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HOMOGENEOUS GORENSTEIN IDEALS AND BOIJ SÖDERBERG DECOMPOSITIONS

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HOMOGENEOUS GORENSTEIN IDEALS AND BOIJ SÖDERBERG DECOMPOSITIONS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Sema Güntürkün
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Director: Dr. Uwe Nagel, Professor of Mathematics
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ABSTRACT OF DISSERTATION

HOMOGENEOUS GORENSTEIN IDEALS AND BOIJ SÖDERBERG DECOMPOSITIONS

This thesis consists of two parts. Part one revolves around a construction for homogeneous Gorenstein ideals and properties of these ideals. Part two focuses on the behavior of the Boij-Söderberg decomposition of lex ideals.

Gorenstein ideals are known for their nice duality properties. For codimension two and three, the structures of Gorenstein ideals have been established by Hilbert-Burch and Buchsbaum-Eisenbud, respectively. However, although some important results have been found about Gorenstein ideals of higher codimension, there is no structure theorem proven for higher codimension cases. Kustin and Miller showed how to construct a Gorenstein ideals in local Gorenstein rings starting from smaller such ideals. A modification of their construction in the case of graded rings is discussed. In a Noetherian ring, for a given two homogeneous Gorenstein ideals, we construct another homogeneous Gorenstein ideal and so we describe the resulting ideal in terms of the initial homogeneous Gorenstein ideals. Gorenstein liaison theory plays a central role in this construction. Using liaison properties, we examine structural relations between the constructed homogeneous ideal and the starting ideals.

Boij-Söderberg theory is a very recent theory. It arose from two conjectures given by Boij and Söderberg and their proof by Eisenbud and Schreyer. It establishes a unique decomposition for Betti diagram of graded modules over polynomial rings. In the second part of this thesis, we focus on Betti diagrams of lex ideals which are the ideals having the largest Betti numbers among the ideals with the same Hilbert function. We describe Boij-Söderberg decomposition of a lex ideal in terms of Boij-Söderberg decompositions of some related lex ideals.

KEYWORDS: Gorenstein, Liaison, free resolutions, Betti diagrams, Boij-Söderberg

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HOMOGENEOUS GORENSTEIN IDEALS AND BOIJ SÖDERBERG DECOMPOSITIONS

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Date: May 2, 2014

To my mom, my dad and Murat and our beloved cat, İrmik.
I love you all dearly.

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Chapter 1 Introduction

Commutative algebra is a branch of abstract algebra that studies commutative rings, its modules and ideals. Other areas such as algebraic geometry, algebraic number theory, invariant theory draw upon it by carrying the structural consequences. This dissertation discusses two topics in commutative algebra. Mainly these topics address two class of ideals; Gorenstein ideals and lex-segment ideals. First, we consider homogeneous Gorenstein ideals, a way of constructing them and structural outcomes of this construction. Second, we examine Boij-Söderberg decomposition of lex-segment ideals. We obtain a pattern for Boij-Söderberg decompositions of lex-segment ideals by using some other related lex-segment ideals.

Gorenstein rings are known as a very important class of rings due to their symmetry properties in commutative algebra and algebraic geometry. We refer to [1] and [20] for more historical background of Gorenstein rings. An ideal I in a Gorenstein ring R is said to be Gorenstein ideal if it is perfect (i.e. $\operatorname{grade} R/I = \operatorname{projdim} R/I$) and the quotient ring R/I is Gorenstein. The minimal free resolutions of R/I is self-dual (i.e. symmetric regarding the ranks of free modules).

Investigating the structure of Gorenstein ideals has been an ongoing effort in the area of commutative area. Two important result have been obtained about the structures of Gorenstein ideals in codimension 2 and 3.

Theorem 1.0.1 (Hilbert-Burch, [8])**.** *Let R be a local ring and I an ideal of codimension 2 in R with a free resolution $0 \to R^n \to^X R^{n+1} \to R \to R/I$. Then I is perfect and $I = aI_n(X)$ ideal generated by n minors of X for an R-regular element a.*

Theorem 1.0.2 (Buchsbaum-Eisenbud, [7])**.** *Let R be a Noetherian local ring and $n \geq 3$ an odd integer. An ideal I of codimension 3 in R is Gorenstein if and only if $I = \operatorname{Pf}_{n-1}(X)$ where X is an $n \times n$ skew-symmetric matrix.*

There are also some results obtained for higher codimension Gorenstein ideals, especially codimension four. Nevertheless, there is no such structural theorem yet. The main motivation for the first part of my thesis is to obtain some structural information about Gorenstein ideals of higher codimension in graded rings.

In [22] Kustin and Miller introduce a construction that produces, for given Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ with grades g and $g-1$, respectively, in a Gorenstein local ring R, a new Gorenstein ideal I of grade g in a larger Gorenstein ring $R[v]$. Here v is a new indeterminate. In [23] they give an interpretation for their construction via liaison theory. These beautiful results prompted us to review their construction for homogeneous Gorenstein ideals in a graded Gorenstein ring. Instead of introducing a new indeterminate, we use a suitable homogeneous element. The construction in [22] does not quite reveal the conditions on that homogenous element. Therefore, we reverse the steps. We use two direct Gorenstein links to produce a new Gorenstein ideal and to describe a generating set of it. Then we adapt the original Kustin-Miller construction suitably in order to produce a graded free resolution of the new

Gorenstein ideal that is often minimal. We also consider the question of when the process can be reversed, that is, when can a Gorenstein ideal be obtained using the construction.

We start Chapter 2 with some definitions, fundamental concepts and then we recall the liaison theory and the mapping cone procedure. Then in Theorem 2.3.1 (in [16]), we present a construction of homogeneous Gorenstein ideals via liaison theory. Given two homogeneous Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ of grades of $g-1$ and g in a graded Gorenstein ring R, by choosing an appropriate homogeneous element f in R we construct a homogeneous Gorenstein ideal $I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*)$ in the original ring R. Here α_{g-1}^* and a_g^* are row vectors derived from comparing the resolutions of $\mathfrak{a}$ and $\mathfrak{b}$ and the second ideal is generated by the entries of the specified row vector.

Using liaison theory, we also get a graded free resolution of I. However, this resolution is never minimal. Adapting the original Kustin-Miller construction and its proof we obtain a smaller resolution that is often minimal (see Theorem 2.4.1). The key is a short exact sequence, which also allows us to interpret the linkage construction in Theorem 2.3.1 as an elementary biliaison from $\mathfrak{a}$ on $\mathfrak{b}$.

As an inevitable question of a construction, we discuss the situations of reversing the construction we introduce.

Question 1.0.3. Is it possible to obtain any homogeneous Gorenstein ideal I from the construction given in Theorem 2.3.1?

We provide some partial answer for this question. First, we obtain a necessary condition on $\mathfrak{a}$ for constructing a given Gorenstein ideal I by such a biliaison (see Corollary 2.5.1). We conclude with an example of a homogeneous Gorenstein ideal that can not be obtained using the construction of Theorem 2.3.1 with a strictly ascending biliaison.

Chapter 2 ends with some examples of our construction. The original Kustin-Miller construction has been used to produce many interesting classes of Gorenstein ideals. In birational geometry it is known as unprojection (see, e.g., [26, 27, 5]). We illustrate the flexibility of our homogeneous construction by producing examples. These include the Artinian Gorenstein ideals with socle degree two as classified by Sally [29] and the ideals of submaximal minors of a generic square matrix that are resolved by the Gulliksen-Negård complex. We also consider some Tom unprojections as studied in [5].

The second part of this thesis focuses on Boij-Söderberg decompositions which is covered in Chapter 3. Boij-Söderberg is very recent theory which addresses the characterization of Betti diagrams of graded modules in polynomial rings. Its origins are in a pair of conjectures by Boij and Söderberg [3], whose proof is given by Eisenbud and Schreyer in [10], see also [4]. The result is a characterization of Betti tables of graded modules up to scalar multiples. For more information about Boij-Söderberg theory, we refers to [12]. There is not much known about the behavior of the Boij-Söderberg decomposition of an ideal in polynomial rings. Any characterization of Boij-Söderberg decompositions that one obtains will also assist to understand and interpret the more structral consequences of this decomposition of the Betti diagrams.

In this chapter, we focus on behavior of the Boij-Söderberg decompositions of lex-segment ideals. Lex-segment ideals have very particular Betti diagrams. The Bigatti-Hulett-Pardue [2, 19, 26] theorem shows that lex-segment ideals have the largest Betti numbers among the ideals with the same Hilbert function. This pivotal property of lex-segment ideals makes their Boij-Söderberg decompositions worthy to study. The main goal is to obtain a pattern for the Boij-Söderbeg decomposition of a lex ideal by using the decompositions of some other related lex-segment ideals. We mainly restrict our attention to the pure Betti diagrams that occur as summands in the decomposition.

Throughout this chapter, let $R = \mathbf{k}[x, y, z]$ be a polynomial ring of 3 variables, with the lexicographic order, $x >_{lex} y >_{lex} z$ and L be a lex-segment ideal in R. The ideal L can be decomposed as $L = x\mathfrak{a} + J$ where $\mathfrak{a}$ is also a lex-segment ideal in R and J is a lex-segment ideal in $\mathbf{k}[y, z]$. We study some relations of the Betti numbers of the ideals L, $\mathfrak{a}$ and J. We describe the entire Betti diagram of the lex ideal L in terms of the Betti numbers of the colon ideal $\mathfrak{a} = L : (x)$ and the stable ideal J. In Theorem 3.2.1 (see [15]), we describe "the beginning of the Boij-Söderberg decomposition" of L in terms of the decomposition of $\mathfrak{a}$. The algorithm of Boij-Söderberg decomposition itself provides a chain of degree sequences. The first degree sequence in the chain is the top degree sequence of the Betti diagram of L. By the algorithm, the second degree sequences is the top degree sequence of the remaining diagram after the subtraction of the first pure diagram with a suitable coefficient from the Betti diagram. It continues until the Betti diagram is decomposed completely. Thus, by saying that "the beginning of the Boij-Söderberg decomposition", we mean the beginning in the order of the chain of degree sequences in of L. Next we show that if there are t degree sequences of the length 3 in the Boij-Söderberg decomposition of $\mathfrak{a} = L : (x)$, we know the first t degree sequences of length 3 in the decomposition of L. We also believe that one could generalize the results shown in Theorem 3.2.1 to the polynomial rings with n variables for finite n.

We also work on pure diagrams of the Boij-Söderberg decomposition of the Betti diagrams of L and (L, x) in the polynomial ring $R = \mathbf{k}[x, y, z]$. Like in Theorem 3.2.1, we notice the similarity of the Boij-Söderberg decompositions of lex ideal L and (L, x). We reveal that the entire part of the Boij-Söderberg decomposition of (L, x) containing all pure diagrams of length less than 3 shows up precisely as the last part of the Boij-Söderberg decomposition of L, that is, all pure diagrams of length less than 3 in Theorem 3.2.3.

One naturally hopes to obtain the description of entire Boij-Söderberg decomposition of lex-segment ideal L. Thus, we conclude this chapter with further observations for a possible way to describe the entire chain of top degree sequences in the Boij-Söderberg decomposition of L. Thanks to Theorems 3.2.1 and 3.2.3, when $R = \mathbf{k}[x, y, z]$, we partly provide a description of the Boij-Söderberg decomposition of lex ideal L in terms of the lex ideals $\mathfrak{a} = L : (x)$ and (L, x). However, most of the time, this description does not cover all pure diagrams in the decomposition of L since there might be some pure diagrams of length 3 which are not described. The lexicographic order $x >_{lex} y >_{lex} z$ makes us to think about the colon ideals $\mathfrak{b} = L : (y)$ and $\mathfrak{c} = L : (z)$. Like for the case $\mathfrak{a} = L : (x)$, one may expect similar results for the lex ideals $\mathfrak{b}$ and $\mathfrak{c}$. Indeed, we see a relation between the Boij-Söderberg decompositions of the lex ideal L and the colon ideals $\mathfrak{b}$ and $\mathfrak{c}$. This allows us to almost give a full description of the pure diagrams appearing in the decomposition of L.

Chapter 2 Homogeneous Gorenstein Ideals

2.1 Preliminaries

This section addresses to the sufficient definitions, fundamental concepts and known results that are required for Chapter 2. We begin with the definitions of the ideals we work on throughout this chapter.

Definition 2.1.1. A ring R is called a *graded ring* if it has a decomposition of abelian groups

$$R = \bigoplus_{i \in \mathbb{N}_0} [R]_i$$

such that

$$[R]_i [R]_j \subset [R]_{i+j} \text{ for all } i, j \in \mathbb{N}_0.$$

Next one could define the graded R-modules.

Definition 2.1.2. Let R be a graded ring. A module M over R is called a *graded R-module* if it has decomposition $M = \bigoplus_{i \in \mathbb{N}_0} [M]_i$ as abelian groups such that $[R]_i [M]_j \subset [M]_{i+j}$ for all $i, j \in \mathbb{N}_0$.

The component $[M]_i$ is called *i-th homogeneous component of* M. Then an element $x \in [M]_i$ is called *homogeneous element of degree i*.

If an ideal I in a graded ring R is generated by homogeneous elements then it is called *homogeneous ideal*.

For any integer s, the module $M(s)$ stands for the module M with the shifted grading given by $[M(s)]_j := [M]_{s+j}$.

Definition 2.1.3. Let M be a graded R-module whose homogeneous components $[M]_j$ have finite dimension. The numerical function $\mathfrak{h} : \mathbb{N}_0 \to \mathbb{N}_0$ with

$$\mathfrak{h}_M(j) := \dim_k [M]_j$$

is called the Hilbert function of M.

Therefore, for an ideal I in R, we define the Hilbert function of the ideal R/I as $\mathfrak{h}_{R/I}(j) := \dim_k [R/I]_j$.

Definition 2.1.4. Let M be a graded module over a graded ring R and $\mathbb{F}$ be a complex of graded free R-modules.

$$\mathbb{F} \quad : \quad \cdots \to F_i \to \cdots \to F_1 \to F_0,$$

where

$$F_i = \oplus_{j \geq 0} R(-j)^{\beta_{i,j}}.$$

$\mathbb{F}$ is called *graded free resolution* of M if $\mathbb{F}$ is exact with homogeneous degree 0 maps and the cokernel of the map $F_1 \to F_0$ is M.

The numbers $\beta_{i,j} \in \mathbb{N}$ are called *Betti Numbers* of the module M, and they are recorded in the *Betti Diagram* of M.

The elements $a \in \mathbb{F}$ are called *homogeneous of degree* i if $a \in F_i$. The ideal generated by these homogneous elements is called *homogeneous ideal*.

Example 2.1.5. Let $R = \mathbf{k}[x_1, ..., x_n]$ be a polynomial ring of n variable over the field $\mathbf{k}$. Thus, $R = \oplus_j R_j$ where $R_j = \{\text{the homogeneous polynomials of degree } j\}$. Clearly, $R_0 = \mathbf{k}$ and $R_i R_j \subset R_{i+j}$.

For a simple example for a homogenous ideal, suppose $n = 3$. Then the ideal $I = (x_1^2, x_2^2, x_3) \subset R$ is homogeneous with (minimal) graded free resolution

$$0 \to R(-5) \xrightarrow{\begin{bmatrix} x_2^2 \\ x_1^2 \\ x_3 \end{bmatrix}} \begin{matrix} R(-4) \\ \oplus \\ R^2(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x_3 & 0 & -x_2^2 \\ 0 & x_3 & x_1^2 \\ -x_1^2 & -x_2^2 & 0 \end{bmatrix}} \begin{matrix} R^2(-2) \\ \oplus \\ R(-1) \end{matrix} \xrightarrow{[x_1^2\, x_2^2\, x_3]} R \to R/I \to 0$$

Hence the Betti diagram for R/I becomes

	0	1	2	3
total	**1**	**3**	**3**	**1**
0	1	1	–	–
1	–	2	2	–
2	–	–	1	1

.

The existance of the graded free resolutions of the graded modules over polynomial rings of finite variables are given by David Hilbert.

Theorem 2.1.6 (Hilbert Sygyzy Theorem)**.** *Let R be a polynomial ring of n variable over a field $\mathbf{k}$, that is, $R = \mathbf{k}[x_1, .., x_n]$. Every finitely generated graded R-module has a graded free resolution of finite length at most n.*

Before defining Gorenstein ideal, we give some necessary definitions from the dimension theory.

Definition 2.1.7. Let R be a Noetherian ring and I an ideal in R. Let $0 \neq M$ be a finitely generated R-module.

(i) The *Krull dimension* of R, $\dim R =$ the supremum of the lengths of chain of prime ideals in R.

(ii) $\operatorname{codim} I = \dim R - \dim R/I$.

(iii) $\operatorname{grade}(I) = \operatorname{grade}(R/I) = \operatorname{grade}(I, R) =$ the length of a maximal R-sequence in $I = \min\{i \mid \operatorname{Ext}_R^i(R/I, R) \neq 0\}$.

(iv) If $(R, \mathfrak{m})$ is local, $\operatorname{depth} M = \operatorname{grade}(\mathfrak{m}, M) =$ maximal lengths of M-sequence in $\mathfrak{m} = \min\{i \mid \operatorname{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\}$.

Definition 2.1.8. Let R be a Noetherian local ring and M an finitely generated R-module. M is called *Cohen-Macaulay module* if $\operatorname{depth} M = \dim M$. So R is called *Cohen Macaulay ring* if it is a Cohen-Macaulay module over itself.

A Cohen-Macaulay ring R is called *Gorenstein* if $\dim \operatorname{Ext}^d_R(R/\mathfrak{m}, R) = 1$ where $d = \operatorname{depth} R$.

We note that if R is Noetherian ring, R is said to be Gorenstein if the local ring $R_\mathfrak{m}$ is Gorenstein for all maximal ideals $\mathfrak{m}$.

Definition 2.1.9. Let R be a Noetherian ring. An ideal $I \subset R$ is called *Gorenstein ideal* of grade g if $\operatorname{projdim} R/I = \operatorname{grade}(I)$ and $\operatorname{Ext}^g_R(R/I, R) \cong R/I$.

The ideal given in Example 2.1.5 is a homogeneous Gorenstein ideal. Clearly, its minimal free resolution is self-dual.

2.2 Liaison theory

In this section, we look at some ideas from liaison theory in details. Liaison theory, which is also known as *linkage theory*, provides a nice classifications for ideals by links. The main idea is to link an ideal to a complete intersection, in some sense, much "nicer" form of ideal. Many of the results in this theory are developed when the links are complete intersections. This type of linkage is called as *complete intersection liaison*, i.e. CI-liaison. The other part of liaison theory is built on Gorenstein links, which sounds more general than complete intersections. This case of linkage is refered as *Gorenstein liaison*, G-liaison. In this thesis, we always work on Gorenstein links.

Throughout this section R denotes a commutative Noetherian ring that is either local with maximal ideal $\mathfrak{m}$ or graded. In the latter case we assume that $R = \oplus_{j \geq 0}[R]_j$ is generated as $[R]_0$-algebra by $[R]_1$ and $[R]_0$ is a field. We denote by $\mathfrak{m}$ its maximal homogenous ideal $\oplus_{j \geq 0}[R]_j$. If R is a graded ring, we consider only homogeneous ideals of R.

Assume that R is a Gorenstein ring.

Definition 2.2.1. An ideal $I \subset R$ is said to be (directly) *linked* to an ideal $J \subset R$ by a Gorenstein ideal $\mathfrak{c} \subset R$ if $\mathfrak{c} \subset I \cap J$ and $\mathfrak{c} : I = J$ and $\mathfrak{c} : J = I$.

Symbolically, we write $I \sim_\mathfrak{c} J$.

Liaison is the equivalence relation generated by linkage. The equivalence classes are called *liaison classes*. We always work in this generality. For a comprehensive introduction to liaison theory we refer to [24].

It is not difficult to show that all complete intersections of a fixed grade are in the same liaison class. Much more is true.

Theorem 2.2.2. *All Gorenstein ideals of R of grade g are in the same liaison class.*

This has been shown in [9] for non-Artinian homogeneous Gorenstein ideals in a polynomial ring. However, the arguments work in this generality.

From now on we focus on graded rings as the results hold analogously for local rings if one forgets the grading.

Let R be a graded Gorenstein ring, and let M be a graded R-module.

Definition 2.2.3. The *canonical module* of M, ω_M is a graded R-module which is defined as $[R]_0$-dual of the local cohomology module $H_{\mathfrak{m}}^{\dim M}(M)$. That is,

$$\omega_M = \operatorname{Hom}_R(H_{\mathfrak{m}}^{\dim M}(M), [R]_0).$$

As R is a graded Gorenstein ring, $\omega_R \cong R(s)$ with some integer s shift. By duality property, there is a graded isomorphism

$$\omega_M \cong \operatorname{Ext}_R^g(M, R)(s), \text{ where } g = \dim R - \dim M.$$

Therefore if I is a Gorenstein ideal in R, then $\omega_{R/I} \cong R/I(u)$ for some integer u by Definition 2.1.9, since $\omega_{R/I} \cong \operatorname{Ext}_R^g(R/I, R) \cong R/I$ where $g = \operatorname{grade}(I)$.

The following lemma gives us a very important short exact sequence.

Lemma 2.2.4. *[25, Lemma 3.5] If the ideals I and J are linked by a Gorenstein ideal $\mathfrak{c}$, there is a short exact sequence*

$$0 \to \mathfrak{c} \hookrightarrow I \to \omega_{R/J}(-s) \to 0, \tag{2.1}$$

where s is the integer such that $\omega_{R/\mathfrak{c}} \cong R/\mathfrak{c}(s)$.

For example, this sequence implies that R/J is Cohen-Macaulay if R/I has this property, and that the mapping cone procedure can be used to derive a free resolution of $\omega_{R/J}$ from the resolutions of I and $\mathfrak{c}$, and thus of J by dualizing (see [28]). Because of its importance we recall the mapping cone procedure (see, e.g., [30]).

Lemma 2.2.5. *Suppose that*

$$\mathbb{F}: \ 0 \to F_n \xrightarrow{d_n^M} F_{n-1} \to \dots \to F_i \xrightarrow{d_i^M} \dots \to F_1 \xrightarrow{d_1^M} F_0 \to M \to 0$$

$$\mathbb{G}: \ 0 \to G_n \xrightarrow{d_n^N} G_{n-1} \to \dots \to G_i \xrightarrow{d_i^N} \dots \to G_1 \xrightarrow{d_1^N} G_0 \to N \to 0$$

are graded free resolutions of M and N, respectively.

Let

$$0 \longrightarrow M \xrightarrow{\alpha} N \longrightarrow K \longrightarrow 0$$

be a short exact sequence of graded R-modules. Then α induces a comparison map $\varphi : \mathbb{F} \to \mathbb{G}$. Its mapping cone *is the following graded free resolution of K:*

$$0 \longrightarrow F_n \xrightarrow{\partial_g} G_n \oplus F_{n-1} \longrightarrow \dots \longrightarrow G_i \oplus F_{i-1} \xrightarrow{\partial_i} \dots$$

$$\longrightarrow G_1 \oplus F_0 \xrightarrow{\partial_1} G_0 \longrightarrow K \longrightarrow 0,$$

where $\partial_i = \begin{bmatrix} d_i^N & \varphi_{i-1} \\ 0 & -d_{i-1}^M \end{bmatrix}$ for $i = 1, 2, ..., n$.

An analysis of the mapping cone procedure implies the following result by Buchsbaum-Eisenbud [7] and Peskine-Szpiro [28].

Lemma 2.2.6. *Let $\mathfrak{c}$ be a Gorenstein ideal of R. Then*

(a) If R/I is Gorenstein and $\mathfrak{c} \subsetneqq I$ with $\operatorname{grade}(\mathfrak{c}) = \operatorname{grade}(I)$, then $J = \mathfrak{c} : I$ is perfect with at most one more minimal generator than $\mathfrak{c}$.

(b) Let $J \subset R$ be a perfect ideal such that $\mathfrak{c} \subsetneqq J$, $\operatorname{grade}(\mathfrak{c}) = \operatorname{grade}(J)$, and all minimal generators of $\mathfrak{c}$ are also minimal generators of J. If J has one more minimal generator than $\mathfrak{c}$, then $I = \mathfrak{c} : J$ is a Gorenstein ideal.

In Case (b), if $\mathfrak{c}$ is a complete intersection, then J is an almost complete intersection, that is, I has $g+1$ minimal generators, where $g = \operatorname{grade} I$.

2.3 Construction of Gorenstein Ideals

In this section we use liaison to produce a homogeneous Gorenstein ideal starting from two given homogeneous Gorenstein ideals. This also allows us to relate the Hilbert functions of the involved ideals.

Let R be a graded Gorenstein ring. Let $\mathfrak{a}$ and $\mathfrak{b} \subset \mathfrak{a}$ be homogeneous Gorenstein ideals in R of grade g and $g-1$, respectively. Let

$$\mathbb{A} : 0 \longrightarrow A_g = R(-v) \xrightarrow{a_g} A_{g-1} \xrightarrow{a_{g-1}} \dots. \longrightarrow A_1 \xrightarrow{a_1} R \longrightarrow 0$$

and

$$\mathbb{B} : 0 \longrightarrow B_{g-1} = R(-u) \xrightarrow{b_{g-1}} \dots. \longrightarrow B_1 \xrightarrow{b_1} R \longrightarrow 0$$

be graded minimal free resolutions of $R/\mathfrak{a}$ and $R/\mathfrak{b}$ respectively. The embedding $\mathfrak{b} \hookrightarrow \mathfrak{a}$ induces the following commutative diagram:

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & B_{g-1} = R(-u) & \xrightarrow{b_{g-1}} & \dots. & \longrightarrow & B_1 & \xrightarrow{b_1} & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha_{g-1}} & & & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0 = \mathrm{id}} & & \\
0 & \longrightarrow & A_g = R(-v) & \xrightarrow{a_g} & A_{g-1} & \xrightarrow{a_{g-1}} & \dots. & \longrightarrow & A_1 & \xrightarrow{a_1} & R & \longrightarrow & 0
\end{array} \tag{2.2}$$

Fixing bases for all the free modules, we identify the maps with their coordinate matrices. Using these assumptions and notation, the main result of this section is as following.

Theorem 2.3.1. *[16] Assume $d = u - v \geq 0$. Let $y \in \mathfrak{a}$ be a homogeneous element such that $\mathfrak{b} : y = \mathfrak{b}$. The embedding $\mu : (\mathfrak{b}, y) \hookrightarrow \mathfrak{a}$ induces an R-module homomorphism $\omega_{R/\mathfrak{a}} \to \omega_{R/(\mathfrak{b},y)}$ that is multiplication by some homogeneous element $\omega \in R$. Its degree is $d + \deg y$.*

Assume there is a homogeneous element $f \in R$ of degree d such that $\mathfrak{b} : (\omega + fy) = \mathfrak{b}$. Consider the ideal I obtained from $\mathfrak{a}$ by the two links

$$\mathfrak{a} \sim_{(\mathfrak{b},y)} J \sim_{(\mathfrak{b},\omega+fy)} I,$$

that is, $I = (\mathfrak{b}, \omega + fy) : [(\mathfrak{b}, y) : \mathfrak{a}]$. Then I is a Gorenstein ideal with the same grade as $\mathfrak{a}$. It can be written as

$$I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*) = (\mathfrak{b}, \alpha_{g-1}^* + (-1)^g f a_g^*),$$

where α_{g-1}^* and a_g^* are interpreted as row vectors and "+" indicates their componentwise sum whose entries, together with generators of $\mathfrak{b}$, generate I.

Proof. As in the proof of the Lemma 2.2.6, we use the mapping cone procedure repeatedly. Multiplication by y induces a short exact sequence

$$0 \to R/\mathfrak{b}(-e) \to R/\mathfrak{b} \to R/(\mathfrak{b}, y) \to 0,$$

where $e = \deg y$. Thus, we obtain a minimal graded free resolution $\mathbb{B}'$ of $(\mathfrak{b}, y)$:

$$\mathbb{B}' : 0 \to R(-u-e) \xrightarrow{d_g} \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-e) \end{matrix} \to \dots \to \begin{matrix} B_1 \\ \oplus \\ R(-e) \end{matrix} \xrightarrow{d_1} R \to (\mathfrak{b}, y) \to 0,$$

where

$$d_1 = \begin{bmatrix} b_1 & y \end{bmatrix},\ d_g = \begin{bmatrix} (-1)^{g-1} y \\ b_{g-1} \end{bmatrix}, \text{ and } d_i = \begin{bmatrix} b_i & (-1)^{i-1} y I_{m_{i-1}} \\ 0 & b_{i-1} \end{bmatrix} \text{ if } 2 \le i < g.$$

Here I_{m_i} denotes the identity matrix with $m_i = \operatorname{rank} B_i$ rows.

Using this resolution, the embedding $\mu : (\mathfrak{b}, y) \hookrightarrow \mathfrak{a}$ induces the following commutative diagram

$$\begin{array}{ccccccccccccc} 0 \to & R(-u-e) & \xrightarrow{d_g} & \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-e) \end{matrix} & \to \dots \to & \begin{matrix} B_i \\ \oplus \\ B_{i-1}(-e) \end{matrix} & \xrightarrow{d_i} & \dots \to & \begin{matrix} B_1 \\ \oplus \\ R(-e) \end{matrix} & \xrightarrow{d_1} & (\mathfrak{b}, y) & \to 0 \\ & \Big\downarrow{\scriptstyle \mu_g} & & \Big\downarrow{\scriptstyle \mu_{g-1}} & & \Big\downarrow{\scriptstyle \mu_i} & & & \Big\downarrow{\scriptstyle \mu_1} & & \Big\downarrow{\scriptstyle \mu} & \\ 0 \to & R(-v) & \xrightarrow{a_g} & A_{g-1} & \to \dots \to & A_i & \xrightarrow{a_i} & \dots \to & A_1 & \xrightarrow{a_1} & \mathfrak{a} & \to 0, \end{array} \tag{2.3}$$

where the maps have the form

$$\mu_i = \begin{bmatrix} \alpha_i & | & r_i \end{bmatrix}_{n_i \times (m_i + m_{i-1})}$$

The commutativity of the diagrams shows that α_i is an $n_i \times m_i$ matrix and r_i is an $n_i \times m_{i-1}$ satisfying $a_i r_i = (-1)^{i-1} y \alpha_{i-1} + r_{i-1} b_{i-1}$. One can notice that

$$\mu_g = \begin{bmatrix} \alpha_g = 0 & | & r_g \end{bmatrix} = r_g \in R \text{ where } a_g r_g = (-1)^{g-1} y \alpha_{g-1} + r_{g-1} b_{g-1}$$

Commutativity of the diagrams also give information about the degrees of the maps μ_i's and r_i's. First $a_1 r_1 = y$ implies $\deg r_1 = e - \deg a_1$. Then, $a_2 r_2 = r_1 b_1 - y \alpha_1$ gives $\deg r_2 = e + \deg b_1 - \deg a_1 - \deg a_2$. When we continue this till we get

$$a_i r_i = (-1)^{i-1} y \alpha_{i-1} + r_{i-1} b_{i-1}.$$

It follows that

$$\deg r_i = e + \sum_{k=1}^{i-1} \deg b_k - \sum_{k=1}^{i} \deg a_k \ \text{ for all } \ i = 1, 2, ..., g.$$

Therefore,

$$\begin{aligned}\deg \mu_g &= e + \sum_{k=1}^{g-1} \deg b_k - \sum_{k=1}^{g} \deg a_k \\ &= e + u - v \\ &= e + d.\end{aligned}$$

Thus the map μ_g is a multiplication by an element of degree $d+e$. We identify this element by ω.

The mapping cone $C(\mu)$ of $\mu : \mathbb{B}' \to \mathbb{A}$ is

$$0 \to R(-u-e) \xrightarrow{\partial_g} \begin{matrix} R(-v) \\ \oplus \\ R(-u) \\ \oplus \\ B_{g-2}(-e) \end{matrix} \to ... \to \begin{matrix} A_{i+1} \\ \oplus \\ B_i \\ \oplus \\ B_{i-1}(-e) \end{matrix} \xrightarrow{\partial_i} ... \to \begin{matrix} A_2 \\ \oplus \\ B_1 \\ \oplus \\ R(-e) \end{matrix} \xrightarrow{\partial_1} A_1,$$

where the maps are

$$\partial_1 = \begin{bmatrix} a_2 & \mu_1 \end{bmatrix} = \begin{bmatrix} a_2 & \alpha_1 & r_1 \end{bmatrix}, \ \partial_g = \begin{bmatrix} \mu_g \\ -d_g \end{bmatrix} = \begin{bmatrix} r_g \\ (-1)^g y \\ -b_{g-1} \end{bmatrix},$$

$$\text{and } \ \partial_i = \begin{bmatrix} a_{i+1} & \mu_i \\ 0 & -d_i \end{bmatrix} = \begin{bmatrix} a_{i+1} & \alpha_i & r_i \\ 0 & -b_i & (-1)^i y I_{m_{i-1}} \\ 0 & 0 & -b_{i-1} \end{bmatrix} \ \text{ if } 2 \le i < g.$$

The Sequence (2.1) shows that $C(\mu)$ gives a free resolution of a shift of the canonical module of R/J. Now, dualizitation of $C(\mu)$ provides the following complex;

$$0 \leftarrow R(u + \deg y) \xleftarrow{\partial_g^*} \begin{matrix} R(v) \\ \oplus \\ R(u) \\ \oplus \\ B_{g-2}^*(\deg y) \end{matrix} \leftarrow ... \leftarrow \begin{matrix} A_{i+1}^* \\ \oplus \\ B_i^* \\ \oplus \\ B_{i-1}^*(\deg y) \end{matrix} \xleftarrow{\partial_i^*} ... \leftarrow \begin{matrix} A_2^* \\ \oplus \\ B_1^* \\ \oplus \\ R(\deg y) \end{matrix} \xleftarrow{\partial_1^*} A_1^* \leftarrow 0$$

Hence, the dualized and shifted complex $C(\mu)^*(-u - \deg y)$

$$0 \to A_1^*(-u-e) \xrightarrow{\partial_1^*} \begin{matrix} A_2^*(-u-e) \\ \oplus \\ B_1^*(-u-e) \\ \oplus \\ R(-u) \end{matrix} \to ... \to \begin{matrix} A_{i+1}^*(-u-e) \\ \oplus \\ B_i^*(-u-e) \\ \oplus \\ B_{i-1}^*(-u) \end{matrix} \xrightarrow{\partial_i^*} ... \to \begin{matrix} R(v-u-e) \\ \oplus \\ R(-e) \\ \oplus \\ B_{g-2}^*(-u) \end{matrix} \xrightarrow{\partial_g^*} R \to 0$$

provides a graded free resolution of $J = (\mathfrak{b}, y, \omega)$. The dual maps are $\partial_i^* = \partial_i^t$, for $i = 1, 2, ..., g$. So $\partial_g^* = \begin{bmatrix} \mu_g & (-1)^g y & -b_{g-1}^t \end{bmatrix}$. So by lemma 2.2.6 part (a), the ideal generated by ∂_g^*, which is $J := (\mathfrak{b}, y, \omega)$, is CM of grade g with a graded free resolution $C(\mu)^*(-u-e)$. One should notice that we do not claim that the stated generating set of J is minimal.

By assumption, there is a homogeneous element $f \in R$ of degree $u - v = d \geq 0$ such that $z := \omega + fy$ is regular in $R/\mathfrak{b}$. Hence, $(\mathfrak{b}, z)$ is a Gorenstein ideal of grade g in J. Consider now the second link

$$J \sim_{(\mathfrak{b}, \omega + fy)} I.$$

We know that $A_i^*(-v) = A_{g-i} = A_i(v)$ and $B_i^*(-u) = B_{g-1-i} = B_i(u)$ since $\mathfrak{a}$ and $\mathfrak{b}$ are Gorenstein of grades g and $g-1$, respectively. Thus, this helps to observe the following comparison map ξ from the resolution of $R/(\mathfrak{b}, z)$ to $C(\mu)^*(-u-e)$. As in the case of the ideal $(\mathfrak{b}, y)$, a mapping cone gives a free resolution of $(\mathfrak{b}, z)$. Thus, the embedding $\xi : (\mathfrak{b}, z) \hookrightarrow J$ induces the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to R(-u-e-d) & \xrightarrow{t_g} & \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-e-d) \end{matrix} & \to \dots \to & \begin{matrix} B_1 \\ \oplus \\ R(-e-d) \end{matrix} & \xrightarrow{t_1} & (\mathfrak{b}, z) \to 0 \\
\Big\downarrow{\scriptstyle \xi_g} & & \Big\downarrow{\scriptstyle \xi_{g-1}} & & \Big\downarrow{\scriptstyle \xi_1} & & \Big\downarrow{\scriptstyle \xi} \\
0 \to A_1^*(-u-e) & \xrightarrow{\partial_1^*} & \begin{matrix} A_2^*(-u-e) \\ \oplus \\ B_1^*(-u-e) \\ \oplus \\ R(-u) \end{matrix} & \to \dots \to & \begin{matrix} A_g^*(-u-e) \\ \oplus \\ B_{g-1}^*(-u-e) \\ \oplus \\ B_{g-2}^*(-u-e) \end{matrix} & \xrightarrow{\partial_g^*} & J \to 0
\end{array}$$

where the maps are

$$t_1 = \begin{bmatrix} b_1 & z \end{bmatrix}, \; t_g = \begin{bmatrix} (-1)^{g-1} z \\ b_{g-1} \end{bmatrix}, \quad \text{and } t_i = \begin{bmatrix} b_i & (-1)^{i-1} z I_{m_{i-1}} \\ 0 & b_{i-1} \end{bmatrix} \text{ if } 2 \leq i < g.$$

Since $J = (\mathfrak{b}, y, \omega + fy)$, we can choose the following coordinate matrix for ξ_1:

$$\xi_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \\ m_1+2 \end{array} \begin{array}{c} \begin{array}{cccc|c} 1 & \dots & \dots & m_1 & m_1+1 \end{array} \\ \left[\begin{array}{c|c} 0 & 1 \\ 0 & (-1)^g f \\ \hline & 0 \\ \gamma_1 & \vdots \\ & \vdots \\ & 0 \end{array}\right] \end{array}$$

where the matrix γ_1 is invertible.

By Sequence (2.1), the mapping cone $C(\xi)$ gives a free resolution of (a shift of) the canonical module R/I. Using the self-duality of the free resolutions $\mathbb{A}$ and $\mathbb{B}$,

$C(\xi)$ can be re-written as

$$0 \to B_{g-1}(-\deg z) \xrightarrow{l_g} \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-\deg z) \\ \oplus \\ A_{g-1}(-\deg z) \end{matrix} \to \ldots \to \begin{matrix} B_1 \\ \oplus \\ R(-\deg z) \\ \oplus \\ A_1(-\deg z) \\ \oplus \\ B_1(-e) \\ \oplus \\ B_2 \end{matrix} \xrightarrow{l_1} \begin{matrix} R(-\deg z) \\ \oplus \\ R(-e) \\ \oplus \\ B_1 \end{matrix}, \quad (2.4)$$

where

$$l_1 = \begin{bmatrix} \partial_g^* & \xi_1 \end{bmatrix}.$$

Since the matrix γ_1 and the upper right entry of ξ_1 are invertible, the cokernel of l_1 is isomorphic to coker $\bar{l}_1$, where

$$\bar{l}_1 = \begin{bmatrix} \alpha_{g-1}^* + (-1)^g f a_g^* & b_{g-1}^* \end{bmatrix} : A_1(-\deg z) \oplus B_1(-e) \oplus B_2 \to R(-e).$$

It follows that the canonical module of R/I has only one minimal generator. Hence, I is a Gorenstein ideal and coker $\bar{l}_1 \cong (R/I)(-e)$. The latter implies the claimed description of a generating set of the ideal I. □

Notice that a "sufficiently general" choice of the element f always gives a desired element $\omega + fy$ in Theorem 2.3.1, at least if the field $k = R/\mathfrak{m}$ is infinite.

We illustrate the result by a simple example.

Example 2.3.2. Consider the complete intersections $\mathfrak{a} = (x, y, z)$ and $\mathfrak{b} = (x^2 - z^2, y^2 - z^2)$ in the polynomial ring $k[x, y, z]$, where k is a field of characteristic zero. Linking $\mathfrak{a}$ by $\mathfrak{b} + (z^2)$, we get as residual $J = \mathfrak{b} + (z^2, xyz)$. Choosing $f = 5z$, we link J by $\mathfrak{b} + (xyz + fz^2)$ to

$$I = \mathfrak{b} + (xf + yz, yf + xz, zf + xy) = (x^2 - z^2, y^2 - z^2, xz, yz, xy + 5z^2).$$

Observe that for the second link we cannot take $f = z$ because $xyz + z^3$ is a zero divisor modulo $\mathfrak{b}$.

The next proposition shows that similar techniques as in Theorem 2.3.1 could help to construct Cohen Macaulay ideals of certain types.

Proposition 2.3.3. *Let $\mathfrak{a}$ be a homogeneous Cohen Macaulay (CM) ideal of type t and $\mathfrak{b} \subset \mathfrak{a}$ be a homogeneous Gorenstein ideal with* $\operatorname{grade}(\mathfrak{a}) = 1 + \operatorname{grade}(\mathfrak{b})$.

Suppose $y \in \mathfrak{a}$ be a regular element in $R/\mathfrak{b}$ and then there is row vector $(w_1, w_2, ..., w_t)$ with $\deg(w_t) \geq \deg(w_i)$ *for all $i = 1, .., t-1$ and* $\deg(w_t) \geq \deg(y)$*. If*

$$\mathfrak{a} \sim_{(\mathfrak{b}, y)} (\mathfrak{b}, y, w_1, w_2, ..., w_t) \sim_{(\mathfrak{b}, w_t + \sum_{i=1}^{t-1} g_i w_i + fy)} I$$

where g_i's and f are homogeneous elements with $\deg(f) = \deg(w_t) - \deg(y)$ *and* $\deg(g_i) = \deg(w_t) - \deg(w_i)$ *such that* $\mathfrak{b} : w_t + \sum_{i=1}^{t-1} g_i w_i + fy = \mathfrak{b}$, *then I is a homogeneous CM ideal of type t.*

Proof. As the statement is a generalization of Theorem 2.3.1 we follow the same path as in the proof of 2.3.1. Say $g = \text{grade}(\mathfrak{a})$. As the ideal $\mathfrak{a}$ is a CM ideal of type t, the last free module of the minimal free resolution of $R/\mathfrak{a}$ is

$$A_g = \oplus_{i=1}^{t} R(-v_i).$$

Say $e = \deg y$. Then the comparison map between the minimal free resolutions of $R/(\mathfrak{b}, y)$ and $R/\mathfrak{a}$ becomes

$$\begin{array}{ccccccccccccc}
0 \to & R(-u-e) & \xrightarrow{d_g} & \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-e) \end{matrix} & \to \dots \to & \begin{matrix} B_i \\ \oplus \\ B_{i-1}(-e) \end{matrix} & \xrightarrow{d_i} & \dots \to & \begin{matrix} B_1 \\ \oplus \\ R(-e) \end{matrix} & \xrightarrow{d_1} & R \\
& \downarrow{\scriptstyle \mu_g} & & \downarrow{\scriptstyle \mu_{g-1}} & & \downarrow{\scriptstyle \mu_i} & & & \downarrow{\scriptstyle \mu_1} & & \| \\
0 \to & \oplus_{i=1}^{t} R(-v_i) & \xrightarrow{a_g} & A_{g-1} & \to \dots \to & A_i & \xrightarrow{a_i} & \dots \to & A_1 & \xrightarrow{a_1} & R
\end{array}$$

Thus, the last map is a column vector,

$$\mu_g = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_t \end{bmatrix}$$

where $\deg w_i = u + \deg(y) - v_i = u + e - v_i$ and $\deg w_t \geq \deg w_i$, for $i = 1, ..., t-1$. Then as in the proof of the Proposition 2.3.1, dual of the mapping cone of μ gives a free resolution for a CM ideal J which has t more minimal generator than $(\mathfrak{b}, y)$. That is,

$$J = (\mathfrak{b}, y, w_1, w_2, ..., w_t).$$

Say

$$z := w_t + \sum_{i=1}^{t-1} g_i w_i + fy$$

where g_i and f are homogeneous elements in R with degrees $\deg(g_i) = \deg(w_t) - \deg(w_i)$ and $\deg(f) = \deg(w_t) - e$ such that $\mathfrak{b} : z = \mathfrak{b}$. Therefore, $\deg z = \deg w_t$.

Now consider the following comparison map ξ;

$$
\begin{array}{ccccccccc}
0 & \to & R(-u-\deg w_t) & \xrightarrow{t_g} & \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-\deg w_t) \end{matrix} & \to \dots \to & \begin{matrix} B_1 \\ \oplus \\ R(-\deg w_t) \end{matrix} & \xrightarrow{t_1} & R/(\mathfrak{b}, z) \\
 & & \Big\downarrow{\scriptstyle \xi_g} & & \Big\downarrow{\scriptstyle \xi_{g-1}} & & \Big\downarrow{\scriptstyle \xi_1} & & \\
0 & \to & A_1^*(-u-e) & \xrightarrow{\partial_1^*} & \begin{matrix} A_2^*(-u-e) \\ \oplus \\ B_{g-2}(-e) \\ \oplus \\ R(-u) \end{matrix} & \to \dots \to & \begin{matrix} \oplus_{i=1}^{t-1} R(v_i-u-e) \\ \oplus \\ R(-\deg w_t) \\ \oplus \\ R(-e) \\ \oplus \\ B_1 \end{matrix} & \xrightarrow{\partial_g^*} & R/J
\end{array}
$$

Then the mapping cone ξ gives a free resolution for the canonical module of the desired ideal I, say $\omega_{R/I}$;

$$
0 \to R(-u-\deg w_t) \xrightarrow{l_g} \begin{matrix} R(-u) \\ \oplus \\ B_{g-2}(-e) \\ \oplus \\ A_1^*(-u-e) \end{matrix} \to \dots . \to \begin{matrix} \cancel{B_1} \\ \oplus \\ \cancel{R(-\deg w_t)} \\ \oplus \\ A_{g-1}^*(-u-e) \\ \oplus \\ B_1(-e) \\ \oplus \\ B_2 \end{matrix} \xrightarrow{l_1} \begin{matrix} \oplus_{i=1}^{t-1} R(-\deg w_i) \\ \oplus \\ \cancel{R(-\deg w_t)} \\ \oplus \\ R(-e) \\ \oplus \\ \cancel{B_1} \end{matrix} \to \omega_{R/I}
$$

As we expected, the cancellations at the beginning are followed by the fact that the ideal J can be also written as

$$
J = (\mathfrak{b}, y, w_1, ..., w_{t-1}, \underbrace{w_t + \sum_{i=1}^{t-1} g_i w_i + fy}_{z}) \supsetneq (\mathfrak{b}, \underbrace{w_t + \sum_{i=1}^{t-1} g_i w_i + fy}_{z}).
$$

Hence the linkage implies that the ideal $I = \operatorname{Im} l_g^* = \operatorname{Im} \left[\xi_g^* \quad (-1)^g z \quad -b_{g-1}\right]$ is CM with the same grade as $\mathfrak{a}$ and I is CM of type t since $\omega_{R/I}$ has t minimal generators. $\square$

Using basic properties of links, we conclude this section by relating the Hilbert function of I to the Hilbert functions of $\mathfrak{a}$ and $\mathfrak{b}$.

Corollary 2.3.4. *Adopt the notation and assumptions of Theorem 2.3.1. Then, for all integers j, the Hilbert function of R/I is given by*

$$
\mathfrak{h}_{R/I}(j) = \mathfrak{h}_{R/\mathfrak{a}}(j-d) + \mathfrak{h}_{R/\mathfrak{b}}(j) - \mathfrak{h}_{R/\mathfrak{b}}(j-d).
$$

Proof. In the proof of Theorem 2.3.1 we have seen that the mapping cone (2.4) gives the following short exact sequence;

$$0 \longrightarrow (R/I)(-e) \longrightarrow R/(\mathfrak{b}, z) \longrightarrow R/J \longrightarrow 0,$$

where $\deg z = d + e$. Furthermore, by symmetry of liaison, the first link provides $(\mathfrak{b}, y) : \omega = \mathfrak{a}$. This implies the short exact sequence

$$0 \longrightarrow (R/\mathfrak{a})(-e-d) \longrightarrow R/(\mathfrak{b}, y) \longrightarrow R/J \longrightarrow 0.$$

Combining the above two sequences we deduce

$$\begin{aligned} \mathfrak{h}_{R/I}(j) &= \mathfrak{h}_{R/(\mathfrak{b},z)}(j+e) - \mathfrak{h}_{R/J}(j+e) \\ &= \mathfrak{h}_{R/(\mathfrak{b},z)}(j+e) - \mathfrak{h}_{R/(\mathfrak{b},y)}(j+e) + \mathfrak{h}_{R/\mathfrak{a}}(j-d) \\ &= -\mathfrak{h}_{R/\mathfrak{b}}(j-d) + \mathfrak{h}_{R/\mathfrak{b}}(j) + \mathfrak{h}_{R/\mathfrak{a}}(j-d), \end{aligned}$$

as claimed. □

2.4 The Kustin-Miller construction in graded rings

In the previous section we have seen that the Complex (2.4) provides a free resolution of the Gorenstein ideal I, constructed in Theorem 2.3.1. However, this resolution is not minimal if $g \geq 3$. In this Section we construct a smaller resolution of I by modifying the approach of Kustin and Miller in [22].

Theorem 2.4.1. *[16] Adopt the notation and assumptions of Theorem 2.3.1. Then there is an short exact sequence of graded R-modules*

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \longrightarrow R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Moreover, the ideal I has a graded free resolution of the form

$$0 \to B_{g-1}(-d) \to \begin{matrix} A_{g-1}(-d) \\ \oplus \\ B_{g-2}(-d) \end{matrix} \to \begin{matrix} B_{g-2} \\ \oplus \\ A_{g-2}(-d) \\ \oplus \\ B_{g-3}(-d) \end{matrix} \to \ldots \to \begin{matrix} B_2 \\ \oplus \\ A_2(-d) \\ \oplus \\ B_1(-d) \end{matrix} \to \begin{matrix} B_1 \\ \oplus \\ A_1(-d) \end{matrix} \to I \to 0,$$

where the maps are described in the proof below.

Proof. We follow the approach in [22], but adjust it suitably. Thus, we focus on the needed modifications and refer for more details to [22].

First, the mapping cone $\mathbb{M}$ of $\alpha : \mathbb{B} \to \mathbb{A}$ gives the exact sequence:

$$\mathbb{M}: \ 0 \to \begin{matrix} A_g \\ \oplus \\ B_{g-1} \end{matrix} \to \ldots \to \begin{matrix} A_{j+1} \\ \oplus \\ B_j \end{matrix} \xrightarrow{\begin{bmatrix} a_{j+1} & \alpha_j \\ 0 & -b_j \end{bmatrix}} \begin{matrix} A_{j+1} \\ \oplus \\ B_j \end{matrix} \to \ldots \to \begin{matrix} A_2 \\ \oplus \\ B_1 \end{matrix} \xrightarrow{\begin{bmatrix} a_2 & \alpha_1 \end{bmatrix}} A_1 \to \mathfrak{a}/\mathfrak{b}. \tag{2.5}$$

Second, by [7, Proposition 1.1], the resolutions $\mathbb{A}$ and $\mathbb{B}$ admit a DGC algebra structure. These induce perfect pairings $B_i \times B_{g-1-i} \to B_g$ and $A_i \times A_{g-i} \to A_g$. We use the former to define the composition

$$\beta : \mathbb{A} \underset{\cong}{\xrightarrow{\gamma}} Hom_R(\mathbb{A}, R)(-v) \xrightarrow{\alpha^*[1]} Hom_R(\mathbb{B}, R)(-v)[-1] \underset{\cong}{\xrightarrow{\rho}} \mathbb{B}(d)[-1]$$

with the following commutative diagrams and ρ and γ are the isomorphisms induced by multiplicative structure of $\mathbb{B}$ and $\mathbb{A}$ respectively.

$$\begin{tikzcd}
0 \arrow[r] & R(-v) \arrow[r, "a_g"] \arrow[d, "\gamma_g"', "\cong"] & A_{g-1} \arrow[r, "a_{g-1}"] \arrow[d, "\gamma_{g-1}"', "\cong"] & \cdots \arrow[r] & A_j \arrow[r, "a_j"] \arrow[d, "\gamma_j"', "\cong"] & \cdots \arrow[r] & A_1 \arrow[r, "a_1"] \arrow[d, "\gamma_1"', "\cong"] & R \arrow[d] \\
0 \arrow[r] & R(-v) \arrow[r, "a_1^*"] \arrow[d, "\alpha_0^*"'] & A_1^*(-v) \arrow[r, "a_2^*"] \arrow[d, "\alpha_1^*"'] & \cdots \arrow[r] & A_{g-j}^*(-v) \arrow[r, "a_{g-j-1}^*"] \arrow[d, "\alpha_{g-j}^*"'] & \cdots \arrow[r] & A_{g-1}^*(-v) \arrow[r, "a_g^*"] \arrow[d, "\alpha_{g-1}^*"'] & R \\
0 \arrow[r] & R(-v) \arrow[r, "b_1^*"] \arrow[d, "\rho_g"', "\cong"] & B_1^*(-v) \arrow[r, "b_2^*"] \arrow[d, "\rho_{g-1}"', "\cong"] & \cdots \arrow[r] & B_{g-j}^*(-v) \arrow[r, "b_{g-j-1}^*"] \arrow[d, "\rho_j"', "\cong"] & \cdots \arrow[r] & B_{g-1}^*(-v) \arrow[d, "\rho_1"', "\cong"] & \\
0 \arrow[r] & R(-v) \arrow[r, "b_{g-1}"] & B_{g-2}(d) \arrow[r, "b_{g-2}"] & \cdots \arrow[r] & B_{j-1}(d) \arrow[r, "b_{j-1}"] & \cdots \arrow[r, "b_1"] & R(d) \arrow[r] & R/\mathfrak{b}(d)
\end{tikzcd} \tag{2.6}$$

where $\beta_i := \rho_i \circ \alpha^*_{g-i} \circ \gamma_i$ for every $i = 1, 2, ..., g$. Therefore, the degree d homomorphisms $\beta_i : A_i \to B_{i-1}(d)$ map $x_i \in A_i$ on the unique element $\beta_i(x_i)$ such that, for all $z_{g-i} \in B_{g-i}$,

$$\beta_i(x_i) \cdot z_{g-i} = (-1)^{i+1} x_i \cdot \alpha_{g-i}(z_{g-i}) \tag{2.7}$$

in $A_g = B_{g-1}(d)$. It follows that

$$\beta_1(x_1) = x_1 \cdot \alpha_{g-1}(1_{B_{g-1}}) \tag{2.8}$$

and that β_g is multiplication by the unit $(-1)^{g+1}$. Using the perfect pairings on $\mathbb{A}$, we also get

$$\beta_i \circ a_{i+1} = b_i \circ \beta_{i+1}. \tag{2.9}$$

Third, there is a map $B_i \otimes B_j \to A_{i+j}$ which maps $z_i \otimes z_j$ to $\alpha_{i+j}(z_i z_j) - (\alpha_i z_i)(\alpha_j z_j)$. This map induces a map of complexes $S_2(\mathbb{B}) \to \mathbb{A}$ which is null homotopic. [22, Lemma 1.1] shows that Diagram (2.2) induces a graded homomorphism of complexes $\xi : \mathbb{B} \otimes \mathbb{B} \to \mathbb{A}[1]$ such that, for all $z_i \in B_i$:

(i) $B_i \otimes B_j \to A_{i+j+1}$ is defined if $i, j \geq 0$,

(ii) $\xi(z_i \otimes z_j) = (-1)^i \xi(z_j \otimes z_i)$,

(iii) $\xi(z_i \otimes z_i) = 0$ if i is odd,

(iv) $\xi(z_0 \otimes z_i) = 0$, and

(v) $\alpha_{i+j}(z_i z_j) - \alpha_i(z_i) \cdot \alpha_j(z_j) = \xi(b_i(z_i) \otimes z_j) + (-1)^i \xi(z_i \otimes b_j(z_j)) + a_{i+j+1}(\xi(z_i \otimes z_j))$.

Finally, we define a degree d homomorphism of complexes $h : \mathbb{B} \to \mathbb{B}(d)$ by mapping $z_i \in B_i$ on the unique element $h_i(z_i)$ such that, for all $z_{g-1-i} \in B_{g-1-i}$,

$$h_i(z_i) \cdot z_{g-1-i} = (-1)^{i+1}\xi(z_i \otimes z_{g-1-i}).$$

Notice that the above Condition (iv) implies $h_0 = h_{g-1} = 0$ and by using Condition (v), we compute that

$$\begin{aligned}
[(b_i \circ h_i + h_{i-1} \circ b_i)(y_i)]y_{g-i} &= (b_i \circ h_i(y_i))y_{g-i} + (h_{i-1} \circ b_i(y_i))y_{g-i} \\
&= (-1)^{i+1}h_i(y_i) \circ b_{g-i}(y_{g-i}) + (h_{i-1} \circ b_i(y_i))y_{g-i} \\
&= (-1)^{i+1}h_i(y_i) \circ b_{g-i}(y_{g-i}) + (-1)^i\xi(b_i(y_i) \otimes y_{g-i}) \\
&= \xi(y_i \otimes b_{g-i}(y_{g-i})) + (-1)^i\xi(b_i(y_i) \otimes y_{g-i}) \\
&= \xi(y_i \otimes b_{g-i}(y_{g-i})) + (-1)^i\alpha_g(y_iy_{g-i}) + (-1)^{i+1}(\alpha_i(y_i)\alpha_{g-i}(y_{g-i})) \\
&\quad -\xi(y_i \otimes b_{g-i}(y_{g-i})) - a_{g+1}\xi(y_i \otimes y_{g-i}) \\
&= (-1)^{i+1}\alpha_i(y_i)\alpha_{g-i}(y_{g-i}).
\end{aligned}$$

We also notice that the isomorphisms γ and ρ can be described as follows;

$$\gamma_i = \begin{cases} s_i^{\mathbb{A}} & \text{if } i \equiv 0 \pmod 4 \\ s_i^{\mathbb{A}} & \text{if } i \equiv 1 \pmod 4 \\ -s_i^{\mathbb{A}} & \text{if } i \equiv 2 \pmod 4 \\ -s_i^{\mathbb{A}} & \text{if } i \equiv 3 \pmod 4 \end{cases} \quad \text{and} \quad \rho_i = \begin{cases} -s_{i-1}^{\mathbb{B}} & \text{if } i \equiv 0 \pmod 4 \\ s_{i-1}^{\mathbb{B}} & \text{if } i \equiv 1 \pmod 4 \\ s_{i-1}^{\mathbb{B}} & \text{if } i \equiv 2 \pmod 4 \\ -s_{i-1}^{\mathbb{B}} & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

where $s_i^{\mathbb{A}} : A_i \to A^*_{g-i}$ and $s_i^{\mathbb{B}} : B^*_{g-1-i} \to B_i$ are isomorphisms induced by multiplications. So the composition $\rho_i \circ \alpha^*_{g-i} \circ \gamma_i$ alternate signs. That is, "+" if i is odd, "−" if i is even. Thus,

$$\begin{aligned}
(\beta_i\alpha_i(y_i))(y_{g-i}) &= (-1)^{i+1}(s_{i-1}^{\mathbb{B}} \circ \alpha^*_{g-i} \circ s_i^{\mathbb{A}}(\alpha_i(y_i)))(y_{g-i}) = (-1)^{i+1}(\alpha^*_{g-i}(s_i^{\mathbb{A}}(\alpha_i(y_i))))y_{g-i} \\
&= (-1)^{i+1}s_i^{\mathbb{A}}(\alpha_i(y_i))(\alpha_{g-i}(y_{g-i})) \\
&= (-1)^{i+1}\alpha_i(y_i)\alpha_{g-i}(y_{g-i}).
\end{aligned}$$

Hence, it is followed that h satisfies

$$\beta_i \circ \alpha_i = h_{i-1} \circ b_i + b_i \circ h_i. \tag{2.10}$$

Consider now the following diagram with the map $\delta : \mathbb{M}[1] \to \mathbb{B}(d)$ as

$$\delta_i := [\beta_i, h_{i-1} + (-1)^{i-1}f.id_{B_{i-1}}] \text{ for } i = 2, ..., g \text{ and } \delta_1 = \beta_1 - fa_1.$$

$$\begin{array}{ccccccccccccc}
0 \to & A_g \oplus B_{g-1} & \to .... \to & A_{j+1} \oplus B_j & \to & .... \to & A_2 \oplus B_1 & \xrightarrow{[a_2,\alpha_1]} & A_1 & \to & \mathfrak{a}/\mathfrak{b} \\
& \downarrow{\scriptstyle [\beta_g,(-1)^g f\,\mathrm{id}]} & & \downarrow{\scriptstyle [\beta_{j+1},h_j+(-1)^{j+1}f\,\mathrm{id}]} & & & \downarrow{\scriptstyle [\beta_2,h_1+f\,\mathrm{id}]} & & \downarrow{\scriptstyle \beta_1+fa_1} & & \\
0 \to & B_{g-1}(d) & \to .... \to & B_j(d) & \xrightarrow{b_j} & .... \to & B_1(d) & \xrightarrow{b_1} & R(d) & \to & (R/\mathfrak{b})(d)
\end{array} \tag{2.11}$$

We easily see that

$$\begin{aligned}
[\beta_j, h_{j-1} + (-1)^{j-1} f.id] \begin{bmatrix} a_{j+1} & \alpha_j \\ 0 & -b_j \end{bmatrix} &= [\beta_j a_{j+1}, \beta_j \alpha_j - h_{j-1} b_j + (-1)^j f b_j] \\
&\overset{(2.9)}{=} [\beta_j a_{j+1}, \beta_j \alpha_j - h_{j-1} b_j + (-1)^j f b_j] \\
&\overset{(2.10)}{=} [\beta_j a_{j+1}, b_j h_j + (-1)^j f b_j] \\
&= b_j [\beta_{j+1}, h_j + (-1)^j f.id]
\end{aligned}$$

Therefore, all the squares commute

$$\begin{array}{ccc}
A_{j+1} \oplus B_j & \xrightarrow{\begin{bmatrix} a_{j+1} & \alpha_j \\ 0 & -b_j \end{bmatrix}} & A_j \oplus B_{j-1} \\
{\scriptstyle [\beta_{j+1}, h_j + (-1)^j f.id]} \downarrow & & {\scriptstyle [\beta_j, h_{j-1} + (-1)^{j-1} f.id]} \downarrow \\
B_j(d) & \xrightarrow{b_j} & B_{j-1}(d)
\end{array}$$

It follows that $\beta_1 + f a_1$ induces a homomorphism $\varphi : \mathfrak{a}/\mathfrak{b} \to (R/\mathfrak{b})(d)$ such that the resulting right-most square in the above diagram also becomes commutative. Thus, the mapping cone gives the chain complex

$$\mathbb{L} : 0 \longrightarrow \begin{matrix} A_g \\ \oplus \\ B_{g-1} \end{matrix} \xrightarrow{l_g} \begin{matrix} B_{g-1}(d) \\ \oplus \\ A_{g-1} \\ \oplus \\ B_{g-2} \end{matrix} \longrightarrow \ldots \longrightarrow \begin{matrix} B_2(d) \\ \oplus \\ A_2 \\ \oplus \\ B_1 \end{matrix} \xrightarrow{l_2} \begin{matrix} B_1(d) \\ \oplus \\ A_1 \end{matrix} \xrightarrow{l_1} R(d), \tag{2.12}$$

where the maps are

$$l_1 = \begin{bmatrix} b_1 & \beta_1 + f a_1 \end{bmatrix}, \; l_2 = \begin{bmatrix} b_2 & \beta_2 & h_1 + f\,\mathrm{id} \\ 0 & -a_2 & -\alpha_1 \end{bmatrix}, \; l_g = \begin{bmatrix} \beta_g & h_{g-1} + (-1)^g f\,\mathrm{id} \\ -a_g & -\alpha_{g-1} \\ 0 & b_{g-1} \end{bmatrix},$$

and

$$l_i = \begin{bmatrix} b_i & \beta_i & h_{i-1} + (-1)^i f\,\mathrm{id} \\ 0 & -a_i & -\alpha_{i-1} \\ 0 & 0 & b_{i-1} \end{bmatrix} \text{ if } 3 \leq i \leq g-1.$$

Using Equation (2.8), it follows that

$$\operatorname{Im} l_1 = \mathfrak{b} + (\alpha_{g-1}^* + f a_g^*).$$

All this remains true if we replace f by $(-1)^g f$. Then Theorem 2.3.1 shows that $I = \mathfrak{b} + (\alpha_{g-1}^* + (-1)^g f a_g^*)$ is a Gorenstein ideal, and Diagram (2.11) yields that I fits into an exact sequence

$$(\mathfrak{a}/\mathfrak{b})(-d) \xrightarrow{\varphi} R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0\cdot$$

It allows us to compute the Hilbert function of $\ker \varphi$. Comparing with Corollary 2.3.4, we deduce that the kernel of φ is trivial. Hence, we obtain the desired short exact sequence

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \xrightarrow{\varphi} R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Now it follows that the above complex $\mathbb{L}$ gives a free resolution of $I(d)$. Since β_g is multiplication by a unit, we can split off the isomorphic free modules A_g and $B_{g-1}(d)$ in the map l_g. After this cancellation we get a complex that is, up to a degree shift, the claimed free resolution of I. $\square$

The free resolution of I we just derived is smaller than the one obtained from the Complex (2.4). In fact, it is often minimal.

Corollary 2.4.2. *If the polynomial f is not a unit and each map α_i is minimal whenever $1 \leq i \leq g-1$, that is, $\operatorname{Im} \alpha_i \subset \mathfrak{m} A_i$, then the resolution of I described in Theorem 2.4.1 is a graded minimal free resolution of I.*

Proof. Since the maps α_i are minimal, the definition of β_i (see Equation (2.7)) implies that also β_i is a minimal map whenever $1 \leq i \leq g-1$. Now the description of the maps in the free resolution obtained in Theorem 2.4.1 shows that all its maps between free modules are minimal. Hence, it is a minimal resolution. $\square$

The short exact sequence in Theorem 2.4.1 allows us to re-interpret Theorem 2.3.1 in terms of liaison theory. To this end we recall the following definition.

Suppose $J \subset I \cap K$ are homogeneous ideals in R with $\operatorname{grade}(I) = \operatorname{grade}(J) + 1$, and J is Cohen-Macaulay and generically Gorenstein. If there is an isomorphism of graded R-modules

$$I/J(-s) \cong K/J,$$

then it is said that K is obtained from I by an *elementary biliaison* on J. It has the same grade as I. (See [21, 24, 17] for more details.)

Using this concept, we get:

Proposition 2.4.3. *The homogeneous Gorenstein ideal $I = (\mathfrak{b}, \alpha_{g-1}^* + (-1)^g f a_g^*)$ in Theorem 2.3.1 is obtained from $\mathfrak{a}$ by an elementary biliaison on $\mathfrak{b}$.*

Proof. Theorem 2.4.1 provides the short exact sequence

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-d) \xrightarrow{\varphi} R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Thus, we get an isomorphism $\mathfrak{a}/\mathfrak{b}(-d) \cong I/\mathfrak{b}$. Since $\mathfrak{b}$ is Gorenstein the claim follows directly from the definition of an elementary biliaison. $\square$

2.5 Decomposing Homogeneous Gorenstein Ideals

So far we have studied the construction of a new homogeneous Gorenstein ideal I of grade g from smaller homogeneous Gorenstein ideals $\mathfrak{b} \subset \mathfrak{a}$ of grades $g-1$ and g, respectively. It is natural to ask when this construction can be reversed. One more precise version of this problem is whether, for given homogeneous Gorenstein ideals I and $\mathfrak{a}$ of grade g, there is a homogeneous Gorenstein ideal $\mathfrak{b} \subset \mathfrak{a}$ of grade $g-1$ such that I can be obtained from $\mathfrak{a}$ by an elementary biliaison on $\mathfrak{b}$. This question has already been considered in the local case in [22]. We now derive a necessary condition in the graded case. Recall that the Castelnuovo-Mumford regularity of a homogenous Gorenstein ideal $I \subset R$ is

$$\operatorname{reg} I = \min\{m \mid [H^i_{\mathfrak{m}}(I)]_j = 0 \text{ whenever } i + j > m\},$$

where $H^i_{\mathfrak{m}}(I)$ denotes the i-th local cohomology module with support in $\mathfrak{m}$. If I has finite projective dimension over R, then its regularity can also be computed from a minimal free resolution as

$$\operatorname{reg} I = \min\{m \mid [\operatorname{Tor}^R_i(I, R/\mathfrak{m})]_j = 0 \text{ whenever } i - j > m\}.$$

Corollary 2.5.1. *Let I and $\mathfrak{a}$ be homogeneous Gorenstein ideals of grade g. If $\operatorname{reg} I - \operatorname{reg}\mathfrak{a}$ is not even, then there is no homogeneous Gorenstein ideal $\mathfrak{b} \subset \mathfrak{a}$ such that I can be obtained from $\mathfrak{a}$ by an elementary biliaison on $\mathfrak{b}$.*

Proof. From the degree shift of the last free module in the minimal free resolution of $\mathfrak{a}$, we see (using the notation in Diagram (2.2)) that

$$\operatorname{reg}\mathfrak{a} = v - g + 1.$$

If I is obtained from $\mathfrak{a}$ and $\mathfrak{b}$ as in Theorem 2.3.1, then the free resolution of I described in Theorem 2.4.1 gives

$$\operatorname{reg} I = u + d - g + 1.$$

It follows that

$$\operatorname{reg} I - \operatorname{reg}\mathfrak{a} = u - v + d = 2d.$$

This implies the assertion. □

Now we give an example whic provides also a partial answer of the question. It shows that given homogeneous Gorenstein ideal cannot be decomposed as in Theorem 2.3.1.

We conclude with an example of a Gorenstein ideal that cannot be produced using the construction of Theorem 2.3.1 with a strictly ascending biliaison.

Example 2.5.2. Let I be a generic Artinian Gorenstein ideal in $R = K[x_1, \ldots, x_5]$ with h-vector $(1, 5, 5, 1)$, where K is an infinite field. It has the least possible Betti numbers. More precisely, its graded minimal free resolution is pure and has the form

$$0 \to R(-8) \to R^{10}(-6) \to R^{16}(-5) \to R^{16}(-3) \to R^{10}(-2) \to I \to 0. \tag{2.13}$$

We claim that there are no Gorenstein ideals $\mathfrak{a}$ and $\mathfrak{b}$ to produce I using a biliaison as in Theorem 2.3.1 that is strictly ascending, i.e., $d > 0$ or, equivalently, $\mathfrak{a}$ has smaller regularity than I.

Indeed, to see this assume such ideals $\mathfrak{a}$ and $\mathfrak{b}$ do exist. Since $\operatorname{reg} I = 4$, this forces $\operatorname{reg}\mathfrak{a} = 2$ by Corollary 2.5.1. It follows that the h-vector of $R/\mathfrak{a}$ must be $(1, 1)$. Hence, possible after a change of coordinates, we may assume

$$\mathfrak{a} = (x_1, x_2, x_3, x_4, x_5^2).$$

Thus, Corollary 2.3.4 gives that $R/\mathfrak{b}$ has h-vector $(1,4,4,1)$. Its graded minimal free resolution has the form

$$0 \longrightarrow R(-7) \longrightarrow \begin{matrix} R^6(-5) \\ \oplus \\ R^c(-4) \end{matrix} \longrightarrow \begin{matrix} R^{5+c}(-4) \\ \oplus \\ R^{5+c}(-3) \end{matrix} \longrightarrow \begin{matrix} R^c(-3) \\ \oplus \\ R^6(-2) \end{matrix} \longrightarrow \mathfrak{b} \longrightarrow 0,$$

where c is some non-negative integer. By Theorem 2.4.1, we have the following short exact sequence of graded R-modules

$$0 \longrightarrow (\mathfrak{a}/\mathfrak{b})(-1) \longrightarrow R/\mathfrak{b} \longrightarrow R/I \longrightarrow 0.$$

Consider now the comparison map between the resolutions of $(\mathfrak{a}/\mathfrak{b})(-1)$ and $R/\mathfrak{b}$ in homological degree two. Using the notation of the proof of Theorem 2.4.1, it is

$$A_3(-1) \oplus B_2(-1) \xrightarrow{[\beta_3, h_2 - f\,\mathrm{id}]} B_2 = R^{5+c}(-4) \oplus R^{5+c}(-3)$$

Since $\deg f = 1$, the map $h_2 - f\,\mathrm{id}$ is minimal. Moreover, notice that $A_3(-1) = R^4(-4) \oplus R^6(-5)$. Considering the map β_3 in degree 4, the mapping cone procedure implies that $[\mathrm{Tor}_2^R(R/I, K)]_4 \neq 0$. Hence I does not have a pure resolution as in (2.13), which completes the argument.

Using our description of the minimal free resolution in Theorem 2.4.1, we show in Example 2.5.2 above that there is a Gorenstein ideal I that cannot be obtained by the construction in Theorem 2.3.1 if $\operatorname{reg} \mathfrak{a} < \operatorname{reg} I$. In general, it is open when a given Gorenstein ideal can be produced by an elementary biliaison as in Theorem 2.3.1.

2.6 Examples

We describe various examples for the construction in Theorem 2.3.1. We begin with the easiest case, where $\mathfrak{a}$ and $\mathfrak{b}$ are complete intersection ideals. It extends Example 2.3.2. This case has also been discussed in the spirit of the original Kustin-Miller construction in [26, Section 4].

Example 2.6.1. Let R be a graded Gorenstein ring, and let $h_1, \ldots, h_g$ and $p_1, \ldots, p_{g-1}$ be regular sequences of homogeneous elements such that

$$\mathfrak{b} = (p_1, \ldots, p_{g-1}) \subset (h_1, \ldots, h_g) = \mathfrak{a}.$$

Then there is a homogeneous $g \times (g-1)$ matrix M such that (as matrices)

$$\begin{pmatrix} p_1 & \ldots & p_{g-1} \end{pmatrix} = \begin{pmatrix} h_1 & \ldots & h_g \end{pmatrix} \cdot M.$$

Setting $u = \sum \deg p_i$ and $v = \sum \deg h_j$, we get the following comparison map between the graded minimal free resolutions of $R/\mathfrak{a}$ and $R/\mathfrak{b}$

$$\begin{array}{ccccccccccccc} & & 0 & \longrightarrow & R(-u) & \xrightarrow{b_{g-1}} & \ldots. & \longrightarrow & B_1 & \xrightarrow{b_1} & R & \longrightarrow & R/\mathfrak{b} \\ & & \downarrow & & \downarrow{\scriptstyle \wedge^{g-1} M} & & & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle =} & & \downarrow \\ 0 & \longrightarrow & R(-v) & \xrightarrow{a_g} & A_{g-1} & \xrightarrow{a_{g-1}} & \ldots. & \longrightarrow & A_1 & \xrightarrow{a_1} & R & \longrightarrow & R/\mathfrak{a} \end{array}$$

Denote by M_i the square matrix obtained by deleting row i of M . Then, by Theorem 2.3.1, for a sufficiently general $f \in R$ of degree $d = v - u \geq 0$, the ideal

$$I = (p_1, \ldots, p_{g-1}, \det M_1 + fh_1, \ldots, \det M_g + fh_g)$$

is a homogeneous Gorenstein ideal of grade g. Moreover, if no entry of the matrix M is a unit, then the graded free resolution of I described in Theorem 2.4.1 is minimal. In particular, then I has $2g-1$ minimal generators.

We can be more explicit in the following special case. Assume $x_1, x_2, \cdots, x_g$ is a regular sequence of homogeneous elements in R. Consider $\mathfrak{b} = (x_1^{m_1}, x_2^{m_2}, \cdots, x_{g-1}^{m_{g-1}}) \subset (x_1^{n_1}, x_2^{n_2}, \cdots, x_g^{n_g}) = \mathfrak{a}$ and assume $d := \sum\limits_{i=1}^{g-1} m_i - \sum\limits_{i=1}^{g} n_i \geq 0$. Then, for a sufficiently general $f \in R$ of degree d,

$$I = (x_1^{m_1}, \cdots, x_{g-1}^{m_{g-1}}, fx_1^{n_1}, \cdots, fx_{g-1}^{n_{g-1}}, c + fx_g^{n_g})$$

is a Gorenstein ideal, where $c = \prod\limits_{j=1}^{g-1} x_j^{m_j - n_j}$. Moreover, if $m_j > n_j$ for each $j = 1, \ldots, g-1$, then the resolution in Theorem 2.4.1 is a minimal free resolution of I.

In the next example we show that all the Gorenstein ideals with socle degree two can be obtained by one elementary biliaison from a complete intersection.

Example 2.6.2. Consider the Artinian Gorenstein ideals $I \subset R = K[x_1, \ldots, x_n]$ with h-vector $(1, n, 1)$, where K is a field. These ideals have been classified by Sally in [29, Theorem 1.1]. Each such ideal is of the form

$$I = (x_i x_j \mid 1 \leq i < j \leq n) + (x_1^2 - c_1 x_n^2, \ldots, x_{n-1}^2 - c_{n-1} x_n^2),$$

where $c_1, \ldots, c_{n-1} \in K$ are suitable units. It can be obtained by an elementary biliaison as in Theorem 2.3.1 from $\mathfrak{a} = (x_1, \ldots, x_n)$ on $\mathfrak{b}R$, where $\mathfrak{b}$ is such a Sally ideal in $n-1$ variables. More precisely, define the ideal $\mathfrak{b}$ as

$$\mathfrak{b} = (x_i x_j \mid 1 \leq i < j \leq n-1) + (x_1^2 - \frac{c_1}{c_{n-1}} x_{n-1}^2, \ldots, x_{n-2}^2 - \frac{c_{n-2}}{c_{n-1}} x_{n-1}^2).$$

Then it is not too difficult to see that there are the following links

$$\mathfrak{a} \sim_{(\mathfrak{b}, x_n)} (\mathfrak{b}, x_n, x_{n-1}^2) \sim_{(\mathfrak{b}, x_{n-1}^2 - c_{n-1} x_n^2)} I.$$

Note that $(\mathfrak{b}, x_n, x_{n-1}^2) = (x_1, \ldots, x_{n-1})^2 + (x_n)$.

The following classical example has been studied from various points of view.

Example 2.6.3. Let $M = (x_{ij})$ be a generic $n \times n$ matrix, where $n \geq 2$.

$$M = \begin{bmatrix} x_1 & \cdots & x_n \\ x_{n+1} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots \\ x_{n(n-1)+1} & \cdots & x_{n^2} \end{bmatrix}_{n \times n}.$$

The ideal $I = \mathrm{I}_{n-1}(M)$ in $K[M]$, generated by the submaximal minors of M is a Gorenstein ideal of grade four. Its graded minimal free resolution is given by the Gulliksen-Negård complex (see [14]):

$$0 \to R(-2n) \to R^{n^2}(-n-1) \to R^{2(n^2-1)}(-n) \to R^{n^2}(-n+1) \to I \to 0.$$

Kustin and Miller show that this resolution can be obtained by using their original construction (see [22, Example 2.4]). Gorla [13] studies these ideals from a liaison-theoretic point of view. Here we make the linkage steps more explicit.

If $n = 2$, then I is a complete intersection. Assume $n \geq 3$, and let N be the generic $(n-1) \times (n-1)$ obtained from M by deleting its last row and column.

$$N = \begin{bmatrix} x_1 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots \\ x_{n(n-2)+1} & \cdots & x_{n(n-1)-1} \end{bmatrix}_{(n-1)\times(n-1)}$$

Its $(n-2) \times (n-2)$ minors generate a homogeneous Gorenstein ideal $\mathfrak{a} = \mathrm{I}_{n-2}(N)$ of grade 4. Denote by $M_{i,j}$ the $(n-1) \times (n-1)$ minor of M obtained by deleting row i and column j. The ideal

$$\mathfrak{b} = (M_{1,n}, M_{2,n}, \cdots, M_{n-1,n}, M_{n,1}, \cdots, M_{n,n-1})$$

is a Gorenstein ideal of grade three (see, e.g, [22, Example 2.4]). Sylvester's identity implies that (see, the proof of Theorem 3.1 in [13]):

$$N_{1,1} \cdot I + \mathfrak{b} = M_{1,1} \cdot \mathfrak{a} + \mathfrak{b}.$$

It follows that there are the following links

$$\mathfrak{a} \sim_{(\mathfrak{b}, N_{1,1})} (\mathfrak{b}, N_{1,1}, M_{1,1}) \sim_{(\mathfrak{b}, M_{1,1})} I.$$

Hence I can be obtained from $\mathfrak{a}$ by an ascending biliaison on $\mathfrak{b}$ as described in Theorem 2.3.1. Repeating the construction, we see that I can be obtained from the complete intersection $(x_{11}, x_{12}, x_{21}, x_{22})$ by $(n-2)$ such ascending biliaisons.

Now we consider some Gorenstein ideals with 9 generators and 16 syzygies. Such Gorenstein ideals are investigated in depth from the point of view of unprojections in [5].

Example 2.6.4. Let $R = K[a, b, c, d, e, f, x, y, z]$ be a polynomial ring in 9 variables over a field K. Consider a generic 3×3 symmetric matrix A and a generic skew-symmetric matrix B:

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

Then, for $\lambda \neq 0$ in K, define a 6×6 skew-symmetric matrix $N = \begin{bmatrix} B & A \\ -A & \lambda B \end{bmatrix}$. It is called "extrasymmetric" in [5, 6] because it is obtained from a generic skew-symmetric matrix by specializing some of the variables. The ideal $\mathfrak{a}$ generated by the 4×4 Pfaffians of N is a homogeneous Gorenstein ideal of grade 4:

$$\begin{aligned}\mathfrak{a} =&(b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2, cd - be + \lambda xz, ce - bf + \lambda yz, \\ &e^2 - df + \lambda z^2, cx - by + az, ex - dy + bz, fx - ey + cz).\end{aligned}$$

It is the defining ideal of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$ and a typical case of a Tom unprojection (see [5, 6, 27]). In particular, $\mathfrak{a}$ is equal to the ideal generated by the 2×2 minors of a 3×3 generic matrix $A + \sqrt{-\lambda}B$. Hence, the Gulliksen and Negård complex gives its minimal free resolution:

$$0 \longrightarrow R(-6) \xrightarrow{a_4} R^9(-4) \xrightarrow{a_3} R^{16}(-3) \xrightarrow{a_2} R^9(-2) \xrightarrow{a_1} \mathfrak{a} \longrightarrow 0.$$

In order to perform the construction of Theorem 2.3.1, we choose the first three listed generators of $\mathfrak{a}$ to define a complete intersection

$$\mathfrak{b} = (b^2 - ad + \lambda x^2, bc - ae + \lambda xy, c^2 - af + \lambda y^2)$$

inside $\mathfrak{a}$. Then we link as follows:

$$\mathfrak{a} \sim_{(\mathfrak{b}, cd-be+\lambda xz)} (\mathfrak{b}, cd - be + \lambda xz, ax) \sim_{(\mathfrak{b}, ax+(cd-be+\lambda xz))} I.$$

Explicitly, the resulting ideal I is

$$\begin{aligned}I =&(e^2 - df - cx + by + az + \lambda z^2, ce - bf + ay + \lambda yz, cd - be + ax + \lambda xz, \\ &c^2 - af + \lambda y^2, bc - ae + \lambda xy, ac + \lambda fx - \lambda ey + \lambda cz, b^2 - ad + \lambda x^2, \\ &ab + \lambda ex - \lambda dy + \lambda bz, a^2 + \lambda cx - \lambda by + \lambda az).\end{aligned}$$

It has the same Betti diagram as $\mathfrak{a}$. In fact, I is again an example of a Tom unprojection. This time the extrasymmetric matrix is

$$M = \begin{bmatrix} 0 & x & y & a & b & c \\ -x & 0 & \frac{1}{\lambda}a + z & b & d & e \\ -y & -\frac{1}{\lambda}a - z & 0 & c & e & f \\ -a & -b & -c & 0 & \lambda x & \lambda y \\ -b & -d & -e & -\lambda x & 0 & a + \lambda z \\ -c & -e & -f & -\lambda y & -a - \lambda z & 0 \end{bmatrix},$$

so $I = \mathrm{Pf}_4(M)$.

Chapter 3 Boij-Söderberg Decompositions

3.1 Background and Preliminaries

Throughout this chapter we assume that R is a graded polynomial ring with 3 variables over a field $\mathbf{k}$ with each variable has degree one. We seek for a description of the Betti diagram $L = x\mathfrak{a} + J$ in terms of the Betti numbers of $\mathfrak{a}$ and J.

Let R be a graded ring and M a graded R-module. We recall the definition 2.1.4 of the minimal graded free resolution of M. Graded minimal free resolution of M is

$$\mathbb{F}: \quad F_n \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_i = \bigoplus_{j\geq 0} R(-j)^{\beta_{i,j}}$. The numbers $\beta_{i,j}$ are the Betti numbers of M and are considered in the Betti diagram $\beta(M)$ of M whose entry in row j and column i is $\beta_{i,i+j}$.

A *degree sequence* $\mathbf{d} = (d_0, d_1, ..., d_n) \in \mathbb{Z}^{n+1}_{\geq 0}$ means a sequence of non-negative integers of length $n+1$ with $d_0 < ... < d_n$.

Definition 3.1.1. The graded free resolution of M is called a *pure resolution of type* $\mathbf{d} = (d_0, ..., d_i, ..., d_n)$ if, for all $i = 0, 1, ..., n$, the i-th syzygy module of M is generated only by elements of degree d_i.

In other words, if a module M has a pure resolution of type $\mathbf{d}$ then all Betti numbers in the Betti diagram are zero except $\beta_{i,d_i}(M)$. Then the Betti diagram of this module is called a *pure diagram of type* $\mathbf{d}$. The formula for the pure diagram associated by $\mathbf{d}$ is based on the Herzog and Kühl equations introduced in [18],

$$\beta_{i,j} = \begin{cases} \lambda \prod\limits_{i=0, i\neq j}^{n} \dfrac{1}{|d_i - d_j|} & \text{if } j = d_i \\ 0 & \text{otherwise} \end{cases} \quad \text{where } \lambda \in \mathbb{Q}.$$

We define a partial order on the degree sequences so that $\mathbf{d}^s < \mathbf{d}^t$ if $d_i^s \leq d_i^t$ for all $i = 0, 1, ..., ns$. The order on the degree sequences induces an order of the pure diagrams $\pi_{\mathbf{d}^s} < \pi_{\mathbf{d}^t}$ if $\mathbf{d}^s < \mathbf{d}^t$. Thus the Boij-Söderberg decomposition of a graded R-module M gives an ordered decomposition of the Betti diagram,

$$\beta(M) = \sum_s a_s \pi_{\mathbf{d}^s} \text{ where } \pi_{\mathbf{d}^s} < \pi_{\mathbf{d}^t} \text{ if } s < t.$$

Example 3.1.2. Let $I = (x^2, xy, xz, y^2)$ be an ideal in $\mathbf{k}[x, y, z]$, the Boij-Söderberg decomposition of R/I is given as following

$$\beta(R/I) = (8)\pi_{\mathbf{d^0}} + (4)\pi_{\mathbf{d^1}} \text{ where}$$

$$\pi_{\mathbf{d^0}} = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{24} & - & - & - \\ 1 & - & \frac{1}{4} & \frac{1}{3} & \frac{1}{8} \end{array} < \pi_{\mathbf{d^1}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{6} & - & - \\ 1 & - & \frac{1}{2} & \frac{1}{3} \end{array} \text{ as } \mathbf{d^0} = (0, 2, 3, 4) < \mathbf{d^1} = (0, 2, 3)$$

NOTATION: Let I a monomial ideal in a polynomial ring R. We denote the set of minimal monomial generators of I with $G(I)$ and then $G(I)_i$ refers to the subset of $G(I)$ containing the minimal generators of degree i. The notation $a(I)$ means the initial degree of the monomials in I and $e^+(I)$ is for the maximum degree of the monomials in $G(I)$ throughout this chapter.

Definition 3.1.3. Let $\mathfrak{m} = x_1^{s_1}...x_n^{s_n}$ and $\mathfrak{n} = x_1^{t_1}...x_n^{t_n}$ be two monomials in $R = \mathbf{k}[x_1, ..., x_n]$. If either $\deg \mathfrak{m} > \deg \mathfrak{n}$ or $\deg \mathfrak{m} = \deg \mathfrak{n}$ and $s_i - t_i > 0$ for the first nonzero index i, then it is said that $\mathfrak{m} >_{lex} \mathfrak{n}$ in *lexicographic order.*

Definition 3.1.4. Let R be a polynomial ring and L be a monomial ideal in R generated by the monomials $m_1, ..., m_l$. The ideal L is called a *lex-segment ideal* (lexicographic ideal, or lex ideal) in R if for each monomial $m \in R$ the existence of some $m_i \in G(L)$ with $m >_{lex} m_i$ and $\deg(m) = \deg(m_i)$ implies $m \in L$.

For simplicitiy, we will use "$>$" for the lex order "$>_{lex}$" unless the order is different than lexicographic order.

In this section, we make some observations about the Betti diagrams of lex-segment ideals. We aim to get some relations between their Betti numbers.

Lemma 3.1.5. *Let L be a lex-segment ideal in $R = \mathbf{k}[x_1, .., x_n]$. Consider the colon ideals $\mathfrak{a}_i = L : (x_i)$, for $i = 1, ..., n$. Then each $\mathfrak{a}_i$ is also lex-segment ideals in R.*

Proof. Let $m' \in \mathfrak{a}_i$ be a monomial, for any $i = 1, ..., n$. Let m be a monomial in R and $\deg m = \deg m'$ and $m >_{lex} m'$. Then $x_i m' \in L$ as $\mathfrak{a}_i = L : (x_i)$, and $x_i m >_{lex} x_i m'$. This implies $x_i m \in L$ and hence $m \in L : (x_i) = \mathfrak{a}_i$. $\square$

Let u be a monomial in $R = \mathbf{k}[x_1, .., x_n]$, we define $m(u)$ to be the largest index i such that x_i divides u. Recall that a monomial ideal I is said to be *stable* if, for every monomial $u \in G(I)$ and all $i < m(u)$, $x_i u / x_{m(u)}$ is also in $G(I)$.

Proposition 3.1.6. ***(Eliahau-Kervaire formula, [11])*** *Let $I \subset R$ be a stable ideal. Then*

(a) $\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{m(u)-1}{i}$;

(b) $\operatorname{projdim} R/I = \max\{m(u) : u \in G(I)\}$;

(c) $\operatorname{reg}(I) = \max\{\deg(u) : u \in G(I)\}$.

From now on, we assume $n = 3$, that is, $R = \mathbf{k}[x, y, z]$.

Lemma 3.1.7. *If L is lex-segment ideal in R, then there are unique monomial ideals $\mathfrak{a} \subset R$ and $J \subset \mathbf{k}[y, z]$ such that*

$$L = x\mathfrak{a} + J.$$

Moreover, the ideal $\mathfrak{a}$ is also a lex-segment ideal since $\mathfrak{a} = L : (x)$ and J is stable in R, and $G(L) = xG(\mathfrak{a}) \uplus G(J)$.

Lemma 3.1.8. *Let $0 \to F_2 \to F_1 \to J$ and $0 \to G_3 \to G_2 \to G_1 \to \mathfrak{a}$ be graded free resolutions for the ideals J and $\mathfrak{a}$. If $L = \mathfrak{a}(x) + J$, then there is a short exact sequence*

$$0 \to J(-1) \to \mathfrak{a}(-1) \oplus J \to L \to 0 \tag{3.1}$$

Moreover,

$$0 \to G_3(-1) \oplus F_2(-1) \to G_2(-1) \oplus F_2 \oplus F_1(-1) \to G_1(-1) \oplus F_1 \to L.$$

is the graded minimal free resolution of L.

Proof. The form of the lex-segment ideal L implies the short exact sequence (3.1). The mapping cone for the short exact sequence provides a free resolution for L. If $m \in G(\mathfrak{a}) \cap G(J)$ then $mx \in G(L)$ and also $m \in G(L)$ but clearly if m is a minimal generator of L then mx cannot be a minimal generator. Therefore the ideals J and $\mathfrak{a}$ do not have common minimal generators. This tells us that there is no cancellation in the mapping cone structure. So the resulting graded free resolution for L is minimal. $\square$

First we analyze the Betti numbers of the ideals L, $\mathfrak{a} = L : (x)$ and J. We know that the lex-segment ideals L and $\mathfrak{a}$ are stable and in addition to this, J is a lex ideal in $\mathbf{k}[y, z]$. Thus, Eliahau-Kervaire formula gives rise to the following decomposition,

$$\begin{aligned}\beta_{i,i+j}(L) &= \sum_{u \in G(L)_j} \binom{m(u)-1}{i} \\ &= \underbrace{\sum_{u \in G(L)_j \text{ and } x \mid u} \binom{m(u)-1}{i}}_{\beta_{i,i+j-1}(\mathfrak{a})} + \underbrace{\sum_{u \in G(L)_j \text{ and } x \nmid u} \binom{m(u)-1}{i}}_{\text{say } D_{i,i+j}}\end{aligned}$$

Let's denote the initial degree of J, $a(J) := k$ and the Betti numbers of $\beta(\mathfrak{a})$ and $\beta(J)$ as

$$a_{i,i+j} := \beta_{i,i+j}(\mathfrak{a}), \qquad c_{i,i+j} := \beta_{i,i+j}(J)$$

The following remark gives some relations and identities about the Betti numbers of L, $\mathfrak{a}$ and J that will help us to describe the entire Betti diagram of L with respect to the Betti numbers of $\mathfrak{a}$ and J.

Remark 3.1.9. Recall that $L = x\mathfrak{a} + J$ in $R = \mathbf{k}[x, y, z]$.

(i) If $a(L) = 1$, then $\mathfrak{a} = 1$. If $a(L) \geq 2$, then $a(L) = a(\mathfrak{a}) + 1$ by stability of ideal L and $\mathfrak{a} = L : (x) \neq 1$.

(ii) We know that $\beta_{i,i+j}(L) = \beta_{i,i+j-1}(\mathfrak{a}) + D_{i,i+j}$. Thus, we observe that
if $j \leq k-1, \quad D_{i,i+j} = 0$,
if $j \geq k, \quad D_{i,i+j} = \beta_{i,i+j}(J,x)$, it implies that $D_{0,j} = c_{0,j}, D_{1,j+1} = c_{0,j} + c_{1,j+1}$, and $D_{2,j+2} = c_{1,j+1}$.

(iii) The Eliahau-Kervaire formula for $\mathfrak{a}$ gives

$$a_{0,j} = \begin{cases} a_{1,j+1} - a_{2,j+2} + 1 & \text{if } j = a(L) - 1 \\ a_{1,j+1} - a_{2,j+2} & \text{if } j > a(L) - 1 \end{cases}$$

and $a_{1,j+1} \geq 2a_{2,j+2}$ for all $j = 1, 2, ..., e^+(\mathfrak{a})$.

(iv) We have the following identities for the Betti numbers of the J

- $c_{0,k} = c_{1,k+1} + 1$,
- $c_{0,j} = c_{1,j+1}$ for all $j \geq k+1$,
- if $c_{0,k} = k+1$ then $c_{1,k+1} = k$ and $c_{i,i+j} = 0$ for all $i = 0, 1$ and $j \geq k+1$.

Remark 3.1.10. $\min\{s | a_{1,s+1} \neq 0\} \leq \min\{s | a_{2,s+2} \neq 0\}$.

Proof. It follows from the fact that $\mathfrak{a}$ is stable. □

Lemma 3.1.11. *If* $a_{0,j-1} = 0$ *then* $\beta_{0,j}(L) = 0$.

Proof. Let $a_{0,j-1} = 0$. Suppose that $c_{0,j} \neq 0$ so $c_{1,j+1} = c_{0,j} - 1 \geq 0$ and by Remark 2 $\beta_{0,j}(L) \neq 0$. Since $a_{0,j-1} = 0$ and $c_{0,j} \neq 0$, no minimal generator of degree j is divisible by x. Thus any minimal generator of degree j is of the form $y^m z^n$ where $m \geq 0, n \geq 0$ and $m + n = j$.
On the other hand, as $e^+(\mathfrak{a}) > j - 1$ there is a minimal generator $v \in G(L)_{e^+(\mathfrak{a})+1}$ such that $x | v$.
Let $v = x^s y^t z^p$ where $s \geq 1$ and $s + t + p = e^+(\mathfrak{a}) + 1 > j$.
Now we can find a monomial such that $x^s y^r z^{j-s-r} \in L$ where $0 \leq r \leq t$ since L is a lex-segment ideal and so $x^s y^r z^{j-s-r} | v$. Hence v cannot be a minimal generator, that is, $a_{0,e^+(\mathfrak{a})} = 0$. This contradicts our assumption. Thus, $c_{0,j} = 0$, then $\beta_{0,j+1} = 0$. □

Lemma 3.1.12. $\min\{s | a_{2,s+1} \neq 0\} < \min\{s | c_{1,s+1} \neq 0\}$

Proof. Say $N := \min\{s | a_{2,s+1} \neq 0\}$and $M := \min\{s | c_{1,s+1} \neq 0\}$. First, recall that $a(L) \geq 2$. Also, recall that $k = a(J)$. Then the Betti diagram for J is

$\beta(J)$	0	1
k	$c_{0,k}$	-
$k+1$	-	-
$\vdots$	$\vdots$	$\vdots$
$M-1$	-	-
M	$c_{0,M}$	$c_{1,M+1} \neq 0$
$M+1$	$c_{0,M+1}$	$c_{1,M+2}$
$\vdots$	$\vdots$	$\vdots$

It shows that there exists at least one minimal generator of the form $y^m z^n \in L$ where $m+n=M$ and $n \geq 1$.
As $xz^{m+n-1} > y^m z^n$, $xz^{m+n-1} \in L$.
If xz^{m+n-1} is a minimal generator in L, then $a_{2,M+1} = \sum\limits_{u \in G(\mathfrak{a})_{M-1}} \binom{m(u)-1}{2} \geq \binom{3-1}{2} \neq 0$.
Therefore, $\min\{s|a_{2,s+1} \neq 0\} \leq M-1 < M = \min\{s|c_{1,s+1} \neq 0\}$.
If xz^{m+n-1} is not a minimal generator, then L contains a minimal generator that divides xz^{m+n-1} and since $x \notin L$.
There is a minimal generator of the form xz^t where $t < m+n-1 = M-1$. Then it follows that ${}_{2,t} \neq 0$ and so $\min\{s|a_{2,s+1} \neq 0\} \leq t < M = \min\{s|c_{1,s+1} \neq 0\}$.

□

Lemma 3.1.13. *If* $a_{1,j} = 0$ *then* $\beta_{1,j+1}(L) = 0$

Proof. First of all, if $a_{1,j} = 0$ then $a_{2,j+1} = 0$.
If $a_{0,j-1} = 0$ then by Lemma (3.1.11) $\beta_{0,j}(L) = 0$, so $\beta_{1,j+1}(L) = 0$.
If $a_{0,j-1} \neq 0$ it is easy to see that the only minimal generator of $\mathfrak{a}$ of degree $j-1$ is x^{j-1} since $a_{1,j} = a_{2,j+1} = 0$. Then, $a(L) = j$. If $c_{0,j} = 0$ then $c_{1,j+1} = 0$ and therefore $\beta_{1,j+1}(L) = a_{1,j} + c_{0,j} + c_{1,j+1} = 0$. Suppose $c_{0,j} \neq 0$, and as $a(L) = j$, $\quad y^j \in G(L)_j$ but also $x^j \in G(L)_j$. Then by lex-order $xy^{j-1} \in G(L)_j$. This contradicts $a_{1,j} = 0$. □

Lemma 3.1.14. $a(J) \geq e^+(\mathfrak{a}) + 1$ *where* $J \neq 0$.

Proof. Say $e^+(\mathfrak{a}) = t$.
Suppose $k = a(J) < t$, then $y^k \in G(L)_k$. So, by lex-order, all monomials u of degree k divisible by x are in L. Thus, u is in the form $x^i y^j z^s$ where $s \geq 1$, $i+j+s = k$. As $e^+(\mathfrak{a}) = t > k$, there is a minimal generator $v \in L$ of degree $t+1$ such that $x|v$. Therefore, v can be written as a product of two monomials w_1 and w_2 such that w_2 is divisible by x and the degree of w_1 is k, and w_2 has degree $t-k$. Since all degree k monomials divisible by x are in L , v cannot be a minimal generator.
Thus $k \geq t$.
Now, we need to show that the equality is not possible. Suppose $k = t$.
So y^k is a minimal generator in L and since $t = k$ we can find at least one minimal generator u of $\mathfrak{a}$ with degree k then xu becomes a minimal generator in L of degree $k+1$. However all monomials v of degree k divisible by x are in L. Then there is a monomial w such that $v = xw$ and $w|u$, but this contradicts that u is a minimal generator of $\mathfrak{a}$.
Hence $k \neq t$. i.e. $k \geq t+1$ □

Lemma 3.1.14 tells us if the Betti diagrams of the ideals $\mathfrak{a}$ and J overlap then they do only at the k^{th} row of the $\beta(L)$. So if we have the following diagrams for $\mathfrak{a}$ and J;

respectively.

Then, the Betti diagram for L appears as following

$\beta(\mathfrak{a})$	0	1	2
1	$a_{0,1}$	$a_{1,2}$	$a_{2,3}$
2	$a_{0,2}$	$a_{1,2}$	$a_{2,4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$a_{0,k-1}$	$a_{1,k}$	$a_{2,k+1}$
k	$a_{0,k}$	$a_{1,k+1}$	$a_{2,k+2}$

and

$\beta(J)$	0	1
k	$c_{0,k}$	$c_{0,k}-1$
$k+1$	$c_{0,k+1}$	$c_{0,k+1}$
$\vdots$	$\vdots$	$\vdots$
$e^+(J)$	$c_{0,e^+(L)}$	$c_{0,e^+(L)}$

,

Table 3.1: Betti diagrams of $\mathfrak{a}$ and J.

$\beta(L)$	0	1	2
2	$a_{0,1}$	$a_{1,2}$	$a_{2,3}$
3	$a_{0,2}$	$a_{1,2}$	$a_{2,4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$a_{0,k-2}$	$a_{1,k-1}$	$a_{2,k}$
k	$a_{0,k-1}+c_{0,k}$	$a_{1,k}+2c_{0,k}-1$	$a_{2,k+1}+c_{0,k}-1$
$k+1$	$c_{0,k+1}$	$2c_{0,k+1}$	$c_{0,k+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$e^+(L)=e^+(J)$	$c_{0,e^+(L)}$	$2c_{0,e^+(L)}$	$c_{0,e^+(L)}$

Table 3.2: Betti diagram of L

3.2 Boij-Söderberg Decomposition of Lex-Segment Ideals

The Boij-Söderberg Decompositions of L and $L:(x)$

In this section we identify the beginning of the Boij-Söderberg decomposition of a lex-segment ideal. More precisely, the next theorem shows that if $\mathbf{d^0} < \mathbf{d^1} < ... < \mathbf{d^i} < ... < \mathbf{d^t}$ is the chain of all length 3 top degree sequences in the Boij-Söderberg decomposition of the Betti diagram of $\mathfrak{a} = L : (x)$ and the chain of the first $t+1$ top degree sequences of the Boij-Söderberg decomposition of the Betti diagram of L is $\bar{\mathbf{d}}^{\mathbf{0}} < \bar{\mathbf{d}}^{\mathbf{1}} < ... < \bar{\mathbf{d}}^{\mathbf{i}} < ... < \bar{\mathbf{d}}^{\mathbf{t}}$ then $\bar{\mathbf{d}}^{\mathbf{i}} = \mathbf{d^i} + \mathbf{1} = (d_0^i+1, d_1^i+1, d_2^i+1)$ for all $i = 0, 1, ..., t$ with exactly the same coefficients, except possibly the coefficient of $\pi_{\bar{\mathbf{d}}^{\mathbf{t}}}$.

Theorem 3.2.1. *[15] Let $R = \mathbf{k}[x, y, z]$ and L be a lex-segment ideal of codimension 3 in R. Suppose $1 \neq \mathfrak{a} = L : (x)$.*

Write the Boij-Söderberg decomposition of $\mathfrak{a}$ as

$$\beta(\mathfrak{a}) = \sum_{i=0}^{t} \alpha_i \pi_{\mathbf{d^i}} + R_{\mathfrak{a}},$$

where $\mathbf{d^0} < \mathbf{d^1} < ... < \mathbf{d^l} < ... < \mathbf{d^t}$ are all top degree sequences of length 3, that is, $\mathbf{d^i} = (d_0^i, d_1^i, d_2^i)$ for $i = 0, 1, ..., t$, and $R_{\mathfrak{a}}$ is the linear combination of the pure *diagrams greater that $\pi_{\mathbf{d^t}}$. Then the Boij-Söderberg decomposition of L has the form*

$$\beta(L) = \sum_{i=0}^{t} \tilde{\alpha}_i \pi_{\mathbf{\bar{d}^i}} + R_L$$

where $\mathbf{\bar{d}^i} = \mathbf{d^i} + \mathbf{1} = (d_0^i+1, d_1^i+1, d_2^i+1)$, and $\tilde{\alpha}_i = \alpha_i$ for $i = 0, 1, ..., t$ and $\tilde{\alpha}_t \geq \alpha_t$, and R_L is a linear combination of pure diagrams greater than $\pi_{\mathbf{\bar{d}^t}}$.

Proof. Recall that, for a given top degree sequence $\mathbf{d} = (d_0, d_1, d_2)$, the "normalized" pure diagram $\pi_{\mathbf{d}}$ can be obtained as following

$$\beta_{i,i+j}(\pi_{\mathbf{d}}) = \begin{cases} 0 & \text{if } i+j \neq d_i \\ \prod\limits_{r=0, r\neq i}^{2} \frac{\lambda}{|d_i - d_r|} & \text{if } i+j = d_i, \quad \text{where } \lambda = \operatorname{lcm}\left(\prod\limits_{r=0, r\neq i}^{2} |d_i - d_r|, i = 0, 1, 2 \right). \end{cases}$$

Thus, this formula provides pure diagrams with integer entries. From now on, we always consider "normalized" pure diagrams, that is, pure diagrams with integer entries.

Let $\mathbf{d^0} = (d_0^0, d_1^0, d_2^0)$ be the top degree sequence for the Betti diagram of $\mathfrak{a}$. if $d_2^0 < k+1$, that is, $d_0^0 < d_1^0 < d_2^0 < k+1$, so $d_0^0 < k-1$. Then we see that $\beta_{i,i+j}(\mathfrak{a}) = \beta_{i,i+j+1}(L)$ for all $j = 0, 1, ..., k-2$ since the Betti diagrams of $\mathfrak{a}$ and J may only overlap on the k-th row in the Betti diagram of L. As $L = x\mathfrak{a} + J$ and degree shift due to multiplication by x the top degree sequence of $\beta(L)$ will be $\mathbf{d^0} + 1$. Thus $\beta(L) - \alpha_0 \pi_{d^0+1}$ becomes the first step of the Boij-Söderberg-decomposition of $\beta(L)$. Actually we generalize this for all degree sequence $\mathbf{d^s}$ such that $d_2^s < k+1$.

Suppose $d_2^s < k+1$ for all $s = 0, 1, ..., l-1$, then we have a chain $d_2^0 < d_2^1 < ... < d_2^{l-1} < k+1$. Therefore, after l steps of the algorithm, we would get the remaining diagram

$$\beta(\mathfrak{a}) - \sum_{s=0}^{l-1} \alpha_s \pi_{\mathbf{d^s}} =: \tilde{\beta}(\mathfrak{a}) \quad \text{and} \quad \beta(L) - \sum_{s=0}^{l-1} \alpha_s \pi_{\mathbf{d^s+1}} =: \tilde{\beta}(L).$$

Let $\mathbf{d^l} = (d_0^l, d_1^l, d_2^l)$ be the next top degree sequence of the Betti diagram for $\mathfrak{a}$ and $d_2^l = k+1$ so above paragraph shows that $\mathbf{d^l} + \mathbf{1}$ becomes the next top degree sequence of Betti diagram for L. Therefore the remaining diagrams after the first l steps of the Boij-Söderberg decompositions for both $\mathfrak{a}$ and Ł look like as following,

and similarly,

By construction of $\beta(L)$, we deduce that

$$\tilde{\beta}_{0,d_0^l+1}(L) = \tilde{\beta}_{0,d_0^l}(\mathfrak{a}) \text{ and } \tilde{\beta}_{1,d_1^l+1}(L) = \tilde{\beta}_{1,d_1^l}(\mathfrak{a}) \text{ as } d_0^l + 1 < k \text{ and } d_1^l < k$$
$$\tilde{\beta}_{2,d_2^l+1}(L) = a_{2,d_2^l} + c_{1,d_2^l} \text{ as } d_2^l - 1 = k.$$

The decomposition algorithm exposes the coefficient of the pure diagram $\pi_{\mathbf{d^l}}$ to be

$$\alpha_l = \min\{ \frac{\tilde{\beta}_{0,d_0^l}(\mathfrak{a})}{\beta_{0,d_0^l}(\pi_{\mathbf{d^l}})}, \frac{\tilde{\beta}_{1,d_1^l}(\mathfrak{a})}{\beta_{1,d_1^l}(\pi_{\mathbf{d^l}})}, \frac{a_{2,d_2^l}}{\beta_{2,d_2^l}(\pi_{\mathbf{d^l}})} \} \tag{3.2}$$

$$\beta(\mathfrak{a}) - \sum_{s=0}^{l-1} \alpha_s \pi_{\mathbf{d^s}} = \begin{array}{c|ccc} \tilde{\beta}(\mathfrak{a}) & 0 & 1 & 2 \\ \hline d_0^l & \tilde{\beta}_{0,d_0^l}(\mathfrak{a}) & - & - \\ \vdots & \vdots & \vdots & \vdots \\ d_1^l - 1 & a_{0,d_1^l-1} & \tilde{\beta}_{1,d_1^l}(\mathfrak{a}) & - \\ \vdots & \vdots & \vdots & \vdots \\ d_2^l - 2 = k-1 & a_{0,d_2^l-2} & a_{1,d_2^l-1} & a_{2,d_2^l} \end{array}$$

Table 3.3: Remaining diagram after l steps for $\beta(\mathfrak{a})$

$$\beta(L) - \sum_{s=0}^{l-1} \alpha_s \pi_{\mathbf{d^s+1}} =$$

$$\begin{array}{c|ccc} \tilde{\beta}(L) & 0 & 1 & 2 \\ \hline d_0^l + 1 & \tilde{\beta}_{0,d_0^l+1}(L) & - & - \\ \vdots & \vdots & \vdots & \vdots \\ d_1^l & a_{0,d_1^l-1} & \tilde{\beta}_{1,d_1^l+1}(L) & - \\ \vdots & \vdots & \vdots & \vdots \\ d_2^l - 1 = k & a_{0,d_2^l-2} + c_{0,d_2^l-1} & a_{1,d_2^l-1} + c_{0,d_2^l-1} + c_{1,d_2^l} & \tilde{\beta}_{2,d_2^l+1}(L) \\ d_2^l & c_{0,d_2^l} & c_{0,d_2^l} + c_{1,d_2^l+1} & c_{1,d_2^l+1} \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Table 3.4: Remaining diagram after l step for $\beta(L)$

and similarly for the Boij-Söderberg-decomposition of $\beta(L)$ there is a rational number $\tilde{\alpha_l}$ as the coefficient of the pure diagram $\pi_{\mathbf{d^l+1}}$ such that

$$\tilde{\alpha_l} = \min\{\frac{\tilde{\beta}_{0,d_0^l}(\mathfrak{a})}{\beta_{0,d_0^l}(\pi_{\mathbf{d^l}})}, \frac{\tilde{\beta}_{1,d_1^l}(\mathfrak{a})}{\beta_{1,d_1^l}(\pi_{\mathbf{d^l}})}, \frac{a_{2,d_2^l} + c_{1,d_2^l}}{\beta_{2,d_2^l}(\pi_{\mathbf{d^l}})}\} \tag{3.3}$$

Hence we just need to look at the k-th row of the Betti diagram of L if $\beta(\mathfrak{a})$ and $\beta(L)$ overlap. Thus, we only need to think about the top degree sequences d^s of length 3 of $\beta(\mathfrak{a})$ such that $d_2^s = k+1$.

CASE I: Let $a_{2,k+1}$ be eliminated in the $(l+1)$-th step of the decomposition algorithm of $\beta(\mathfrak{a})$. In other words, $\mathbf{d^l} = (d_0^l, d_1^l, d_2^l)$ is of length 3, but $\mathbf{d^{l+1}} = (d_0^{l+1}, d_1^{l+1})$ has length 2. It shows that $\mathbf{d^0} < \mathbf{d^1} < ... < \mathbf{d^i} < ... < \mathbf{d^l}$ are all length 3 degree sequences in the decomposition of $\beta(\mathfrak{a})$. Hence, Boij-Söderberg-decomposition of $\beta(\mathfrak{a})$ is as

$$\beta(\mathfrak{a}) = \sum_{s=0}^{l} \alpha_s \pi_{\mathbf{d^s}} + [\text{all pure diagrams of length less than 3}].$$

So we do not need to pay attention to the $(l+2)$-th step in the decomposition. Besides the diagram 3.4 already shows that $\mathbf{d^l+1}$ is top degree sequence of the remaining diagram of L, $\tilde{\beta}(L)$. Therefore the first $(l+1)$-th top degree sequences of Boij-Söderberg decomposition of $\beta(L)$ is

$$\mathbf{d^0+1} < \mathbf{d^1+1} < \ldots < \mathbf{d^l+1}$$

where the coefficients $\tilde{\alpha}_i = \alpha_i$ for $i = 0, 1, \ldots, l-1$.

CASE II: Suppose that $a_{2,k+1}$ is not eliminated in the $(l+1)$-th step of the decomposition of $\beta(\mathfrak{a})$. Moreover we assume that it will vanish in the $(t+1)$-th step for some $t > l$. That is, the chain of the degree sequences in the Boij-Söderberg decomposition of $\beta(\mathfrak{a})$ is

$$\mathbf{d^0} < \mathbf{d^1} < \ldots < \mathbf{d^l} < \ldots < \mathbf{d^t} < \ldots < \mathbf{d^n}$$

where,

- for $s = 0, 1, \ldots, l-1$, $\mathbf{d^s} = (d_0^s, d_1^s, d_2^s)$ has length 3 such that $d_2^s < k+1$,
- for $s = l, \ldots, t$, $\mathbf{d^s} = (d_0^s, d_1^s, d_2^s)$ has length 3 such that $d_2^s = k+1$ and
- for $s = t+1, \ldots, n$, $\mathbf{d^s} = (d_0^s, d_1^s)$ has length 2.

As the entries only above the $(k-1)$-th row are eliminated until the $(t+1)$-th step of the decomposition, it is not difficult to guess the remaining diagram of L.

In the previous section we have seen that the entries above the k-th in $\beta(\mathfrak{a})$ are the same entries in $\beta(L)$. Let the remaining diagram of $\beta(\mathfrak{a})$ after subtracting the first t pure diagrams be

$$\beta(\mathfrak{a}) - \sum_{s=0}^{t-1} \alpha_s \pi_{\mathbf{d^s}} = \begin{array}{c|ccc} \tilde{\beta}(\mathfrak{a}) & 0 & 1 & 2 \\ \hline d_0^t & \tilde{\beta}_{0,d_0^t}(\mathfrak{a}) & - & - \\ \vdots & \vdots & \vdots & \vdots \\ d_1^t - 1 & a_{0,d_1^t-1} & \tilde{\beta}_{1,d_1^t}(\mathfrak{a}) & - \\ \vdots & \vdots & \vdots & \vdots \\ d_2^t - 2 = k-1 & a_{0,d_2^t-2} & a_{1,d_2^t-1} & \tilde{\beta}_{2,d_2^t}(\mathfrak{a}) \end{array}$$

where $\tilde{\beta}_{i,d_i^t}(\mathfrak{a}) = \beta_{i,d_i^t}(\mathfrak{a}) - \sum_{s=0}^{t-1} \alpha_s \beta_{i,d_i^t}(\pi_{\mathbf{d^s}})$, for $i = 0, 1, 2$.

Furthermore, as in (3.2) and (3.3), we can observe similar relations between the coefficients in both Boij-Söderberg decomposition of $\beta(\mathfrak{a})$ and $\beta(L)$ during their first t steps. The coefficients of the pure diagrams $\pi_{\mathbf{d^s}}$ in the decomposition of $\beta(\mathfrak{a})$ for $s = l, \ldots, t-1$ is

$$\alpha_s = \min\{\frac{\tilde{\beta}_{i,d_i^s}(\mathfrak{a})}{\beta_{i,d_i^s}(\pi_{\mathbf{d^s}})}, \quad \text{for} \quad i = 0, 1, 2\}.$$

Similarly, the corresponding coefficient $\tilde{\alpha_s}$ of the pure diagram $\pi_{\mathbf{d^s+1}}$ in the decomposition of $\beta(L)$ becomes

$$\begin{aligned} \tilde{\alpha_s} &= \min\{\frac{\tilde{\beta}_{i,d_i^s+1}(L)}{\beta_{i,d_i^s+1}(\pi_{\mathbf{d^s+1}})}, \quad \text{for} \quad i = 0, 1, 2\} \\ &= \min\{\frac{\tilde{\beta}_{0,d_0^s}(\mathfrak{a})}{\beta_{0,d_0^s}(\pi_{\mathbf{d^s}})}, \frac{\tilde{\beta}_{1,d_1^s}(\mathfrak{a})}{\beta_{1,d_1^s}(\pi_{\mathbf{d^s}})}, \frac{\tilde{\beta}_{2,d_2^s}(\mathfrak{a}) + c_{1,d_2^s}}{\beta_{2,d_2^s}(\pi_{\mathbf{d^s}})}\} \end{aligned}$$

We assume that any of the entries corresponding to d_i^s for $i = 0, 1$ would be eliminated where $s = l, .., t-1$. Thus

$$\alpha_s < \frac{\tilde{\beta}_{2,d_2^s}(\mathfrak{a})}{\beta_{2,k+1}(\pi_{d^s})}, \quad \text{where} \quad d_2^s = k+1.$$

So it follows that

$$\frac{\tilde{\beta}_{2,d_2^s}(\mathfrak{a})}{\beta_{2,k+1}(\pi_{\mathbf{d^s}})} < \frac{\tilde{\beta}_{2,d_2^s}(\mathfrak{a}) + c_{1,k+1}}{\beta_{2,k+1}(\pi_{\mathbf{d^s}})}.$$

Hence $\tilde{\alpha}_s = \alpha_s$ for $s = l, ..., t-1$. However, this situation may change for the coefficients α_t and $\tilde{\alpha}_t$ since $\tilde{\beta}_{2,d_2^t}(\mathfrak{a})$ will be eliminated in the next step. So $\alpha_t \leq \tilde{\alpha}_t$.
Hence the remaining diagram of $\beta(L)$ is

$$\tilde{\beta}(L) \quad := \quad \beta(L) - \sum_{s=0}^{t-1} \alpha_s \pi_{\mathbf{d^s+1}}$$

$$= \quad \begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
d_0^t + 1 & \tilde{\beta}_{0,d_0^t+1}(L) & - & - \\
\vdots & \vdots & \vdots & \vdots \\
d_1^t & a_{0,d_1^t-1} & \tilde{\beta}_{1,d_1^t+1}(L) & - \\
\vdots & \vdots & \vdots & \vdots \\
d_2^t - 1 = k & a_{0,d_2^t-2} + c_{0,d_2^t-1} & a_{1,d_2^t-1} + c_{0,d_2^t-1} + c_{1,d_2^t} & \tilde{\beta}_{2,d_2^t}(L) \\
k+1 & c_{0,k+1} & c_{0,k+1} + c_{1,k+2} & c_{1,k+2} \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

where

$$\tilde{\beta}_{0,d_0^t+1}(L) = \tilde{\beta}_{0,d_0^t}(\mathfrak{a}) \;\text{ and }\; \tilde{\beta}_{1,d_1^t+1}(L) = \tilde{\beta}_{1,d_1^t}(\mathfrak{a}) \;\text{ as }\; d_0^t + 1 < k \;\text{ and }\; d_1^t < k$$

$$\beta_{2,d_2^t+1}(L) = \tilde{\beta}_{2,d_2^t}(\mathfrak{a}) + c_{1,d_2^t} \;\text{ as }\; d_2^t - 1 = k.$$

This will bring us back to Case I, $\mathbf{d^t} = (d_0^t, d_1^t, d_2^t)$ is the last top degree sequence of length 3 in the Boij-Söderberg decomposition fo $\beta(\mathfrak{a})$. Above remaining diagram clearly shows us that $\mathbf{d^t} + \mathbf{1} = (d_0^t + 1, d_1^t + 1, d_2^t + 1)$ is a degree sequence in the Boij-Söderberg decomposition fo $\beta(L)$.

As a summary, if $\mathbf{d^0} < \mathbf{d^1} < ... < \mathbf{d^t}$ is the chain of the all top degree sequences of length 3 in the Boij-Söderberg-decomposition of $\beta(\mathfrak{a})$ with coefficients α_s for $s = 0, 1, .., t$. Then $\mathbf{d^0} + \mathbf{1} < \mathbf{d^1} + \mathbf{1} < ... < \mathbf{d^t} + \mathbf{1}$ becomes the initial t top degree sequences of length 3 in the Boij-Söderberg-decomposition of $\beta(L)$ with $\tilde{\alpha}_s = \alpha_s$ if $s < t$ and $\tilde{\alpha}_t \geq \alpha_t$.

□

Remark 3.2.2. We believe that this result can be generalized to the lex-ideals in $\mathbf{k}[x_1, ..., x_n]$.

Let $L = (x_1)\mathfrak{a} + J$ in $R = \mathbf{k}[x_1, x_2, ..., x_n]$ be a lex-segment ideal, then $\mathfrak{a}$ is also lex-segment ideal in R and J turns out to be a stable ideal of codim $n-1$ in $\mathbf{k}[x_2, ..., x_n]$. Suppose

$$F_{n-1} \to \cdots \to F_i \to \cdots \to F_1 \to J \to 0$$

$$G_n \to \cdots \to G_i \to \cdots \to G_1 \to \mathfrak{a} \to 0$$

are the minimal free resolutions of J and $\mathfrak{a}$, respectively. We get the same short exact sequence (3.1) like in Lemma 3.1.8, then by mapping cone we have the following minimal free resolution for L

$$0 \to G_n(-1) \oplus F_{n-1}(-1) \to ... \to G_2(-1) \oplus F_2 \oplus F_1(-1) \to G_1(-1) \oplus F_1 \to L.$$

So it yields

$$\beta_{i,i+j}(L) = \begin{cases} \beta_{i,i+j-1}(\mathfrak{a}) & \text{if} \quad i+j < a(J), \\ \beta_{i,i+j-1}(\mathfrak{a}) + \sum\limits_{t=i-1}^{i} \beta_{t,j+t}(J) & \text{if} \quad i+j \geq a(J), \end{cases}$$

where $\quad i = 0, 1, ..., n-1$.

By using lex-order properties of L and a, as we did in case $n = 3$, we conclude that the Betti diagrams of $\mathfrak{a}$ and J either overlap only on the $a(J)$-th row of the Betti diagram of L or do not overlap at all. Identify $k := a(J)$. Therefore, the Betti diagram of L in $\mathbf{k}[x_1, ..., x_n]$ is

$\beta(L)$	0	1	...	n-1
2	$a_{0,1}$	$a_{1,2}$	...	$a_{n-1,n}$
3	$a_{0,2}$	$a_{1,3}$	...	$a_{n-1,n+1}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$k-1$	$a_{0,k-2}$	$a_{1,k-1}$	...	$a_{n-1,k+n-3}$
k	$a_{0,k-1} + c_{0,k}$	$a_{1,k} + c_{0,k} + c_{1,k+1}$	...	$a_{n-1,k+n-2} + c_{n-1,k+n-1}$
$k+1$	$c_{0,k+1}$	$c_{0,k+1} + c_{1,k+2}$	...	$c_{n-1,k+n-1}$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$
$e^+(L) = e^+(J)$	$c_{0,e^+(L)}$	$c_{0,e^+(L)} + c_{1,e^+(L)+1}$	...	$c_{n-1,e^+(L)+n-1}$

Table 3.5: Betti diagram of L in $\mathbf{k}[x_1, ..., x_n]$

We believe that the proof of Theorem 3.2.1 can be modified for the polynomial ring of n variables. Hence one could conclude that if $\pi_{\mathbf{d}^0} < \pi_{\mathbf{d}^1} < ... < \pi_{\mathbf{d}^t}$ are all pure diagrams of length n in the Boij-Söderberg decomposition of $\mathfrak{a}$, where $\mathbf{d}^i = (d_0^i, d_1^i, ..., d_{n-1}^i)$ for $i = 0, 1, ..., t$. Then the chain of pure diagrams

$$\pi_{\bar{\mathbf{d}}^0} < \pi_{\bar{\mathbf{d}}^1} < ... < \pi_{\bar{\mathbf{d}}^t}$$

appears in the beginning of the Boij-Söderberg decomposition of L such that $\bar{\mathbf{d}}^{\mathbf{i}} = \mathbf{d}^{\mathbf{i}} + \mathbf{1} = (d_0^i + 1, d_1^i + 1, .., d_{n-1}^i + 1)$.

The Boij-Söderberg Decomposition for (L, x)

Previously we depicted the beginning of the chain of the degree sequences in the Boij-Söderberg decomposition of $\beta(L) = x\mathfrak{a} + J$ in terms of the decomposition of $\beta(\mathfrak{a})$. Now we aim to give a description of the end of the Boij-Söderberg decomposition of L in $R = \mathbf{k}[x, y, z]$.

We conjecture that all degree sequences of length less than 3 in the decomposition of $\beta(L, x) = \beta(J, x)$ occur precisely as all degree sequences of length less that 3 in the decomposition for L.

We give the proof of this statement for all Artinian lex-segment ideals $L = \mathfrak{a}(x)+J$ except the ones of the form $L = x(x, y, z^t) + J$ where J is different that $(y, z)^{a(J)}$ and $1 < t < k-1$. Actually we believe that the statement is also true of this particular case of L, however proof of this particular case requires a case analyzing which becomes infeasible.

Theorem 3.2.3. *[15] Let $L \subset R = \mathbf{k}[x, y, z]$ be an Artinian lex-segment ideal of codimension* 3*. Suppose that L is not decomposed as $L = x(x, y, z^t) + J$ where J is different that $(y, z)^{a(J)}$ and $1 < t < k - 1$.*

Let $\mathfrak{a} = L : (x)$ be a lex-segment ideal of R. Then $L = x\mathfrak{a} + J$ where $J \in \mathbf{k}[y, z]$ is a stable ideal of codim 2*. The ideal $(J, x) = (L, x)$ is also a* codim 3 *Artinian, lex-segment ideal in R.*

$$\beta(L, x) = R_{(L,x)} + \sum_{i=t+1}^{n} \alpha_i \pi_{\mathbf{d^i}}$$

where $\mathbf{d}^{t+1} < \mathbf{d}^{t+2} < \ldots < \mathbf{d}^n$ are all top degree sequences of length less than 3*, with the coefficients α_i, $i = t + 1, \ldots, n$. $R_{(L,x)}$ is the linear combination of the pure diagrams associated with the degree sequences of length* 3*.*

Then the Boij-Söderberg decomposition of L is

$$\beta(L) = R_L + \sum_{i=t+1}^{n} \alpha_i \pi_{\mathbf{d^i}}$$

the chain $\mathbf{d^{t+1}} < \mathbf{d^{t+2}} < \ldots < \mathbf{d^n}$ of degree sequences of length 2 *and* 1 *exactly with the same coefficients α_i and R_L is the linear combination of the pure diagrams associated with the degree sequences of length* 3*.*

Proof. First let's observe the decomposition of the Betti diagram of (L, x). Say $e^+(L, x) = e^+(L) = n$. Suppose $k = a(J) > 2$ and $n \geq k + 1$. So the diagram has the following form;

$$
\begin{array}{c|ccc}
\beta(L,x) & 0 & 1 & 2 \\ \hline
1 & \boxed{1} & - & - \\
2 & - & - & - \\
\vdots & \vdots & \vdots & \vdots \\
k-1 & - & - & - \\
k & c_{0,k} & \boxed{2c_{0,k}-1} & \boxed{c_{0,k}-1} \\
k+1 & c_{0,k+1} & 2c_{0,k+1} & c_{0,k+1} \\
\vdots & \vdots & \vdots & \vdots \\
n & c_{0,n} & 2c_{0,n} & c_{0,n}
\end{array}
$$

We aim to show that before the entry $\beta_{2,k+3}(L,x) = c_{0,k} - 1$ gets eliminated, $\beta_{0,1}(L,x) = 1$ is eliminated.
First degree sequence is $\bar{\mathbf{d}}^{\mathbf{0}} = (1, k+1, k+2)$, then we have $\beta(L,x) - \gamma_0 \pi_{\bar{\mathbf{d}}^{\mathbf{0}}}$ where $\pi_{\bar{\mathbf{d}}^{\mathbf{0}}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 1 & \mathbf{1} & - & - \\ \vdots & \vdots & \vdots & \vdots \\ k & - & \mathbf{k+1} & \mathbf{k} \end{array}$ and $\gamma_0 = \min\{1, \frac{2c_{0,k}-1}{k+1}, \frac{c_{0,k}-1}{k}\} = \frac{c_{0,k}-1}{k}$. The next degree sequence becomes $\bar{\mathbf{d}}^{\mathbf{1}} = (1, k+1, k+3)$ and then the pure diagram is $\pi_{\bar{\mathbf{d}}^{\mathbf{1}}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 1 & \mathbf{2} & - & - \\ \vdots & \vdots & \vdots & \vdots \\ k & - & \mathbf{k+2} & - \\ k+1 & - & - & \mathbf{k} \end{array}$.
Then the coefficient can be obtained as $\gamma_1 = \min\{\frac{1}{2} - \frac{c_{0,k}-1}{2k}, \frac{c_{0,k}(k+1)-1}{k+2}, \frac{c_{0,k+1}}{k}\} = \frac{1}{2} - \frac{c_{0,k}-1}{2k}$. Hence we have eliminated the entry $\beta_{0,1}(L,x)$, then the remaining diagram $\beta(L,x) - \gamma_0\pi_{\bar{\mathbf{d}}^{\mathbf{0}}} - \gamma_1\pi_{\bar{\mathbf{d}}^{\mathbf{1}}}$ is

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
k & \boxed{c_{0,k}} & \boxed{(k+2)\dfrac{c_{0,k}-1}{k} - k(\dfrac{1}{2} - \dfrac{c_{0,k}-1}{2k})} & - \\
k+1 & c_{0,k+1} & 2c_{0,k+1} & \boxed{c_{0,k+1} - k(\dfrac{1}{2} - \dfrac{c_{0,k}-1}{2k})} \\
\vdots & \vdots & \vdots & \vdots \\
n & c_{0,n} & 2c_{0,n} & c_{0,n}
\end{array}
$$

Next, $\bar{\mathbf{d}}^{\mathbf{2}} = (k, k+1, k+3)$ and $\pi_{\bar{\mathbf{d}}^{\mathbf{2}}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline k & \mathbf{2} & \mathbf{3} & - \\ k+1 & - & - & \mathbf{1} \end{array}$. The corresponding coefficient is $\gamma_2 = \min\{\frac{3(c_{0,k-1})-k}{6}, c_{0,k+1} - \frac{k+1-c_{0,k}}{2}\}$. This brings us to two separate cases;

(i) If $c_{0,k+1} < \frac{k}{3}$, then $\gamma_2 = c_{0,k+1} - \frac{k+1-c_{0,k}}{2}$. Then,

$$
\beta(L,x) - \sum_{i=0}^{2} \gamma_i \pi_{\bar{\mathbf{d}}^{\mathbf{i}}} = \begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
k & k+1-2c_{0,k} & k-3c_{0,k+1} & \text{-} \\
k+1 & c_{0,k+1} & 2c_{0,k+1} & \text{-} \\
\vdots & \vdots & \vdots & \vdots \\
n & c_{0,n} & 2c_{0,n} & c_{0,n}
\end{array} \tag{3.4}
$$

(ii) If $\frac{k}{3} < c_{0,k+1}$, then $\gamma_2 = \dfrac{3(c_{0,k-1}) - k}{6}$. So, we obtain

$$\beta(L,x) - \sum_{i=0}^{2} \gamma_i \pi_{\bar{\mathbf{d}}^{\mathbf{i}}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline k & \frac{k}{3}+1 & - & - \\ k+1 & c_{0,k+1} & 2c_{0,k+1} & c_{0,k+1} - \frac{k}{3} \\ \vdots & \vdots & \vdots & \vdots \\ n & c_{0,n} & 2c_{0,n} & c_{0,n} \end{array} \tag{3.5}$$

Now we examine the Boij-Söderberg decomposition of the lex ideal L. First of all, as a trivial case, we must notice that if $a(L) = 1$, then the statement is vacuously true since $L = (L, x)$.

We induct on the the difference of the initial degrees $a(J) - a(\mathfrak{a}) \geq 1$.

Base Step: In this step, we show that the statement is true for the lex ideals $L = x\mathfrak{a} + J$ when $a(J) - a(\mathfrak{a}) = 1$. That is, if $a(J) = k \geq 2$ then $a(\mathfrak{a}) = k - 1$. So $\mathfrak{a} = (x, y, z)^{k-1}$ since L is a lex ideal, .

Thus we modify the Betti diagram of L in the Table (3.1) to this case,

$$\beta(L) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline k & \boxed{\dfrac{k(k+1)}{2} + c_{0,k}} & \boxed{(k-1)(k+1) + 2c_{0,k} - 1} & \boxed{\dfrac{k(k-1)}{2} + c_{0,k} - 1} \\ k+1 & c_{0,k+1} & 2c_{0,k+1} & c_{0,k+1} \\ \vdots & \vdots & \vdots & \vdots \\ n & c_{0,n} & 2c_{0,n} & c_{0,n} \end{array}$$

Then obviously $\mathbf{d^0} = (k, k+1, k+2)$ and $\alpha_0 = \dfrac{k(k-1)}{2} + c_{0,k} - 1$. Then $\mathbf{d^1} = (k, k+1, k+3)$ becomes the next degree sequence with the coefficient $\alpha_1 = \min\{\frac{k}{3}, c_{0,k+1}\}$.

(i) If $\alpha_1 = c_{0,k+1} < \frac{k}{3}$, then

$$\begin{aligned} \beta(L) - \sum_{i=0}^{1} \alpha_i \pi_{\mathbf{d^i}} &= \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline k & k+1-2c_{0,k+1} & k-3c_{0,k+1} & - \\ k+1 & c_{0,k+1} & 2c_{0,k+1} & - \\ \vdots & \vdots & \vdots & \vdots \\ n & c_{0,n} & 2c_{0,n} & c_{0,n} \end{array} \\ \text{by 3.4} &= \beta(L,x) - \sum_{i=0}^{2} \gamma_i \pi_{\bar{\mathbf{d}}^{\mathbf{i}}} \end{aligned}$$

(ii) If $\alpha_1 = \frac{k}{3} < c_{0,k+1}$ then the remaining diagram of of $\beta(L)$ after three steps becomes

$$\beta(L) - \sum_{i=0}^{2} \alpha_i \pi_{\mathbf{d^i}} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline k & \frac{k}{3}+1 & \text{-} & \text{-} \\ k+1 & c_{0,k+1} & 2c_{0,k+1} & c_{0,k+1} - \frac{k}{3} \\ \vdots & \vdots & \vdots & \vdots \\ n & c_{0,n} & 2c_{0,n} & c_{0,n} \end{array}$$

$$\text{by 3.5} = \beta(L,x) - \sum_{i=0}^{2} \gamma_i \pi_{\bar{\mathbf{d}}^{\mathbf{i}}}$$

Thus, $\beta(L)$ and $\beta(L,x)$ have exactly the same remaining diagrams in the decomposition. Hence, the statement holds for the case of $a(J) - a(\mathfrak{a}) = 1$.

Induction Hypothesis: Let the statement be true for all lex ideals $L = x\mathfrak{a} + J$ satisfying $a(J) - a(\mathfrak{a}) = N \geq 1$. We need to show that it is also true for the lex ideals satisfying $a(J) - a(\mathfrak{a}) = N + 1$. We identify the initial degrees of J and $\mathfrak{a}$ by $a(J) = k$ and $a(\mathfrak{a}) = m$.

Suppose that $L = x\mathfrak{a} + J$ is a lex ideal such that $k - m = N + 1$. So $k - m = N + 1 \geq 2$. The proof is given in two cases.

<u>*CASE I:*</u> If $y^m \notin \mathfrak{a}$. Since $\mathfrak{a}$ is a lex ideal, we write $\mathfrak{a} = x\mathfrak{b} + I$. Then we notice that $a(I) \neq k$ otherwise it contradicts to $y^k \in G(J)$. Thus $k \gneq a(I) \gneq m$ as $y^m \notin \mathfrak{a}$.

Define $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ as the ideal containing all monomials of $\mathfrak{a}$ of degree greater or equal to $m + 1$. One can easily check that $\tilde{\mathfrak{a}}$ is also a lex ideal with $a(\tilde{\mathfrak{a}}) = m + 1$. Define $\tilde{L} = x\tilde{\mathfrak{a}} + J$ and it is a lex ideal with $a(J) - a(\tilde{\mathfrak{a}}) = k - (m + 1) = k - m - 1 = N + 1 - 1 = N$. Therefore by the induction hypothesis, $\beta(\tilde{L})$ and $\beta(L,x)$ have the same ends in their Boij-Söderberg decompositions, i.e. same pure diagrams of length less than 3 with same coefficients,

$$\beta(\tilde{L}) - \sum_{\substack{\tilde{\mathbf{d}}^i,\\ \text{all length 3}\\ \text{degree seqs.}}} \tilde{\alpha}_i \pi_{\tilde{\mathbf{d}}^i} = \beta(L,x) - \sum_{\substack{\mathbf{d^i},\\ \text{all length 3}\\ \text{degree seqs.}}} \gamma_i \pi_{\mathbf{d}^i}.$$

On the other hand, $\tilde{\mathfrak{a}}$ can be decomposed as $\tilde{\mathfrak{a}} = x\tilde{\mathfrak{b}} + \tilde{I}$. It is easy to see that $\tilde{I} = I$ as $y^m \notin \mathfrak{a}$ and $a(\tilde{\mathfrak{b}}) = m$. Clearly, $a(I) - a(\tilde{\mathfrak{b}}) \leq (k-1) - m = N$. Thus, again by the hypothesis Boij-Söderberg decompositions of $\beta(\tilde{\mathfrak{a}})$ and $\beta(I,x)$ have the same ends.

Recall that $\mathfrak{a} = x\mathfrak{b} + I$, so we get $a(I) - a(\mathfrak{b}) \leq (k-1) - (m-1) = k - m = N + 1$. Suppose that $a(I) - a(\mathfrak{b}) < N + 1$, then the hypothesis provide the results, that is, Boij-Söderberg decompositions of $\mathfrak{a}$ and (I,x) have the same ends, so do $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$. That is,

$$D := \beta(\mathfrak{a}) - \sum \text{all length 3 pure diagrams} = \beta(\tilde{\mathfrak{a}}) - \sum \text{all length 3 pure diagrams}.$$

Also using the Theorem 3.2.1 Boij-Söderberg decompositions for the ideals L and $\tilde{L}$

can be observed as following;

$$\beta(L) = \sum_{\mathbf{d^i} \text{ with } l(\mathbf{d^i})=3} \alpha_i \pi_{\mathbf{d^i}} \quad + \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 2 & & \text{Remaining} & \\ \vdots & & \text{diagram, } \mathbf{D} & \\ \hline k & & & \\ \vdots & & \beta_{i,i+j}(L,x),\ i \geq k & \\ & & & \end{array}$$

$$\beta(\tilde{L}) = \sum_{\tilde{\mathbf{d}}^{i}+1 \text{ with } l(\tilde{\mathbf{d}}^i)=3} \tilde{\alpha}_i \pi_{\tilde{\mathbf{d}}^i} \quad + \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 2 & & \text{Remaining} & \\ \vdots & & \text{diagram, } \mathbf{D} & \\ \hline k & & & \\ \vdots & & \beta_{i,i+j}(L,x),\ i \geq k & \\ & & & \end{array}$$

This shows that $\beta(L)$ and $\beta(\tilde{L})$ have same ends but we also know that $\beta(\tilde{L})$ and $\beta(L,x)$ have same ends. Hence the statement is true.

However, we must still explain the case when $a(I) - a(\mathfrak{b}) = (k-1) - (m-1) = k - m = N+1$, which means $a(I) = k-1$. It follows $I = (y,z)^{k-1}$ since $a(J) = k$. Then $\mathfrak{a} = x\mathfrak{b} + I$ and $\mathfrak{b} = x\overline{\mathfrak{b}} + \overline{I}$ where $a(\overline{I}) - a(\overline{\mathfrak{b}}) \geq (k-2) - (m-2) \geq N+1$. If it is a strict inequality then same process as we have done for L can be applied to $\mathfrak{a}$ to prove the statement. If there is an equality, we end up with the same situation. $L = x\mathfrak{a} + J$ where $a(J) = k$, $a(\mathfrak{a}) = m$ and $k - m = N+1$, and $\mathfrak{a} = x\mathfrak{b} + I$ where $I = (y,z)^{k-1}, a(\mathfrak{b}) = m-1$, and $\mathfrak{b} = x\overline{\mathfrak{b}} + \overline{I}$ where $\overline{I} = (y,z)^{k-2}, a(\overline{\mathfrak{b}}) = m-2$. We repeat this until we get

$$\mathfrak{c} = x(x, y, z^{t-1}) + K \quad \text{where } K = (y,z)^s, s = k - m + 1, 1 \leq t \leq k - m.$$

For this form of the lex ideal, one can check the Boij-Söderberg decomposition of the ideal $\mathfrak{c}$.

$$\beta(\mathfrak{c}) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 2 & 2 & 1 & - \\ \vdots & \vdots & \vdots & \vdots \\ t & 1 & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ s & s+1 & 2s+1 & s \end{array}$$

$$= \frac{1}{t}\begin{bmatrix} 2: & t-1 & t & - \\ t: & - & - & 1 \end{bmatrix} + \frac{1}{t}\begin{bmatrix} 2: & 1 & - & - \\ t: & - & t-1 & t \end{bmatrix}$$

$$+ \frac{1}{s}\begin{bmatrix} 2: & s-t+1 & - & - \\ t: & - & s & - \\ s: & - & - & t-1 \end{bmatrix} + \frac{t-1}{s}\begin{bmatrix} 2: & 1 & - & - \\ s: & - & s & s-1 \end{bmatrix}$$

$$+ 1\begin{bmatrix} t: & 1 & - & - \\ s: & - & s-t+2 & s-t+1 \end{bmatrix} + \underbrace{s\begin{bmatrix} s: & 1 & 1 \end{bmatrix} + 1\begin{bmatrix} s: & 1 \end{bmatrix}}_{\text{same end as in the decomposition of } (L,x)}$$

Therefore the statement is true for the ideal $\mathfrak{c}$. So we may assume that, without loss of generality, $\mathfrak{a}$ can be assumed as in the form of $\mathfrak{c}$, that is, $a(I) - a(\mathfrak{a}) < N + 1$. This observation completes the proof for Case I.

<u>*CASE II:*</u> Let $y^m \in \mathfrak{a}$.

(i) If $m \notin 1$; we write $\mathfrak{a} = x\mathfrak{b} + I$ and $a(I) = m > 1$. This implies that $\mathfrak{b} = (x, y, z)^{m-1}$. Consider $\tilde{\mathfrak{a}} = (\mathfrak{a}, x) = x(1) + I$. Clearly $\tilde{\mathfrak{a}} = (\tilde{\mathfrak{a}}, x)$, so the statement is trivially true for the ideal $\tilde{\mathfrak{a}}$. Moreover, $a(I) - a(\mathfrak{a}) = m - (m-1) = 1$. By the base case, the decompositions of $\beta(\mathfrak{a})$ and $\beta(I, x)$ have the same ends. Hence,

$$\begin{aligned} \beta(\mathfrak{a}) - \sum \text{all length 3 pure diagrams} &= \beta(I, x) - \sum \text{all length 3 pure diagrams} \\ &= \beta(\tilde{\mathfrak{a}}) - \sum \text{all length 3 pure diagrams.} \end{aligned}$$

Similar to the Case I, consider the lex ideal $\tilde{L} = x\tilde{\mathfrak{a}} + J$ and $y^{a(\tilde{\mathfrak{a}})} = y \notin \tilde{\mathfrak{a}}$. Thus by the result of the Case I, the statement is true for $\tilde{L}$. We do exactly the same trick as in Case I to show that $\beta(L)$ and $\beta(\tilde{L})$ have the same ends and it follows that the statement holds for L.

(ii) If $m = 1$; that is, $\mathfrak{a} = (x, y, z^t)$ where $1 \le t \le k - 1$. In the Case I we have already shown that the Boij-Söderberg decomposition of the $\beta(L)$ satisfy the statement if $L = x(x, y, z^t) + J$ where $J = (y, z)^k$. Nevertheless, for more general stable ideal $J \subset \mathbf{k}[y, z]$ we had already assumed that L cannot be in that form in the statement.

$\square$

Conjecture 3.2.4. *The statement of Theorem 3.2.3 holds for all Artinian lex-ideals in* $\mathbf{k}[x, y, z]$.

Theorem 3.2.3 shows that the ends of the Boij-Söderbeg decompositions of L and $(L, x) = (J, x)$ are exactly the same for all Artinian lex ideals L in R except the ones in the form of $L = x(x, y, z^t) + J$ where J is different that $(y, z)^{a(J)}$ and $1 < t < k - 1$. However, based on the observations we have done using the `BoijSoederberg` packages of the computer algebra software `Macaulay2`, we believe that this result is also true for the lex ideals in that particular form.

3.3 Further Observations and Examples

For an Artinian lex ideal $L \subset \mathbf{k}[x, y, z]$ of codimension 3, we have shown that the summands of length 3 pure diagrams of the Boij-Söderberg decomposition of $\mathfrak{a}$ where $\mathfrak{a} = L : (x)$, and the summands of pure diagrams of length less than 3 in the Boij-Söderberg decomposition of (L, x) appear in the decomposition of the ideal $L = \mathfrak{a}(x) + J$ in the beginning and the end, respectively.

$$\beta(L) = \begin{bmatrix} \text{length 3 degree} \\ \text{sequences coming} \\ \text{from } \mathfrak{a} \end{bmatrix} + \begin{bmatrix} \text{extra length 3} \\ \text{degree sequences} \end{bmatrix} + \begin{bmatrix} \text{all length} < 3 \text{ degree} \\ \text{sequences coming} \\ \text{from } (L, x) \end{bmatrix}$$

There might be also some other pure diagrams of length 3 other than the ones coming from the Boij-Söderberg decomposition of $\mathfrak{a}$. However, how this middle part containing pure diagrams of length 3 comes out is not quite clear. One might ask whether or not the ideals $\mathfrak{b} = L : (y)$ and $\mathfrak{c} = L : (z)$ help to describe the middle part. In fact, examples show that there is a quite strong relation between them. Nevertheless, there are some cases, the diagrams obtained from the decompositions of $\beta(\mathfrak{b})$ and $\beta(\mathfrak{c})$ do not cover the entire middle part of the decomposition of $\beta(L)$ or the Boij-Söderberg decomposition of $\mathfrak{b}$ and $\mathfrak{c}$ may have pure diagrams which do not appear in the decomposition of $\beta(L)$.

Now in this section we illustrate the possible relation between the Boij-Söderberg decompositions of the ideals $\mathfrak{b}$, $\mathfrak{c}$ and L via examples.

Example 3.3.1. $L = (x^2, xy^2, xyz, xz^2, y^8, y^7z, y^6z^2, y^5z^3, y^4z^4, y^3z^5, y^2z^6, yz^7, z^8)$ is a lex segment ideal in R. Then

$$\mathfrak{a} = L : x = (x, y^2, yz, z^2)$$

is lex segment ideal such that $L = x\mathfrak{a} + J$ where $J = (y, z)^8$ is stable in R and lex segment in $\mathbf{k}[y, z]$.
Similarly the ideals

$$\mathfrak{b} = L : y = (x^2, xy, xz, y^7, y^6z, y^5z^2, y^4z^3, y^3z^4, y^2z^5, yz^6, z^7) = L : z = \mathfrak{c}$$

are lex segment ideals such that $L = y\mathfrak{b} + I = z\mathfrak{c} + K$ where $I = (x^2, xz^2, z^8)$ and $K = (x^2, xy^2, y^8)$.

Ww construct similar short exact sequences like (3.1) for the ideals $\mathfrak{b}$ and $\mathfrak{c}$. Unlike the case for $\mathfrak{a}$, we might have cancellations in the mapping cone of the short exact sequences for ideals. It means we can have cancellations in the Betti diagram since the mapping cone structure may not yield the minimal free resolution This situation causes different degree sequences that do not appear in Boij-Söderberg decomposition of L.

Now,first we notice that $\mathfrak{b} = \mathfrak{c}$ and find the Boij-Söderberg decomposition of $\beta(\mathfrak{a})$

$$\beta(\mathfrak{a}) = (1)\pi_{(1,3,4)} + [\text{pure diags. of length} < 3],$$

Then we consider the short exact sequence for the ideal $\mathfrak{b}$

$$\begin{array}{ccccccc}
0 \to & I(-1) & \to & \mathfrak{b}(-1) \oplus I & \to & L & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \begin{matrix} R(-3) \oplus R(-4) \\ \oplus R(-9) \end{matrix} & & \begin{matrix} R(-2) \oplus R^4(-3) \\ \oplus R^9(-8) \end{matrix} & & \begin{matrix} R(-2) \oplus \cancel{R^4(-3)} \\ \oplus R^9(-8) \end{matrix} & \\
& \downarrow & & \downarrow & & \downarrow & \\
& \begin{matrix} R(-5) \\ \oplus \\ R(-10) \end{matrix} & & \begin{matrix} R^4(-4) \\ \oplus \\ R^{16}(-9) \end{matrix} & & \begin{matrix} \cancel{R(-3)} \oplus R^5(-4) \\ \oplus R^{17}(-9) \end{matrix} & \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & R(-5) \oplus R^7(-10) & & R^2(-5) \oplus R^8(-10) & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 &
\end{array}$$

The mapping cone of the short exact sequence for ideal $\mathfrak{b}$ (so the same for $\mathfrak{c}$) ends up with "one" cancellation in the first degree. So we interpret this as ignoring one pure diagram at the beginning, which is the one corresponding to the degree sequence $(2, 3, 4)$ at the beginning of the decomposition of $\beta(\mathfrak{b})$. Therefore,

$$\beta(\mathfrak{b}) = \beta(\mathfrak{c}) = \cancel{(1)\pi_{(2,3,4)}} + (\frac{1}{7})\pi_{(2,3,9)} + (\frac{8}{7})\pi_{(2,8,9)} + [\text{pure diags. length} < 3].$$

The pure diagrams of length less than 3 are coming from the ideal

$$\beta(L, x) = [\text{length 3 pure diags.}] + (8)\pi_{(8,9)} + (1)\pi_{(8)}.$$

Hence we claim that the summands (with coefficients) in the Boij-Söderberg decomposition of $\beta(L)$ are

$$\beta(L) \approx \underbrace{(1)\pi_{(2,4,5)}}_{\text{from } \mathfrak{a}(-1)} + (\alpha_2) \underbrace{\pi_{(3,4,10)}}_{\text{from } \mathfrak{b}(-1)} + (\alpha_3) \underbrace{\pi_{(3,9,10)}}_{\text{from } \mathfrak{b}(-1)} + \underbrace{(8)\pi_{(8,9)} + (1)\pi_{(8)}}_{\text{from } (L,x)},$$

for some coefficients α_2, α_3 in $\mathbb{Q}$.

Indeed, the Boij-Söderberg decomposition of L is,

$$\beta(L) = (1)\pi_{(2,4,5)} + (\frac{2}{7})\pi_{(3,4,10)} + (\frac{9}{7})\pi_{(3,9,10)} + (8)\pi_{(8,9)} + (1)\pi_{(8)}.$$

The impressive point of this example is that one might expect to deduce a structural meaning from the description of the chain of degree sequences in Boij-Söderberg decomposition of L from the ones from $\mathfrak{a}$, $\mathfrak{b}$ and (L, x) because we are able to describe the entire chain of degree sequences of L from its the colon ideals $\mathfrak{a}$, $\mathfrak{b}$ and the ideal (L, x).

Example 3.3.2. This example will show that some different situations might occur other than the previous example.

Let $L = (x^2, xy^2, xyz, xz^2, y^4, y^3z, y^2z^2, yz^6, z^9)$ be lex-segment ideal in R. Then

$$\mathfrak{a} = L : x = (x, y^2, yz, z^2),$$

$$\mathfrak{b} = L : y = (x^2, xy, xz, y^3, y^2z, yz^2, z^6), \text{ and}$$

$$\mathfrak{c} = L : (z) = (x^2, xy, xz, y^3, y^2z, yz^5, z^8).$$

We observe that one cancellation occurs in the mapping cone process of each ideal $\mathfrak{b}$ and $\mathfrak{c}$. Boij-Söderberg decompositions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and (L, x) are

$$\beta(\mathfrak{a}) = 1\pi_{(1,3,4)} + [\text{pure diags. of length } < 3],$$

$$\begin{aligned}\beta(\mathfrak{b}) &= \cancel{1\pi_{(2,3,4)}} + \frac{1}{3}\pi_{(2,3,5)} + \frac{5}{6}\pi_{(2,4,5)} + \frac{1}{4}\pi_{(2,4,8)} + \frac{7}{20}\pi_{(3,4,8)} + \frac{1}{10}\pi_{(3,7,8)} \\ &+ [\text{pure diags. of length } < 3],\end{aligned}$$

$$\begin{aligned}\beta(\mathfrak{c}) &= \cancel{1\pi_{(2,3,4)}} + \frac{1}{3}\pi_{(2,3,5)} + \frac{1}{3}\pi_{(2,4,5)} + \frac{1}{2}\pi_{(2,4,8)} + \frac{1}{10}\pi_{(3,4,8)} + \frac{1}{10}\pi_{(3,7,8)} \\ &+ \frac{3}{14}\pi_{(3,7,10)} + \frac{1}{42}\pi_{(3,9,10)} + [\text{pure diags. of length } < 3], \text{ and}\end{aligned}$$

$$\beta(L, x) = [\text{pure diags. of length } 3] + 1\pi_{(4,10)} + 1\pi_{(7,10)} + 1\pi_{(9)}.$$

So, the Boij-Söderberg decomposition for the ideal L is likely to be

$$\begin{aligned}\beta(L) &\approx 1\pi_{(2,4,5)} + \alpha_2\pi_{(3,4,6)} + \alpha_3\pi_{(3,5,6)} + \alpha_4\pi_{(3,5,9)} + \alpha_5\pi_{(4,5,9)} + \alpha_6\pi_{(4,8,9)} + \alpha_7\pi_{(4,8,11)} \\ &+ \alpha_{\mathbf{8}}\pi_{(\mathbf{4,10,11})} + 1\pi_{(4,10)} + 1\pi_{(7,10)} + 1\pi_{(9)}, \quad \text{where } \alpha_i \in \mathbb{Q}, \quad i = 2, ...8.\end{aligned}$$

Thus it seems that we almost obtain the actual Boij-Söderberg decomposition for L which is

$$\begin{aligned}\beta(L) \;=\;& \underbrace{1\pi_{(2,4,5)}}_{\text{from } \mathfrak{a}(-1)} + \frac{2}{3}\underbrace{\pi_{(3,4,6)}}_{\text{from } \mathfrak{b}(-1) \text{ and } \mathfrak{c}(-1)} + \frac{2}{3}\underbrace{\pi_{(3,5,6)}}_{\text{from } \mathfrak{b}(-1) \text{ and } \mathfrak{c}(-1)} \\ &+ \frac{1}{2}\underbrace{\pi_{(3,5,9)}}_{\text{from } \mathfrak{b}(-1) \text{ and } \mathfrak{c}(-1)} + \frac{3}{10}\underbrace{\pi_{(4,5,9)}}_{\text{from } \mathfrak{b}(-1) \text{ and } \mathfrak{c}(-1)} + \frac{1}{20}\underbrace{\pi_{(4,8,9)}}_{\text{from } \mathfrak{b}(-1) \text{ and } \mathfrak{c}(-1)} \\ &+ \frac{1}{4}\underbrace{\pi_{(4,8,11)}}_{\text{from } \mathfrak{c}(-1)} + \underbrace{1\pi_{(4,10)} + 1\pi_{(7,10)} + 1\pi_{(9)}}_{\text{from } (L,x)}.\end{aligned}$$

Apparently, the Boij-Söderberg decomposition of $\mathfrak{c}$ provides an additional pure diagram, $\pi_{(4,10,11)}$, which does not appear in the Boij-Söderberg decomposition of L. Nevertheless it still supports the idea of the connection of the middle part of the decomposition of $\beta(L)$ and the decompositions of $\beta(\mathfrak{b})$ and $\beta(\mathfrak{c})$.

Example 3.3.3. In the previous example we saw that one can obtain a longer chain of the degree sequences for L that the actual chain of the degree sequences via the Boij-Söderberg decomposition of the ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and (L, x). This example shows

Consider the lex-segment ideal $L = (x^2, xy, xz^2, y^6, y^5z, y^4z^3, y^3z^4, y^2z^5, yz^6, z^9) \subset R$. Then the colon ideals are

$$\mathfrak{a} = L : x = (x, y, z^2),$$

$$\mathfrak{b} = L : y = (x, y^5, y^4z, y^3z^3, y^2z^4, yz^5, z^6), \text{ and}$$

$$\mathfrak{c} = L : z = (x^2, xy, xz, y^5, y^4z^2, y^3z^3, y^2z^4, yz^5, z^8).$$

The mapping cone for the ideal $\mathfrak{c}$ requires two cancellations, so we ignore the first two degree sequences. Then,

$$\beta(\mathfrak{a}) = \frac{1}{3}\pi_{(1,2,4)} + \frac{1}{3}\pi_{(1,3,4)} + [\text{pure diags. of length } < 3],$$

$$\beta(\mathfrak{b}) = \frac{1}{5}\pi_{(1,6,7)} + \frac{9}{35}\pi_{(1,6,8)} + \frac{2}{7}\pi_{(\mathbf{1},\mathbf{7},\mathbf{8})} + \frac{1}{2}\pi_{(\mathbf{5},\mathbf{7},\mathbf{8})} + [\text{pure diags. of length } < 3],$$

$$\begin{aligned}\beta(\mathfrak{c}) \;=\;& \cancel{1\pi_{(2,3,4)}} + \cancel{\frac{1}{6}\pi_{(2,3,8)}} + \frac{1}{3}\pi_{(2,6,8)} + \frac{19}{30}\pi_{(2,7,8)} + \frac{1}{15}\pi_{(\mathbf{2},\mathbf{7},\mathbf{10})} + \frac{1}{3}\pi_{(5,7,10)} \\ &+ [\text{pure diags. of length } < 3],\end{aligned}$$

$$\text{and } \beta(L,x) = [\text{pure diags. length } 3] + \frac{1}{2}\pi_{(6,8)} + 2\pi_{(7,8)} + 2\pi_{(7,10)} + 1\pi_{(9)}.$$

Then, we get the following chain of degree sequences in order to set up the approximate Boij-Söderberg decomposition for L

$$\begin{aligned}\beta(L) \;\approx\;& \underbrace{(2,3,5) < (2,4,5)}_{\text{from } \mathfrak{a}(-1)} < \underbrace{(2,7,8) < (2,7,9) < (\mathbf{2},\mathbf{8},\mathbf{9}) < (\mathbf{6},\mathbf{8},\mathbf{9})}_{\text{from } \mathfrak{b}(-1)} \\ &< \underbrace{(3,7,9) < (3,8,9) < (\mathbf{3},\mathbf{8},\mathbf{11}) < (6,8,11)}_{\text{from } \mathfrak{c}(-1)} < \underbrace{(7,9) < (8,9) < (8,11) < (10)}_{\text{from } (L,x)}.\end{aligned}$$

However, the degree sequences in the decomposition must be a partial ordered chain, so we have to eliminate the ones that violate the partial order. From the decomposition of $\beta(\mathfrak{c})$, we get $(3,7,9)$ as the first degree sequence, but $(2,8,9)$ and $(6,8,9)$ cannot be before $(3,7,9)$. So we have to ignore the sequences $(2,8,9)$ and $(6,8,9)$. Then we get an approximate decomposition such as

$$\begin{aligned}\beta(L) \;&\approx\; \frac{1}{3}\pi_{(2,3,5)} + \frac{1}{3}\pi_{(2,4,5)} + \alpha_3\pi_{(2,7,8)} + \alpha_4\pi_{(2,7,9)} + \alpha_7\pi_{(3,7,9)} + \alpha_8\pi_{(3,8,9)} + \alpha_9\pi_{(3,8,11)}\\ &+\; \alpha_{10}\pi_{(6,8,11)} + \frac{1}{2}\pi_{(6,8)} + \frac{1}{2}\pi_{(7,9)} + 2\pi_{(8,9)} + 2\pi_{(8,11)} + 1\pi_{(10)}.\end{aligned}$$

The Boij-Söderberg decomposition of $\beta(L)$ is

$$\begin{aligned}\beta(L) \;&=\; \frac{1}{3}\pi_{(2,3,5)} + \frac{1}{3}\pi_{(2,4,5)} + \boldsymbol{\frac{1}{3}\pi_{(2,4,8)}} + \frac{2}{15}\pi_{(2,7,8)} + \frac{1}{10}\pi_{(2,7,9)} + \frac{1}{2}\pi_{(3,7,9)} + \frac{1}{2}\pi_{(3,8,9)}\\ &+\; \frac{1}{2}\pi_{(6,8,11)} + \frac{1}{2}\pi_{(6,8)} + \frac{1}{2}\pi_{(7,9)} + 2\pi_{(8,9)} + 2\pi_{(8,11)} + 1\pi_{(10)}.\end{aligned}$$

The degree sequence $(3,8,11)$ associated with $(2,7,10)$, which is coming from the decomposition of $\beta(\mathfrak{c})$, does not show up in the decomposition of $\beta(L)$, similar to the situation in Example (3.3.2). Moreover, for this lex-segment ideal L, we realize another different situation. The degree sequence $(2,4,8)$ shows up in the chain of the Boij-Söderberg decomposition of $\beta(L)$, but $(2-1,4-1,8-1) = (1,3,7)$ does not appear in any of the decompositions of $\beta(\mathfrak{a})$, $\beta(\mathfrak{b})$ and $\beta(\mathfrak{c})$. Hence we get the entire chain of degree sequences.

We see that $(2,4,5)$ is the last degree sequence coming from $\mathfrak{a}(-1)$ and the next degree sequence $(2,7,8)$ is from $\mathfrak{b}(-1)$. If we assume that there is no other degree sequence between $(2,4,5)$ and $(2,7,8)$, it implies that simultaneous elimination of the entries in the positions of $\beta_{1,4}$ and $\beta_{2,5}$ in the Betti diagram of L by the algorithm of Boij-Söderberg decomposition. However, this is not possible because otherwise there would not be a pure diagram of length 2 in the Boij-Söderberg decomposition of $\mathfrak{a}$. Hence again by the partial order, it must be $(2,4,5) < (\mathbf{2},\mathbf{4},\mathbf{8}) < (2,7,8)$.

This research about Boij-Söderberg decomposition of lex-segment ideals continues with further directions.

Remark 3.3.4. The curiosity about a full description for the Boij-Söderberg decomposition of $\beta(L)$ in terms of $\beta(L,x)$, $\beta(\mathfrak{a})$, $\beta(\mathfrak{b})$, and $\beta(\mathfrak{c})$ is inevitable. So far we characterized the Boij-Söderberg decomposition of any (Artinian) lex-segment ideal L through the Boij-Söderberg decompositions of other lex ideals $\mathfrak{a} = L : x$ and (L,x) in terms of the pure diagrams in the decompositions. Moreover the examples showed that if we know the Boij-Söderberg decompositions of the colon ideals $\mathfrak{b} = L : y$ and $\mathfrak{c} = L : z$, they help us to reveal almost the entire chain of the degree sequences of the decomposition for the lex-segment ideal L. However,the examples (3.3.2) and (3.3.3) also showed that the Boij-Söderberg decompositions of $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and (L,x) may not be enough to provide the entire chain of degree sequences in the Boij-Söderberg decomposition of L. There might be some gaps and redundant degree sequences.

With the explanations, such as the cancellations in mapping cone, the necessity of the order of the chain of the degree sequences, we step closer to the entire chain of degree sequences in the Boij-Söderberg decomposition of L. In other words, we hope to formalize the full chain of degree sequences of the Boij-Söderberg decomposition of the ideal L by using the Boij-Söderberg decompositions of the colon ideals $\mathfrak{a}$,$\mathfrak{b}$ $\mathfrak{c}$ and the lex ideal (L, x).

Furthermore, in order to get a full description of the Boij-Söderberg decomposition of L, we must also examine the coefficients of the pure diagrams in the decompositions. Even though we focus on the pure diagrams in the Boij-Söderbeg decomposition, the Theorems 3.2.1 and 3.2.3 show the relations of the coefficients of the pure diagrams as well. However the relation of the coefficients of the pure diagrams in the Boij-Söderberg decomposition of the colon ideals $\mathfrak{b}$ and $\mathfrak{c}$ with the coefficients of the corresponding pure diagrams in the decomposition of L has not been studied yet.

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