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On the Dimension of a Certain Measure Arising from a Quasilinear Elliptic Partial Differential Equation

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ON THE DIMENSION OF A CERTAIN MEASURE ARISING FROM A QUASILINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
Murat Akman
Lexington, Kentucky

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ABSTRACT OF DISSERTATION

ON THE DIMENSION OF A CERTAIN MEASURE ARISING FROM A QUASILINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

We study the Hausdorff dimension of a certain Borel measure associated to a positive weak solution of a quasilinear elliptic partial differential equation in a simply connected domain. We also assume that the solution vanishes on the boundary of the domain. Then it is shown that the Hausdorff dimension of this measure is less than one, equal to one, greater than one depending on the homogeneity of the certain function. This work generalizes the work of Makarov when the partial differential equation is the usual Laplaces equation and the work of Lewis and his coauthors when it is the p-Laplace's equation.

KEYWORDS: Hausdorff dimension, Harmonic measure, P-Harmonic measure, Hausdorff measure, Quasiregular Mapping

Author's signature: Murat Akman

Date: April 28, 2014

ON THE DIMENSION OF A CERTAIN MEASURE ARISING FROM A QUASILINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

By
Murat Akman

Director of Dissertation: John Lewis

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Date: April 28, 2014

To my family, my wife Sema, & my cat İrmik

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Chapter 1 Introduction

In this dissertation we study the Hausdorff dimension of a finite positive Borel measure associated to a positive weak solution of a certain quasi linear elliptic partial differential equations with certain boundary conditions. To explain the problem we shall give some definitions.

1.1 Definitions

Let Ω' denote a bounded region in the complex plane $\mathbb{C}$. Given p, $1 < p < \infty$, let $z = z_1 + \mathrm{i}z_2$ denote points in $\mathbb{C}$ and let $W^{1,p}(\Omega')$ denote equivalence classes of functions $h : \mathbb{C} \to \mathbb{R}$ with distributional gradient $\nabla h = h_{z_1} + \mathrm{i}h_{z_2}$ and Sobolev norm

$$\|h\|_{W^{1,p}(\Omega')} = \left(\int_{\Omega'} (|h|^p + |\nabla h|^p) \mathrm{d}\nu \right)^{\frac{1}{p}} < \infty \tag{1.1}$$

where $\mathrm{d}\nu$ denotes two dimensional Lebesgue measure. The space $W^{1,p}_{\text{loc}}(\Omega')$ is defined in the obvious manner; $h \in W^{1,p}_{\text{loc}}(\Omega')$ if and only if $h \in W^{1,p}(U)$ for every open $U \Subset \Omega'$, i.e compactly conatined in Ω'.

Let $C_0^\infty(\Omega')$ denote infinitely differentiable functions with compact support in Ω' and let $W_0^{1,p}(\Omega')$ denote the closure of $C_0^\infty(\Omega')$ in the norm of $W^{1,p}(\Omega')$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{C}$.

Fix p, $1 < p < \infty$ and let $f : \mathbb{C} \setminus \{0\} \to (0, \infty)$ be homogeneous of degree p on $\mathbb{C} \setminus \{0\}$. That is,

$$f(\eta) = |\eta|^p f(\frac{\eta}{|\eta|}) > 0 \text{ when } \eta \in \mathbb{C} \setminus \{0\}. \tag{1.2}$$

We also assume that ∇f is $\delta-$monotone on $\mathbb{C}$ for some $0 < \delta \leq 1$. By definition, this means that $f \in W^{1,1}(B(0, R))$ for each $R > 0$ and for almost every $\eta, \eta' \in \mathbb{C}$ (with respect to two dimensional Lebesgue measure)

$$\langle \nabla f(\eta) - \nabla f(\eta'), \eta - \eta' \rangle \geq \delta |\nabla f(\eta) - \nabla f(\eta')| |\eta - \eta'|. \tag{1.3}$$

Next we give an example.

Example 1.1.1.

$$f(\eta_1, \eta_2) = |\eta|^p (1 + \varepsilon \frac{\eta_1}{|\eta|}) = |\eta|^p + \varepsilon \eta_1 |\eta|^{p-1}$$

is homogeneous of degree p on $\mathbb{C} \setminus \{0\}$ and $\delta-$monotone on $\mathbb{C}$ for some $\delta > 0$ provided ε is small enough. Homogeneity of f and $f \in W^{1,1}(B(0, R))$ for $R > 0$ are easily checked. Also if $\eta \in \mathbb{C} \setminus \{0\}$ and $\xi \in \mathbb{C}$,

$$p \min(p-1, 1)|\eta|^{p-2}\,|\xi|^2 \leq \sum_{j,k=1}^{2} (|\eta|^p)_{\eta_j \eta_k} \xi_j \xi_k \leq p \max(p-1, 1)|\eta|^{p-2}|\xi|^2. \tag{1.4}$$

Now second derivatives of f are clearly continuous when $|\eta| = 1$ and homogeneous of degree $p-2$. Using this observation and (1.4) we deduce for $\varepsilon > 0$ small that there exists $M = M(p)$, $1 \leq M < \infty$, with

$$\tfrac{1}{M}|\eta|^{p-2}|\xi|^2 \leq \sum_{j,k=1}^{2} f_{\eta_j \eta_k}(\eta) \xi_j \xi_k \leq M|\eta|^{p-2}|\xi|^2. \tag{1.5}$$

(1.5) implies (1.3). In fact (1.3) for a degree p homogeneous f locally in $W^{1,1}$ and (1.5) are equivalent as we show in Chapter 2.

Next, given $h \in W^{1,p}(\Omega')$ let $\mathfrak{A} = \{h + \phi : \phi \in W_0^{1,p}(\Omega')\}$. It is well known from [17, Chapter 5] that there is $u' \in \mathfrak{A}$ satisfying

$$\inf_{w \in \mathfrak{A}} \int_{\Omega'} f(\nabla w) \mathrm{d}\nu = \int_{\Omega'} f(\nabla u') \mathrm{d}\nu \;\text{ for some }\; u' \in \mathfrak{A}. \tag{1.6}$$

Also u' is a weak solution at $z \in \Omega'$ to the Euler-Lagrange equation,

$$\begin{aligned} 0 = \nabla \cdot (\nabla f(\nabla u'(z))) &= \sum_{k=1}^{2} \frac{\partial}{\partial z_k} \left(\frac{\partial f}{\partial \eta_k}(\nabla u'(z)) \right) \\ &= \sum_{k,j=1}^{2} f_{\eta_k \eta_j}(\nabla u'(z))\, u'_{z_k z_j}(z) \end{aligned} \tag{1.7}$$

That is, $u' \in W^{1,p}(\Omega')$ and

$$\int_{\Omega'} \langle \nabla f(\nabla u'(z)), \nabla \phi(z) \rangle \mathrm{d}\nu = 0 \;\text{ whenever }\; \phi \in W_0^{1,p}(\Omega'). \tag{1.8}$$

Next, suppose $\Omega \subset \mathbb{C}$ is a bounded simply connected domain, N is a neighborhood of $\partial\Omega$, and $u > 0$ is a weak solution to the Euler Lagrange equation in (1.7) with $\Omega' = \Omega \cap N$, $u' = u$. Also assume that $u = 0$ on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ sense. More specifically, let $u \equiv 0$ on $N \setminus \Omega$. Then $u\zeta \in W_0^{1,p}(\Omega)$ whenever $\zeta \in C_0^\infty(\Omega)$. Under this scenario it follows from [17, Chapter 21] that there exists a unique finite positive Borel measure μ with support on $\partial\Omega$ satisfying

$$\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \phi \rangle \mathrm{d}\nu = -\int_{\partial\Omega} \phi \mathrm{d}\mu \tag{1.9}$$

whenever $\phi \in C_0^\infty(N)$ and $\phi \geq 0$.

In case u has a continuous extension to $\Omega \cap N$, to prove the existence of μ, it is enough to show the map,

$$\phi \to -\int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \phi \rangle \mathrm{d}\nu \geq 0. \tag{1.10}$$

for every nonnegative admissible function ϕ. Then it follows from the Riesz representation theorem and basic Caccioppoli inequalities that the integral can be represented by a measure.

A proof of (1.10) is based on choosing the right admissible function. To this end, let $\phi \in C_0^\infty(B(z,r))$, $\phi = 1$ on $B(z, r/2)$ with the support of $\phi \subset B(z,r)$ for some $z \in \partial\Omega$. Define

$$\psi = ((\varepsilon + \max(u - \varepsilon, 0))^\delta - \varepsilon^\delta)\phi \text{ for small } \ \delta, \varepsilon \geq 0.$$

Then the support of $\psi \subset \partial\Omega$ and ψ is an admissible function so we obtain

$$\begin{aligned}
0 &= \int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \psi \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla(((\varepsilon + \max(u - \varepsilon, 0))^\delta - \varepsilon^\delta)\phi) \rangle \mathrm{d}\nu \\
&= \int_{\mathbb{C}} ((\varepsilon + \max(u - \varepsilon, 0))^\delta - \varepsilon^\delta) \langle \nabla f(\nabla u(z)), \nabla \phi \rangle \mathrm{d}\nu \\
&+ \int_{\{z:\ u(z) \geq \varepsilon\}} \delta\phi(\varepsilon + \max(u - \varepsilon))^{\delta-1} \langle \nabla f(\nabla u(z)), \nabla u \rangle \mathrm{d}\nu \\
&= I + II
\end{aligned}$$

By homogeneity of f we have

$$\begin{aligned}
II &= \int_{\{z:\ u(z) \geq \varepsilon\}} \delta\phi(\varepsilon + \max(u - \varepsilon))^{\delta-1} \langle \nabla f(\nabla u(z)), \nabla u \rangle \mathrm{d}\nu \\
&= p \int_{\{z:\ u(z) \geq \varepsilon\}} \delta\phi(\varepsilon + \max((u - \varepsilon), 0))^{\delta-1} f(\nabla u) \mathrm{d}\nu \\
&\geq 0
\end{aligned}$$

Hence $I \leq 0$. Moreover, we show in the Chapter 3 that $|\nabla f(\nabla u)| \leq c|\nabla u|^{p-1}$. Using this, we have

$$((\varepsilon + \max(u - \varepsilon, 0))^\delta - \varepsilon^\delta) \langle \nabla f(\nabla u(z)), \nabla \eta \rangle \leq c\|\nabla \eta\|_{L^\infty} |\nabla u|^{p-1}.$$

As $|\nabla u| \in L^{p-1}(\Omega \cap N)$ we can first send $\delta \to 0$ then $\varepsilon \to 0$ and after that use the dominated convergence theorem to interchange the order of limits and integration. We get

$$\phi \to - \int_{\mathbb{C}} \langle \nabla f(\nabla u(z)), \nabla \phi \rangle \mathrm{d}\nu \geq 0.$$

Remark 1.1.2. We remark from (1.9) that if $\partial\Omega$ and ∇u are smooth enough then

$$\mathrm{d}\mu = p \, \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1|_{\partial\Omega}.$$

We next introduce the notions of Hausdorff measure and Hausdorff dimension of μ associated with a positive weak solution u to (1.7) in $\Omega \cap N$.

Let $\lambda > 0$ be defined on $(0, r_0)$ with $\lim\limits_{r \to 0} \lambda(r) = 0$ for some fixed r_0. We define the H^λ measure of a set $E \subset \mathbb{C}$ as follows;

For fixed $0 < \delta < r_0$, let $\{B(z_i, r_i)\}$ be a cover of E with $0 < r_i < \delta$, $i = 1, 2, \ldots$, and set

$$\phi_\delta^\lambda(E) = \inf \sum_i \lambda(r_i).$$

where the infimum is taken over all possible covers of E.

Then the Hausdorff measure of E, denoted by $H^\lambda(E)$, is

$$H^\lambda(E) = \lim_{\delta \to 0} \phi_\delta^\lambda(E).$$

When $\lambda(r) = r^\alpha$ we write H^α for H^λ. Next we define the Hausdorff dimension of the measure μ obtained in (1.9) as

$$\text{H-dim}\, \mu = \inf\{\alpha : \ \exists \, \text{Borel set } \ E \subset \partial\Omega \ \text{ with } \ H^\alpha(E) = 0 \ \text{ and } \ \mu(E \setminus \partial\Omega) = 0\}.$$

1.2 History on the study of Hausdorff dimension measures

When $f(\eta) = |\eta|^2$ then it turns out that the pde in (1.7) becomes the usual Laplace equation. In this case, if u is the Greens function for Laplaces equation with pole at some $z_0 \in \Omega$, then the measure associated to this function u as in (1.9) is harmonic measure relative to z_0 and will be denoted by ω.

The Hausdorff dimension of ω has been extensively studied in the last 30 years in planar domains. In particular, Carleson showed in [7] that

Theorem 1.2.1. *H-dim $\omega = 1$ when $\partial\Omega$ is a snowflake and H-dim $\omega \leq 1$ when Ω is any self similar cantor set.*

In [26], Makarov proved that

Theorem 1.2.2. *Let Ω be simply connected and $\mu = \omega$ in (1.9) be harmonic measure with respect to a point in Ω, and let*

$$\lambda(r) = r \, \exp\{A \sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}}\}, \quad 0 < r < 10^{-6}$$

then

a) There exists an absolute constant $A > 0$ such that harmonic measure ω is absolutely continuous with respect to the H^λ measure.

b) ω is concentrated on a set of $\sigma-$finite H^1 measure.

In [18], Jones and Wolff proved that

Theorem 1.2.3. *H-dim $\omega \le 1$ for an arbitrary domain Ω in the plane when ω exists.*

Later Wolff in [29] proved that

Theorem 1.2.4. *Harmonic measure ω is concentrated on a set of $\sigma-$finite H^1 measure.*

Batakis in [4], Kaufmann and Wu in [19], and Volberg in [28] independently showed that

Theorem 1.2.5. *For certain fractal domains and domains whose complements are Cantor sets,*

$$\text{Hausdorff dimension of } \ \partial\Omega = \inf\{\alpha : \ H^\alpha(\partial\Omega) = 0\} > \textit{H-dim}\,\omega$$

In [5], Bennewitz and Lewis obtained the following result for μ defined as in (1.9) for fixed p, $1 < p < \infty$, relative to $f(\nabla u) = |\nabla u|^p$. In this case the corresponding pde in (1.7) becomes

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0, \tag{1.11}$$

which is called the $p-$Laplace equation. Moreover a positive weak solution of (1.11) is called a $p-$harmonic function.

Theorem 1.2.6. *Let $\Omega \subset \mathbb{C}$ be a domain bounded by a quasi circle and let N be a neighborhood of $\partial\Omega$. Fix $p \neq 2$, $1 < p < \infty$, and suppose u is p-harmonic in $\Omega \cap N$ with boundary value 0 in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. If μ is the measure corresponding to u as in (1.9) relative to $f(\nabla u) = |\nabla u|^p$, then H-dim $\mu \le 1$ for $2 < p < \infty$ while H-dim $\mu \ge 1$ for $1 < p < 2$. Moreover, if $\partial\Omega$ is the von Koch snowflake then strict inequality holds for H-dim μ.*

In [25], Lewis, Nyström, and Poggi-Corradini proved that

Theorem 1.2.7. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and N a neighborhood of $\partial\Omega$. Fix $p \neq 2, 1 < p < \infty$, and let u be p harmonic in $\Omega \cap N$ with boundary value 0 on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. Let μ be the measure corresponding to u as in (1.9), relative to $f(\nabla u) = |\nabla u|^p$ and put*

$$\lambda(r) = r \, \exp\left[A \sqrt{\log\frac{1}{r} \, \log\log\frac{1}{r}} \right] \ \text{ for } \ 0 < r < 10^{-6}.$$

a) If $p > 2$, there exists $A = A(p) \le -1$ such that μ is concentrated on a set of $\sigma-$finite H^λ measure.

b) If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^{λ} measure.

In the recent paper [24], Lewis proved that

Theorem 1.2.8. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and N be a neighborhood of $\partial\Omega$. Fix $p \neq 2$, $1 < p < \infty$, and let u be $p-$harmonic in $\Omega \cap N$ with boundary value 0 on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. Let μ be the measure corresponding to u as in (1.9), relative to $f(\nabla u) = |\nabla u|^p$ and put*

$$\tilde{\lambda}(r) = r \exp\left[A\sqrt{\log\frac{1}{r}\,\log\log\log\frac{1}{r}}\right] \quad \text{for } 0 < r < 10^{-6}.$$

a) If $p > 2$, then μ is concentrated on a set of $\sigma-$finite H^1 measure.

b) If $1 < p < 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to $H^{\tilde{\lambda}}$ measure. Moreover $A(p)$ is bounded on $(3/2, 2)$.

This theorem is the complete extension of Makarov's theorem to the p-harmonic setting.

In [17], it was shown that the measure associated to a positive weak solution u with 0 boundary values as in (1.9) exists for a large class of quasilinear elliptic PDE. In [5][Closing remarks 10], the authors pointed out this fact and asked for what PDE can one obtain dimension estimates on the associated measure.

In this dissertation we try to give an answer to this problem. More specifically, we show that

Main Theorem 1.2.9. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and N be a neighborhood of $\partial\Omega$. Fix p, $1 < p < \infty$, let f be homogeneous of degree p and let ∇f be δ monotone for some $0 < \delta \leq 1$. Let $u > 0$ be a weak solution to (1.7) in $\Omega \cap N$ with boundary value 0 on $\partial\Omega$ in the $W^{1,p}(\Omega \cap N)$ Sobolev sense. Let μ be the measure corresponding to u as in (1.9) and put*

$$\lambda(r) = r \exp\left[A\sqrt{\log\frac{1}{r}\,\log\log\frac{1}{r}}\right] \quad \text{for } 0 < r < 10^{-6}.$$

a) If $p \geq 2$, there exists $A = A(p) \leq -1$ such that μ is concentrated on a set of $\sigma-$finite H^{λ} measure.

b) If $1 < p \leq 2$, there exists $A = A(p) \geq 1$, such that μ is absolutely continuous with respect to H^{λ} measure.

Note that the Main Theorem 1.2.9 and the definition of H-dim μ imply the following corollary.

Corollary 1.2.10. *Given p, $1 < p < \infty$, let u, μ be as in the Main Theorem 1.2.9, and suppose $\Omega \subset \mathbb{C}$ is a simply connected domain. Then H-dim $\mu \leq 1$ for $2 \leq p < \infty$, while H-dim $\mu \geq 1$ for $1 < p \leq 2$.*

The pde (1.7) we consider is more complicated and has less regularity than the p-Laplacian. Thus one has to overcome numerous procedural difficulties not encountered in p-harmonic setting.

This thesis is organized as follows. In chapter 2 we first introduce some notation which we use throughout this thesis and state some well known theorems: Sobolev's theorem and standard theorems on quasiregularity, (regularity properties of a quasiregular function, Stoilow factorization theorem). Second we derive some regularity properties of f satisfying (1.2) and (1.3) suitable for use in elliptic regularity theory.

In chapter 3 we study a variational problem and indicate some properties of weak solutions to the corresponding Euler Lagrange equation: maximum principle, Harnack inequality, interior Hölder continuity of a solution, and Hölder continuity near the boundary of Ω. After that we study the behavior of u near $\partial\Omega$ and the relationship between u and μ as in (1.9). Using this relationship we obtain that H-dim μ is independent of the corresponding u. Moreover, we use elliptic and quasiregularity theory to derive more advanced regularity properties of u: quasiregulariy of u_z, Hölder continuity of ∇u, and ∇u locally in $W^{1,2}$, so u is almost everywhere a pointwise solution to (1.7). We also show for a certain u that $\nabla u \neq 0$ near $\partial\Omega$.

In chapter 4 we outline a proof in [25] which shows in our situation that for a certain u as in Theorem (1.2.9) ∇u satisfies a certain inequality. Using this inequality and results from chapters 2, 3, 4 we obtain first that u and ∇u are weak solutions to a certain pde and then that $\log f(\nabla u)$ is a weak sub, super or solution to this pde, depending on whether $p > 2, < 2$, or $= 2$.

In chapter 5 we give a proof for Theorem 1.2.9.

Chapter 2 K Quasiregularity and δ Monotonicity

2.1 Notation and Terminology

Throughout this thesis various positive constants are denoted by c and they may differ even on the same line. The dependence on parameters is expressed, for example, by $c = c(p, f)$. We use $\approx$ symbols for example $g \approx h$ to mean that there is constant c such that

$$\tfrac{1}{c}h \leq g \leq c\,h.$$

If $\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ is a 2 x 1 column matrix, let $\eta^{\mathrm{T}} = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}$ denote the transpose of η. Let $B(z, r)$ denote the disk in $\mathbb{R}^2$ or $\mathbb{C}$ with center z and radius r and let ν be two dimensional Lebesgue measure. We specifically denote the unit disk, $B(0, 1)$, by $\mathbb{D}$. Ω will always denote an open set and often Ω is a simply connected domain. That is Ω is an open connected domain whose complement is connected.

We are now ready to define quasiregular mappings.

Definition 2.1.1. A mapping $h : \Omega \to \mathbb{C}$ is called K**-quasiregular** in Ω if $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\left|\frac{\partial h}{\partial \bar{z}}\right| \leq k \left|\frac{\partial h}{\partial z}\right| \ \nu \text{ a.e. in } \Omega \tag{2.1}$$

where we use complex derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) \ \text{ and } \ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right),$$

and

$$k = \frac{K-1}{K+1}.$$

Next regarding $h = h_1 + \mathrm{i}h_2$ as a mapping from $\Omega \to \mathbb{R}^2$ we let $Dh(z)$ be the matrix whose jk entry is defined by $Dh(z) = \left(\frac{\partial h_j}{\partial z_k}\right)$, $1 \leq j, k \leq 2$, and Jacobian determinant $J_h(z) = \det Dh(z)$ at the points $z = z_1 + \mathrm{i}z_2$ in Ω where all partials of h_1, h_2 exist. Then a second equivalent definition of $K-$quasiregularity is to require that $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is sense preserving and for a.e $z = z_1 + \mathrm{i}z_2 \in \Omega$

$$\|Dh(z)\|^2 = \max_{|\xi|=1} |Dh(z)\xi|^2 \leq K\, J_h(z), \tag{2.2}$$

where ξ is regarded as a column matrix.

A third equivalent definition of $K-$quasiregularity is to require that $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$, is sense preserving and

$$\|Dh(z)\| \leq K \min_{|\xi|=1} |Dh(z)\xi| \ \nu \text{ a.e in } \Omega. \tag{2.3}$$

To see why the definitions are equivalent suppose $h(z) = Az + B\bar{z}$ with $|A| > |B|$. Then it is easily checked that

$$\frac{\partial h}{\partial z} = A, \ \ \frac{\partial h}{\partial \bar{z}} = B, \ \text{and} \ \min_{|\xi|=1} |Dh(0)\xi| = |A| - |B|.$$

Also

$$J_h(0) = |A|^2 - |B|^2 \text{ and } \|Dh(0)\| = |A| + |B|.$$

Thus

$$\|Dh(0)\|^2 = (|A| + |B|)^2 = \frac{|A| + |B|}{|A| - |B|} J_h(0) = K\, J_h(0)$$

and

$$\|Dh(0)\| = K \min_{|\xi|=1} |Dh(0)\xi| \text{ while } \left|\frac{\partial h}{\partial \bar{z}}\right| = k \left|\frac{\partial h}{\partial z}\right|.$$

h is said to be quasiregular if h is $K-$quasiregular for some K. We now give the definition of a quasiconformal mapping.

Definition 2.1.2. A mapping $\phi : \Omega \to \mathbb{R}^2$ is called a **quasiconformal** mapping if ϕ is a quasiregular homeomorphism onto $\phi(\Omega)$.

In the next theorem, we see that in the plane quasiregular mappings are not more general than analytic functions.

Theorem 2.1.3 (Stoïlov's Decomposition Theorem)**.** *Suppose that $h : \Omega \to \mathbb{R}^2$ is a $K-$quasiregular mapping. Then $h = g \circ \phi$ where ϕ is a $K-$quasiconformal mapping and g is an analytic function in $\phi(\Omega)$.*

A proof of this theorem is in [22, Chapter VI]. An immediate consequence of the Stoïlov's decomposition theorem and well-known fact that the zeros of an analytic function are isolated is the following corollary.

Corollary 2.1.4. *The zeros of a nonconstant quasiregular mapping are isolated.*

Next we recall the definition of $\delta-$monotonicity of a function.

Definition 2.1.5 (δ-Monotone)**.** A mapping $h : \tilde{\Omega} \to \mathbb{C}$ is called $\delta-$**monotone** if

$$\langle h(\tilde{\eta}) - h(\hat{\eta}), \tilde{\eta} - \hat{\eta}\rangle \geq \delta |h(\tilde{\eta}) - h(\hat{\eta})||\tilde{\eta} - \hat{\eta}| \tag{2.4}$$

holds for all $\tilde{\eta}, \hat{\eta} \in \tilde{\Omega}$ and for some $\delta \in (0, 1]$.

$\delta-$Monotonicity has been studied by Kovalev in [20] in Hilbert space. See also the book [3, Chapter 3.11] for more details about $\delta-$monotonicity.

Geometrically $\delta-$monotonicity of a function h means that the angle between $\tilde{\eta} - \hat{\eta}$ and $\nabla h(\tilde{\eta}) - \nabla h(\hat{\eta})$ is at most $\cos^{-1}(\delta) < \pi/2$. Following [20] we see that every non-constant δ-monotone function in the domain $\tilde{\Omega}$ is indeed $K-$quasiconformal where

$$K = \frac{1 + \sqrt{1 - \delta^2}}{1 - \sqrt{1 - \delta^2}}.$$

We first give a sketch of Kovalev's proof for $\delta-$monotonicity implies K-quasiconformality in $\tilde{\Omega}$ To this end, let h be δ-monotone in $\tilde{\Omega}$ and let $\tilde{\eta}, \hat{\eta}, \zeta$ be distinct points in $\tilde{\Omega}$ with

$$\langle \zeta - \tilde{\eta}, \hat{\eta} - \tilde{\eta} \rangle \geq \sigma |\zeta - \tilde{\eta}||\zeta - \hat{\eta}| \text{ and } \langle \zeta - \hat{\eta}, \tilde{\eta} - \hat{\eta} \rangle \geq \sigma |\zeta - \tilde{\eta}||\zeta - \hat{\eta}|.$$

Using $\delta-$monotonicity of h one can show that

$$|h(\zeta) - h(\tilde{\eta})| + |h(\zeta) - h(\hat{\eta})| \leq \frac{2}{\delta} |h(\tilde{\eta}) - h(\hat{\eta})|$$

where $\sigma = 1 - \delta^2/8$. Thus if $h(\tilde{\eta}) = h(\hat{\eta})$ then $h(\zeta) = h(\tilde{\eta})$ for all ζ in the above sectorial type region. Continuing this argument throughout $\tilde{\Omega}$ one gets that $h \equiv h(\tilde{\eta})$. We conclude that h is either a homeomorphism of $\tilde{\Omega}$ or identically constant. Using the above inequality Kovalev also shows that a non-constant $\delta-$monotone mapping h satisfies for some $1 \leq H = H(\delta) < \infty$

$$|h(\zeta) - h(\tilde{\eta})| \leq H |h(\zeta) - h(\hat{\eta})| \;\; \tilde{\eta},\, \hat{\eta},\, \zeta \in B(w, r),\, |\zeta - \tilde{\eta}| \leq |\zeta - \hat{\eta}|, \tag{2.5}$$

whenever $B(w, 2r) \subset \tilde{\Omega}$.

Then using injectivity, (2.5) and [15, Theorem 11.14] Kovalev obtains that h is $K-$quasiconformal.

2.2 Basic Regularity Results for f

Let f be as in (1.2), (1.3). Then ∇f has a representative in $L^1(\mathbb{C})$ (also denoted by ∇f) that is $\delta-$monotone on $\mathbb{C}$. From homogeneity of f and Kovalev's theorem in [20] we see that ∇f is in fact a $K-$quasiconformal mapping. In this section we give an alternate proof for $K-$quasiregularity of ∇f and in the process we obtain some elliptic type inequalities which will be useful in our study of regularity for weak solutions to (1.7).

Let $\theta(z)$ be the standard mollifier, i.e;

$$\theta(z) = \begin{cases} c \exp(\frac{1}{|z|^2 - 1}) & \text{if } |z| < 1 \\ 0 & \text{if } |z| \geq 1 \end{cases}$$

and the constant c is selected so that

$$\int_{\mathbb{C}} \theta(z) \mathrm{d}\nu = 1.$$

Moreover, we define

$$\theta_\varepsilon(z) := \tfrac{1}{\varepsilon^2} \theta(\tfrac{z}{\varepsilon}).$$

Then $\theta_\varepsilon(z) \in C^\infty(\mathbb{C})$,

$$\int_{\mathbb{C}} \theta_\varepsilon(z) \mathrm{d}\nu = 1 \text{ and support of } \theta_\varepsilon \subset \overline{B(0, \varepsilon)}.$$

We first define $f_\varepsilon = f * \theta_\varepsilon$ by

$$f_\varepsilon(z) = \int\limits_{\mathbb{C}} \theta_\varepsilon(z-w)f(w)\mathrm{d}w = \int\limits_{B(0,\varepsilon)} \theta_\varepsilon(w)f(z-w)\mathrm{d}w \tag{2.6}$$

for $z \in \mathbb{C}$. Next we show that ∇f_ε is also $\delta-$monotone.

Since ∇f is δ-monotone we see that

$$\begin{aligned}
\langle \nabla f_\varepsilon(\tilde{\eta}) - \nabla f_\varepsilon(\hat{\eta}), \tilde{\eta} - \hat{\eta}\rangle &= \langle \int \nabla\left(f(\tilde{\eta}-z) - f(\hat{\eta}-z)\right)\theta_\varepsilon(z)\mathrm{d}\nu, \tilde{\eta}-\hat{\eta}\rangle \\
&= \int \langle \nabla(f(\tilde{\eta}-z) - f(\hat{\eta}-z)), (\tilde{\eta}-z) - (\hat{\eta}-z)\rangle\theta_\varepsilon(z)\mathrm{d}\nu \\
&\geq \delta \int |\nabla(f(\tilde{\eta}-z) - f(\hat{\eta}-z))|\,|\tilde{\eta}-\hat{\eta}|\theta_\varepsilon(z)\mathrm{d}\nu \\
&\geq \delta | \int \nabla(f(\tilde{\eta}-z) - f(\hat{\eta}-z))\theta_\varepsilon(z)\mathrm{d}\nu||\tilde{\eta}-\hat{\eta}| \\
&= \delta|\nabla f_\varepsilon(\tilde{\eta}) - \nabla f_\varepsilon(\hat{\eta})|\,|\tilde{\eta}-\hat{\eta}|.
\end{aligned} \tag{2.7}$$

Using smoothness of f_ε and $\delta-$monotonicity of ∇f_ε we have

$$\langle \frac{\nabla f_\varepsilon(z+t\eta) - \nabla f_\varepsilon(z)}{t}, \eta\rangle \geq \delta|\frac{\nabla f_\varepsilon(z+t\eta) - \nabla f_\varepsilon(z)}{t}| \tag{2.8}$$

where $\eta \in \mathbb{R}^2$ with $|\eta| = 1$ and $t \in (0,\infty)$. If we let $t \to 0$ we obtain in the matrix notation that

$$\eta^{\mathrm{T}}\, D^2 f_\varepsilon(z)\,\eta \geq \delta\,|D^2 f_\varepsilon(z)\,\eta| \tag{2.9}$$

where $D^2 f_\varepsilon(z)$ is 2 x 2 matrix whose jk entry is $\frac{\partial^2 f_\varepsilon}{\partial z_j \partial z_k}(z)$ and η is column matrix.

Let $\eta^{\mathrm{T}} = \eta_1, \eta_2$ with $\eta_1 = \cos(\theta)$ and $\eta_2 = \sin(\theta)$, and let O be an orthonormal matrix such that $O^T\, D^2 f_\varepsilon(z)\, O$ is a diagonal matrix, with diagonals λ_1, λ_2. As O is orthonormal we see that

$$\langle O^T\, D^2 f_\varepsilon(z)\, O\eta, \eta\rangle \geq \delta|D^2 f_\varepsilon(z)\,\eta|. \tag{2.10}$$

Then we see that (2.10) is indeed equivalent to

$$\lambda_1\eta_1^2 + \lambda_2\eta_2^2 \geq \delta\left(\lambda_1^2\eta_1^2 + \lambda_2^2\eta_2^2\right)^{\frac{1}{2}}. \tag{2.11}$$

Suppose first that we have $\lambda_1 = 0$. Then (2.11) becomes $\lambda_2\eta_2^2 \geq \delta|\lambda_2||\eta_2|$. But for θ near zero we have $\theta \geq \delta$ which is contradiction unless $\lambda_2 = 0$ in which case $D^2 f_\varepsilon(z) \equiv 0$. Otherwise suppose $0 < \lambda_1 \leq \lambda_2$. Dividing both sides of (2.10) by λ_1 we may assume that $\lambda_1 = 1$, $\lambda_2 = \lambda \geq 1$, and $D^2 f_\varepsilon(z)$ is a diagonal matrix with entries 1 and λ on the diagonal. Then

$$D^2 f_\varepsilon(z)\eta = \begin{pmatrix} \eta_1 \\ \lambda\eta_2 \end{pmatrix} = \eta_1 + \mathrm{i}\lambda\eta_2 = \tfrac{1}{2}(1+\lambda)e^{\mathrm{i}\theta} + \tfrac{1}{2}(1-\lambda)e^{-\mathrm{i}\theta}$$

in complex notation. Moreover

$$\begin{aligned} \eta^{\mathrm{T}}\, D^2 f_\varepsilon(z)\, \eta &= \mathrm{Re}\, \tfrac{1}{2}(1+\lambda) + \tfrac{1}{2}(1-\lambda)e^{-2\mathrm{i}\theta}, \\ |D^2 f_\varepsilon(z)\, \eta| &= |\tfrac{1}{2}(1+\lambda) + \tfrac{1}{2}(1-\lambda)e^{-2\mathrm{i}\theta}|. \end{aligned} \tag{2.12}$$

If $\lambda > 1$ we put

$$A = \frac{\lambda+1}{\lambda-1}, \quad \zeta = A - e^{-2\mathrm{i}\theta}$$

and use (2.12) to rewrite (2.10) after division by $\lambda - 1$ as

$$\mathrm{Re}\, \zeta \geq \delta |\zeta|. \tag{2.13}$$

Letting θ vary from 0 to π we see that ζ describes a circle, say C of radius 1 with center at A. From high school geometry it is easily seen that $\mathrm{Re}\, \zeta/|\zeta|$, $\zeta \in C$ has a minimum at a point $\tilde{\zeta}$ on C with the property that the line from the origin to $\tilde{\zeta}$ is tangent to C.

Using this fact it follows that equality holds in (2.13) when $\tilde{\zeta} = (\delta + \mathrm{i}\sqrt{1-\delta^2})(A^2 - 1)^{1/2}$. Since $\tilde{\zeta} - A = -\sqrt{1-\delta^2} + \mathrm{i}\delta$ we conclude that

$$\frac{\delta}{\sqrt{1-\delta^2}} = \sqrt{A^2-1} \text{ or } A = \frac{1}{\sqrt{1-\delta^2}}$$

Solving for λ it follows that

$$\lambda \leq \frac{1+\sqrt{1-\delta^2}}{1-\sqrt{1-\delta^2}} = K. \tag{2.14}$$

We conclude for K as in (2.14) that ∇f_ε is $K-$quasiregular mapping in $\mathbb{C}$.

From $K-$quasiregularity of ∇f_ε and $f \in W^{1,1}B(0,R)$ we see that $\nabla f_\varepsilon \in W^{1,2}_{\mathrm{loc}}(B(0,R))$ for each $R > 0$ with norm bounds that are independent of ε. Using these facts, $f_\varepsilon \to f$ in $W^{1,1}B(0,R)$, and taking a weak limit of second derivatives of f_ε we see that $\nabla f \in W^{1,2}B(0,R)$ for each $R > 0$. From properties of mollifiers it follows first that

$$\frac{\partial^2 f_\varepsilon}{\partial z_j \partial z_k} = \theta_\varepsilon * \frac{\partial^2 f}{\partial z_j \partial z_k} \text{ for } 1 \leq j,k \leq 2$$

and thereupon from the Lebesgue differentiation theorem that second derivatives of f_ε converge pointwise to second derivatives of f. Using this fact and (2.9) we get for a.e $z \in \mathbb{C}$,

$$f_{\eta\eta} = \langle D^2 f(z)\eta, \eta\rangle = \delta |D^2 f(z)\eta|. \tag{2.15}$$

We can now use $\nabla f \in W^{1,2}B(0,R)$ for each $R > 0$ and repeat the argument after (2.9) in order to conclude that ∇f is $K-$quasiregular where K is as (2.14). So the eigenvalues of the Hessian matrix of ∇f either both exist and are zero or have ratios bounded above by K and below by $1/K$.

As f is homogeneous of degree p, i.e $f(\eta) = |\eta|^p f(\eta/|\eta|)$, if we introduce polar coordinates; $r = |\eta|$, $\tan(\theta) = \eta_2/\eta_1$, then

$$f(r,\theta) = r^p f(\cos(\theta), \sin(\theta)).$$

Hence first and second derivatives of f along rays through the origin are

$$f_r = pr^{p-1} f(\cos(\theta), \sin(\theta)) \text{ and } f_{rr} = p(p-1)r^{p-2} f(\cos(\theta), \sin(\theta)). \tag{2.16}$$

From $K-$quasiregularity of ∇f we see that f is continuous in $\mathbb{C}$. Since $f > 0$ it follows that $f(\cos(\theta), \sin(\theta))$ is bounded above and below by constants $1 \leq M$ and $1/M$ respectively. We conclude from this fact and (2.16) that

$$\tfrac{1}{M} p(p-1)r^{p-2} \leq f_{rr} \leq M\, p(p-1)r^{p-2}. \tag{2.17}$$

From $K-$quasiregularity of ∇f we find for a.e $\eta \in \mathbb{C}$ and all ξ with $|\xi| = 1$ that

$$\tfrac{1}{MK} p(p-1)|\eta|^{p-2} \leq f_{\xi\xi}(\eta) = \xi^{\mathrm{T}}\, D^2 f\, \xi \leq MK\, p(p-1)|\eta|^{p-2}. \tag{2.18}$$

Using homogeneity of f and (2.18) we also have for some $M' \geq 1$ that

$$\tfrac{1}{M'}|\eta|^p \leq \min\{f(\eta), |\eta||\nabla f(\eta)|\} \leq \max\{f(\eta), |\eta||\nabla f(\eta)|\} \leq M'|\eta|^p. \tag{2.19}$$

We next prove various inequalities that we need later. We first see that

$$\begin{aligned}
\langle \nabla f(\eta) - \nabla f(\eta'), \eta - \eta' \rangle &= \int_0^1 \langle \frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla f(\eta' + t(\eta - \eta'))\right), (\eta - \eta')\rangle \mathrm{d}t \\
&= \int_0^1 \langle D^2 f(\eta' + t(\eta - \eta'))(\eta - \eta'), (\eta - \eta')\rangle \mathrm{d}t \qquad (2.20) \\
&= \int_0^1 \langle (\eta - \eta')^T D^2 f(\eta' + t(\eta - \eta'))(\eta - \eta')\rangle \mathrm{d}t
\end{aligned}$$

Then using (2.18) we get for a.e $\eta, \eta' \in \mathbb{R}^2$,

$$\begin{aligned}
\tfrac{1}{MK}|\eta - \eta'|^2 \int_0^1 |\eta' + t(\eta - \eta')|^{p-2} \mathrm{d}t &\leq \int_0^1 \langle D^2 f(\eta' + t(\eta - \eta'))(\eta - \eta'), (\eta - \eta')\rangle \mathrm{d}t \\
&\leq MK|\eta - \eta'|^2 \int_0^1 |\eta' + t(\eta - \eta')|^{p-2} \mathrm{d}t.
\end{aligned} \tag{2.21}$$

It follows from [6] that

$$\tfrac{1}{c}(|\eta| + |\eta'|)^{p-2} \leq \int_0^1 |\eta' + t(\eta - \eta')|^{p-2} \mathrm{d}t \leq c(|\eta| + |\eta'|)^{p-2}. \tag{2.22}$$

We then conclude from (2.20)-(2.22) that

$$\tfrac{1}{c}(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2 \leq \langle \nabla f(\eta) - \nabla f(\eta'), \eta - \eta' \rangle \leq c(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2. \tag{2.23}$$

For later use we note that (2.23) easily implies

$$\tfrac{1}{c}(|\eta| + |\eta'| + \varepsilon)^{p-2}|\eta - \eta'|^2 \leq \langle \nabla f_\varepsilon(\eta) - \nabla f_\varepsilon(\eta'), \eta - \eta' \rangle \leq c\,(|\eta| + |\eta'| + \varepsilon)^{p-2}|\eta - \eta'|^2. \tag{2.24}$$

Indeed if $|\eta| + |\eta'| \geq 2\varepsilon$ then (2.24) follows easily from (2.23) and the definition of f_ε. Otherwise, using (2.23) we deduce

$$\begin{aligned}
\langle \nabla f_\varepsilon(\eta) - \nabla f_\varepsilon(\eta'), \eta - \eta' \rangle &= \int\limits_{\mathbb{C}} \langle \nabla f_\varepsilon(\eta - z) - \nabla f_\varepsilon(\eta' - z), \eta - z - \eta' + z \rangle \theta_\varepsilon(z)\mathrm{d}z \\
&\geq \tfrac{1}{c}|\eta - \eta'|^2 \int\limits_{C} (|\eta - z| + |\eta' - z|)^{p-2}\,\theta_\varepsilon(z)\mathrm{d}z \\
&\geq \tfrac{1}{c^2}\varepsilon^{p-2}|\eta - \eta'|^2 \\
&\geq \tfrac{1}{c^3}\left(|\eta| + |\eta'| + \varepsilon\right)^{p-2}|\eta - \eta'|^2
\end{aligned} \tag{2.25}$$

for some $c \geq 1$. Here we used the fact that for some $c' \geq 1$,

$$\min\{|\eta - z|, |\eta' - z|\} \geq \tfrac{\varepsilon}{c'} \text{ and } \theta_\varepsilon(z) \geq \frac{1}{(c'\varepsilon)^2}$$

on a disk of radius $\varepsilon/4$.

Note that we have used (2.18) to show that (2.23) holds for a.e $\eta, \eta' \in \mathbb{R}^2$ and (2.23) clearly implies $\delta-$monotonicity of ∇f when $f \in W^{1,1}(B(0,R))$ for each $R > 0$. Also we have shown that $\delta-$monotonicity of a homogeneous degree p function implies (2.18). Thus $\delta-$monotonicity and (2.18) are equivalent under the scenario mentioned in the introduction.

Chapter 3 Advanced Regularity Results

3.1 Variational Problem

In this chapter we study the variational problem (1.6) and indicate some properties of weak solutions to the corresponding Euler-Lagrange equation. To this end, let $\tilde{f}$ be homogenous of degree p on $\mathbb{C} \setminus \{0\}$ and suppose that $\nabla \tilde{f}$ is $\delta-$monotone on $\mathbb{C}$ for some $0 < \delta \leq 1$. We also put $\tilde{f}(0) = 0$. Then (2.18) and (2.19) hold for $\tilde{f}$. Using these inequalities we see that $\tilde{f}$ satisfies the hypothesis in [17, Chapter 5] or [8, Theorem 3.13] which guarantee that the problem of finding a minimizer;

$$\inf_{w \in \mathfrak{A}} \left\{ \int_{\tilde{\Omega}} \tilde{f}(\nabla w) \mathrm{d}\nu \right\} \tag{3.1}$$

has a solution when $h \in W^{1,p}(\tilde{\Omega})$ and

$$\mathfrak{A} = \{h + \phi : \ \phi \in W_0^{1,p}(\tilde{\Omega})\}.$$

That is, there is $\tilde{u} \in \mathfrak{A}$ satisfying

$$\inf_{w \in \mathfrak{A}} \int_{\tilde{\Omega}} \tilde{f}(\nabla w) \mathrm{d}\nu = \int_{\tilde{\Omega}} \tilde{f}(\nabla \tilde{u}) \mathrm{d}\nu \ \text{ for some } \ \tilde{u} \in \mathfrak{A}. \tag{3.2}$$

Also $\tilde{u}$ is a weak solution to the following Euler-Lagrange equation in $\tilde{\Omega}$,

$$\begin{aligned} 0 = \nabla \cdot (\nabla \tilde{f}(\nabla \tilde{u})) &= \sum_{k=1}^{2} \frac{\partial}{\partial z_k} \left(\frac{\partial \tilde{f}}{\partial \eta_k} (\nabla \tilde{u}) \right) \\ &= \sum_{k,j=1}^{2} \tilde{f}_{\eta_k \eta_j} (\nabla \tilde{u}) \, \tilde{u}_{z_k z_j}. \end{aligned} \tag{3.3}$$

More specifically,

$$\int_{\tilde{\Omega}} \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \phi \rangle \mathrm{d}\nu = 0 \text{ whenever } \phi \in W_0^{1,p}(\tilde{\Omega}). \tag{3.4}$$

Lemma 3.1.1 (Weak Maximum Principle)**.** *Let $\tilde{u}, \hat{u}$ be weak solutions to (3.3) in $\tilde{\Omega}$ with $\tilde{u} = \hat{u}$ on $\partial \tilde{\Omega}$ in the Sobolev sense. Then $\hat{u} = \tilde{u}$ a.e in $\tilde{\Omega}$.*

Proof. Since $\tilde{u}, \hat{u}$ are solutions and $\hat{u} - \tilde{u} \in W_0^{1,p}(\tilde{\Omega})$, and (2.23) holds we see that

$$\begin{aligned} 0 &= \int_{\tilde{\Omega}} \langle \nabla f(\nabla \hat{u}) - \nabla f(\nabla \tilde{u}), \nabla \hat{u} - \nabla \tilde{u} \rangle \mathrm{d}\nu \\ &\geq \tfrac{1}{c} \int_{\tilde{\Omega}} (|\nabla \hat{u}| + |\nabla \tilde{u}|)^{p-2} \, |\nabla \hat{u} - \nabla \tilde{u}|^2 \mathrm{d}\nu \\ &\geq 0. \end{aligned} \tag{3.5}$$

Then $\nabla\hat{u} - \nabla\tilde{u} = 0$ a.e in $\tilde{\Omega}$. As $\hat{u} = \tilde{u}$ on $\partial\tilde{\Omega}$ in the $W_0^{1,p}(\tilde{\Omega})$ Sobolev sense we have $\hat{u} = \tilde{u}$ a.e in $\tilde{\Omega}$. $\square$

Remark 3.1.2 (Variational Problem has a unique solution)**.** Using Lemma 3.1.1 we remark that the variational problem (3.1) has a unique solution. That is,

$$\inf_{w\in\mathfrak{A}} \int\limits_{\tilde{\Omega}} \tilde{f}(\nabla w)\mathrm{d}\nu = \int\limits_{\tilde{\Omega}} \tilde{f}(\nabla\tilde{u})\mathrm{d}\nu \text{ for some unique } \tilde{u} \in \mathfrak{A}.$$

Some Well-Known Theorems

In this subsection we first give some well known theorems that will be useful in our proof of the Main Theorem 1.2.9. To do so we shall need some notation.

If ϕ is integrable on $B(z,r)$ let

$$\phi_B = \fint_{B(z,r)} \phi\,\mathrm{d}\nu = \frac{1}{\nu(B(z,r))} \int\limits_{B(z,r)} \phi\,\mathrm{d}\nu.$$

The **Oscillation** of a function h on a set D is defined as

$$\operatorname*{osc}_D h = \operatorname*{ess\,sup}_{\tilde{z},\hat{z}\in D} |h(\tilde{z}) - h(\hat{z})| = \operatorname*{ess\,sup}_D h - \operatorname*{ess\,inf}_D h.$$

Theorem 3.1.3 (Sobolev's Inequality)**.** *Fix p, $1 < p < \infty$. Then there is a constant $\tau > 1$ and $c = c(p) > 0$ such that*

$$\left(\fint_{B(z,r)} |\phi - \phi_B|^{\tau p}\mathrm{d}\nu\right)^{\frac{1}{\tau p}} \leq r \left(\fint_{B(z,r)} |\nabla\phi|^p \mathrm{d}\nu\right)^{\frac{1}{p}} \tag{3.6}$$

whenever $\phi - \phi_B \in W^{1,p}(B(z,r))$ and where

$$\tau = \begin{cases} \frac{2}{2-p} & \text{if } 1 < p < 2 \\ 2 & \text{if } 2 \leq p < \infty. \end{cases}$$

A proof of Sobolev's Inequality can be found in [10, Chapter 5.6].

Theorem 3.1.4 (Morrey's Inequality)**.** *Let $\phi \in W^{1,p}(\mathbb{R}^2)$ and $p > 2$. Then there is a constant $c = c(n)$ such that for a.e $z, z' \in \mathbb{R}^2$,*

$$|\phi(z) - \phi(z')| \leq c\, r^{1-\frac{n}{p}} \left(\int\limits_{B(z,2r)} |\nabla\phi(z)|^p \mathrm{d}\nu\right)^{\frac{1}{p}} \tag{3.7}$$

where $r = |z - z'|$.

A proof of Morrey's Inequality can be found in [21, Chapter 18].

We next give the Poincaré inequality.

Theorem 3.1.5 (Poincaré's Inequality)**.** *There is a constant $c = c(p) > 0$ such that*

$$\int\limits_{B(z,r)} |\phi - \phi_B|^p \mathrm{d}\nu \le cr^p \int\limits_{B(z,r)} |\nabla \phi|^p \mathrm{d}\nu \tag{3.8}$$

whenever $\phi \in W^{1,p}(B(z,r))$.

A proof of the Poincare inequality can be found in [10, Chapter 5.8].

Theorem 3.1.6 (Coarea Formula)**.** *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be Lipschitz continuous and suppose that $h : \mathbb{R}^2 \to \mathbb{R}$ is integrable. Then*

$$\int\limits_{\mathbb{R}^2} h|\nabla w| \,\mathrm{d}\nu = \int\limits_{-\infty}^{\infty} \left(\int\limits_{\{t:\, w(z)=t\}} h \,\mathrm{d}H^1 \right) \mathrm{d}t.$$

For a proof of and more on the coarea formula see [11, Chapter 3].
We next define capacity of a set as it is given in [17, Chapter 2].

Definition 3.1.7 (p capacity of a set)**.** Let K be a compact subset of $\tilde{\Omega}$ and let

$$\mathfrak{W}(K, \tilde{\Omega}) = \{h \in C_0^\infty(\tilde{\Omega});\ \ h \ge 1 \ \text{ on } \ K\}.$$

Define

$$\operatorname{cap}_p(K, \tilde{\Omega}) = \inf_{h \in \mathfrak{W}(K,\tilde{\Omega})} \int\limits_{\tilde{\Omega}} |\nabla h|^p \mathrm{d}\nu.$$

If U is an open subset of $\tilde{\Omega}$ then

$$\operatorname{cap}_p(U, \tilde{\Omega}) = \sup_{\substack{K \subset U \\ K \ \text{ is compact}}} \operatorname{cap}_p(K, \tilde{\Omega}),$$

and for an arbitrary set $E \subset \tilde{\Omega}$, we define p capacity of E as

$$\operatorname{cap}_p(E, \tilde{\Omega}) = \inf_{\substack{E \subset U \subset \tilde{\Omega} \\ U \ \text{ is open}}} \operatorname{cap}_p(U, \tilde{\Omega}).$$

Following [17, Chapter 6] we give the definition of p "thickness" of a set E.

Definition 3.1.8 (p thickness, Wiener Criterion)**.** A set E is called $p-$thick at z if

$$\int\limits_0^1 \left(\frac{\operatorname{cap}_p(E \cap B(z,t), B(z,2t))}{\operatorname{cap}_p(B(z,t), B(z,2t))} \right) \frac{\mathrm{d}t}{t} = \infty. \tag{3.9}$$

Finally we state a lemma which is a direct consequence of (2.18) and (2.19) for $\tilde{u} \in W^{1,1}(\tilde{\Omega})$.

Lemma 3.1.9. *For some constants $c, c', c'' \geq 1$ depending only on $\tilde{f}$, we have for a.e $z \in \tilde{\Omega}$,*

$$\frac{1}{c}|\nabla \tilde{u}|^p \leq \tilde{f}(\nabla \tilde{u}) \leq c|\nabla \tilde{u}|^p, \tag{3.10}$$
$$\frac{1}{c'}|\nabla \tilde{u}|^{p-1} \leq |\nabla \tilde{f}(\nabla \tilde{u})| \leq c'|\nabla \tilde{u}|^{p-1}, \tag{3.11}$$
$$\frac{1}{c''}|\nabla \tilde{u}|^{p-2} \leq \|D^2 \tilde{f}(\nabla \tilde{u})\| \leq c''|\nabla \tilde{u}|^{p-2}. \tag{3.12}$$

We are now ready to begin our study of regularity for weak solutions to (3.3). In Lemmas 3.1.10-3.1.14 we suppose $0 < r < \operatorname{diam} \partial\tilde{\Omega} < \infty$, and that $\tilde{u} \geq 0$ is a weak solution to (3.3) in $B(w, 4r) \cap \tilde{\Omega}$ with $\tilde{u} \equiv 0$ on $\partial\tilde{\Omega} \cap B(w, 4r)$ in the Sobolev sense. Let $\tilde{u} = 0$ on $B(w, 4r) \setminus \tilde{\Omega}$.

Lemma 3.1.10. *For fixed p, $1 < p < \infty$, let $\tilde{u}, \tilde{f}, \tilde{\Omega}, w, r$ be defined as above. Then*

$$\tfrac{1}{c} r^{p-2} \int\limits_{B(w,\frac{r}{2})} \tilde{f}(\nabla \tilde{u}) \mathrm{d}\nu \leq \operatorname*{ess\,sup}_{B(w,r)} \tilde{u}^p \leq c \frac{1}{r^2} \int\limits_{B(w,r)} \tilde{u}^p \mathrm{d}\nu. \tag{3.13}$$

Proof. The first part in the lemma follows by a sub solution type argument. To this end, let $\eta \in C_0^\infty(B(w, 2r))$, $0 \leq \eta \leq 1$, with $\eta = 1$ in $B(w, r)$ and $|\nabla \eta| \leq c/r$. Then one can easily show that $\tilde{u}\eta^p \in W_0^{1,p}(\tilde{\Omega} \cap B(w, 4r))$. Thus by (3.4) with $\tilde{\Omega}$ replaced by $\tilde{\Omega} \cap B(w, 4r)$,

$$\begin{aligned} 0 &= \int\limits_{\tilde{\Omega} \cap B(w,4r)} \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla(\tilde{u}\eta^p)\rangle \mathrm{d}\nu \\ &= \int\limits_{B(w,2r)} \eta^p \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \tilde{u}\rangle \mathrm{d}\nu + p \int\limits_{B(w,2r)} \tilde{u}\eta^{p-1} \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \eta\rangle \mathrm{d}\nu. \end{aligned} \tag{3.14}$$

By homogeneity of $\tilde{f}$ and Euler's identity we have $\langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \tilde{u}\rangle = p\tilde{f}(\nabla \tilde{u})$. Using this equality and rearranging (3.14) we see that

$$\begin{aligned} p \int\limits_{B(w,2r)} \eta^p \tilde{f}(\nabla \tilde{u}) \mathrm{d}\nu &= \int\limits_{B(w,2r)} \eta^p \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \tilde{u}\rangle \mathrm{d}\nu \\ &= -p \int\limits_{B(w,2r)} \tilde{u}\eta^{p-1} \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \eta\rangle \mathrm{d}\nu \\ &\leq p \int\limits_{B(w,2r)} \tilde{u}\eta^{p-1} |\nabla \tilde{f}(\nabla \tilde{u})||\nabla \eta| \mathrm{d}\nu. \end{aligned} \tag{3.15}$$

Then using (3.10), (3.11) and Hölder's inequality in (3.15) we get that

$$
\begin{aligned}
\int\limits_{B(w,2r)} \eta^p \tilde{f}(\nabla\tilde{u})\mathrm{d}\nu &\le \int\limits_{B(w,2r)} \tilde{u}\eta^{p-1}|\nabla\tilde{f}(\nabla\tilde{u})||\nabla\eta|\mathrm{d}\nu \\
&\le c \int\limits_{B(w,2r)} \tilde{u}\eta^{p-1}|\nabla\tilde{u}|^{p-1}|\nabla\eta|\mathrm{d}\nu \\
&\le c \left(\int\limits_{B(w,2r)} \tilde{u}^p|\nabla\eta|^p\mathrm{d}\nu\right)^{\frac{1}{p}} \left(\int\limits_{B(w,2r)} \eta^p|\nabla\tilde{u}|^p\mathrm{d}\nu\right)^{\frac{p-1}{p}} \\
&\le c \left(\int\limits_{B(w,2r)} \tilde{u}^p|\nabla\eta|^p\mathrm{d}\nu\right)^{\frac{1}{p}} \left(\int\limits_{B(w,2r)} \eta^p\tilde{f}(\nabla\tilde{u})\mathrm{d}\nu\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.16}
$$

Then it follows from (3.16) that

$$
\int\limits_{B(w,2r)} \eta^p \tilde{f}(\nabla\tilde{u})\mathrm{d}\nu \le c \int\limits_{B(w,2r)} \tilde{u}^p|\nabla\eta|^p\mathrm{d}\nu \tag{3.17}
$$

From (3.17) we obtain the left hand inequality of Lemma 3.1.10,

$$
\begin{aligned}
\int\limits_{B(w,r)} \tilde{f}(\nabla\tilde{u})\mathrm{d}\nu \le \int\limits_{B(w,2r)} \eta^p\tilde{f}(\nabla\tilde{u})\mathrm{d}\nu &\le c \int\limits_{B(w,2r)} |\nabla\eta|^p\tilde{u}^p\mathrm{d}\nu \\
&\le c\, r^{2-p} \operatorname*{ess\,sup}_{B(w,r)} \tilde{u}^p.
\end{aligned} \tag{3.18}
$$

For the second display, we first need to obtain (3.19) and then we employ Moser iteration to get the desired estimate.

For $q \ge 0$ and $\eta \in C_0^\infty(\tilde{\Omega})$ there exists $c = c(\tilde{f}, p)$ such that

$$
\int\limits_{\tilde{\Omega}} \tilde{u}^q\tilde{f}(\nabla\tilde{u})\eta^p\mathrm{d}\nu \le c \int\limits_{\tilde{\Omega}} \tilde{u}^{p+q}|\nabla\eta|^p\mathrm{d}\nu. \tag{3.19}
$$

The case $q = 0$ follows as in (3.18). Also for a positive ϕ-measurable function $\tilde{u}$ and for $0 < q < \infty$ we have

$$
\int\limits_{\tilde{\Omega}} \tilde{u}^q\mathrm{d}\nu = q\int\limits_0^\infty t^{q-1}\phi\{z:\ \tilde{u}(z) > t\}\mathrm{d}t. \tag{3.20}
$$

Using (3.20), Lemma 3.1.9, as well as (3.19) with $q = 0$ and $\tilde{u}$ replaced by $\max(\tilde{u}-t, 0)$, we obtain that

$$\begin{aligned}
\int_{\tilde{\Omega}} \tilde{u}^q \tilde{f}(\nabla\tilde{u})\eta^p \mathrm{d}\nu &\le c\, q \int_0^\infty t^{q-1} \int_{\{z:\ \tilde{u}(z)>t\}} |\nabla\tilde{u}|^p \eta^p \mathrm{d}\nu \mathrm{d}t \\
&= c\, q \int_0^\infty t^{q-1} \int_{\{z:\ \tilde{u}(z)>t\}} |\nabla(\tilde{u}-t)|^p \eta^p \mathrm{d}\nu \mathrm{d}t \\
&\le c^2 \int_0^\infty t^{q-1} \int_{\{z:\ \tilde{u}(z)>t\}} |\tilde{u}-t|^p |\nabla\eta|^p \mathrm{d}\nu \mathrm{d}t \\
&\le c^2 \int_0^\infty t^{q-1} \int_{\{z:\ \tilde{u}(z)>t\}} \tilde{u}^p |\nabla\eta|^p \mathrm{d}\nu \mathrm{d}t \\
&= c^3 \int_{\tilde{\Omega}} \tilde{u}^{p+q} |\nabla\eta|^p \mathrm{d}\nu.
\end{aligned}$$

Therefore we have (3.19). Next we employ Moser iteration in (3.19) to get the desired result.

Define $r_i = \frac{1}{2} + 2^{-i-1}$ for $i = 0, 1, 2, \ldots$. Let $0 \le \phi_i \le 1$ such that

$$\phi_i \in C_0^\infty(B(w, r_i r)), \quad \phi_i = 1 \text{ in } B(w, r_{i+1} r), \quad \text{and} \quad |\nabla\phi_i| \le \frac{2^{i+3}}{r}.$$

Fixed $t \ge 0$ and set

$$h_i = \left(\tilde{u}^{1+\frac{t}{p}}\right)\phi_i.$$

Using (3.10), (3.19) and the estimate for $|\nabla\phi|$ we have that

$$\begin{aligned}
\left(\int_{B(w,r_i r)} |\nabla h_i|^p \mathrm{d}\nu\right)^{\frac{1}{p}} &\le c \left(\int_{B(w,r_i r)} \tilde{u}^t |\nabla\tilde{u}|^p \phi_i^p \mathrm{d}\nu\right)^{\frac{1}{p}} + \left(\int_{B(w,r_i r)} \tilde{u}^{p+t} |\nabla\phi_i|^p \mathrm{d}\nu\right)^{\frac{1}{p}} \\
&\le c \left(\int_{B(w,r_i r)} \tilde{f}(\nabla\tilde{u})\tilde{u}^t \phi_i^p \mathrm{d}\nu\right)^{\frac{1}{p}} + \left(\int_{B(w,r_i r)} \tilde{u}^{p+t} |\nabla\phi_i|^p \mathrm{d}\nu\right)^{\frac{1}{p}} \\
&\le c^2 \left(\int_{B(w,r_i r)} \tilde{u}^{p+t} |\nabla\phi_i|^p \mathrm{d}\nu\right)^{\frac{1}{p}} \le c^3 \frac{2^{i+1}}{r} \left(\int_{B(w,r_i r)} \tilde{u}^{p+t} \mathrm{d}\nu\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.21}$$

Using Sobolev's inequality 3.1.3 and (3.21) we obtain

$$\begin{aligned}\left(\fint_{B(w,r_ir)} |h_i|^{\tau p}\mathrm{d}\nu\right)^{\frac{1}{\tau p}} &\le c r_i r \left(\fint_{B(w,r_ir)} |\nabla h_i|^{p}\mathrm{d}\nu\right)^{\frac{1}{p}} \\ &\le c\,(2^i+1)\left(\fint_{B(w,r_ir)} \tilde{u}^{p+t}\mathrm{d}\nu\right)^{\frac{1}{p}}\end{aligned} \tag{3.22}$$

where τ is the constant defined in Sobolev's inequality 3.1.3. (3.21) and (3.22) yield

$$\left(\fint_{B(w,r_{i+1}r)} \tilde{u}^{\tau(p+t)}\mathrm{d}\nu\right)^{\frac{1}{\tau(p+t)}} \le c\,2^{-\frac{p}{p+t}}2^{\frac{pi}{p+t}}(p+t)^{\frac{p}{p+t}}\left(\fint_{B(w,rir)} \tilde{u}^{p+t}\mathrm{d}\nu\right)^{\frac{1}{(p+t)}}. \tag{3.23}$$

As (3.23) holds for every exponent $p+t>p$ we can apply it to $p\tau^i$ for $i=0,1,\dots$. Iterating (3.23) we finally obtain the desired estimate,

$$\begin{aligned}\operatorname*{ess\,sup}_{B(w,\frac{r}{2})} \tilde{u} &\le \lim_{i\to\infty}\left(\fint_{B(w,r_ir)} \tilde{u}^{p\tau^{i+1}}\mathrm{d}\nu\right)^{\frac{1}{p\tau^{i+1}}} \\ &\le c\left(\fint_{B(w,r)} \tilde{u}^{p}\mathrm{d}\nu\right)^{\frac{1}{p}}.\end{aligned} \tag{3.24}$$

Hence if follows from (3.24) that

$$\operatorname*{ess\,sup}_{B(w,\frac{r}{2})} \tilde{u}^p \le c r^{-2}\int\limits_{B(w,r)} \tilde{u}^p\mathrm{d}\nu$$

which finishes the proof for the right hand inequality of Lemma 3.1.10. □

Lemma 3.1.11 (Harnack's Inequality)**.** *Let $\tilde{u},\tilde{\Omega},r,w$ be as in Lemma 3.1.10. Then there is a constant $c=c(p,f)$ such that*

$$\operatorname*{ess\,sup}_{B(\tilde{w},s)} \tilde{u} \le c\operatorname*{ess\,inf}_{B(\tilde{w},s)} \tilde{u}. \tag{3.25}$$

whenever $B(\tilde{w},2s)\subset B(w,4r)\cap\tilde{\Omega}$.

Proof. We may assume that $\tilde{u}>0$ in $B(w,4r)\cap\tilde{\Omega}$ (otherwise we can consider $\tilde{u}+\varepsilon$ for small $\varepsilon>0$). First assume that $B(\tilde{w},4s)\subset B(w,4r)\cap\tilde{\Omega}$. Then by (3.24) and [17, Theorem 3.41] we see for small $t>0$

$$\operatorname*{ess\,sup}_{B(\tilde{w},s)} \tilde{u} \le c\left(\fint_{B(\tilde{w},2s)} \tilde{u}^t\mathrm{d}\nu\right)^{\frac{1}{t}} \le c\operatorname*{ess\,inf}_{B(\tilde{w},s)} \tilde{u}. \tag{3.26}$$

To finish the proof when $B(\tilde{w},2s)\subset B(w,4r)\cap\tilde{\Omega}$, we use the fact that for any two points $\hat{w}',\hat{w}''\in\overline{B(\tilde{w},s)}$ there is a chain of finitely many balls $B(\tilde{w}_1,s_1),\dots,B(\tilde{w}_n,s_n)$ such that $\hat{w}'\in B(z_1,s_1)$, $\hat{w}''\in B(\tilde{w}_n,s_n)$, $B(\tilde{w}_i,4s_i)\subset B(w,4r)\cap\tilde{\Omega}$ for $i=1,\dots,n$ and $B(\tilde{w}_i,s_i)\cap B(\tilde{w}_{i+1},s_{i+1})\neq\emptyset$ for $i=1,\dots,n-1$. Then a repeated use of (3.26) proves Lemma 3.1.11. □

Next we show local Hölder continuity which follows from an iteration argument using Harnack's inequality.

Lemma 3.1.12. *Let $\tilde{u}, \tilde{\Omega}, w, r$ be as in Lemma 3.1.10. Let $0 < s_0 < \infty$ and suppose that $B(w_0, s_0) \subset B(w, 4r) \cap \tilde{\Omega}$. Then for $0 < s < s_0$ there is a constant $0 < \alpha = \alpha(p, f) \leq 1$ such that*

$$\operatorname*{osc}_{B(w_0,s)} \tilde{u} \leq 2^\alpha \left(\tfrac{s}{s_0}\right)^\alpha \operatorname*{osc}_{B(w_0,s_0)} \tilde{u}. \tag{3.27}$$

Note that Lemma 3.1.12 implies for every $\tilde{w}, \hat{w} \in B(w_0, s)$ that

$$|\tilde{u}(\tilde{w}) - \tilde{u}(\hat{w})| \leq \operatorname*{osc}_{B(w_0,s)} \tilde{u} \leq 2^\alpha \left(\tfrac{s}{s_0}\right)^\alpha \operatorname*{osc}_{B(w_0,s_0)} \tilde{u} \leq c|\tfrac{\tilde{w}-\hat{w}}{s}|^\alpha \operatorname*{ess\,sup}_{B(w_0,s_0)} \tilde{u}.$$

Thus,

Corollary 3.1.13 (Interior Hölder Continuity)**.** *For fixed p, $1 < p < \infty$, let $\tilde{u}, \tilde{\Omega}, w, r$ be as in Lemma 3.1.10. Then $\tilde{u}$ has an $\alpha-$Hölder continuous representative in $B(w_0, s)$ for every $0 < s < s_0 < \infty$ whenever $B(w_0, s_0) \subset B(w, 4r) \cap \tilde{\Omega}$.*

Proof of Lemma 3.1.12. Let $0 < s < s_0$. As $\left(\tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u}\right)$ is a non-negative weak solution to (3.3), we can apply Harnack's inequality to this function on $B(w_0, s/2)$ to get

$$\operatorname*{ess\,sup}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \leq c \left(\operatorname*{ess\,inf}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right). \tag{3.28}$$

Assume we have

$$\operatorname*{ess\,inf}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \leq \tfrac{1}{c} \left(\operatorname*{ess\,sup}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right). \tag{3.29}$$

Then using (3.28) we have

$$\begin{aligned}
\operatorname*{osc}_{B(w_0,\frac{s}{2})} \tilde{u} &= \operatorname*{ess\,sup}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} + \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,\frac{s}{2})} \tilde{u} \\
&\leq (c-1) \left(\operatorname*{ess\,inf}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right) \\
&\leq \tfrac{c-1}{c} \left(\operatorname*{ess\,sup}_{B(w_0,s)} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right) \\
&\leq \tfrac{c-1}{c} \operatorname*{osc}_{B(w_0,s)} \tilde{u}.
\end{aligned}$$

If (3.29) is false, then we have

$$\operatorname*{ess\,inf}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} > \tfrac{1}{c}\left(\operatorname*{ess\,sup}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right). \tag{3.30}$$

Using (3.28) we see that

$$\begin{aligned}
\operatorname*{osc}_{B(w_0,\frac{s}{2})} \tilde{u} &= \operatorname*{ess\,sup}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} - \left(\operatorname*{ess\,inf}_{B(w_0,\frac{s}{2})} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right) \\
&< (1-\frac{1}{c}) \left(\operatorname*{ess\,sup}_{B(w_0,s)} \tilde{u} - \operatorname*{ess\,inf}_{B(w_0,s)} \tilde{u} \right) \\
&\leq \tfrac{c-1}{c} \operatorname*{osc}_{B(w_0,s)} \tilde{u}.
\end{aligned}$$

In both cases (3.29), (3.30) we have

$$\operatorname*{osc}_{B(w_0,\frac{s}{2})} \tilde{u} \leq \tfrac{c-1}{c} \operatorname*{osc}_{B(w_0,s)} \tilde{u}. \tag{3.31}$$

If we iterate (3.31) we see that

$$\begin{aligned}
\operatorname*{osc}_{B(w_0,s)} \tilde{u} &\leq (\tfrac{c-1}{c})^{m-1} \operatorname*{osc}_{B(w_0,2^{m-1}s)} \tilde{u} \\
&\leq (\tfrac{c-1}{c})^{m-1} \operatorname*{osc}_{B(w_0,s_0)} \tilde{u}
\end{aligned} \tag{3.32}$$

where m is an integer ≥ 1 with $2^{m-1} \leq s_0/s \leq 2^m$. Then for some α we have the desired result. □

Next we show Hölder continuity of $\tilde{u}$ near $B(w,4r) \cap \partial\tilde{\Omega}$.

Lemma 3.1.14 (Behavior of $\tilde{u}$ near the boundary)**.** *Let $\tilde{u}$ be as in Lemma 3.1.10 and $w \in \partial\tilde{\Omega}$. Then there is $\alpha = \alpha(p,f) > 0$ such that $\tilde{u}$ has a Hölder continuous representative in $B(w,r)$ and if $\tilde{w}, \hat{w} \in B(w,r)$ then*

$$|\tilde{u}(\tilde{w}) - \tilde{u}(\hat{w})| \leq c \left(\frac{|\tilde{w} - \hat{w}|}{r} \right)^{\alpha} \operatorname*{ess\,sup}_{B(w,2r)} \tilde{u}. \tag{3.33}$$

Proof. The proof for $p > 2$ follows from Lemma 3.1.10 and Morrey's Theorem 3.1.4.

For $1 < p \leq 2$ we note that there is a continuum$\subset \overline{B}(w,t) \setminus \tilde{\Omega}$ connecting w to $\partial B(w,t)$ as follows from simply connectivity of $\tilde{\Omega}$. We also note that this continuum is uniformly fat in the sense of $p-$capacity (see [23] for the definition of uniformly fat set). That is, the $p-$capacity of this continuum is $\geq c^{-1}$ times the $p-$capacity of $B(w,r)$.

We can find $h \in W^{1,p}(B(w,2r) \cap C(\overline{B(w,2r)})$ with $h = 0$ on $\partial\tilde{\Omega} \cap B(w,r)$ and $h = 1$ in $\Omega \setminus B(w,2r)$. Then using this in the Wiener integral in [17][Theorem 6.18] we obtain for $0 < \rho \leq r/2$

$$
\begin{aligned}
\operatorname*{osc}_{B(w,\rho)\cap\tilde{\Omega}} u \leq & \operatorname*{osc}_{\partial\tilde{\Omega}\cap\overline{B(w,2\rho)}} \left(\operatorname*{ess\,sup}_{B(w,2r)} \tilde{u} \right) h \\
& + \operatorname*{osc}_{\partial\tilde{\Omega}} \left(\operatorname*{ess\,sup}_{B(w,2\rho)} \tilde{u} \right) h \, \exp \left[-c \int_{\rho}^{2\rho} \left(\frac{\operatorname{cap}_p((\mathbb{C}\setminus\tilde{\Omega}) \cap B(w,t), B(w,2t))}{\operatorname{cap}_p(B(w,t), B(w,2t))} \right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t} \right] \\
& \leq 2^{-c} \operatorname*{ess\,sup}_{B(w,2\rho)} \tilde{u}
\end{aligned}
\tag{3.34}
$$

for some $c = c(p, f, \tilde{\Omega}) > 0$. Then using (3.34) we have

$$
\operatorname*{ess\,sup}_{B(w,\rho)} \tilde{u} \leq c \operatorname*{ess\,sup}_{B(w,2\rho)} \tilde{u} \tag{3.35}
$$

for some $0 < c = c(f, p) < 1$ and $0 < 4\rho < r$. Then iterating (3.35) we obtain Lemma 3.1.14 for $1 < p \leq 2$ when $\tilde{w}$ or $\hat{w}$ in $B(w, 4r) \cap \partial\tilde{\Omega}$. Other values of $\tilde{w}, \hat{w}$ in (3.33) are handled by using this estimate and the interior estimate in Lemma 3.1.12 □

Lemma 3.1.15. *For fixed p, $1 < p < \infty$, and let $\tilde{u}, \tilde{\Omega}, w, r$ be as in Lemma 3.1.10. Let μ be the measure corresponding to $\tilde{u}$ as in (1.9).*

Then

$$
\tfrac{1}{c}\, r^{p-2}\mu(B(w, \tfrac{r}{2})) \leq \operatorname*{ess\,sup}_{B(w,r)} \tilde{u}^{p-1} \leq c\, r^{p-2}\mu(B(w,2r)). \tag{3.36}
$$

Proof. Let $0 < s < t < 10\rho$, and $\phi \in C_0^\infty(B(w,t))$ with $\phi = 1$ on $B(w,s)$ and $|\nabla\phi| \leq c/(t-s)$. Then as in (3.18)

$$
\begin{aligned}
\int_{B(w,s)} \tilde{f}(\nabla\tilde{u}) \mathrm{d}\nu &\leq c \int_{B(w,t)} \tilde{u}^p |\nabla\phi|^p \mathrm{d}\nu \\
&\leq c \tfrac{t^2}{(t-s)^p} (\operatorname*{ess\,sup}_{B(w,t)} \tilde{u})^p
\end{aligned}
\tag{3.37}
$$

Using (1.9) and (3.37) with $s = \rho$, $t = \frac{3}{2}\rho$, as well as Lemma 3.1.9 and Lemma 3.1.10

we obtain that

$$
\begin{aligned}
\mu(B(w,\rho)) &\le \int\limits_{B(w,\frac{3}{2}\rho)} \phi \, \mathrm{d}\mu = - \int\limits_{B(w,\frac{3}{2}\rho)} \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \phi \rangle \mathrm{d}\nu \\
&\le c \int\limits_{B(w,\frac{3}{2}\rho)} |\nabla \tilde{u}|^{p-1} |\nabla \phi| \mathrm{d}\nu \\
&\le c \left(\int\limits_{B(w,\frac{3}{2}\rho)} \tilde{f}(\nabla \tilde{u}) \mathrm{d}\nu \right)^{\frac{p-1}{p}} \left(\int\limits_{B(w,\frac{3}{2}\rho)} |\nabla \phi|^p \mathrm{d}\nu \right)^{\frac{1}{p}} \\
&\le c \left(\int\limits_{B(w,\frac{3}{2}\rho)} \tilde{f}(\nabla \tilde{u}) \mathrm{d}\nu \right)^{\frac{p-1}{p}} \rho^{\frac{2}{p}-1} \\
&\le c \rho^{2-p} \left(\operatorname*{ess\,sup}_{B(w,2\rho)} \tilde{u} \right)^{p-1}
\end{aligned} \tag{3.38}
$$

Since this estimate holds with $\rho = r/2$, we obtain the left inequality of Lemma 3.1.15.

To obtain the right hand inequality of the Lemma 3.1.15 we will divide the proof into two parts, we first prove it for $p \ge 2$ and then for $1 < p < 2$.

For $p \ge 2$, let h be a weak solution to (3.3) in $\Omega' = B(w, 2r)$ with $h = \tilde{u}$ on $\partial B(w, 2r)$ in the Sobolev sense. That is $h - \tilde{u} \in W_0^{1,p}(B(w, 2r))$. Then by the maximum principle 3.1.1 we have $0 \le \tilde{u} \le h$ (for details see [16]). Then using once again Harnack's inequality for h we have

$$
\operatorname*{ess\,sup}_{B(w,r)} \tilde{u} \le \operatorname*{ess\,sup}_{B(w,r)} h \le c h(w_1) \le c^2 h(w) \tag{3.39}
$$

whenever $w_1 \in B(w, r)$.

It follows from boundary Hölder continuity of $\tilde{u}$ in Lemma 3.1.14 that

$$
\operatorname*{ess\,sup}_{B(w,r_1)} \tilde{u} \le c \left(\frac{r_1}{r_2} \right)^{\alpha} \operatorname*{ess\,sup}_{B(w,r_2)} \tilde{u} \;\text{ whenever }\; 0 < r_1 < r_2 < 2r \tag{3.40}
$$

for some $0 < \alpha = \alpha(f, p) < 1$. Hence

$$
\operatorname*{ess\,sup}_{B(w,c_1 r)} \tilde{u} \le \tfrac{1}{2} \underline{\min_{B(w,r)}} h. \tag{3.41}
$$

for some $c_1 = c_1(f, p) > 0$. From (3.41) we conclude that there are constants c_2, c_3 depending on f, p and for every $w_1 \in B(w, c_1 r)$ that

$$
c_2 h(w) \le (h - \tilde{u})(w_1) \le c_3 h(w). \tag{3.42}
$$

We are now ready to prove the right hand inequality for $p \geq 2$ using Poincaré's inequality, (1.9), that h is a weak solution to (1.8), and Lemma 3.1.9. Let $\phi = \min\{h - \tilde{u}, c_3 h(w)\}$ and let D be the set where $\nabla\phi$ exists and is nonzero. We have

$$
\begin{aligned}
(c_2 h(w))^p (c_1 r)^2 &\leq c \int_D \phi^p \mathrm{d}\nu \\
&\leq c r^p \int_D |\nabla\phi|^p \mathrm{d}\nu \\
&\leq c r^p \int_D (|\nabla h| + |\nabla \tilde{u}|)^{p-2} |\nabla h - \nabla \tilde{u}|^2 \mathrm{d}\nu \\
&\leq c r^p \int_D \langle (\nabla \tilde{f}(\nabla h) - \nabla f(\nabla \tilde{u})), (\nabla h - \nabla \tilde{u}) \rangle \mathrm{d}\nu \\
&= -c r^p \int_D \langle \nabla \tilde{f}(\nabla \tilde{u}), \nabla \phi \rangle \mathrm{d}\nu \\
&= c r^p \int \phi \mathrm{d}\mu \\
&\leq c_3 h(w) r^p \mu(B(w, 2r)).
\end{aligned}
\tag{3.43}
$$

To prove the right hand inequality for $1 < p < 2$ let $r \leq s \leq 2r$, and $h(w, s)$ be a weak solution to (3.3) in $\tilde{\Omega} = B(w, s)$ and $h - \tilde{u} \in W_0^{1,p}(B(w, s))$. Then as in (3.39) for $0 < s_1 < s$ we obtain that

$$
\operatorname*{ess\,sup}_{B(w,s_1)} \tilde{u} \leq c \left(\frac{s_1}{s - s_1} \right)^{\alpha} h(w_1, s) \text{ for } w_1 \in B(w, s_1). \tag{3.44}
$$

For large enough c, we can choose $s_1 = \delta s$ (δ small enough) so that

$$
c_4 h(w, s) \leq (h(w_1, s) - \tilde{u}(w_1)) \leq c_5 h(w, s) \text{ for } w_1 \in B(w, s_1) \tag{3.45}
$$

where c_4, c_5 depend only on f and p. Once again set $\phi = \min\{h - \tilde{u}, c_5 h(w, s)\}$ and let D be the the set where $\nabla\phi$ is nonzero and exists. Using Poincaré's inequality, Lemma 3.1.9, and that h is a weak solution to (3.3) in $\Omega' = B(w, s)$ we find

$$
\begin{aligned}
(c_2 h(w, s))^p (d_1 s_1)^2 &\leq c \int_D \phi^p \mathrm{d}\nu \\
&\leq c s^p \int_D |\nabla \phi|^p \mathrm{d}\nu \\
&\leq c s^p \, A \times B
\end{aligned}
\tag{3.46}
$$

where

$$
A = \left(\int_D (|\nabla h| + |\nabla \tilde{u}|)^{p-2} |\nabla h - \nabla \tilde{u}|^2 \mathrm{d}\nu \right)^{\frac{p}{2}}
$$

and

$$B = \left(\int_D (|\nabla h| + |\nabla \tilde{u}|)^p \, \mathrm{d}\nu \right)^{\frac{2-p}{p}} .$$

As in the proof of the $p > 2$ case,

$$\begin{aligned} \int_D (|\nabla h| + |\nabla \tilde{u}|)^{p-2} |\nabla h - \nabla \tilde{u}|^2 \mathrm{d}\nu &\le c \int \langle (\nabla \tilde{f}(\nabla h) - \nabla \tilde{f}(\nabla \tilde{u})), (\nabla h - \nabla \tilde{u}) \rangle \mathrm{d}\nu \\ &= c \int \phi \, \mathrm{d}\mu \\ &\le c_5 h(w,s) \mu(B(w,s)). \end{aligned} \tag{3.47}$$

To finish the proof we need to obtain a similar estimate for the second part of (3.46). To this end we use the fact that h is a weak solution to (3.3), $h - u$ is an admissible function, (3.2), and Lemma 3.1.9 to obtain

$$\int_{B(w,s)} |\nabla h|^p \mathrm{d}\nu \le c \int_{B(w,s)} |\nabla \tilde{u}|^p \mathrm{d}\nu \tag{3.48}$$

Using (3.46)-(3.48), and an iteration argument (a more general argument can be found in [9]) we obtain the desired result. □

3.2 Relation Between Measures

Using Lemma 3.1.15 we prove that H-dim μ is independent of the corresponding u. Indeed, we show that $\tilde{\mu}, \hat{\mu}$ corresponding to $\tilde{u}, \hat{u}$ respectively as in (1.9) are mutually absolutely continuous. Hence H-dim $\tilde{\mu}$ = H-dim $\hat{\mu}$. We also conclude from mutual absolute continuity of $\tilde{\mu}, \hat{\mu}$ that our Main Theorem 1.2.9 holds for a measure $\tilde{\mu}$ if and only if it holds for $\hat{\mu}$.

To this end, let ν be a compactly supported finite Borel measure, not identically zero and suppose that there is a Borel set $E \subset \mathbb{C}$ with $\nu(E) > 0$. Suppose also for $z \in E$ that there exists r_0 depending on z such that

$$\nu(B(z,r)) \le \delta \nu(B(z,100r)) \text{ for } 0 < r < r_0. \tag{3.49}$$

Then iterating (3.49) we obtain for δ small enough and some $M < \infty$ that

$$\nu(B(z,r)) \le M r^4, \ 0 < r < r_0. \tag{3.50}$$

Then (3.50) implies that

$$\lim_{r \to 0} \frac{\nu(B(z,r))}{r^3} = 0 \ \text{ whenever } \ z \in E. \tag{3.51}$$

From a covering argument it then follows that $\nu(E) \leq cH^3(E) = 0$ which is a contradiction. Therefore we have that for every compactly supported finite Borel measure ν, not identically zero, on $\mathbb{C}$, there is a $c < \infty$ which is independent of w, r so that if

$$\Gamma = \left\{ z \in \text{ support of } \nu : \ \liminf_{r \to 0} \frac{\nu(B(z,100r))}{\nu(B(z,r))} \leq c \right\}. \tag{3.52}$$

then $\nu(\mathbb{C} \setminus \Gamma) = 0$.

To compare dimension of measures we first define the capacitary function relative to a point. That is choose a point $\tilde{z} \in \tilde{\Omega}$ and define $\tilde{D} := \tilde{\Omega} \setminus \overline{B(\tilde{z}, d(\tilde{z}, \partial\tilde{\Omega})/2)}$. h is called a capacitary function if h is a weak solution to (3.3) in $\tilde{D}$ with boundary values $h \equiv 0$ on $\partial\tilde{\Omega}$ and $h \equiv 1$ on $\partial B(\tilde{z}, d(\tilde{z}, \partial\tilde{\Omega})/2)$. Then again there is a finite Borel measure ν with support on $\partial\tilde{\Omega}$ satisfying (1.9).

Lemma 3.2.1. *Let $\tilde{u}$ be as in non-negative weak solution to (3.3) with $\tilde{u} = 0$ on $\partial\tilde{\Omega}$ in the $W^{1,p}(\tilde{\Omega})$ Sobolev sense and let $\hat{u}$ be a capacitary function defined as above. Let $\tilde{\mu}, \hat{\mu}$ be the measures corresponding to $\tilde{u}, \hat{u}$, respectively as in (1.9). Then $\tilde{\mu}, \hat{\mu}$ are mutually absolutely continuous.*

Proof. Let N be a neighborhood of $\partial\tilde{\Omega}$ with $\partial\tilde{\Omega} \subset N \subset \overline{N}$. Since both $\tilde{u}$ and $\hat{u}$ are continuous we can find $c < \infty$ with

$$\tilde{u} \leq c\hat{u} \leq c^2\tilde{u} \quad \text{on} \quad \partial N \cap \tilde{\Omega}. \tag{3.53}$$

Then by the maximum principle, Lemma 3.1.1 and compactness we have

$$\tilde{u} \leq c\hat{u} \leq c^2\tilde{u} \quad \text{in} \quad N \cap \tilde{\Omega}. \tag{3.54}$$

Observe also that support of $\tilde{\mu}, \hat{\mu} = \partial\tilde{\Omega}$ by (3.36). Then by (3.36) and (3.54) there is r_0 and $c_2 < \infty$ such that

$$\tilde{\mu}(B(z,r)) \leq c_2\hat{\mu}(B(z,2r)) \leq c_2^2\tilde{\mu}(B(z,4r)) \tag{3.55}$$

whenever $z \in \partial\tilde{\Omega}$ and $0 < r < r_0$.

Suppose that there is a Borel set $K_1 \subset \partial\tilde{\Omega}$ with $\tilde{\mu}(K_1) = 0$ and $\hat{\mu}(K_1) > 0$. Then by the observation in (3.52) for $\nu = \hat{\mu}$ there is Γ and a compact set K_2 with $K_2 \subset K_1 \cap \Gamma$ and $\hat{\mu}(K_2) > 0$. On the other hand, given $\varepsilon > 0$ we can find an open set K_3 with $K_1 \subset K_3$ and $\tilde{\mu}(K_3) < \varepsilon$.

By Vitalli's covering argument we can find $\{z_i, r_i\}$ with $z_i \in \partial\tilde{\Omega}$ satisfying $K_2 \subset \bigcup_i B(z_i, 100r_i)\} \subset K_3$, and $\{B(z_i, 10r_i)\}$ are pairwise disjoint. Using (3.52) for $\nu = \hat{\mu}$ we may also assume

$$\hat{\mu}(B(z_i, 100r_i)) \leq c_3\hat{\mu}(B(z_i, r_i)).$$

With these observations and (3.55) we see that

$$\begin{aligned} \hat{\mu}(K_1) \leq \hat{\mu}(\bigcup_i B(z_i, 100r_i)) &\leq \sum_i \hat{\mu}(B(z_i, 100r_i)) \\ &\leq c_3 \sum_i \hat{\mu}(B(z_i, r_i)) \leq c \sum_i \tilde{\mu}(B(z_i, 10r_i)) \\ &\leq c\tilde{\mu}(K_3) \leq c\varepsilon \end{aligned} \tag{3.56}$$

where $c = c(c_2, c_3)$ is independent of ε. Since ε is arbitrary we have that $\hat{\mu}(K_1) = 0$ which is a contradiction. Therefore for every Borel set E if $\tilde{\mu}(E) = 0$ then $\hat{\mu}(E) = 0$. Thus $\hat{\mu}$ is absolutely continuous with respect to $\tilde{\mu}$. Reversing the role of $\tilde{\mu}$ and $\hat{\mu}$ we obtain $\tilde{\mu}$ is absolutely continuous with respect to $\hat{\mu}$. $\square$

3.3 More Advanced Regularity Results

In this section we study more advanced regularity properties of a weak solution u to (3.3).

We first obtain regularity results for ∇u. To this end, assume that $B(w, 4r) \subset \tilde{\Omega}$ and let u_ε be a weak solution to

$$0 = \nabla \cdot (\nabla f_\varepsilon(\nabla u_\varepsilon(w))) = \sum_{k=1}^{2} \frac{\partial}{\partial z_k} \left(\frac{\partial f_\varepsilon}{\partial \eta_k} (\nabla u_\varepsilon(w)) \right) \tag{3.57}$$

in $B(w, 2r)$ with $u_\varepsilon - u \in W_0^{1,p}(B(w, 2r))$ where $f_\varepsilon = f * \theta_\varepsilon$ defined in chapter 2. Let $f_{kj}^\varepsilon = (f_\varepsilon)_{\eta_k \eta_j}(\nabla u_\varepsilon)$.

The De Giorgi method can be used to obtain that the directional derivative $\zeta = (u_\varepsilon)_\xi$ is in $W^{1,2}(B(w, r))$ and satisfies a uniformly elliptic equation in divergence form (for more details see [14]). To find this equation recall that

$$0 = \int\limits_{B(w,2r)} \langle \nabla f_\varepsilon(\nabla u_\varepsilon), \nabla \phi \rangle \mathrm{d}\nu. \tag{3.58}$$

for every admissible function $\phi \in C_0^\infty(B(w, 2r))$. Then taking $(\phi)_\xi$ as an admissible function and after an integration by parts, we get

$$\begin{aligned} 0 &= \int\limits_{B(w,2r)} \langle \nabla f_\varepsilon(\nabla u_\varepsilon), \nabla \phi_\xi \rangle \mathrm{d}\nu \\ &= - \int\limits_{B(w,2r)} \langle \nabla_\xi \left(\nabla f_\varepsilon(\nabla u_\varepsilon) \right), \nabla \phi \rangle \mathrm{d}\nu \\ &= - \int\limits_{B(w,2r)} \sum_{k,j=1}^{2} f_{kj}^\varepsilon \left((u_\varepsilon)_\xi \right)_{z_j} \phi_{z_k} \, \mathrm{d}\nu \\ &= \int\limits_{B(w,2r)} \sum_{k,j=1}^{2} \frac{\partial}{\partial z_k} \left(f_{kj}^\varepsilon \left((u_\varepsilon)_\xi \right)_{z_j} \right) \phi \, \mathrm{d}\nu \end{aligned} \tag{3.59}$$

It follows from (3.59) and pde theory that $\zeta = (u_\varepsilon)_\xi$ is a weak solution to

$$0 = \sum_{k,j=1}^{2} \frac{\partial}{\partial z_k} \left(f_{kj}^\varepsilon \frac{\partial \zeta}{\partial z_j} \right). \tag{3.60}$$

in $B(w,2r)$. Here ellipticity constants and $W^{2,2}$ norm of u_ε depend on ε.

On the other hand, u_ε also satisfies a nondivergence form equation (3.57) after division by $(|\nabla u_\varepsilon|+\varepsilon)^{p-2}$. That is, $\zeta = u_\varepsilon$ is a weak solution to

$$0 = \frac{1}{(|\nabla u_\varepsilon| + \varepsilon)^{p-2}} \sum_{j,k=1}^{2} f^{\varepsilon}_{kj} \zeta_{z_j z_k}. \tag{3.61}$$

in $B(w,2r)$. It follows from (2.24) that ellipticity constants are independent of ε.

Now we argue as in [13, Chapter 5] to get that ∇u_ε is a $K-$quasiregular mapping where K is independent of ε. To this end, let

$$\begin{aligned}
&\mathfrak{h}^\varepsilon = \langle (u_\varepsilon)_{z_1}, -(u_\varepsilon)_{z_2}\rangle \\
&\mathfrak{a}^\varepsilon = \frac{(f_\varepsilon)_{\eta_1\eta_1}(\nabla u_\varepsilon)}{(|\nabla u_\varepsilon|+\varepsilon)^{p-2}}, \quad \mathfrak{b}^\varepsilon = \frac{(f_\varepsilon)_{\eta_1\eta_2}(\nabla u_\varepsilon)}{(|\nabla u_\varepsilon|+\varepsilon)^{p-2}}, \quad \mathfrak{c}^\varepsilon = \frac{(f_\varepsilon)_{\eta_2\eta_2}(\nabla u_\varepsilon)}{(|\nabla u_\varepsilon|+\varepsilon)^{p-2}}, \\
&\mathfrak{p}^\varepsilon = (u_\varepsilon)_{z_1}, \quad \mathfrak{q}^\varepsilon = (u_\varepsilon)_{z_2}.
\end{aligned}$$

We show that $\mathfrak{h}^\varepsilon$ satisfies (2.2) for some finite constant K. We can write (3.57) for ν a.e in $B(w,2r)$ as

$$0 = \frac{\mathfrak{a}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{p}^\varepsilon_{z_1} + \frac{2\mathfrak{b}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{p}^\varepsilon_{z_2} + \mathfrak{q}^\varepsilon_{z_2}, \tag{3.62}$$

and

$$0 = \mathfrak{p}^\varepsilon_{z_1} + \frac{2\mathfrak{b}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{q}^\varepsilon_{z_1} + \frac{\mathfrak{c}^\varepsilon}{\mathfrak{a}^\varepsilon}\mathfrak{q}^\varepsilon_{z_2}. \tag{3.63}$$

Then $\mathfrak{p}^\varepsilon = (u_\varepsilon)_{z_1}$ is a weak solution to a uniformly elliptic partial differential equation in divergence form,

$$\begin{aligned}
0 &= (\frac{\mathfrak{a}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{p}^\varepsilon_{z_1} + \frac{2\mathfrak{b}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{p}^\varepsilon_{z_2})_{z_1} + (\mathfrak{q}^\varepsilon_{z_2})_{z_1} \\
&= (\frac{\mathfrak{a}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{p}^\varepsilon_{z_1} + \frac{2\mathfrak{b}^\varepsilon}{\mathfrak{c}^\varepsilon}\mathfrak{p}^\varepsilon_{z_2})_{z_1} + (\mathfrak{p}^\varepsilon_{z_1})_{z_2}.
\end{aligned}$$

Similarly $\mathfrak{q}^\varepsilon$ is a weak solution to

$$0 = \mathfrak{q}^\varepsilon_{z_1} + (\frac{2\mathfrak{b}^\varepsilon}{\mathfrak{a}^\varepsilon}\mathfrak{q}^\varepsilon_{z_1} + \frac{\mathfrak{c}^\varepsilon}{\mathfrak{a}^\varepsilon}\mathfrak{q}^\varepsilon_{z_2})_{z_2}.$$

Multiplying (3.62) by $\mathfrak{c}^\varepsilon\mathfrak{p}^\varepsilon_{z_1}$ and using $\mathfrak{p}^\varepsilon_{z_2} = \mathfrak{q}^\varepsilon_{z_1}$ we obtain for ν a.e in $B(w,2r)$,

$$\begin{aligned}
\frac{1}{c}\left((\mathfrak{p}^\varepsilon_{z_1})^2 + (\mathfrak{p}^\varepsilon_{z_2})^2\right) &\leq \tfrac{1}{c}\left(\mathfrak{a}^\varepsilon(\mathfrak{p}^\varepsilon_{z_1})^2 + 2\mathfrak{b}^\varepsilon(\mathfrak{p}^\varepsilon_{z_1})(\mathfrak{p}^\varepsilon_{z_2}) + \mathfrak{c}^\varepsilon(\mathfrak{p}^\varepsilon_{z_2})^2\right) \\
&= \frac{1}{c}\left(\mathfrak{a}^\varepsilon(\mathfrak{p}^\varepsilon_{z_1})^2 + 2\mathfrak{b}^\varepsilon(\mathfrak{p}^\varepsilon_{z_1})(\mathfrak{p}^\varepsilon_{z_2}) + \mathfrak{c}^\varepsilon(\mathfrak{p}^\varepsilon_{z_2})^2 + \mathfrak{c}^\varepsilon\mathfrak{p}^\varepsilon_{z_1}\mathfrak{q}^\varepsilon_{z_2} - \mathfrak{c}^\varepsilon\mathfrak{p}^\varepsilon_{z_1}\mathfrak{q}^\varepsilon_{z_2}\right) \\
&= \frac{1}{c}\left(\left[\mathfrak{a}^\varepsilon(\mathfrak{p}^\varepsilon_{z_1})^2 + 2\mathfrak{b}^\varepsilon(\mathfrak{p}^\varepsilon_{z_1})(\mathfrak{p}^\varepsilon_{z_2}) + \mathfrak{c}^\varepsilon\mathfrak{p}^\varepsilon_{z_1}\mathfrak{q}^\varepsilon_{z_2}\right] + \mathfrak{c}^\varepsilon\left[(\mathfrak{p}^\varepsilon_{z_2})^2 - \mathfrak{p}^\varepsilon_{z_1}\mathfrak{q}^\varepsilon_{z_2}\right]\right) \\
&= \frac{1}{c}(0 + \mathfrak{c}^\varepsilon J_{\mathfrak{h}^\varepsilon}) = \frac{\mathfrak{c}^\varepsilon}{c}J_{\mathfrak{h}^\varepsilon}.
\end{aligned} \tag{3.64}$$

where $J_{\mathfrak{h}^\varepsilon} = \mathfrak{p}^\varepsilon_{z_2}\mathfrak{q}^\varepsilon_{z_1} - \mathfrak{p}^\varepsilon_{z_1}\mathfrak{q}^\varepsilon_{z_2}$ is the Jacobian determinant of $\mathfrak{h}^\varepsilon$. Similarly we have,

$$\tfrac{1}{c}\left((\mathfrak{q}^\varepsilon_{z_1})^2 + (\mathfrak{q}^\varepsilon_{z_2})^2\right) \le \frac{\mathfrak{a}^\varepsilon}{c} J_{\mathfrak{h}^\varepsilon} \text{ for } \nu \text{ a.e in } B(w, 2r). \tag{3.65}$$

Then adding (3.64) and (3.65) we deduce

$$\|D\mathfrak{h}^\varepsilon\|^2 \le K\, J_{\mathfrak{h}^\varepsilon}(z),\ \nu \text{ a.e in } B(w, 2r), \tag{3.66}$$

for some constant K which depends on the constant c in (2.24).

Therefore $\mathfrak{h}^\varepsilon$ is a $K-$quasiregular mapping. Then $u_\varepsilon \in W^{2,2}(B(w, 2r))$ with norm independent of ε. Also ∇u_ε is $\alpha-$Hölder continuous where $\alpha = K - \sqrt{K^2-1}$ with constant independent of ε (see [22]).

Since $\nabla u_\varepsilon \to \nabla u$ in $W^{1,p}(B(w, 2r))$, then for some subsequence, $\varepsilon^i \to 0$ we have $\nabla u_{\varepsilon_i} \to \nabla u$ a.e in $B(w, 2r)$. $\{\nabla u_{\varepsilon_i}\}$ is equicontinuous as $\{\nabla u_{\varepsilon_i}\}$ is uniformly Hölder continuous with constant independent of ε. We may redefine ∇u in a set of measure zero if needed, $\nabla u_{\varepsilon_{i_k}} \to \nabla u$ on compact subsets of $B(w, 2r)$. Then it follows from [22] that ∇u is a $K-$quasiregular mapping. From quasiregularity we also have

$$\|\nabla u\|_{W^{1,2}(B(w,r)\cap\tilde\Omega)} \le c\|\nabla u\|_{L^2(B(w,\frac{3r}{2})\cap\tilde\Omega)} \tag{3.67}$$

where $c = c(p)$, and ∇u is $\alpha-$Hölder continuous. Using these facts and basic Cacciopoli type estimates for u_ξ we deduce the following lemma,

Lemma 3.3.1 (Local interior regularity for ∇u)**.** *Let $u, f, \tilde\Omega$ be as in Lemma 3.1.10 with $\tilde u, \tilde f$ replaced by u, f respectively. If $B(\tilde w, 4s) \subset B(w, 4r)\cap\tilde\Omega$, then u has a representative with Hölder continuous derivatives in $B(\tilde w, 2s)$ (also denoted u). Moreover there exists σ, $0 < \sigma < 1$, and $c \ge 1$, depending only on f and p, with*

$$|\nabla u(\tilde z) - \nabla u(\hat z)| \le c\left(\frac{|\tilde z - \hat z|}{s}\right)^\sigma \operatorname*{ess\,sup}_{B(\tilde w,s)} |\nabla u| \le \frac{c}{s}\left(\frac{|\tilde z - \hat z|}{s}\right)^\sigma \operatorname*{ess\,sup}_{B(\tilde w,s)} u. \tag{3.68}$$

Also if $\nabla u \ne 0$ in $B(\tilde w, 2s)$, then

$$\int_{B(\tilde w,s)} |\nabla u|^{p-2}\sum_{k,j=1}^{2}(u_{z_k z_j})^2 \mathrm{d}\nu \le \frac{c}{(t-s)^2}\int_{B(\tilde w,t)} |\nabla u|^p \mathrm{d}\nu \tag{3.69}$$

for $s < t < 2s$.

Lemma 3.3.2. *Let $u, f, \tilde\Omega, w, r$ be as in Lemma 3.1.10 with u, f replaced by $\tilde u, \tilde f$ respectively. Let $\nabla u \ne 0$ in $B(\tilde w, 4s) \subset B(w, 4r)\cap\tilde\Omega$.*

Then $h(z) = \log|\nabla u|(z)$ is a weak solution to a uniformly elliptic divergence form partial differential equation for which a Harnack inequality holds.

Proof. As ∇u is a $K-$quasiregular mapping and by assumption $\nabla u \ne 0$ in $B(\tilde w, 4s)$ then $h(z)$ is well-defined in $B(\tilde w, 4s)$ and is a weak solution to

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial z_i}\left(\mathcal{A}_{ij}h_{z_j}\right) = 0 \text{ in } B(\tilde w, 4s) \tag{3.70}$$

where $(\mathcal{A}_{ij}) = \mathcal{A}$, $D^2u = (\frac{\partial^2 u}{\partial z_j \partial z_i})$, and

$$\mathcal{A} = \begin{cases} \det D^2u \, \left(D^2u^{\mathrm{T}} \, D^2u\right)^{-1} & \text{if } D^2u \text{ is invertible,} \\ \text{Identity matrix} & \text{otherwise} \end{cases}$$

(for more details see [17, Chapter 14]). It follows from an observation in [17, Theorem 14.61] and $K-$quasiregularity of ∇u that

$$\tfrac{1}{c}|\eta|^2 \le \mathcal{A}\eta \cdot \eta \le c|\eta|^2 \text{ a.e in } B(\tilde{w}, 4s) \text{ and for all } \eta \in \mathbb{R}^2.$$

Therefore $h = \log|\nabla u|$ is a weak solution to a uniformly elliptic partial differential equation in divergence form in $B(\tilde{w}, 4s)$ from which we conclude Harnack's inequality can be applied to h in $B(\tilde{w}, 4s)$ when $h > 0$. □

3.4 A Proof of $\nabla u \neq 0$ in D using the Principle of the Arguments

In this section, using the principle of the argument we give a proof that $\nabla u \neq 0$ in D. Indeed, we use the principle of the argument for a $K-$quasiregular mapping.

Let u be the capacitary function and Ω as in Lemma 3.4 where $\tilde{u}, \tilde{\Omega}$ replaced u, Ω respectively. Since (1.7) is invariant under dilation and translation we may assume that $0 \in \Omega$, $D = \Omega \setminus \overline{B(0,1)}$, and $d(0, \partial\Omega) = 4$. Let $u_z = u_{z_1} - \mathrm{i}u_{z_2}$. Then from section 3.3 we have u_z is a $K-$quasiregular mapping and therefore the zeros of u_z are isolated and countable in D. Therefore, there exist $0 < t_0 < t_1 < 1$ with t_0 is arbitrarily close to 0 and t_1 is arbitrarily close to 1. Moreover, we have $u_z \neq 0$ on $\gamma_j = \{z \in D;\ u(z) = t_j\}$ for $j = 0, 1$. $K-$quasiregularity of u_z implies that u_z is $\alpha-$Hölder continuous for some $0 < \alpha < 1$. Then γ_j, $j = 0, 1$, is a $C^{1,\alpha}$ Jordan curve and without loss of generality we can assume that γ_j is oriented counterclockwise for $j = 0, 1$.

Let $\Gamma_j = u_z(\gamma_j)$ for $j = 0, 1$. We claim that

$$\frac{1}{2\pi \mathrm{i}} \left(\int_{\Gamma_0} \frac{\mathrm{d}w}{w} - \int_{\Gamma_1} \frac{\mathrm{d}w}{w} \right) = \#\, \text{of zeros of } u_z \text{ in } \{z \in D;\ t_0 < u_z < t_1\} \tag{3.71}$$

Indeed, (3.71) is well-known if u_z is analytic as follows from the "principle of the argument".

We prove (3.71) using this idea and the Stoïlov factorization theorem 2.1.3, that is

$$u_z(z) = h \circ g(z),\ \ z \in D \tag{3.72}$$

where h is an analytic function in $g(D)$ and g is a $K-$quasiconformal mapping of D. Then

$$\partial g(\{z \in D;\ t_0 < u(z) < t_1\}) = \tau_0 \cup \tau_1 = (g \circ \gamma_0) \cup (g \circ \gamma_1) \tag{3.73}$$

where $\tau_j = g \circ \gamma_j$ is a C^β Jordan curve for some $0 < \beta < 1$, oriented counterclockwise for $j = 0, 1$. Applying the principle of the argument to h as in (3.71) we claim that

$$\frac{1}{2\pi \mathrm{i}} [\triangle \arg (h \circ \tau_0) - \triangle \arg (h \circ \tau_1)] = \#\, \text{of zeros of } h \text{ in } g(\{z \in D;\, t_0 < |z| < t_1\}). \tag{3.74}$$

Here $\triangle \arg (h \circ \tau_j)$, $j = 0, 1$, denotes the change in the argument of $h \circ \tau_j$ as τ_j is traversed counterclockwise.

To prove claim (3.74) we first show that $\triangle \arg h(\tau_j) = \triangle \arg h(\sigma_j)$ for $j = 0, 1$ where σ_j is a Jordan curve. That is homotopic to τ_j by a deformation which does not pass through the zeros of u_z. Choosing σ_j sufficiently smooth we get (3.74) from this observation and the principle of the argument in [1]. (3.71) follows from the fact that g^{-1} is a homeomorphism of $\mathbb{C}$ onto $\mathbb{C}$ and (3.74).

Moreover, let $z_j(s)$, $0 \le s \le 1$ be a parametrization of γ_j for $j = 0, 1$. Since γ_j is $C^{1,\alpha}$ we have

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}s}(t_j) &= \frac{\mathrm{d}}{\mathrm{d}s}(u(z_j(s))) \\ &= u_z \frac{\mathrm{d}z_j(s)}{\mathrm{d}s} + u_{\overline{z}} \frac{\mathrm{d}\overline{z_j}(s)}{\mathrm{d}s} \\ &= 2\,\mathrm{Re}[u_z \frac{\mathrm{d}z_j(s)}{\mathrm{d}s}]. \end{aligned} \tag{3.75}$$

Therefore, $u_z \frac{\mathrm{d}z_j(s)}{\mathrm{d}s}$ is always pure imaginary on γ_j, $j = 0, 1$, and so

$$\begin{aligned} 0 &= \triangle \arg [u_z \frac{\mathrm{d}z_j(s)}{\mathrm{d}s}] \\ &= \triangle \arg u_z(\gamma_j) + \triangle \arg \frac{\mathrm{d}z_j(s)}{\mathrm{d}s}. \end{aligned} \tag{3.76}$$

From (3.76) we see that

$$\triangle \arg u_z(\gamma_j) = -\triangle \arg \frac{\mathrm{d}z_j(s)}{\mathrm{d}s} \tag{3.77}$$

Finally, as γ_j, $j = 0, 1$ is a Jordan curve oriented counterclockwise, it follows from the Gauss-Bonnet Theorem that

$$\frac{1}{2\pi} \triangle \arg \frac{\mathrm{d}z_j}{\mathrm{d}s} = 1 \text{ for } j = 1, 2. \tag{3.78}$$

One way to prove (3.78) using analytic function theory is to use the Riemann mapping theorem to first get ψ_j mapping $\{z \ |z| < 1\}$ onto $G_j =$: inside of γ_j, $j = 0, 1$. As in [27] it follows that ψ_j extends to a $C^{1,\beta}$ homeomorphism of $\{z \ |z| \le 1\}$ onto $\overline{G_j}$. Then we can put

$$z_j(s) = \psi_j(e^{2\pi \mathrm{i} s}),\ 0 \le s \le 1, \tag{3.79}$$

and observe that

$$\frac{\mathrm{d}z_j(s)}{\mathrm{d}s} = 2\pi\mathrm{i}\psi_j'(e^{2\pi\mathrm{i}s})e^{2\pi\mathrm{i}s}. \tag{3.80}$$

Then on $\{z;\ |z| = 1\}$ we have

$$\begin{aligned} \triangle \arg \frac{\mathrm{d}z}{\mathrm{d}s} &= \triangle \arg \psi_j'(z) + \triangle \arg z \\ &= 0 + 2\pi = 2\pi. \end{aligned} \tag{3.81}$$

In view of (3.77), (3.78), (3.72), and (3.74) we conclude $u_z \neq 0$ in $G_1 \setminus G_0$, i.e between the level sets γ_0 and γ_1. Using this observation and letting $t_0 \to 0$, and $t_1 \to 1$ in (3.71) we have the desired result, $u_z \neq 0$ in D.

Chapter 4 A Pointwise Estimate and A Weak Solution Argument

4.1 Fundamental Inequality

In this section we prove the so called fundamental inequality. We mimic the proof given in [25] and modify it to our case.

We first give the definitions of the hyperbolic metric and the quasi-hyperbolic metric.

Definition 4.1.1 (The Hyperbolic Distance)**.** Let $z_1, z_2 \in \mathbb{D}$ and let Γ be the set of all arcs in $\mathbb{D}$ connecting z_1 to z_2.

The **hyperbolic distance** $\rho_{\mathbb{D}}(z_1, z_2)$ from z_1 to z_2 is

$$\rho_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma \in \Gamma} \int\limits_{\gamma} \frac{|\mathrm{d}z|}{1 - |z|^2}. \tag{4.1}$$

As the hyperbolic distance is conformally invariant, it can be defined in a simply connected domain $\Omega \neq \mathbb{C}$ by moving back to $\mathbb{D}$ via a conformal map $\phi : \mathbb{D} \to \Omega$. That is

$$\rho_\Omega(w_1, w_2) = \rho_{\mathbb{D}}(z_1, z_2)$$

when $w_i = \phi(z_i)$, for $i = 1, 2$. Moreover, $\rho_\Omega(w_1, w_2)$ is independent of the choice of the conformal map ϕ.

Definition 4.1.2 (Quasi-hyperbolic Distance)**.** Let $w_1, w_2 \in \Omega$ and let Γ' be the set of all arcs in Ω connecting w_1 to w_2.

The **Quasi-hyperbolic distance** $Q_\Omega(w_1, w_2)$ between w_1 and w_2 is

$$Q_\Omega(w_1, w_2) = \inf_{\gamma \in \Gamma'} \int\limits_{\gamma} \frac{|\mathrm{d}w|}{\operatorname{dist}(w, \partial\Omega)}. \tag{4.2}$$

We next give two estimates of Koebe (A proof of these estimates can be found in [12, Theorem 4.3 and Theorem 4.5]).

Theorem 4.1.3 (Koebe's estimate)**.** *Let $h(z)$ be a conformal mapping from the unit disc $\mathbb{D}$ onto a simply connected domain Ω. Then for all $z \in \mathbb{D}$*

$$\tfrac{1}{4}|h'(z)|(1 - |z|^2) \leq \operatorname{dist}(h(z), \partial\Omega) \leq |h'(z)|(1 - |z|^2). \tag{4.3}$$

It follows from Koebe's estimate that the hyperbolic distance is comparable to the quasi-hyperbolic distance. That is, if we let $w = h(z)$ then (4.3) can be written as

$$\frac{4|\mathrm{d}z|}{1 - |z|^2} \geq \frac{|h'(z)||\mathrm{d}z|}{d(h(z), \partial\Omega)} = \frac{|\mathrm{d}w|}{d(w, \partial\Omega)} \geq \frac{|\mathrm{d}z|}{1 - |z|^2} \tag{4.4}$$

Then from (4.4) we see

$$\rho_\Omega \leq Q_\Omega \leq 4\rho_\Omega. \tag{4.5}$$

Theorem 4.1.4 (Koebe's Theorem)**.** *Let $h(z)$ be a univalent function, that is h is analytic and one-to-one in $\mathbb{D}$. Assume also that $h(0) = 0$ and $h'(0) = 1$. Then*

$$\frac{1-|z|}{(1+|z|)^3} \leq |h'(z)| \leq \frac{1+|z|}{(1-|z|)^3}. \tag{4.6}$$

We give a distortion estimate which is an application of Koebe's Theorem 4.1.4. That is, for every conformal mapping $h : B(0,1) \to \mathbb{C}$, and for every $z_1, z_2 \in \Omega \setminus B(z_0, r)$, where $2r = d(z_0, \partial\Omega)$, then

$$\rho_\Omega(z_1, z_2) \leq c_1 \implies |h'(h^{-1}(z_2))| \leq c_2 |h'(h^{-1}(z_1))| \tag{4.7}$$

for some $c_2 = c_2(c_1)$.

Next we give the so called fundamental inequality.

Fundamental Inequality 4.1.5. *Fix p, $1 < p < \infty$, let u be a capacitary function defined before Lemma 3.2.1 for $D = \Omega \setminus \overline{B(z_0, d(z_0, \partial\Omega)/4)}$. Then there is a constant $c = c(f, p)$ such that*

$$\frac{1}{c}\frac{u(z)}{d(z, \partial\Omega)} \leq |\nabla u(z)| \leq c\frac{u(z)}{d(z, \partial\Omega)} \tag{4.8}$$

whenever $z \in D$ and $d(z, \partial\Omega) \geq \frac{d(z_0, \partial\Omega)}{2}$.

Proof. As u is a capacitary function then by subsection 3.4 we have $\nabla u \neq 0$ in D. To prove (4.8) we show that for $\tilde{z} \in D \setminus B(z_0, d(z_0, \partial\Omega)/2)$ there is $z^\star \in D$ such that

$$u(z^\star) = \frac{u(\tilde{z})}{2} \text{ with } \rho_\Omega(\tilde{z}, z^\star) \leq c \text{ for some } c = c(p, f). \tag{4.9}$$

We postpone the proof of existence of such a point $z^\star$ in (4.9)to the next subsection. At the moment, suppose that we have such a $z^\star$, and let Γ be the hyperbolic geodesic connecting $\tilde{z}$ to $z^\star$.

Define

$$\gamma = \begin{cases} \Gamma & \text{if } \Gamma \cap B(z_0, 3d(z_0, \partial\Omega)/8) = \emptyset \\ \gamma_1 + \gamma_2 + \gamma_3 & \text{otherwise} \end{cases} \tag{4.10}$$

where γ_1 is the subarc of Γ joining $\tilde{z}$ to the first point, P_1 on $\partial\Omega(z_0, 3d(z_0, \partial\Omega)/8)$, and γ_2 is the shortest arc of $\partial B(z_0, 3d(z_0, \partial\Omega)/8)$ joining P_1 to P_2 where P_2 is the last point of intersection of $\partial B(z_0, 3d(z_0, \partial\Omega)/8)$ and Γ. γ_3 is the subarc of Γ joining P_2 to $z^\star$(see figure 4.1).

Using Koebe's distortion theorem 4.1.4 there is $c = c(f, p)$ such that

$$H^1(\gamma) \leq cd(\tilde{z}, \partial\Omega) \text{ and } \tfrac{1}{c}d(\tilde{z}, \partial\Omega) \leq d(\gamma, \partial\Omega). \tag{4.11}$$

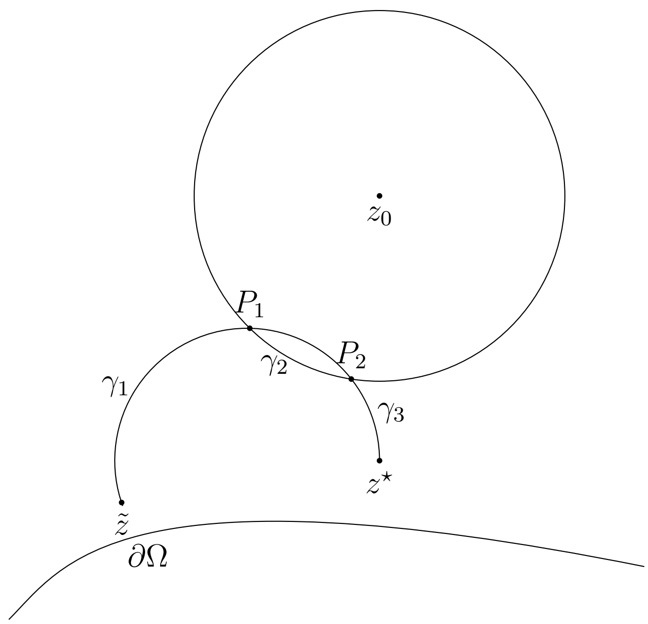

Figure 4.1: The hyperbolic geodesic γ from $\tilde{z}$ to $z^\star$ when $\Gamma \cap B(z_0, 3d(z_0, \partial\Omega)/8) \neq \emptyset$

Using (4.11) we see

$$\begin{aligned} \tfrac{1}{2}u(\tilde{z}) &= u(\tilde{z}) - u(z^\star) \\ &\leq \int_\gamma |\nabla u(z)||\mathrm{d}z| \\ &\leq c\, H^1(\gamma) \operatorname*{ess\,sup}_\gamma |\nabla u| \\ &\leq c\, d(\tilde{z}, \partial\Omega) \operatorname*{ess\,sup}_\gamma |\nabla u|. \end{aligned} \tag{4.12}$$

Thus for some $\hat{z} \in \gamma$ we have

$$\tfrac{1}{c}\frac{u(\tilde{z})}{d(\tilde{z}, \partial\Omega)} \leq |\nabla u(\hat{z})|. \tag{4.13}$$

where $c = c(c, p) > 1$. Moreover, it follows from Vitali's covering argument and (4.11) that there is $\{z_i, r_i\}_{i=1}^n$ with $z_i \in \gamma$ such that

$$\begin{aligned} &\gamma \subset \bigcup_i^n B(z_i, r_i), \\ &B(z_i, r_i) \bigcap B(z_{i+1}, r_{i+1}) \neq \emptyset \text{ for } i = 1 \ldots, n-1, \text{ and} \\ &r_i \approx d(B(z_i, 4r_i), \partial\Omega) \approx d(\tilde{z}, \partial\Omega) \end{aligned} \tag{4.14}$$

where $n = n(f,p)$ is an absolute constant. By Harnack's inequality applied to u, we get $u(z) \approx u(\tilde{z})$ for $z \in \cup_{i=1}^{n} B(z_i, 4r_i)$. Using Lemma 3.3.1 and (4.14) we see

$$|\nabla u(z)| \leq c \frac{u(\tilde{z})}{d(\tilde{z}, \partial\Omega)} \text{ for } z \in \bigcup_{i=1}^{n} B(z_i, 2r_i). \tag{4.15}$$

Then for large $c = c(p,f) \geq 1$ we have

$$h(z) = \log\left(\frac{cu(\tilde{z})}{d(\tilde{z}, \partial\Omega)|\nabla u(z)|}\right) > 0 \text{ whenever } z \in \cup_{i=1}^{n} B(z_i, 2r_i). \tag{4.16}$$

Choose $j \in \{1, \ldots, n\}$ such that $\hat{z} \in B(z_j, 2r_j)$. Then from (4.13) we have $h(\tilde{z}) \leq c$. Also from Lemma 3.3.2 and (4.16) it follows that h is a positive weak solution to a uniformly elliptic divergence form partial differential equation for which a Harnack's inequality holds. Therefore we can apply Harnack's inequality in $B(z_j, 2r_j)$ to get

$$\begin{aligned} h(z) &\leq \operatorname*{ess\,sup}_{B(z_j, r_j)} h(z) \\ &\leq \tilde{c} \operatorname*{ess\,inf}_{B(z_j, r_j)} h \\ &\leq \hat{c} \end{aligned} \tag{4.17}$$

From (4.17) we have

$$\frac{u(\tilde{z})}{d(\tilde{z}, \partial\Omega)} \leq c|\nabla u(z)| \text{ for every } z \in B(z_j, r_j). \tag{4.18}$$

It follows from (4.14) and finitely many repeated use of (4.18) the Harnack's inequality as in (4.17) to $\tilde{z}$ that we have (4.8) for $\tilde{z} \in D \setminus B(z_0, d(z_0, \partial\Omega)/2)$.

To finish the proof of Lemma 4.1.5 , it remains to show that such a $z^\star$ defined in (4.9) exists. □

Existence of $z^\star$

We can approximate Ω by a sequence of domains $\{\Omega_n\}$ with $\Omega_1 \subset \Omega_2 \ldots \subset \Omega_n \subset \ldots \subset \Omega$ such that $\partial\Omega_i$ is Jordan curve for $i = 1, 2, \ldots,$. Moreover, if we define u_n as the capacitary function for $D_n = \Omega_n \setminus B(z_0, r)$ where $2r = d(z_0, \partial\Omega)$. If we let $n \to \infty$ then it follows from the Hölder continuity of u_n that there are subsequences $u_{n_k} \to u$ and $\nabla u_{n_k} \to \nabla u$ uniformly on compact subsets of Ω. Therefore, without loss of generality we can assume that $\partial\Omega$ is an analytic Jordan curve.

Let $h : \mathbb{H} \to \Omega$ be a Riemann mapping which has a continuous extension from $\overline{\mathbb{H}}$ to $\overline{\Omega}$. We can choose h so that for some s, $0 < s < 1$, $h(\mathrm{i}s) = \tilde{z}$ and $h(\mathrm{i}) = z_0$. If $\mathfrak{U} = u \circ h$ then $\mathfrak{U}$ satisfies a maximum principle and Harnack's inequality since u has these properties.

Define the box

$$B(\mathrm{i}s) = \{z = w_1 + \mathrm{i}w_2 : -s \leq w_1 \leq s, \text{ and } 0 < w_2 < s\}. \tag{4.19}$$

In [25] it is shown that $B(\mathrm{i}s)$ can be shifted to a nearby box $\tilde{B}(\mathrm{i}s)$ with some nice properties. To outline these properties let the boundary of $\tilde{B}(\mathrm{i}s)$ on the upper half plane be Υ which consists of line segment from $w_1 + \mathrm{i}s$ to $w_2 + \mathrm{i}s$ and line segments from $w_1 + \mathrm{i}s$ to w_1 and $w_2 + \mathrm{i}s$ to w_2 for some $-s < w_1 < -s/2$ and $s/2 < w_2 < s$ (see Figure 4.2).

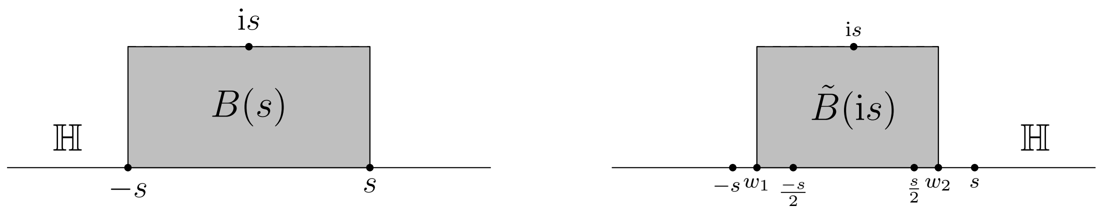

Figure 4.2: The Boxes $B(s)$ and $\tilde{B}(\mathrm{i}s)$.

In [25] the authors show that $\mathfrak{U} \leq c\,\mathfrak{U}(\mathrm{i}s)$ on Υ. Then by the maximum principle we see that $\mathfrak{U} \leq c\,\mathfrak{U}(\mathrm{i}s)$ in $\tilde{B}(\mathrm{i}s)$ for some $c = c(p, f)$. It follows from this observation that

$$u \leq C\,u(\tilde{z}) \text{ in } Q(s). \tag{4.20}$$

where $Q(s) = h(\tilde{B}(\mathrm{i}s))$. Also if $\Phi = h(\Upsilon)$, then proceeding as in [25], we deduce that there is an absolute constant c_1 such that

$$H^1(\Phi) \leq c_1\, d(\tilde{z}, \partial\Omega). \tag{4.21}$$

and there is also c_2 and w_0 such that $-s/4 < w_0 < s/4$ satisfying

$$|h(w_0) - \tilde{z}| \leq c_2\, d(\tilde{z}, \partial\Omega) \text{ and } \tfrac{1}{c_2}\, d(\tilde{z}, \partial\Omega) \leq d(h(w_0), \Phi). \tag{4.22}$$

Finally, there is a Lipschitz curve $\Lambda : [0,1] \to Q(s)$ and c_3 with $\Lambda(0) = \tilde{z}$ and $\Lambda(1) = h(w_0)$ satisfying

$$\min\{H^1(\Lambda[0,t]), H^1(\Lambda[t,1])\} \leq c_3 \min\{d(\tilde{z}, \partial\Omega), d(\Lambda(t), \partial\Omega)\}. \tag{4.23}$$

Let Γ be parametrized by $[0,1]$ with $\Gamma(0) = \mathrm{i}s$ and $\Gamma(1) = w_0$ (see figure 4.3). If we set

$$t^\star = \max\{t : \mathfrak{U}(\Gamma(t)) = \tfrac{1}{2}\mathfrak{U}(\mathrm{i}s)\} \tag{4.24}$$

then $z^\star = h(\Gamma(t^\star))$. By definition, we get $u(z^\star) = u(\tilde{z})/2$.

It remains to show that $\rho_\Omega(\tilde{z}, z^\star) \leq c$ for some absolute constant $c = c(p)$. To this end, set $\tilde{r} = d(h(w_0), \Phi)$. Then by definition of Φ we have $B(h(w_0), \tilde{r}) \cap \Omega \subset \Phi$. Using Hölder continuity of u restricted to Φ (see Corollary 3.1.13) and (4.21)-(4.23) we obtain

$$\begin{aligned} \tfrac{1}{2}u(\tilde{z}) = u(z^\star) &\leq c \left(\frac{d(z^\star, \partial\Omega)}{\tilde{r}}\right)^{\alpha} \operatorname*{ess\,sup}_{B(h(w_0),\tilde{r}))\cap\Omega} u \\ &\leq \tilde{c} \left(\frac{d(z^\star, \partial\Omega)}{d(\tilde{z}, \partial\Omega)}\right)^{\alpha} u(\tilde{z}). \end{aligned} \tag{4.25}$$

From (4.25) for some constant $c = c(p)$ we see that

$$d(\tilde{z}, \partial\Omega) \leq c\, d(z^\star, \partial\Omega). \tag{4.26}$$

It follows from (4.23) and (4.26) that there is a chain of at most $c = c(p, f)$ balls connecting $\tilde{z}$ to $z^\star$. Using this observation and (4.5) we obtain $\rho_\Omega(\tilde{z}, z^\star) \leq c$.

We next give a proof for (4.20) by a contradiction argument. That is, suppose for suitably large $\tilde{C} = \tilde{C}(p)$ that $u(\tilde{z}) < \tilde{C}\, u$ on Φ. Let $\tilde{B}(\mathrm{i}s)$ be the shifted box and set $b_{1,1} = b_{1,1}(s) = w_1 + \mathrm{i}\delta_\star s$ and $b_{2,1} = b_{2,1}(s) = w_2 + \mathrm{i}\delta_\star s$ for some fixed constant $\delta_\star$ depending on p. Observe that $b_{i,1}$ is on the vertical side of Φ for $i = 1, 2$. There exist $\tilde{B}(b_{i,1})$ generated by $b_{i,1}$ for $i = 1, 2$. Then these boxes $\tilde{B}(b_{i,1})$ will generate one more box $\tilde{B}(b_{i,2})$ for $i = 1, 2$, and so on.

We choose a polygonal path $\Gamma_{1,1}$ from $b_{1,1}$ to some point in $w_{11} \in [\operatorname{Re} b_{1,1} - \operatorname{Im} b_{1,1}, \operatorname{Re} b_{1,1} + \operatorname{Im} b_{1,1}]$ in a way that the path $\Gamma_{1,1}$ is in the left half plane $\{\operatorname{Re} z < \operatorname{Re} b_{1,1}\}$.

Similarly, we choose a polygonal path $\Gamma_{2,1}$ from $b_{2,1}$ to some point in $w_{21} \in [\operatorname{Re} b_{2,1} - \operatorname{Im} b_{2,1}, \operatorname{Re} b_{2,1} + \operatorname{Im} b_{2,1}]$ in a way that the path $\Gamma_{2,1}$ is in the plane $\{\operatorname{Re} z > \operatorname{Re} b_{2,1}\}$ (See figure 4.3 for the construction).

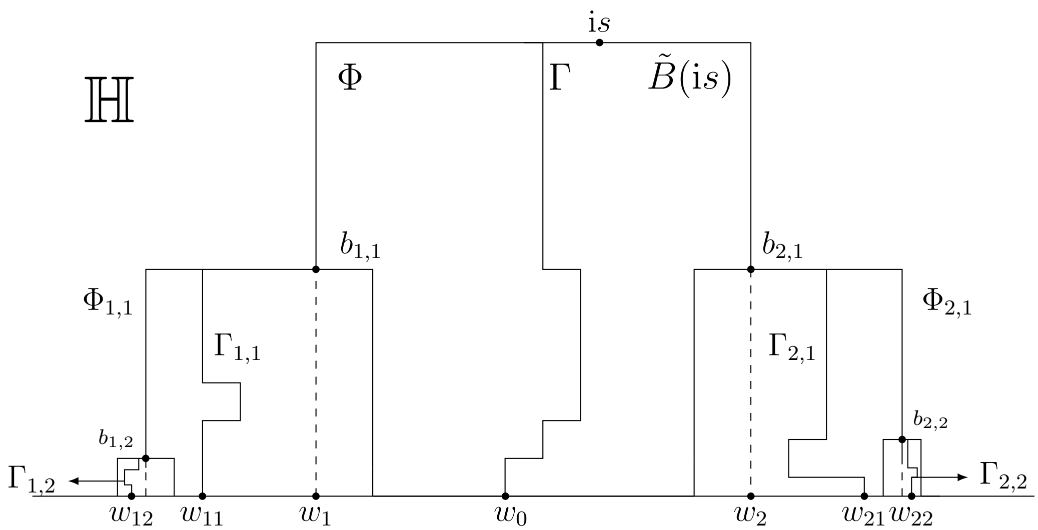

Figure 4.3: Recursive Construction of the Boxes

It follows from Harnack's inequality that there is a constant c' depending only on p and f such that

$$c' u(h(\mathrm{i}s)) \leq u(h(z)) \text{ whenever } z \in \Phi \text{ and } \delta_\star\, s \leq \operatorname{Im}\, z. \tag{4.27}$$

The constant $\tilde{C}$ is assumed to be so large that $\tilde{C} > c'$. Therefore we have $C\,\mathfrak{U}(\mathrm{i}s) < \mathfrak{U}(z)$ for some $z \in \Gamma_{1,1} \cup \Gamma_{2,1}$.

Without loss of generality assume that $z \in \Gamma_{1,1}$ (proof for $z \in \Gamma_{2,1}$ is exactly the same). We see that when $\tilde{C}$ gets larger, then z gets closer to the real axis. That is, if $K^n < \tilde{C}$ for some sufficiently large K and integer n we have $\operatorname{Im} z \le \delta_\star^n$. Now it follows from the construction of $\Gamma_{1,1}$ as in [25] that there a is δ depending only on p, such that if $\delta^\star = e^{-\frac{c}{\delta}}$ for some constant c, then

$$|h(z) - h(w_{11})| \le \tilde{c}\delta^{k-1} d(h(b_{1,1}), \partial\Omega).$$

Proceeding as in finding $z^\star$, let $\Phi_{1,1}$ be the boundary of $\tilde{B}(b_{1,1})$ which stays in $\mathbb{H}$ and if we set $\tilde{r}_{1,1} = d(f(w_{11}), f(\Gamma_{1,1}))$ then $B(f(w_{1,1}), \tilde{r}_{1,1}) \cap \Omega \subset f(\tilde{B}(b_{1,1}))$. Then it follows from Hölder continuity of u on $\Phi_{1,1}$ (see Lemma 3.1.13) that

$$(u \circ h)(z) \le \tilde{C}\delta^{\alpha n} \operatorname*{ess\,sup}_{\tilde{B}(b_{1,1})} (u \circ h).$$

Choose n_0 to be the least positive integer satisfying $\tilde{C}\delta^{\alpha n} < K^{-1}$. For this choice of $\tilde{C} > K^{n_0}$ we see

$$\tilde{C}K\,\mathfrak{U}(\mathrm{i}s) < K\mathfrak{U}(z) < \operatorname*{ess\,sup}_{\Gamma_{1,1}} \mathfrak{U}. \tag{4.28}$$

It follows from (4.28) and $\mathfrak{U}(b_{1,1}) \le K\mathfrak{U}(\mathrm{i}s)$ that we can repeat the same argument replacing $\tilde{B}(\mathrm{i}s)$ with $\tilde{B}(b_{1,1})$. So we find $b_{1,2}$ on the vertical side of $\Phi_{1,1}$ in $\mathbb{H}$ with $\operatorname{Im} b_{1,2} = \delta_\star^2\, s$ and a box $\tilde{B}(b_{1,2})$ with boundary $\Gamma_{1,2}$ satisfying

$$\tilde{C}\,\mathfrak{U}(b_{1,2}) \le K^2\tilde{C}\,\mathfrak{U}(\mathrm{i}s) \le \operatorname*{ess\,sup}_{\Gamma_{1,2}} \mathfrak{U}. \tag{4.29}$$

If we recursively continue then we get a contradiction as $\mathfrak{U} = 0$ continuously on $\mathbb{R}$. Therefore (4.20) holds.

4.2 $\log f(\nabla u)$ is a weak sub solution, solution or super solution to L

We first need some definitions.

Let $\mathcal{A}$ be a elliptic equations of the form

$$\nabla \cdot \mathcal{A}(\nabla u) = 0 \tag{4.30}$$

where $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping satisfying certain structural assumptions.

Definition 4.2.1 (Weak Solution)**.** A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a **weak solution** of (4.30) in Ω if

$$-\int_\Omega \langle \mathcal{A}(\nabla u), \nabla\phi\rangle \mathrm{d}\nu = 0$$

whenever $\phi \in W^{1,p}_0(\Omega)$.

Definition 4.2.2 (Weak sub solution)**.** A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a **weak sub solution** of (4.30) in Ω if

$$\nabla \cdot \mathcal{A}(\nabla u) \ge 0$$

weakly in Ω, i.e

$$-\int_{\Omega} \langle \mathcal{A}(\nabla u), \nabla \phi \rangle \mathrm{d}\nu \geq 0.$$

for every nonnegative $\phi \in W_0^{1,p}(\Omega)$.

Definition 4.2.3 (Weak super solution)**.** A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a **weak super solution** of (4.30) in Ω if

$$\nabla \cdot \mathcal{A}(\nabla u) \leq 0$$

weakly in Ω, i.e

$$-\int_{\Omega} \langle \mathcal{A}(\nabla u), \nabla \phi \rangle \mathrm{d}\nu \leq 0.$$

for every nonnegative $\phi \in W_0^{1,p}(\Omega)$.

When f in (1.2) is smooth enough and homogeneous of degree p in $\mathbb{C} \setminus \{0\}$ and u is smooth enough as well as pointwise solution to (4.31), then in [2, Theorem 1] it is shown that $\log f(\nabla u)$ is a sub solution, solution or super solution to a partial differential equation respectively when $2 < p < \infty$, $p = 2$ or $1 < p < 2$. In this section we show that $\log f(\nabla u)$ is a weak sub solution, solution or super solution to a partial differential equation $L\zeta = 0$ when

$$L\zeta \ = \sum_{k,j=1}^{2} \frac{\partial}{\partial z_k} \left(f_{\eta_j \eta_k}(\nabla u) \frac{\partial \zeta}{\partial z_j} \right). \tag{4.31}$$

and u as in Theorem 3.2.1 is a p capacitary function. Using Lemmas 3.1.9, 3.3.1 as well as the fundamental inequality 4.1.5 we obtain

$$\begin{aligned} 0 &= \int_{\Omega} \langle \nabla f(\nabla u), \nabla \phi_{z_l} \rangle \mathrm{d}\nu \\ &= -\int_{\Omega} \sum_{k=1}^{2} \frac{\partial (f_{\eta_k}(\nabla u))}{\partial z_l} \, \phi_{z_k} \mathrm{d}\nu \\ &= -\int_{\Omega} \sum_{k,j=1}^{2} f_{\eta_k \eta_j}(\nabla u))(u_{z_l})_{z_j} \phi_{z_k} \mathrm{d}\nu. \end{aligned} \tag{4.32}$$

From (4.32) we see that for $l = 1, 2$, $\zeta = u_{z_l}$ is a weak solution to $L\zeta = 0$.

We next show that $\zeta = u$ is also a weak solution to $L\zeta = 0$. To this end, by homogeneity of f and Euler's formula we have

$$\sum_{j=1}^{2} \eta_j f_{\eta_k \eta_j}(\eta) = (p-1) f_{\eta_k}(\eta) \ \text{ and } \ \sum_{j=1}^{2} \eta_j f_{\eta_j}(\eta) = p f(\eta) \tag{4.33}$$

for $k = 1, 2$ and for a.e. η. Then it follows from (4.32) and (4.33) that

$$\int_\Omega \sum_{k,j=1}^{2} f_{\eta_j \eta_k}(\nabla u) u_{z_j} \phi_{z_k} \, \mathrm{d}\nu = (p-1) \int_\Omega \sum_{k=1}^{2} f_{\eta_k}(\nabla u) \phi_{z_k} \mathrm{d}\nu = 0. \tag{4.34}$$

Therefore $\zeta = u$ is also a weak solution to $L\zeta = 0$. We note also that since $u, f \in W^{2,2}_{\text{loc}}(D)$ thanks to Lemmas 3.1.9, 3.3.1, and the fundamental inequality 4.1.5

$$\begin{aligned} 0 &= \int_\Omega \sum_{k=1}^{2} f_{\eta_k}(\nabla u) \phi_{z_k} \mathrm{d}\nu \\ &= - \int_\Omega \sum_{k,l=1}^{2} f_{\eta_k \eta_l}(\nabla u) u_{z_k z_l} \phi \mathrm{d}\nu \end{aligned} \tag{4.35}$$

As (4.35) holds for every $\phi \in W^{1,p}_0(\Omega)$, for ν a.e $z \in \Omega$ we have

$$0 = \sum_{k,l=1}^{2} f_{\eta_k \eta_l}(\nabla u(z)) u_{z_k z_l}(z). \tag{4.36}$$

Using both u and u_{z_l} for $l = 1, 2$ are weak solutions to $L\zeta = 0$ we show that $\zeta = \log f(\nabla u)$ is a weak sub solution, solution or super solution to $L\zeta = 0$ in (4.31) respectively when $p > 2$, $p = 2$, $p < 2$. To this end, let $v = \log f(\nabla u)$, $\mathfrak{b}_{ij} = f_{\eta_i \eta_j}(\nabla u)$ and observe that

$$\mathfrak{b}_{kj} v_{z_j} = \frac{1}{f(\nabla u)} \sum_{n=1}^{2} f_{\eta_n}(\nabla u) \mathfrak{b}_{kj} u_{z_n z_j}. \tag{4.37}$$

Using (4.37) we see

$$\begin{aligned} \int_\Omega \sum_{k,j=1}^{2} \mathfrak{b}_{kj} v_{z_j} \phi_{z_k} \, \mathrm{d}\nu &= \int_\Omega \sum_{k,j=1}^{2} \frac{1}{f(\nabla u)} \sum_{n=1}^{2} \mathfrak{b}_{kj} f_{\eta_n}(\nabla u) u_{z_n z_j} \phi_{z_k} \, \mathrm{d}\nu \\ &= - \int_\Omega \sum_{n,k,j=1}^{2} \frac{\partial}{\partial z_k} \left(\frac{f_{\eta_n}(\nabla u)}{f(\nabla u)} \right) \mathfrak{b}_{kj} u_{z_n z_j} \, \phi \, \mathrm{d}\nu \end{aligned} \tag{4.38}$$

where to get the last line in (4.38) we have used

$$0 = \int_\Omega \sum_{n,k,j=1}^{2} \mathfrak{b}_{kj} u_{z_n z_j} \frac{\partial}{\partial z_k} \left(\frac{f_{\eta_n}(\nabla u)}{f(\nabla u)} \phi \right) \mathrm{d}\nu. \tag{4.39}$$

(4.39) is a consequence of (4.32) and the fact that

$$\frac{f_{\eta_k}(\nabla u)}{f(\nabla u)} \in W^{1,2}_{\text{loc}}(\Omega),$$

thanks to Lemmas 3.1.9, 3.3.1, and the fundamental inequality 4.1.5.

From (4.38) we have

$$\begin{aligned}\int\limits_{\Omega}\sum_{k,j=1}^{2}\mathfrak{b}_{kj}v_{z_j}\phi_{z_k}\,\mathrm{d}\nu &= -\int\limits_{\Omega}\sum_{n,k,j=1}^{2}\frac{\partial}{\partial z_k}\left(\frac{f_{\eta_n}(\nabla u)}{f(\nabla u)}\right)\mathfrak{b}_{kj}u_{z_nz_j}\,\phi\,\mathrm{d}\nu\\ &= -\int\limits_{\Omega}(I'+I'')\phi\mathrm{d}\nu\end{aligned}\tag{4.40}$$

where

$$I' = \sum_{n,j,k,l=1}^{2}\frac{1}{f(\nabla u)}\,\mathfrak{b}_{nl}\mathfrak{b}_{kj}u_{z_lz_k}u_{z_nz_j}\tag{4.41}$$

and

$$I'' = -\frac{1}{f^2(\nabla u)}\sum_{n,j,k,l=1}^{2}\mathfrak{b}_{kj}f_{\eta_n}(\nabla u)f_{\eta_l}(\nabla u)u_{z_lz_k}u_{z_nz_j}.\tag{4.42}$$

We can rewrite (4.41) and (4.42) using matrix notation. Let

$$D^2f = D^2f(\nabla u(z)) = \begin{bmatrix} f_{\eta_1\eta_1} & f_{\eta_1\eta_2}\\ f_{\eta_2\eta_1} & f_{\eta_2\eta_2}\end{bmatrix} = \begin{bmatrix}\mathfrak{b}_{11} & \mathfrak{b}_{12}\\ \mathfrak{b}_{21} & \mathfrak{b}_{22}\end{bmatrix}$$

and

$$D^2u = D^2u(z) = \begin{bmatrix} u_{z_1z_1} & u_{z_1z_2}\\ u_{z_2z_1} & u_{z_2z_2}\end{bmatrix}.$$

Let

$$\nabla u = \nabla u(z) = \begin{bmatrix} u_{z_1}\\ u_{z_2}\end{bmatrix}\ \text{ and } Df = Df(\nabla u) = \begin{bmatrix} f_{\eta_1}\\ f_{\eta_2}\end{bmatrix}$$

be column vectors and set $f = f(\nabla u)$. The homogeneity conditions (4.33) can be written in the following form

$$D^2f\cdot\nabla u = (p-1)\,Df\ \text{ and }\ p(p-1)f = (\nabla u)^{\mathrm{T}}\cdot D^2f\cdot\nabla u.\tag{4.43}$$

Then (4.36) becomes

$$\operatorname{tr}\left(D^2f\cdot D^2u\right) = 0 \text{ for } \nu \text{ a.e } z \text{ in } \Omega.\tag{4.44}$$

It follows from (4.44) that there exists $\mathfrak{m},\mathfrak{n},\mathfrak{l}$ such that

$$D^2f\cdot D^2u = \begin{bmatrix}\mathfrak{m} & \mathfrak{n}\\ \mathfrak{l} & -\mathfrak{m}\end{bmatrix}\ \text{ for } \nu \text{ a.e } z \text{ in } \Omega.\tag{4.45}$$

Squaring both sides of (4.45) gives that

$$(D^2 f \cdot D^2 u)^2 = (\mathfrak{m}^2 + \mathfrak{n}\mathfrak{l}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -\det(D^2 f \cdot D^2 u) I \text{ for } \nu \text{ a.e } z \text{ in } \Omega. \tag{4.46}$$

Using (4.45), we can write (4.41) as

$$\begin{aligned} I' &= \frac{1}{f} \operatorname{tr}\left((D^2 f \cdot D^2 u)^2\right) \\ &= -\frac{\det(D^2 f \cdot D^2 u)}{f} \operatorname{tr}(I) \\ &= -2 \frac{\det(D^2 f \cdot D^2 u)}{f}. \end{aligned} \tag{4.47}$$

To handle (4.42) note from symmetry of $D^2 u$ and $D^2 f$ that

$$\sum_{k,j=1}^{2} \mathfrak{b}_{kj} u_{z_l z_k} u_{z_n z_j}$$

is the ln element of $D^2 u \cdot D^2 f \cdot D^2 u$.

So using (4.43) for ν a.e z in Ω we obtain

$$\begin{aligned} I'' &= -\frac{1}{f^2} \operatorname{tr}\left(Df \cdot (Df)^{\mathrm{T}} \cdot D^2 u \cdot D^2 f \cdot D^2 u\right) \\ &= -\frac{1}{f^2} \operatorname{tr}\left(\tfrac{1}{(p-1)^2}\, D^2 f \cdot \nabla u \cdot (D^2 f \cdot \nabla u)^{\mathrm{T}} \cdot D^2 u \cdot D^2 f \cdot D^2 u\right) \\ &= -\frac{1}{f^2} \operatorname{tr}\left(\tfrac{1}{(p-1)^2} D^2 f \cdot \nabla u \cdot (\nabla u)^{\mathrm{T}} \cdot \left(D^2 f \cdot D^2 u\right)^2\right) \\ &= \tfrac{1}{(p-1)^2} \frac{\det(D^2 f \cdot D^2 u)}{f^2} \operatorname{tr}\left(D^2 f \cdot \nabla u \cdot (\nabla u)^{\mathrm{T}}\right) \\ &= \tfrac{1}{(p-1)^2} \frac{\det(D^2 f \cdot D^2 u)}{f^2} (\nabla u)^{\mathrm{T}} \cdot D^2 f \cdot \nabla u \\ &= \tfrac{p(p-1)}{(p-1)^2} \frac{f \, \det(D^2 f \cdot D^2 u)}{f^2} \\ &= \tfrac{p}{(p-1)} \frac{\det(D^2 f \cdot D^2 u)}{f} \end{aligned} \tag{4.48}$$

where we have used

$$\operatorname{tr}(D^2 f \cdot \nabla u \cdot (\nabla u)^{\mathrm{T}}) = \sum_{l,k=1}^{2} \mathfrak{b}_{lk} u_{z_l} u_{z_k} = (\nabla u)^{\mathrm{T}} \cdot D^2 f \cdot \nabla u \tag{4.49}$$

Note that (4.47) and (4.48) imply for ν a.e z in Ω

$$\begin{aligned} I' + I'' &= -2 \frac{\det(D^2 f \cdot D^2 u)}{f} + \frac{p}{(p-1)} \frac{\det(D^2 f \cdot D^2 u)}{f} \\ &= -\left(\frac{p-2}{p-1}\right) \frac{\det(D^2 f \cdot D^2 u)}{f}. \end{aligned} \tag{4.50}$$

It follows from (2.23) that $\det(D^2 f)$ is positive for ν a.e. z in Ω. Moreover, since $\mathfrak{b}_{11} = f_{\eta_1\eta_1}$ is positive and by (3.12) $\frac{b_{11}}{|\nabla u|^{p-2}} \approx c''$ for ν a.e z in Ω, consider

$$\frac{\mathfrak{b}_{11}}{|\nabla u|^{p-2}} \det(D^2 u) = \frac{1}{|\nabla u|^{p-2}} \left(\mathfrak{b}_{11} u_{z_1 z_1} u_{z_2 z_2} - \mathfrak{b}_{11} u_{z_1 z_2}^2 \right). \tag{4.51}$$

We can rewrite (4.36) as

$$\begin{aligned} 0 &= \operatorname{tr}(D^2 f \cdot D^2 u) \\ &= \mathfrak{b}_{11} u_{z_1 z_1} + 2\mathfrak{b}_{12} u_{z_1 z_2} + \mathfrak{b}_{22} u_{z_2 z_2} \end{aligned} \tag{4.52}$$

for ν a.e z in Ω. Rearranging (4.52) for ν a.e z in Ω we have

$$\begin{aligned} \frac{\mathfrak{b}_{11}}{|\nabla u|^{p-2}} \det(D^2 u) &= -\frac{1}{|\nabla u|^{p-2}} \left((2\mathfrak{b}_{12} u_{z_1 z_2} + \mathfrak{b}_{22} u_{z_2 z_2}) u_{z_2 z_2} - \mathfrak{b}_{11} u_{z_1 z_2}^2 \right) \\ &= -\frac{1}{|\nabla u|^{p-2}} (\nabla u_{z_2})^{\mathrm{T}} \cdot D^2 f \cdot \nabla u_{z_2}. \end{aligned} \tag{4.53}$$

Likewise,

$$\frac{\mathfrak{b}_{22}}{|\nabla u|^{p-2}} \det(D^2 u) = -\frac{1}{|\nabla u|^{p-2}} \left((\nabla u_{z_1})^{\mathrm{T}} \cdot D^2 f \cdot \nabla u_{z_1} \right). \tag{4.54}$$

It follows once again from (2.23) that for ν a.e z in Ω

$$-\frac{\mathfrak{b}_{11}}{|\nabla u|^{p-2}} \det(D^2 u) \geq 0 \text{ and } -\frac{\mathfrak{b}_{22}}{|\nabla u|^{p-2}} \det(D^2 u) \geq 0. \tag{4.55}$$

Now from (4.40) and (4.50) we see that

$$\begin{aligned} \int_\Omega \sum_{k,j=1}^{2} \mathfrak{b}_{kj} v_{z_j} \phi_{z_k} \, \mathrm{d}\nu &= \left(\frac{p-2}{p-1} \right) \int_\Omega \frac{\det(D^2 u \cdot D^2 f)}{f} \phi \, \mathrm{d}\nu \\ &\approx -\left(\frac{p-2}{p-1} \right) \int_\Omega \frac{-\mathfrak{b}_{11} \det(D^2 u)}{|\nabla u|^{p-2}} \frac{\det(D^2 f)}{f} \phi \, \mathrm{d}\nu. \end{aligned} \tag{4.56}$$

From (4.56) for $p = 2$ we have $\zeta = v = \log(f(\nabla u))$ is a weak solution to (4.31), $L\zeta = 0$. Similarly, $\zeta = v$ is a weak sub solution or super solution to $L\zeta = 0$ respectively when $2 < p < \infty$ or $1 < p < 2$.

Moreover, we note, for later use, that combining (4.53), (4.54) and using Lemma 3.1.9 we have

$$Lv = (p-2)\mathcal{F} \text{ weakly} \tag{4.57}$$

where $\mathcal{F} \approx |\nabla u|^{p-4} \sum_{i,j=1}^{2} (u_{z_i z_j})^2$.

Chapter 5 Proof of The Main Theorem 1.2.9

In this chapter, we first obtain Lemma 5.1.1, and then using this lemma we prove Main Theorem 1.2.9 for fix p when $1 < p \leq 2$ and $2 \leq p < \infty$ separately. To this end, we shall give the definitions of $u, \hat{u}, r, \Omega, z_0, w$ again.

Let Ω be a bounded simply connected domain in the plane. Let $w \in \partial\Omega$, and $0 < r < \operatorname{diam}\partial\Omega$. Let $\hat{u}$ be a positive weak solution to (3.3) in $B(w, 4r) \cap \Omega$ with $\hat{u} = 0$ in the Sobolev sense on $B(w, 4r) \cap \partial\Omega$. Extend $\hat{u}$ by putting $\hat{u} \equiv 0$ on $B(w, 4r) \setminus \Omega$. Let $\hat{\mu}$ be the corresponding finite positive Borel measure in (1.9).

Moreover, we choose $z_0 \in \Omega$ and let $D = \Omega \setminus \overline{B(z_0, d(z_0, \partial\Omega)/2)}$. Let u be a capacitary function for D. That is, u is a positive weak solution to (3.3) in D with continuous boundary values, $u \equiv 0$ on $\partial\Omega$ and $u \equiv 1$ on $\partial B(z_0, d(z_0, \partial\Omega)/2)$.

From section 3.2 we have H-dim μ = H-dim $\hat{\mu}$. Therefore, it suffices to prove Main Theorem 1.2.9 when u is a capacitary function and μ is the measure corresponding to u as in (1.9).

Let D be as above and let $2\tilde{s} = d(z_0, \partial\Omega)$ and set $\Xi(z) = z_0 + \hat{s}z$. Then it follows from the fact that (3.3) is invariant under translation and dilation that $\tilde{u} = u(\Xi(z))$ for $\Xi(z) \in D$ is also a weak solution to (3.3) in $\Xi^{-1}(D)$. Let $\tilde{\mu}$ be the measure corresponding to $\tilde{u}$ in (1.9). It can be easily shown from (3.3) that

$$\tilde{\mu}(E) = \hat{s}^{p-2}\mu(\Xi(E)) \text{ whenever } E \subset \mathbb{R}^2 \text{ is a Borel set.} \tag{5.1}$$

Clearly, (5.1) implies that H-dim $\tilde{\mu}$ = H-dim μ. Therefore without loss of generality we can assume that $z_0 = 0$ and $d(z_0, \partial\Omega) = 4$, and $D = \Omega \setminus \overline{B(0, 1)}$.

We first need a lemma.

5.1 A Lemma

Let u be a capacitary function for $D = \Omega \setminus \overline{B(0, 1)}$ corresponding to f, and let μ be the corresponding Borel measure. Define

$$w(z) = \begin{cases} \max(v(z), 0) & \text{when } 1 < p < 2 \\ \max(-v(z), 0) & \text{when } 2 < p < \infty \end{cases}$$

for $z \in D$ where $v(z) = \log(f(\nabla u)(z))$ as defined in section 4.2.

Lemma 5.1.1. *Let m be a non negative integer. There exists $c = c(f, p) \geq 1$ such that for $0 < t < 1/2$,*

$$\int\limits_{\{z:\, u(z)=t\}} \frac{f(\nabla u)}{|\nabla u|} w^{2m} dH^1(z) \leq c^{m+1} m! [\log \frac{1}{t}]^m. \tag{5.2}$$

Proof. Define $g(z) = \max(w(z) - c', 0)$, $z \in D$ where c' is large enough so that $g \equiv 0$ in $\overline{B(0, 2)} \cap D$. Since u is continuous in D, there is such a c'.

Set $g \equiv 0$ in $\overline{B(0,1)}$, so that g is continuously defined in Ω. Set $\mathfrak{b}_{ij} = f_{\eta_i \eta_j}(\nabla u)$ and let L be as in section 4.2.

Let $\Omega(t) = \{z \in D : u(z) > t\}$ for $0 < t < 1/2$ and let $\tilde{u} = \max(u - t, 0)$. Note that $g^2 \in W^{2,\infty}(\Omega(t))$.

Fix p, $1 < p \leq 2$ until further notices. From section 4.2 $\zeta = v = \log f(\nabla u)$ is a weak super solution to (4.31), $L\zeta = 0$ in D. Using $g^{2m-1}\tilde{u} \geq 0$ as a test function in (4.31) for $\zeta = g$ and the fact that $g \equiv 0$ in $\overline{B(0,1)}$, we get

$$
\begin{aligned}
0 &\leq 2m \int_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} \frac{\partial}{\partial z_j}(\log f(\nabla u)) \frac{\partial}{\partial z_k}\left(g^{2m-1} u\right) \mathrm{d}\nu \\
&= 2m \int_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} \frac{\partial}{\partial z_k}\left(g^{2m-1}\tilde{u}\right) \mathrm{d}\nu \\
&= 2m(2m-1) \int_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} g_{z_k} g^{2m-2} \tilde{u} \mathrm{d}\nu + 2m \int_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} g^{2m-1} \tilde{u}_{z_k} \mathrm{d}\nu \\
&= II' + II''.
\end{aligned}
\tag{5.3}
$$

We first handle II''. To this end, let $\psi \in C_0^\infty(\{z : u(z) > t - \varepsilon\})$ with $\psi = 1$ on $\overline{\Omega(t)}$. Then since $\zeta = u$ is a weak solution to (4.31) and using $g^{2m}\psi$ as a test function, we obtain

$$
\begin{aligned}
0 &= \int_{\Omega(t-\varepsilon)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} \tilde{u}_{z_k} \frac{\partial}{\partial z_j}\left(\psi g^{2m}\right) \mathrm{d}\nu \\
&= 2m \int_{\Omega(t-\varepsilon)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} \tilde{u}_{z_k} g^{2m-1} g_{z_j} \psi \mathrm{d}\nu + \int_{\Omega(t-\varepsilon)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} \tilde{u}_{z_k} g^{2m} \psi_{z_j} \mathrm{d}\nu \\
&= II_1'' + II_2''.
\end{aligned}
\tag{5.4}
$$

Letting $\varepsilon \to 0$ and using the Lebesgue dominated convergence give $II_1'' \to II''$. We now show that for H^1 a.e t and properly chosen ψ that

$$
II_2'' \to \int_{\{z:\, u(z)=t\}} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g^{2m} \tilde{u}_{z_k} \frac{\tilde{u}_{z_j}}{|\nabla \tilde{u}|} \quad \text{as } \varepsilon \to 0. \tag{5.5}
$$

To this end let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying $0 \leq \phi \leq 1$, and $|\phi'| \leq c/\varepsilon$ such that

$$
\phi(s) = \begin{cases} 1 & \text{when } s \geq 1 \\ 0 & \text{when } s \leq 1 - \varepsilon \end{cases}
$$

If we set $\psi = \phi(u(z)/t)$ in II_2'' and use the coarea formula (Theorem 3.1.6) we see that

$$
\begin{aligned}
II_2'' &= \int\limits_{\Omega(t-\varepsilon)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj}\tilde{u}_{z_k} g^{2m} \psi_{z_j} \mathrm{d}\nu \\
&= \int\limits_{\Omega(t(1-\varepsilon))} \sum_{k,j=1}^{2} \mathfrak{b}_{kj}\tilde{u}_{z_k} g^{2m} \left(\phi\left(\frac{u(z)}{t}\right)\right)_{z_j} \mathrm{d}\nu \\
&= \frac{1}{t} \int\limits_{\Omega(t(1-\varepsilon))} \sum_{k,j=1}^{2} \mathfrak{b}_{kj}\tilde{u}_{z_k} g^{2m} \phi'\left(\frac{u(z)}{t}\right) u_{z_j} \mathrm{d}\nu \\
&= \frac{1}{t} \int\limits_{t(1-\varepsilon)}^{t} \phi'(\frac{\tau}{t}) \left(\int\limits_{\{z:\, u(z)=\tau\}} \sum_{k,j=1}^{2} \mathfrak{b}_{kj}\tilde{u}_{z_k} g^{2m} \frac{u_{z_j}}{|\nabla u|} \mathrm{d}H^1 \right) \mathrm{d}\tau.
\end{aligned}
\tag{5.6}
$$

Let

$$
\Theta(\tau) = \int\limits_{\{z:\, u(z)=\tau\}} \sum_{k,j=1}^{2} \mathfrak{b}_{kj}\tilde{u}_{z_k} g^{2m} \frac{u_{z_j}}{|\nabla u|} \mathrm{d}H^1.
$$

Then using

$$
\frac{1}{t} \int\limits_{t(1-\varepsilon)}^{t} \phi'\left(\frac{\tau}{t}\right) \mathrm{d}\tau = \phi(1) - \phi(1-\varepsilon) = 1
$$

we have

$$
II_2'' = \frac{1}{t} \int\limits_{t(1-\varepsilon)}^{t} \phi'(\frac{\tau}{t}) \left(\Theta(\tau) - \Theta(t)\right) \mathrm{d}\tau + \Theta(t) \tag{5.7}
$$

for almost every $t \in (0, 1/2)$. If we let $\varepsilon \to 0$ it follows from the strong form of the Lebesgue Differentiation theorem that

$$
\lim_{\varepsilon\to 0} \Big|\frac{1}{t} \int\limits_{t(1-\varepsilon)}^{t} \phi'(\frac{\tau}{t}) \left(\Theta(\tau) - \Theta(t)\right) \mathrm{d}\tau\Big| \le \lim_{\varepsilon\to 0} \frac{1}{t\varepsilon} \int\limits_{t(1-\varepsilon)}^{t} |\Theta(\tau) - \Theta(t)| \mathrm{d}\tau = 0 \tag{5.8}
$$

for H^1 a.e $t \in (0, 1/2)$. From (5.8) and (5.6) for H^1 a.e $t \in (0, 1/2)$ we have

$$
II_2'' \to \Theta(t) \text{ as } \varepsilon \to 0. \tag{5.9}
$$

Thus (5.5) is true. Hence using (5.9) in (5.5) and then (5.5) and (5.4) in (5.3) we see that

$$
\int\limits_{\{t:\, u(z)=t\}} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g^{2m} u_{z_k} \frac{u_{z_j}}{|\nabla u|} dH^1(z) \le 2m(2m-1) \int\limits_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} g_{z_k} g^{2m-2} (u-t) \mathrm{d}\nu \tag{5.10}
$$

Similarly, for fixed p, $2 < p < \infty$ from section 4.2, $\zeta = v = \log f(\nabla u)$ is a weak sub solution to (4.31), $L\zeta = 0$ in D. Using this observation and $g^{2m-1}\tilde{u} \geq 0$ as a test function and the fact that $g \equiv 0$ on $\overline{B(0,1)}$, we have

$$
\begin{aligned}
0 &\geq 2m \int\limits_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} \frac{\partial}{\partial z_j}(\log f(\nabla u)) \frac{\partial}{\partial z_k}\left(g^{2m-1}u\right) \mathrm{d}\nu \\
&= -2m \int\limits_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} \frac{\partial}{\partial z_k}\left(g^{2m-1}\tilde{u}\right) \mathrm{d}\nu \\
&= -2m(2m-1) \int\limits_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} g_{z_k} g^{2m-2} \tilde{u} \mathrm{d}\nu + 2m \int\limits_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} g^{2m-1} \tilde{u}_{z_k} \mathrm{d}\nu \\
&= -(III' + III'').
\end{aligned}
\tag{5.11}
$$

Arguing as in the previous case we have (5.10) when $p > 2$. Therefore, for fixed p, $1 < p < \infty$, (5.10), Lemma 3.1.9, and Euler's formula for a homogenous function yield

$$
\begin{aligned}
p(p-1) \int\limits_{\{z:\, u(z)=t\}} g^{2m} \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1(z) &= \int\limits_{\{z:\, u(z)=t\}} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} \frac{u_{z_k} u_{z_j}}{|\nabla u|} g^{2m} \mathrm{d}H^1(z) \\
&\leq 2m(2m-1) \int\limits_{\Omega(t)} \sum_{k,j=1}^{2} \mathfrak{b}_{kj} g_{z_j} g_{z_k} g^{2m-2}(u-t) \mathrm{d}\nu \\
&\leq c\, 2m(2m-1) \int\limits_{\Omega(t)} |\nabla u|^{p-2} |\nabla g|^2 g^{2m-2} u \, \mathrm{d}\nu.
\end{aligned}
\tag{5.12}
$$

Let $\{Q_i\}$ be a closed Whitney cube decomposition of $\Omega(t)$ and let z_i be the center of Q_i for $i = 1, \ldots$. Let R_i be the union of cubes that have a common point in the boundary with Q_i.

Note that the definition of g and Lemma 3.1.9 yield for a.e $z \in \Omega$

$$
|\nabla g| \leq c \frac{|\nabla f(\nabla u)||D^2 u|}{f(\nabla u)} \approx \frac{|D^2 u|}{|\nabla u}. \tag{5.13}
$$

Moreover, it can be easily deduce from Lemma 3.3.1 that

$$
\int\limits_{Q_i} |\nabla u|^{p-2} \sum_{k,j} \left(u_{z_k z_j}\right)^2 \mathrm{d}\nu \leq c \int\limits_{R_i} \frac{|\nabla u|^p}{d(z, \partial\Omega(t))} \mathrm{d}\nu \tag{5.14}
$$

for every $i = 1, \ldots$.

Using (5.13), (5.14), Lemma 3.1.9, and Fundamental Inequality 4.1.5 in (5.12) on the Whitney cubes Q_i we see that

$$
\begin{aligned}
\int\limits_{\{z:\, u(z)=t\}} g^{2m}\frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1 &\le c'\, m^2 \int\limits_{\Omega(t)} u|\nabla u|^{p-2}|\nabla g|^2 g^{2m-2}\mathrm{d}\nu \\
&\le c'\, m^2 \sum_i \operatorname*{ess\,sup}_{Q_i}\left(\frac{u}{|\nabla u|^2}g^{2m-2}\right)\int\limits_{Q_i}|\nabla u|^p|\nabla g|^2\mathrm{d}\nu \\
&\le c'\, m^2 \sum_i \operatorname*{ess\,sup}_{Q_i}\left(\frac{u}{|\nabla u|^2}g^{2m-2}\right)\int\limits_{Q_i}|\nabla u|^p\frac{|D^2u|^2}{|\nabla u|^2}\mathrm{d}\nu \\
&\le c'\, m^2 \sum_i \operatorname*{ess\,sup}_{Q_i}\left(\frac{u}{|\nabla u|^2}g^{2m-2}\right)\int\limits_{Q_i}|\nabla u|^{p-2}|D^2u|^2\mathrm{d}\nu \\
&\le c'\, m^2 \sum_i \operatorname*{ess\,sup}_{Q_i}\left(\frac{u}{|\nabla u|^2}g^{2m-2}\right)\int\limits_{R_i}\frac{|\nabla u|^p}{(d(z,\partial\Omega))^2}\mathrm{d}\nu \\
&\le c'\, m^2 \sum_i \operatorname*{ess\,sup}_{Q_i}\left(g^{2m-2}\right)\int\limits_{R_i}u|\nabla u|^{p-2}\frac{1}{(d(z,\partial\Omega))^2}\mathrm{d}\nu \\
&\le c\,\tfrac{2m(2m-1)}{p(p-1)}\sum_i \operatorname*{ess\,sup}_{Q_i}\left(g^{2m-2}\right)\int\limits_{R_i}u|\nabla u|^{p-2}\frac{|\nabla u|^2}{u^2}\mathrm{d}\nu \\
&\le c\,\tfrac{2m(2m-1)}{p(p-1)}\sum_i \operatorname*{ess\,sup}_{Q_i}\left(g^{2m-2}\right)\int\limits_{R_i}\frac{|\nabla u|^p}{u}\mathrm{d}\nu \\
&\le c\,\tfrac{2m(2m-1)}{p(p-1)}\int\limits_{\Omega(t)}(g+c')^{2m-2}\frac{f(\nabla u)}{u}\mathrm{d}\nu.
\end{aligned}
\tag{5.15}
$$

Here we have used the fact that Q_i intersects with finitely many R_i which allows us to interchange freely R_i and Q_i.

Moreover, the fundamental inequality 4.1.5 and Lemma 3.1.13 yield

$$
\log f(\nabla u) \approx \log|\nabla u| \approx \log\left(\frac{u(z)}{d(z,\partial\Omega(z))}\right) \le \log(u^{\frac{1}{\alpha}-1}) \le (1-\tfrac{1}{\alpha})\,\log\left(\frac{1}{t}\right) \tag{5.16}
$$

whenever $z \in \{\tilde{z};\, u(\tilde{z}) = t\}$ and $0 < t < 1/2$. Therefore, for $z \in \{\tilde{z};\, u(\tilde{z}) = t\}$ and $0 < t < 1/2$ we see from the fundamental inequality 4.1.5 and Lemma 3.1.11 that

$$
(g+\tilde{c})^{2m-2} = (g^2 + 2g\tilde{c} + \tilde{c}^2)^{m-1} \le (g^2 + c\log 1/t)^{m-1}. \tag{5.17}
$$

whenever $0 < t < 1/2$. Using the Binomial theorem and (5.17) we can write

$$
(g^2 + c\log 1/t)^{m-1} = \sum_{k=0}^{m-1} \tfrac{(m-1)!}{k!(m-k-1)!} g^{2k}\left(c\log\frac{1}{t}\right)^{m-1-k}. \tag{5.18}
$$

Let

$$I_m(t) = \int\limits_{\{z\colon u(z)=t\}} g^{2m}\frac{f(\nabla u)}{|\nabla u|}dH^1(z) \text{ for } 0<t<\frac{1}{2}.$$

Then using the Coarea formula (Theorem 3.1.6), (5.12), (5.15) and (5.18) we obtain

$$\begin{aligned}
I_m(t) &= \int\limits_{\{z\colon u(z)=t\}} g^{2m}\frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z)\\
&\le c'\,m^2 \int\limits_{\Omega(t)} (g+c)^{2m-2}\frac{f(\nabla u)}{u}\mathrm{d}\nu\\
&= c'\,m^2 \int\limits_t^1 \frac{1}{\tau}\left(\int\limits_{\{z\colon u(z)=\tau\}} (g+c)^{2m-2}\frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z)\right)\mathrm{d}\tau\\
&\le c'\,m^2 \int\limits_t^1 \frac{1}{\tau}\left(\int\limits_{\{z\colon u(z)=\tau\}} \sum_{k=0}^{m-1}\tfrac{(m-1)!}{k!(m-k-1)!}g^{2k}(c\log\frac{1}{\tau})^{m-1-k}\frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z)\right)\mathrm{d}\tau\\
&\le c'\,m^2\left[\sum_{k=0}^{m-1}\tfrac{(m-1)!}{k!(m-k-1)!}\int\limits_t^1 \frac{(c\log\frac{1}{\tau})^{m-1-k}}{\tau}\left(\int\limits_{\{z\colon u(z)=\tau\}} g^{2k}\frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z)\right)\mathrm{d}\tau\right]\\
&\le c'\,m^2\sum_{k=0}^{m-1}\tfrac{(m-1)!}{k!(m-k-1)!}\left[\int\limits_t^1 \frac{(c\log\frac{1}{\tau})^{m-1-k}}{\tau}I_k\mathrm{d}\tau\right].
\end{aligned} \tag{5.19}$$

It easily follows from $\nabla\cdot\nabla f(\nabla u(z))=0$ for a.e z in Ω and the divergence theorem that

$$I_0(t) = \int\limits_{\{z\colon u(z)=t\}} \frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z) = \text{constant} = c(p,f) \text{ for } 0<t<1. \tag{5.20}$$

In fact, using the divergence theorem we have

$$\int\limits_{\{z\colon u(z)=t_0\}} \frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z) = \int\limits_{\{z\colon u(z)=t_1\}} \frac{f(\nabla u)}{|\nabla u|}\mathrm{d}H^1(z) \ \text{ for }\ 0<t_0,t_1<1. \tag{5.21}$$

One can use an induction argument on m in the following way; by (5.20) we have $I_0\le c'$ for $0<t<1$, and next assume that we have

$$I_k \le c_*^{k+1}k![\log\frac{1}{t}]^k \text{ when } 0<t<\frac{1}{2} \text{ and for every } 1\le k\le m-1, \tag{5.22}$$

where $1 \leq c_*$, then for $k = m$ we have

$$
\begin{aligned}
I_m(t) &\leq c\,\frac{2m(2m-1)}{p(p-1)} \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} \left[\int_t^1 \frac{(c_0 \log \frac{1}{\tau})^{m-1-k}}{\tau} I_k \mathrm{d}\tau\right] \\
&\leq c\,\frac{2m(2m-1)}{p(p-1)} \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} \left[\int_t^1 \frac{(c_0 \log \frac{1}{\tau})^{m-1-k}}{\tau} c_*^{k+1} k! (\log(\frac{1}{\tau}))^k \mathrm{d}\tau\right] \\
&\leq c\,\frac{2m(2m-1)}{p(p-1)} \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} c_0^{m-k-1} c_*^{k+1} k! \left[\int_t^1 \frac{(\log \frac{1}{\tau})^{m-1}}{\tau} \mathrm{d}\tau\right] \\
&\leq c\,\frac{2m(2m-1)}{p(p-1)} \sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} c_0^{m-k-1} c_*^{k+1} k! \frac{(\log(\frac{1}{t}))^m}{m} \\
&\leq 4c\,\frac{1}{p(p-1)} c_*^m m! (\log \frac{1}{t})^m \left(\sum_{k=0}^{m-1} \frac{1}{(m-k-1)!}\right) \\
&\leq c_{**}^{m+1} m! (\log \frac{1}{t})^m .
\end{aligned}
\tag{5.23}
$$

for $0 < t < 1/2$.

Hence by (5.23), Lemma 5.1.1 is true for w replaced by h. It follows from $w \leq h + c'$ that Lemma 5.1.1 is true for w. □

By Lemma 5.1.1 we get for $0 < t < 1/2$

$$\int_{\{z:\, u(z)=t\}} \frac{f(\nabla u)}{|\nabla u|} \frac{w^{2m}}{(2c_{**})^m m! [\log \frac{1}{t}]^m} \mathrm{d}H^1(z) \leq 2^{-m} c_{**}. \tag{5.24}$$

Summing over m in (5.24) yields for $0 < t < 1/2$

$$\int_{\{z:\, u(z)=t\}} \frac{f(\nabla u)}{|\nabla u|} \exp\left[\frac{w^2}{2c_{**} \log \frac{1}{t}}\right] \mathrm{d}H^1(z) \leq 2c_{**}. \tag{5.25}$$

Define

$$\alpha(t) = \sqrt{4c_{**} \left(\log \frac{1}{t}\right) \left(\log \log \frac{1}{t}\right)} \ \text{ for } \ 0 < t < e^{-2}, \tag{5.26}$$

and

$$\beta(t) = \{z;\, u(z) = t \text{ and } \ w(z) \geq \alpha(t)\}. \tag{5.27}$$

Then by (5.25) we have

$$
\begin{aligned}
2c_{**} &\geq \int\limits_{\{z:\,u(z)=t\}} \frac{f(\nabla u)}{|\nabla u|} \exp\left[\frac{w^2}{2c_{**}\log\frac{1}{t}}\right] \mathrm{d}H^1(z) \\
&\geq \int\limits_{\beta(t)} \frac{f(\nabla u)}{|\nabla u|} \exp\left[\frac{w^2}{2c_{**}\log\frac{1}{t}}\right] \mathrm{d}H^1(z) \\
&\geq \int\limits_{\beta(t)} \frac{f(\nabla u)}{|\nabla u|} \exp\left[\frac{\alpha^2}{2c_{**}\log\frac{1}{t}}\right] \mathrm{d}H^1(z) \\
&= \int\limits_{\beta(t)} \frac{f(\nabla u)}{|\nabla u|} (-\log t)^2 \mathrm{d}H^1(z).
\end{aligned}
\tag{5.28}
$$

We conclude from (5.28) that

$$
\int\limits_{\beta(t)} \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1(z) \leq \frac{2c_{**}}{\left(\log\frac{1}{t}\right)^2}. \tag{5.29}
$$

For a fixed and large a, we define the Hausdorff measure H^λ as follows;

Let

$$
\lambda(r) = \begin{cases} r\mathrm{e}^{a\alpha(r)} & \text{when } 1 < p \leq 2 \text{ for section 5.2} \\ r\mathrm{e}^{-a\alpha(r)} & \text{when } 2 \leq p < \infty \text{ for section 5.3.} \end{cases} \tag{5.30}
$$

Let H^λ be Hausdorff measure and Hausdorff dimension of a measure be as defined before Theorem 1.2.1 relative λ as in (5.30).

In the following subsections 5.2 and 5.3 we shall follow closely the argument in [25, Section 3].

5.2 Proof of The Main Theorem 1.2.9 for $1 < p \leq 2$

In this subsection we show that for a large a,

$$
\mu \text{ is absolutely continuous with respect to } H^\lambda \text{ measure when } 1 < p \leq 2. \tag{5.31}
$$

Proof of (5.31). Fix p, $1 < p \leq 2$, let $E \subset \partial\Omega$ be a Borel set with $H^\lambda(E) = 0$. Let $E = E_1 \cup E_2$ where

$$
E_1 := \{z \in E;\ \limsup_{r\to 0} \frac{\mu(B(z,r))}{\lambda(r)} < \infty\}, \tag{5.32}
$$

and

$$
E_2 := \{z \in E;\ \limsup_{r\to 0} \frac{\mu(B(z,r))}{\lambda(r)} = \infty\}. \tag{5.33}
$$

Note that $E_2 = E \setminus E_1$ for μ a.e z in $\partial\Omega$. We first show that $\mu(E_1) = 0$. It easily follows from the definition of H^λ measure and a covering argument that $\mu(E_1) = 0$.

We next show that $\mu(E_2) = 0$. To this end, given $0 < r_0 < 10^{-100}$ we show that for every $z \in E_2$ there is $s = s(z)$ with $0 < s/100 < r_0$ such that

$$\mu(B(z, 100s)) \leq 10^9 \mu(B(z, s)) \text{ and } \lambda(100s) \leq \mu(B(z, s)). \tag{5.34}$$

Let s be the last point on the interval $(0, r_0)$ satisfying

$$10^{20} \min\{1, \frac{\mu(B(z, r_0))}{\lambda(r_0)}\} \leq \frac{\mu(B(z, s))}{\lambda(s)}. \tag{5.35}$$

Existence of s follows from (5.33). Moreover,

$$\lambda(100r) \leq 200\lambda(r) \text{ for } 0 < r < r_0/100 \tag{5.36}$$

as we see from the definition of λ. It follows from (5.35) and (5.36) that there is such s satisfying (5.34).

Observe that by Vitali's covering argument we can find $\{r_i < \delta, z_i \in E_2\}$ such that

$$\begin{aligned} &B(z_i, 10r_i) \text{ are disjoint balls,} \\ &\{B(z_i, 100r_i)\} \text{ covers } E_2, \\ &(5.34) \text{ holds for } z = z_i \text{ and } s = r_i \text{ for every } i. \end{aligned} \tag{5.37}$$

Choose $\zeta_i \in \partial B(z_i, 2r_i)$ such that $u(\zeta_i) = \max u$ on $\overline{B(z_i, 2r_i)}$. From (5.34) and Lemma 3.1.15 we know that the maximum of u on $\overline{B(z_i, 2r_i)}$ and the maximum of u on $\overline{B(z_i, 5r_i)}$ are proportional. Therefore, $u(\zeta_i)$ can not be too small in comparison to the maximum of u on $\overline{B(z_i, 5r_i)}$. Thus, these observations and Lemma 3.1.13 yiled $d(\zeta_i, \partial\Omega) \approx r_i$.

Moreover, using $d(\zeta_i, \partial\Omega) \approx r_i$ and Lemma 3.1.15 we see for fixed i that

$$\frac{\mu(B(z_i, 10r_i))}{r_i} \approx \left(\frac{u(\zeta_i)}{d(\zeta_i, \partial\Omega)}\right)^{p-1} \approx \frac{f(\nabla u(z))}{|\nabla u(z)|} \tag{5.38}$$

whenever $z \in B(\zeta_i, d(\zeta_i, \partial\Omega)/2)$. Choose m so that $2^{-m} \leq u(\zeta_i) \leq 2^{-m+1}$, and let η_i be the first point on the line segment from ζ_i to a point on $\partial\Omega \cap \partial B(\zeta_i, d(\zeta_i, \partial\Omega))$ satisfying $u(\eta_i) = 2^{-m}$. Then we see that (5.38) holds with ζ_i replaced by η_i. That is,

$$\begin{aligned} &u(\eta_i) = 2^{-m} \text{ and } d(\eta_i, \partial\Omega) \approx r_i, \\ &\frac{\mu(B(z_i, 10r_i))}{r_i} \approx \left(\frac{u(\eta_i)}{d(\eta_i, \partial\Omega)}\right)^{p-1} \approx \frac{f(\nabla u(z))}{|\nabla u(z)|} \approx |\nabla u(z)|^{p-1} \end{aligned} \tag{5.39}$$

whenever $z \in B(\eta_i, d(\eta_i, \partial\Omega)/2)$.

From (5.37) and (5.39) for $z \in B(\eta_i, d(\eta_i, \partial\Omega)/2)$ we have

$$\begin{aligned} a\alpha(100r_i) &= \log\left(\frac{\lambda(100r_i)}{100r_i}\right) \\ &\leq \log\left(\frac{\mu(B(z_i, r_i))}{100r_i}\right) \\ &\leq c\log\left(\frac{\mu(B(z_i, 10r_i))}{r_i}\right) \\ &\leq c\log|\nabla u|^{p-1} \approx \log f(\nabla u) = w(z). \end{aligned} \tag{5.40}$$

where a is as in (5.30) and $c = c(p, f) \geq 1$. It follows from the maximum principle and geometry of level sets of u, $\{z;\, u(z) = t\}$, and a connectivity argument that

$$H^1[B(\eta_i, d(\eta_i, \partial\Omega)/2) \cap \{z:\ u(z) = 2^{-m}\}] \geq \frac{d(\eta_i, \partial\Omega)}{2}. \tag{5.41}$$

Using Lemma 3.1.13 we can estimate 2^{-m} above in terms of r_i. We can also estimate 2^{-m} below in terms of r^i using the first line in (5.37). That is, there exist $c' = c(p, f)$ and $\beta = \beta(p, f) < 1$ such that

$$r_i \leq c'(2^{-m})^{\beta} \ \text{ and } \ 2^{-m} \leq c' r_i^{\beta}. \tag{5.42}$$

From (5.29), (5.40)-(5.42) we have,

$$\mu[B(z_i, 10r_i)] \leq c \int\limits_{\beta(2^{-m})\cap B(z_i, 10r_i)} \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1(z) \tag{5.43}$$

For large a, (5.42), (5.43), and (5.29) yield

$$\begin{aligned} \mu(E_2) &\leq \mu\left(\bigcup_i (B(z_i, 100r_i)\right) \\ &\leq 10^9 \sum_i \mu(B(z_i, 10r_i)) \\ &\leq c \sum_{m=m_0} \int\limits_{\beta(2^{-m})} \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1(z) \\ &\leq c^2 \sum_{m=m_0} m^{-2} \leq \frac{c^3}{m_0}. \end{aligned} \tag{5.44}$$

where $2^{-m_0\beta} = c r_0^{\beta^2}$. As $r_0 \to 0$ we have $\mu(E_2) \to 0$. So we have (5.31).

5.3 Proof of The Main Theorem 1.2.9 for $2 \leq p < \infty$

To finish the proof of the Main Theorem 1.2.9, it remains to show that for $2 \leq p < \infty$, μ is concentrated on a set of $\sigma-$finite H^λ measure. To obtain this, by definition, we show that there is a Borel set $K \subset \partial\Omega$ having $\sigma-$finite H^λ measure satisfying $\mu(K) = \mu(\partial\Omega)$.

We first show that $\mu(K') = 0$ where

$$K' := \{z \in \partial\Omega; \lim_{r\to 0} \frac{\mu(B(z,r)}{\lambda(r)} = 0\}. \tag{5.45}$$

Then we show that $\mu(K) = \mu(\partial\Omega)$ where

$$K = \{z \in \partial\Omega; \limsup_{r\to 0} \frac{\mu(B(z,r)}{\lambda(r)} > 0\}.$$

Let r_0 be sufficiently small. We can argue as in the proof of (3.52) to get that for each $z \in K'$ there is $r = r(z)$ with $0 < r/100 < r_0$ satisfying

$$\mu(B(z,100r)) \le c\mu(B(z,r)) \text{ and } \mu(B(z,100r)) \le \lambda(r) \tag{5.46}$$

where the constant is independent of z and r.

Observe once again that by Vitali's covering argument we can find $\{r_i < \delta, z_i \in K'\}$ such that

$$\begin{aligned} &B(z_i, 10r_i) \text{ are disjoint balls,} \\ &\{B(z_i, 100r_i)\} \text{ covers } K', \\ &(5.46) \text{ holds for } z = z_i \text{ and } r = r_i \text{ for every } i. \end{aligned} \tag{5.47}$$

Let I' be the set of all indexes i for which $r_i^3 \le \mu(B(z_i, 100r_i))$ and let I'' be the indexes where this inequality does not hold. By (5.47) and (5.46) we see that

$$\begin{aligned} \mu(K') \le \mu(&\bigcup_{i\in I'} B(z_i, 100r_i)) \\ &+ \mu(\bigcup_{i\in I'} B(z_i, 100r_i)) + \mu(\bigcup_{i\in I''} B(z_i, 100r_i)) \\ &\le \mu(\bigcup_{i\in I'} B(z_i, 100r_i)) + \sum_{i\in I''} r_i^3 \\ &\le \mu(\bigcup_{i\in I'} B(z_i, 100r_i)) + c' r_0 H^2(\Omega). \end{aligned} \tag{5.48}$$

When $i \in I'$ we can repeat the argument for $1 < p \le 2$ to get first (5.38) and then (5.39), (5.40). As $i \in I'$, we get the left hand inequality. Once again the right hand inequality follows from Lemma 3.1.13. From these observations we see (5.43) holds for sufficiently large a as in (5.30) and

$$
\begin{aligned}
\mu(K') - c' r_0 H^2(\Omega) &\leq \mu(\bigcup_{i\in I'} B(z_i, 100r_i)) \\
&\leq c \sum_{i\in I'} \mu(B(z_i, 10r_i)) \\
&\leq c \sum_{m=m_0} \int_{\beta(2^{-m})} \frac{f(\nabla u)}{|\nabla u|} \mathrm{d}H^1 \\
&\leq c^2 \sum_{m=m_0} m^{-2} \leq \frac{c^3}{m_0}.
\end{aligned}
\tag{5.49}
$$

Hence $2^{-m_0\beta} = c r_0^{\beta^2}$. Since r_0 can be arbitrarily small, we can let $r_0 \to 0$ from which we conclude that $\mu(K') = 0$.

It remains to show that $\mu(K) = \mu(\partial\Omega)$ and K has σ finite H^λ measure. To this end let K_i, for positive integer i, be the set of points in K with the property that

$$
K_i = \{z \in \partial\Omega; \limsup_{r\to 0} \frac{\mu(B(z,r)}{\lambda(r)}\} \geq \frac{1}{i}.
$$

From a covering argument it follows that

$$
H^\lambda(K_i) \leq 100 i \mu(K_i)
$$

from which we can conclude that K_i has finite H^λ measure. Since $\bigcup_i K_i = K$, we conclude that K has $\sigma-$finite H^λ measure. which finishes the proof for $p > 2$. □

The proof of Main Theorem 1.2.9 is now complete.

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- (with John Lewis and Andrew Vogel) $\sigma-$finiteness of a certain measure arising from a positive weak solution to a divergence type PDE in space, in preparation.
- (with John Lewis and Andrew Vogel) Hausdorff dimension and $\sigma-$ finiteness of $p-$harmonic measures in space when $p \geq n$, *arXiv:1306.5617*.
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Honors

- JMM-AMS Graduate Travel Grant, January 2014
- AMS Graduate Student Travel Grant, Spring 2013, Fall 2014
- Dissertation Enhancement Award, Fall 2013
- Summer Research Assistantship Award, Department of Mathematics, University of Kentucky, 2010-2013
- Graduate Student Travel Support, University of Kentucky, 2011-2013
- Honor Undergraduate Award, Middle East Technical University, June 2006