Q(x)$ decays as $x \to \pm\infty$ then the effect of this perturbation should dissipate out as $x \to \pm\infty$. Motivated by this heuristic argument, we seek solutions of the ZS-AKNS system with the same asymptotics as the free solution. We will prove that for $w \in X$, there exist unique solutions $\Psi_\pm$ to (2.2) satisfying the respective asymptotic conditions

$$\lim_{x\to\pm\infty} \left|\Psi_\pm(x,z) - e^{ixz\sigma_3}\right|_2 = 0. \tag{2.3}$$

These solutions are called the *left and right Jost solutions*. In slightly different terms, the Jost solutions are the matrix eigenfunctions for the ZS-AKNS that are asymptotic at either $+\infty$ or $-\infty$ to the exponential families $\exp(ixz\sigma_3)$.

The following relationship between the left and right Jost solutions is the key to the direct scattering theory of the ZS-AKNS equation.

Proposition 2.2.4. *Suppose for all $z \in \mathbb{R}$ the solutions $\Psi_\pm(x,z)$ exist and are unique. Then for each $z \in \mathbb{R}$ there is a matrix $R(z)$ such that*

$$\Psi_+(x,z) = \Psi_-(x,z)R(z). \tag{2.4}$$

Moreover, $R(z)$ has the form

$$R(z) = \begin{bmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix},$$

where

$$|a(z)|^2 - |b(z)|^2 = 1.$$

Proof. Since $\Psi_\pm$ are solutions to ZS-AKNS, we can apply part 1 of Proposition 2.2.1 and the asymptotic conditions on $\Psi_\pm$ to conclude that

$$\det \Psi_\pm = \det \begin{bmatrix} e^{ixz} & 0 \\ 0 & e^{-ixz} \end{bmatrix} = 1.$$

Using part 3 of Proposition 2.2.1, define

$$R(z) = \Psi_-(x,z)^{-1}\Psi_+(x,z).$$

A priori, the matrix R has the form

$$R(z) = \begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}.$$

For real z, part 2 of Proposition 2.2.1 implies that

$$\tilde{\Psi}_\pm(x,z) = \sigma_1 \overline{\Psi_\pm(x,z)} \sigma_1$$

are also solutions to ZS-AKNS. By the asymptotic conditions

$$\overline{\Psi_\pm(x,z)} \sim e^{-ixz\sigma_3},$$

and we can compute that

$$\sigma_1 e^{-ixz\sigma_3}\sigma_1 = \sigma_1 \begin{bmatrix} e^{-ixz} & 0 \\ 0 & e^{ixz} \end{bmatrix} \sigma_1 = e^{ixz\sigma_3}.$$

Uniqueness of $\Psi_\pm$ gives

$$\Psi_\pm = \sigma_1 \overline{\Psi_\pm} \sigma_1.$$

Noting that $\sigma_1\sigma_1 = \mathbf{1}$, compute

$$\begin{aligned} R(z) &= \Psi_-(x,z)^{-1}\Psi_+(x,z) \\ &= (\sigma_1 \overline{\Psi_-(x,z)}\sigma_1)^{-1}\sigma_1 \overline{\Psi_+(x,z)}\sigma_1 \\ &= \sigma_1 \overline{\Psi_-(x,z)^{-1}\Psi_+(x,z)}\sigma_1 \\ &= \sigma_1 \overline{R(z)}\sigma_1. \end{aligned}$$

This gives the symmetry

$$\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix} = \begin{bmatrix} \overline{a}_{22}(z) & \overline{a}_{21}(z) \\ \overline{a}_{12}(z) & \overline{a}_{11}(z) \end{bmatrix}.$$

Set $a = a_{11}, b = a_{21}$ and use the fact that $\det \Psi_\pm = 1$ to conclude that $R(z)$ has the form given in the proposition. ∎

The Reflection Coefficients

Suppose for the moment that the solutions $\Psi_\pm(x,z)$ exist and are unique for any $w \in X$. Then for each fixed potential $w \in X$, we may apply the previous proposition to define the coefficients

$$r_-(z) = \frac{b(z)}{a(z)} \qquad r_+(z) = -\frac{\bar{b}(z)}{a(z)} \tag{2.5}$$

for $z \in \mathbb{R}$. Note that $|a(z)|^2 - |b(z)|^2 = 1$ so that

$$|a(z)|^2 \geq 1$$

and

$$\|r_\pm\|_{L^\infty(\mathbb{R})} < 1.$$

Hence, the mappings

$$\mathcal{D}_P^\pm : w \mapsto r_\pm$$

are well-defined on X, provided that we can establish the existence and uniqueness of $\Psi_\pm$. Before attending to the technical existence and uniqueness issue, let us first provide a physical characterization of the reflection coefficients $r_\pm(z)$.

Observe that we can use relation (2.4) to obtain the asymptotic formulas

$$\begin{aligned} x \to +\infty : \qquad & \Psi_+(x,z) \sim \exp(ixz\sigma_3) \\ x \to -\infty : \qquad & \Psi_+(x,z) \sim \exp(ixz\sigma_3)R(z). \end{aligned}$$

Let

$$t(z) = \frac{1}{a(z)}$$

and

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If $\Psi_+^{(1)}$ denotes the first column of Ψ_+ then

$$\begin{aligned} x \to +\infty : \qquad & \frac{\Psi_+^{(1)}(x,z)}{a(z)} \sim t(z)\mathbf{e}_1\, e^{ixz} \\ x \to -\infty : \qquad & \frac{\Psi_+^{(1)}(x,z)}{a(z)} \sim \mathbf{e}_1\, e^{ixz} + r_-(z)\mathbf{e}_2\, e^{-ixz} \end{aligned} \tag{2.6}$$

These equations depict a hypothetical process where a wave e^{ixz} tuned to frequency z travels in from the left and interacts with the potential Q. The potential partially transmits and partially scatters the incident wave. The transmitted part corresponds to $t(z)\mathbf{e}_1 e^{ixz}$, while the reflected part corresponds to $r_-(z)\mathbf{e}_2 e^{-ixz}$ (see the diagram in Figure 2.2). By a similar computation, the second column $\Psi_+^{(2)}$ of Ψ_+ corresponds to a process where a wave incident on the right interacts with $Q(x)$. In this case, the coefficient $\overline{r_-}(z)$ corresponds to the reflection of the incident wave.

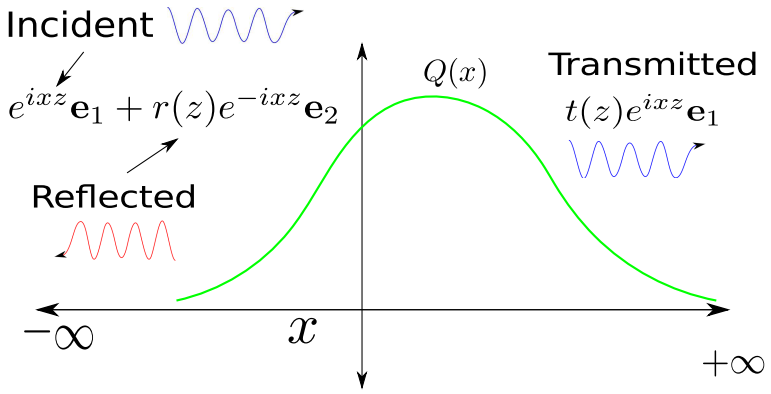

Figure 2.1: In this cartoon diagram of scattering for the ZS-AKNS system, a wave incident on the left interacts with the potential. The incident wave is partly transmitted and partly reflected.

Of course, we could also take the limit as $x \to +\infty$ in the relation (2.4) to obtain the asymptotic formulas

$$x \to +\infty : \qquad \exp(ixz\sigma_3)R(z)^{-1} \sim \Psi_-(x,z).$$

If $\Psi_-^{(2)}$ is the second of column of Ψ_-, we then write

$$\begin{aligned} x \to -\infty : &\qquad \Psi_-^{(2)}(x,z) \sim \mathbf{e}_2 e^{ixz} \\ x \to +\infty : &\qquad \Psi_-^{(2)}(x,z) \sim -\bar{b}(z)\mathbf{e}_1 e^{ixz} + a(z)\mathbf{e}_2 e^{-ixz}. \end{aligned}$$

In this case, it is natural to define the reflection coefficient

$$r_+(z) = -\frac{\bar{b}(z)}{a(z)}.$$

The relationship between left and right scattering is well-known; see, for example, [12, 50]. Since $|a(z)|^2 - |b(z)|^2 = 1$ for $z \in \mathbb{R}$ and $\|r_\pm\|_{L^\infty(\mathbb{R})} < 1$, it is easy to see that

$$|a(z)|^2 = \frac{1}{1 - |r_\pm(z)|^2}.$$

As we shall see, $a(z)$ extends to a bounded analytic function in $\mathbb{C}^+$ satisfying $a(z) \to 1$ as $z \to +\infty$ in $\mathbb{C}^+$. Also, $a(z)$ is non-vanishing on $\overline{\mathbb{C}^+}$ (see Lemma 2.3.1, below). We may therefore recover $a(z)$ in $\mathbb{C}^+$ from its modulus by the formula

$$a(z) = \exp\left(\mathcal{C}(\log\,|a|)(z)\right)$$

where $\mathcal{C}$ is the Cauchy operator on $L^2(\mathbb{R})$ (see Appendix C). On the line, we then recover $a(z)$ from the boundary values

$$a(z) = \exp\left(\mathcal{C}_+(\log\,|a|)(z)\right).$$

Since we can always recover the coefficient $a(z)$ from the modulus of either reflection coefficient $r_\pm$, we can use the formula

$$b(z) = a(z)r_-(z) = -\overline{a(z)r_+(z)},$$

to recover $b(z)$. Therefore, knowing either reflection coefficient is equivalent to knowing both reflection coefficients.

Existence and Uniqueness of the Jost Solutions

It remains to prove that the Jost solutions $\Psi_\pm$ exist and are unique for all $z \in \mathbb{R}$. We present an argument similar to the proof developed in [21], and also refer to [20] where many of the results of [21] are worked out in detail. The first step is to formulate the ZS-AKNS system as a system of integral equations, and for this it is convenient to introduce a factorization. Following [50], factor out the leading singularity at $\pm\infty$ of the free solution to ZS-AKNS by setting

$$\Psi = \Phi e^{ixz\sigma_3}. \tag{2.7}$$

Now differentiate this factorization to obtain

$$iz\sigma_3\Psi + Q(x)\Psi = \Phi_x e^{ixz\sigma} + iz\Phi\sigma_3 e^{ixz\sigma},$$

so that the matrix Φ satisfies

$$\Phi_x = iz\sigma_3\Phi - iz\Phi\sigma_3 + Q(x)\Phi. \tag{2.8}$$

Let ad_{σ_3} be the linear operator on $M_2(\mathbb{C})$ defined by

$$\mathrm{ad}_{\sigma_3}(A) = [\sigma_3, A] = \sigma_3 A - A\sigma_3.$$

Write the system (2.8) as

$$\Phi_x - iz\ \mathrm{ad}_{\sigma_3}\Phi = Q(x)\Phi. \tag{2.9}$$

The Jost solutions $\Psi_\pm$ to ZS-AKNS will correspond to the solutions $\Phi_\pm$ of the system (2.9) satisfying the asymptotic conditions

$$\lim_{x\to\pm\infty} |\Phi_\pm(x,z) - \mathbf{1}| = 0. \tag{2.10}$$

We can integrate (2.9) with either of the conditions (2.10) by introducing an exponential operator of the form $\exp(izx\ \mathrm{ad}_{\sigma_3})$. This operator is determined from the eigenbasis of the linear operator ad_{σ_3}. It is easy to compute that ad_{σ_3} has eigenvalues $\lambda = -2,\ 0,\ 2$, so that the action of $\exp(ixz\mathrm{ad}_{\sigma_3})$ on $M_2(\mathbb{C})$ is given by

$$e^{izx\ \mathrm{ad}_{\sigma_3}} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & e^{2ixz}b \\ e^{-2ixz}c & d \end{bmatrix}.$$

The following properties of the exponential operator $\exp(\zeta\mathrm{ad}_{\sigma_3})$ can be readily verified by direct computation.

Proposition 2.2.5. *Let $A, B \in M_2(\mathbb{C})$. For any $\zeta \in \mathbb{C}$:*

1. $\exp(\zeta \ \mathrm{ad}_{\sigma_3})(A \cdot B) = (\exp(\zeta \mathrm{ad}_{\sigma_3})A) \cdot (\exp(\zeta \mathrm{ad}_{\sigma_3})B)$
2. $\exp(\zeta \ \mathrm{ad}_{\sigma_3})A = e^{\zeta\sigma_3} A e^{-\zeta\sigma_3}$.

Proof. The first identity follows from the second. The second identity holds because both sides of the equation are solutions to

$$A'(\xi) = \mathrm{ad}_{\sigma_3} \ A(\xi)$$

with $A(0) = A$. □

Applying the operator $\exp(-ixz\mathrm{ad}_{\sigma_3})$ to equation (2.9), we obtain

$$e^{-izx \ \mathrm{ad}_{\sigma_3}} \Phi_x - ize^{-ixz \ \mathrm{ad}_{\sigma_3}} \mathrm{ad}_{\sigma_3} \Phi = e^{-ixz \ \mathrm{ad}_{\sigma_3}} Q(x)\Phi,$$

which simplifies to

$$\left(e^{-ixz \ \mathrm{ad}_{\sigma_3}} \Phi\right)_x = e^{-ixz \ \mathrm{ad}_{\sigma_3}} Q(x)\Phi.$$

We integrate this equation from x to $\pm\infty$ and have:

$$\left[\lim_{y\to+\infty} e^{-izy \ \mathrm{ad}_{\sigma_3}} \Phi(y,z)\right] - e^{-ixz\mathrm{ad}_{\sigma_3}} \Phi(x,z) = \int_x^\infty e^{-izy \ \mathrm{ad}_{\sigma_3}} Q(y)\Phi(y,z) \ dy$$

$$e^{-ixz\mathrm{ad}_{\sigma_3}} \Phi(x,z) - \left[\lim_{y\to-\infty} e^{-izy \ \mathrm{ad}_{\sigma_3}} \Phi(y,z)\right] = \int_{-\infty}^x e^{-izy \ \mathrm{ad}_{\sigma_3}} Q(y)\Phi(y,z) \ dy$$

Now we apply the asymptotic conditions (2.10) and rearrange to arrive at the integral equations

$$\Phi_+(x,z) = \mathbf{1} - \int_x^{+\infty} e^{iz(x-y) \ \mathrm{ad}_{\sigma_3}} Q(y)\Phi_+(y,z) \ dy \tag{2.11}$$

$$\Phi_-(x,z) = \mathbf{1} + \int_{-\infty}^x e^{iz(x-y) \ \mathrm{ad}_{\sigma_3}} Q(y)\Phi_-(y,z) \ dy \tag{2.12}$$

Let Y be the Banach space of $M_2(\mathbb{C})$-valued functions $f : \mathbb{R} \to M_2(\mathbb{C})$ continuous in the supremum norm

$$\|f\|_Y = \sup_{x\in\mathbb{R}} \ |f(x)|_2.$$

Define the operators $T_\pm$ on Y by

$$\left(T_\pm f\right)(x) = \mp \int_x^{\pm\infty} e^{iz(x-y) \ \mathrm{ad}_{\sigma_3}} Q(y) f(y) \ dy. \tag{2.13}$$

and recast (2.11)-(2.12) as

$$\Phi_\pm = \mathbf{1} + T_\pm \Phi_\pm.$$

Formally, the solutions to these equations are

$$\Phi_\pm = (I - T_\pm)^{-1} \mathbf{1}.$$

As in [21], we will develop explicit Volterra series expressions for the operators $(I - T_\pm)^{-1}$.

Lemma 2.2.1 (Permutation Symmetry Lemma)**.** *For $n \in \mathbb{N}$:*

$$\int_{x \le y_1 \le \ldots \le y_n \le +\infty} |w(y_1)| \cdots |w(y_n)| \, dy_1 \cdots dy_n = \frac{1}{n!} \left(\int_x^\infty |w(y)| \, dy \right)^n$$
$$\int_{-\infty \le y_n \le \ldots \le y_1 \le x} |w(y_1)| \cdots |w(y_n)| \, dy_1 \cdots dy_n = \frac{1}{n!} \left(\int_{-\infty}^x |w(y)| \, dy \right)^n$$

Proof. The intuition behind this useful fact is the following. Label the permutations of the integer indices $(1, 2, ..., n)$ by $\sigma_j = (i_1, i_2, ..., i_n)$ for $j = 1, ..., n!$ with σ_1 the identity permutation. Define the sets

$$R_{\sigma_j}(x) = \{(y_1, y_2, .., y_n) \in \mathbb{R}^n : x < y_{i_1} \le y_{i_2} \le \cdots \le y_{i_n} < +\infty\}.$$

Observe that

$$H(x) = \bigcup_{j=1}^{n!} R_{\sigma_j}(x) = \{(y_1, y_2, ..., y_n) \in \mathbb{R}^n : y_i > x \text{ for } i = 1, ..., n\}.$$

and $R_{\sigma_i} \cap R_{\sigma_j}$ has measure zero for $i \neq j$.

Let

$$I_{\sigma_j}(x) = \int_{R_{\sigma_j}} |w(y_{i_1})| \cdots |w(y_{i_N})| \, dy_{i_1} \cdots dy_{i_n}$$

By the change of variables formula, we see that for any permutation

$$I_{\sigma_1}(x) = I_{\sigma_j}(x).$$

Then we have

$$n! I_{\sigma_1}(x) = \sum_{j=1}^{n!} I_{\sigma_j}(x) = \int_{H(x)} |w(y_1)||w(y_2)| \cdots |w(y_n)| \, dy_1 \cdots dy_n$$

and clearly

$$I_{\sigma_1}(x) = \frac{1}{n!} \int_{H(x)} |w(y_1)||w(y_2)| \cdots |w(y_n)| \, dy_1 \cdots dy_n = \frac{\left(\int_x^\infty |w(y)| \, dy \right)^n}{n!}.$$

A symmetric argument applies to the second inequality of the lemma. A proper proof follows directly from a straightforward induction argument. □

Lemma 2.2.2. *For $z \in \mathbb{R}$ and $w \in L^1(\mathbb{R})$, the operators $(I - T_\pm)^{-1}$ exist as bounded linear operators on Y and satisfy the estimates*

$$\|(I - T_\pm)^{-1}\|_{Y \to Y} \le \exp\left(\|w\|_{L^1(\mathbb{R})} \right).$$

The operators $(I - T_\pm)^{-1}$ have convergent series expansions

$$(I - T_\pm)^{-1} = \sum_{n=0}^{\infty} T_\pm^n.$$

Proof. First, we show that the iterates of $T_\pm$ satisfy

$$|(T_+^n f)(x)| \leq \left(\int_{x \leq y_1 \leq \ldots \leq y_n < +\infty} |w(y_1)| \cdots |w(y_n)| \, dy_1 \cdots dy_n \right) \|f\|_Y$$
$$|(T_-^n f)(x)| \leq \left(\int_{-\infty \leq y_n \leq \ldots \leq y_1 \leq x} |w(y_1)| \cdots |w(y_n)| \, dy_1 \cdots dy_n \right) \|f\|_Y.$$

Recall that the matrix 2-norm satisfies the sub-multiplicative property

$$|AB|_2 \leq |A|_2 \, |B|_2 \, .$$

Observe in addition that for any $\lambda, \upsilon \in \mathbb{C}$,

$$\left| \begin{bmatrix} 0 & \upsilon \\ \lambda & 0 \end{bmatrix} \right|_2 = \max \{|\lambda|, |\upsilon|\} \, .$$

During the rest of the proof, we suppress the subscript on the $|\cdot|_2$ norm. For fixed $z \in \mathbb{R}$ and $f \in Y$, we estimate

$$\begin{aligned}
|T_\pm f(x)| &\leq \pm \int_x^{\pm\infty} \left| e^{iz(x-y) \, \mathrm{ad}_{\sigma_3}} Q(y) f(y) \right| \, dy \\
&\leq \pm \int_x^{\pm\infty} \left| e^{iz(x-y) \, \mathrm{ad}_{\sigma_3}} Q(y) \right| |f(y)| \, dy \\
&\leq \int_x^{\pm\infty} \left(\sup_{t \geq y} |f(t)| \right) \left| \begin{bmatrix} 0 & e^{2iz(x-y)} w(y) \\ e^{-2iz(x-y)} w(y) & 0 \end{bmatrix} \right| \, dy \\
&\leq \pm \int_x^{\pm\infty} \left(\sup_{t \geq y} |f(t)| \right) |w(y)| \, dy
\end{aligned}$$

This completes the $n = 1$ base case of an inductive proof. Suppose that the result holds for n. We estimate for T_+ that

$$\begin{aligned}
\left| T_+^{n+1} f(x) \right| &\leq \int_x^{+\infty} \left(\sup_{t \geq x} |T_+^n f(t)| \right) |w(y)| \, dy \\
&\leq \|f\|_Y \int_{x \leq y_1 \leq \ldots \leq y_{n+1} < +\infty} |w(y_1)| \cdots |w(y_{n+1})| \, dy_1 \cdots dy_{n+1} \\
&\leq \frac{1}{(n+1)!} \left(\int_x^{\infty} |w(y)| \, dy \right)^{n+1} \|f\|_Y,
\end{aligned}$$

where the last line follows from the Permutation Symmetry Lemma (Lemma 2.2.1). A completely analogous computation proves the result for T_-^n. Thus, the Volterra series

$$(I - T_\pm)^{-1} = \sum_{n=0}^{\infty} T_\pm^n$$

converge and the resolvents $(I - T_\pm)^{-1}$ are bounded linear operators on Y with

$$\|(I - T_\pm)^{-1}\|_{Y \to Y} \leq \exp(\|w\|_1).$$

□

Now we can prove the existence and uniqueness of the Jost solutions $\Psi_\pm$ to ZS-AKNS.

Proposition 2.2.6. *For $w \in X$ and $z \in \mathbb{R}$, $\Psi_\pm$ exist and are unique.*

Proof. Fix $z \in \mathbb{R}$. By the previous lemma, $(I - T_\pm)^{-1}$ exist as bounded linear operators on the space Y. Therefore, the functions

$$\Phi_\pm(x, z) = (I - T_\pm)^{-1}\mathbf{1}$$

are continuous solutions to the integral equation

$$\Phi_\pm = \mathbf{1} + T_\pm \Phi_\pm.$$

Iterate this integral equation once to write

$$\Phi_\pm = \mathbf{1} + T_\pm (I - T_\pm)^{-1}\mathbf{1}.$$

Now for $f \in Y$, we have

$$\begin{aligned} T_\pm f(x) &= \mp \int_x^{\pm\infty} \exp^{iz(x-y)\mathrm{ad}_{\sigma_3}} \begin{bmatrix} 0 & w(y) \\ w(y) & 0 \end{bmatrix} f(y)\, dy \\ &= \mp \int_x^{\pm\infty} \begin{bmatrix} 0 & w(y)e^{2iz(x-y)} \\ w(y)e^{-2iz(x-y)} & 0 \end{bmatrix} f(y)\, dy \end{aligned}$$

When $f \in Y$ is bounded in each component, the integrand in $T_\pm$ is an integrable function. The bound

$$\|\Phi_\pm(\cdot, z)\|_Y \leq \exp(\|w\|_{L^1})$$

implies $\Phi_\pm$ is uniformly bounded. For each fixed z, $\Phi_\pm$ can be written in the form

$$\Phi_\pm = \mathbf{1} + E_\pm(x) \int_x^{+\infty} F_\pm(y)\, dy,$$

where the entries of $E_\pm$ are complex exponential factors and the entries of $F_\pm$ are $L^1(\mathbb{R})$ functions. It follows that $\Phi_\pm$ are absolutely continuous functions for each fixed z, as claimed.

Finally, suppose that $\Phi_1, \Phi_2 \in Y$ both satisfy

$$\Phi = \mathbf{1} + T_+ \Phi.$$

Then $\Phi = \Phi_1 - \Phi_2$ satisfies

$$(I - T_+)\Phi = 0.$$

Since $I - T_+$ is invertible on Y, it follows that $\Phi = 0$, and so, $\Phi_1 = \Phi_2$. An identical argument applies to T_-. We conclude that $\Phi_\pm$ are unique. Set

$$\Psi_\pm = \Phi_\pm e^{ixz\sigma_3}$$

to obtain the unique Jost solutions to the ZS-AKNS equation. □

2.3 The Fourier Representation Theorem for the Scattering Map

In the previous section, we established that the Jost solutions to the ZS-AKNS system for $z \in \mathbb{R}$ satisfy

$$\Psi_+(x,z) = \Psi_-(x,z)R(z), \tag{2.14}$$

where

$$R(z) = \begin{bmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix} \qquad |a(z)|^2 - |b(z)|^2 = 1.$$

Now we use the Neumann series for $\Psi_\pm$ together with relation (2.14) to develop Fourier transform formulas for the reflection coefficients $r_-(z) = b(z)/a(z)$ and $r_+(z) = -\bar{b}(z)/a(z)$. We will analyze how the support of the potential w determines analyticity of the coefficients $a(z), b(z), r_-(z)$, and $r_+(z)$. Throughout this section, we assume that $w \in X$ has support on $[\alpha, \beta]$ where $\alpha, \beta \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.[3]

The idea behind our approach is to use the series expressions for $\Psi_\pm$ to evaluate (2.14) at $x = \alpha$ or $x = \beta$, and then solve (2.14) for the reflection coefficients. This is precisely what is done for w supported on the whole line in [21], where the authors prove

$$r_-(z) = \mathcal{F}C_1(z)$$
$$r_+(z) = \mathcal{F}^{-1}C_2(z)$$

for some $C_1, C_2 \in X$.

Series Expansions for the Jost Solutions

To study $\Psi_\pm = \Phi_\pm e^{ixz\sigma_3}$, we follow [21] and introduce a second factorization. Returning to equation (2.9), we factor out the operator $\exp(ixz\mathrm{ad}_{\sigma_3})$ by defining

$$\Phi(x,z) = \exp(ixz\mathrm{ad}_{\sigma_3})\Theta(x,z). \tag{2.15}$$

By (2.10), it follows that

$$\lim_{x\to\pm\infty} |\Theta_\pm(x,z) - \mathbf{1}| = 0 \tag{2.16}$$

where $\Phi_\pm = \exp(ixz\mathrm{ad}_{\sigma_3})\Theta_\pm$.

From equation (2.9) and the factorization (2.15), we compute

$$\begin{aligned}\Phi_x &= iz\mathrm{ad}_{\sigma_3}\left(\exp(ixz\mathrm{ad}_{\sigma_3})\,\Theta\right) + \exp(ixz\mathrm{ad}_{\sigma_3})\Theta_x \\ &= iz\mathrm{ad}_{\sigma_3}\left(\exp(ixz\mathrm{ad}_{\sigma_3})\,\Theta\right) + Q(x)\cdot\left(\exp(ixz\mathrm{ad}_{\sigma_3})\,\Theta\right).\end{aligned}$$

Therefore,

$$\Theta_x = \exp(-ixz\mathrm{ad}_{\sigma_3})Q(x)\cdot\left(\exp(ixz\mathrm{ad}_{\sigma_3})\,\Theta\right)$$

Recalling Part 1 of Proposition 2.2.5, we have

$$\exp(-ixz\mathrm{ad}_{\sigma_3})Q(x)\cdot\left(\exp(ixz\mathrm{ad}_{\sigma_3})\,\Theta\right) = \left(\exp(-ixz\mathrm{ad}_{\sigma_3})Q(x)\right)\Theta.$$

[3] For simplicity, we abuse notation slightly by using the closed interval notation $[\alpha, \beta]$ even when α, β may be infinite.

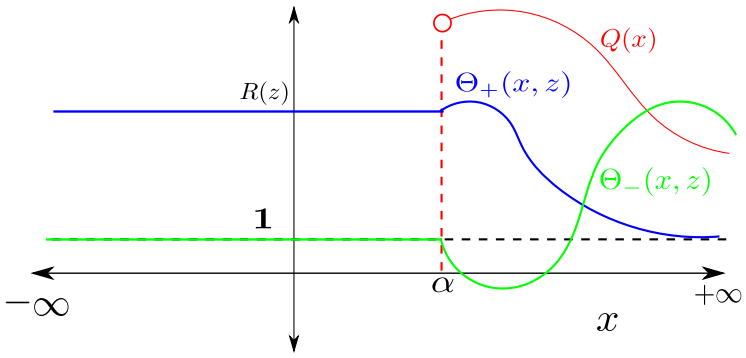

Figure 2.2: This sketch characterizes the relationship between Θ_+ and Θ_- in the case where w is supported on $[\alpha, +\infty)$ for α finite. The sketch is only a mnemonic device, as $\Theta_\pm, Q(x), R(z)$ are complex matrices.

Hence

$$\Theta_x(x,z) = (\exp(-ixz\mathrm{ad}_{\sigma_3})Q(x))\,\Theta(x,z)$$

We integrate this equation with the conditions (2.16) to obtain the pair of integral equations

$$\Theta_\pm(x,z) = \mathbf{1} - \int_x^{\pm\infty} (\exp(-iyz\mathrm{ad}_{\sigma_3})\,Q(y))\,\Theta_\pm(y,z)\,dy. \tag{2.17}$$

Now

$$\Psi_\pm = (\exp(ixz\,\mathrm{ad}_{\sigma_3})\Theta_\pm)\cdot e^{ixz\sigma_3}.$$

Recalling Part 2 of Proposition 2.2.5

$$\Psi_\pm = e^{ixz\sigma_3}\Theta_\pm.$$

The relation (2.14) implies

$$\Theta_+(x,z) = \Theta_-(x,z)R(z).$$

We now solve for $R(z)$ in (2.14) with these formulas. Recall that $\Psi_\pm(x,z) = (\exp(ixz\mathrm{ad}_{\sigma_3})\Theta_\pm)e^{ixz\sigma_3}$ and that $\Theta_\pm \to \mathbf{1}$ as $x \to \pm\infty$. The Jost solutions to the ZS-AKNS problem with $Q(x) = 0$ (the free problem) are $\Psi_\pm = e^{ixz\sigma_3}$. For $w(x)$ supported on $[\alpha,\beta] \subseteq \overline{\mathbb{R}}$, it follows by uniqueness of the Jost solutions that $\Psi_-(x,z) = e^{ixz\sigma_3}$ for any $x \le \alpha$. Similarly, $\Psi_+(x,z) = e^{ixz\sigma_3}$ for any $x \ge \beta$. We therefore have that $\Theta_+(x,z) = \mathbf{1}$ for $x \ge \beta$ and $\Theta_-(x,z) = \mathbf{1}$ for $x \le \alpha$. Thus,

$$\begin{aligned}\Theta_+(\alpha,z) &= R(z)\\ \Theta_-(\beta,z) &= R(z)^{-1}.\end{aligned}$$

The heuristic sketch appearing in Figure 2.2 illustrates the relationship between Θ_+ and Θ_- for α finite.

We summarize the computation of the matrix $R(z)$ from the solutions $\Theta_\pm$ in the following proposition.

Proposition 2.3.1. *Let $w \in X$ have support on $[\alpha, \beta] \subseteq \overline{\mathbb{R}}$. The limits*

$$\Theta_+(\alpha, z) := \lim_{x \to \alpha^-} \Theta_+(x, z)$$
$$\Theta_-(\beta, z) := \lim_{x \to \beta^+} \Theta_-(x, z)$$

are well-defined for all $z \in \mathbb{R}$. Moreover,

$$R(z) = \Theta_+(\alpha, z)$$
$$R(z)^{-1} = \Theta_-(\beta, z),$$

and in particular

$$\begin{aligned} a(z) &= [\Theta_+(\alpha, z)]_{11} = [\Theta_-(\beta, z)]_{22} \\ b(z) &= [\Theta_+(\alpha, z)]_{21} = -[\Theta_-(\beta, z)]_{21} \end{aligned} \tag{2.18}$$

for $z \in \mathbb{R}$.

Proof. For finite α, β, the result is an immediate consequence of the continuity of $\Psi_\pm$ established by Proposition 2.2.6. For α or β infinite, the result will follow from the series expressions derived for $\Theta_\pm$ below. □

Now we will derive explicit Neumann series expressions for $\Theta_\pm$.

$$\Theta_\pm(x, z) = \mathbf{1} + \sum_{n=1}^{\infty} S_\pm^n \mathbf{1}(x; z)$$

where for each fixed z the operators $S_\pm$ are defined by

$$S_\pm f(x; z) = \int_x^{\pm\infty} (\exp(-iyz\mathrm{ad}_{\sigma_3})\, Q(y))\, f(y)\, dy.$$

From Lemma 2.2.2 it readily follows that these series converges for each $x, z \in \mathbb{R}$.

The first few terms of the series for $\Theta_\pm - \mathbf{1}$ are

$$S_\pm \mathbf{1}(x; z) = \int_x^{\pm\infty} \begin{bmatrix} 0 & e^{-2iy_1 z} w(y_1) \\ e^{2iy_1 z} w(y_1) & 0 \end{bmatrix} dy_1$$
$$S_\pm^2 \mathbf{1}(x; z) = \int_x^{\pm\infty} \int_{y_1}^{\pm\infty} \begin{bmatrix} e^{2iz(y_2 - y_1)} w(y_1) w(y_2) & 0 \\ 0 & e^{-2iz(y_2 - y_1)} w(y_1) w(y_2) \end{bmatrix} dy_2 dy_1$$

We now introduce some additional notation to produce general formulas for $S_\pm^n \mathbf{1}$. Let

$$\vec{y} = (y_1, ..., y_n) \in \mathbb{R}^n$$

and

$$W_n(\vec{y}) = w(y_1) \cdots w(y_n).$$

Define $\vec{\alpha}_n$ as the n-component vector with k-th component equal to $(-1)^{n-k}$, i.e. for n even

$$\vec{\alpha}_n = (-1, 1, \ldots, -1, 1)$$

and for n odd

$$\vec{\alpha}_n = (1, -1, 1, \ldots, -1, 1).$$

Also define the half-spaces

$$H_n^+(x) = \{\vec{y} \in \mathbb{R}^n : x \leq y_1 \leq y_2 \leq ... \leq y_n\}$$

and

$$H_n^-(x) = \{\vec{y} \in \mathbb{R}^n : x \geq y_1 \geq y_2 \geq ... \geq y_n\},$$

and let $\chi_{H_n^{\pm}(x)}$ denote their respective characteristic functions.

Finally, we let

$$\mathcal{F}_n f\left(\vec{\zeta}\right) = \int_{\mathbb{R}^n} e^{2i\vec{x}\cdot\vec{\zeta}} f(\vec{x})\, d\vec{x}$$

denote the n-dimensional Fourier transform.

With this notation, we easily obtain the following formulas by induction.

Proposition 2.3.2.

$$S_{\pm}^n \mathbf{1}(x; z) = \begin{bmatrix} \mathcal{F}_n\left[\chi_{H_n^{\pm}(x)} \cdot W_n\right](z\vec{\alpha}_n) & 0 \\ 0 & \mathcal{F}_n\left[\chi_{H_n^{\pm}(x)} \cdot W_n\right](-z\vec{\alpha}_n) \end{bmatrix}$$

for n even, and

$$S_{\pm}^n \mathbf{1}(x; z) = \begin{bmatrix} 0 & \mathcal{F}_n\left[\chi_{H_n^{\pm}(x)} \cdot W_n\right](-z\vec{\alpha}_n) \\ \mathcal{F}_n\left[\chi_{H_n^{\pm}(x)} \cdot W_n\right](z\vec{\alpha}_n) & 0 \end{bmatrix}$$

for n odd.

We will now argue that the n-dimensional Fourier transform terms in the preceding proposition may be re-expressed in terms of a 1-dimensional Fourier transform. Recall the *Projection Slice Theorem* which says that slicing the n-dimensional Fourier transform of f along a line ℓ through the origin is equivalent to projecting f on to ℓ and taking a 1-dimensional Fourier transform.

Theorem 2.3.1 (Projection Slice Theorem, [35])**.** *Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let $\vec{\omega}$ be a fixed unit vector in $\mathbb{R}^n$. Then*

$$\mathcal{F}_n f(\rho\vec{\omega}) = \left(\mathcal{F}_1 P_{\vec{\omega}} f\right)(\rho) \tag{2.19}$$

where

$$P_{\vec{\omega}} f(t) = \int_{\vec{\omega}\cdot\vec{y}=t} f(\vec{y}) dS_n(\vec{y})$$

and dS_n is the $n-1$ dimensional Hausdorff measure on the hyperplane defined by $\vec{\omega}\cdot\vec{y} = t$ for $\vec{y} \in \mathbb{R}^n$.

Clearly, $W_n \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for any n. Since $|\alpha_n| = \sqrt{n}$, the Projection Slice Theorem gives

$$\begin{aligned}\mathcal{F}_n\left[\chi_{H_n^{\pm}(x)} \cdot W_n\right](z\,\vec{\alpha}_n) &= \int_{-\infty}^{\infty} e^{2i(z\sqrt{n})\sigma} \left(\int_{\vec{y}\cdot\vec{\alpha}_n = z\sqrt{n}} \chi_{H_n^{\pm}} \cdot W_n \, dS_n(\vec{y}) \right) d\sigma \\ &= \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} e^{2iz\xi} \left(\int_{\substack{\vec{y}\cdot\vec{\alpha}_n = \xi \\ \vec{y}\in H_n^{\pm}(x)}} W_n(\vec{y}) dS_n(\vec{y}) \right) d\xi\end{aligned}$$

Some Geometry for the Projection Terms

We briefly consider some geometric properties of the surface integrals

$$A_n^{\pm}(x;\xi) = \frac{1}{\sqrt{n}} \int_{\substack{\vec{y}\cdot\vec{\alpha}_n = \xi \\ \vec{y}\in H_n^{\pm}(x)}} W_n(\vec{y}) dS(\vec{y}).$$

For $n = 1$, it is not hard to see that

$$\begin{aligned}A_1^+(x;\xi) &= w(\xi)\chi_{[x,+\infty)}(\xi) \\ A_1^-(x;\xi) &= w(\xi)\chi_{(-\infty,x]}(\xi)\end{aligned}$$

in the L^2 sense.

Then for each fixed x, $A_1^+(x;\xi)$ has ξ-support in

$$\operatorname{supp}(w) \cap [x,+\infty) = (\alpha,\beta) \cap [x,+\infty),$$

and $A_1^-(x;\xi)$ has ξ-support in

$$\operatorname{supp}(w) \cap (-\infty,x] = [\alpha,\beta] \cap (-\infty,x].$$

Next, we observe that

$$\begin{aligned}A_2^+(x;\xi) &= \frac{1}{\sqrt{2}} \int_{\substack{y_2-y_1=\xi \\ x\le y_1\le y_2}} w(y_1)w(y_2)\, dS_2(\vec{y}) \\ A_2^-(x;\xi) &= \frac{1}{\sqrt{2}} \int_{\substack{y_2-y_1=\xi \\ x\ge y_1\ge y_2}} w(y_1)w(y_2)\, dS_2(\vec{y}),\end{aligned}$$

which are line integrals in $\mathbb{R}^2$. It is easy to see from Figure 2.3 that $A_2^+(x;\xi)$ has ξ-support in $[0,+\infty)$. A similar diagram can be drawn for A_2^- which shows that A_2^- has ξ-support in $(-\infty,0]$.

We now study the ξ-support properties of $A_n^{\pm}$ for general n; a summary of the results appears in Table 2.1.

General even n When n is even and $\vec{y} \in H_n^+(x)$ then

$$\xi = \vec{\alpha_n} \cdot \vec{y} = (y_2 - y_1) + \ldots + (y_n - y_{n-1}) \ge 0,$$

Similarly, if $\vec{y} \in H_n^-(x)$ then

$$\xi = \vec{\alpha_n} \cdot \vec{y} = (y_2 - y_1) + \ldots + (y_n - y_{n-1}) \le 0.$$

It follows that for n even $A_n^+(x;\xi)$ has support in $\xi \in [0,+\infty)$ and $A_n^-(x;\xi)$ has support in $\xi \in (-\infty,0]$.

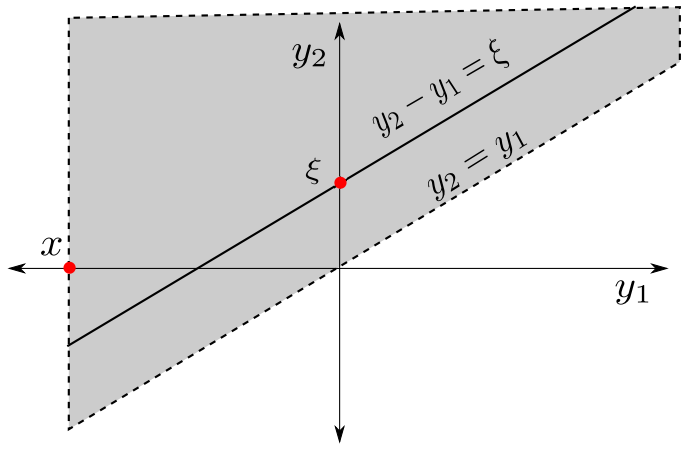

Figure 2.3: This diagram depicts the region of support for the integral defining $A_2^+(x;\xi)$. The gray region is the half-plane $H_2^+(x)$. The part of the line $y_2 - y_1 = \xi$ that lies in $H_2^+(x)$ is the integral support of $A_2(x;\xi)$. The integral support is empty if $\xi < 0$.

	n is even	n is odd
$A_n^+(x;\xi)$	$[0, +\infty)$	$[\alpha, \beta] \cap [x, +\infty)$
$A_n^-(x;\xi)$	$(-\infty, 0]$	$[\alpha, \beta] \cap (-\infty, x]$

Table 2.1: This table summarizes the ξ-support properties of $A_n(x;\xi)$ for each fixed x with w supported in $[\alpha, \beta] \subseteq \overline{\mathbb{R}}$.

General odd n Now when n is odd, for $\vec{y} \in H_n^+(x)$

$$\xi = \vec{\alpha_n} \cdot \vec{y} = y_1 + (y_3 - y_2) + ... + (y_n - y_{n-1}) \geq y_1 \geq x,$$

and for $\vec{y} \in H_n^-(x)$

$$\xi = \vec{\alpha_n} \cdot \vec{y} = y_1 + (y_3 - y_2) + ... + (y_n - y_{n-1}) \leq y_1 \leq x.$$

Since $w(y_1)$ vanishes when the coordinate y_1 lies outside of $[\alpha, \beta]$, $A_n^+(x;\xi)$ must also vanish for $\xi < \alpha$. Similarly, $A^-(x;\xi)$ must vanish for $\xi > \beta$.

Also, observe that if $\vec{y} \in H_n^+(x)$

$$\begin{aligned}\xi &= y_1 + (y_3 - y_2) + ... + (y_n - y_{n-1}) \\ &\leq y_2 + (y_3 - y_2) + ... + (y_n - y_{n-1}) \\ &= y_3 + (y_5 - y_4) + ... + (y_n - y_{n-1}) \\ &\vdots \\ &\leq y_n.\end{aligned}$$

Similarly, if $\vec{y} \in H_n^-(x)$ we can estimate that

$$\xi \geq y_n.$$

Since $w(y_n)$ vanishes when the coordinate y_n is outside of $[\alpha, \beta]$, it follows that $A_n^+(x;\xi)$ has ξ-support in $[\alpha, \beta] \cap [x, +\infty)$ and $A_n^-(x;\xi)$ has ξ-support in $[\alpha, \beta] \cap (-\infty, x]$.

Fourier Formulas for $R(z)$

Applying the results of Table 2.1, we obtain the following formulas.

For n even:

$$\mathcal{F}_n \left[\chi_{H_n^\pm(x)} \cdot W_n\right](z\,\vec{\alpha}_n) = \frac{\pm 1}{\sqrt{n}} \int_0^{\pm\infty} e^{2iz\xi} A_n^\pm(x;\xi)\, d\xi$$

and for n odd:

$$\mathcal{F}_n \left[\chi_{H_n^\pm(x)} \cdot W_n\right](z\,\vec{\alpha}_n) = \frac{\pm 1}{\sqrt{n}} \int_x^{\pm\infty} e^{2iz\xi} A_n^\pm(x;\xi)\, d\xi.$$

Returning to the formulas of Proposition 2.3.2, we write

$$S_\pm^n \mathbf{1}(x;z) = \frac{\pm 1}{\sqrt{n}} \int_0^{\pm\infty} e^{2iz\xi\sigma_3} A_n^\pm(x;\xi) d\xi,$$

for n even, and

$$S_\pm^n \mathbf{1}(x;z) = \frac{\pm \sigma_1}{\sqrt{n}} \int_x^{\pm\infty} e^{2iz\xi\sigma_3} A_n^\pm(x;\xi)\, d\xi$$

for n odd. Note that $e^{2iz\xi\sigma_3}$, and hence $S_\pm^n \mathbf{1}$, are matrix-valued.

Next we show that the series

$$\begin{aligned}A_\pm(x;\xi) &= \sum_{k=1}^{\infty} A_{2k}^\pm(x;\xi) \\ B_\pm(x;\xi) &= \sum_{k=1}^{\infty} A_{2k-1}^\pm(x;\xi).\end{aligned} \tag{2.20}$$

converge uniformly for each x and that $A(x;\cdot), B(x;\cdot) \in X$. We will then be able to exchange the summation and integral in the series

$$\Theta_{\pm}(x;\xi) = \mathbf{1} + \sum_{k=1}^{\infty} S_{\pm}^{k}\mathbf{1}(x;z)$$

and obtain the Fourier formulas

$$\begin{aligned}\Theta_{\pm}(x,z) = \mathbf{1} \pm \int_0^{\pm\infty} e^{2i\xi z\sigma_3} A_{\pm}(x;\xi)\, d\xi \\ \pm \sigma_1 \int_x^{\pm\infty} e^{2i\xi z\sigma_3} B_{\pm}(x;\xi)\, d\xi.\end{aligned} \tag{2.21}$$

The following estimates also appear in [21].

Proposition 2.3.3. *For each fixed $x \in \mathbb{R}$, $A_n^{\pm}(x,\cdot) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with*

$$\|A_n^{\pm}(x,\cdot)\|_{L^1(\mathbb{R})} \leq \frac{1}{n!}\left(\int_x^{\infty} |w(\xi)|\, d\xi\right)^n$$

and

$$\|A_n^{\pm}(x,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{(n-1)!}\left(\int_x^{\infty} |w(\xi)|\, d\xi\right)^{n-1}\left(\int_x^{\infty} |w(\xi)|^2\, d\xi\right)^{1/2}.$$

Proof. We prove the result for $A_n^{+}(x;\xi)$ only, as the computation for $A_n^{-}(x;\xi)$ is quite similar.

Recall the coarea formula for $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ Lipschitz and $g \in L^1$:

$$\int_{\Omega} g(x)|\nabla u(x)|\, dx = \int_{-\infty}^{\infty}\left(\int_{\xi=u(x)} g(x)\, dS_n(x)\right) d\xi$$

where dS_n is the $(n-1)$-dimensional Hausdorff measure on the hypersurface defined by $u(x) = \xi$. Also, note that

$$dS_n(\vec{y})d\xi = d\vec{y}$$

where $d\vec{y}$ denotes the n-dimensional Lebesgue measure and that

$$|\nabla_{\vec{y}}\,(\vec{\alpha}_n \cdot \vec{y})\,| = \sqrt{n}.$$

We compute directly:

$$\begin{aligned}\int_{-\infty}^{\infty} |A_n(x,\xi)|d\xi &\leq \frac{1}{\sqrt{n}}\int_{-\infty}^{\infty}\int_{\substack{\vec{\alpha}_n\cdot\vec{y}=\xi \\ x<y_1<\cdots<y_n}} |W_n(\vec{y})|\, dS_n(\vec{y})d\xi \\ &= \int_{x<y_1<\cdots<y_n} |W_n(\vec{y})|\, d\vec{y} \\ &\leq \frac{\|w\|_{L^1(\mathbb{R})}^n}{n!}.\end{aligned}$$

Now suppose that $g \in L^2(\mathbb{R})$. We compute

$$\begin{aligned}
\int_{-\infty}^{\infty} A_n(x,\xi)g(\xi)\,d\xi &\leq \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \left(\int_{\substack{\vec{\alpha}_n \cdot \vec{y} = \xi \\ x<y_1<\cdots<y_n}} |W_n(\vec{y})|\, dS_n(\vec{y}) \right) g(\xi)\,d\xi \\
&\leq \int_{x<y_1<\cdots<y_n} |W_n(\vec{y})||g(\vec{\alpha}_n \cdot \vec{y})|\,d\vec{y} \\
&\leq \int_{x<y_1<\cdots<y_{n-1}} |W_{n-1}(\vec{y})| \int_{y_{n-1}}^{\infty} |w(y_{n-1})||g(\vec{\alpha}_n \cdot \vec{y})|\,dy_n dy_1 \cdots dy_{n-1} \\
&\leq \|g\|_{L^2(\mathbb{R})} \left(\int_x^{\infty} |w(\xi)|^2\,d\xi \right)^{1/2} \left(\frac{\int_x^{\infty} |w(\xi)|\,d\xi}{(n-1)!} \right).
\end{aligned}$$

Apply the extremal version of Hölder's inequality to finish the proof. □

From Proposition 2.3.3, we obtain the estimates

$$\begin{aligned}
\|A_\pm(x;\cdot)\|_1 &\leq \exp(\|w\|_1) \\
\|A_\pm(x;\cdot)\|_2 &\leq \exp(\|w\|_1)\|w\|_2 \\
\|B_\pm(x;\cdot)\|_1 &\leq \exp(\|w\|_1) \\
\|B_\pm(x;\cdot)\|_2 &\leq \exp(\|w\|_1)\|w\|_2,
\end{aligned} \tag{2.22}$$

holding for any fixed x. Since the series for $A_\pm(x;\cdot), B_\pm(x;\cdot)$ converge uniformly in L^1, we can exchange the summation and the integral in the series for $\Theta_\pm$ and obtain the formulas in (2.21).

These estimates together with the result of the previous proposition show that the limits $A_+(\alpha,\xi), B_+(\alpha,\xi)$ and $A_-(\beta,\xi), B_-(\beta,\xi)$ are well-defined, even in the cases where α, β are infinite. Recalling Table 2.1, we have

$$\begin{aligned}
\Theta_+(\alpha,z) &= \mathbf{1} + \int_0^{\infty} e^{2i\xi z\sigma_3} A_+(\alpha;\xi)\,d\xi + \sigma_1 \int_\alpha^{\beta} e^{2i\xi z\sigma_3} B_+(\alpha;\xi)\,d\xi \\
\Theta_-(\beta,z) &= \mathbf{1} + \int_{-\infty}^{0} e^{2i\xi z\sigma_3} A_-(\beta;\xi)\,d\xi + \sigma_1 \int_\alpha^{\beta} e^{2i\xi z\sigma_3} B_-(\beta;\xi)\,d\xi
\end{aligned} \tag{2.23}$$

We have proved:

Theorem 2.3.2. *For w supported on $[\alpha,\beta] \subseteq \overline{\mathbb{R}}$,*

$$\begin{aligned}
R(z) &= \mathbf{1} + \int_0^{\infty} e^{2i\xi z\sigma_3} A_+(\alpha;\xi)\,d\xi + \sigma_1 \int_\alpha^{\beta} e^{2i\xi z\sigma_3} B_+(\alpha;\xi)\,d\xi \\
R(z)^{-1} &= \mathbf{1} + \int_{-\infty}^{0} e^{2i\xi z\sigma_3} A_-(\beta;\xi)\,d\xi + \sigma_1 \int_\alpha^{\beta} e^{2i\xi z\sigma_3} B_-(\beta;\xi)\,d\xi
\end{aligned}$$

In particular,

$$\begin{aligned}
a(z) &= 1 + \int_0^{\infty} e^{2iz\xi} A_+(\alpha;\xi)\,d\xi = 1 + \int_{-\infty}^{0} e^{-2iz\xi} A_-(\beta;\xi)\,d\xi \\
b(z) &= \int_\alpha^{\beta} e^{2iz\xi} B_+(\alpha;\xi)\,d\xi = -\int_\alpha^{\beta} e^{2iz\xi} B_-(\beta;\xi)\,d\xi
\end{aligned} \tag{2.24}$$

Coefficient	α, β infinite	α finite	β finite	α, β finite
$a(z)$	$\mathbb{C}^+$	$\mathbb{C}^+$	$\mathbb{C}^+$	$\mathbb{C}$
$b(z)$	None	$\mathbb{C}^+$	$\mathbb{C}^-$	$\mathbb{C}$
$r_-(z)$	None	$\mathbb{C}^+$	None	$\mathbb{C}$
$r_+(z)$	None	None	$\mathbb{C}^+$	$\mathbb{C}$

Table 2.2: This table gives the region of analyticity for the continuations of $a(z)$, $b(z)$, $r_-(z)$, and $r_+(z)$ under the various possible hypotheses on the support of w. Here, supp $(w) = [\alpha, \beta] \subseteq \overline{\mathbb{R}}$.

Remark 2.3.1. *By uniqueness of the Fourier transform, these formulas imply the symmetries*

$$A_+(\alpha;\xi) = A_-(\beta;-\xi)$$
$$B_+(\alpha;\xi) = -B_-(\beta;\xi).$$

The following L^1-continuity result for the mappings $w \mapsto A$, $w \mapsto B$ appears in [21]; the proof is a straightforward extension of the proof of Proposition 2.3.3.

Proposition 2.3.4. *Let $w, \tilde{w} \in X$. Define A, B for potential w and $\tilde{A}, \tilde{B}$ for the potential $\tilde{w}$ as above. For any $x \in \mathbb{R}$:*

$$\left\| A_\pm(x;\cdot) - \tilde{A}_\pm(x;\cdot) \right\| \leq C \exp\left(\|w\|_{L^1(\mathbb{R})} + \|\tilde{w}\|_{L^1(\mathbb{R})} \right) \|w - \tilde{w}\|_{L^1(\mathbb{R})}$$
$$\left\| B_\pm(x;\cdot) - \tilde{B}_\pm(x;\cdot) \right\| \leq C \exp\left(\|w\|_{L^1(\mathbb{R})} + \|\tilde{w}\|_{L^1(\mathbb{R})} \right) \||w - \tilde{w}\|_{L^1(\mathbb{R})}$$

Analyticity Properties of $R(z)$

Until this point, all our results have been valid only for $z \in \mathbb{R}$. We now consider the circumstances under which $a(z), b(z), r_-(z)$, and $r_+(z)$ admit analytic extensions.

From (2.24), $a(z) - 1$ is an element of the Hardy space $H^2(\mathbb{C}^+)$. Therefore, $a(z)$ always continues to a bounded analytic function in $\mathbb{C}^+$, a well-known result in scattering theory. The possibility of continuing $b(z)$ analytically depends on at least one of α, β being finite. We analyze the possible cases in the sections to follow. A summary of our results appears in Table 2.2.

Half-line Support Cases

When α is finite, we write

$$b(z) = e^{2i\alpha z} \int_0^\infty e^{2i\xi z} B_+(\alpha; \xi - \alpha)\, d\xi,$$

and see that $b(z)$ continues to an analytic function in $\mathbb{C}^+$. In this case, $e^{-2i\alpha z} b(z) \in H_2(\mathbb{C}^+)$ and $b(z)$ satisfies the exponential order condition

$$|b(R + iT)| \leq C e^{-2\alpha T}, \qquad T \geq 0.$$

When β is finite, we write

$$\bar{b}(z) = e^{-2i\beta z} \int_{-\infty}^{0} e^{-2i\xi z} B_-(\beta;\xi)\, d\xi,$$

and see that $\bar{b}(z)$ continues to an analytic function in $\mathbb{C}^+$. In this case, $e^{2i\xi z}\bar{b}(z) \in H_2(\mathbb{C}^+)$ and $\bar{b}(z)$ satisfies the exponential order condition

$$|\bar{b}(R+iT)| \leq Ce^{2\beta T}, \qquad T \geq 0.$$

Now we consider the analyticity properties of the reflection coefficients

$$r_-(z) = \frac{b(z)}{a(z)} \qquad r_+(z) = -\frac{\bar{b}(z)}{a(z)}.$$

For this, we equip the set

$$\mathcal{A} = \left\{ \chi = \lambda + \int_0^\infty e^{2izx} f(x)\, dx \ : \ f \in X,\ \lambda \in \mathbb{C} \right\}$$

with the norm

$$\left\| \lambda + \int_0^\infty e^{2iz\zeta} f(\zeta)\, d\zeta \right\|_{\mathcal{A}} = |\lambda| + \|f\|_{L^2(\mathbb{R})}.$$

From [22] (see also Remark 3.2 and the appendix of [21]), $\mathcal{A}$ is a Banach algebra under pointwise multiplication. The following theorem from [22], Chapter III, §18.7 provides necessary and sufficient conditions for an element in $\mathcal{A}$ to have an inverse in $\mathcal{A}$.

Theorem 2.3.3. *The element*

$$\chi(z) = \lambda + \int_0^\infty e^{2izx} f(x)\, dx \in \mathcal{A}$$

has an inverse in $\mathcal{A}$ if and only if $\lambda \neq 0$ and $\chi(z) \neq 0$ for all $z \in \overline{\mathbb{C}^+}$.

We now verify that $a(z)$ satisfies the conditions of this theorem and conclude that $a(z)$ has an inverse in $\mathcal{A}$.

Lemma 2.3.1. *$a(z)$ extends to an analytic function in $\mathbb{C}^+$ with $a(z) \neq 0$ on $\overline{\mathbb{C}^+}$.*

Proof. The determinant relation

$$|a(z)|^2 - |b(z)|^2 = 1$$

gives $a(z) \neq 0$ on $\mathbb{R}$. To see that $a(z) \neq 0$ in $\mathbb{C}^+$, we recall that the operator

$$P = -i \begin{bmatrix} d/dx & -w \\ w & -d/dx \end{bmatrix}$$

is self-adjoint and has no L^2 eigenvalues (see [28]). Then we suppose $z_0 \in \overline{\mathbb{C}^+}$ and that $a(z_0) = 0$.

From (2.21), it follows that $\Psi_+^{(1)}(x,z)$ extends to an analytic function on $\mathbb{C}^+$. A similar argument can be made to show that $\Psi_-^{(2)}(x,z)$ also extends to an analytic function in $\mathbb{C}^+$.

Define the fundamental solution matrix

$$\tilde{M}(x,z) = \begin{bmatrix} \Phi_+^{(1)}(x,z) & \Phi_-^{(2)}(x,z) \end{bmatrix}$$

for $z \in \overline{\mathbb{C}^+}$.

Now we have

$$\begin{aligned} a(z) &= \phi_{11}^+(0,z)\phi_{22}^-(0,z) - \phi_{21}^+(0,z)\phi_{12}^{-1}(0,z) \\ &= \det \begin{bmatrix} \Phi_+^{(1)}(0,z) & \Phi_-^{(2)}(0,z) \end{bmatrix} \\ &= \det \tilde{M}(0,z) \end{aligned}$$

Recalling Proposition 2.2.1,

$$\det \tilde{M}(x,z_0) = \det \tilde{M}(0,z_0) = a(z_0) = 0.$$

The columns of $\tilde{M}(x,z_0)$ are linearly dependent as functions of x, and so

$$\Psi_+^{(1)}(x,z_0) = c\Psi_-^{(2)}(x,z_0)$$

for some constant c. Then the column

$$\psi(x,z_0) = \begin{cases} c\Psi_-^{(2)}(x,z_0) & x < 0 \\ \Psi_+^{(1)}(x,z_0) & x \geq 0 \end{cases}$$

is a solution to

$$P\psi = z\psi.$$

As $x \to \pm\infty$, ψ decays exponentially and is hence an L^2-eigenvalue of P. This is impossible.

An alternative proof of this fact may be found in [1]. □

Now we can establish Theorem 2.1.1.

Proof of Theorem 2.1.1. Suppose that $w(x)$ has support on $[\alpha, +\infty)$ with α finite and let $r_-(z) = \mathcal{D}_P^- w$. Because $a(z)$ is nonvanishing in $\mathbb{C}^+$, $a(z)$ has an inverse in $\mathcal{A}$. Set

$$a(z)^{-1} = 1 + \int_0^\infty e^{2iz\xi} D(\xi)\, d\xi$$

where $D \in X$. Since α is finite, there is a function $B(\xi) = B_+(\alpha;\xi)$ in X so that

$$b(z) = \int_\alpha^\infty e^{2iz\xi} B(\xi)\, d\xi$$

Compute

$$\begin{aligned} r_-(z) &= b(z)a(z)^{-1} \\ &= \int_\alpha^\infty e^{2iz\xi} B(\xi)\, d\xi \left(1 + \int_0^\infty e^{2iz\xi'} D(\xi')\, d\xi'\right) \\ &= \int_\alpha^\infty e^{2iz\xi} B(\xi)\, d\xi + \int_\alpha^\infty \int_0^\infty e^{2iz(\xi+\xi')} B(\xi) D(\xi')\, d\xi' d\xi \\ &= \int_\alpha^\infty e^{2iz\xi} B(\xi)\, d\xi + \int_\alpha^\infty \int_\xi^\infty e^{2izy} B(y) D(y-\xi)\, dy d\xi \\ &= \int_\alpha^\infty e^{2iz\xi} B(\xi)\, d\xi + \int_\alpha^\infty \int_\alpha^\xi e^{2izy} B(y) D(y-\xi)\, d\xi dy. \end{aligned}$$

We define

$$C(\xi) = B(\xi) + \int_\alpha^\xi B(\xi') D(\xi - \xi')\, d\xi'$$

Since $B, D \in X$, it is easy to estimate that

$$\begin{aligned} \|C\|_{L^1(\mathbb{R})} &\leq \|B\|_{L^1(\mathbb{R})} + \|B\|_{L^1((\mathbb{R})} \|D\|_{L^1((\mathbb{R})} \\ \|C\|_{L^2(\mathbb{R})} &\leq \|B\|_{L^2(\mathbb{R})} + \|B\|_{L^1((\mathbb{R})} \|D\|_{L^2((\mathbb{R})}. \end{aligned}$$

Thus $C \in X$ (as must be the case in order for $\mathcal{A}$ to be closed). Since B is supported on $[\alpha, +\infty)$ so is C. We conclude that $C \in \mathcal{K}_\alpha$, which completes the proof of Theorem 2.1.1. □

Next, recall that if $\Psi(x, z)$ is a solution to the ZS-AKNS equation with real potential $Q(x)$, then

$$\tilde{\Psi}(x, z) = \Psi(-x, -z)$$

is a solution to the ZS-AKNS equation with potential $\tilde{Q}(x) = -Q(-x)$. Evidently,

$$\tilde{\Psi}_\pm(x, z) = \Psi_\mp(-x, -z)$$

are the Jost solutions to the $\tilde{Q}$ potential ZS-AKNS equation. For $z \in \mathbb{R}$, we have

$$\tilde{\Psi}_+(x, z) = \tilde{\Psi}_-(x, z)\tilde{R}(z)$$

with

$$\tilde{R}(z) = \begin{bmatrix} \tilde{a}(z) & \bar{\tilde{b}}(z) \\ \tilde{b}(z) & \bar{\tilde{a}}(z) \end{bmatrix} \qquad |\tilde{a}(z)|^2 - |\tilde{b}(z)|^2 = 1.$$

Proposition 2.3.5. *Let r_+ be the right reflection coefficient for the ZS-AKNS equation with real potential $Q(x)$ and let $\tilde{r}_-$ be the left reflection coefficient for the ZS-AKNS equation with potential $\tilde{Q}(x) = -Q(-x)$. Then*

$$\tilde{r}_-(z) = r_+(z).$$

Proof. Let $\widetilde{\Psi}_\pm$ be the Jost solutions to the $\tilde{Q}$ potential equation. By Proposition 2.2.3, $\Psi_\pm(-x,-z)$ are also solutions to the $\tilde{Q}(x)$ potential equation. By the asymptotics of $\Psi_\pm$ and the uniqueness of the Jost solutions (Proposition 2.2.6), we conclude that

$$\widetilde{\Psi}_\pm(x,z) = \Psi_\pm(-x,-z).$$

From Proposition 2.2.4,

$$\begin{aligned} \Psi_+(x,z) &= \Psi_-(x,z)R(z) \\ \Psi_-(-x,-z) &= \Psi_+(-x,-z)\tilde{R}(z), \end{aligned} \tag{2.25}$$

where

$$R(z) = \begin{bmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix}, \qquad \tilde{R}(z) = \begin{bmatrix} \tilde{a}(z) & \bar{\tilde{b}}(z) \\ \tilde{b}(z) & \bar{\tilde{a}}(z) \end{bmatrix}.$$

From (2.25), it follows that

$$R(-z)^{-1} = \tilde{R}(z)$$

for $z \in \mathbb{R}$.

Because Q is real-valued $\overline{R(z)} = R(-z)$ for $z \in \mathbb{R}$. We conclude that

$$\begin{aligned} a(z) &= \tilde{a}(z) \\ \bar{b}(z) &= -\tilde{b}(z). \end{aligned}$$

Therefore,

$$\tilde{r}_-(z) = r_+(z).$$

□

Now we can prove Corollary 2.1.1.

Proof of Corollary 2.1.1. If Q is supported on $(-\infty, \beta]$ then $\tilde{Q}$ is supported on $[-\beta, +\infty)$ and

$$\tilde{r}_-(z) = \int_{-\beta}^{\infty} e^{2iz\xi} C(\xi)\, d\xi.$$

Hence,

$$r_+(z) = \int_{-\beta}^{\infty} e^{2iz\xi} C(\xi)\, d\xi$$

for some $C \in X$. Note that this result can also be obtained by a direct computation similar to the one carried out for the $[\alpha, +\infty)$ support case. We have

$$|r_+(R+iT)| \leq e^{2\beta T} \|C\|_{L^1(\mathbb{R})}, \qquad T \geq 0$$

and $e^{2i\beta z} r_+(z) \in H_2(\mathbb{C}^+)$. □

We now consider some additional special cases.

Finite Interval Support Case

Suppose that both α and β are finite. We can easily verify that for n even $A_n^+(\alpha;\xi)$ has ξ-support in $(0,\beta-\alpha)$ and $A_n^-(x;\xi)$ has ξ-support in $(\alpha-\beta,0)$. See the diagram for A_2 in Figure 2.4. It follows that $a(z)$ is entire. From (2.24) $b(z)$ is also entire. The argument made in Lemma 2.3.1 will also apply to $a(z)$ for $z \in \mathbb{C}^-$ when this quantity is defined. Therefore, both r_+, r_- are entire functions. We have shown:

Proposition 2.3.6. *If $w \in X$ has compact support than $r_-(z)$ is an entire function.*

This fact has been known since [1]. We can be a bit more explicit, however, on the form of $r_\pm$. Since α is finite, we may write

$$r_-(z) = \int_\alpha^\infty e^{2iz\xi} C(\xi)\, d\xi$$

where

$$C(\xi) = B(\xi) + \int_\alpha^\xi B(\xi')D(\xi-\xi')\, d\xi'.$$

By Table 2.2, it follows that $B(\alpha,\xi)$ is supported on $[\alpha,\beta]$. With a change of variables, the convolutional term becomes

$$\int_\alpha^\xi B(\xi')D(\xi-\xi')\, d\xi' = \int_{\xi-\beta}^{\xi-\alpha} B(\xi-u)D(u)\, du.$$

Since D is supported on $[0,+\infty)$ and $\xi-\beta < \xi-\alpha$, the convolution vanishes whenever $\xi-\beta<0$, i.e. for $\xi<\beta$. It follows that

$$r_-(z) = \int_\alpha^\beta e^{2iz\xi} C(\xi)\, d\xi,$$

where $C \in X$. A similar result holds for r_+.

Whole Line Support Case

Finally, if both α,β are infinite then $b(z)$ does not admit analytic continuation to either half plane. As before, $a(z) \neq 0$ in $\overline{\mathbb{C}^+}$ and it follows that $a(z)$ has an inverse in $\mathcal{A}$. As in [21],

$$r_-(z) = \int_{-\infty}^\infty C(\xi) e^{2iz\xi}\, d\xi$$

for $C \in X$, and a similar expansion can be made for r_+. It is generally not possible to continue r_- or r_+ analytically to either half-plane.

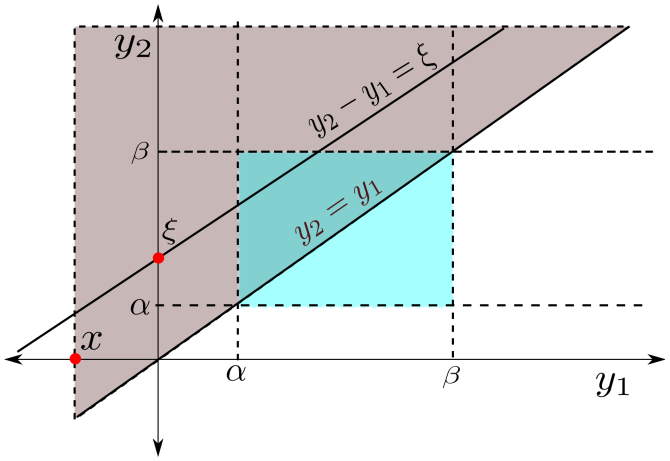

Figure 2.4: This diagram depicts the integral support of $A_2^+(x;\xi)$ when α, β are both finite. The gray region is the half-plane $H_2^+(x)$. For fixed x, $A_2^+(x;\xi)$ is supported in $0 < \xi < \beta - \alpha$.

Chapter 3 Inverse Scattering and the Nonlinear Paley-Wiener theorem

3.1 Overview

In this chapter, we prove the inverse part of Theorem 1.2.1:

Theorem 3.1.1. *If $r_- = \mathcal{D}_P^- w \in \widehat{\mathcal{K}_\alpha}$ then $w \in \mathcal{K}_\alpha$.*

We will again appeal to the symmetry in Proposition 2.3.5, to obtain the corollary:

Corollary 3.1.1. *If $r_+ = \mathcal{D}_P^+ w \in \widehat{\mathcal{K}^\beta}$ then $w \in \mathcal{K}^\beta$.*

Now if $r_- \in \widehat{\mathcal{K}_\alpha}$ then according to the discussion in the introduction to Chapter 2:

1. $\mathcal{F}^{-1} r_-(x) \in X$,
2. $r_-(z)$ extends to an analytic in function in $\mathbb{C}^+$,
3. For $T \geq 0$: $|r_-(R+iT)| < Ce^{-2\alpha T}$ for some constant C,
4. $\|r_-\|_{L^\infty(\mathbb{R})} < 1$.

On the other hand, if $r_-(z)$ satisfies conditions (1)-(4) then $r_-(z) \in \widehat{\mathcal{K}_\alpha}$ by the Two-Sided Paley-Wiener Theorem 1.1.2. Therefore, conditions (1)-(4) are equivalent to $r_-(z) \in \widehat{\mathcal{K}_\alpha}$. A similar statement can be made for r_+. Hence, by the end of this chapter we will also have established the main result of this thesis, the Nonlinear Paley-Wiener Theorem 1.1.1 for the ZS-AKNS scattering transforms.

Our proof of Theorem 3.1.1 will use the Riemann-Hilbert approach to inverse scattering of the ZS-AKNS equation developed by Beals and Coifman in [2]. After formulating the inverse problem as an equivalent Riemann-Hilbert problem, we will factorize the Riemann-Hilbert jump matrix so that the dependence of $\mathrm{supp}(w)$ on the analyticity properties of $r_-(z) = \mathcal{D}_P^-(w)$ is explicit. We begin with a recapitulation of the inverse scattering theory for the ZS-AKNS equation over X via the Riemann-Hilbert method.

3.2 Inverse Scattering with the Riemann-Hilbert Method.

Recall the ZS-AKNS system

$$\begin{cases} \dfrac{d}{dx}\Psi = iz\sigma_3\Psi + Q(x)\Psi, \qquad z \in \mathbb{C}, x \in \mathbb{R} \\ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & w(x) \\ w(x) & 0 \end{bmatrix} \quad , \\ w \in X = L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \end{cases} \tag{3.1}$$

In the inverse scattering problem for this system, the reflection coefficients $r_\pm$ are known but the potential w is unknown. To solve the problem, we must construct the inverse scattering maps

$$\mathcal{I}_P^\pm : r_\pm \to w.$$

There are two main approaches to this problem. The classical approach described, for example, in [1] uses the structure of the ZS-AKNS equation to formulate the inverse scattering problem as a Gelfand-Levitan-Marchenko system of integral equations. The authors of [21] apply the classical method to show that $\mathcal{D}_P^\pm : X \to \widehat{X_\mp}$ are bijections.

The modern approach, pioneered by Beals and Coifman in [2], formulates the inverse scattering problem as a family of Riemann-Hilbert problems. This approach has been well-developed for the scattering of the ZS-AKNS equation. See for example [12, 14, 50]. Here, we will follow the presentation of the Riemann-Hilbert method as found in Deift and Zhou [12].[1] The advantage of the Riemann-Hilbert method is that it will allow us to prove the nonlinear Paley-Wiener theorem by making a shift of contours in the Riemann-Hilbert problem. The approach has the intuitive appeal of being quite similar to standard shift of contour techniques used to establish a number of identities in Fourier theory.

Preliminaries

Let us briefly describe a simple scalar model for the type of Riemann-Hilbert problems relevant to ZS-AKNS scattering.

Problem 3.2.1 (Scalar Riemann-Hilbert Model Problem)**.** *Let $f \in L^2(\mathbb{R})$. Determine a function F satisfying all of the following properties.*

1. *F is analytic on $\mathbb{C}\backslash\mathbb{R}$ and continuous on $\overline{\mathbb{C}^+}, \overline{\mathbb{C}^-}$*
2. *$F(z) \to 1$ as $|z| \to +\infty$*
3. *If the boundary values of F at $\mathbb{R}$ are given by the pointwise limits*

$$F_+(z) = \lim_{\Im z \to 0^+} F(z) \qquad F_-(z) = \lim_{\Im z \to 0^-} F(z),$$

then

$$f(z) = F_+(z) - F_-(z).$$

[1] Note, however, the ZS-AKNS theory developed in Section 3 of [12] is primarily focused on the scattering for potentials smoother than ours, namely those belonging to the weighted Sobolev space

$$H^{1,1}(\mathbb{R}) = \left\{ w \in L^2(\mathbb{R}) : xw, xw' \in L^2(\mathbb{R}) \right\}.$$

These authors use the convention $r_-(z) = -\frac{\bar{b}(z)}{\bar{a}(z)}$ whereas our reflection coefficient is $r_-(z) = \frac{b(z)}{a(z)}$, and also use the matrix $\sigma = \frac{1}{2}\sigma_3$ where we use only σ_3.

Intuitively, the solution $F(z)$ to this problem is an analytic function in $\mathbb{C}^+$ and $\mathbb{C}^-$ but which has a "jump"

$$f(z) = F_+(z) - F_-(z)$$

across the the real axis. Since the difference of two solutions to this problem is an entire function vanishing at infinity, the solution $F(z)$ to this problem is unique by Liouville's theorem.

The Cauchy integral operator on $L^2(\mathbb{R})$ plays an important role in constructing the solution to Riemann-Hilbert problems. We define the Cauchy integral of $f \in L^2(\mathbb{R})$ as

$$\mathcal{C}f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(\zeta) \frac{1}{\zeta - z} \, d\zeta$$

for $z \in \mathbb{C} \backslash \mathbb{R}$. Using Morera's theorem, it is easy to verify that $\mathcal{C}f(z)$ extends f to an analytic function on $\mathbb{C} \backslash \mathbb{R}$. The *Cauchy boundary values* for $f \in L^2(\mathbb{R})$ are obtained for $z \in \mathbb{R}$ by

$$\mathcal{C}_\pm f(z) = \lim_{\epsilon \downarrow 0} \mathcal{C}f(z \pm i\epsilon).$$

Without further mention, we will extend the operators $\mathcal{C}, \mathcal{C}_\pm$ to act component-wise on $L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$ functions when needed.

As shown in Appendix C, the projection operators $\mathcal{C}_\pm$ act as cutoff functions in Fourier space:

$$\mathcal{C}_+ f(z) = \int_0^\infty \check{f}(x) e^{2ixz} \, dx$$
$$\mathcal{C}_- f(z) = -\int_{-\infty}^0 \check{f}(x) e^{2ixz} \, dx.$$

We can also readily check that

$$\|\mathcal{C}_\pm f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}. \tag{3.2}$$

and

$$\mathcal{C}_+ - \mathcal{C}_- = I. \tag{3.3}$$

From these facts, it follows that the unique solution to Problem 3.2.1 is given by

$$F(z) = 1 + \mathcal{C}f(z).$$

Outline of the Riemann-Hilbert Method

We now outline the Riemann-Hilbert approach to inverse scattering for the ZS-AKNS equation following [12]. We focus on the reconstruction from the left reflection coefficient $r_-(z)$, as in our case the left-side result implies the right-side result by a symmetry argument. Further details on the right-side reconstruction may be found in [12].

Recall that by introducing the factorization

$$\Psi = \Phi e^{ixz\sigma_3}$$

into (3.1), we obtain the factorized ZS-AKNS equation

$$\frac{d}{dx}\Phi = iz\mathrm{ad}_{\sigma_3}\Phi + Q(x)\Phi. \tag{3.4}$$

As we will show, for each $w \in X$ there exists a unique family of solutions $M(x,z)$ to the following problem.

Problem 3.2.2 (Beals-Coifman Problem [2])**.** *Determine the matrix $M(x,z)$ satisfying the factorized ZS-AKNS equation (3.4) and all of the following properties:*

1. *For each fixed $x \in \mathbb{R}$:*
   - *a) $M(x,z)$ is an analytic function of z in $\mathbb{C}\backslash\mathbb{R}$*
   - *b) $M(x,z)$ has the continuous boundary values*

$$M_\pm(x,z) = \lim_{\pm\Im z\downarrow 0} M(x,z).$$

   - *c) $M_\pm(x,\cdot) - \mathbf{1} \in L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$*

2. *For each fixed $z \in \mathbb{C}\backslash\mathbb{R}$:*
   - *a) $M(x,z) \to \mathbf{1}$ as $x \to -\infty$.*
   - *b) $|M(x,z)|$ is bounded as $x \to +\infty$.*

The solution family $M(x,z)$ to this problem is called the *Beals-Coifman solution.* Suppose that w belongs to the Schwartz class

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in \mathbb{C}^\infty(\mathbb{R}) : \left\| x^\mu \frac{d^\nu f}{dx^\nu} \right\|_{L^\infty(\mathbb{R})} < \infty, \quad \nu, \mu \in \{0,1,2,...\} \right\}.$$

Since

$$\lim_{z\to\infty} M(x,z) = \mathbf{1},$$

it can be shown (see [12]) that M has the large-z asymptotics

$$M(x,z) = \mathbf{1} + \frac{M_1(x)}{z} + o\left(\frac{1}{z}\right)$$

for each fixed x.

Inserting this expansion into the factorized ZS-AKNS equation (3.4), we obtain

$$w(x) = i\left[M_1(x)\right]_{21}. \tag{3.5}$$

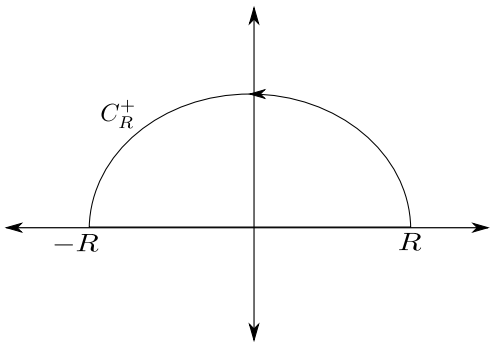

Figure 3.1: The contour C_R used to recover $w(x)$ from $M_+(x,z)$.

Since $M(x,z)$ is analytic in $\mathbb{C}^+$ and continuous on $\overline{\mathbb{C}^+}$, we may integrate the asymptotic expansion over the semi-circular contour C_R^+ depicted in Figure 3.1. Sending $R \to +\infty$ and applying the Cauchy Integral Theorem, we obtain

$$w(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} [M_+]_{21}(x,z)\, dz. \tag{3.6}$$

This gives a formula to recover the Schwartz class potential w from the Beals-Coifman solution $M(x,z)$. Arguing by density, (3.6) holds in the L^2-sense for general $w \in X$. Therefore, w may be recovered from $M(x,z)$ by formula (3.6). We again refer to [12] and the references therein for details.

On the other hand, the Beals-Coifman solution $M(x,z)$ for given $w \in X$ is also a solution to the following x-parametrized family of multiplicative matrix Riemann-Hilbert problems.

Problem 3.2.3. *For each fixed x, determine the 2×2 matrix function $M(x,z)$ satisfying*

$$\begin{cases} M(x,z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\ M(x,z) \text{ continuous on } \overline{\mathbb{C}^+}, \overline{\mathbb{C}^-} \\ M(x,z) \to \mathbf{1} \text{ uniformly as } |z| \to \infty \text{ in } \overline{\mathbb{C}_+} \cup \overline{\mathbb{C}^-} \\ M_+(x,z) = M_-(x,z) V_x(z) \text{ for } z \in \mathbb{R} \end{cases} \tag{3.7}$$

where $V_x(z)$ is the multiplicative jump matrix

$$V_x(z) = \exp(ixz\mathrm{ad}_{\sigma_3}) \begin{bmatrix} 1 - |r_-(z)|^2 & -\overline{r_-}(z) \\ r_-(z) & 1 \end{bmatrix}. \tag{3.8}$$

Observe that Problem 3.2.3 contains a multiplicative jump condition across the real axis, a situation somewhat different than the additive jump considered in the model problem. To solve Problem 3.2.3, one factorizes the jump matrix and transforms the multiplicative problem to an additive one (see Figure 3.2). This additive Riemann-Hilbert problem can then be cast in the form of a uniquely solvable singular integral equation, as we will describe below.

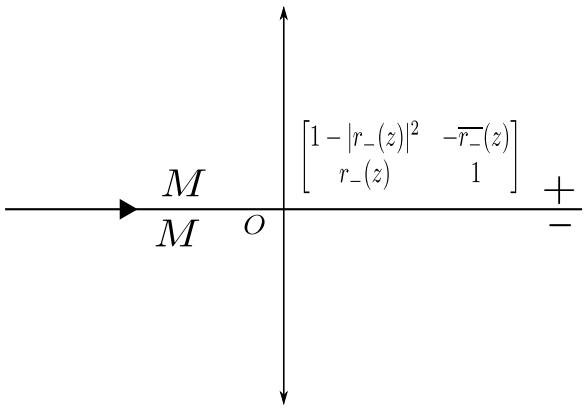

Figure 3.2: $M(x,z)$ has a multiplicative jump along the contour $\mathbb{R}$. We have suppressed the x dependence in the diagram for simplicity.

Given the data $r_- \in \widehat{X_+}$, the solution family $M(x,z)$ to the family of Riemann-Hilbert problems in Problem 3.2.3 exists and is unique. As it turns out, the same family $M(x,z)$ also uniquely solves Problem 3.2.2 with the initial data $w \in X$ such that $\mathcal{D}_P^- w = r_-$. To recover w from its corresponding reflection coefficient r_-, we therefore first solve Problem 3.2.3 for $M(x,z)$. Then we use (3.6) to extract w from $M(x,z)$.

The Riemann-Hilbert approach to ZS-AKNS scattering is well-established and can be applied in considerably more generality than we require here (for example, see [3, 14, 50]). For $w \in X$, results from the previous chapter allow us to give relatively simple proofs of the ideas outlined above. In the sections to follow, we present some details of the development of the Riemann-Hilbert framework specialized to the case of real potentials $w \in X$.

We begin by proving that a unique Beals-Coifman solution family $M(x,z)$ exists for each fixed potential $w \in X$. Then we show that this solution family $M(x,z)$ also uniquely solves the family of Riemann-Hilbert problems (3.2.3) with initial data $r_- = \mathcal{D}_P^- w$. Finally, we show that every family of Riemann-Hilbert problems (3.2.3) with initial data $r_- \in \widehat{X_+}$ has a unique solution. By [21], $\mathcal{D}_P^-$ is a bijection between X and $\widehat{X_+}$. Hence, for any $r_- \in \widehat{X_+}$ we may uniquely recover $w \in X$ such that $r_- = \mathcal{D}_P^- w$ by solving the family of Riemann-Hilbert problems 3.2.3 to obtain $M(x,z)$.

The Beals-Coifman Solutions

We begin by establishing the existence of a unique Beals-Coifman solution family for each potential $w \in X$.

Theorem 3.2.1. *For each $w \in X$, a unique solution family $M(x,z)$ to Problem 3.2.2 exists.*

Proof. The proof is constructive.

Let $\Psi_\pm = \Phi_\pm e^{ixz\sigma_3}$ be the Jost solutions to the ZS-AKNS equation with potential $w \in X$.

Recall that $\Phi_\pm$ are the unique solutions to equation (3.4), satisfying the respective asymptotic conditions

$$\lim_{x\to\pm\infty} |\Phi_\pm(x,z) - \mathbf{1}|_2 = 0.$$

From the previous chapter,

$$\Phi_\pm(x,z) = \exp(ixz\mathrm{ad}_{\sigma_3})\Theta_\pm(x,z),$$

where by the formula (2.21)

$$\Theta_\pm(x,z) = \mathbf{1} \pm \int_0^{\pm\infty} e^{2i\xi z\sigma_3} A_\pm(x;\xi)\, d\xi \pm \sigma_1 \int_x^{\pm\infty} e^{2i\xi z\sigma_3} B_\pm(x;\xi)\, d\xi.$$

Here A, B are scalar functions with $A_\pm(x,\cdot), B_\pm(x,\cdot) \in X$ for each fixed x and by (2.22):

$$\begin{aligned} \|A_\pm(x,\cdot)\|_1 &\le \exp\left(\|w\|_{L^1(\mathbb{R})}\right) \\ \|B_\pm(x,\cdot)\|_1 &\le \exp\left(\|w\|_{L^1(\mathbb{R})}\right). \end{aligned}$$

Using the properties of $\exp(ixz\ \mathrm{ad}_{\sigma_3})$ collected in Proposition 2.2.5, we compute

$$\begin{aligned} \Phi_\pm(x,z) &= \mathbf{1} \pm \int_0^{\pm\infty} e^{2i\xi z\sigma_3} A_\pm(x;\xi)\, d\xi \\ &\qquad \pm \int_x^{\pm\infty} \exp(ixz\mathrm{ad}_{\sigma_3}) \left(\sigma_1 e^{2i\xi z\sigma_3}\right) B_\pm(x;\xi)\, d\xi \\ &= \mathbf{1} \pm \int_0^{\pm\infty} e^{2i\xi z\sigma_3} A_\pm(x;\xi)\, d\xi \\ &\qquad \pm \sigma_1 \int_x^{\pm\infty} e^{2iz(\xi-x)\sigma_3} B_\pm(x;\xi)\, d\xi. \end{aligned}$$

Changing variables in the last line, we obtain the representation formula

$$\Phi_\pm(x,z) = \mathbf{1} + \int_0^{\pm\infty} e^{2iz\sigma_3} \left[A_\pm(x;\xi) + \sigma_1 B_\pm(x;\xi+x)\right]\, d\xi. \tag{3.9}$$

By this formula, it follows that for each fixed x the first column $\Phi_+^{(1)}(x,z)$ of Φ_+ and the second column $\Phi_-^{(2)}$ of Φ_- extend to analytic functions of $z \in \mathbb{C}^+$. Moreover, we have the uniform bounds

$$\begin{aligned} \left|\Phi_+^{(1)}(x,z)\right| &\le 1 + \exp(\|w\|_{L^1(\mathbb{R})}) \\ \left|\Phi_-^{(2)}(x,z)\right| &\le 1 + \exp(\|w\|_{L^1(\mathbb{R})}) \end{aligned} \tag{3.10}$$

holding for $z \in \overline{\mathbb{C}^+}$.

Motivated by the asymptotics in (2.6) and recalling Lemma 2.3.1, we define

$$M(x,z) = \begin{bmatrix} \dfrac{\Phi_+^{(1)}(x,z)}{a(z)} & \Phi_-^2(x,z) \end{bmatrix}. \tag{3.11}$$

which is a fundamental solution to (3.4) for $z \in \overline{\mathbb{C}^+}$.

From equation (3.9) and Theorem 2.3.3, it follows that for each fixed x there is $\tilde{M}_x \in X \otimes M_2(\mathbb{C})$ so that

$$M(x,z) = \mathbf{1} + \int_0^\infty e^{2iz\xi} \tilde{M}_x(\xi)\, d\xi. \tag{3.12}$$

From this representation, we readily conclude that for each fixed x:

- $M(x,\cdot) - \mathbf{1} \in L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$
- $M(x,z)$ extends to an analytic function of $z \in \mathbb{C}^+$
- $M(x,z)$ has continuous boundary values as $\Im z \downarrow 0$.

We now show that for each fixed $z \in \overline{\mathbb{C}^+}$, $M(x,z) \to \mathbf{1}$ as $x \to -\infty$ and $|M(x,z)|$ remains bounded as $x \to +\infty$.

Recalling that $\Psi_+ = \Phi_+ e^{ixz}$, from the asymptotics (2.6) it is clear that

$$\lim_{x \to -\infty} \left| \phi_{11}^+(x,z) - a(z) \right| = 0 \tag{3.13}$$

holds for $z \in \mathbb{R}$.

Now from the integral equation (2.11), we have

$$\begin{aligned} \phi_{11}^+(x,z) &= 1 - \int_x^\infty w(y)\phi_{21}^+(y,z)\, dy \\ \phi_{21}^+(x,z) &= -\int_x^\infty e^{-2iz(x-y)} w(y)\phi_{11}^+(y,z)\, dy \end{aligned} \tag{3.14}$$

From (3.14), it follows that

$$a(z) = 1 - \int_{-\infty}^\infty w(y)\phi_{21}^+(y,z)\, dy \tag{3.15}$$

holds for $z \in \mathbb{R}$. Since $\phi_{21}(x,z)$ extends analytically to $\mathbb{C}^+$ and satisfies the uniform bound in (3.10), the integral on the right in equation (3.15) is well-defined for $z \in \overline{\mathbb{C}^+}$. Hence, the relation (3.15) extends to hold for $z \in \overline{\mathbb{C}^+}$. It follows that (3.13) holds for $z \in \mathbb{C}^+$. We easily apply equation (3.14) and the bounds (3.10), to obtain that

$$M(x,z) \to \mathbf{1}$$

as $x \to -\infty$.

By definition $\Phi_+^{(1)}(x,z) \to \mathbf{e}_1$ as $x \to +\infty$ and $\Phi_-^{(2)}(x,z)$ is bounded uniformly in z as $x \to +\infty$ by (3.10). Hence, $M(x,z)$ is bounded as $x \to +\infty$. In fact since $|a(z)| \geq 1$, we have the uniform bound

$$|M(x,z)| \leq 1 + \exp(\|w\|_{L^1(\mathbb{R})}) \tag{3.16}$$

from (3.10).

Finally, we extend $M(x,z)$ to an analytic function in $\mathbb{C}^-$ by defining

$$M(x,z) = \sigma_1 \overline{M(x,\bar{z})} \sigma_1 \qquad \Im z < 0. \tag{3.17}$$

An easy computation verifies that this extension satisfies all the required properties of the theorem. Uniqueness follows directly from the uniqueness of the Jost solutions $\Psi_\pm$ (see Proposition 2.2.6). □

The Riemann-Hilbert Problem

Now we show that the Beals-Coifman solution family $M(x,z)$ for potential $w \in X$ is also the solution to the family of Riemann-Hilbert problems in Problem 3.2.3 with the initial data $r_- = \mathcal{D}_P^- w$.

Theorem 3.2.2. *Let $M(x,z)$ be the Beals-Coifman solution family for Problem 3.2.2 with potential $w \in X$. Then $M(x,z)$ solves the Riemann-Hilbert Problem 3.2.2 with the data $r_- = \mathcal{D}_P^-(w)$.*

Proof. Fix any $x \in \mathbb{R}$. By uniqueness, $M(x,z)$ is given by (3.11) with the extension defined in (3.17). It follows that $M(x,z)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and continuous to the boundary at $\mathbb{R}$. For each fixed x, apply the Riemann-Lebesgue Lemma to (3.12) to see that $M(x,z) \to \mathbf{1}$ pointwise in $\overline{\mathbb{C}^+}$. Using the extension (3.17) the result also holds for $z \in \overline{\mathbb{C}^-}$. This convergence is uniform in x using the uniform bound (3.16).

Next we verify the jump condition for $M(x,z)$ across the contour $\mathbb{R}$, oriented from $-\infty$ to $+\infty$ (Figure 3.2). By Proposition 2.2.2,

$$\Psi_+(x,z) = \Psi_-(x,z) R(z)$$

where $\Psi_\pm = \Phi_\pm e^{ixz}$ are the Jost solutions, and

$$R(z) = \begin{bmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix}.$$

It is easy to check that

$$\Phi_+(x,z) = \Phi_-(x,z) e^{ixz\sigma_3} R(z) e^{-ixz\sigma_3} = \Phi_-(x,z) \exp(ixz \mathrm{ad}_{\sigma_3}) R(z).$$

On the other hand, we also have that $M_+(x,z)e^{ixz\sigma_3}$ and $M_-(x,z)e^{ixz\sigma_3}$ are solutions to the same unfactorized ZS-AKNS equation (3.1).[2] It follows that for each fixed z, these boundary values are related by

$$M_+(x,z)e^{ixz\sigma_3} = M_-(x,z)e^{ixz\sigma_3} V(z)$$

[2]Note carefully that the $\pm$ subscript on $M_\pm(x,z)$ denotes boundary values of $M(x,z)$, *and not a normalization of a solution at $\pm\infty$.*

for some matrix $V(z)$.

With $r_-(z) = b(z)/a(z)$, we can write:

$$\begin{aligned} M_+(x,z) &= \begin{bmatrix} \frac{1}{a(z)}\Phi_+^{(1)} & \Phi_-^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} \Phi_- \begin{bmatrix} 1 \\ r_-(z)e^{-2ixz} \end{bmatrix} & \Phi_-^{(2)} \end{bmatrix} \\ &= \Phi_- \begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix}. \end{aligned}$$

Then by (3.17)

$$M_-(x,z) = \sigma_1 \overline{M_+(x,z)}\sigma_1 = \sigma_1 \overline{\Phi_-} \begin{bmatrix} 1 & 0 \\ \overline{r_-}e^{2ixz} & 1 \end{bmatrix} \sigma_1.$$

Hence,

$$\Phi_- \begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix} e^{ixz\sigma_3} = \sigma_1 \overline{\Phi_-} \begin{bmatrix} 1 & 0 \\ \overline{r_-}e^{2ixz} & 1 \end{bmatrix} \sigma_1 e^{ixz\sigma_3} V(z).$$

Using the asymptotics of Φ_- as $x \to -\infty$ and solving for $V(z)$:

$$\begin{aligned} V(z) &= e^{-ixz\sigma_3}\sigma_1 \begin{bmatrix} 1 & 0 \\ -\bar{r}_-(z)e^{2ixz} & 1 \end{bmatrix} \sigma_1 \begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix} e^{ixz\sigma_3} \\ &= \begin{bmatrix} 1-|r_-(z)|^2 & -\bar{r}_-(z) \\ r_-(z) & 1 \end{bmatrix}. \end{aligned}$$

□

Finally, we prove the existence and uniqueness of the solutions $M(x,z)$ to the Riemann-Hilbert Problem 3.2.3 for data $r_- \in \widehat{X_+}$ and give an explicit procedure for computing $M(x,z)$.

To state the result, we will require some additional notation. Upper-lower factorize the jump matrix (3.8) as

$$V_x(z) = V_{-,x}(z)^{-1} V_{+,x}(z) \tag{3.18}$$

where

$$\begin{aligned} V_{-,x}(z) &= \begin{bmatrix} 1 & \overline{r_-}(z)e^{2ixz} \\ 0 & 1 \end{bmatrix} \\ V_{+,x}(z) &= \begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix}. \end{aligned} \tag{3.19}$$

Then let

$$\begin{aligned} \theta_{-,x}(z) &= V_{-,x}(z) - \mathbf{1} = \begin{bmatrix} 0 & 0 \\ r_-(z)e^{-2ixz} & 0 \end{bmatrix} \\ \theta_{+,x}(z) &= \mathbf{1} - V_{+,x}(z) = \begin{bmatrix} 0 & -\overline{r_-}(z)e^{2ixz} \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.20}$$

Define the integral operator $\mathcal{C}_{\theta,x}$ by

$$\mathcal{C}_{\theta,x} f(z) = \mathcal{C}_-[f\theta_{+,x}](z) + \mathcal{C}_+[f\theta_{-,x}](z). \tag{3.21}$$

for $f \in L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$.

The following is essentially Proposition 2.11 of [12].

Theorem 3.2.3. *Let $r_- \in \widehat{X_+}$. The unique solution to Problem 3.2.3 with data r_- is*

$$M(x,z) = \mathbf{1} + \mathcal{C}\left(\mu(x,\cdot)\left[\theta_{-,x}(\cdot) + \theta_{+,x}(\cdot)\right]\right)(z),$$

where for each fixed x, $\mu(x,z)$ is the solution to

$$(I - \mathcal{C}_{\theta,x})\mu(x,z) = \mathbf{1}. \tag{3.22}$$

Proof. We begin by converting the multiplicative Riemann-Hilbert Problem 3.2.3 to an additive Riemann-Hilbert problem, a matrix analog of Problem 3.2.1.

Use the factorization (3.18) to write the jump relation $M_+ = M_- V_x$ of Problem 3.2.3 as

$$M_+(x,z)V_{+,x}(z)^{-1} = M_-(x,z)V_{-,x}(z)^{-1}.$$

Then define

$$\mu(x,z) = M_+(x,z)V_{+,x}(z)^{-1} = M_-(x,z)V_{-,x}(z)^{-1}. \tag{3.23}$$

We compute the additive jump condition

$$\begin{aligned} M_+(x,z) - M_-(x,z) &= \mu(x,z)V_{+,x}(z) - \mu(x,z)V_{-,x}(z) \\ &= \mu(x,z)\left(V_{+,x}(z) - V_{-,x}(z)\right) \\ &= \mu(x,z)\left(\theta_{-,x}(z) + \theta_{+,x}(z)\right). \end{aligned}$$

For each x,

$$M(x,z) = \mathbf{1} + \mathcal{C}\left(\mu(x,\cdot)(\theta_{-,x}(\cdot) + \theta_{+,x}(\cdot))\right)(z)$$

defines the extension of $M(x,z)$ off of the contour $\mathbb{R}$. Sending $\Im z \downarrow 0$, we obtain

$$M_+(x,z) = \mathbf{1} + \mathcal{C}_+\left(\mu(x,\cdot)(\theta_{-,x}(\cdot) + \theta_{+,x}(\cdot))\right)(z). \tag{3.24}$$

Since

$$M_+ = \mu\left(\mathbf{1} + \theta_{+,x}\right),$$

we can write (3.24) as

$$\mu(x,z) = \mathbf{1} - \mu(x,z)\theta_{+,x}(z) + \mathcal{C}_+\left(\mu(x,\cdot)(\theta_{-,x}(\cdot) + \theta_{+,x}(\cdot))\right)(z).$$

Then by (3.3)

$$[(\mathcal{C}_+ - \mathcal{C}_-)\mu(x,\cdot)\theta_{+,x}(\cdot)](z) = \mu(x,z)\theta_{+,x}(z),$$

and we obtain the singular integral equation (3.22).

Formally, the solution to this equation (3.22) is

$$\mu = (I - \mathcal{C}_{\theta,x})^{-1}\,\mathbf{1}.$$

Equip $M_2(\mathbb{C})$ with the Frobenius norm

$$|A|_F = \left(\sum_{i,j=1,2} |A_{ij}|^2\right)^{1/2}$$

Recall that the projection $\mathcal{C}_\pm$ on $f \in L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$ satisfy (3.2). We then estimate

$$\begin{aligned} \|\mathcal{C}_{\theta,x} f\|_{L^2(\mathbb{R})\otimes M_2(\mathbb{C})} &\leq \|r\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})\otimes M_2(\mathbb{C})} \\ &< \|f\|_{L^2(\mathbb{R})\otimes M_2(\mathbb{C})}. \end{aligned} \tag{3.25}$$

Since $\|r\|_{L^\infty(\mathbb{R})} < 1$, the resolvent operator $(I - \mathcal{C}_{\theta,x})^{-1}$ exists as a bounded linear operator on $L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$ and

$$\left\|(I - \mathcal{C}_{\theta,x})^{-1}\right\|_{L^2(\mathbb{R})\otimes M_2(\mathbb{C}) \to L^2(\mathbb{R})\otimes M_2(\mathbb{C})} \leq \frac{1}{1 - \|r_-\|_\infty}.$$

Since $\mathbf{1} \notin L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$, we iterate once to obtain

$$\mu(x,z) = \mathbf{1} + (I - \mathcal{C}_{\theta,x})^{-1}\,\mathcal{C}_{\theta,x}\mathbf{1}.$$

We note that

$$\mathcal{C}_{\theta,x}\mathbf{1} = \mathcal{C}_+\theta_{-,x}(\cdot) + \mathcal{C}_-\theta_{+,x}(\cdot) \in L^2(\mathbb{R}) \otimes M_2(\mathbb{C}) \tag{3.26}$$

since $r_- \in L^2(\mathbb{R})$.

It follows that

$$M(x,z) = \mathbf{1} + \mathcal{C}\left(\mu(x,\cdot)(\theta_{+,x}(\cdot) + \theta_{-,x}(\cdot))\right)(z)$$

is a well-define solution to the Riemann-Hilbert Problem (3.2.3).

The uniqueness is straightforward but requires some additional computation not otherwise needed for our work. We refer to the proof in [12]. □

Reconstruction

Let us close this section with a summary of the reconstruction of w from the scattering coefficient $r_-(z)$. Because $\mathcal{D}_P^-$ is a bijection, for any $r_- \in \widehat{X_+}$ there is $w \in X$ so that $r_- = \mathcal{D}_P^- w$. We solve the singular integral equation (3.22) with the data r_- to obtain the solution $M(x,z)$ to the family of Riemann-Hilbert problems in (3.2.3). $M(x,z)$ is also the solution to the family of problems in (3.2.2) with potential w. Since $M_+(x,z) = \mu(x,z)v_{+,x}(z)$, recalling equation (3.6) we obtain the reconstruction formula

$$w(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} r_-(z)\mu_{22}(x,z)e^{-2ixz}\,dz. \tag{3.27}$$

Note that to leading order $\mu_{22}(x,z) = 1$ and (3.27) is simply the inverse Fourier transform of $r_-(z)$. When

$$r_-(z) = \int_\alpha^\infty C(\xi) e^{2iz\xi}\, d\xi,$$

to first order approximation we have

$$w(x) \approx \chi_{[\alpha,+\infty)}(x) C(x). \tag{3.28}$$

3.3 Shifting Contours in the Riemann-Hilbert Problem

Now let r_- be a reflection coefficient in the space $\widehat{\mathcal{K}_\alpha}$. Then

$$\begin{aligned} r_-(z) &= \int_\alpha^\infty C(\xi) e^{2iz\xi}\, d\xi \\ &= e^{2iz\alpha} r_0(z), \end{aligned}$$

where $C \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$r_0(z) = \int_0^\infty C(\xi+\alpha) e^{2iz\xi}\, d\xi.$$

Recalling the symmetry

$$\overline{r_-}(z) = r_-(-z),$$

it follows that

$$\begin{aligned} \overline{r_-}(z) &= \int_\alpha^\infty C(\xi) e^{-2iz\xi}\, d\xi \\ &= e^{-2iz\alpha} \overline{r_0}(z). \end{aligned}$$

Since $C \in X$, it is easy to see that $r_0 \in H^2(\mathbb{C}^+) \cap H^\infty(\mathbb{C}^+)$ and $\overline{r_0} \in H^2(\mathbb{C}^-) \cap H^\infty(\mathbb{C}^-)$. We note explicitly that $r_-(z)$ continues analytically to $\mathbb{C}^+$, while $\overline{r_-}(z)$ continues to $\mathbb{C}^-$. We also have the estimates

$$\begin{aligned} |r_-(z)| &\le e^{-2\alpha \Im z} |r_0(z)| \qquad z \in \overline{\mathbb{C}^+} \\ |\overline{r_-}(z)| &\le e^{2\alpha \Im z} |\overline{r_0}(z)| \qquad z \in \overline{\mathbb{C}^-}. \end{aligned} \tag{3.29}$$

Our strategy for proving Theorem 3.1.1 is to make a change of contours in the Riemann-Hilbert problem (3.2.3) so that the support of w may be read off directly from the asymptotics of the transformed Riemann-Hilbert solution. In particular, we will use the transformed problem to show that $\mu(x,z) = 1$ for all $x < \alpha$ and $z \in \mathbb{R}$. In this case, the approximation (3.28) is exact for $x < \alpha$ so that $w(x) = 0$ when $x < \alpha$.

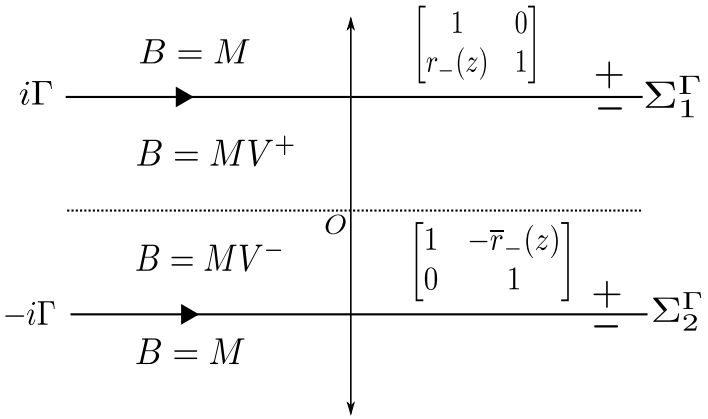

Figure 3.3: $B(x, z; \Gamma)$ has jumps along the contours $\Im(z) = \pm i\Gamma$. We have suppressed the x dependence for simplicity.

The Transformed Problem

Let $M(x, z)$ be the solution to the Riemann-Hilbert Problem (3.2.3) with the data r_-.

For each $\Gamma > 0$, define the family

$$B(x, z; \Gamma) = \begin{cases} M(x, z) & \Im z > \Gamma \\ M(x, z) \begin{bmatrix} 1 & 0 \\ -r_-(z)e^{-2ixz} & 1 \end{bmatrix} & 0 \leq \Im z < \Gamma \\ M(x, z) \begin{bmatrix} 1 & -\overline{r_-}(z)e^{2ixz} \\ 0 & 1 \end{bmatrix} & -\Gamma \leq \Im z < 0 \\ M(x, z) & \Im z < -\Gamma \end{cases} \tag{3.30}$$

$B(x, z; \Gamma)$ is analytic in z off of the lines $\Im z = \pm\Gamma$. It is continuous in z from above and below $\Im z = \pm\Gamma$, but has a multiplicative jump as z crosses these lines. See Figure 3.3. Clearly, this shift is only possible because $r_-(z)$ and $\overline{r_-}(z)$ extend analytically to $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. Also observe from (3.23) that

$$B(x, z; \Gamma) = \mu(x, z) \tag{3.31}$$

for $z \in \mathbb{R}$ and any $\Gamma > 0$.

Define the contours

$$\begin{aligned} \Sigma_1^\Gamma &= \{z \in \mathbb{C} : \Im(z) = \Gamma\} \\ \Sigma_2^\Gamma &= \{z \in \mathbb{C} : \Im(z) = -\Gamma\} \\ \Sigma^\Gamma &= \Sigma_1^\Gamma \cup \Sigma_2^\Gamma \end{aligned}$$

and orient these in the direction of increasing real part as in Figure 3.3.

We now compute the jump matrices for $B(x,z;\Gamma)$. Along the contour Σ_1^Γ, we have

$$\begin{aligned} B_+(x,z;\Gamma) &= M(x,z) \\ B_-(x,z;\Gamma) &= M(x,z)\begin{bmatrix} 1 & 0 \\ -r_-(z)e^{-2ixz} & 1 \end{bmatrix} \end{aligned}$$

so that

$$B_+(x,z;\Gamma) = B_-(x,z;\Gamma)\begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix} \tag{3.32}$$

Similarly, along the contour Σ_2^Γ, we have

$$\begin{aligned} B_+(x,z;\Gamma) &= M(x,z)\begin{bmatrix} 1 & -\overline{r_-}(z)e^{2ixz} \\ 0 & 1 \end{bmatrix} \\ B_-(x,z;\Gamma) &= M(x,z) \end{aligned}$$

so that

$$B_+(x,z;\Gamma) = B_-(x,z;\Gamma)\begin{bmatrix} 1 & -\overline{r_-}(z)e^{2ixz} \\ 0 & 1 \end{bmatrix}. \tag{3.33}$$

Proposition 3.3.1. *Fix $x < \alpha$ and $\Gamma > 0$. $B(x,z;\Gamma)$ defined by (3.30) is the solution to the Riemann-Hilbert problem*

$$\begin{cases} B(x,z;\Gamma) \text{ is analytic for } z \in \mathbb{C}\backslash\Sigma^\Gamma, \\ B_+(x,z;\Gamma) = B_-(x,z;\Gamma)U_x(z) \text{ for } z \in \Sigma^\Gamma, \\ \lim\limits_{|z|\to+\infty} B(x,z;\Gamma) = \mathbf{1} \end{cases} \tag{3.34}$$

where

$$U_x(z) = \begin{cases} V_{x,+}(z)^{-1} & z \in \Sigma_1 \\ V_{x,-}(z)^{-1} & z \in \Sigma_2 \end{cases}.$$

Proof. Fix $x < \alpha, \Gamma > 0$. $B(x,z;\Gamma)$ is analytic for $z \in \mathbb{C}\backslash\Sigma^\Gamma$, since $M(x,z)$ is analytic for $z \in \mathbb{C}\backslash\mathbb{R}$ and $r_-(z)$, $\overline{r_-}(z)$ extend analytically to the upper and lower half-planes respectively. By the computations in (3.32) and (3.33), $B(x,z;\Gamma)$ also satisfies the required jump conditions. It remains to compute the pointwise limit of $B(x,z;\Gamma)$ as $|z| \to +\infty$.

For this, we first recall that M is normalized to satisfy

$$\lim_{|z|\to+\infty} M(x,z) = \mathbf{1}.$$

Then using the estimates in (3.29), for $z \in \mathbb{C}^+$

$$\begin{aligned} |r_-(z)e^{2ixz}| &\leq e^{2(x-\alpha)\Im(z)}|r_0(z)| \\ &\leq Ce^{2(x-\alpha)\Im(z)}. \end{aligned} \tag{3.35}$$

Similarly, for $z \in \mathbb{C}^-$

$$\begin{aligned} |\overline{r_-(z)}e^{-2ixz}| &\leq e^{2(\alpha-x)\Im(z)}|\overline{r_0}(z)| \\ &\leq Ce^{2(\alpha-x)\Im(z)} \end{aligned} \tag{3.36}$$

For $z \in \mathbb{R}$,

$$|r_-(z)e^{2ixz}| = |r_0(z)|,$$

and by the Riemann-Lebesgue Lemma,

$$\lim_{\substack{|z|\to+\infty \\ z\in\mathbb{R}}} |r_0(z)| = 0.$$

Hence

$$\lim_{\substack{|z|\to+\infty \\ z\in\overline{\mathbb{C}^+}}} |r_-(z)e^{2ixz}| = \lim_{\substack{|z|\to+\infty \\ z\in\overline{\mathbb{C}^-}}} |\overline{r_-(z)}e^{-2ixz}| = 0.$$

From (3.30) we conclude that

$$\lim_{|z|\to+\infty} B(x,z;\Gamma) = \mathbf{1}.$$

□

Analysis of the Transformed Inverse Problem

Following the treatment of the inhomogeneous Riemann-Hilbert problems in [12], we now derive the solution formula for the Riemann-Hilbert in (3.34) for $x < \alpha$.

We introduce a second change of variables to convert the multiplicative jump problem to an additive one. Define

$$\begin{aligned} z \in \Sigma_1 : \quad \nu_1(x,z) &:= B_+(x,z;\Gamma) = B_-(x,z;\Gamma)\begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix} \\ z \in \Sigma_2 : \quad \nu_2(x,z) &:= B_+(x,z;\Gamma) = B_-(x,z;\Gamma)\begin{bmatrix} 1 & -\overline{r_-}(z)e^{2ixz} \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Then for $z \in \Sigma_1^\Gamma$

$$\begin{aligned} B_+(x,z;\Gamma) - B_-(x,z;\Gamma) &= \nu_1(x,z) - \nu_1(x,z)\begin{bmatrix} 1 & 0 \\ r_-(z)e^{-2ixz} & 1 \end{bmatrix} \\ &= \nu_1(x,z)\begin{bmatrix} 0 & 0 \\ -r_-(z)e^{-2ixz} & 0 \end{bmatrix}, \end{aligned}$$

and for $z \in \Sigma_2^\Gamma$

$$\begin{aligned} B_+(x,z;\Gamma) - B_-(x,z;\Gamma) &= \nu_2(x,z) - \nu_2(x,z)\begin{bmatrix} 1 & -\overline{r_-}(z)e^{2ixz} \\ 0 & 1 \end{bmatrix} \\ &= \nu_2(x,z)\begin{bmatrix} 0 & \overline{r_-}(z)e^{2ixz} \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The Cauchy operator on $L^2(\Sigma^\Gamma)$ is

$$\mathcal{C}f(z) = \frac{1}{2\pi i}\int_{\Sigma^\Gamma} \frac{f(\zeta)}{z-\zeta}\, d\zeta.$$

The Cauchy boundary value operators on $L^2(\Sigma^\Gamma)$ are the projections

$$\mathcal{C}_\pm f(z) = \lim_{\epsilon \downarrow 0} \mathcal{C}f(z \pm i\epsilon),$$

with

$$\|\mathcal{C}_\pm f\|_{L^2(\Sigma^\Gamma)} \leq \|f\|_{L^2(\Sigma^\Gamma)} \tag{3.37}$$

and

$$\mathcal{C}_+ - \mathcal{C}_- = I. \tag{3.38}$$

See, for example, [3, 12]. Again, we extend these operators to act on $L^2(\Sigma^\Gamma) \otimes M_2(\mathbb{C})$ functions component-wise.

Now set

$$\nu(x,z;\Gamma) = \begin{cases} \nu_1(x,z) & z \in \Sigma_1^\Gamma \\ \\ \nu_2(x,z) & z \in \Sigma_2^\Gamma \end{cases}$$

and

$$d(x,z;\Gamma) = \begin{cases} d_1(x,z) & z \in \Sigma_1^\Gamma \\ \\ d_2(x,z) & z \in \Sigma_2^\Gamma \end{cases}$$

where

$$d_1(x,z;\Gamma) = \begin{bmatrix} 0 & 0 \\ -r_-(z)e^{-2ixz} & 0 \end{bmatrix}, \qquad d_2(x,z;\Gamma) = \begin{bmatrix} 0 & \overline{r_-}(z)e^{2ixz} \\ 0 & 0 \end{bmatrix}.$$

For $z \in \Sigma^\Gamma$,

$$B_+(x,z;\Gamma) - B_-(x,z;\Gamma) = \nu(x,z;\Gamma)d(x,z;\Gamma)$$

The analytic extension of $B(x,z;\Gamma)$ off of the contours Σ^Γ is

$$B(x,z) = \mathbf{1} + \mathcal{C}[\nu(x,\cdot)d(x,\cdot)](z), \tag{3.39}$$

where for readability we have temporarily suppressed the dependence of B, ν, and d on the parameter Γ.

Taking boundary values from above $\Sigma_1^\Gamma, \Sigma_2^\Gamma$ in this expression yields

$$B_+(x,z) = \mathbf{1} + \mathcal{C}_+[\nu(x,\cdot)d(x,\cdot)](z).$$

On the other hand,

$$B_+(x,z) = \nu(x,z).$$

Hence, we obtain the integral equation

$$\nu(x,z) = \mathbf{1} + \mathcal{C}_d[\nu(x,\cdot)](z), \tag{3.40}$$

where the operator $\mathcal{C}_d$ on $L^2(\Sigma^\Gamma) \otimes M_2(\mathbb{C})$ is defined by

$$\mathcal{C}_d f(z) = \mathcal{C}_+[f(\cdot)d(x,\cdot)](z). \tag{3.41}$$

Proposition 3.3.2. *For Γ sufficiently large, the singular integral equation (3.40) has a unique solution for all $x < \alpha$.*

Proof. Fix $\Gamma > 0$ and $x < \alpha$.

Formally, the solution to (3.40) is

$$\nu(x,z) = (\mathbf{1} - \mathcal{C}_d)^{-1}\mathbf{1}.$$

Note however that $\mathbf{1} \notin L^2(\Sigma)$, so we iterate once to obtain the putative solution formula

$$\nu(x,z) = \mathbf{1} + (\mathbf{1} - \mathcal{C}_d)^{-1}\mathcal{C}_d\mathbf{1}. \tag{3.42}$$

Define the norm

$$\|A(z)\|_{L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))} = \| |A|_F \|_{L^2(\Sigma^\Gamma)}.$$

on the Banach space $L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))$.

Notice that the factor $d(x,z)$ can be trivially rewritten as

$$d(x,z) = d_1(x,z)\chi_{\Sigma_1}(z) + d_2(x,z)\chi_{\Sigma_2}(z).$$

For $z \in \Sigma^\Gamma$, it follow from the estimates (3.35) and (3.36) that

$$\left| d_i(x,z)\chi_{\Sigma_i^\Gamma} \right|_F \leq Ce^{2(x-\alpha)\Gamma}$$

when $i = 1, 2$.

Then if $f \in L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))$, we apply the sub-multiplicative property of the Frobenius norm to estimate

$$\begin{aligned}
\|\mathcal{C}_d f\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})} &= \left\| \left| f \cdot d_1 \cdot \chi_{\Sigma_1^\Gamma} + f \cdot d_2 \cdot \chi_{\Sigma_2^\Gamma} \right|_F \right\|_{L^2(\Sigma^\Gamma)} \\
&\leq \| \, |f \cdot d_1|_F \|_{L^2(\Sigma_1^\Gamma)} + \| \, |f \cdot d_2|_F \|_{L^2(\Sigma_2^\Gamma)} \\
&\leq \| \, |f|_F \, |d_1|_F \|_{L^2(\Sigma_1^\Gamma)} + \| \, |f|_F \, |d_2|_F \|_{L^2(\Sigma_2^\Gamma)} \\
&\leq Ce^{2(x-\alpha)\Gamma} \, \|f\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})} \, .
\end{aligned}$$

When the contour parameter Γ is sufficiently large, $\mathcal{C}_d$ has small $L^2(\Sigma^\Gamma) \otimes M_2(\mathbb{C})$ operator norm and $(\mathbf{1} - \mathcal{C}_d)^{-1}$ is a bounded operator on $L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))$. Note that this result is somewhat different from the $\Sigma = \mathbb{R}$ Riemann-Hilbert problem for ZS-AKNS where one can estimate the norm of the singular integral operator independently of x.

Now let us verify that $\mathcal{C}_d\mathbf{1} \in L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))$. Compute

$$\begin{aligned}
\|\mathcal{C}_d\mathbf{1}\|_{L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))} &\leq \|\mathcal{C}_+ d_1(x,\cdot)\|_{L^2(\Sigma^\Gamma \otimes M_2(\mathbb{C}))} + \|\mathcal{C}_+ \left[d_2(x,\cdot) \right]\|_{L^2(\Sigma_1^\Gamma)} \\
&\leq \|r_-(\cdot)e^{-2ix(\cdot)}\|_{L^2(\Sigma_1^\Gamma)} + \|\overline{r_-}(\cdot)e^{2ix(\cdot)}\|_{L^2(\Sigma_2^\Gamma)}
\end{aligned}$$

Using the estimates in (3.29), compute:

$$\begin{aligned}
\|r_-(\cdot)e^{-2ix(\cdot)}\|^2_{L^2(\Sigma_1^\Gamma)} &= \int_{-\infty}^{\infty} \left|r_-(R+i\Gamma)e^{-2ix(R+i\Gamma)}\right|^2 \, dR \\
&\leq \int_{-\infty}^{\infty} \left[e^{-2\alpha\Gamma} r_0(R+i\Gamma)\left|e^{-2ix(R+i\Gamma)}\right|\right]^2 \, dR \\
&\leq e^{4(x-\alpha)\Gamma} \|r_0\|_{H_2(\mathbb{C}^+)} \\
\|\overline{r_-}(\cdot)e^{2ix(\cdot)}\|_{L^2(\Sigma_2^\Gamma)} &= \int_{-\infty}^{\infty} \left|\overline{r_-}(R+i(-\Gamma))e^{2ix(R+i(-\Gamma))}\right|^2 \, dR \\
&\leq \int_{-\infty}^{\infty} \left[e^{2\alpha(-\Gamma)}\overline{r_0}(R-i\Gamma)\left|e^{2ix(R-i\Gamma)}\right|\right]^2 \, dR \\
&\leq e^{4(x-\alpha)\Gamma} \|\overline{r_0}\|_{H_2(\mathbb{C}^-)}
\end{aligned}$$

Thus $\mathcal{C}_d\mathbf{1} \in L_2(\Sigma^\Gamma)$ for any Γ. Together with the boundedness of the resolvent, this proves the solvability of the integral equation (3.40) for $x < \alpha$ and the validity of the putative solution formula (3.42). □

For each fixed $x < \alpha$, we have from (3.31) that

$$\mu(x,z) = \mathbf{1} + \mathcal{C}\left[\nu(x,\cdot\,;\Gamma)d(x,\cdot\,;\Gamma)\right](z)$$

for $z \in \mathbb{R}$ and Γ sufficiently large.

We will now take $\Gamma \to +\infty$ in this expression and show that

$$\mu(x,z) = \mathbf{1}$$

holds pointwise for any $x < \alpha$. Let $\mathbf{0}$ be the 2×2 zero matrix.

Proposition 3.3.3. *For $z \in \mathbb{C}\backslash\Sigma^\Gamma$*

$$\lim_{\Gamma\to+\infty} \mathcal{C}\left[\nu(x,\cdot\,;\Gamma)d(x,\cdot\,;\Gamma)\right](z) = \mathbf{0}.$$

Proof. We have

$$\begin{aligned}
\mathcal{C}\left[\nu(x,\cdot\,;\Gamma)d(x,\cdot\,;\Gamma)\right](z) &= \int_{\Sigma^\Gamma} \frac{\nu(x,\zeta;\Gamma)d(x,\zeta;\Gamma)}{\zeta - z}\, d\zeta \\
&= T_1(x,z;\Gamma) + T_2(x,z;\Gamma),
\end{aligned}$$

where

$$\begin{aligned}
T_1(x,z;\Gamma) &= \int_{-\infty}^{\infty} \frac{\nu_1(x,\xi+i\Gamma;\Gamma)d_1(x,\xi+i\Gamma;\Gamma)}{\xi+i\Gamma-z}\, d\zeta \\
T_2(x,z;\Gamma) &= \int_{-\infty}^{\infty} \frac{\nu_2(x,\xi-i\Gamma;\Gamma)d_2(x,\xi-i\Gamma;\Gamma)}{\xi-i\Gamma-z}\, d\zeta.
\end{aligned}$$

We estimate only the T_1 term, as T_2 is quite similar.

$$
\begin{aligned}
|T_1(x,z;\Gamma)|_F &\le \int_{-\infty}^{\infty} \frac{|\nu_1(x,\xi+i\Gamma;\Gamma)|_F\,|d_1(x,\xi+i\Gamma;\Gamma)|_F}{|\xi-i\Gamma-z|}\,d\xi \\
&\le e^{2(x-\alpha)\Gamma}\int_{-\infty}^{\infty} \frac{|\nu_1(x,\xi+i\Gamma;\Gamma)|_F\,|r_0(\xi+i\Gamma)|}{|\xi-i\Gamma-z|}\,d\xi \\
&\le e^{2(x-\alpha)\Gamma}\left(\int_{-\infty}^{\infty} |\nu_1(x,\xi+i\Gamma;\Gamma)|_F^2\,|r_0(\xi+i\Gamma)|^2\,d\xi\right)^{1\backslash 2} \\
&\qquad\qquad \times\left(\int_{-\infty}^{\infty}\frac{1}{|\xi-i\Gamma-z|^2}\,d\xi\right)^{1\backslash 2} \\
&= \frac{\pi e^{2(x-\alpha)\Gamma}}{|\Gamma-\Im z|}\left(\int_{-\infty}^{\infty} |\nu_1(x,\xi+i\Gamma;\Gamma)|_F^2\,|r_0(\xi+i\Gamma)|^2\,d\xi\right)^{1\backslash 2}
\end{aligned}
$$

To estimate

$$
G(x;\Gamma)=\int_{-\infty}^{\infty} |\nu_1(x,\xi+i\Gamma;\Gamma)|_F^2\,|r_0(\xi+i\Gamma)|^2\,d\xi,
$$

we write

$$
\nu_1(x,z;\Gamma)=\mathbf{1}+\tilde{\nu}_1(x,z;\Gamma)
$$

where

$$
\tilde{\nu}(x,z;\Gamma)=(\mathbf{1}-\mathcal{C}_d)^{-1}\,\mathcal{C}_d\mathbf{1}.
$$

Now $\tilde{\nu}_1(x,\cdot\,;\Gamma)\in L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})$ for Γ sufficiently large, with

$$
\begin{aligned}
\|\tilde{\nu}_1(x,\cdot\,;\Gamma)\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})} &\le \|(\mathbf{1}-\mathcal{C}_d)^{-1}\mathcal{C}_d\mathbf{1}\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})} \\
&\le \frac{1}{1-\|\mathcal{C}_d\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})}}\|\mathcal{C}_d\mathbf{1}\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})} \\
&\le e^{4(x-\alpha)\Gamma}\left(\|r_0\|_{H_2(\mathbb{C}^+)}+\|\overline{r_0}\|_{H_2(\mathbb{C}^-)}\right) \\
&= 2e^{4(x-\alpha)\Gamma}\|r_0\|_{H_2(\mathbb{C}^+)}.
\end{aligned}
$$

Then

$$\begin{aligned}
|G(x;\Gamma)| &= \int_{-\infty}^{\infty} |\mathbf{1} + \tilde{\nu}_1(x, \xi + i\Gamma; \Gamma)|_F^2 \, |r_0(\xi + i\Gamma)|^2 \, d\xi \\
&\leq \int_{-\infty}^{\infty} (1 + |\tilde{\nu}_1(x, \xi + i\Gamma; \Gamma)|_F^2 + 2\,|\tilde{\nu}_1(x, \xi + i\Gamma; \Gamma)|) \\
&\qquad\qquad \times |r_0(\xi + i\Gamma)|^2 \, d\xi \\
&\leq \|r_0\|_{H_2(\mathbb{C}^+)}^2 + \|r_0\|_{H_\infty(\mathbb{C}^+)}^2 \|\tilde{\nu}_1(x, \xi + i\Gamma; \Gamma)\|_{L^2(\Sigma_1^\Gamma)\otimes M_2(\mathbb{C})}^2 \\
&\quad + 2\|r_0\|_{H_\infty(\mathbb{C}^+)} \int_{-\infty}^{\infty} |\tilde{\nu}_1(x, \xi + i\Gamma; \Gamma)|_F |r_0(\xi + i\Gamma)| \, d\xi \\
&\leq \|r_0\|_{H_2(\mathbb{C}^+)}^2 + 2\,\|r_0\|_{H_2(\mathbb{C}^+)}\, \|r_0\|_{H_\infty(\mathbb{C}^+)}^2 e^{4(x-\alpha)\Gamma} \\
&\quad + 2\|r_0\|_{H_2(\mathbb{C}^+)} \|r_0\|_{H_\infty(\mathbb{C}^+)} \|\tilde{\nu}_1(x, \cdot; \Gamma)\|_{L^2(\Sigma^\Gamma)\otimes M_2(\mathbb{C})} \\
&\leq \|r_0\|_{H_2(\mathbb{C}^+)}^2 + 2\|r_0\|_{H_2(\mathbb{C}^+)} \|r_0\|_{H_\infty(\mathbb{C}^+)}^2 e^{4(x-\alpha)\Gamma} \\
&\qquad + 4\|r_0\|_{H_2(\mathbb{C}^+)}^2 \|r_0\|_{H_\infty(\mathbb{C}^+)} e^{4(x-\alpha)\Gamma} \\
&\leq K(r_0),
\end{aligned}$$

where

$$K(r_0) = \|r_0\|_{H_2(\mathbb{C}^+)}^2 + 2\|r_0\|_{H_2(\mathbb{C}^+)} \|r_0\|_{H_\infty(\mathbb{C}^+)}^2 + 4\|r_0\|_{H_2(\mathbb{C}^+)}^2 \|r_0\|_{H_\infty(\mathbb{C}^+)}.$$

Thus $G(x, \Gamma)$ may be uniformly bounded in Γ. Estimating T_2 in a completely similar way, we obtain

$$\left|\mathcal{C}\left[\nu(x, \cdot\,; \Gamma) d(x, \cdot\,; \Gamma)\right](z)\right|_F \leq \frac{2\pi e^{2(x-\alpha)\Gamma}}{|\Gamma - \Im z|} K(r_0).$$

Sending $\Gamma \to +\infty$ completes the proof. □

We may now prove Theorem 3.1.1, the main result of this chapter.

Proof of Theorem 3.1.1. Let $r_- = \mathcal{D}_P^- w \in \widehat{\mathcal{K}_\alpha}$ where $w \in X$.

By the previous proposition, we may conclude that $\mu(x, z) = \mathbf{1}$ for $z \in \mathbb{R}$ and $x < \alpha$.

Then for each $x < \alpha$, the reconstruction formula

$$w(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mu_{22}(x, z) r_-(z) e^{-2ixz} \, dz$$

reduces to

$$w(x) = \frac{1}{\pi} \int_{0}^{\infty} r_-(z) e^{-2ixz} \, dz.$$

Since

$$r_-(z) = \int_{-\infty}^{\infty} \chi_{[\alpha, +\infty)}(\xi) C(\xi) e^{2iz\xi} \, d\xi$$

for some $C \in X$, we have

$$w(x) = 0 \text{ for } x < \alpha$$

by the Fourier Inversion Formula. Therefore, $w \in \mathcal{K}_\alpha$. □

Now we use Theorem 3.1.1 and a ZS-AKNS symmetry to prove Corollary 3.1.1.

Proof of Corollary 3.1.1. If $r_+(z) = \mathcal{D}_P^+ w \in \widehat{K^\beta}$ then by changing variables

$$r_+(z) = \int_{-\beta}^{\infty} C(\xi) e^{2ix\xi}\, d\xi$$

for some $C \in X$. Let $\tilde{r}_- = \mathcal{D}_P^- \tilde{w}$ where $\tilde{w}(x) = -w(-x)$. By Proposition 2.3.5

$$\tilde{r}_-(z) = r_+(z).$$

Thus $r_-(z) \in \widehat{\mathcal{K}_{-\beta}}$. From Theorem 3.1.1, it follows that $\tilde{w}(x) \in \mathcal{K}_{-\beta}$ so that $w(x) \in \mathcal{K}^\beta$. □

Chapter 4 Localization of the Reconstruction

4.1 Introduction

The previous chapter described how to reconstruct a ZS-AKNS potential $w \in \mathcal{K}_\alpha$ from its reflection coefficient $r_-(z)$. In this chapter, we elaborate on the relationship between $r_-(z)$ and the local behavior of w. The main result of the chapter is:

Theorem 4.1.1. *If*

$$r_-(z) = \int_\alpha^\infty e^{2iz\xi} C(\xi)\, d\xi$$

for $w \in \mathcal{K}_\alpha$ then for a.e. x, $w(x)$ may be recovered from the values of C on $[\alpha, x]$.

As a direct consequence of this result, we have:

Corollary 4.1.1. *Let $w, \tilde{w} \in \mathcal{K}_\alpha$ and let*

$$r_-(z) = D_P^- w(z) = \int_\alpha^\infty e^{2iz\xi} C(\xi)\, d\xi$$
$$\tilde{r}_-(x) = D_P^- \tilde{w}(z) = \int_\alpha^\infty e^{2iz\xi} \tilde{C}(\xi)\, d\xi$$

where $C, \tilde{C} \in X$. If $C(\xi) = \tilde{C}(\xi)$ for a.e. $\xi \in [\alpha, \beta]$, then $w(x) = \tilde{w}(x)$ a.e. on $[\alpha, \beta]$.

Notation and Preliminaries

Let $*$ denote the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t) g(t)\, dt.$$

One easily computes the Fourier convolution formulas:

$$\mathcal{F} f(z) \cdot \mathcal{F} g(z) = \mathcal{F}\left(f * g\right)(z)$$
$$\mathcal{F}^{-1} f(x) \cdot \mathcal{F}^{-1} g(x) = \frac{1}{\pi} \mathcal{F}^{-1}\left(f * g\right)(x).$$

Denote by T_c the translation

$$T_c f(x) = f(x - c),$$

and by R the reflection

$$Rf(x) = f(-x).$$

From the definitions, we easily check that

$$e^{2icz} \mathcal{F}\left(f\right)(z) = \mathcal{F}\left(T_c f\right)(z)$$
$$e^{2icx} \mathcal{F}^{-1}\left(f\right)(x) = \mathcal{F}\left(T_{-c} f\right)(x).$$

Observe that if $r_- \in \widehat{X}$ then

$$r_- = \mathcal{F}C$$
$$\overline{r_-} = \mathcal{F}RC$$

for some $C \in X$.

The Cauchy boundary operators $\mathcal{C}_\pm$ act as half-line cutoffs in Fourier space:

$$\mathcal{C}_+ f(z) = \mathcal{F}\left[\chi_+ \cdot \mathcal{F}^{-1} f\right](z)$$
$$\mathcal{C}_- f(z) = -\mathcal{F}\left[\chi_- \cdot \mathcal{F}^{-1} f\right](z)$$

(see Appendix C). The symbols $\chi_\pm$ are a shorthand for the cutoff functions

$$\chi_+(x) = \begin{cases} 0 & x \in (-\infty, 0] \\ 1 & x \in (0, +\infty) \end{cases} \qquad \chi_-(x) = \begin{cases} 1 & x \in (-\infty, 0] \\ 0 & x \in (0, +\infty) \end{cases}$$

The following simple proposition will be very useful.

Proposition 4.1.1. *Let f be supported on $[\alpha, \beta] \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and g on $[\gamma, \delta] \subseteq \overline{\mathbb{R}}$. Then*

$$\text{supp}\,(f * g) \subseteq [\alpha + \gamma, \beta + \delta],$$

with the convention that $a + (+\infty) = +\infty$ and $b + (-\infty) = -\infty$ for any $a, b \in \mathbb{R}$.

Proof. Suppose that f is supported on $[\alpha, \beta]$ and g is supported on $[\gamma, \delta]$. Then

$$\begin{aligned}
(f * g)(x) &= \int_{\mathbb{R}} f(x - t) g(t)\, dt \\
&= \int_\gamma^\delta f(x - t) g(t)\, dt \\
&= -\int_{x-\gamma}^{x-\delta} f(u) g(x - u)\, du \\
&= \int_{[x-\delta, x-\gamma]} \chi_{[\alpha,\beta]}(u) f(u) g(x - u)\, du \\
&= \int_{[x-\delta, x-\gamma] \cap [\alpha, \beta]} f(u) g(x - u)\, du
\end{aligned}$$

The convolution vanishes when $[x - \delta, x - \gamma] \cap [\alpha, \beta] = \emptyset$. There are two ways for this to happen; either $x - \delta > \beta$ or $x - \gamma < \alpha$. Thus when $x > \beta + \delta$ or $x < \alpha + \gamma$, the convolution vanishes. □

4.2 Localization of the reflection coefficients

Recall from equation (3.27) of the previous chapter that the integral

$$w(x) = \frac{1}{\pi} \int_{\mathbb{R}} r_-(z) \mu_{22}(x, z) e^{-2ixz}\, dz. \tag{4.1}$$

inverts the scattering operator $\mathcal{D}_P$ on X. From Theorem 3.2.3, the function $\mu(x,z)$ is the solution to the family of x-parametrized integral equations

$$(I - \mathcal{C}_x)\mu(x,z) = \mathbf{1}, \tag{4.2}$$

where the action of $\mathcal{C}_x$ on a matrix valued function $\Psi \in X$ is given by

$$\mathcal{C}_x\Psi(z) = \begin{bmatrix} \mathcal{C}_-\left[r_-(\circ)e^{-2ix\circ}\psi_{12}(\circ)\right](z) & \mathcal{C}_+\left[\overline{r_-}(\circ)e^{2ix\circ}\psi_{11}(\circ)\right](z) \\ \mathcal{C}_-\left[r_-(\circ)e^{-2ix\circ}\psi_{22}(\circ)\right](z) & \mathcal{C}_+\left[\overline{r_-}(\circ)e^{2ix\circ}\psi_{21}(\circ)\right](z) \end{bmatrix}. \tag{4.3}$$

$\mu(x,z)$ may be expanded in a Neumann series as

$$\mu(x,z) = \mathbf{1} + \sum_{k=1}^{\infty} \mathcal{C}_x^k \mathbf{1} \tag{4.4}$$

where $\mu(x,z) - \mathbf{1} \in L^2(\mathbb{R}) \otimes M_2(\mathbb{C})$.

Let

$$e_x(z) = e^{2ixz}$$

and observe that

$$\mathcal{C}_x^2\Psi(z) = \begin{bmatrix} \mathcal{C}_-\left[r_- e_{-x}\mathcal{C}_+(\overline{r_-}e_+\psi_{11})\right] & \mathcal{C}_+\left[\overline{r_-}e_x\mathcal{C}_-(re_{-x}\psi_{12})\right] \\ \mathcal{C}_-\left[r_- e_{-x}\mathcal{C}_+(\overline{r_-}e_-\psi_{21})\right] & \mathcal{C}_+\left[\overline{r_-}e_x\mathcal{C}_-(re_{-x}\psi_{22})\right] \end{bmatrix}(z).$$

Hence, the operator $\mathcal{C}_x^2$ acts component-wise on the entries of the matrix Ψ. It follows that the iterate $\mathcal{C}_x^k\mathbf{1}$ is diagonal when k is even, and off-diagonal when k is odd. Only the even terms of the series in (4.4) contribute to $\mu_{22}(x,z)$. In particular,

$$\mu_{22} = (I - \mathcal{S}_x)^{-1}1 = 1 + \sum_{k=1}^{\infty} \mathcal{S}_x^k 1$$

where

$$\mathcal{S}_x f = \mathcal{C}_+(\overline{r_-}e_{-x}\mathcal{C}_+(re_x f)). \tag{4.5}$$

The operator $\mathcal{S}_x$ is defined for $f \in L^2(\mathbb{R})$ but may be extended to act on constant functions $f = \kappa$ as follows. Recall that $\mathcal{C}_\pm$ are projections on $L^2(\mathbb{R})$ with

$$\|\mathcal{C}_\pm f\|_{L^2(\mathbb{R})} < c\|f\|_{L^2(\mathbb{R})}.$$

Also, note that $r_- \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\|r_-\|_\infty < 1$. By taking $f = \kappa$ in definition (4.5), we easily estimate that

$$\|\mathcal{S}_x 1\|_{L^2(\mathbb{R})} < |\kappa|c\|r_-\|_{L^\infty(\mathbb{R})}\|r\|_{L^2(\mathbb{R})}$$

(c.f. equation (3.26) of the previous chapter). Similarly, for $f \in L^2(\mathbb{R})$, we estimate

$$\begin{aligned} \|\mathcal{S}_x f\|_{L^2(\mathbb{R})} &\leq c\|r_-\|_{L^\infty(\mathbb{R})}\|\mathcal{C}_-\left(r_- e_x f\right)\|_{L^2(\mathbb{R})} \\ &\leq c\|r_-\|^2_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})} \end{aligned}$$

(c.f. equation (3.25)). Hence the iterates $\mathcal{S}_x^k 1$ are well-defined with

$$\|\mathcal{S}_x^k 1\|_{L^2(\mathbb{R})} \leq c\|r_-\|_{L^\infty(\mathbb{R})}^{2k-1}\|r_-\|_{L^2(\mathbb{R})}. \tag{4.6}$$

Set

$$\nu_x(z) = [\mu(x,z)]_{22} - 1 = \sum_{k=1}^{\infty} \mathcal{S}_x^k 1. \tag{4.7}$$

Since $\nu_x(\cdot) \in L^2(\mathbb{R})$ for each x, we may also define

$$N_x(y) = \mathcal{F}^{-1}(\nu_x)(y). \tag{4.8}$$

If $r_- \in X$ has the Fourier representation $r_- = \mathcal{F}C$, by putting (4.7) into (4.1) we compute that

$$w(x) = C(x) + (C(\circ) * N_x(\circ))(x). \tag{4.9}$$

The inversion formula (4.1) for $r_- \in \widehat{X}$ therefore decomposes into a linear term and a nonlinear convolution term. Heuristically, the linear $C(x)$ term arises because the inverse scattering transform $\mathcal{I}_P^-$ is Fourier inversion to first order approximation, while the lower-order convolution term captures the nonlinearity of $\mathcal{I}_P^-$. When $r_- \in \widehat{\mathcal{K}_\alpha}$, the nonlinearity $C * N_x$ in (4.9) has a simple finite support structure. In particular, we will prove:

Lemma 4.2.1 (Localization Lemma)**.** *Let $r_- \in \widehat{\mathcal{K}_\alpha}$ have the Fourier representation $r_- = \mathcal{F}C$ for $C \in \mathcal{K}_\alpha$.*

1. *N_x is supported on $[0, x-\alpha]$.*
2. *$\nu_x = \mathcal{F}N_x$ is completely determined by the values of C on $[\alpha, x]$.*

The proof of this result is somewhat technical and we postpone it until the end of the chapter. Assuming the result for the moment, let us show how the main results of this chapter are simple consequences of Lemma 4.2.1.

Proof of Theorem 4.1.1. Since $N_x(\xi)$ is supported on $[0, x-\alpha]$, we first observe that

$$\begin{aligned} C * N_x(x) &= \int_{-\infty}^{\infty} C(\tau) N_x(x-\tau)\, d\tau \\ &= \int_{\alpha}^{x} C(\tau) N_x(x-\tau)\, d\tau. \end{aligned}$$

For each fixed x, the function ν_x is completely determined by the values of C on $[\alpha, x]$. Therefore, $N_x = \mathcal{F}^{-1}\nu_x$ is also determined by C on $[\alpha, x]$. It follows that the convolution $C * N_x(x)$ is completely determined by the range of C on $[\alpha, x]$.

Hence

$$w(x) = C(x) + C * N_x(x)$$

is completely determined by C on $[\alpha, x]$. □

Proof of Corollary 4.1.1. Suppose that $r_-^{(1)}, r_-^{(2)} \in \widehat{\mathcal{K}}_\alpha$ have the Fourier representations $r_-^{(1)} = \mathcal{F}C_1$, $r_-^{(2)} = \mathcal{F}C_2$ for $C_1, C_2 \in \mathcal{K}_\alpha$ with $C_1 = C_2$ a.e. on $[\alpha, \beta]$. By Proposition 4.2.1, $\nu_x^{(1)}, \nu_x^{(2)}$ are completely determined by the values of $C_1 = C_2$ on $[\alpha, x]$, and so

$$\nu_x^{(1)} = \nu_x^{(2)}$$

for any $x \in [\alpha, \beta]$. Then for any $x \in [\alpha, \beta]$, $N_x^{(1)} = N_x^{(2)}$, and we have from (4.9) that

$$\begin{aligned} w_1(x) - w_2(x) &= \left(C_1 * N_x^{(1)}\right)(x) - \left(C_2 * N_x^{(2)}\right)(x) \\ &= \left([C_1(\circ) - C_2(\circ)] * N_x^{(1)}\right)(x). \end{aligned}$$

Now $C_1(\circ) - C_2(\circ)$ is supported on $[\beta, +\infty)$ and $N_x^{(1)}$ is supported on $[0, x-\alpha]$. By Proposition 4.1.1, the support of the convolution $[C_1(\circ) - C_2(\circ)] * N_x^{(1)}$ is contained in $[\beta, +\infty)$. We conclude that $w_1(x) = w_2(x)$ a.e. on $(-\infty, \beta]$. □

4.3 Proof of the Localization Lemma

The proof of Lemma 4.2.1 use elementary techniques, but requires carefully tracking how the support of $w(x)$ propagates through the nonlinear transformation $\mathcal{D}_P^-$. Though the computation is rather involved, the underlying idea is quite simple. If $w(x)$ is supported on $[\alpha, +\infty)$ then so is C, the inverse Fourier transform of r_-. The function $\nu_x(z)$ is obtained from the iteration of the operator $\mathcal{S}_x$ on 1. The operator $\mathcal{S}_x$ composes the Cauchy boundary value operators $\mathcal{C}_\pm$ with multiplication by $r_- e_{-x}, \overline{r_-} e_x$. The combination of the $[\alpha, +\infty)$ support of C and the cutoff action of $\mathcal{C}_\pm$ leads to the finite support properties claimed in Lemma 4.2.1.

We first prove that N_x is supported on $[0, x-\alpha]$ when C is supported on $[\alpha, +\infty)$, which is Part (1) of Lemma 4.2.1. Recall that

$$N_x(y) = \mathcal{F}^{-1}\left(\sum_{k=1}^{\infty} \mathcal{S}_x^k 1\right)(y). \tag{4.10}$$

We need the following lemma.

Lemma 4.3.1. *Suppose $C \in \mathcal{K}_\alpha$. Then $\mathcal{F}^{-1}(\mathcal{S}_x\psi)$ is supported on $[0, x-\alpha]$ for any $\psi \in L^2(\mathbb{R})$ or ψ a constant function.*

Proof. Recall that $r_- \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. If $\psi \in L^2(\mathbb{R})$ or ψ is a constant function, we may compute

$$\begin{aligned} \mathcal{F}^{-1}(\mathcal{S}_x\psi) &= \mathcal{F}^{-1}\left(\mathcal{C}_+\left(\overline{r_-} e_x \mathcal{C}_-\left(r_- e_{-x}\psi\right)\right)\right) \\ &= -\chi_+ \mathcal{F}^{-1}\left(\overline{r_-} e_x \cdot \mathcal{F}\left(\chi_- \mathcal{F}^{-1}\left(r_- e_{-x}\psi\right)\right)\right). \end{aligned}$$

Set $f = r_- e_x$ and $g = \mathcal{F}\left[\chi_- \mathcal{F}^{-1}(r e_{-x}\psi)\right]$. It is easily verified that $f, g \in L^2(\mathbb{R})$ for $\psi \in L^2(\mathbb{R})$ or ψ a constant function.

Then

$$\begin{aligned}\mathcal{F}^{-1}(\mathcal{S}_x\psi) &= \chi_+\mathcal{F}^{-1}\left(f\cdot g\right)\\ &= \pi\chi_+\left(\mathcal{F}^{-1}f\right)*\left(\mathcal{F}^{-1}g\right).\end{aligned}$$

Note that

$$\begin{aligned}\mathcal{F}^{-1}f &= \mathcal{F}^{-1}\left[(\mathcal{F}RC)e_x\right]\\ &= \mathcal{F}^{-1}\mathcal{F}(T_xRC)\\ &= T_xRC\end{aligned}$$

has support in $(-\infty, x-\alpha]$. On the other hand,

$$\mathcal{F}^{-1}g = \chi_-\mathcal{F}^{-1}(r_-e_{-x}\psi)$$

has support on $(-\infty, 0]$. By Proposition 4.1.1, the convolution $(\mathcal{F}^{-1}f)*(\mathcal{F}^{-1}g)$ is supported on $(-\infty, x-\alpha]$. Because of the cutoff χ_+, it follows that $\mathcal{F}^{-1}(\mathcal{S}_x\psi)$ is supported in $[0, x-\alpha]$ for any $\psi\in L^2(\mathbb{R})$ or ψ a constant function. □

Now we prove the first part of Lemma 4.2.1.

Proof of Lemma 4.2.1, Part 1. To see that $N_x(y)$ is supported on $[0, x-\alpha]$, we must argue that the inverse Fourier transform in (4.10) may be brought under the summation.

Let

$$Y_k(\xi) = \mathcal{F}^{-1}\left(\mathcal{S}_x^k 1\right).$$

Using the Plancherel identity and the estimate (4.6), we have

$$\|Y_k\|_{L^2(\mathbb{R})} \le c\|r_-\|_{L^\infty(\mathbb{R})}^{2k-1}\|r_-\|_{L^2(\mathbb{R})}.$$

Since Y_k is supported on $[0, x-\alpha]$, we also have that $Y_k\in L^1(\mathbb{R})$ with the bound

$$\|Y_k\|_{L^1(\mathbb{R})} \le c|x-\alpha|^{1/2}\|r_-\|_{L^\infty(\mathbb{R})}^{2k-1}\|r_-\|_{L^2(\mathbb{R})}.$$

$Y_k(\xi)$ is defined for a.e. ξ so that the sequences of partial sums

$$F_N(\xi) = \sum_{i=1}^{N} Y_{2k}(\xi)e^{2iz\xi},$$

and

$$G_N(\xi) = \sum_{i=1}^{N} |Y_{2k}(\xi)|$$

are also defined for a.e. ξ.

Then

$$\|G_N\|_{L^1(\mathbb{R})} \le c|x-\alpha|\sum_{k=1}^{N}\left(\|r\|_{L^\infty(\mathbb{R})}\right)^k \le c\frac{|x-\alpha|}{1-\|r\|_\infty}.$$

Let

$$G(\xi) = \lim_{N\to\infty} G_N(\xi).$$

The Lebesgue Monotone Convergence Theorem applies to show that G is measurable and belongs to $L^1(\mathbb{R})$. The series for $G(\xi)$ therefore converges pointwise for a.e. ξ. Thus, the series

$$N(\xi) = \sum_{k=1}^{\infty} Y_{2k}(\xi) e^{2iz\xi}$$

is pointwise absolutely convergent for a.e. ξ. Hence, $F_N \to N$ pointwise for a.e ξ. Since

$$|F_N(\xi)| \leq G_N(\xi) \leq G(\xi),$$

an application of the Lebesgue Dominated Convergence Theorem completes the proof. □

A priori, if $r_- = \mathcal{F}C$ with $C \in \mathcal{K}_\alpha$ and $x \in \mathbb{R}$, we would expect in the reconstruction formula (4.9) that ν_x depends on the values of the C on $[\alpha, +\infty)$. In fact, we will now prove that ν_x is determined by the values of C over the finite interval $[\alpha, x]$. This is part 2 of Proposition 4.2.1. To this end, we define

$$\widetilde{C}(y) = \chi_{[\alpha,x]}(y) C(y).$$

Recall that

$$\mathcal{S}_x f = \mathcal{C}_+ \left(\overline{r_-} e_x \mathcal{C}_- \left(r e_{-x} f \right) \right).$$

Working from the inside out, we have

$$\begin{aligned} \mathcal{C}_- \left(r e_{-x} \cdot f \right) &= -\mathcal{F} \left(\chi_- \mathcal{F}^{-1} \left(\mathcal{F}C \, e_{-x} \cdot f \right) \right) \\ &= -\mathcal{F} u_1 \end{aligned}$$

where

$$u_1 = \chi_- \mathcal{F}^{-1} \left(\mathcal{F}(T_{-x} C) \cdot f \right).$$

Compute

$$\begin{aligned} \mathcal{S}_x f &= \mathcal{C}_+ \left(\overline{r_-} e_x \cdot \mathcal{F} u_1 \right) \\ &= \mathcal{F} \left[\chi_+ \mathcal{F}^{-1} \left(\mathcal{F}(RC) \, e_x \cdot \mathcal{F} u_1 \right) \right] \\ &= \mathcal{F} \left[\chi_+ \mathcal{F}^{-1} \left(\mathcal{F} \left(T_x RC \right) \cdot \mathcal{F} u_1 \right) \right] \\ &= \mathcal{F} \left[\chi_+ (T_x RC) * u_1 \right]. \end{aligned}$$

Thus,

$$\mathcal{S}_x f = -\mathcal{F} \left[\chi_+ \left\{ (T_x RC) * \left(\chi_- \mathcal{F}^{-1} \left(\mathcal{F}(T_{-x} C) \, f \right) \right) \right\} \right].$$

This motivate us to define

$$\widetilde{\mathcal{S}}_x f = -\mathcal{F} \left[\chi_+ \left\{ \left(T_x R \widetilde{C} \right) * \left(\chi_- \mathcal{F}^{-1} \left(\mathcal{F}(T_{-x} \widetilde{C}) \, f \right) \right) \right\} \right]$$

We now prove part 2 of Lemma 4.2.1. We will show that $\nu_x = \widetilde{\nu}_x$ where

$$\widetilde{\nu}_x = \left(1 - \widetilde{\mathcal{S}}_x\right)^{-1} \tilde{S}_x 1,$$

by establishing that

$$\mathcal{S}_x^k 1 = \tilde{\mathcal{S}}_x^k 1$$

for each k.

Lemma 4.3.2. *If either:*

1. *ψ is a constant function*
2. *$\psi \in L^2(\mathbb{R})$ with $\mathcal{F}^{-1}\psi$ supported in $[0, x - \alpha]$*

then

$$\mathcal{S}_x \psi = \tilde{\mathcal{S}}_x \psi.$$

Proof. Define

$$\begin{aligned} f_1 &= T_x R C \\ g_1 &= \chi_- \mathcal{F}^{-1}\left(\mathcal{F}(T_{-x}C)\psi\right) \\ f_2 &= T_x R \widetilde{C} \\ g_2 &= \chi_- \mathcal{F}^{-1}\left(\mathcal{F}(T_{-x}\widetilde{C})\psi\right) \end{aligned}$$

We must show that

$$\chi_+(f_1 * g_1) = \chi_+(f_2 * g_2),$$

or equivalently that

$$\chi_+(f_1 - f_2) * g_1 - \chi_+ f_2 * (g_1 - g_2) = 0. \tag{4.11}$$

Observe that $f_1 - f_2 = T_x R(C - \tilde{C})$ has its support in $(-\infty, 0]$ since $C - \tilde{C}$ has its support in $[x, +\infty)$. Now g_1 has its support in $(-\infty, 0]$ because of the cutoff χ_-. Therefore,

$$\chi_+(f_1 - f_2) * g_1 = 0$$

by Proposition 4.1.1.

If $\psi = c$ is constant then

$$g_1 - g_2 = c\chi_- T_{-x}\left(C - \widetilde{C}\right).$$

The support of this difference is empty since $T_{-x}(C - \widetilde{C})$ is supported on $[0, +\infty)$ and χ_- is supported on $(-\infty, 0]$. Equation (4.11) follows immediately.

If $\psi \in L^2(\mathbb{R})$, we compute that

$$\begin{aligned} g_1 - g_2 &= \chi_- \left(\mathcal{F}^{-1}\left(F[(T_{-x}(C - \tilde{C})]\psi\right)\right) \\ &= \chi_- \left(T_{-x}(C - \widetilde{C})\right) * \mathcal{F}^{-1}\psi \end{aligned}$$

$T_{-x}(C-\widetilde{C})$ is supported on $[0,+\infty)$. By assumption $\mathcal{F}^{-1}\psi$ is supported on $[0,x-\alpha]$. By Proposition 4.1.1, the support of g_1-g_2 is empty, so that $g_1-g_2=0$. Equation (4.11) also holds in this case. □

We can now complete the proof of the second part of Lemma 4.2.1.

Proof of Lemma 4.2.1, Part 2. We claim that

$$\mathcal{S}_x^k 1 = \widetilde{\mathcal{S}}_x^k 1$$

holds for any $k=1,2,....$

For $k=1$, the result follows by applying Lemma 4.3.2 with $\psi=1$.

For $\psi \in L^2(\mathbb{R})$, let

$$f = \mathcal{S}_x \psi$$
$$\tilde{f} = \widetilde{\mathcal{S}}_x \psi$$

By Lemma 4.3.1, $\mathcal{F}^{-1}f, \mathcal{F}^{-1}\tilde{f}$ have support on $[0,x-\alpha]$. By Lemma 4.3.2, it follows that

$$f = \tilde{f}.$$

In particular,

$$\tilde{S}_x^k 1 = \widetilde{S}_x^k 1$$

for any $k=2,3,...,$.

It follows that $\nu_x = \widetilde{\nu}_x$ and hence that ν_x is depends only on the values of C on $[\alpha,x]$. □

Chapter 5 An Application to Singular Potential Schrödinger Equations

5.1 Introduction

In this final chapter, we describe an application of our results to an inverse spectral problem for a class of Schrödinger operators with distributional potentials.

Miura Potentials

Let $H^{-1}(\mathbb{R})$ be the dual to the Sobolev space

$$H^1(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R}) \right\}.$$

Following the seminal work of Miura in [34], define the *Miura map* from $L^2_{\text{loc}}(\mathbb{R}) \to H^{-1}_{\text{loc}}(\mathbb{R})$ by

$$\mathcal{B}(w) = w' + w^2.$$

$\mathcal{B}$ is well-known in the theory of Korteweg-de Vries (KdV) equation as the map which takes classical solutions of the modified KdV equation to classical solutions of the KdV equation (see [33, 34]). Kappeler, Perry, Shubin, and Topolav have characterized the range and geometry of this mapping in [26]. Following these authors, we call a distribution $q \in H^{-1}_{\text{loc}}(\mathbb{R})$ in the range of $\mathcal{B}$ a *Miura potential* and a function $w \in L^2_{\text{loc}}$ a *Riccati representative* for q. If q is in the class of Miura potentials, the self-adjoint Schrödinger operator

$$L_q = -\frac{d^2}{dx^2} + q(x)$$

can be factorized as

$$L_q = \left(\frac{d}{dx} - w\right)^* \left(\frac{d}{dx} + w\right)$$

where * is the adjoint in the L^2 inner product (see [25]). Moreover, the range conditions in [26] show that Miura potential Schrödinger operator L_q is nonnegative definite, and that $L_q y = 0$ has a strictly positive solution in $H^1_{\text{loc}}(\mathbb{R})$. Thus, the Miura class is a natural one over which to study the spectral and scattering theory of L_q, and new results for singular potential problems using the framework of [26] appear in [21, 24, 27].

To make precise the definition of the self-adjoint operator L_q on $L^2(\mathbb{R})$ for q a Miura potential, we recall that there is a bijective correspondence between semi-bounded, self-adjoint operators on $L^2(\mathbb{R})$ and symmetric quadratic, semi-bounded, and closed quadratic forms on $L^2(\mathbb{R})$ (see [29], VI §2, part 1). In particular, for $q \in H^{-1}(\mathbb{R})$ define

$$\mathfrak{t}_{\mathfrak{q}}(\phi) = \int_{-\infty}^{\infty} |\phi'(x)|^2 \, dx + \left\langle q, |\phi|^2 \right\rangle \tag{5.1}$$

for $\phi \in C_0^\infty(\mathbb{R})$. Here $\langle \cdot, \cdot \rangle$ is the dual space pairing of distribution and test function. It is possible to show that this symmetric quadratic form is closed and nonnegative

(see [43] and the Appendix of [26]). Hence, there exists a unique nonnegative, self-adjoint operator T such that:

1. $\mathfrak{D}(T) \subset \mathfrak{D}(\mathfrak{t}_\mathfrak{q})$ and
$$\mathfrak{t}_\mathfrak{q}(\phi, \psi) = (T\phi, \psi)$$
holds for all $\phi, \psi \in \mathfrak{D}(T)$.

2. If $\phi \in \mathfrak{D}(\mathfrak{t}_\mathfrak{q})$ and there exists $\chi \in L^2(\mathbb{R})$ with
$$\forall \psi \in \mathfrak{D}(\mathfrak{t}_\mathfrak{q}), \quad \mathfrak{t}_\mathfrak{q}(\phi, \psi) = (\chi, \psi)$$
then $T\phi = \chi$.

By this identification, the formal Schrödinger equation (5.3) with Miura potential q is a well-defined, self-adjoint operator on $L^2(\mathbb{R})$.

Overview

Following the work of [21] on the whole-line scattering for L_q, we consider Miura potential Schrödinger operators

$$L_q = -\frac{d^2}{dx^2} + q(x) \tag{5.2}$$

on $L^2(\mathbb{R})$ where $q \in H^{-1}(\mathbb{R})$ has the form

$$q = \mathcal{B}(w)$$

for $w \in \mathcal{K}_\alpha$. In general, a Miura potential may have more than one Riccati representation, however when $q = \mathcal{B}(w)$ with $w \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then w is unique [26]. Some explicit examples of the Miura representation of distributional potentials may be found in [21, 26].

For each α, we define the subclass of the Miura potentials

$$\mathcal{M}_\alpha = \{q = \mathcal{B}(w) : w \in \mathcal{K}_\alpha\}.$$

There is a simple relation between the Schrödinger equation

$$L_q u = z^2 u \qquad z \in \mathbb{C}^+ \tag{5.3}$$

with $q = w' + w^2 \in \mathcal{M}_\alpha$, and the ZS-AKNS equation with potential $w \in \mathcal{K}_\alpha$. In particular, if $\Psi_+^{(1)}(x, z)$ is the first column of the Jost solution to the ZS-AKNS equation, then

$$u(x, z) = \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \Psi_+^{(1)}(x, z)$$

is an L^2 solution to (5.3). In the remainder of the chapter, we exploit this simple relation to develop a singular potential analog of the classical direct and inverse spectral theory of the Schrödinger operator. In particular, we will an analog of the classical Weyl-Titchmarsh m-function the operator L_q with Miura potential q. We

will then use the direct and inverse scattering theory of the ZS-AKNS equation to study the direct spectral mapping

$$\mathcal{D}_L : q \mapsto m$$

and its inverse

$$\mathcal{I}_L : m \mapsto q.$$

We close the chapter by establishing a local Borg-Marchenko type result for the $\mathcal{M}_\alpha$ potential Schrödinger equation, a result which follows quite naturally from Theorem 4.1.1 of the previous chapter.

5.2 The Direct and Inverse Spectral Problem for L_q

We begin with a brief summary of the classical direct and inverse spectral theory of the operator L_q. Our presentation primarily follows the standard treatments in [10, 30].

The Classical Spectral Theory of the Half-Line Schrödinger Equation

In the classical spectral theory of the 1D Schrödinger operator, we assume that $q \in L^1_{\text{loc}}(\mathbb{R})$ and consider the classical operator L_q on $L^2[\alpha, +\infty)$ formally defined by (5.2).

The solutions $\theta(x, z)$ and $\xi(x, z)$ of

$$L_q u = z^2 u \qquad z \in \mathbb{C}^+$$

satisfying

$$\begin{cases} \theta(\alpha, z) = \sin\eta \\ \theta'(\alpha, z) = -\cos\eta \end{cases} \qquad \begin{cases} \xi(\alpha, z) = \cos\eta \\ \xi'(\alpha, z) = \sin\eta \end{cases}$$

form a basis for the solution space of (5.3). In our convention $z^2 \in \mathbb{C}/[0, +\infty)$ is the spectral parameter and we parameterize solution families of the Schrödinger equation by $z \in \mathbb{C}^+$.

Now impose a boundary condition on $u(x, z)$ at $x = \beta > \alpha$:

$$\cos\gamma u(\beta, z) + \sin\gamma u'(\beta, z) = 0. \tag{5.4}$$

For $z \in \mathbb{C}^+$, every solution to $L_q u = z^2 u$ satisfying (5.4) may be written as

$$u(x, z) = \theta(x, z) + m\, \xi(x, z), \tag{5.5}$$

up to a multiplicative constant. Solving for m, we find that

$$m(z^2; \beta) = -\frac{\cot\gamma\, \theta(\beta, z) + \theta'(\beta, z)}{\cot\gamma\, \xi(\beta, z) + \xi'(\beta, z}. \tag{5.6}$$

Sending $\beta \to +\infty$, which corresponds to imposing a boundary condition at $x = +\infty$, a simple geometric argument shows that the coefficient $m(z^2; \beta)$ either converges to a single point or a circle of points (see, for example, Chapter 9.2 of [10]). Hence, we

have the limit-point/limit-circle dichotomy for the half-line problem. In the limit-point case, there is only one independent $L^2(\mathbb{R})$ solution to $L_q u = z^2 u$ for $z \in \mathbb{C}^+$; in the limit-circle case any solution is $L^2(\mathbb{R})$. This is a rather difficult distinction to work with in practice, but we note that a sufficient condition for the equation to be in the limit-point case is that

$$q(x) > -kx^2$$

a.e. for some constant $k > 0$ [38]. We restrict our attention to the limit-point case and refer to [10] for further discussion of the limit-circle case. Taking the limit as $\beta \to +\infty$ in (5.6), we obtain the celebrated *Weyl-Titchmarsh m-function*, which we denote by $m(z^2)$. To understand how the m-function relates to the spectral decomposition of L_q, we must first state the following theorem, due originally to Weyl and which may be found, for example, in [10, 30].

Theorem 5.2.1. *There exists a monotone increasing function ρ so that the transform*

$$\mathcal{F}_\rho(\lambda) = \int_\alpha^{+\infty} f(x)\xi(x,\lambda)\, dx$$

defines a unitary mapping between $L^2[\alpha, +\infty)$ and $L^2(\mathbb{R}, d\rho)$.

The function ρ of this theorem is called the *spectral function* for the operator L_q. For $\sin\eta \neq 0$, it is related to the m-function by the transformation

$$m(z) = -\cot\eta + \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{z-\lambda}$$

(see [30, 31] where the case $\sin\eta = 0$ is also treated). Observe also that the m-function is meromorphic in the spectral parameter z^2 on the slit plane $\mathbb{C}/[0,\infty)$, with simple poles on $[0, +\infty)$ coinciding with any eigenvalues of L_q [31]. This makes the m-function a natural and convenient object to consider in the study of the spectral theory of the self-adjoint operator L_q.

For any $z \in \mathbb{C}/\mathbb{R}$, we may solve (5.3) with asymptotic conditions

$$\begin{aligned} \lim_{x\to+\infty} \left|u(x,z) - e^{ixz}\right| &= 0 \\ \lim_{x\to+\infty} \left|u'(x,z) - ize^{ixz}\right| &= 0. \end{aligned} \tag{5.7}$$

We call such a solution a *Jost solution* to the Schrödinger equation. Because we are in the limit point case, up to multiplicative constants there is only one L^2 solution to (5.3), and hence

$$\begin{aligned} cu(x,z) &= \theta(x,z) + m(z^2)\xi(x,z) \\ cu'(x,z) &= \theta'(x,z) + m(z^2)\xi(x,z). \end{aligned}$$

Using a spectral argument, one can show that the Jost solution $u(x,z)$ to (5.3) is nonvanishing (see, for example, [19]). Hence the quotient

$$\frac{u'(\alpha,z)}{u(\alpha,z)} = \frac{-\cos\eta + m(z^2)\ \sin\eta}{\sin\eta + m(z^2)\ \cos\eta}$$

is well-defined for each z. Note that if $\eta = \pi/2$, then the m-function is simply:

$$m(z^2) = \left.\frac{u'(x,z)}{u(x,z)}\right|_{x=\alpha}, \tag{5.8}$$

and more generally the m-function is a fractional linear transformation of this derivative. Without loss of generality, we may therefore take the definition of the m-function to be (5.8). This gives a simple way to resolves the direct problem; given q, we can find the Jost solution and evaluate (5.8).

There are several distinct approaches to the inverse spectral problem for L_q. The classical approach detailed in Chapter 2 of [30] is to formulate the inverse problem as a Gelfand-Levitan-Marchenko integral equation. More recent approaches are Remling's de Branges space technique [39] and Simon's A-function method [42]. Remling [39, 40] applies de Branges space characterization results from [11] to establish a bijection between Schrödinger operators and families of de Branges spaces. His approach has been extended to consider the case of measure potentials in [4].

For q supported $[0, +\infty)$, Simon [42] proves that the m-function for $k \in \mathbb{R}$ has the form

$$m(-k^2) = -k - \int_0^\infty A(z) e^{-2kz}\, dz \tag{5.9}$$

for $A \in L^1[0, +\infty)$. He then proves that the A-function in this expansion serves as the initial data $A(z, 0) = A(z)$ for the uniquely solvable PDE

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial z} + \int_0^z A(\zeta, x) A(z - \zeta, x)\, d\zeta, \tag{5.10}$$

where $A(0, x) = q(x)$. To solve the inverse problem by this method, one first inverts the Laplace transform in (5.9) and then evolves $A(z)$ to $q(x)$ through the PDE (5.10). The A-function determines the potential q locally in the sense that $q(x)$ may be recovered from the values of $A(\xi)$ on $[0, x]$ via this framework. More precisely, if A_1, A_2 are A-functions for q_1, q_2, and $A_1(\xi) = A_2(\xi)$ for a.e. $\xi \in [0, \beta]$ then $q_1(x) = q_2(x)$ for a.e. $x \in [0, \beta]$.

We recall that $f(x) = \widetilde{o}\,(g(x))$ when $g(x) \to 0$ as $|x| \to +\infty$ and

$$\forall \epsilon > 0, \qquad \lim_{x \to +\infty} \frac{|f(x)|}{|g(x)|^{1-\epsilon}} = 0. \tag{5.11}$$

Appendix 2 of [42] gives an elementary proof of the following Paley-Wiener type result for the Laplace transform.

Proposition 5.2.1. *If*

$$f(k) = \int_0^\infty e^{-2k\xi} C(\xi)\, d\xi = \tilde{o}\left(e^{-2k\beta}\right)$$

the $C(\xi)$ is supported on $[0, \beta]$.

Because A on $[0, x]$ determines $q(x)$, Simon applies this proposition to prove the following local Borg-Marchenko Theorem:

Theorem 5.2.2. *Suppose $q_1, q_2 \in L^1_{\mathrm{loc}}[0,+\infty)$ and let L_{q_1}, L_{q_2} have the m-functions m_1, m_2. If*

$$m_1(-k^2) - m_2(-k^2) = \tilde{o}\left(e^{-2k\beta}\right)$$

as $k \to +\infty$ then $q_1 = q_2$ on $[0,\beta]$.

As mentioned in Chapter 1, this theorem is an improvement of the results of Borg in [7] and Marchenko in [32], which show that

$$m_1(z) = m_2(z) \Rightarrow q_1 = q_2.$$

Simon's elegant technique has received considerable elaboration in [23, 37, 40, 49].

Let us make explicit how the results of Simon [42] translate to the half-line $[\alpha,+\infty)$.

Proposition 5.2.2. *Let $q, \tilde{q} \in L^1_{\mathrm{loc}}[\alpha,+\infty)$. Consider the m-functions m and $\tilde{m}$ defined for the self-adjoint operators L_q and $L_{\tilde{q}}$ by equation (5.8). Then:*

1. *There is $A \in L^1(\mathbb{R})$ so that*

$$m(-k^2) = -k - \int_0^\infty e^{-2k\xi} A(\xi)\, d\xi.$$

   For $x \geq \alpha$, the values of A on $[0, x-\alpha]$ determine q on $[\alpha, x]$.

2. *If*

$$m(-k^2) - \tilde{m}(-k^2) = \tilde{o}\left(e^{-2k(\beta-\alpha)}\right)$$

   then $q(x) = \tilde{q}(x)$ a.e. on $[\alpha,\beta]$.

Proof. Suppose $q(x)$ has support on $[\alpha,+\infty)$. Then if we set $q_\alpha(x) = q(x+\alpha)$, the operator L_{q_α} is a self-adjoint operator on $L^2[0,+\infty)$ whenever L_q is a self-adjoint operator on $L^2[\alpha,+\infty)$. Moreover, if $u(x,z)$ is the Jost solution to $L_q u = z^2 u$ then $u_\alpha(x,z) = u(x+\alpha,z)$ is the Jost solution to $L_{q_\alpha} u = z^2 u$. We have

$$\begin{aligned} m(-z^2;q) &= \left.\frac{u'(x,z)}{u(x,z)}\right|_{x=\alpha} \\ &= \left.\frac{u'_\alpha(x,z)}{u_\alpha(x,z)}\right|_{x=0} \\ &= m(-z^2;q_\alpha) \\ &= -k - \int_0^\infty A(\xi) e^{-2k\xi}\, d\xi. \end{aligned}$$

The function $A(\xi)$ on $[0,x]$ determines $q_\alpha(x)$ and hence $A(\xi)$ on $[0,x-\alpha]$ determines $q(x)$.

Next suppose that

$$m(-k^2) - \tilde{m}(-k^2) = \int_0^\infty \left(A(\xi) - \tilde{A}(\xi)\right) e^{-2k\xi}\, d\xi = \tilde{o}\left(e^{-2k(\beta-\alpha)}\right)$$

as $k \to +\infty$. By Proposition 5.2.1, it follows that $A = \tilde{A}$ a.e. on $[0,\beta-\alpha]$. Because A and $\tilde{A}$ on $[0,x-\alpha]$ determine $q(x)$ and $\tilde{q}(x)$ for almost every $x < \beta$, it follows that $q(x) = \tilde{q}(x)$ a.e. on $[\alpha,\beta]$. □

The Direct Problem For the Singular Potential Equation

Suppose that $q \in \mathcal{M}_\alpha$ so that $q = w' + w^2$ for some $w \in \mathcal{K}_\alpha$. We will now introduce an analog of the classical Weyl-Titchmarsh m-function for the Schrödinger operator L_q and show how this quantity is related to the reflection coefficient $r_-(z)$ for the w-potential ZS-AKNS equation. The basic idea of the approach is to *define* the m-function for the Miura potential equation by (5.8). The problem with this definition is that, although L^2 solutions to $L_q u = z^2 u$ exist, pointwise evaluation of (5.8) no longer makes sense. To remedy this, we first define the quasiderivative of $u \in H^1[\alpha, +\infty)$ by

$$u^{[1]} = u' - wu. \tag{5.12}$$

This type of regularization is a standard tool in the spectral theory of singular potential Sturm-Liouville operators, see [43].

The Schrödinger equation

$$-u'' + (w' + w^2)u = z^2 u \tag{5.13}$$

is equivalent by a direct computation to the first-order system

$$\frac{d}{dx}\begin{bmatrix} u \\ u^{[1]} \end{bmatrix} = \begin{bmatrix} w & 1 \\ -z^2 & -w \end{bmatrix}\begin{bmatrix} u \\ u^{[1]} \end{bmatrix}; \tag{5.14}$$

see [26] where this computation is carried out in detail. We can readily prove that this first-order system with the prescribed initial condition has its solution in the absolutely continuous functions so that u and $u^{[1]}$ have meaningful pointwise definition.

Lemma 5.2.1. *Any Jost solution to the Schrödinger equation $L_q u = z^2 u$ is nonvanishing for $z \in \mathbb{C}^+$.*

Proof. The operator L_q admits the factorization $L_q = D^* D$ where

$$D = \left(\frac{d}{dx} - w\right).$$

Suppose that $u \in L^2[0, +\infty)$ satisfies $L_q u = z^2 u$ and the Dirichlet condition $u(0, z) = 0$. Then on the one hand

$$(u, L_q u) = z^2 \|u\|_{L^2(\mathbb{R})}.$$

On the other,

$$(u, L_q u) = (u, D^* D u) = (Du, Du) = \|Du\|^2_{L^2(\mathbb{R})}.$$

This can only happen if $z^2 \in [0, +\infty)$, i.e. if $z \in \mathbb{R}$. □

Remark 5.2.1. *One can extend the argument of [19] to show that $L_q u = z^2 u$ has no L^2 eigenvalues for $z \in [0, +\infty)$ as well, but we will not require this fact.*

In analogy to (5.8), we may use the Lemma to define the *modified m-function* for the singular problem by

$$m(z^2) = \left.\frac{u^{[1]}(x, z)}{u(x, z)}\right|_{x=\alpha}, \tag{5.15}$$

where u satisfies $L_q u = z^2 u$ and the Jost conditions (5.7).

Remark 5.2.2.

1. *We stress that the definition of the modified m-function has been made only based on the analogy with (5.8). The operator L_q with $q \in \mathcal{M}_\alpha$ is meant as an operator on $L^2(\mathbb{R})$ with potential supported on $[\alpha, +\infty)$, where as the classical m-function is defined with regards to a self-adjoint operator on $L^2[\alpha, +\infty)$.*

2. *By direct computation, it follows that, if w is also continuous, then both the classical and modified m-function are defined and*

$$m_{\text{classical}}(z^2) = m(z^2) - w(\alpha). \tag{5.16}$$

Given the correspondence between the Jost solution of the w-potential ZS-AKNS equation and the $q = w' + w^2$ Schrödinger equation described in the introduction, we easily obtain the following proposition.

Proposition 5.2.3. *The modified m-function for the $q = w' + w^2$ potential Schrödinger equation (5.3) is given by*

$$m(z^2) = iz \frac{\psi_{11}^+(x,z) - \psi_{21}^+(x,z)}{\psi_{11}^+(x,z) + \psi_{21}^+(x,z)}\bigg|_{x=\alpha} \tag{5.17}$$

where

$$\Psi_+^{(1)}(x,z) = \begin{bmatrix} \psi_{11}^+(x,z) \\ \psi_{21}^+(x,z) \end{bmatrix}$$

is the first column of the solution to the w-potential ZS-AKNS equation (1.1).

Proof. From the ZS-AKNS equation (1.1),

$$\begin{aligned} \psi_{11}'^+ &= iz\psi_{11}^+ + w\psi_{21}^+ \\ \psi_{21}'^+ &= -iz\psi_{21}^+ + w\psi_{11}^+ \end{aligned}$$

we compute

$$\begin{aligned} u^{[1]}(x,z) &= (\psi_{11}'^+ + \psi_{21}'^+) - w(\psi_{11}^+(x,z) + \psi_{21}^+(x,z)) \\ &= (iz\psi_{11}^+(x,z) + w(x)\psi_{21}^+(x,z) - iz\psi_{21}^+(x,z) + w(x)\psi_{11}^+(x,z)) \\ &\qquad - w(\psi_{11}^+(x,z) + \psi_{21}^+(x,z)) \\ &= iz(\psi_{11}^+(x,z) - \psi_{21}^+(x,z)). \end{aligned}$$

□

Recall from Chapter 2 that for $z \in \mathbb{R}$

$$r_-(z) = \frac{b(z)}{a(z)}$$

where

$$\Psi_+(\alpha, z) = \Psi_-(\alpha, z)R(z)$$

and

$$R(z) = \begin{bmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{bmatrix}.$$

It follows that

$$a(z) = e^{i\alpha z}\psi_{11}^{+}(\alpha, z) \qquad b(z) = e^{-i\alpha z}\psi_{21}^{+}(\alpha, z),$$

and we note that $\psi_{11}(\alpha, z), \psi_{21}(\alpha, z)$ extend analytically to $\mathbb{C}^+$. We write

$$m(z^2) = iz\frac{1 - e^{-2i\alpha z}r_-(z)}{1 + e^{-2i\alpha z}r_-(z)}. \tag{5.18}$$

There are several ways to look at this formula. Given $q = w' + w^2$, we can determine the Jost solutions to the ZS-AKNS equation, construct the reflection coefficient $r_-(z)$, and then use (5.18) to determine m. This solves the direct spectral problem for L_q.

Recall also that

$$r_-(z) = \int_\alpha^\infty e^{2iz\xi}C(\xi)\, d\xi, \tag{5.19}$$

and hence

$$1 + e^{-2iz\alpha}r_-(z) = 1 + \int_0^\infty e^{2iz\xi}C(\xi + \alpha)\, d\xi.$$

From Lemma 5.2.1, this quantity is nonvanishing for $z \in \mathbb{C}^+$ and is therefore an invertible element of the Banach algebra $\mathcal{A}$. We use Theorem 2.3.3 to write

$$\left(1 + \int_0^\infty e^{2iz\xi}C(\xi + \alpha)\, d\xi\right)^{-1} = 1 + \int_0^\infty e^{2iz\xi}E(\xi)\, d\xi \tag{5.20}$$

where $E \in X$.

Then we compute

$$\begin{aligned} m(z^2) &= iz\left(1 + \int_0^\infty e^{2iz\xi}C(\xi + \alpha)\, d\xi\right)\left(1 + \int_0^\infty e^{2iz\xi}E(\xi)\, d\xi\right)^{-1} \\ &= iz + iz\int_0^\infty e^{2iz\xi}G(\xi)\, d\xi. \end{aligned} \tag{5.21}$$

with

$$G(\xi) = C(\xi + \alpha) + E(\xi) + \int_0^\xi C(\tau + \alpha)E(\xi - \tau)\, d\tau.$$

Taking $z = ik$ in this formula, we obtain the singular potential analog of (5.9):

$$m(-k^2) = -k - k\int_0^\infty G(\xi)e^{-2k\xi}\, d\xi.$$

In [42], the A-function and the potential have the same degree of singularity. The additional factor of k in our analogous expression arises because our potentials are more singular than Simon's. A formal integration by parts argument using (5.16) shows that when both the A-function and the G-function exist, then

$$G'(\xi) = 2A(\xi)$$

up to an additive constant. We also make note of the fact that G is supported on $[0, \infty)$, independent of α.

5.3 The Inverse Spectral Problem

Given $m(z^2)$ the modified m-function for L_q, we may solve (5.18) for r_- to obtain

$$r_-(z) = e^{2i\alpha z}\frac{iz - m(z^2)}{iz + m(z^2)}. \tag{5.22}$$

We can recover the potential using the reconstruction procedure described in Chapter 3, which is a straightforward way to solve the inverse problem for the modified m-function.

We now argue that the modified m-function determines the potential locally. We shall prove:

Theorem 5.3.1. *Suppose that $q, \tilde{q} \in \mathcal{M}_\alpha$. Let $m(-k^2), \tilde{m}(-k^2)$ be the modified m-functions for the operator L_q and $L_{\tilde{q}}$, respectively. If*

$$m(-k^2) - \tilde{m}(-k^2) = \tilde{o}(e^{-2k(\beta-\alpha)})$$

then $q = \tilde{q}$ a.e. on $[\alpha, \beta]$.

Proof. For $q = w^2 + w'$ and $\tilde{q} = \tilde{w}^2 + \tilde{w}'$, we let $r_-(z)$ and $\tilde{r}_-(z)$ denote the left reflection coefficients for the w and $\tilde{w}$ potential ZS-AKNS problem, respectively. Taking $z = ik$ with $k > 0$ in (5.22), we compute

$$\begin{aligned} r_-(ik) - \tilde{r}_-(ik) &= -2ke^{-2\alpha k}\frac{\tilde{m}(-k^2) - m(-k^2)}{(-k + m(-k^2))(-k + \tilde{m}(-k^2))} \\ &= -2ke^{-2\alpha k}\frac{\tilde{m}(-k^2) - m(-k^2)}{(-2k + m_0(-k^2))(-2k + \tilde{m}_0(-k^2))}, \end{aligned} \tag{5.23}$$

where

$$m_0(z^2) = m(z^2) - iz \qquad \tilde{m}_0(z^2) = \tilde{m}(z^2) - iz.$$

We make a change of variables in formula (5.19) to write $r_-(z) = e^{2i\alpha z}r_0(z)$ and $\tilde{r}_-(z) = e^{2i\alpha z}\tilde{r}_0(z)$, where

$$\begin{aligned} r_0(z) &= \int_0^\infty C(\xi + \alpha)e^{2iz\xi}\, d\xi \\ \tilde{r}_0(z) &= \int_0^\infty \tilde{C}(\xi + \alpha)e^{2iz\xi}\, d\xi. \end{aligned}$$

Then using (5.23), we have

$$r_0(ik) - \tilde{r}_0(ik) = \frac{\tilde{m}(-k^2) - m(-k^2)}{\tau(k)} \tag{5.24}$$

where

$$\tau(k) = \left(1 - \frac{m_0(-k^2)}{2k}\right)\left(1 - \frac{\tilde{m}_0(-k^2)}{2k}\right).$$

By (5.21):

$$m_0(-k^2) = -k \int_0^\infty G(\xi) e^{-2k\xi}\, d\xi,$$

and by the Cauchy-Schwarz Inequality

$$|m_0(-k^2)| \leq \sqrt{k}\, \frac{\|G\|_{L^2(\mathbb{R})}}{2}.$$

A similar estimate can be made for $\tilde{m}_0$, so we deduce that

$$\tau(k) = 1 + \mathcal{O}\left(\frac{1}{\sqrt{k}}\right).$$

Using definition (5.11), it readily follows that

$$\frac{f(k)}{h(k)} = \tilde{o}(g(k))$$

whenever $f(k) = \tilde{o}(g(k))$ and $h(k) = 1 + \mathcal{O}(1/\sqrt{k})$. Since by hypothesis

$$\tilde{m}(-k^2) - m(-k^2) = \tilde{o}\left(e^{-2k(\beta-\alpha)}\right),$$

equation (5.24) implies

$$r_0(ik) - \tilde{r}_0(ik) = \tilde{o}\left(e^{-2k(\beta-\alpha)}\right).$$

Then by Proposition 5.2.1,

$$C(\xi + \alpha) = \tilde{C}(\xi + \alpha)$$

for $\xi \in [0, \beta - \alpha]$. That is, $C(\xi) = \tilde{C}(\xi)$ for $\xi \in [\alpha, \beta]$. By Corollary 4.1.1, $w(x) = \tilde{w}(x)$ for a.e. $x \in [\alpha, \beta]$. Conclude that $q(x) = \tilde{q}(x)$ for a.e. $x \in [\alpha, \beta]$. □

Appendix A: Notation Index

Appendix B: Analyticity and the Fourier Transform

In this appendix, we collect some useful facts about the 1D Fourier transform and prove the "two-sided" version of the linear Paley-Wiener Theorem.

Recall that our convention on the direct and inverse Fourier transform of $f \in L^2(\mathbb{R})$ is

$$\mathcal{F}f(z) = \hat{f}(z) = \int_{-\infty}^{\infty} f(x)e^{2ixz}\,dx$$
$$\mathcal{F}^{-1}f(x) = \check{f}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)e^{-2ixz}\,dz.$$

The nonstandard factor of 2 in the exponential of our definition accounts for the nonstandard factor of 2 appearing in the statements to follow.

Preliminaries

Following [15], we shall say a function f analytic in the region Ω has *exponential type* K *in* Ω if there exist constants $C > 0$ and $K < +\infty$ so that

$$|f(z)| \leq Ce^{2K|z|}$$

holds for all $z \in \Omega$.

For functions satisfying an exponential type T condition, it is often possible to extend the Maximum Modulus Principle to certain unbounded domains Ω. A result of this kind is called a *Phragmén-Lindelöff Theorem*. We recall the following sectorial version of the Phragmén-Lindelöff Theorem.

Theorem 1 (The Phragmén-Lindelöff Theorem, [15])**.** *Let f be analytic on the sector*

$$\Omega = \{z \in \mathbb{C} : \alpha < \arg z < \beta\}$$

with $\beta - \alpha < \pi$ and continuous on the closure $\overline{\Omega}$. Suppose in addition that f has exponential type K on Ω and satisfies

$$|f(z)| \leq M$$

on the boundary $\partial\Omega$ of Ω. Then

$$|f(z)| \leq M$$

on $\overline{\Omega}$.

We refer to the proof in section 3.1.7 of [15].

The Linear Paley-Wiener Theorem

A standard statement of the Paley-Wiener theorem is along the lines of the one found, for example, in [41]:

Theorem 2 (Standard Paley-Wiener Theorem)**.** *A function $f \in L^2(\mathbb{R})$ has its support in $[-A, A]$ if and only if its Fourier transform $\hat{f}$ extends to an entire function with*

$$|\hat{f}(z)| \leq Ce^{2A|z|}.$$

For our purposes, we need a somewhat more specific version which allows for consideration of half-line supports and half-plane extensions.

Theorem 3 (Two-Sided Paley-Wiener Theorem for the Fourier Transform)**.** *Suppose that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then*

1. *supp $f \subseteq [\alpha, +\infty)$ if and only if $\hat{f}$ extends to an analytic function on $\mathbb{C}^+$ satisfying*
$$|\hat{f}(R + iT)| \leq Ce^{-2\alpha T} \qquad T \geq 0.$$
2. *supp $f \subseteq (-\infty, \beta]$ if and only if $\hat{f}$ extends to an analytic function on $\mathbb{C}^-$ satisfying*
$$|\hat{f}(R + iT)| \leq Ce^{-2\beta T} \qquad T \leq 0.$$

Proof. We consider only the first case, since the second follows from the first by the symmetry

$$\hat{f}(-z) = \overline{\hat{f}(z)}.$$

Suppose f is supported on $[\alpha, +\infty)$. Then we estimate

$$\begin{aligned} |\hat{f}(R + iT)| &= \left| \int_\alpha^\infty f(x) e^{2ixR} e^{-2xT} \, dx \right| \\ &\leq e^{-2\alpha T} \|f\|_1 \end{aligned}$$

so that $\hat{f}(z)$ for any $z \in \mathbb{C}^+$ is well-defined. Note that such a bound cannot be made for $T < 0$, and in general the integral may fail to converge in $\mathbb{C}^-$.

To prove that $\hat{f}$ is analytic on $\mathbb{C}^+$, we apply Morera's theorem. For this, let $\gamma \subset \mathbb{C}^+$ be any simple closed contour. Estimate that

$$\begin{aligned} \int_\gamma \int_\alpha^{+\infty} \left| f(x) e^{2ixz} \right| \, dz &\leq \int_\gamma e^{-2\alpha \Im z} \|f\|_1 \, dz \\ &\leq \text{length}(\gamma) \left(\max_{z \in \gamma} e^{-2\alpha \Im z} \right) \|f\|_1 < +\infty. \end{aligned}$$

Fubini's theorem applies and we may compute

$$\int_\gamma \hat{f}(z) \, dz = 0.$$

Since γ is arbitrary, it follows that $\hat{f}$ is analytic in $\mathbb{C}^+$. This completes the proof of the forward implication.

To prove the reverse implication, we adapt the clever proof of the standard Paley-Wiener Theorem that appears in Chapter 3.3 of [15]. Suppose that $\hat{f}$ extends to an analytic function on $\mathbb{C}^+$ satisfying

$$|\hat{f}(R+iT)| \leq Ce^{-2\alpha T} \qquad T \geq 0.$$

For $y \in \overline{\mathbb{C}^+}$, define

$$h(y) = \frac{1}{2}\int_{-1}^{1} \hat{f}(y+z)\, dz.$$

On the one hand, we readily compute

$$\begin{aligned} h(z) &= \frac{1}{2}\int_{-1}^{1} \hat{f}(z+y)\, dy \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-\infty}^{\infty} f(x)e^{2ix(z+y)}\, dx\, dy \\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)e^{2ixz}\int_{-1}^{1} e^{2ixy}\, dy\, dx \\ &= \int_{-\infty}^{\infty} f(x)\frac{\sin 2x}{2x}e^{2ixz}\, dx \\ &= \widehat{\left[f(x)\frac{\sin 2x}{2x}\right]}(z). \end{aligned}$$

On the other hand, we will verify below that

$$\check{h}(x) = 0 \tag{25}$$

for a.e. $x < \alpha$.

It follows that

$$\sin(2x)f(x) \equiv 0$$

holds for a.e $x < \alpha$, and hence that $f(x) = 0$ a.e. on $(-\infty, \alpha]$.

To prove (25), we first estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |h(R)|^2\, dR &\leq \frac{1}{4}\int_{-\infty}^{\infty}\left(\int_{-1}^{1} |\hat{f}(R+z)|\, dz\right)^2 dR \\ &\leq \frac{1}{2}\int_{-\infty}^{\infty}\int_{-1}^{1} |\hat{f}(R+z)|^2\, dz\, dR \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-\infty}^{\infty} |\hat{f}(R+z)|^2\, dR\, dz \\ &= \|\hat{f}\|_2^2. \end{aligned}$$

For $T \geq 0, R \in \mathbb{R}$, we also have the bound

$$\begin{aligned} |h(R+iT)|^2 &\leq \int_{-1}^{1} |\hat{f}(R+iT+y)|^2 \, dy \\ &\leq Ce^{-4\alpha T}. \end{aligned}$$

Now, fix $\xi < \alpha$. For $y = R + iT \in \overline{\mathbb{C}^+}$ define

$$g(y) = e^{-2i\xi y} h(y).$$

Then we have

$$\begin{aligned} |g(R+iT)| &\leq e^{2\xi T} |h(R+iT)| \\ &\leq \frac{Ce^{2\xi T}}{2} \int_{-1}^{1} \left| \hat{f}(R+iT+z) \right| dz \\ &\leq Ce^{2(\xi-\alpha)T} \end{aligned}$$

so that g is of exponential order $(\xi - \alpha)$ in $\mathbb{C}^+$. In particular, we have

$$\begin{aligned} |g(iT)| &\leq Ce^{2(\xi-\alpha)T} \leq C \\ |g(R)| &\leq C. \end{aligned}$$

Now g is analytic in the upper-half plane. Because $f \in L^1(\mathbb{R})$, $\hat{f}$ is continuous on $\mathbb{R}$ and hence g is continuous on $\overline{\mathbb{C}^+}$. We may then apply the method of Phrágmen and Lindelöf (Theorem 1) to extend the bound holding for g on the real and the positive imaginary axes to all of $\mathbb{C}^+$. We do this by applying the method in the first and second quadrants separately. We then have the bound $g(z) \leq C$, holding for all $z \in \overline{\mathbb{C}^+}$.

Since $h(z) = g(z)e^{2i\xi z}$, we have

$$\begin{aligned} |h\left(\rho e^{i\theta}\right)| &< C \left| \exp\left(2i\xi\rho e^{i\theta}\right) \right| \\ &\leq C \exp(-2\xi\rho \sin\theta), \end{aligned}$$

holding for $0 \leq \theta \leq \pi$ and $\rho \geq 0$.

Then for fixed $A > 0$ and $x < \xi < \alpha$, define the function

$$v_A(z) = \frac{h(z)}{1 - iAz}.$$

Let $C_R \subset \overline{\mathbb{C}^+}$ be the semicircular contour of points

$$[-R, R] \cup \left\{ Re^{i\theta} : R > 0, \theta \in (0, \pi) \right\},$$

oriented counterclockwise. Because $v_A(z)$ is analytic in $\mathbb{C}^+$, Cauchy's Integral Theorem implies

$$\int_{-R}^{R} e^{-2ixz} v_A(z) \, dz = -iR \int_0^{\pi} e^{i\theta} \exp\left(-2ixRe^{i\theta}\right) v_A(Re^{i\theta}) \, d\theta.$$

Then we estimate

$$\begin{aligned}\left|\int_{-R}^{R} e^{-2ixz} v_A(z)\, dz\right| &\leq R\int_0^{\pi} \frac{|\exp\left(-2ixRe^{i\theta}\right)||h\left(Re^{i\theta}\right)|}{|1-iARe^{i\theta}|}\, d\theta \\ &\leq CR\int_0^{\pi} \frac{\exp\left(2xR\sin\theta\right)\exp\left(-2\xi R\sin\theta\right)}{AR+1}\, d\theta \\ &\leq \frac{CR}{AR+1}\int_0^{\pi} \exp\left(-2R(\xi-x)\sin\theta\right)\, d\theta.\end{aligned}$$

Since for $\theta \in [0, \pi/2]$ we have

$$\sin\theta \geq \frac{2\theta}{\pi},$$

we obtain the bound

$$\begin{aligned}\left|\int_{-R}^{R} e^{-2ixz} v_A(z)\, dz\right| &\leq \frac{2CR}{AR+1}\int_0^{\frac{\pi}{2}} \exp\left(-4R(\xi-x)\theta/\pi\right)\, d\theta \\ &= \frac{CR\pi}{2R(AR+1)(\xi-x)}\left[1-\exp\left(-4R(\xi-x)\right)\right]\end{aligned}$$

Since $R(\xi - x) > 0$, the integral vanishes as $R \to +\infty$. But this is just to say that the inverse Fourier transform of $v_A(z)$ vanishes when $x < \xi$. Now observe that for $R \in \mathbb{R}$

$$|v_A(R)| \leq \left|\frac{1}{1-iAR}\right| |h(R)| < |h(R)|$$

and by dominated convergence $v_A(R) \to h(R)$ in $L^2(\mathbb{R})$ as $A \downarrow 0$. Therefore, $\check{h}(x) = 0$ holds for $x < \xi < \alpha$. Since $\xi < \alpha$ is arbitrary, it follows that $\check{h}(x)$ and hence $f(x)$ vanish a.e. on $(-\infty, \alpha]$. □

Appendix C: The Cauchy Operator on $L^2(\mathbb{R})$

Here we show that the Cauchy boundary operators on $L^2(\mathbb{R})$ act as cutoff-functions in Fourier space. For the treatment of the Cauchy operators on $L^2(\Sigma)$ for $\Sigma \subset \mathbb{C}$ a general contour, see [3].

Proposition 1. *For $f \in L^2(\mathbb{R})$,*

$$\begin{aligned} \mathcal{C}_+ f(z) &= \int_0^\infty \check{f}(x) e^{2ixz}\, dx \\ \mathcal{C}_- f(z) &= -\int_{-\infty}^0 \check{f}(x) e^{2ixz}\, dx. \end{aligned} \tag{26}$$

Define the contours

$$\begin{aligned} C_R^+ &= [-R, R] \cup \left\{ Re^{i\theta} : \theta \in (0, \pi) \right\} \\ C_R^- &= [-R, R] \cup \left\{ Re^{i\theta} : \theta \in (\pi, 2\pi) \right\} \\ \gamma_R^+ &= \left\{ Re^{i\theta} : \theta \in (0, \pi) \right\} \\ \gamma_R^- &= \left\{ Re^{i\theta} : \theta \in (\pi, 2\pi) \right\}, \end{aligned}$$

where C_R^-, γ_R^- are oriented clockwise and C_R^+, γ_R^+ are oriented counter-clockwise.

For the proof of Proposition 1, we need the following version of Jordan's Lemma

Lemma 1.

1. *($\mathbb{C}^+$ version) Suppose that $f(\zeta) = e^{-2ix\zeta} g(\zeta)$ with $x < 0$. Then*

$$\left| \int_{C_R^+} f(z)\, dz \right| \leq \frac{\pi}{2x} \max_{z \in C_R^+} |g(z)|.$$

2. *($\mathbb{C}^-$ version) Suppose that $f(\zeta) = e^{-2ix\zeta} g(\zeta)$ with $x > 0$. Then*

$$\left| \int_{C_R^-} f(z)\, dz \right| \leq \frac{\pi}{2x} \max_{z \in C_R^+} |g(z)|.$$

For a proof of this fact, consult an elementary text on complex variables such as [44].

Proof of Proposition 1. For each fixed z, set

$$g_z(\zeta) = \frac{1}{\zeta - z},$$

and write

$$\mathcal{C} f(z) = \frac{1}{2\pi i} \int_{-\infty}^\infty f(\zeta) g_z(\zeta)\, d\zeta = \frac{1}{2\pi i} \left(f, \bar{g}_z \right)_{L^2(\mathbb{R})}.$$

With the given Fourier convention, apply the Fourier convolution theorem to compute the Plancherel relation

$$\begin{aligned}(f,g)_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx \\ &= \pi\left[\frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\bar{g}(z)e^{-2i0z}\,dz\right] \\ &= \pi\mathcal{F}^{-1}\left[f\bar{g}\right](0) \\ &= \pi\int_{-\infty}^{\infty}\check{f}(x)\check{\bar{g}}(0-x)\,dx \\ &= \pi\int_{-\infty}^{\infty}\check{f}(x)\check{\bar{g}}(-x)\,dx\end{aligned}$$

Thus

$$\mathcal{C}f(z) = \frac{1}{2i}\int_{-\infty}^{\infty}\check{f}(x)\check{g_z}(-x)\,dx.$$

We must compute

$$\check{g_z}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\zeta - z}e^{-2ix\zeta}\,d\zeta$$

We consider four cases.

Case # 1. Suppose that $\Im z < 0$ and $x < 0$. By Cauchy's theorem

$$\int_{\gamma_R^+}\frac{1}{\zeta - z}e^{-2ix\zeta}\,d\zeta = 0$$

so that

$$\int_{-R}^{R}\frac{1}{\zeta - z}e^{-2ix\zeta}\,d\zeta = -\int_{C_R^+}\frac{1}{\zeta - z}e^{-2ix\zeta}\,d\zeta$$

For each fixed $x < 0$, apply the $\mathbb{C}^+$ version of Jordan's Lemma, and take $R \to +\infty$ to obtain

$$\int_{-\infty}^{\infty}\frac{1}{\zeta - z}e^{-2ix\zeta}\,d\zeta = 0.$$

Case # 2. Suppose that $\Im z < 0$ and $x > 0$. By the Residue Theorem

$$\int_{\gamma_R^-}\frac{1}{\zeta - z}e^{-2ix\zeta}\,d\zeta = 2\pi i\,\mathrm{Res}\left(\frac{1}{\zeta - z}e^{-2ix\zeta},\ \zeta = z\right).$$

Compute

$$\mathrm{Res}\left(\frac{1}{\zeta - z}e^{-2ix\zeta},\ \zeta = z\right) = \lim_{\zeta\to z}(\zeta - z)\cdot\left(\frac{1}{\zeta - z}e^{-2ix\zeta}\right) = e^{-2ixz}.$$

For each fixed $x > 0$, apply the $\mathbb{C}^-$ version of Jordan's Lemma, and take $R \to +\infty$ to obtain

$$\int_{-\infty}^{\infty}\frac{1}{\zeta - z}e^{-2ix\zeta}\,\zeta = -2\pi i e^{-2ixz}$$

The negative sign comes from the orientation of the contour.

Case # 3. Suppose that $\Im z > 0$ and $x > 0$. By Cauchy's Theorem

$$\int_{\gamma_R^-} \frac{1}{\zeta - z} e^{-2ix\zeta}\, d\zeta = 0.$$

Apply the $\mathbb{C}^-$ version of Jordan's Lemma to obtain

$$\int_{-\infty}^{\infty} \frac{1}{\zeta - z} e^{-2ix\zeta}\, d\zeta = 0.$$

Case # 4. Suppose that $\Im z > 0$ and $x < 0$. By the Residue Theorem

$$\int_{\gamma_R^+} \frac{1}{\zeta - z} e^{-2ix\zeta}\, d\zeta = 2\pi i e^{-2ixz}.$$

Apply the $\mathbb{C}^+$ version of Jordan's Lemma to obtain

$$\int_{-\infty}^{\infty} \frac{1}{\zeta - z} e^{-2ix\zeta}\, \zeta = 2\pi i e^{-2ixz}.$$

Combining the four cases, we have

$$\check{g_z}(x) = 2i \begin{cases} \begin{cases} 0 & x < 0 \\ -e^{-2ixz} & x > 0 \end{cases} & \Im z < 0 \\ \begin{cases} e^{-2ixz} & x < 0 \\ 0 & x > 0 \end{cases} & \Im z > 0 \end{cases}$$

With $z = R + iT$, we write

$$\check{g_z}(-x) = 2i \begin{cases} \begin{cases} 0 & x > 0 \\ -e^{2ix(R+iT)} & x < 0 \end{cases} & T < 0 \\ \begin{cases} e^{2ix(R+iT)} & x > 0 \\ 0 & x < 0 \end{cases} & T > 0 \end{cases}$$

Then for $T > 0$:

$$\mathcal{C}f(R + iT) = \int_0^{\infty} \check{f}(x) e^{2ix(R+iT)}\, dx.$$

and for $T < 0$:

$$\mathcal{C}f(R + iT) = -\int_{-\infty}^{0} \check{f}(x) e^{2ix(R+iT)}\, dx.$$

Taking $T \to 0$ and applying the Lebesgue Dominated Convergence Theorem, we conclude (26).

□

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