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Global Well-posedness for the Derivative Nonlinear Schrödinger Equation Through Inverse Scattering

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DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

Global Well-posedness for the Derivative Nonlinear Schrödinger Equation Through Inverse Scattering

We study the Cauchy problem of the derivative nonlinear Schrödinger equation in one space dimension. Using the method of inverse scattering, we prove global well-posedness of the derivative nonlinear Schrödinger equation for initial conditions in a dense and open subset of weighted Sobolev space that can support bright solitons.

KEYWORDS: nonlinear dispersive equations, solitons, inverse scattering, Riemann-Hilbert problem

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Dedicated to my parents

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Baori accompanied me when this dissertation is written. Thank you for your cheerfulness and patience.

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Chapter 1 Introduction

The Derivative Nonlinear Schrödinger equation (DNLS)

$$iu_t + u_{xx} = i\varepsilon(|u|^2 u)_x \tag{1.0.1}$$

where $\varepsilon = \pm 1$, is a canonical dispersive equation derived from the Magneto-Hydrodynamic equations in the presence of the Hall effect. The equation models the dynamics of Alfvén waves propagating along an ambient magnetic field in a long-wave, weakly nonlinear scaling regime [5, 18]. There is a Hamiltonian form for the DNLS equation

$$\begin{bmatrix} u_t \\ \overline{u}_t \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \mathrm{grad} H$$

with the Hamiltonian

$$H = \int_{-\infty}^{\infty} -iu\overline{u}_x - \frac{1}{2}u^2\overline{u}^2 dx$$

and

$$\mathrm{grad} H = \begin{bmatrix} -i\overline{u}_x - u\overline{u}^2 \\ iu_x - u^2\overline{u} \end{bmatrix}$$

In terms of scaling properties, DNLS is invariant under the transformation

$$u \to u_\lambda = \lambda^{-1/2} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right).$$

In particular, it is L^2-critical in the sense that

$$\|u_\lambda\|_{L^2} = \|u\|_{L^2}.$$

It was proved by Hayashi and Ozawa [10] that solutions exist locally in time in the Sobolev space $H^1(\mathbb{R})$ and they can be extended for all time if the L^2-norm of the initial condition is small enough, namely if $\|u_0\|_{L^2} < \sqrt{2\pi}$. Recently, this upper bound has been improved to $\sqrt{4\pi}$ by Wu [24].

A central property of DNLS, discovered by Kaup and Newell [12], is that it is solvable through the inverse scattering method. In this pioneering work, the authors establish the main elements of the inverse scattering analysis. In particular, they find the Lax pair, analyze the linear spectral flow and derive the soliton solutions. A key observation is that the associated spectral problem is of second order in terms of the spectral parameter. This is in contrast with the ZS-AKNS system [1, 25] (the linear spectral problem associated to NLS) which is of first order in the spectral parameter. The study of the DNLS equation was the object of Lee's thesis [13] and his subsequent work [14]. The spectral analysis also provides an infinite number of conserved quantities [12].

The present dissertation is devoted to a rigorous analysis of the direct and inverse scattering map, with the goal of establishing global well-posedness for the DNLS equation and providing building blocks for proving the soliton resolution conjecture, which will appear in a subsequent paper [11]. In the rest of the introduction, we present the framework for the inverse scattering approach, building on seminal works by Beals-Coifman [2] and Deift and Zhou [7, 8, 26, 27]. We first review the Lax representation for DNLS, the spectral problem that defines the direct scattering map, and the Riemann-Hilbert problem (RHP) that defines the inverse scattering map (Section 1.1). We then use symmetry reduction to give a more precise and analytically tractable definition of the direct and inverse scattering maps (Section 1.2; see Definitions 1.2.1 and 1.2.2). The introduction ends with a summary of our results (Section 1.3). These include a Lipschitz continuity property of the direct and scattering maps in weighted Sobolev spaces (Theorems 1.3.7, 1.3.8). Due to the simple time-evolution of the scattering data (see (1.1.10)), this analysis provides a construction of a global solution of the DNLS equation with generic initial conditions in $H^{2,2}(\mathbb{R})$ (excluding spectral singularities, a notion that we will define precisely later). Finally, we mention that the analysis of the direct and inverse scattering maps and well-posedness of DNLS is also the subject of recent work by Pelinovsky, Saalmann and Shimabukoro [19, 21].

1.1 DNLS as an Integrable System

As the solution spaces of (1.0.1) with $\varepsilon = 1$ and $\varepsilon = -1$ are connected by the simple mapping $u(x,t) \mapsto u(-x,t)$, we will now fix $\varepsilon = -1$ and consider only this case for the rest of the dissertation. This choice of sign will induce the symmetry required for proving the existence of solution to the inverse scattering problem. So equation (1.0.1) becomes

$$iu_t + u_{xx} + i(|u|^2u)_x = 0 \tag{1.1.1}$$

A gauge transformation

$$q(x,t) = u(x,t)\exp\left(i\int_{-\infty}^{x}|u(y,t)|^2dy\right) \tag{1.1.2}$$

maps solutions of (1.1.1) into solutions of

$$iq_t + q_{xx} - iq^2\bar{q}_x + \frac{1}{2}|q|^4q = 0. \tag{1.1.3}$$

We will actually solve (1.1.3) by inverse scattering and use the inverse of the gauge transformation (1.1.2) to obtain the solution of (1.1.1). The advantage of this formulation manifests itself when analyzing the inverse scattering map through the Riemann-Hilbert problem (RHP), allowing us to write appropriately normalized, piecewise analytic solutions.

The integrable equations (1.1.1) and (1.1.3) each admit a Lax representation

$$L_t - A_x + [L, A] = 0$$

for suitable operators L and A. Equivalently, (1.1.1) and (1.1.3) are the compatibility conditions for the system of equations

$$\psi_x = L\psi, \quad \psi_t = A\psi. \tag{1.1.4}$$

The operators A and L for (1.1.1) and (1.1.3) and their equivalence through the gauge transformation are given in Section 8.1. The flow defined by (1.1.3) with $\varepsilon = -1$ is linearized by the scattering transform associated with the linear problem

$$\frac{d}{dx}\Psi = -i\zeta^2\sigma\Psi + \zeta Q(x)\Psi + P(x)\Psi, \tag{1.1.5}$$

where Ψ is a 2×2 matrix-valued function of x and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & q(x) \\ -\overline{q(x)} & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p_1(x) & 0 \\ 0 & p_2(x) \end{pmatrix}$$

with

$$p_1(x) = (i/2)|q(x)|^2, \quad p_2(x) = -(i/2)|q(x)|^2.$$

To describe the scattering transform, we recall the Jost solutions and their associated scattering data. First, observe that if $q = 0$, (1.1.5) admits the solutions $\Psi_0(x, \zeta) = \exp(-ix\zeta^2\sigma)$. These solutions are bounded for $\zeta \in \Sigma$ where

$$\Sigma = \left\{\zeta \in \mathbb{C} : \operatorname{Im}(\zeta^2) = 0\right\}.$$

From this observation, we are led to consider bounded solutions $\Psi^\pm(x, \zeta)$ of (1.1.5) with $\zeta \in \Sigma$, asymptotic as $x \to \pm\infty$ to $\Psi_0(x, \zeta)$. For such ζ, we denote by $\Psi^\pm(x, \zeta)$ the unique solutions of (1.1.5) with

$$\lim_{x\to\pm\infty} \Psi^\pm(x, \zeta)e^{ix\zeta^2\sigma} = \mathbf{1}.$$

Here and in what follows, $\mathbf{1}$ denotes the 2×2 identity matrix. The $\Psi^\pm$ are called *Jost solutions*. Analytically, it is more convenient to work with the *normalized Jost solutions*

$$m^\pm(x, \zeta) = \Psi^\pm(x, \zeta)e^{ix\zeta^2\sigma} \tag{1.1.6}$$

These functions solve the equation

$$\frac{d}{dx}m = -i\zeta^2 \operatorname{ad}\sigma(m) + \zeta Q(x)m + P(x)m, \quad \operatorname{ad}\sigma(A) \equiv \sigma A - A\sigma, \tag{1.1.7}$$

with asymptotic condition

$$\lim_{x\to\pm\infty} m^\pm(x, \zeta) = \mathbf{1}.$$

It follows from (1.1.5) that $\det\Psi(x)$ is constant for any solution Ψ, and that any two solutions Ψ_1 and Ψ_2 of (1.1.5) with nonvanishing determinant are related by $\Psi_1 = \Psi_2 A$ for a constant nonsingular matrix A. Hence, $\det m(x)$ is constant for any solution of (1.1.7), and any two solutions m_1 and m_2 of (1.1.7) with nonvanishing determinant are related by $m_1 = m_2 e^{-ix\zeta^2 \operatorname{ad}\sigma}A$ for a nonsingular constant matrix A.

In particular, for all x the Jost solutions $\Psi^\pm$ obey the relation

$$\Psi^+(x, \zeta) = \Psi^-(x, \zeta)T(\zeta), \quad T(\zeta) = \begin{pmatrix} a(\zeta) & \breve{b}(\zeta) \\ b(\zeta) & \breve{a}(\zeta) \end{pmatrix} \tag{1.1.8}$$

The matrix $T(\zeta)$ is called the *transition matrix*, and the functions $a(\zeta)$, $b(\zeta)$, $\breve{a}(\zeta)$, $\breve{b}(\zeta)$ are called the *scattering data*. By asymptotic condition we have

$$\det T(\zeta) = a(\zeta)\breve{a}(\zeta) - b(\zeta)\breve{b}(\zeta) = 1 \tag{1.1.9}$$

and $(a, \breve{a}, b, \breve{b})$ obey the symmetry relations (1.2.5) described below. Roughly and informally, the map $q \mapsto (a, \breve{a}, b, \breve{b})$ is the direct scattering map.

The crucial result of inverse scattering theory for (1.1.3) is the following: suppose that $a(\zeta,t)$, $\breve{a}(\zeta,t)$, $b(\zeta,t)$, $\breve{b}(\zeta,t)$ are the scattering data of a solution $q(x,t)$ of (1.1.3). It follows from the spatial asymptotics of $\Psi^{\pm}$ and the equations (1.1.4) that

$$\dot{a}(\zeta,t) = \dot{\breve{a}}(\zeta,t) = 0, \quad \dot{b}(\zeta,t) = 4i\zeta^4 b(\zeta,t), \quad \dot{\breve{b}}(\zeta,t) = -4i\zeta^4\breve{b}(\zeta,t) \tag{1.1.10}$$

(see [12, eq. (34)]). Hence, to solve the Cauchy problem (1.1.3), it suffices to compute the scattering data, solve the linear evolution equations (1.1.10), and invert the time-evolved scattering data to recover $q(x,t)$.

Let

$$\Omega^{\pm} = \{z \in \mathbb{C} : \pm \operatorname{Im}(z^2) > 0\}$$

and

$$\Sigma = \bigcup_{i=1}^{4} \Sigma_i$$

(see Figure 1.2 below). In Chapter 4 we will show that the function a extends analytically to Ω^-, while $\breve{a}$ extends analytically to Ω^+. It follows from Theorem A in [13] that any zeros of a are contained in a bounded set. The zero set respects the symmetries $\zeta \to \overline{\zeta}$ and $\zeta \to -\zeta$. In addition, zeros of $a(\zeta)$ and $\breve{a}(\zeta)$ cannot occur on the imaginary axis. This is a consequence of the symmetry conditions (1.2.5) and the fact that the determinant of the transition matrix $T(z)$ is equal to 1 (see Figure 1.1 below).

Zeros of a in Ω^- correspond to L^2 eigenfunctions and give rise to bright (exponentially decaying) solitons. Zeros of a on Σ are called spectral singularities and give rise to algebraic solitons. In the following, we exclude initial conditions q for which $a(\zeta)$ has zeros on Σ. Due to (1.1.10), this property persists for all time.

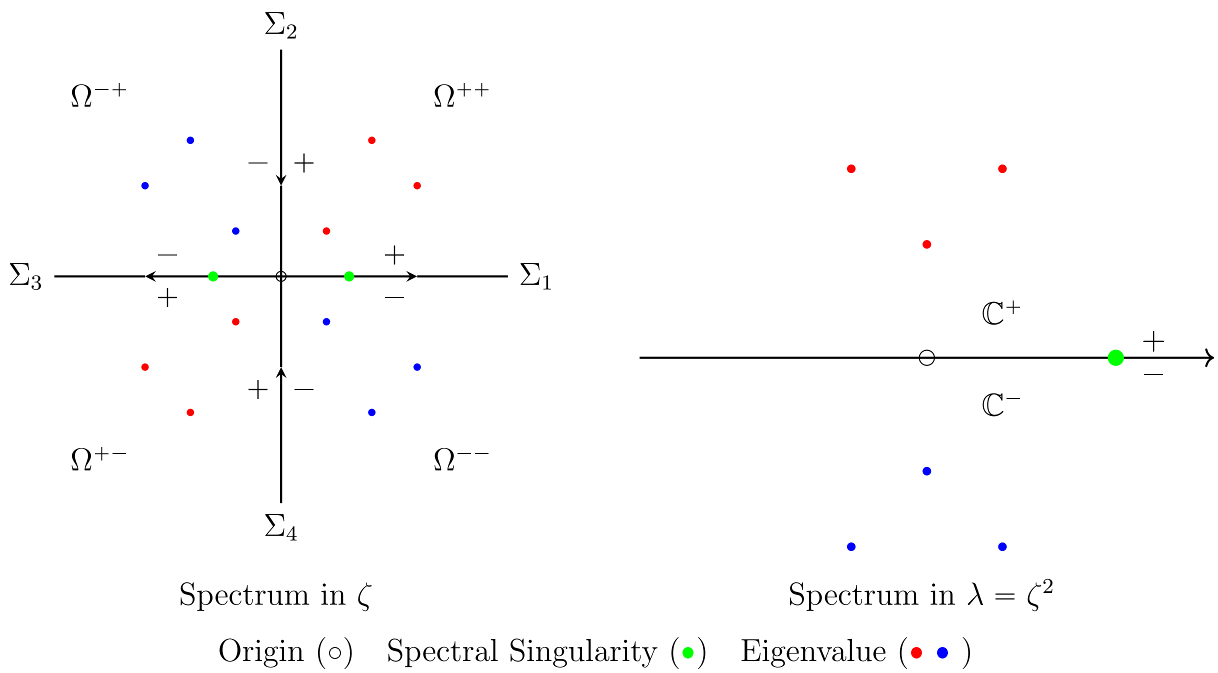
The class of initial data considered here (see Theorems 1.3.7 and 1.3.8) excludes initial data with spectral singularities. In Chapter 3, we will show that the set of initial conditions with this property is open and dense in the weighted Sobolev space used here. We discuss this further in Remark 1.3.1 of what follows.

We denote by $\psi_{i,j}$ $i,j = 1,2$ the components of Ψ. From (1.1.8) we deduce that

$$\breve{a}(\zeta) = \det \begin{pmatrix} \psi_{11}^{-}(x,\zeta) & \psi_{12}^{+}(x,\zeta) \\ \psi_{21}^{-}(x,\zeta) & \psi_{22}^{+}(x,\zeta) \end{pmatrix}. \tag{1.1.11}$$

$$a(\zeta) = \det \begin{pmatrix} \psi_{11}^{+}(x,\zeta) & \psi_{12}^{-}(x,\zeta) \\ \psi_{21}^{+}(x,\zeta) & \psi_{22}^{-}(x,\zeta) \end{pmatrix}. \tag{1.1.12}$$

Figure 1.1: Spectral Singularities in the ζ- and λ-planes



Suppose that $\breve{a}(\zeta_i) = 0$ for some $\zeta_i \in \Omega^{++}$, $i = 1, 2, ..., N$ then we have the linear dependence of the columns of (1.1.11)

$$\begin{bmatrix} \psi_{11}^-(x,\zeta_i) \\ \psi_{21}^-(x,\zeta_i) \end{bmatrix} = b_i \begin{bmatrix} \psi_{12}^+(x,\zeta_i) \\ \psi_{22}^+(x,\zeta_i) \end{bmatrix} \tag{1.1.13}$$

Definition 1.1.1. We call initial data q_0 generic if spectral singularities do not occur and zeros of $\breve{a}$ are of order one.

Using normalization (1.1.6) we get

$$\begin{bmatrix} m_{11}^-(x,\zeta_i) \\ m_{21}^-(x,\zeta_i) \end{bmatrix} = b_i \begin{bmatrix} m_{12}^+(x,\zeta_i) \\ m_{22}^+(x,\zeta_i) \end{bmatrix} e^{2ix\zeta_i^2} \tag{1.1.14}$$

We set the *norming constant* to be

$$c_i = \frac{b_i}{\breve{a}'(\zeta_i)} \tag{1.1.15}$$

In Section 1.2 we show $\breve{a}(\zeta) = \overline{a(\overline{\zeta})}$, which implies that $a(\overline{\zeta_i}) = 0$. Thus we also have

$$\begin{bmatrix} m_{12}^-(x,\overline{\zeta}_i) \\ m_{22}^-(x,\overline{\zeta}_i) \end{bmatrix} = \breve{b}_i \begin{bmatrix} m_{11}^+(x,\overline{\zeta}_i) \\ m_{21}^+(x,\overline{\zeta}_i) \end{bmatrix} e^{-2ix\overline{\zeta_i}^2} \tag{1.1.16}$$

and set

$$\breve{c}_i = \frac{\breve{b}_i}{a'(\overline{\zeta_i})}$$

To formulate the inverse scattering map, we recall that equation (1.1.7) admits *Beals-Coifman solutions* $M(x,z)$, piecewise analytic for $z \in \mathbb{C}\backslash\Sigma$, with the following spatial normalizations:

(i) The "right-hand" Beals-Coifman solutions are normalized so that

$$\lim_{x\to\infty} M(x,z) = \mathbf{1}$$

for $z \in \mathbb{C}\backslash\Sigma$, and $M(x,z)$ is bounded as $x \to -\infty$, for each such z, while

(ii) The "left-hand" Beals-Coifman solutions are normalized so that

$$\lim_{x\to-\infty} M(x,z) = \mathbf{1}$$

for $z \in \mathbb{C}\backslash\Sigma$, and $M(x,z)$ is bounded as $x \to +\infty$ for each such z.

Denoting by $M(x,z)$ either the left or right Beals-Coifman solution for $x \in \mathbb{R}$ and $z \in \mathbb{C}\backslash\Sigma$, there exist boundary values $M_\pm(x,\zeta)$ as $\pm \operatorname{Im}(z^2) \downarrow 0$ which obey a jump relation of the form

$$M_+(x,\zeta) = M_-(x,\zeta) e^{-ix\zeta^2 \operatorname{ad}\sigma} v(\zeta), \quad \zeta \in \Sigma. \tag{1.1.17}$$

The *jump matrix* $v(\zeta)$ is determined by the scattering data a, b, $\breve{a}$, and $\breve{b}$. For the right-hand Beals-Coifman solution,

$$v_r(\zeta) = \begin{pmatrix} 1 - b(\zeta)\breve{b}(\zeta)/a(\zeta)\breve{a}(\zeta) & \breve{b}(\zeta)/a(\zeta) \\ -b(\zeta)/\breve{a}(\zeta) & 1 \end{pmatrix}, \tag{1.1.18}$$

while for the left-hand Beals-Coifman solution,

$$v_\ell(\zeta) = \begin{pmatrix} 1 & \breve{b}(\zeta)/\breve{a}(\zeta) \\ -b(\zeta)/a(\zeta) & 1 - b(\zeta)\breve{b}(\zeta)/a(\zeta)\breve{a}(\zeta) \end{pmatrix}. \tag{1.1.19}$$

M has simple poles at each ζ_i and $\overline{\zeta_i}$. For the right-hand Beals-Coifman solution,

$$\operatorname{Res}_{\zeta=\zeta_i} M_r(x,\zeta) = \lim_{\zeta\to\zeta_i} M_r(x,\zeta) \begin{pmatrix} 0 & 0 \\ \dfrac{b_i}{\breve{a}'(\zeta_i)} e^{2i\zeta_i^2 x} & 0 \end{pmatrix}$$

$$\operatorname{Res}_{\zeta=\overline{\zeta}_i} M_r(x,\zeta) = \lim_{\zeta\to\overline{\zeta}_i} M_r(x,\zeta) \begin{pmatrix} 0 & \dfrac{\breve{b}_i}{a'(\overline{\zeta_i})} e^{-2i\overline{\zeta}_i^2 x} \\ 0 & 0 \end{pmatrix}$$

and for the left-hand Beals-Coifman solution,

$$\operatorname{Res}_{\zeta=\zeta_i} M_l(x,\zeta) = \lim_{\zeta\to\zeta_i} M_l(x,\zeta) \begin{pmatrix} 0 & \dfrac{1}{b_i \breve{a}'(\zeta_i)} e^{-2i\zeta_i^2 x} \\ 0 & 0 \end{pmatrix}$$

$$\operatorname{Res}_{\zeta=\zeta_i} M_l(x,\zeta) = \lim_{\zeta\to\overline{\zeta}_i} M_l(x,\zeta) \begin{pmatrix} 0 & 0 \\ \dfrac{1}{\breve{b}_i a'(\overline{\zeta}_i)} e^{2i\overline{\zeta}_i^2 x} & 0 \end{pmatrix}$$

These jump conditions, together with the large-z asymptotics

$$M(x,z) \sim \mathbf{1} + \frac{M_{-1}(x)}{z} + o\left(z^{-1}\right) \tag{1.1.20}$$

suffices to determine the Beals-Coifman solutions uniquely. Using this asymptotic expansion in (1.1.7), it is easy to see that the potential $Q(x)$ may be recovered from the formula

$$Q(x) = i \operatorname{ad}\sigma \left[M_{-1}(x)\right]$$

which implies that

$$q(x) = 2i \lim_{z\to\infty} z M_{12}(x,z).$$

We may take the Riemann-Hilbert problem (RHP) (1.1.17), (1.1.20) with given scattering data

$$\mathcal{D} = (a, \breve{a}, b, \breve{b}, \{\pm\zeta_i\}, \{\pm\overline{\zeta}_i\}, \{b_i\}, \{\breve{b}_i\})$$

as a starting point to recover q from the scattering data. In practice, the RHP with jump matrix (1.1.18) gives a stable reconstruction of $q(x)$ for x in half-lines $[c,\infty)$, while the RHP with jump matrix (1.1.19) gives a stable reconstruction of the potential $q(x)$ for x in half-lines $(-\infty, c]$. Roughly and informally, the map $\mathcal{D} \to q$ defined by these RHPs is the inverse scattering map.

1.2 Symmetry Reduction

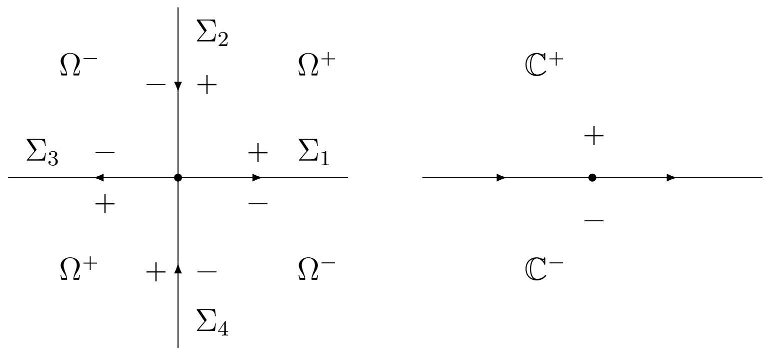
To give a precise formulation of the direct and inverse maps, we reduce by symmetry from scattering data on the oriented contour Σ to scattering data on the oriented contour $\mathbb{R}$. Both contours with orientation are shown in Figure 1.2. The contour Σ can be viewed as the boundary of the regions

$$\Omega^{\pm} = \left\{\zeta \in \mathbb{C} : \pm \operatorname{Im}\zeta^2 > 0\right\}.$$

The map $\zeta \mapsto \zeta^2$ maps Σ onto $\mathbb{R}$, Ω^+ onto $\mathbb{C}^+$, Ω^- onto $\mathbb{C}^-$, and induces the natural orientation on the contour $\mathbb{R}$.

Even functions on Σ determine and are determined by functions on $\mathbb{R}$. This observation allows us to reduce the scattering map defined by (1.1.5) to a map involving functions on the real line.

Figure 1.2: The Contours Σ and $\mathbb{R}$



The Contour Σ The Contour $\mathbb{R}$

We can reduce the spectral problem (1.1.7) from Σ to $\mathbb{R}$ by noting that the maps

$$m(x,\zeta) \mapsto \begin{pmatrix} m_{11}(x,-\zeta) & -m_{12}(x,-\zeta) \\ -m_{21}(x,-\zeta) & m_{22}(x,-\zeta) \end{pmatrix} \tag{1.2.1}$$

and

$$m(x,\zeta) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{m(x,\overline{\zeta})} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.2.2}$$

preserves the solution space of (1.1.7). It follows from the unicity of normalized Jost solutions $m^{\pm}$ and stability of the solution space under (1.2.1) that

$$\begin{aligned} m_{11}^{+}(x,-\zeta) &= m_{11}^{+}(x,\zeta), & m_{12}^{+}(x,-\zeta) &= -m_{12}^{+}(x,-\zeta) \\ m_{21}^{+}(x,-\zeta) &= -m_{21}^{+}(x,\zeta), & m_{22}^{+}(x,-\zeta) &= m_{22}^{+}(x,\zeta) \end{aligned} \tag{1.2.3}$$

and similarly for $m_{-}(x,\zeta)$. In an analogous way, it follows from unicity of $m^{\pm}$ and stability of the solution space under (1.2.2), that

$$m_{22}^{+}(x,\zeta) = \overline{m_{11}^{+}(x,\overline{\zeta})}, \quad m_{12}^{+}(x,\zeta) = -\overline{m_{21}^{+}(x,\overline{\zeta})} \tag{1.2.4}$$

and similarly for $+$ replaced by $-$.

Equations (1.1.8), (1.2.3), and (1.2.4) imply the symmetry relations

$$\breve{a}(\zeta) = \overline{a(\overline{\zeta})}, \quad \breve{b}(\zeta) = -\overline{b(\overline{\zeta})}, \quad a(-\zeta) = a(\zeta), \quad b(-\zeta) = -b(\zeta) \tag{1.2.5}$$

for the continuous scattering data. For the discrete scattering data, we have that

$$\breve{b}_i = -\overline{b}_i, \quad \breve{c}_i = -\overline{c}_i \tag{1.2.6}$$

Using these relations, we can now reduce the scattering problem (1.1.7) with $\zeta \in \Sigma$ to scattering problem for $\lambda = \zeta^2 \in \mathbb{R}$ to define the direct scattering map. Let

$$m^{\sharp}(x,\zeta) = \begin{pmatrix} m_{11}(x,\zeta) & \zeta^{-1} m_{12}(x,\zeta) \\ \zeta m_{21}(x,\zeta) & m_{22}(x,\zeta) \end{pmatrix}.$$

By (1.2.3), $m^\sharp$ is an even function of ζ. Then define

$$\lambda = \zeta^2, \quad n(x,\lambda) = m^\sharp(x,\zeta) \tag{1.2.7}$$

The map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \zeta^{-1}b \\ \zeta c & d \end{pmatrix}$$

is an automorphism of 2×2 matrices and commutes with differentiation in x. It follows that the functions $n^\pm$ obtained from $M_\pm$ by this map obey

$$\frac{dn^\pm}{dx} = -i\lambda \operatorname{ad}\sigma(n^\pm) + \begin{pmatrix} 0 & q \\ -\lambda\overline{q} & 0 \end{pmatrix} n^\pm + Pn^\pm \tag{1.2.8a}$$

$$\lim_{x\to\pm\infty} n^\pm(x,\lambda) = \mathbf{1} \tag{1.2.8b}$$

and are related by

$$\begin{aligned} n^+(x,\lambda) &= n^-(x,\lambda)e^{-i\lambda x \operatorname{ad}\sigma} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \\ \lambda\breve{\beta}(\lambda) & \breve{\alpha}(\lambda) \end{pmatrix} \\ &= n^-(x,\lambda)e^{-i\lambda x \operatorname{ad}\sigma} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \\ -\lambda\overline{\beta(\lambda)} & \overline{\alpha(\lambda)} \end{pmatrix} \end{aligned} \tag{1.2.9}$$

where for $\lambda = \zeta^2$

$$\alpha(\lambda) = a(\zeta), \quad \beta(\lambda) = \zeta^{-1}\breve{b}(\zeta) \tag{1.2.10}$$

and the relation

$$|\alpha(\lambda)|^2 + \lambda|\beta(\lambda)|^2 = 1$$

holds. We used the symmetry relations (1.2.5) to compute the form of the transition matrix in (1.2.9). Denoting by n one of n^+ or n^-, we also have

$$n_{22}(x,\lambda) = \overline{n_{11}(x,\lambda)}, \quad n_{12}(x,\lambda) = -\lambda^{-1}\overline{n_{21}(x,\lambda)} \tag{1.2.11}$$

so that one column or row of $n(x,\lambda)$ determines $n(x,\lambda)$ completely. Finally we define the continuous scattering data:

$$\rho(\lambda) = \frac{\beta(\lambda)}{\alpha(\lambda)}. \tag{1.2.12}$$

By setting $B_k = b_i/\zeta_i$ and $\lambda_i = \zeta_i^2$, we follow the change of variable $m(x,\zeta) \mapsto n(x,\lambda)$ to get

$$\begin{bmatrix} n_{11}^-(x,\lambda_i) \\ n_{21}^-(x,\lambda_i) \end{bmatrix} = B_i\lambda_i \begin{bmatrix} n_{12}^+(x,\lambda_i) \\ n_{22}^+(x,\lambda_i) \end{bmatrix} e^{2ix\lambda_i} \tag{1.2.13}$$

and

$$\begin{bmatrix} n_{12}^-(x,\overline{\lambda}_i) \\ n_{22}^-(x,\overline{\lambda}_i) \end{bmatrix} = -\overline{B}_i \begin{bmatrix} n_{11}^+(x,\overline{\lambda}_i) \\ n_{21}^+(x,\overline{\lambda}_i) \end{bmatrix} e^{-2ix\overline{\lambda}_i} \tag{1.2.14}$$

We also set

$$C_i = \frac{B_i}{\breve{\alpha}'(\lambda_i)}. \tag{1.2.15}$$

and define the discrete scattering data

$$\{\lambda_i, C_i\}_{i=1}^n \tag{1.2.16}$$

From (1.2.10) and (1.2.12) it is easy to see that

$$r(\zeta) = \zeta\rho(\zeta^2)$$

Definition 1.2.1. The map $q \mapsto \{\rho, \{C_i, \lambda_i\}_{i=1}^n\}$ defined by (1.2.8), (1.2.9), (1.2.12) and (1.2.15) is called the *direct scattering map* and denoted $\mathcal{R}$.

Similarly, one can reduce the RHP (1.1.17) with contour Σ and jump matrix (1.1.18) to a RHP with contour $\mathbb{R}$ using symmetry 1.2.3. It follows from the parity properties of the scattering data (see (1.2.5)) and the jump relation (1.1.17) that the mapping

$$M(x,\zeta) \mapsto \begin{pmatrix} M_{11}(x,-\zeta) & -M_{12}(x,-\zeta) \\ \\ -M_{21}(x,-\zeta) & M_{22}(x,-\zeta) \end{pmatrix}$$

preserves the solution space of the RHP. This fact, together with unicity of the solution (Lemma 5.2.9), implies that the diagonal entries of $M_\pm$ are even under the reflection $\zeta \mapsto -\zeta$, while the off-diagonal entries are odd under this reflection.

Let

$$M^\sharp(x,\zeta) = \begin{pmatrix} M_{11}(x,\zeta) & \zeta^{-1}M_{12}(x,\zeta) \\ \\ \zeta M_{21}(x,\zeta) & M_{22}(x,\zeta) \end{pmatrix}$$

and define

$$N(x,\zeta^2) = M^\sharp(x,\zeta).$$

One arrives at the RHP

$$N_+(x,\lambda) = N_-(x,\lambda)e^{-i\lambda x \operatorname{ad}\sigma}J(\lambda) \tag{1.2.17}$$

$$J(\lambda) = \begin{pmatrix} 1+\lambda|\rho(\lambda)|^2 & \rho(\lambda) \\ \\ \lambda\overline{\rho(\lambda)} & 1 \end{pmatrix}$$

corresponding to the RHP with jump matrix (1.1.18), where $\rho(\lambda) = \zeta^{-1}\breve{b}(\zeta)/a(\zeta)$. A similar computation for the RHP with jump matrix (1.1.19) leads to a RHP in the λ variable with $\rho(\lambda)$ replaced by $\tilde{\rho}(\lambda) = \zeta^{-1}\breve{b}(\zeta)/\breve{a}(\zeta)$.

N has simple poles at each λ_i and $\overline{\lambda_i}$. Suppose that $\lambda \in \{\lambda_i, \overline{\lambda_i}\}_{i=1}^n$

$$\operatorname{Res}_{z=\lambda} N(x,z) = \lim_{z\to\lambda} N(x,z)J_{Res}(\lambda)$$

with $J_{Res}(\lambda)$ given as follows:

$$J_{Res}(\lambda_i) = \begin{pmatrix} 0 & 0 \\ C_i \lambda_i e^{2i\lambda_i x} & 0 \end{pmatrix}, \qquad J_{Res}(\overline{\lambda_i}) = \begin{pmatrix} 0 & -\overline{C_i} e^{-2i\overline{\lambda_i} x} \\ 0 & 0 \end{pmatrix}.$$

with $C_i = 2c_i$.

However, this RHP is not properly normalized: a careful computation shows that the piecewise analytic function $N(x,z)$ on $\mathbb{C}\backslash\mathbb{R}$ has the asymptotics

$$N(x,z) \sim \begin{pmatrix} 1 & 0 \\ -(i/2)\overline{q(x)} & 1 \end{pmatrix} + \mathcal{O}\left(z^{-1}\right)$$

as $z \to \infty$. It is more effective to consider the row-wise RHP

$$\begin{aligned} \mathbf{N}_+(x,\lambda) &= \mathbf{N}_-(x,\lambda) e^{-i\lambda x \, \mathrm{ad}(\sigma)} J(\lambda) \\ \mathrm{Res}_{z=\lambda} N(x,z) &= \lim_{z\to\lambda} N(x,z) J_{Res}(\lambda) \\ \lim_{\lambda\to\infty} \mathbf{N}_\pm(x,\lambda) &= (1,0). \end{aligned} \tag{1.2.18}$$

where

$$\mathbf{N}(x,\lambda) = \left(N_{11}(x,\lambda), N_{12}(x,\lambda)\right).$$

A similar problem occurs in the study of KdV in a small dispersion limit [6].

One recovers q from the relation

$$q(x) = 2i \lim_{z\to\infty} z N_{12}(x,z) \tag{1.2.19}$$

where $z \to \infty$ in $\mathbb{C}\backslash\mathbb{R}$. As we show in §5.3 (see Proposition 5.3.1 and (5.3.13)), one can also compute the limit in (1.2.19) from the integral formula

$$q(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \nu_{11}(x,s)\rho(s) e^{-2isx} ds - \sum_{k=1}^{n} 2i \nu_{11}(x, \overline{\lambda_k}) \overline{C_k} e^{-2i\overline{\lambda_k} x} \tag{1.2.20}$$

where $\nu = (\nu_{11}, \nu_{12})$ solves the Beals-Coifman integral equation (5.1.2) associated with the RHP 5.1.1.

Definition 1.2.2. The mapping $\{\rho, \{C_k, \lambda_k\}_{k=1}^n\} \mapsto q$ determined by the RHP (1.2.18) and the asymptotic formula (1.2.19) is called the *inverse scattering map* and denoted $\mathcal{I}$.

From the evolution equations (1.1.10), we see that[1]

$$\dot{\rho}(\lambda,t) = -4i\lambda^2 \rho(\lambda,t) \tag{1.2.21a}$$

$$\dot{\lambda}_k = 0, \quad k = 1, ..., n \tag{1.2.21b}$$

[1]This equation differs by a minus sign from Kaup-Newell [12, eq. (34)] since Kaup and Newell take $\rho(\lambda) = \zeta^{-1} b(\zeta)/a(\zeta)$ whereas we take $\rho(\lambda) = \zeta^{-1}\breve{b}(\zeta)/a(\zeta)$.

$$\dot{C}_i = -4i\lambda_k^2 C_k \tag{1.2.21c}$$

In Lee's thesis, it is shown that for generic Schwartz class initial condition q_0 the formula

$$q(x,t) = \mathcal{I}\left[e^{-4i(\diamond)^2 t}\left(\mathcal{R}q_0\right)(\diamond)\right](x) \tag{1.2.22}$$

gives a classical solution to (1.1.3). Lee's class includes $\mathcal{S}(\mathbb{R}) \cap U$ where U is the open set in Theorem 1.3.2 below. We give a self-contained proof of the solution formula (1.3.4) in section 7 and Section 8.2.

1.3 Summary of Results

Given the solution formula (1.2.22) for Schwartz class data, the key to obtaining a globally defined solution map with good continuity properties is to prove precise continuity properties of the maps $\mathcal{R}$ and $\mathcal{I}$ in a natural function space in which the map $\rho \mapsto e^{-4it(\diamond)^2}\rho(\diamond)$ is continuous.

Let $H^{2,2}(\mathbb{R})$ be the completion of $\mathcal{S}(\mathbb{R})$ in the norm

$$\|q\|_{H^{2,2}(\mathbb{R})} = \left\|[1+(\,\cdot\,)^2]q(\,\cdot\,)\right\|_2 + \|q''\|_2 .$$

Definition 1.3.1. The set U consists, by definition, of those $q \in H^{2,2}(\mathbb{R})$ for which the scattering data $\breve{\alpha}(\lambda)$ is everywhere nonzero on $\mathbb{R}$ and has an analytic extension $\breve{\alpha}(z)$ to $\mathbb{C}^+$ with finitely many first order zeros. We can further write

$$U = \bigcup_{n=0}^{\infty} U_n$$

where $q \in U_n$ implies that the scattering data $\breve{\alpha}(\lambda)$ associated to q has n simple zeros in $\mathbb{C}^+$.

From the relation

$$|\alpha(\lambda)|^2 + \lambda|\beta(\lambda)|^2 = 1$$

we deduce that

$$(\alpha(\lambda)\breve{\alpha}(\lambda))^{-1} = |\breve{\alpha}(\lambda)|^{-2} = 1 + \lambda|\rho(\lambda)|^2$$

then we have the following observations:

$$1 + \lambda|\rho(\lambda)|^2 \leqslant \frac{1}{\inf|\breve{\alpha}(\lambda)|^2} < \infty \tag{1.3.1}$$

$$1 + \lambda|\rho(\lambda)|^2 \geqslant \frac{1}{\sup|\breve{\alpha}(\lambda)|^2} > c > 0 \tag{1.3.2}$$

The first observation follows from the fact that $\breve{\alpha}(\lambda)$ is everywhere nonzero on $\mathbb{R}$ while the second observation is the consequence of the boundedness of q in $H^{2,2}$ norm.

The main result of this dissertation is expressed in the following theorem:

Theorem 1.3.2. *There is a spectrally determined open subset U of $H^{2,2}(\mathbb{R})$ containing a neighborhood of* 0 *so that the solution map* (1.3.4) *for* (1.1.3)

$$\begin{aligned} H^{2,2}(\mathbb{R}) \times \mathbb{R} &\longrightarrow H^{2,2}(\mathbb{R}) \\ (q_0, t) &\mapsto q(\,\cdot\,, t) \end{aligned}$$

is continuous, and Lipschitz continuous in q_0 for each t.

Lee [13] proved that the set of such potentials q is open and dense in $\mathcal{S}(\mathbb{R})$. His proof is based on a general argument of Beals and Coifman [2] In Chapter 3 we give a more precise functional analytic argument inspired by analogous results in Schrödinger scattering theory (see the manuscript of Dyatlov and Zworski [9, Chapter 2, Theorem 2.2]. Following the work of Tovbis-Venakides [22], in [11] we construct potentials for which the spectral data can be fully calculated. In particular, we find sufficient conditions ensuring that the discrete spectrum be empty and the L^2 norm of the potential arbitrarily large at the same time.

Since the gauge transformation (1.1.2) defines a Lipschitz continuous self-mapping $\mathcal{G}$ of $H^{2,2}(\mathbb{R})$ onto itself with $\mathcal{G}(0) = 0$, we immediately obtain:

Corollary 1.3.3. *There is an open subset of $H^{2,2}(\mathbb{R})$ containing a neighborhood of* 0 *so that the solution map for* (1.1.1)

$$\begin{aligned} H^{2,2}(\mathbb{R}) \times \mathbb{R} &\longrightarrow H^{2,2}(\mathbb{R}) \\ (u_0, t) &\mapsto u(\,\cdot\,, t) \end{aligned}$$

is continuous, and Lipschitz continuous in u_0 for each t.

The technical core of this dissertation consists of the following two continuity results for the direct and inverse scattering maps. We begin with the following definitions:

Definition 1.3.4. By S we denote a subset of $H^{2,2}(\mathbb{R})$ where $\rho \in S$ satisfies the conditions given in (1.3.1)-(1.3.2).

Definition 1.3.5. By V we denote the disjoint union

$$V = \bigcup_{0}^{\infty} V_n$$

where $V_0 = S$ and for $n \geqslant 1$

$$V_n = S \times (\mathbb{C}_\times \times \mathbb{C}^+)^n$$

We call a subset of V_n bounded if there is a constant C with

$$\|\rho\|_{H^{2,2}} + \sup_i |C_i| + \sup_i |\lambda_i| \leqslant C$$

and a constant $c > 0$ so that $\inf_{1 \leqslant i \leqslant n} |\operatorname{Im} \lambda_i| \geqslant c$ for all data $(\rho, \{\lambda_i, C_i\}_{i=1}^n)$ in that subset.

Definition 1.3.6. From (1.2.21a)-(1.2.21c) we define flow

$$\Phi_t(\rho, \{\lambda_k, C_k\}) = \begin{cases} e^{-4i\lambda^2 t}\rho(\lambda) \\ \lambda_k, \quad k = 1, ..., n \\ e^{-4i\lambda_k^2 t} C_k, \quad k = 1, ..., n \end{cases} \tag{1.3.3}$$

Theorem 1.3.7. *There is a spectrally determined open and dense subset U of $H^{2,2}(\mathbb{R})$ containing a neighborhood of 0 so that for $n = 0, 1, ...$ the direct scattering map $\mathcal{R}$*

$$\mathcal{R} : U_n \longrightarrow S \times (\mathbb{C}_\times \times \mathbb{C}^+)^n$$
$$q \mapsto (\rho, \{C_i, \lambda_i\}_{i=1}^n)$$

is a Lipschitz continuous map from bounded subsets of U_n into bounded subsets of V_n. Moreover, $\mathcal{R}(U)$ is invariant under the flow $\Phi_t : V_n \to V_n$, and also contains an open neighborhood of 0 in $S \times (\mathbb{C}_\times \times \mathbb{C}^+)^n$.

Theorem 1.3.8. *For $n = 0, 1, ...$ the inverse scattering map $\mathcal{I}$*

$$\mathcal{I} : V_n \longrightarrow U_n$$
$$(\rho, \{C_i, \lambda_i\}_{i=1}^n) \mapsto q$$

is a Lipschitz continuous map from bounded subsets of V_n to bounded subsets of U_n with the property that $\mathcal{R} \circ \mathcal{I}$ is the identity map on the open set V_n and $\mathcal{I} \circ \mathcal{R}$ is the identity map on the open set U_n of Theorem 1.3.7.

We emphasize that results from Lee's thesis [13] already imply that the direct scattering map is continuous from $\mathcal{S}(\mathbb{R}) \cap U$ into $\mathcal{S}(\mathbb{R})$, and that the inverse map is continuous from $\mathcal{S}(\mathbb{R}) \cap V$ to $\mathcal{S}(\mathbb{R})$. Our contribution is to prove sharp continuity estimates between weighted Sobolev spaces.

The space $H^{2,2}(\mathbb{R})$ is invariant under the Fourier transform, and, for $\rho \in H^{2,2}(\mathbb{R})$, the map $t \mapsto e^{-4it\lambda^2}\rho(\lambda)$ describes a continuous curve in $H^{2,2}(\mathbb{R})$. Since the nonlinear maps $\mathcal{R}$ and $\mathcal{I}$ linearize respectively to the direct and inverse Fourier transform, the space $H^{2,2}(\mathbb{R})$ is well-suited to study the map (1.3.4).

Given Theorems 1.3.7 and 1.3.8, the proof of Theorem 1.3.2 is straightforward. The solution map $\mathcal{M}$ defined by

$$(q_0, t) \mapsto \mathcal{I}\left[\Phi_t \circ (\mathcal{R}q_0)(\diamond)\right](\,\cdot\,) \tag{1.3.4}$$

has the claimed continuity properties by Theorems 1.3.7 and 1.3.8. Thus, from Theorem 1.3.8 that $\mathcal{M}(q_0, 0) = q_0$. Moreover, the solution map gives a classical solution of (1.1.3) by Theorem 7.0.2. The result for $q \in H^{2,2}(\mathbb{R})$ now follows from Lipschitz continuity of $\mathcal{R}$ and $\mathcal{I}$.

Finally, we mention that in [16] and [11] J. Liu, P. Perry and C. Sulem established the large-time asymptotics of the solution $q(x, t)$ for (1.1.3) using the $\overline{\partial}$ version [4] of the Deift-Zhou nonlinear steepest descent method [7, 8] for a spectrally determined subset of the initial data in $H^{2,2}(\mathbb{R})$.

The dissertation is organized as follows. In Chapter 2, we present some useful tools of functional and complex analysis. They include Volterra integral equation that will be used in the analysis on the direct map, as well as basic properties of Cauchy projectors onto the lower and upper half complex planes that come into play in the analysis of the RHP. We also recall the Beals-Coifman formulation of the RHP which shows that the RHP is equivalent to the Beals-Coifman integral equation. Chapter 3 is devoted to Lipschitz continuity properties of the direct scattering map defined on U. We discuss the construction of Beals-Coifman solutions in Chapter 4. In Chapter 5, we study the RHP that defines the inverse scattering map and in Chapter 6, we prove Lipschitz continuity of the inverse scattering map defined on an open subset V of $H^{2,2}(\mathbb{R}) \times (\mathbb{C}_{\times})^n \times (\mathbb{C}^+)^n$. In Chapter 7, we give a self-contained proof that the formula (1.3.4) gives a classical solution of (1.1.3) if the initial data belong to $\mathcal{S}(\mathbb{R}) \cap U$. For sake of completeness, several technical calculations and proofs are presented in Chapter 8. In Section 8.1, we formulate the Lax pairs for (1.1.1) and (1.1.3) and show their equivalence through the gauge transformation (1.1.2). Finally, in Section 8.2, we supply some technical computations needed in Chapter 7.

We close this introduction with a table of notations used for various solutions of the linear systems defining the direct scattering map and the Riemann-Hilbert problem defining the inverse scattering map.

Notation	Summary
m, $m^\sharp$	Solutions to the linear system (1.1.7).
n	Solution to the linear system (1.2.8)
$\pm$ superscripts	Jost solutions obeying a boundary condition at $\pm\infty$.
Boldface $\mathbf{n}$	Renormalized first column of n (see (3.1.17)).
M	Solution of the Riemann-Hilbert problems 1.1.17 and 5.1.4.
N	Solution of the Riemann-Hilbert problem 1.2.17.
Boldface $\mathbf{N}$	The first row of N. The RHP for $\mathbf{N}$ is formulated precisely as Problem 5.1.1.
μ	The 2×2 matrix-valued solution for the Beals-Coifman integral equation corresponding to Problem 5.1.4.
ν	The row vector-valued solution to the Beals-Coifman integral equation for Problem 5.1.1.

The Beals-Coifman equation for $\nu = (\nu_{11}, \nu_{12})$ can be reduced for a scalar integral equation for ν_{11} which is studied in depth in Chapter 6.

We start this chapter with some functional analysis results, namely estimates for Volterra-type integral equations (Section 2.1.2) useful in the analysis of the direct scattering map. Since the spectral problem and the RHP are formulated for matrix-valued functions, we present in Section 2.2 some classical operations on matrices. We then turn to complex analysis tools that are central for the study of the inverse scattering map and recall some properties of Cauchy integrals and Cauchy operators in Section 2.3. We present a useful change-of-variables formula for the Cauchy projectors in Section 2.4. Finally, we discuss the key ideas leading to the reduction of the RHP to the so-called Beals-Coifman integral equation in Section 2.5.

2.1 Some Tools from Functional Analysis

2.1.1 Notations

If X and Y are Banach spaces, we denote by $\mathcal{B}(X,Y)$ the Banach space of bounded linear operators from X to Y. We write $\mathcal{B}(X)$ for $\mathcal{B}(X,X)$. If A is a Hilbert-Schmidt operator on a Hilbert space $\mathcal{H}$, we denote by $\|A\|_{\mathrm{HS}}$ the Hilbert-Schmidt norm of A. If I is an interval on the real line, $C^0(I,X)$ denotes the space of continuous functions on I taking values in X. It is equipped with the norm

$$\|f\|_{C^0(I,X)} = \sup_{x\in I} \|f(x)\|_X .$$

We write $C^0(I)$ if there is no possibility of confusion.

We denote by D the operator $-i(d/dx)$, by $\langle x\rangle$ the smooth function $(1+x^2)^{1/2}$. Note that $\|\langle x\rangle u'\|_2 \leqslant C\,\|u\|_{H^{2,2}}$. We normalize the Fourier transform as follows:

$$\begin{aligned}
\widehat{f}(\lambda) &:= (\mathcal{F}f)(\lambda) = \int_{-\infty}^{\infty} e^{-2i\lambda x} f(x)\,dx \\
\breve{g}(x) &:= \left(\mathcal{F}^{-1}g\right)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{2i\lambda x} g(\lambda)\,d\lambda.
\end{aligned}$$

2.1.2 Volterra Integral Equations

Lemma 2.1.1. *Suppose that X is a Banach space and consider the Volterra-type integral equation*

$$u(x) = f(x) + (Tu)(x) \tag{2.1.1}$$

on the space $C^0(\mathbb{R}^+,X)$, where $f\in C^0(\mathbb{R}^+,X)$ and T is an integral operator on $C^0(\mathbb{R}^+,X)$. Let $f^*(x)=\sup_{y\geqslant x}\|f(y)\|_X$, and assume there is a nonnegative function $h\in L^1(\mathbb{R}^+)$ such that

$$(Tf)^*(x) \leqslant \int_x^{\infty} h(t) f^*(t)\,dt. \tag{2.1.2}$$

Then for each f, equation (2.1.1) has a unique solution. Moreover, the resolvent $(I-T)^{-1}$ obeys the bound

$$\|(I-T)^{-1}\|_{\mathcal{B}(C^0(I,X))} \leqslant \exp\left(\int_0^\infty h(t)\,dy\right). \tag{2.1.3}$$

Proof. Estimate (2.1.3) is obtained by expanding $(I-T)^{-1}$ in powers of T and using (2.1.2) iteratively in the form

$$(T^n f)^*(x) \leqslant \frac{1}{n!}\left(\int_x^\infty h(y)\,dy\right)^n f^*(x).$$

to get a convergent series. □

Remark 2.1.2. There is an obvious analogue of Lemma 2.1.1 for the negative half-line.

2.2 Matrix Operations

For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we denote its Frobenius norm $|A| = \sqrt{a^2+b^2+c^2+d^2}$. Let

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define $\operatorname{ad}\sigma(A) = [\sigma, A] = \sigma A - A\sigma$. We have

$$\operatorname{ad}\sigma(A) = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}$$

and $\operatorname{ad}\sigma(A_{\text{diag}}) = 0$. If A is off-diagonal,

$$(\operatorname{ad}\sigma)^{-1}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$$

The exponential operator $e^{i\theta\operatorname{ad}\sigma}$ acts linearly on 2×2 matrices:

$$e^{i\theta\operatorname{ad}\sigma}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & e^{2i\theta}b \\ e^{-2i\theta}c & d \end{pmatrix}.$$

2.3 Cauchy Projections and Hilbert Transform

2.3.1 Contours

Figure 1.2 displays the oriented contours under consideration. Integration on the oriented contour Σ may be parameterized as follows:

$$\int_\Sigma f(\zeta)\,d\zeta = \int_0^\infty f(t)\,dt - i\int_0^\infty f(it)\,dt - \int_0^\infty f(-t)\,dt + i\int_0^\infty f(-it)\,dt \tag{2.3.1}$$

and integration with respect to arc length is parameterized as

$$\int_{\Sigma} f(\zeta)\ |d\zeta| = \int_0^{\infty} \left(f(t) + f(-t) + f(it) + f(-it)\right)\ dt.$$

Denote by $L^p(\Sigma)$ the space of measurable functions on Σ with norm

$$\|f\|_{L^p(\Sigma)} \equiv \left(\int_{\Sigma} |f(\zeta)|^p\ |d\zeta|\right)^{1/p}$$

finite. We say that $f \in L^1(\Sigma)$ is *even* if $f(-\zeta) = f(\zeta)$ and *odd* if $f(-\zeta) = -f(\zeta)$. It is easy to see that the integral of an even function is zero while the integral of an odd function is given by

$$\int_{\Sigma} f(\zeta)\, d\zeta = 2\int_0^{\infty} f(t)\, dt - 2i\int_0^{\infty} f(it)\, dt.$$

A short computation using (2.3.1) shows that for any function $f \in H^1(\Sigma)$,

$$\int_{\Sigma} f(\zeta)\, d\zeta = \int_{\mathbb{R}} \left(f(\sqrt{u}) - f(-\sqrt{u})\right) \frac{du}{2\sqrt{u}}. \tag{2.3.2}$$

where $\mathbb{R}$ is given the usual orientation.

2.3.2 Cauchy Projectors

We recall some basic facts about the Cauchy transform and the Cauchy projectors. See, for example, Deift-Zhou [8, Section 2] and Trogdon-Olver [23, Chapter 2] for details and references.

Let Λ denote an oriented contour in the complex plane which can be either Σ or $\mathbb{R}$ (as plotted in Figure 1.2). $\Omega^{\pm}$ denotes the region $\pm\operatorname{Im}(\zeta^2) > 0$ if $\Lambda = \Sigma$, and the region $\pm\operatorname{Im}(\lambda) > 0$ if $\Lambda = \mathbb{R}$.

For $f \in L^p(\Lambda)$, $p \in (1, \infty)$, the Cauchy integral

$$(Cf)(z) = \frac{1}{2\pi i}\int_{\Lambda} \frac{1}{s-z} f(s)\, ds$$

defines a function bounded and analytic in $\mathbb{C}\backslash\Lambda$. The nontangential limits

$$(C^{\pm}f)(\zeta) = \lim_{z\to\zeta,\ z\in\Omega^{\pm}} (Cf)(z)$$

exist for almost every $z \in \Lambda$, and the estimate

$$\left\|C^{\pm}f\right\|_p \leqslant c_p \|f\|_p$$

holds. We have the Plemelj-Sokhotski formula

$$C^{\pm}f = \pm\frac{1}{2}f - \frac{1}{2}Hf \tag{2.3.3}$$

where H is the Hilbert transform

$$(Hf)(z) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi i} \int_{\Lambda \cap \{|s-z|>\varepsilon\}} \frac{1}{z-s} f(s)\, ds$$

From this, it follows that $C^+ - C^- = I$ on $L^p(\Lambda)$. If $p=2$, $C^\pm$ are orthogonal projections. In particular, if the contour is $\mathbb{R}$, the Cauchy projectors $C^\pm$ are simply defined via the Fourier transform :

$$\left(C^+ f\right)(\lambda) = \frac{1}{\pi} \int_0^\infty e^{2i\lambda\zeta} \widehat{f}(\zeta)\, d\zeta \tag{2.3.4}$$

$$\left(C^- f\right)(\lambda) = \frac{1}{\pi} \int_0^{-\infty} e^{2i\lambda\zeta} \widehat{f}(\zeta)\, d\zeta \tag{2.3.5}$$

A short computation using (2.3.1) shows that the Hilbert transform H on Σ preserves the subspaces of odd and even functions on Σ. We will use this fact in the analysis of direct and inverse scattering map. We will also make use of the following commutator identities.

Lemma 2.3.1. *Suppose that m is a nonnegative integer and $f \in L^{2,m}(\mathbb{R})$. Then*

$$\zeta^m C^\pm [f](\zeta) = C^\pm [(\,\cdot\,)^m f(\,\cdot\,)] - \sum_{j=0}^{m-1} \zeta^{m-1-j} \frac{f_j}{2\pi i}, \quad f_j \equiv \int_\Sigma s^j f(s)\, ds. \tag{2.3.6}$$

Proof. The case $m=1$ follows from (2.3.3) and the commutator identity

$$[z, H]f = \frac{1}{\pi i} \int_\Sigma f.$$

The general formula is derived by induction. □

2.4 Change of Variables in Cauchy Projectors

The following change of variables formula for the Cauchy transform appears in Lee [13, §8]. We reproduce it here for the reader's convenience.

Let $\Sigma = \{\zeta \in \mathbb{C} : \Im(\zeta^2) = 0\}$ with the orientation shown in Figure 1.2. The mapping $\zeta \mapsto \zeta^2$ maps Σ onto $\mathbb{R}$ and induces the usual orientation. Let C_Σ and $C_\mathbb{R}$ be the respective Cauchy integrals for Σ and $\mathbb{R}$. For $u \in \mathbb{R}$ we denote by $\sqrt{u}$ the principal branch of the square root function, so that, referring to Figure 1.2:

- $u \mapsto \sqrt{u}$ maps $\mathbb{R}$ onto $\Sigma_1 \cup \Sigma_2$, and
- $u \mapsto -\sqrt{u}$ maps $\mathbb{R}$ onto $\Sigma_3 \cup \Sigma_4$.

For $f \in H^1(\Sigma)$, define

$$g(u) = \frac{1}{2}\left(f\left(\sqrt{u}\right) + f(-\sqrt{u})\right),$$

$$h(u) = \frac{1}{2\sqrt{u}}\left(f(\sqrt{u}) - f(-\sqrt{u})\right).$$

Lemma 2.4.1. *Let $f \in H^1(\Sigma)$ and $z \in \mathbb{C}\backslash\Sigma$. The identities*

$$\left(C_\Sigma f\right)(z) = \left(C_\mathbb{R} g\right)(z^2) + z\left(C_\mathbb{R} h\right)(z^2) \tag{2.4.1}$$

hold. Moreover, for any $\zeta \in \Sigma$,

$$\left(C_\Sigma^\pm f\right)(\zeta) = \left(C_\mathbb{R}^\pm g\right)(\zeta^2) + \zeta\left(C_\mathbb{R}^\pm h\right)(\zeta^2) \tag{2.4.2}$$

Proof. Using (2.3.2), we compute

$$\begin{aligned}\int_\Sigma \frac{f(s)}{s-z}\frac{ds}{2\pi i} &= \frac{1}{2\pi i}\int_\mathbb{R}\left(\frac{f(\sqrt{u})}{\sqrt{u}-z} - \frac{f(-\sqrt{u})}{-\sqrt{u}-z}\right)\frac{du}{2\sqrt{u}} \\ &= -\frac{1}{2}\int_\mathbb{R}\frac{f(\sqrt{u})+f(-\sqrt{u})}{z^2-u}\frac{du}{2\pi i} \\ &\quad - z\left(\frac{1}{2}\int_\mathbb{R}\frac{f(\sqrt{u})-f(-\sqrt{u})}{\sqrt{u}}\frac{1}{z^2-u}\frac{du}{2\pi i}\right).\end{aligned}$$

This gives the formula for $C_\Sigma f$. Observe that the quadratic mapping $z \mapsto z^2$ takes the regions $\Omega^\pm$ to the half-planes $\mathbb{C}^\pm$, and paths approaching Σ non-tangentially from Ω^+ (resp. Ω^-) are mapped to paths approaching $\mathbb{R}$ non-tangentially from $\mathbb{C}^+$ (resp. $\mathbb{C}^-$). Formula (2.4.2) is now an immediate consequence of (2.4.1). □

Remark 2.4.2. From Lemma 2.4.1, we easily deduce that if f is an odd function on Σ and $h(u) = f(\sqrt{u})/\sqrt{u}$ then

$$\left(C_\Sigma^\pm f\right)(\zeta) = \zeta\left(C_\mathbb{R}^\pm h\right)(\zeta^2). \tag{2.4.3}$$

On the other hand, if f is an even function on Σ and $g(u) = f(\sqrt{u})$, then

$$\left(C_\Sigma^\pm f\right)(\zeta) = \left(C_\mathbb{R}^\pm g\right)(\zeta^2). \tag{2.4.4}$$

2.5 Riemann-Hilbert Problem and Beals-Coifman Integral Equation

We recall briefly the Beals-Coifman [2] approach to RHPs: see, for example, [8, Section 2] for a detailed exposition and further references. Let Λ be an oriented contour (for our purpose, a finite union of oriented lines) that divides $\mathbb{C}\backslash\Lambda$ into disjoint open sets Ω^+ and Ω^-. Suppose given a 2×2 measurable, matrix-valued function v on Λ with $v, v^{-1} \in L^\infty(\Lambda)$. Formally, the normalized RHP (Λ, v) is stated as follows:

Problem 2.5.1. Find a piecewise analytic function $M(z)$ on $\mathbb{C}\backslash\Lambda$ so that

- $M(z) \to \mathbf{1}$ as $|z| \to \infty$, and
- the boundary values $M_\pm(\zeta)$ obey the jump relation $M_+(\zeta) = M_-(\zeta)v(\zeta)$.

To formulate this notion more rigorously, we say that a pair of measurable functions (f_+, f_-) on Λ belong to $\partial C_\Lambda(L^p)$ if there is a function $h \in L^p(\Lambda)$ with the property that $f_\pm = C_\Lambda^\pm h$. In this case, $f_\pm$ are boundary values of the piecewise analytic function

$$F(z) = \frac{1}{2\pi i}\int_\Lambda \frac{1}{s-z}h(s)\,ds.$$

Here $p \in (1, \infty)$; in the sequel, we will be concerned exclusively with the case $p = 2$. We now reformulate the normalized RHP (Λ, v) as follows:

Problem 2.5.2. Find a pair of matrix-valued functions (M_+, M_-) with

- $M_\pm - \mathbf{1} \in \partial C_\Lambda(L^p)$, and
- $M_+(\zeta) = M_-(\zeta)v(\zeta)$ for a.e. $\zeta \in \Lambda$

Given a solution of the RHP (Λ, v_x), we can then recover the piecewise analytic function $M(z)$ through the Cauchy transform of the function h with $M_\pm - \mathbf{1} = C_\Lambda^\pm h$:

$$M(z) = \mathbf{1} + \int_\Lambda \frac{h(s)}{s - z} \frac{ds}{2\pi i}.$$

Note that

$$h = C^+h - C^-h = M_+ - M_-.$$

To derive the Beals-Coifman integral equation, we assume that the jump matrix $v(\zeta)$ admits a matrix factorization of the form

$$v(\zeta) = (\mathbf{1} - w^-(\zeta))^{-1}(\mathbf{1} + w^+(\zeta))$$

for weight functions $w^\pm \in L^\infty(\Lambda) \cap L^p(\Lambda)$. If we set

$$\mu(\zeta) = M_+(\zeta)(\mathbf{1} + w^+(\zeta))^{-1} = M_-(\zeta)(\mathbf{1} - w^-(\zeta))^{-1}$$

assuming that $M_\pm$ solve the RHP, it follows that the additive jump $M_+ - M_-$ is given by

$$M_+(\zeta) - M_-(\zeta) = \mu(\zeta)(w^+(\zeta) + w^-(\zeta))$$

so that the piecewise analytic function $M(z)$ is given by

$$M(z) = \mathbf{1} + \int_\Lambda \frac{1}{s - z}\left[\mu(s)(w^+(s) + w^-(s))\right] \frac{ds}{2\pi i}.$$

Taking boundary values from Ω^+, we find that

$$M_+(\zeta) = \mu(\zeta)(\mathbf{1} + w^+(\zeta)) = \mathbf{1} + C_\Lambda^+\left[\mu(\,\cdot\,)(w^+(\,\cdot\,) + w^-(\,\cdot\,)\right](\zeta).$$

Using $I = C_\Lambda^+ - C_\Lambda^-$, we conclude that μ obeys the *Beals-Coifman integral equation*

$$\mu = \mathbf{1} + \mathcal{C}_w\mu \tag{2.5.1}$$

where, for any 2×2 matrix-valued function $h \in L^p(\Lambda)$,

$$\mathcal{C}_w(h) = C_\Lambda^+(hw^-) + C_\Lambda^-(hw^+). \tag{2.5.2}$$

In (2.5.2), the operators $C_\Lambda^\pm$ act componentwise on matrix-valued functions. Note that $\mathcal{C}_w$ is a bounded operator from $L^p(\Lambda)$ to itself for any $p \in (1, \infty)$ since $w^\pm \in L^\infty(\Lambda) \cap L^p(\Lambda)$ and $C_\Lambda^\pm$ are bounded operators on $L^p(\Lambda)$. Also note that $C_w 1 \in L^p(\Lambda)$. An important result of the theory is the following (see for example [8, Proposition 2.6]).

Proposition 2.5.3. *The operator $(I - \mathcal{C}_w)$ has trivial kernel as an operator on $L^p(\Lambda)$ if and only if there exists a unique solution for the RHP (Λ, v) on L^p .*

In applications, $(I - \mathcal{C}_w)$ will be a Fredholm operator on $L^p(\Lambda)$.

Chapter 3 The Direct Scattering Map

This section is devoted to the Lipschitz continuity of the scattering data $(\rho, \{\lambda_k, C_k\}_{k=1}^N)$ with respect to the potential q. The Lipschitz continuity of the reflection coefficient ρ in terms of q is given in Proposition 3.1.1. The Lipschitz continuity of the discrete scattering data in terms of q presented in Section 3.4. For this purpose, we make the notion of generic potential precise and prove in Section 3.3 that the set of potentials supporting at most finitely many solitons and having no spectral singularities is open and dense in $H^{2,2}(\mathbb{R})$. We also prove that the coefficients α and $\breve{\alpha}$ are analytic in the lower (resp. upper) complex plane, and the location of their zeros in a compact set of $\mathbb{C}$, (Propositions 3.2.4 and 3.2.5).

3.1 Lipschitz Continuity of the Continuous Scattering Data

To study the Jost solutions it is convenient to set

$$m^{\pm}(x,\zeta) = \Psi^{\pm}(x,\zeta)e^{ix\zeta^2\sigma}, \lim_{x\to\pm\infty} m^{\pm}(x,\zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We recall from (1.2.1)-(1.2.2) that the off-diagonal components of m are odd functions of ζ and the on-diagonal components are even in ζ. Because of this symmetry, the change of variables

$$n^{\pm}(x,\zeta^2) = \begin{pmatrix} m_{11}(x,\zeta) & \zeta^{-1}m_{12}(x,\zeta) \\ \zeta m_{21}(x,\zeta) & m_{22}(x,\zeta) \end{pmatrix}$$

yields well-defined functions $n^{\pm}(x,\lambda)$ obeying the differential equation

$$\frac{d}{dx}n^{\pm} = -i\lambda\,\mathrm{ad}(\sigma)n^{\pm} + \left(\begin{pmatrix} 0 & q(x) \\ -\lambda\overline{q(x)} & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} |q(x)|^2 & 0 \\ 0 & -|q(x)|^2 \end{pmatrix}\right)n^{\pm}, \tag{3.1.1}$$

the respective asymptotic conditions

$$\lim_{x\to\pm\infty} n^{\pm}(x,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.1.2}$$

and the relation

$$n^{+}(x,\lambda) = n^{-}(x,\lambda)e^{-i\lambda x\,\mathrm{ad}(\sigma)}T(\lambda) \tag{3.1.3}$$

where $T(\lambda)$ is the transition matrix

$$T(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \lambda\breve{\beta}(\lambda) & \breve{\alpha}(\lambda) \end{pmatrix} \tag{3.1.4}$$

and $\alpha(\zeta^2) = a(\zeta)$ and $\beta(\zeta^2) = \zeta^{-1}b(\zeta)$ are well-defined functions of $\lambda = \zeta^2$ owing to the symmetries (1.2.5). We also recall from (1.2.9) that

$$\breve{\alpha}(\lambda) = \overline{\alpha(\lambda)}, \;\; \breve{\beta}(\lambda) = -\overline{\beta(\lambda)}.$$

Using the relation (3.1.3) and the asymptotic condition (3.1.2) we can write α and β in terms of Jost solutions:

$$\alpha(\lambda) = n_{11}^{+}(0,\lambda)\overline{n_{11}^{-}(0,\lambda)} + \lambda^{-1}\overline{n_{21}^{-}(0,\lambda)}n_{21}^{+}(0,\lambda), \tag{3.1.5}$$

$$\beta(\lambda) = \frac{1}{\lambda}\left(-\overline{n_{11}^{-}(0,\lambda)n_{21}^{+}(0,\lambda)} + \overline{n_{11}^{+}(0,\lambda)n_{21}^{-}(0,\lambda)}\right). \tag{3.1.6}$$

and this reduces the analysis of α and β to the study of the normalized Jost functions $n^{\pm}$.

Recall that $\rho(\lambda) = \beta(\lambda)/\alpha(\lambda)$. By showing the map $q \mapsto (\alpha - 1, \beta)$ is Lipschitz continuous and using the quotient rule we will prove:

Proposition 3.1.1. *The map $q \to \rho$ is Lipschitz continuous from the open subset*

$$U_n = \left\{ q \in H^{2,2}(\mathbb{R}) : \breve{\alpha} \text{ has } n \text{ simple zeros in } \mathbb{C}^{+} \text{ and } \inf_{\lambda \in \mathbb{R}} |\breve{\alpha}(\lambda)| > 0 \right\}$$

of $H^{2,2}(\mathbb{R})$ into $H^{2,2}(\mathbb{R})$.

To prove Proposition 3.1.1, we need estimates on the solutions of (3.1.9) and their derivatives in λ as $L^2(\mathbb{R})$-valued functions of x. We have from (3.1.1) that $n_{11}^{\pm}$ and $n_{12}^{\pm}$ obey the integral equations

$$n_{11}^{\pm}(x,\lambda) = 1 - \int_x^{\pm\infty} q(y)n_{21}^{+}(y)\,dy - \int_x^{\pm\infty} \frac{i}{2}|q(y)|^2 n_{11}^{\pm}(y,\lambda)\,dy \tag{3.1.7}$$

$$n_{21}^{\pm}(x,\lambda) = \int_x^{\pm\infty} e^{2i\lambda(x-y)}\lambda\overline{q(y)}n_{11}^{\pm}(y,\lambda)\,dy + \frac{i}{2}\int_x^{\pm\infty} e^{2i\lambda(x-y)}|q(y)|^2 n_{21}^{\pm}(y,\lambda)\,dy. \tag{3.1.8}$$

For λ in a bounded interval I_0, we can study the equations (3.1.7)-(3.1.8) directly. In subsection 3.1.1, we will prove:

Proposition 3.1.2. *Let I_0 be a bounded interval in $\mathbb{R}$. The maps*

$$q \to n_{11}^{\pm}(0,\lambda), \quad q \to n_{21}^{\pm}(0,\lambda)/\lambda,$$

defined for $q \in U_n$, are Lipschitz continuous maps from $H^{2,2}(\mathbb{R})$ into $H^2(I_0)$.

To obtain uniform estimates for large λ, we begin with some simple algebraic manipulations on the solutions of (1.2.8). Define $e_\lambda(x) = e^{-2i\lambda x}$ and use the identity

$$(-2i\lambda)^{-1}(d/dx)e_\lambda(x) = e_\lambda(x)$$

and integrating by parts in (3.1.9b), we may remove the factor of λ at the expense of taking derivatives of q. Inserting (1.2.8a) to evaluate the derivative of $n_{11}^{\pm}$ that occurs in the computation,we observe some cancellations and obtain that

$$n_{11}^{\pm}(x,\lambda) = 1 + \frac{i}{2}\int_x^{\pm\infty} q(y)\int_y^{\pm\infty} e_\lambda(z-y)q^{\sharp}(z)n_{11}^{\pm}(z,\lambda)\,dz\,dy \tag{3.1.9a}$$

$$n_{21}^{\pm}(x,\lambda) = -\frac{i}{2}\overline{q(x)}n_{11}^{\pm}(x,\lambda) - \frac{i}{2}\int_x^{\pm\infty} e_\lambda(y-x)q^{\sharp}(y)n_{11}^{\pm}(y,\lambda)\,dy \tag{3.1.9b}$$

where

$$q^\sharp(x) = \overline{q'(x)} + \overline{q(x)}p_1(x) = \overline{q'(x)} + \frac{i}{2}|q(x)|^2\overline{q(x)}. \tag{3.1.10}$$

Note that $n_{21}^\pm$ does not appear in the equation for $n_{11}^\pm$. We first solve the integral equation (3.1.9a) for $n_{11}^\pm$, and then use its solution to compute $n_{21}^\pm$.

It is helpful to extract the leading order behavior of $n_{11}^\pm$ and $n_{21}^\pm$ for large λ by setting

$$\eta_{11}^\pm(x,\lambda) = n_{11}^\pm(x,\lambda) - 1, \quad \eta_{21}^\pm(x,\lambda) = n_{21}^\pm(x,\lambda) + \frac{i}{2}\overline{q(x)}. \tag{3.1.11}$$

From (3.1.9) and (3.1.11), we conclude that

$$\eta_{11}^\pm(x,\lambda) = F_\pm(x,\lambda) + \left(T_\pm \eta_{11}^\pm\right)(x,\lambda) \tag{3.1.12a}$$

$$\eta_{21}^\pm(x,\lambda) = G_\pm(x,\lambda) - \frac{i}{2}\overline{q(x)}\eta_{11}^\pm - \frac{i}{2}\int_x^{\pm\infty} e_\lambda(y-x)q^\sharp(y)\eta_{11}^\pm(y,\lambda)\,dy \tag{3.1.12b}$$

where

$$F_\pm(x,\lambda) = -\int_x^{\pm\infty} q(y)G_\pm(y,\lambda)\,dy, \tag{3.1.13a}$$

$$G_\pm(x,\lambda) = -\frac{i}{2}\int_x^{\pm\infty} e_\lambda(y-x)q^\sharp(y)\,dy \tag{3.1.13b}$$

$$\left(T_\pm f\right)(x,\lambda) = \frac{i}{2}\int_x^{\pm\infty} q(y)\int_y^{\pm\infty} e_\lambda(z-y)q^\sharp(z)f(z)\,dz \tag{3.1.13c}$$

In terms of the solutions $\eta_{11}^\pm$ and $\eta_{21}^\pm$, the functions $\alpha(\lambda)$ and $\beta(\lambda)$ defined in (3.1.5) and (3.1.6) are expressed as

$$\alpha(\lambda) - 1 = \alpha_1(\lambda) + \frac{1}{\lambda}\alpha_2(\lambda) \tag{3.1.14}$$

$$\lambda\beta(\lambda) = \beta_1(\lambda) + \beta_2(\lambda) \tag{3.1.15}$$

where

$$\alpha_1(\lambda) = \eta_{11}^+(0,\lambda) + \overline{\eta_{11}^-(0,\lambda)} + \eta_{11}^+(0,\lambda)\overline{\eta_{11}^-(0,\lambda)}$$

$$\alpha_2(\lambda) = \frac{|q(0)|^2}{4} + \frac{i}{2}q(0)\eta_{21}^+(0,\lambda) - \frac{i}{2}\overline{q(0)\eta_{21}^-(0,\lambda)} + \eta_{21}^+(0,\lambda)\overline{\eta_{21}^-(0,\lambda)}$$

and

$$\beta_1(\lambda) = \left(-\overline{\eta_{21}^+(0,\lambda)} + \frac{i}{2}q(0)\overline{\eta_{11}^+(0,\lambda)}\right) + \left(\overline{\eta_{21}^-(0,\lambda)} - \frac{i}{2}q(0)\overline{\eta_{11}^-(0,\lambda)}\right)$$

$$\beta_2(\lambda) = -\overline{\eta_{11}^-(0,\lambda)\eta_{21}^+(0,\lambda)} + \overline{\eta_{11}^+(0,\lambda)\eta_{21}^-(0,\lambda)}$$

Let $\eta^\pm = \left(\eta_{11}^\pm, \eta_{21}^\pm\right)$ and $I_\infty \equiv \{\lambda \in \mathbb{R} : |\lambda| > 1\}$. In Subsection 3.1.2, we will prove:

Proposition 3.1.3. *The maps*

$$q \to \eta^{\pm}(0,\lambda), \quad q \to \eta^{\pm}_{\lambda}(0,\lambda), \quad q \to \lambda^{-1}\eta^{\pm}_{\lambda\lambda}(0,\lambda),$$

defined for $q \in U_n$, are Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $L^2(I_\infty)$.

Proof of Proposition 3.1.1, given Propositions 3.1.2 and 3.1.3. Propositions 3.1.2, 3.1.3, and Sobolev embedding show that $q \mapsto \eta(0, \cdot\,)$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ into $H^1(\mathbb{R})$. It follows from this fact and (3.1.14) that $q \mapsto \alpha - 1$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $H^1(\mathbb{R})$. Since $\alpha \equiv 1$ if $q = 0$, there is an open neighborhood U of zero in $H^{2,2}(\mathbb{R})$ so that $\inf_{\lambda\in\mathbb{R}} |\alpha(q)(\lambda)| > 0$ for all $q \in U$. The map $q \to 1/\alpha - 1$ is locally Lipschitz continuous from U into $H^1(\mathbb{R})$.

It follows from Proposition 3.1.2 and (3.1.5)–(3.1.6) that the map $q \to \rho$ is Lipschitz from U to $H^2(I_0)$ for any bounded interval I_0. To show that $q \mapsto \rho$ is also Lipschitz from U to $H^{2,2}(I_\infty)$, we need to show that the maps $q \to \lambda^2\rho$ and $q \to \rho''$ are Lipschitz continuous on U.

To prove that $q \to \lambda^2\rho$ is Lipschitz continuous from U to $L^2(I_\infty)$, it suffices to show that $q \to \lambda^2\beta$ has the same Lipschitz continuity. From (3.1.15), we compute

$$\begin{aligned}\lambda^2\beta(\lambda) = \lambda\left(-\overline{\eta^{+}_{21}(0,\lambda)} + \frac{i}{2}q(0)\overline{\eta^{+}_{11}(0,\lambda)}\right) + \lambda\left(\overline{\eta^{-}_{21}(0,\lambda)} - \frac{i}{2}q(0)\overline{\eta^{-}_{11}(0,\lambda)}\right)\\ + \lambda\left(-\overline{\eta^{-}_{11}(0,\lambda)\eta^{+}_{21}(0,\lambda)} + \overline{\eta^{+}_{11}(0,\lambda)\eta^{-}_{21}(0,\lambda)}\right)\end{aligned}$$

To estimate the three right-hand terms, we rewrite (3.1.12b) as

$$\eta^{\pm}_{21} = -\frac{i}{2}\overline{q}\eta^{\pm}_{11} - \frac{i}{2}\int_x^{\pm\infty} e^{-2i\lambda(y-x)}q^\sharp(1+\eta^{\pm}_{11})\,dy.$$

Setting $x = 0$ and integrating by parts to remove the power of λ we obtain

$$\begin{aligned}\lambda\left(\eta^{\pm}_{21}(0,\lambda) + \frac{i}{2}\overline{q}(0)\eta^{\pm}_{11}(0,\lambda)\right) &= -\frac{i\lambda}{2}\int_0^{\pm\infty} e^{-2i\lambda y}q^\sharp(1+\eta^{\pm}_{11})\,dy\\ &= -\frac{1}{4}q^\sharp(0) - \frac{1}{4}q^\sharp(0)\eta^{\pm}_{11}(0,\lambda) + R^{\pm}(\lambda)\end{aligned}$$

where

$$\begin{aligned}R^{\pm}(\lambda) = &-\frac{1}{4}\int_0^{\pm\infty} e^{-2i\lambda y}(q^\sharp)'(1+\eta^{\pm}_{11})dy\\ &-\frac{1}{4}\int_0^{\pm\infty} e^{-2i\lambda y}q^\sharp\left[q\eta^{\pm}_{21} + \frac{i}{2}|q|^2\eta^{\pm}_{11}\right]dy.\end{aligned}$$

We can then compute

$$\begin{aligned}\lambda^2\beta(\lambda) = &-\overline{R^{+}(\lambda)} + \overline{R^{-}(\lambda)} - \overline{\eta^{-}_{11}(0,\lambda)R^{+}(\lambda)} + \overline{\eta^{+}_{11}(0,\lambda)R^{-}(\lambda)}\\ &+ \frac{1}{4}\overline{q}^\sharp\left[\overline{\eta^{+}_{11}(0,\lambda)} - \overline{\eta^{-}_{11}(0,\lambda)}\right]\end{aligned}$$

Since $q \mapsto \eta_{11}^{\pm}(0,\lambda)$ is Lipschitz from $H^{2,2}(\mathbb{R})$ to $C^0(I_\infty)$ by Sobolev embedding, it suffices to show that $q \mapsto R^{\pm}(\lambda)$ is Lipschitz from $H^{2,2}(\mathbb{R})$ to $L^2(I_\infty)$. This follows from the estimate

$$\begin{aligned}\left\|R^{\pm}\right\|_{L^2(I_\infty)} &\leqslant \left\|(q^\sharp)_x\right\|_2 \left(1 + \left\|\eta_{11}^{\pm}\right\|_{L^2(\mathbb{R}^\pm \times I_\infty)}\right) \\ &\quad + \left(\left\|q^\sharp q\right\|_2 + \left\|q^\sharp |q|^2\right\|_2\right)\left(1 + \left\|\eta_{11}^{\pm}\right\|_{L^2(\mathbb{R}^\pm \times I_\infty)} + \left\|\eta_{21}^{\pm}\right\|_{L^2(\mathbb{R}^\pm \times I_\infty)}\right).\end{aligned}$$

To prove that $q \to \rho''$ is Lipschitz continuous on U, we exploit the identity

$$\left(\frac{\beta}{\alpha}\right)'' = \frac{(\beta)''}{\alpha} - 2\frac{(\beta)'(\alpha)'}{(\alpha)^2} + \frac{\beta}{\alpha}\left(-\frac{\alpha''}{\alpha} + \frac{2(\alpha')^2}{\alpha^2}\right).$$

From this identity, it suffices to show that the maps

$$q \mapsto \beta''(\lambda), \quad q \mapsto \beta'(\lambda), \quad q \mapsto \lambda\beta'(\lambda), \quad q \mapsto \alpha'(\lambda), \quad q \mapsto \lambda^{-1}\alpha''(\lambda), \tag{3.1.16}$$

are Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $L^2(I_\infty)$, and that the map $q \mapsto \lambda^{-1}\alpha'(\lambda)$ is continuous from $H^{2,2}(\mathbb{R})$ to $C^0(I_\infty)$. This last fact will follow from Lipschitz continuity of the maps $q \mapsto \alpha'(\lambda)$ and $q \mapsto \lambda^{-1}\alpha''(\lambda)$ from $H^{2,2}(\mathbb{R})$ to $L^2(I_\infty)$ and Sobolev embedding. Lipschitz continuity of the maps (3.1.16) is easily deduced from (3.1.14), (3.1.15), and Proposition 3.1.3. □

3.1.1 Small-λ Estimates

In this subsection we prove Proposition 3.1.2. We give the proofs for n_{11}^{+} and n_{21}^{+} since the others are similar. We set

$$\mathbf{n} = (n_{11}^{+} - 1, \lambda^{-1} n_{21}^{+})^T \tag{3.1.17}$$

so that (3.1.7)-(3.1.8) become

$$\mathbf{n} = \mathbf{n}_0 + T_0\mathbf{n}, \quad \mathbf{n}_0 \equiv T_0\mathbf{e}_1 \tag{3.1.18}$$

where

$$(T_0 h)(x) = \int_x^\infty K_0(x,y,\lambda)h(y)\,dy \tag{3.1.19}$$

and

$$K_0(x,y,\lambda) = \begin{pmatrix} -p_1(y) & -q(y) \\ e_\lambda(y-x)\overline{q(y)} & -p_2(y) \end{pmatrix} \tag{3.1.20}$$

so that

$$\mathbf{n}_0 = \int_x^\infty \begin{pmatrix} -p_1(y) \\ e_\lambda(y-x)\overline{q(y)} \end{pmatrix} dy. \tag{3.1.21}$$

We will establish existence, uniqueness, and estimates on $\mathbf{n}$ by studying (3.1.18) as a Volterra integral equation. To study λ-derivatives of the solution, we will solve the integral equations

$$\mathbf{n}_\lambda = \mathbf{n}_1 + T_0(\mathbf{n}_\lambda), \qquad \mathbf{n}_1 \equiv (\mathbf{n}_0)_\lambda + (T_0)_\lambda\mathbf{n} \tag{3.1.22}$$

$$\mathbf{n}_{\lambda\lambda} = \mathbf{n}_2 + T_0(\mathbf{n}_{\lambda\lambda}), \qquad \mathbf{n}_2 \equiv (\mathbf{n}_0)_{\lambda\lambda} + (T_0)_{\lambda\lambda}\mathbf{n} + 2(T_0)_\lambda \mathbf{n}_\lambda \tag{3.1.23}$$

We will prove Proposition 3.1.2 in the following steps. Let I_0 denote a bounded interval of $\mathbb{R}$, which we'll finally set to $I = (-2, 2)$. We will write T_0 as $T_0(\lambda)$ or $T_0(\lambda, q)$ to emphasize its dependence on $\lambda \in I$ and $q \in H^{2,2}(\mathbb{R})$. First, we obtain basic estimates on $\mathbf{n}_0$ and its derivatives (Lemma 3.1.4) and obtain mapping properties of the operators T_0, $(T_0)_\lambda$, and $(T_0)_{\lambda\lambda}$ (Lemma 3.1.5). Second, we show that the family of operators $(I - T_0(\lambda))^{-1} - I$ indexed by $\lambda \in I_0$ induces bounded operators

$$\widehat{L_0} : C^0(\mathbb{R}^+, L^2(I_0)) \to C^0(\mathbb{R}^+, L^2(I_0)), \qquad \widehat{L_0} : L^2(\mathbb{R}^+ \times I_0) \to L^2(\mathbb{R}^+ \times I_0)$$

(Lemmas 3.1.6, 3.1.7 and Remark 3.1.8). Third, we solve (3.1.18) to prove that

$$\mathbf{n} \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$$

(Lemma 3.1.9). Fourth, we use this result to show that

$$\mathbf{n}_1 \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$$

and solve (3.1.22) to show that

$$\mathbf{n}_\lambda \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$$

(Lemma 3.1.10). Fifth, we use this result to show that $\mathbf{n}_2 \in C^0(\mathbb{R}^+, L^2(I_0))$ and solve (3.1.23) to prove that

$$\mathbf{n}_{\lambda\lambda} \in C^0(\mathbb{R}^+, L^2(I_0))$$

(Lemma 3.1.11). Combining these results, we conclude that $\mathbf{n}(0, \lambda) \in H^2(I_0)$. Lipschitz continuity of the map $q \mapsto \mathbf{n}(0, \lambda)$ follows from resolvent bounds established on $(I - T_0)^{-1}$ and the second resolvent formula.

In what follows, we define

$$\gamma_1(y) = 2|q(y)| + 2|p_1(y)|.$$

Note that $\|q\|_{H^{2,2}}$ bounds $\|\gamma_1\|_{L^1}$ and $\|\gamma_1\|_{L^{2,2}}$.

(1) **Estimates on $\mathbf{n}_0$ and T_0.** Let

$$\begin{aligned} g_1(x, y, \lambda) &= 2i(x - y)e_\lambda(y - x)\overline{q(y)}, \\ g_2(x, y, \lambda) &= -4(x - y)^2 e_\lambda(y - x)\overline{q(y)} \end{aligned}$$

Then

$$(\mathbf{n}_0)_\lambda = \begin{pmatrix} 0 \\ \displaystyle\int_x^\infty g_1(x, y, \lambda)\, dy \end{pmatrix}, \quad (\mathbf{n}_0)_{\lambda\lambda} = \begin{pmatrix} 0 \\ \displaystyle\int_x^\infty g_2(x, y, \lambda)\, dy \end{pmatrix} \tag{3.1.24}$$

while the integral kernels of $(T_0)_\lambda$ and $(T_0)_{\lambda\lambda}$ are

$$(K_0)_\lambda(x,y,\lambda) = \begin{pmatrix} 0 & 0 \\ g_1(x,y,\lambda) & 0 \end{pmatrix}, \tag{3.1.25}$$

$$(K_0)_{\lambda\lambda}(x,y,\lambda) = \begin{pmatrix} 0 & 0 \\ g_2(x,y,\lambda) & 0 \end{pmatrix}. \tag{3.1.26}$$

Lemma 3.1.4. *Let I_0 be a bounded interval. The following estimates hold.*

$$\|\mathbf{n}_0\|_{C^0(\mathbb{R}^+,L^2(I_0))} \leqslant \|p_1\|_{L^1} + \|q\|_{L^2}\,, \quad \|\mathbf{n}_0\|_{L^2(\mathbb{R}^+\times I_0)} \leqslant \|q\|_{L^{2,1/2}}\,. \tag{3.1.27}$$

$$\|(\mathbf{n}_0)_\lambda\|_{C^0(\mathbb{R}^+,L^2(I_0))} \leqslant \|q\|_{L^{2,1/2}}\,, \quad \|(\mathbf{n}_0)_\lambda\|_{L^2(\mathbb{R}^+\times I)} \lesssim \|q\|_{L^{2,1}} \tag{3.1.28}$$

$$\|(\mathbf{n}_0)_{\lambda\lambda}\|_{C^0(\mathbb{R}^+,L^2(I_0))} \leqslant \|q\|_{L^{2,1}}\,. \tag{3.1.29}$$

Proof. To prove (3.1.27), we note that the first component is independent of λ, bounded by $\|p_1\|_{L^1}$, and continuous. To bound the second component, let $\varphi \in C_0^\infty(I_0)$, compute

$$\int_I \varphi(\lambda) \int_x^\infty e_\lambda(y-x)\overline{q(y)}\,dy = \int_x^\infty \widehat{\varphi}(y-x)\overline{q(y)}\,dy$$

so that

$$\left\| \int_x^\infty e_{(\,\cdot\,)}(y-x)\overline{q(y)}\,dy \right\|_{L^2(I_0)} \leqslant \left(\int_x^\infty |q(y)|^2\,dy \right)^{1/2}.$$

The first estimate is immediate and the second follows by integration in x.

A similar argument shows that

$$\left\| \int_x^\infty g_1(x,y,\,\cdot\,)\,dy \right\|_{L^2(I_0)} \leqslant \left(\int_x^\infty y|q(y)|^2\,dy \right)^{1/2}$$

from which (3.1.28) follows.

Similarly,

$$\left\| \int_x^\infty g_2(x,y,\,\cdot\,)\,dy \right\|_{L^2(I_0)} \leqslant \left(\int_x^\infty y^2|q(y)|^2\,dy \right)^{1/2}.$$

□

The operator $(T_0)_\lambda$ induces linear mappings $L^2(\mathbb{R}^+\times I_0) \to L^2(\mathbb{R}^+\times I_0)$ and $L^2(\mathbb{R}^+\times I_0) \to C^0(\mathbb{R}^+, L^2(I_0))$ by the formula $g(x,\lambda) = (T_0)_\lambda(f(\,\cdot\,,\lambda))(x)$, and similarly for $(T_0)_{\lambda\lambda}$. We will need the following estimates on these induced maps.

Lemma 3.1.5. *Suppose that $q \in H^{2,2}(\mathbb{R})$. The following operator bounds hold uniformly in $q \in H^{2,2}(\mathbb{R})$, and the operators are Lipschitz functions of q.*

(i) $\|(T_0)_\lambda\|_{L^2(\mathbb{R}^+\times I_0)\to L^2(\mathbb{R}^+\times I_0)} \lesssim \|q\|_{L^{2,3/2}}$,

(ii) $\|(T_0)_\lambda\|_{L^2(\mathbb{R}^+\times I_0)\to C^0(\mathbb{R}^+,L^2(I_0))} \lesssim \|q\|_{L^{2,2}}$

(iii) $\|(T_0)_{\lambda\lambda}\|_{L^2(\mathbb{R}^+\times I_0)\to C^0(\mathbb{R}^+,L^2(I_0))} \lesssim \|q\|_{L^{2,2}}$

Proof. For an operator $T(\lambda)$ with integral kernel $k(x,y,\lambda)$ satisfying the estimate $\sup_{\lambda\in I}|k(x,y,\lambda)| \leqslant h(y)$ and satisfying $k(x,y,\lambda)=0$ if $x>y$, the $\mathcal{B}(L^2(\mathbb{R}^+\times I_0))$-norm is controlled by

$$\left(\int_0^\infty \int_x^\infty h(y)^2\,dy\,dx\right)^{1/2} = \left(\int_0^\infty y\,h(y)^2\,dy\right)^{1/2}$$

and the $\mathcal{B}(L^2(\mathbb{R}^+\times I_0), C^0(\mathbb{R}^+,L^2(I_0)))$-norm is controlled by

$$\sup_x \left(\int_0^\infty h(y)^2\,dy\right)^{1/2}.$$

The conclusions follow from this observation and the estimates

$$|g_1(x,y,\lambda)| \leqslant |y||q(y)|, \quad |g_2(x,y,\lambda)| \leqslant y^2|q(y)|$$

true for $x\leqslant y$. Since these operators are linear in q the Lipschitz continuity is immediate. $\square$

(2) **Resolvent estimates**. Our construction of the resolvent is based on the estimate (see Lemma 2.1.1 and (2.1.2))

$$(T_0 f)^*(x) \leqslant \int_x^\infty \gamma_1(y) f^*(y)\,dy. \tag{3.1.30}$$

which is an easy consequence of (3.1.20).

Lemma 3.1.6. *For each $\lambda\in\mathbb{R}$ and $q\in H^{2,2}(\mathbb{R})$, the operator $(I-T_0)^{-1}$ exists as a bounded operator from $C^0(\mathbb{R}^+)\otimes\mathbb{C}^2$ to itself. Moreover, $(I-T_0)^{-1}-I$ is an integral operator with continuous integral kernel $L_0(x,y,\lambda)$, $L_0(x,y,\lambda)=0$ for $x>y$. The integral kernel $L_0(x,y,\lambda)$ satisfies the estimate*

$$|L_0(x,y,\lambda)| \leqslant \exp\left(\|\gamma_1\|_{L^1}\right)\gamma_1(y) \tag{3.1.31}$$

Proof. Because T_0 is a Volterra operator, we deduce from Lemma 2.1.1 that $(I-T_0)^{-1}$ exists as a bounded operator on $C^0(\mathbb{R}^+)\otimes\mathbb{C}^2$. We can obtain rather precise estimates on the resolvent through the Volterra series. The integral kernel $K_0(x,y,\lambda)$ obeys the estimate $|K_0(x,y,\lambda)|\leqslant\gamma_1(y)$ where on the left, $|\cdot|$ denotes the operator norm on 2×2 matrices. The operator

$$L_0 \equiv (I-T_0)^{-1} - I$$

is an integral operator with integral kernel $L_0(x,y,\lambda)$ given by

$$L_0(x,y,\lambda) = \begin{cases} \sum_{n=1}^\infty K_n(x,y,\lambda), & x\leqslant y \\ 0, & x>y \end{cases}$$

where

$$K_n(x,y,\lambda) = \int_{x \leqslant y_1 \leqslant \cdots \leqslant y_{n-1}} K_0(x,y_1,\lambda) K_0(y_1,y_2,\lambda) \ldots K_0(y_{n-1},y,\lambda)\, dy_{n-1} \, \ldots \, dy_1$$

and the estimate

$$|K_n(x,y,\lambda)| \leqslant \frac{1}{(n-1)!} \left(\int_x^\infty \gamma_1(t) \right)^{n-1} \gamma_1(y)$$

holds. The estimate (3.1.31) follows. $\square$

Now suppose that $f \in C^0(\mathbb{R}^+, L^2(I_0))$ and let

$$g(x,\lambda) = \int_x^\infty L_0(x,y,\lambda) f(y,\lambda)\, dy. \tag{3.1.32}$$

Denote by $\widehat{L}_0$ the map $f \to g$. We will prove:

Lemma 3.1.7. *The estimates*

$$\left\| \widehat{L}_0 \right\|_{\mathcal{B}(C^0(\mathbb{R}^+, L^2(I_0)))} \leqslant e^{\|\gamma_1\|_{L^1}} \left\| \gamma_1 \right\|_{L^1}. \tag{3.1.33}$$

and

$$\left\| \widehat{L}_0 \right\|_{\mathcal{B}(L^2(\mathbb{R}^+ \times I_0))} \leqslant e^{\|\gamma\|_{L^1}} \left\| \gamma_1 \right\|_{L^{2,2}} \tag{3.1.34}$$

hold.

Proof. Suppose that $g \in C^0(\mathbb{R}^+, L^2(I_0))$. Then f belongs to $C^0(\mathbb{R}^+, L^2(I_0))$ since

$$|g(x,\lambda)| \leqslant e^{\|\gamma_1\|_{L^1}} \int_x^\infty \gamma_1(y) |f(y,\lambda)|\, dy$$

and we may conclude from Minkowski's integral equality that

$$\|g(x,\,\cdot\,)\|_{L^2(I_0)} \leqslant e^{\|\gamma_1\|_{L^1}} \int_x^\infty \gamma_1(y) \left\| f \right\|_{C^0(\mathbb{R}^+, L^2(I_0))}.$$

It follows that L induces a bounded mapping $\widehat{L}_0$ from $C^0(\mathbb{R}^+, L^2(I_0))$ to itself obeying the estimate (3.1.33).

Similarly, suppose that $f \in L^2(\mathbb{R}^+ \times I_0)$. Defining g as in (3.1.32), we estimate

$$|g(x,\lambda)| \leqslant \left(\int_x^\infty |L_0(x,y,\lambda)|^2 \, dy \right)^{1/2} \left(\int_x^\infty |g(y,\lambda)|^2 \, dy \right)^{1/2}$$

so that

$$\begin{aligned}
\|g\|^2_{L^2(\mathbb{R}^+ \times I_0)} &\leqslant \int_0^\infty \left(\sup_{\lambda \in I} \int_x^\infty |L_0(x,y,\lambda)|^2 \, dy \right) \left(\int_I \left(\int_x^\infty |g(y',\lambda)|^2 \, dy' \right) d\lambda \right) dx \\
&\leqslant \left(\int_0^\infty \int_x^\infty \gamma_1(y) e^{\|\gamma_1\|_{L^1}} \, dy \, dx \right) \|g\|_{L^2(\mathbb{R}^+ \times I_0)} \\
&\leqslant e^{\|\gamma_1\|_{L^1}} \left\| \gamma_1 \right\|_{L^{2,2}} \|g\|_{L^2(\mathbb{R}^+ \times I_0)}.
\end{aligned}$$

so that the operator bound (3.1.34) holds. $\square$

Remark 3.1.8. As an immediate consequence of Lemma 3.1.7, we see that $(I - T_0)^{-1}$ induces bounded operators on $\mathcal{B}(C^0(\mathbb{R}^+, L^2(I_0)))$ and $\mathcal{B}(L^2(\mathbb{R} \times I_0))$ with respective norms bounded by $1 + \|\gamma_1\|_{L^1} \exp \|\gamma_1\|_{L^1}$ and $1 + \|\gamma_1\|_{L^1} \exp \|\gamma_1\|_{L^{2,2}}$.

(3) Solving for n. We can now use these resolvent estimates to solve (3.1.18).

Lemma 3.1.9. *Suppose that $q \in H^{2,2}(\mathbb{R})$ and let $I_0 \subset \mathbb{R}$ be a bounded interval. There exists a unique solution of (3.1.18) for each $\lambda \in I_0$ so that $\mathbf{n} \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$. Moreover the map $q \to \mathbf{n}$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$.*

Proof. An immediate consequence of Lemma 3.1.4, (3.1.27), Lemma 3.1.7, and Remark 3.1.8. ☐

(4) Solving for $\mathbf{n}_\lambda$. Next, we estimate $\mathbf{n}_\lambda$ by controlling $\mathbf{n}_1$ and solving (3.1.22).

Lemma 3.1.10. *Suppose that $q \in H^{2,2}(\mathbb{R})$ and let $I_0 \subset \mathbb{R}$ be a bounded interval. There exists a unique solution of (3.1.22) belonging to $\mathbf{n} \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$. Moreover, the map $q \to \mathbf{n}$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$.*

Proof. From Lemma 3.1.9, estimate (3.1.28) of Lemma 3.1.4, and Lemma 3.1.5(i) and (ii), we may conclude that $\mathbf{n}_1 \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$ and is Lipschitz continuous in q. We may then solve (3.1.22) for $\mathbf{n}_\lambda \in C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$ using Lemma 3.1.7 and Remark 3.1.8. The map $q \to \mathbf{n}_\lambda$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_0)) \cap L^2(\mathbb{R}^+ \times I_0)$ since $q \to \mathbf{n}_1$ has this continuity and the resolvents are Lipschitz continuous as operator-valued functions. ☐

(5) Solving for $\mathbf{n}_{\lambda\lambda}$. Finally, we control $\mathbf{n}_2$ and solve (3.1.23) to estimate $\mathbf{n}_{\lambda\lambda}$.

Lemma 3.1.11. *Suppose that $q \in H^{2,2}(\mathbb{R})$ and $I_0 \subset \mathbb{R}$ is a bounded interval. There exists a unique solution of (3.1.23) in $C^0(\mathbb{R}^+, L^2(I_0))$. Moreover, the map $q \to \mathbf{n}_{\lambda\lambda}$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $C(\mathbb{R}^+, L^2(I_0))$.*

Proof. From Lemma 3.1.4, eq. 3.1.29, Lemma 3.1.9, Lemma 3.1.10, and Lemma 3.1.5(ii), (iii), we deduce that $\mathbf{n}_2 \in C^0(\mathbb{R}^+; L^2(I_0))$ with $q \to \mathbf{n}_2$ Lipschitz as a map from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_0))$. We now use Lemma 3.1.7 and Remark 3.1.8 to solve for $\mathbf{n}_{\lambda\lambda}$ as before. ☐

Proof of Proposition 3.1.2. An immediate consequence of Lemmas 3.1.9, 3.1.10, 3.1.11, and the fact that the restriction map $f \to f(0)$ from $C^0(\mathbb{R}^+; L^2(I_0))$ to $L^2(I_0)$ is continuous. ☐

3.1.2 Large-λ Estimates

In this subsection we prove Proposition 3.1.3. To study η_{11}^+, $(\eta_{11}^+)_\lambda$ and $(\eta_{11}^+)_{\lambda\lambda}$, we solve (3.1.12a) and the derived equations

$$(\eta_{11}^+)_\lambda = (F_+)_\lambda + (T_+)_\lambda [\eta_{11}] + T_+ \left[(\eta_{11}^+)_\lambda\right] \tag{3.1.35}$$

$$(\eta_{11}^+)_{\lambda\lambda} = (F_+)_{\lambda\lambda} + 2(T_+)_\lambda [(\eta_{11})_\lambda] + (T_+)_{\lambda\lambda} [\eta_{11}] + T_+ \left[(\eta_{11}^+)_{\lambda\lambda}\right]. \tag{3.1.36}$$

With good estimates in hand for η_{11}^+ and its derivatives, it will be a simple matter to prove the corresponding estimates on η_{21}^+ using (3.1.12b).

In the rest of this section, we will drop the $\pm$ and obtain estimates η_{11}^{+} and η_{21}^{+} since the analogous estimates for η_{11}^{-} and η_{21}^{-} are similar. We will write η_{11} for η_{11}^{+}, F for F_{+}, T for T^{+}, etc. We recall that $I_{\infty} = \{\lambda \in \mathbb{R} : |\lambda| > 1\}$.

Overall, we follow a strategy similar to that of section 3.1.1 to study the scalar equation (3.1.12a) for η_{11}, and then use these results to obtain comparable estimates on η_{21}. First, we will obtain estimates on F and G and derivatives of these functions in λ (Lemmas 3.1.12, 3.1.13, 3.1.15, and 3.1.16). Second, we will obtain resolvent estimates for $(I - T)^{-1}$ by a method similar to that used in the previous subsection (Lemmas 3.1.17 and 3.1.18). Third, we will solve (3.1.12a) for η_{11} (Lemma 3.1.19). Fourth, we'll solve (3.1.35) for $\partial\eta_{11}/\partial\lambda$ (Lemma 3.1.20). Fifth, we'll solve (3.1.36) for $\partial^2\eta_{11}/\partial\lambda^2$ (Lemma 3.1.21). Finally we will use (3.1.12b) to obtain estimates on η_{21} (Lemma 3.1.22).

(1) Estimates on F, G, and T.

Lemma 3.1.12. *Suppose $q \in H^{2,2}(\mathbb{R})$. The following define Lipschitz maps from $H^{2,2}(\mathbb{R})$ into $C^0(\mathbb{R}^+, L^2(I_\infty)) \cap L^2(\mathbb{R}^+ \times I_\infty)$:*

$$\text{(i) } G, \quad \text{(ii) } F, \quad \text{(iii) } \frac{\partial G}{\partial \lambda}, \quad \text{(iv) } \frac{\partial F}{\partial \lambda}.$$

The following define Lipschitz maps from $H^{2,2}(\mathbb{R})$ into $C^0(\mathbb{R}^+, L^2(I_\infty))$:

$$\text{(v) } \lambda^{-1}\frac{\partial^2 G}{\partial \lambda^2}, \quad \text{(vi) } \lambda^{-1}\frac{\partial^2 F}{\partial \lambda^2}$$

Proof. Observing that

$$\begin{aligned}\|F\|_{C^0(\mathbb{R}^+,L^2(I_\infty))} &\leqslant \|q\|_{L^1}\|G\|_{C^0(\mathbb{R}^+,L^2(I_\infty))},\\ \|F\|_{L^2(\mathbb{R}^+ I_\infty)} &\leqslant \|q\|_{L^{2,1/2}}\|G\|_{L^2(\mathbb{R}^+\times I_\infty)}\end{aligned}$$

we see that (i) $\Rightarrow$ (ii). To prove (i) we pick $\varphi \in C_0^\infty(I_\infty)$ and mimic the proof of Lemma 3.1.4. The Lipschitz continuity follows from the fact that G is linear in q and F is bi linear in q.

One can similarly check that (iii) $\Rightarrow$ (iv), so it suffices to prove (iii). We do so by applying mimicking the proof of Lemma 3.1.4 for the function

$$\frac{\partial G}{\partial \lambda} = \int_x^\infty (x-y)e_\lambda(y-x)q^\sharp(y)\,dy.$$

Finally, it is easy to see that (v) $\Rightarrow$ (vi). To prove (v), we recall $q^\sharp = \overline{q}' - \dfrac{i}{2}|q|^2\overline{q}$ and split

$$\frac{\partial^2 G}{\partial \lambda^2}(x,\lambda) = h_1(x,\lambda) + h_2(x,\lambda)$$

where

$$h_1(x,\lambda) = 2i\int_x^\infty (y-x)^2 e_\lambda(y-x)\overline{q'(y)}\,dy$$

$$h_2(x,\lambda) = -\int_x^\infty (y-x)^2 e_\lambda(y-x)\overline{q(y)}|q(y)|^2\,dy$$

We can estimate h_2 as before but for h_1 we integrate by parts to obtain

$$h_1(x,\lambda) = \int_x^\infty \left(2i\lambda(y-x)^2 + 2(y-x)\right) q(x) e_\lambda(y-x)$$

We can now use previous techniques to bound $\lambda^{-1}h_1(x,\lambda)$ for I_∞. □

The operator T defined in (3.1.13c) has the integral kernel

$$K_+^q(x,y,\lambda) = \begin{cases} \left(\int_x^y e_\lambda(y-z)q(z)\,dz\right) q^\sharp(y), & x<y, \\ 0 & x>y. \end{cases} \tag{3.1.37}$$

From this computation, we can prove:

Lemma 3.1.13. *The Volterra estimate*

$$(Tf)^*(x) \leqslant \left(\|q\|_1 \int_x^\infty |q^\sharp(y)|\,dy\right) f^*(x) \tag{3.1.38}$$

holds. Moreover

$$\sup_{x\in\mathbb{R}^+} \left|\int_x^{\pm\infty} |K_+^q(x,y,\lambda)|^2\,dy\right|^{1/2} \lesssim \lambda^{-1}\|q\|_{H^{2,2}} \left|\int_x^{\pm\infty} |q^\sharp(y)|^2\,dy\right|^{1/2} \tag{3.1.39}$$

$$\left(\int_{\mathbb{R}^+\times\mathbb{R}^+} |K_+^q(x,y,\lambda)|^2\,dx\,dy\right)^{1/2} \lesssim \lambda^{-1}\|q\|_{H^{2,2}} \left|\int_0^{+\infty} |y||q^\sharp(y)|^2\,dy\right|^{1/2} \tag{3.1.40}$$

$$\left(\int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^+} |K_+^q(x,y,\lambda)|\,dy\right)^2 dx\right)^{1/2} \lesssim \|q\|_{L^{2,1}} \left\|q^\sharp\right\|_{L^{2,1}}. \tag{3.1.41}$$

We omit the proof.

Remark 3.1.14. It follows respectively from (3.1.38), (3.1.39), (3.1.40), and (3.1.41) that T is a bounded operator from $C^0(\mathbb{R}^+)$ to itself, from $L^2(\mathbb{R}^+)$ to $C^0(\mathbb{R}^+)$, and from $C^0(\mathbb{R}^+)$ to $L^2(\mathbb{R}^+)$. The map $q \mapsto T$ is bilinear and Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to the corresponding Banach spaces of bounded operators with constants uniform in $\lambda \in I_\infty$.

Lemma 3.1.15. *Let $\delta \in [0,1/2)$. Then, the continuity estimates*

$$\|T_{q,\lambda_1} - T_{q,\lambda_2}\|_{\mathcal{B}(C^0)} \lesssim_\delta |\lambda_1-\lambda_2|^\delta \left\|q^\sharp\right\|_{L^{2,1}} \|q\|_{L^1}, \tag{3.1.42}$$

$$\|T_{q_1,\lambda} - T_{q_2,\lambda}\|_{\mathcal{B}(C^0)} \lesssim \left\|q_1^\sharp - q_2^\sharp\right\|_{L^1} \|q_1\|_{L^1} + \left\|q_2^\sharp\right\|_{L^1} \|q_1-q_2\|_{L^1}, \tag{3.1.43}$$

$$\|T_{q,\lambda_1} - T_{q,\lambda_2}\|_{\mathcal{B}(L^2)} \lesssim |\lambda_1-\lambda_2|^\delta \left(\int_0^\infty |y|^{1+2\delta}|q^\sharp(y)|^2\,dy\right)^{1/2} \tag{3.1.44}$$

$$\|T_{q_1,\lambda} - T_{q_2,\lambda}\|_{\mathcal{B}(L^2)} \lesssim \left|\int_0^{\pm\infty} |y||q_1^\sharp - q_2^\sharp|^2\,dy\right|^{1/2} \|q_1\|_{L^1} + \left|\int_0^{\pm\infty} |y||q_2^\sharp|^2\,dy\right|^{1/2} \|q_1 - q_2\|_{L^1} \tag{3.1.45}$$

hold, where the implied constants in (3.1.43) *and* (3.1.45) *are uniform in* λ *with* I_∞*.*

Finally we need mapping properties of the operators T_λ and $T_{\lambda\lambda}$.

Lemma 3.1.16. *The estimates*

(i) $\|T_\lambda\|_{\mathcal{B}(L^2(\mathbb{R}^+\times I_\infty))} \leqslant \|q^\sharp\|_{L^{2,1}} \|q\|_{L^1}$,

(ii) $\|T_\lambda\|_{\mathcal{B}(L^2(\mathbb{R}^+\times I_\infty), C^0(\mathbb{R}^+, L^2(I_\infty)))} \leqslant \|q^\sharp\|_{L^{2,1}} \|q\|_{L^{2,2}}$,

(iii) $\|\lambda^{-1} T_{\lambda\lambda}[h]\|_{C^0(\mathbb{R}^+, L^2(I_\infty))} \lesssim \|q\|_{H^{2,2}}^2 \left(\|h\|_{L^2(\mathbb{R}^+\times I_\infty)} + \|h_x\|_{L^2(\mathbb{R}^+\times I_\infty)}\right)$.

hold.

Proof. (i), (ii) From the formula

$$\frac{\partial T}{\partial\lambda}[h](x,\lambda) = \int_x^\infty q(y) \int_y^\infty (z-y) e_\lambda(z-y) q^\sharp(z) h(z,\lambda)\,dz\,dy.$$

we may estimate

$$\left|\frac{\partial T}{\partial\lambda}[h](x,\lambda)\right| \leqslant \|q^\sharp\|_{L^{2,1}} \|h(\,\cdot\,,\lambda)\|_{L^2(\mathbb{R}^+)} \int_x^\infty |q(y)|\,dy.$$

We easily conclude that

$$\begin{aligned} \left\|\frac{\partial T}{\partial\lambda}[h]\right\|_{C^0(\mathbb{R}^+, L^2(I_\infty))} &\leqslant \|q^\sharp\|_{L^{2,1}} \|q\|_{L^1} \|h\|_{L^2(\mathbb{R}^+\times I_\infty)}, \\ \left\|\frac{\partial T}{\partial\lambda}[h]\right\|_{L^2(\mathbb{R}^+\times I_\infty)} &\lesssim \|q^\sharp\|_{L^{2,1}} \|q\|_{L^{2,2}} \|h\|_{L^2(\mathbb{R}^+\times I_\infty)} \end{aligned}$$

which imply (i) and (ii). The maps are Lipschitz since they are bilinear in q.

(iii) From the formula

$$\begin{aligned} \frac{\partial^2 T}{\partial\lambda^2}[h](x,\lambda) &= -2i \int_x^\infty q(y) \int_y^\infty (z-y)^2 e_\lambda(z-y) q^\sharp(z) h(z,\lambda)\,dz\,dy \\ &= I_1 + I_2 \end{aligned}$$

where

$$I_1 = -2i \int_x^\infty q(y) \int_y^\infty (z-y)^2 e_\lambda(z-y) q'(z) h(z,\lambda)\,dz\,dy,$$

$$I_2 = \int_x^\infty q(y) \int_y^\infty (z-y)^2 e_\lambda(z-y) |q(z)|^2 h(z,\lambda)\, dz\, dy.$$

Since $z^2|q(z)|^2 \in L^{2,1}$ for $q \in H^{2,2}(\mathbb{R})$, we can estimate I_2 using the same techniques used for (i), (ii). The expression I_1 makes sense for $q \in \mathcal{S}(\mathbb{R})$ but we must integrate by parts to obtain an expression that is meaningful for arbitrary $q \in H^{2,2}(\mathbb{R})$. We compute

$$\begin{aligned} I_1 &= 2i \int_x^\infty q(y) \int_y^\infty e_\lambda(z-y) q(z) \left[-2i\lambda (z-y)^2 h(z,y) + 2(z-y) h(z,y) \right] dz\, dy \\ &\quad + 2i \int_x^\infty q(y) \int_y^\infty e_\lambda(z-y) q(z) \left[(z-y)^2 h_z(z,\lambda) \right] dz\, dy. \end{aligned}$$

from which (iii) follows. □

(2) Resolvent Estimates. As before we exploit Volterra estimates to construct the resolvent, obtain an integral kernel, and extend the resolvent to a bounded operator on the spaces $C^0(\mathbb{R}^+, L^2(I_\infty))$ and $L^2(\mathbb{R}^+ \times I_\infty)$.

Lemma 3.1.17. *Suppose that $q \in H^{2,2}(\mathbb{R})$. The resolvent $(I-T)^{-1}$ exists as a bounded operator in $C^0(\mathbb{R}^+)$ and the operator $L \equiv (I-T)^{-1} - I$ is an integral operator with integral kernel $L(x,y,\lambda)$ so that $L(x,y,\lambda) = 0$ for $x > y$, $L(x,y,\lambda)$ continuous in (x,y,λ) for $x < y$, and obeying the estimates*

$$|L(x,y,\lambda)| \leqslant \exp\left(\|q\|_{L^1} \|q^\sharp\|_{L^1} \right) \|q\|_{L^1} |q^\sharp(y)|.$$

Moreover, the map $q \to \widehat{L}$ is Lipschitz continuous from $H^{2,2}(\mathbb{R})$ into

$$\mathcal{B}(C^0(\mathbb{R}^+, L^2(I_\infty)) \cap \mathcal{B}(L^2(\mathbb{R}^+ \times I_\infty)).$$

We omit the proof, which is very similar to the proof of Lemma 3.1.6. The integral kernel L defines an operator $\widehat{L}$ much as the integral kernel L_0 defined an operator $\widehat{L}_0$ in (3.1.32) and Lemma 3.1.7. Following that analysis, one has:

Lemma 3.1.18. *The estimates*

$$\|\widehat{L}\|_{\mathcal{B}(C^0(\mathbb{R}^+, L^2(I_\infty)))} \leqslant \|q\|_{L^1} \|q^\sharp\|_{L^1} \exp\left(\|q\|_{L^1} \|q^\sharp\|_{L^1} \right) \tag{3.1.46}$$

and

$$\|\widehat{L}\|_{\mathcal{B}(L^2(\mathbb{R}^+, L^2(I_\infty)))} \leqslant \|q\|_{L^{2,1}} \|q^\sharp\|_{L^{2,1}} + \|q\|_{H^{2.2}}^6 \exp\left(\|q\|_{L^1} \|q^\sharp\|_{L^1} \right) \tag{3.1.47}$$

Proof. The estimate (3.1.47) follows from (3.1.46), the formula

$$(I-T)^{-1} - I = T + T(I-T)^{-1} T,$$

and the bounds on $T : L^2 \to C^0$, $T : C^0 \to L^2$, and $T : L^2 \to L^2$ obtained in Lemma 3.1.13 and Remark 3.1.14. The estimate (3.1.46) follows from the Volterra estimate (3.1.38) and the same argument used to in the proof of Lemma 3.1.7 to prove (3.1.33). □

(3) Solving for η_{11}. From the resolvent construction above, we can solve for η_{11}.

Lemma 3.1.19. *For each $q \in H^{2,2}(\mathbb{R})$ and $\lambda \in I_\infty$, the equation (3.1.12a) admits a unique solution $\eta_{11} \in C^0(\mathbb{R}^+, L^2(I_\infty)) \cap L^2(\mathbb{R}^+ \times I_\infty)$. Moreover, $q \to \eta_{11}$ is Lipschitz continuous as a map from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_\infty)) \cap L^2(\mathbb{R}^+ \times I_\infty)$.*

Proof. A direct consequence of Lemma 3.1.12(ii) and Lemma 3.1.18. □

(4) Solving for $\partial\eta_{11}/\partial\lambda$. By controlling the inhomogeneous term in (3.1.35), we can estimate $(\eta_{11})_\lambda$.

Lemma 3.1.20. *For each $q \in H^{2,2}(\mathbb{R})$ and $\lambda \in I_\infty$, the equation (3.1.35) admits a unique solution $(\eta_{11})_\lambda \in C^0(\mathbb{R}^+, L^2(I_\infty)) \cap L^2(\mathbb{R}^+ \times I_\infty)$. Moreover, $q \to (\eta_{11})_\lambda$ is Lipschitz continuous as a map from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_\infty)) \cap L^2(\mathbb{R}^+ \times I_\infty)$.*

Proof. By (3.1.35) and Lemma 3.1.18, it suffices to show that the inhomogeneous term

$$F_\lambda + T_\lambda [\eta_{11}]$$

belongs to $C^0(\mathbb{R}^+, L^2(I_\infty)) \cap L^2(\mathbb{R}^+ \times I_\infty)$. This follows from Lemma 3.1.12(iv), Lemma 3.1.16(i),(ii), and Lemma 3.1.19. □

(5) Solving for $\partial^2\eta_{11}/\partial\lambda^2$. Next we obtain estimates on $(\eta_{11})_{\lambda\lambda}$ using (3.1.36).

Lemma 3.1.21. *For each $q \in H^{2,2}(\mathbb{R})$, $\lambda \in I_\infty$, equation (3.1.36) admits a unique solution $(\eta_{11})_{\lambda\lambda}$ with $\lambda^{-1}(\eta_{11})_{\lambda\lambda} \in C^0(\mathbb{R}^+, L^2(I_\infty))$. Moreover, $q \to \lambda^{-1}(\eta_{11})_{\lambda\lambda}$ is Lipschitz continuous as a map from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_\infty))$.*

Proof. By (3.1.36) and Lemma 3.1.18, it suffices to show that the inhomogeneous term

$$\lambda^{-1} F_{\lambda\lambda} + 2\lambda^{-1} T_\lambda [(\eta_{11})_\lambda] + \lambda^{-1} T_{\lambda\lambda} [\eta_{11}]$$

belongs to $C^0(\mathbb{R}^+, L^2(I_\infty))$. For the first term, this follows from Lemma 3.1.12(vi), for the second term from Lemma 3.1.16(i), (ii), Lemma 3.1.12a, and for the third term from Lemma 3.1.16(iii) and Lemmas 3.1.19 and 3.1.20. □

(6) Estimates on η_{21} and its derivatives. It is now a simple matter to use (3.1.12b), estimates on G, and results already proved for η_{11} to obtain Lipschitz continuity of η_{21} and its derivatives.

Lemma 3.1.22. *The maps $q \to \eta_{21}$, $q \to (\eta_{21})_\lambda$, and $q \to \lambda^{-1}(\eta_{21})_{\lambda\lambda}$ are Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $C^0(\mathbb{R}^+, L^2(I_\infty))$*

Proof. Referring to (3.1.12b) and dropping the $\pm$ signs, the term $G(x,\lambda)$ has the required properties by Lemma 3.1.12(i), (iii), (v), and the second right-hand term of (3.1.12b) has the required properties since q is bounded and η_{11} has the correct mapping properties by Lemmas 3.1.19, 3.1.20, and 3.1.21. Thus, it remains to analyze the third term of (3.1.12b) (dropping the + sign on η_{11})

$$W(x,\lambda) = \frac{i}{2}\int_x^\infty e_\lambda(y-x)q^\sharp(y)\eta_{11}(y,\lambda)\,dy.$$

It is easy to see that

$$\begin{aligned}
\|W\|_{C^0(\mathbb{R}^+,L^2(I_\infty))} &\leqslant \frac{1}{2}\left\|q^\sharp\right\|_{L^2}\|\eta_{11}\|_{L^2(\mathbb{R}^+\times I_\infty)}\\
\|W_\lambda\|_{C^0(\mathbb{R}^+,L^2(I_\infty))} &\leqslant \frac{1}{2}\left(\left\|q^\sharp\right\|_{L^{2,1}}\|\eta_{11}\|_{L^2(\mathbb{R}^+\times I_\infty)} + \left\|q^\sharp\right\|_{L^2}\|(\eta_{11})_\lambda\|_{L^2(\mathbb{R}^+\times I_\infty)}\right)
\end{aligned}$$

which shows that $q\to W$ and $q\to W_\lambda$ have the required properties.

To analyze $W_{\lambda\lambda}$, recall (3.1.10) to write $W = W_1 + W_2$ where

$$\begin{aligned}
W_1(x,\lambda) &= \frac{i}{2}\int_x^\infty e_\lambda(y-x)\overline{q'(y)}\eta_{11}(y,\lambda)\,dy,\\
W_2(x,\lambda) &= \frac{1}{4}\int_x^\infty e_\lambda(y-x)\overline{q(y)}|q(y)|^2\eta_{11}(y,\lambda)\,dy.
\end{aligned}$$

We first control W_1. Differentiating in λ we have

$$(W_1)_{\lambda\lambda}(x,\lambda) = W_{11}(x,\lambda) + W_{12}(x,\lambda)$$

where

$$\begin{aligned}
W_{11}(x,\lambda) &= -2i\int_x^\infty e_\lambda(y-x)(y-x)^2\overline{q'(y)}\eta_{11}(y,\lambda)\,dy\\
W_{12}(x,\lambda) &= \frac{i}{2}\int_x^\infty e_\lambda(y-x)\overline{q'(y)}(\eta_{11})_{\lambda\lambda}(y,\lambda)\,dy
\end{aligned}$$

It is easy to see that

$$\sup_{x\geqslant 0}\left\|(\,\cdot\,)^{-1}W_{12}(x,\,\cdot\,)\right\|_{L^2(I_\infty)} \leqslant \|q'\|_{L^2}\left\|(\diamond)^{-1}(\eta_{11})_\lambda(\,\cdot\,,\diamond)\right\|_{L^2(\mathbb{R}^+\times I_\infty)}$$

so that $q\to\lambda^{-1}(W_{12})$ has the correct mapping property. Turning to W_{11}, we integrate by parts to remove the derivative on q and obtain

$$\begin{aligned}
W_{11}(x,\lambda) = 2i&\int_x^\infty e_\lambda(y-x)\overline{q(y)}\left(2(y-x)\eta_{11}(x,\lambda) + (y-x)^2(\eta_{11})_x(y,\lambda)\right)\,dy \\
&-4\lambda\int_x^\infty e_\lambda(y-x)(y-x)^2\,\overline{q(y)}\eta_{11}(y,\lambda)\,dy.
\end{aligned} \tag{3.1.48}$$

The first right-hand term in (3.1.48) has $C^0(\mathbb{R}^+,L^2(I_\infty))$-norm bounded by

$$\|q\|_{L^{2,1}}\|\eta_{11}\|_{L^2(\mathbb{R}^+\times I_\infty)}\,. \tag{3.1.49}$$

Since, by (3.1.9a) and (3.1.11),

$$(\eta_{11})_x = (n_{11})_x = q(y)\left(\eta_{21}(x,\lambda) - \frac{i}{2}\overline{q(x)}\right) + \frac{i}{2}|q(y)|^2\left(1+\eta_{11}(x,\lambda)\right),$$

we can reexpress the second right-hand term in (3.1.48) as

$$2i\int_x^\infty e_\lambda(y-x)(y-x)^2|q(y)|^2\left(-\frac{i}{2}\overline{q(y)} + \eta_{21}(y,\lambda)\right)dy$$
$$+\,2i\int_x^\infty e_\lambda(y-x)(y-x)^2\left(\frac{i}{2}\overline{q(y)}|q(y)|^2\left(1+\eta_{11}(y,\lambda)\right)\right)dy$$

which has $C^0(\mathbb{R}^+, L^2(I_\infty))$-norm bounded by constants times

$$\|q\|^3_{H^{2,2}}\left(1 + \|\eta_{11}\|_{L^2(\mathbb{R}^+\times I_\infty)} + \|\eta_{21}\|_{L^2(\mathbb{R}^+\times I_\infty)}\right). \tag{3.1.50}$$

Finally, dividing by λ in the third term, we can estimate the $C^0(\mathbb{R}^+, L^2(I_\infty))$ norm of the quotient by

$$\|q\|_{L^{2,2}}\|\eta_{11}\|_{L^2(\mathbb{R}^+\times I_\infty)}. \tag{3.1.51}$$

Combining (3.1.49), (3.1.50), and (3.1.51), we see that

$$\begin{aligned}\left\|(\diamond)^{-1}W_{11}(\,\cdot\,,\diamond)\right\|_{C^0(\mathbb{R}^+,I_\infty)} &\lesssim \left(1+\|q\|^3_{H^{2,2}}\right) \\ &\quad\times\left(1 + \|\eta_{11}\|_{L^2(\mathbb{R}^+\times I_\infty)} + \|\eta_{21}\|_{L^2(\mathbb{R}^+\times I_\infty)}\right).\end{aligned} \tag{3.1.52}$$

which shows that W_1 has the required mapping property.

Now we turn to W_2. Since

$$\begin{aligned}(W_2)_{\lambda\lambda}(x,\lambda) = \frac{1}{4}\int_x^\infty e_\lambda(y-x)\overline{q(y)}|q(y)|^2\big([(\eta_{11})_{\lambda\lambda}(y,\lambda) \\ -4i(y-x)(\eta_{11})_\lambda(y,\lambda) - 4(y-x)^2\eta_{11}(y,\lambda)\big]\,dy,\end{aligned}$$

we may estimate

$$\begin{aligned}\left\|(\diamond)^{-1}(W_2)_{\lambda\lambda}(\,\cdot\,,\diamond)\right\|_{C^0(\mathbb{R}^+,L^2(I_\infty))} \leqslant \|q\|^2_{H^{2,2}}\Big(&\left\|\lambda^{-1}(\eta_{11})_\lambda\right\|_{C^0(\mathbb{R}^+,L^2(I_\infty))} \\ &+\left\|(\eta_{11})_\lambda\right\|_{C^0(\mathbb{R}^+,L^2(I_\infty))} \\ &+\|\eta_{11}\|_{C^0(\mathbb{R}^+,L^2(I_\infty))}\Big).\end{aligned} \tag{3.1.53}$$

This shows that $q \to W_2$ has the correct mapping properties.

Combining (3.1.52) and (3.1.53), we conclude that $q \mapsto W_{\lambda\lambda}$ has the correct mapping property, and hence, also, $q \mapsto (\diamond)^{-1}\eta_{\lambda\lambda}(\,\cdot\,,\diamond)$. □

Proof of Proposition 3.1.3. An immediate consequence of Lemmas 3.1.19, 3.1.20, 3.1.21, and 3.1.22 and the fact that the restriction map $f \to f(0)$ from $C^0(\mathbb{R}^+; L^2(I_\infty))$ to $L^2(I_\infty)$ is continuous. □

3.2 Discrete Scattering Data

Lemma 3.2.1. *$n_{11}^{-}(x,\lambda)$, $n_{21}^{-}(x,\lambda)$, $n_{12}^{+}(x,\lambda)$ and $n_{22}^{+}(x,\lambda)$ all have analytic continuation into $\mathbb{C}^{+}$.*

Proof. For reference convenience we state the following Volterra integral equations

$$n_{11}^{\pm}(x,\lambda) = 1 + \frac{i}{2}\int_{x}^{\pm\infty} q(y)\int_{y}^{\pm\infty} e_{\lambda}(z-y)q^{\sharp}(z)n_{11}^{\pm}(z,\lambda)\,dz\,dy \tag{3.2.1}$$

$$n_{21}^{\pm}(x,\lambda) = -\frac{i}{2}\overline{q(x)}n_{11}^{\pm}(x,\lambda) - \frac{i}{2}\int_{x}^{\pm\infty} e_{\lambda}(y-x)q^{\sharp}(y)n_{11}^{\pm}(y,\lambda)\,dy \tag{3.2.2}$$

where

$$q^{\sharp}(x) = \overline{q'(x)} + \frac{i}{2}|q(x)|^{2}\overline{q(x)}$$

Recall from (3.1.12a) that

$$n_{11}^{-}(x,\lambda) - 1 = F_{-}(x,\lambda) + T_{-}(n_{11}^{-} - 1)$$

where F is given by (3.1.13a) and T is given by (3.1.13c). Using that $\left|e^{2i\lambda(y-z)}\right| \leqslant 1$ for $\operatorname{Im}\lambda \geqslant 0$ and $z \leqslant y$, the estimate

$$|(T_{-}f)(x)| \leqslant \gamma(x)\sup_{y<x}|f(y)|$$

holds with

$$\gamma(x) = \int_{x}^{-\infty}|q(y)|\int_{y}^{-\infty}|q^{\sharp}(z)|\,dz\,dy.$$

we claim that each term of the resolvent operator

$$(I - T_{-})^{-1} = \sum_{n=0}^{\infty} T_{-}^{n}$$

is analytic in λ. To see this, we deduce from (3.1.13a)-(3.1.13c) that

$$\left(T_{-}^{n}f\right)(x,\lambda) = \left(\frac{i}{2}\right)^{n}\int_{x}^{-\infty} q(x_{n-1})G(x_{n-1},\lambda)\cdots\left(\int_{x_{1}}^{-\infty} q(y)G(y,\lambda)f(y)\,dy\right)\cdots dx_{n-1}$$

Direct calculation gives

$$\sup\left|T_{-}^{n}f\right| \leqslant \frac{\|q\|_{1}\left\|q^{\sharp}\right\|_{1}}{n!}\sup_{y<x}|f(y)|$$

Analyticity of $(T_{-}^{n}f)$ in $\mathbb{C}^{+}$ follows from changing the order of integration and an application of Morera's theorem. Using Neumann series we write

$$n_{11}^{-}(x,\lambda) - 1 = \sum_{n=0}^{\infty}\left(T_{-}^{n}F\right)(x,\lambda).$$

and analyticity of n_{11}^- follows from the fact that uniform limit of analytic functions is analytic. By (3.2.2) the analyticity of n_{21}^- follows from the analyticity of n_{11}^-. We can show that n_{11}^+ and n_{21}^+ have analytic continuation into $\mathbb{C}^-$ using the same argument above. And the analyticity of $n_{12}^+(x,\lambda)$ and $n_{22}^+(x,\lambda)$ in $\mathbb{C}^+$ follow from the symmetry relation

$$n_{12}^+(x,\lambda) = \lambda^{-1}\overline{n_{21}^+(x,\overline{\lambda})}, \quad n_{22}^+(x,\lambda) = \overline{n_{11}^+(x,\overline{\lambda})}$$

□

Proposition 3.2.2. *$\breve{\alpha}$ (α) has analytic continuation into $\mathbb{C}^+$ ($\mathbb{C}^-$).*

Proof. An easy consequence of the fact that

$$\breve{\alpha}(\lambda) = \begin{vmatrix} n_{11}^- & n_{12}^+ \\ n_{21}^- & n_{22}^+ \end{vmatrix}$$

and

$$\alpha(\lambda) = \begin{vmatrix} n_{11}^+ & n_{12}^- \\ n_{21}^+ & n_{22}^- \end{vmatrix}$$

□

Lemma 3.2.3. *There exist unique solutions of (3.2.1)–(3.2.2) with*

$$\sup_{\operatorname{Im}\lambda \geqslant 0} \left|n_{11}^-(x,\lambda)\right| \leqslant \exp\left(\frac{1}{2}\|q\|_{L^1}\|q^\sharp\|_{L^1}\right)$$

and

$$\sup_{\operatorname{Im}\lambda \geqslant 0} \left|n_{21}^-(x,\lambda)\right| \leqslant \exp\left(\frac{1}{2}\|q\|_{L^1}\|q^\sharp\|_{L^1}\right)\left(\|q\|_{L^1} + \|q^\sharp\|_{L^1}\right).$$

Moreover,

$$\begin{aligned} &\left|n_{11}^-(x,\lambda;q_1) - n_{11}^-(x,\lambda;q_2)\right| \\ &\quad \leqslant \exp\left[C\left(\|q_1\|_{L^1}\|q_1^\sharp\|_{L^1} + \|q_2\|_{L^1}\|q_2^\sharp\|_{L^1}\right)\right]\left(\|q_1 - q_2\|_{L^1}\|q_1^\sharp\|_{L^1} + \|q_2\|_{L^1}\|q_1^\sharp - q_2^\sharp\|_{L^1}\right) \end{aligned} \tag{3.2.3}$$

where C is independent of λ with $\operatorname{Im}(\lambda) \geqslant 0$.

Proof. From the standard theory of Volterra integral equations, we have that

$$\left\|(I - T_-)^{-1}\right\|_{C(\mathbb{R}^+)\to C(\mathbb{R}^+)} \leqslant \exp\left(\frac{1}{2}\|q\|_{L^1}\|q^\sharp\|_{L^1}\right), \tag{3.2.4}$$

uniformly in λ with $\operatorname{Im}\lambda \geqslant 0$.Since $\|q\|_{L^1}$ and $\|q^\sharp\|_{L^1}$ are controlled by $\|q\|_{H^{2,2}}$, we have $\left\|n_{11}^-\right\|_{C(\mathbb{R}^+)}$ and (from (3.2.2)) also $\left\|n_{21}^-\right\|_{C(\mathbb{R}^+)}$ bounded uniformly in λ, $\operatorname{Im}\lambda \leqslant 0$ and in q for q in a bounded subset of $H^{2,2}(\mathbb{R})$. Finally (3.2.3) follows from the resolvent estimate (3.2.4) and the second resolvent formula. □

Similar estimates are obtained for n_{11}^- and n_{21}^-. From these estimates and the Wronskian formula (3.1.5), we conclude:

Proposition 3.2.4. *The function α satisfies*

$$|\alpha(\lambda; q_1) - \alpha(\lambda; q_2)| \leqslant \exp C \left(\left\|q_1^2\right\|_{H^{2,2}} + \left\|q_2^2\right\|_{H^{2,2}} \right) \left(\|q_1\|_{H^{2,2}} + \|q_2\|_{H^{2,2}} \right) \left(\|q_1 - q_2\|_{H^{2,2}} \right).$$

where the constants are uniform in λ with $\operatorname{Im} \lambda \leqslant 0$.

It is important that the zeros of α lie in a compact set of $\mathbb{C}^-$, more precisely, in $\mathbb{C}^- \cap \{|z| \leqslant R\}$ where $R > 0$ depends only on $\|q\|_{H^{2,2}(\mathbb{R})}$. This is the object of the next proposition.

Proposition 3.2.5. *The function α satisfies*

$$\lim_{R \to \infty} \sup_{|\lambda| \geqslant R, \operatorname{Im} \lambda \leqslant 0} |\alpha(\lambda) - 1| = 0 \tag{3.2.5}$$

where the convergence is uniform in q in a bounded subset of $H^{2,2}(\mathbb{R})$.

Proof. From the Wronskian formula (3.1.5) for α and the uniform bounds on n_{21}^- and n_{21}^+, estimate (3.2.5) will follow from

$$\lim_{R \to \infty} \sup_{|\lambda| \geqslant R} |n_{11}^{\pm}(0, \lambda) - 1| = 0. \tag{3.2.6}$$

We now sketch the proof of (3.2.6) for n_{11}^+, the proof for n_{11}^- is similar.

From (3.2.1) and an integration by parts we see that

$$\begin{aligned} n_{11}^+(x, \lambda) - 1 = &\frac{i}{2} \int_x^\infty q(y) \int_y^\infty e^{-2i\lambda(z-y)} q^\sharp(z) \, dz \, dy \\ &- \frac{1}{4\lambda} \int_x^\infty q(y) \left[G_1(y, \lambda) + G_2(y, \lambda) + G_3(y, \lambda) \right] dy, \end{aligned} \tag{3.2.7}$$

where

$$\begin{aligned} G_1(x, \lambda) &= -q^\sharp(x) \left(n_{11}^+(x, \lambda) - 1 \right) \\ G_2(x, \lambda) &= -\int_x^\infty e^{-2i\lambda(y-x)} \left(q^\sharp \right)'(y) \left(n_{11}^+(y, \lambda) - 1 \right) \, dy \\ G_3(x, \lambda) &= -\int_x^\infty e^{-2i\lambda(y-x)} q^\sharp(y) \frac{\partial n_{11}^+}{\partial x}(y, \lambda) \, dy. \end{aligned}$$

Reversing the orders of integration in the first right-hand term of (3.2.7) and integrating by parts we may estimate

$$\left| \int_x^\infty q(y) \int_y^\infty e^{-2i\lambda(z-y)} q^\sharp(z) \, dz \, dy \right| \leqslant \frac{1}{|\lambda|} \|q^\sharp\|_{L^1} \left(\|q\|_{L^\infty} + \|q'\|_{L^1} \right).$$

From Lemma 3.2.3 we have $|G_1(x, \lambda)| \lesssim 1$ where the implied constants depend only on $\|q\|_{L^1}$ and $\|q^\sharp\|_{L^1}$. Differentiating (3.2.1) to compute $\partial n_{11}^+ / \partial x$ we may similarly estimate $|G_3(x, \lambda)|$. To estimate $G_2(x, \lambda)$, we need to show that $\left\| n_{11}^+(\,\cdot\,, \lambda) - 1 \right\|_{L^2(\mathbb{R}^+)}$ is bounded uniformly in λ with $\operatorname{Im} \lambda \leqslant 0$ and q in a bounded subset of $H^{2,2}(\mathbb{R})$. This is carried out in Lemma 3.2.7 below. ∎

To prove the L^2 estimate on $n_{11}^+(x,\lambda)-1$, we return to the integral equation (**??**) and note that the operator S is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^+)$ uniformly in λ for $\operatorname{Im}\lambda \leqslant 0$. Indeed its integral kernel is given by

$$K(x,z) = \begin{cases} \displaystyle\int_x^z q(y) e^{-2i\lambda(z-y)} q^\sharp(z)\, dy, & x < z \\ \\ 0, & x > z \end{cases} \tag{3.2.8}$$

with

$$\|K\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+)} \leqslant \left\|q^\sharp\right\|_{L^{2,1/2}} \|q\|_{L^1}.$$

One checks that

$$\ker_{L^2(\mathbb{R}^+)}(I-T_+) \subset \ker_{C(\mathbb{R}^+)}(I-T_+) = \{0\}$$

where the last equality follows from the existence of the resolvent $(I-T_+)^{-1}$ on $C(\mathbb{R}^+)$. Writing $T_+ = T_+(\lambda)$ to display the dependence of the operator T_+ on λ, we can show that

$$\lim_{|\lambda|\to\infty} \|T_+(\lambda)\|_{HS} = 0 \tag{3.2.9}$$

uniformly in λ with $\operatorname{Im}\lambda \leqslant 0$ and q in a bounded subset of $H^{2,2}(\mathbb{R})$. This follows from the integration by parts

$$\int_x^z q(y) e^{-2i\lambda(z-y)}\, dy = \frac{1}{2i\lambda}\left[q(z) - q(x) + \int_x^z e^{-2i\lambda(z-y)} q(y)\, dy\right]$$

and a straightforward estimate of the Hilbert- Schmidt norm using (3.2.8). Writing $K = K(\lambda, q)$, we may also estimate

$$\|K(\lambda,q_1) - K(\lambda,q_2)\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+)} \leqslant \|q_1-q_2\|_{L^{2,1/2}} \|q_1\|_{L^1} + \|q_2\|_{L^{2,1/2}} \|q_1-q_2\|_{L^1}$$

uniformly in λ with $\operatorname{Im}\lambda \leqslant 0$. On the other hand, it follows from the Dominated Convergence Theorem that $\|K(\lambda_1,q) - K(\lambda_2,q)\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+)} \to 0$ as $\lambda_1 \to \lambda_2$ for any fixed $q \in L^1 \cap L^{2,1/2}$. Writing $T_+ = T_+(\lambda, q)$, we now use a 'continuity-compactness argument' as well as (3.2.9) to prove:

Lemma 3.2.6. *The resolvent $(I - T_+(\lambda,q))^{-1}$ exists as a bounded operator on $L^2(\mathbb{R}^+)$ and for any $M > 0$,*

$$\sup_{\operatorname{Im}\lambda\leqslant 0,\, \|q\|_{H^{2,2}}\leqslant M} \left\|(I-T_+(\lambda,q))^{-1}\right\|_{L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)} < \infty.$$

Proof. For any $M > 0$, $R > 0$, the identity map takes the set

$$\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \leqslant 0, |\lambda| \leqslant R\} \times \left\{q \in H^{2,2}(\mathbb{R}) : \|q\|_{H^{2,2}} \leqslant M\right\}$$

into a subset of $\mathbb{C} \times (L^{2,1/2} \cap L^1)$ with compact closure. By the second resolvent formula, the map $(\lambda, q) \mapsto (I - S(\lambda,q))^{-1}$ is continuous into the bounded operators on $L^2(\mathbb{R}^+)$. It follows by compactness and continuity that the set

$$\left\{(I-T_+(\lambda,q))^{-1} : \operatorname{Im}\lambda \leqslant 0, |\lambda| \leqslant R, \|q\|_{H^{2,2}} \leqslant M\right\}$$

is compact in $\mathcal{B}(L^2(\mathbb{R}^+))$, hence bounded. On the other hand, for sufficiently large R depending on M, we have from (3.2.9) that $\sup_{|\lambda|\geqslant R} \|(I - T_+(\lambda, q))^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^+))} \leqslant 2$ for any q with $\|q\|_{H^{2,2}} \leqslant M$. $\square$

We can now prove:

Lemma 3.2.7. *If $q \in H^{2,2}(\mathbb{R})$, the estimate*

$$\left\|n_{11}^+(\,\cdot\,, \lambda) - 1\right\|_{L^2(\mathbb{R}^+)} \lesssim 1$$

holds.

Proof. The function $\eta = n_{11}^+ - 1$ obeys the integral equation $\eta = F_+(x, \lambda) + T_+(\eta)$ where

$$F_+(x, \lambda) = \frac{i}{2} \int_x^\infty q(y) \int_z^\infty e^{-2i\lambda(z-y)} q^\sharp(z)\, dz\, dy.$$

We may estimate

$$\|F_+\|_{L^2(\mathbb{R}^+)} \leqslant \langle x \rangle^{-3/2} \|q\|_{L^{2,2}} \|q^\sharp\|_{L^1}$$

which shows that $F_+(x, \lambda) \in L^2(\mathbb{R}^+)$ uniformly in λ with $\operatorname{Im}\lambda \leqslant 0$. The desired bound is obtained using Lemma 3.2.6 on n_{11}^+. $\square$

3.3 Generic Properties of Spectral Data

Lee [13] showed that generic potentials q in the Schwartz class have at most finitely many simple zeros of α and no spectral singularities. His proof is based on a general argument of Beals and Coifman [2]. Here we give a more precise functional analytic argument inspired by analogous results in Schrödinger scattering theory (see the manuscript of Dyatlov and Zworski [[9] Chapter 2, Theorem 2.2]DZ17). We will prove:

Theorem 3.3.1. *The set of potentials q supporting at most finitely many solitons and having no spectral singularities is open and dense in $H^{2,2}(\mathbb{R})$.*

Our strategy is to study the dense set of $q \in C_0^\infty(\mathbb{R})$ and prove that any such q can be perturbed by an arbitrarily small amount in $H^{2,2}$-norm to a potential with the desired properties. We then use continuity of spectral quantities to show that the set is open as well as dense. These steps are carried out in Propositions 3.3.5 and 3.3.6 below which together give the proof of Theorem 3.3.1.

We begin with the study of C_0^∞ potentials. The following fact is well-known and easy to prove; see for example Chapter 2 of Lee's thesis [13].

Lemma 3.3.2. *Suppose that $q \in C_0^\infty(\mathbb{R})$. Then $\alpha(\lambda; q)$ is analytic in $\mathbb{C}$ and has at most finitely many zeros in $\operatorname{Im}\lambda \leqslant c$ for any $c \in \mathbb{R}$.*

Using this fact, a perturbation argument, and Rouché's theorem, we will construct a dense set of potentials in $H^{2,2}(\mathbb{R})$ for which α has at most finitely many simple zeros in $\mathbb{C}^-$ and no zeros on $\mathbb{R}$. We will then exploit Proposition 3.2.4 to show that this set is also open.

To construct the dense set, we need two perturbation lemmas. The first concerns perturbation from the zero potential for which $\alpha(\lambda) \equiv 1$ and $\beta(\lambda) \equiv 0$.

Lemma 3.3.3. *Suppose that* $\varphi \in C_0^\infty(-R,R)$*,* $\lambda \neq 0$*, and* μ *is a small parameter. Let* $q = \mu\varphi$*. Then the associated transition matrix has the form*

$$T(\lambda, q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mu c_\varphi \\ -\lambda\overline{\mu c_\varphi} & 0 \end{pmatrix} + \mathcal{O}\left(\mu^2\right) \tag{3.3.1}$$

where

$$c_\varphi = -\int_{-\infty}^{\infty} e^{2i\lambda y}\varphi(y)\, dy \tag{3.3.2}$$

and the correction term depends on $\|\varphi\|_{H^{1,1}}$*.*

Proof. It suffices to show that

$$\alpha(\lambda) \sim 1 + \mathcal{O}\left(\mu^2\right), \tag{3.3.3}$$

$$\lambda\breve{\beta}(\lambda) \;= \lambda\overline{\mu}\int e^{-2i\lambda y}\overline{\varphi(y)}\, dy + \mathcal{O}\left(\mu^2\right). \tag{3.3.4}$$

First, we recall from Lemma 3.2.7 that, for $\lambda \in \mathbb{R}$, we have the uniform estimate

$$\left|\left(n_{11}^+(x,\lambda), n_{21}^+(x,\lambda)\right)\right| \lesssim 1$$

where the implied constants depend only on $\|q\|_{H^{1,1}}$ (the key issue is that the large-λ behavior is controlled despite the λ-dependence of the perturbing term in (3.1.1); see equations (3.2.1)–(3.2.2) for the integration by parts that removes this term). Taking limits as $x \to -\infty$ in equations (3.2.1)–(3.2.2) for n^+ (and as $x \to -\infty$ in the corresponding equations for n^-) and using the relation (3.1.3), we deduce that (3.3.3) and (3.3.4) hold. ☐

The next lemma will give a mechanism for splitting multiple poles and perturbing zeros on the real axis.

Lemma 3.3.4. *Suppose that* q_1 *and* q_2 *are compactly supported potentials with disjoints supports, and that the support of* q_2 *on the real line is to the left of the support of* q_1*. Then:*

(i) The identity

$$T(\lambda, q_1 + q_2) = T(\lambda, q_2)T(\lambda, q_1)$$

holds.

(ii) If $q_1 \in C_0^\infty((-R,R))$ *and* $q_2 = \mu\varphi$ *with* $\varphi \in C_0^\infty((-2R,-R))$*, the formula*

$$T(\lambda, q_1 + \mu\varphi) = \begin{pmatrix} 1 & \mu c_\varphi \\ -\lambda\overline{\mu c_\varphi} & 1 \end{pmatrix} T(\lambda, q_1) + \mathcal{O}\left(\mu^2\right) \tag{3.3.5}$$

holds.

Proof. Consider the normalized solution $n^+(x,\lambda,q)$. It is not difficult to see that

$$n^+(x,\lambda,q_1+q_2) = n^+(x,\lambda,q_2)n^+(x,\lambda,q_1).$$

We may now compute

$$T(\lambda, q_1+q_2) = \lim_{x\to-\infty} e^{i\lambda x\,\mathrm{ad}(\sigma_3)}\left[n^+(x,\lambda,q_2)n^+(x,\lambda,q_1)\right] = T(\lambda,q_2)T(\lambda,q_1)$$

The second assertion is an immediate consequence of the first. ☐

Suppose that q_1 and q_2 are chosen as in Lemma 3.3.4(ii). To simplify the notation, let us write $\alpha(\lambda, \mu)$ to denote $\alpha(\lambda, q_1 + \mu\varphi)$. It follows from (3.3.5) that

$$\alpha(\lambda, \mu) = \alpha(\lambda, 0) + \mu c_\varphi \lambda \breve{\beta}(\lambda) + \mathcal{O}\left(\mu^2\right). \tag{3.3.6}$$

where c_φ is given by (3.3.2). In the next proposition, we will expand the above formula near $\lambda = \lambda_0$:

$$\alpha(\lambda, \mu) = \alpha(\lambda, 0) + \mu c_{\varphi_0} \lambda_0 \breve{\beta}(\lambda_0) + C_0(\lambda - \lambda_0)\mu + \mathcal{O}\left(\mu^2\right). \tag{3.3.7}$$

where

$$c_{\varphi_0} = -\int_{-\infty}^{\infty} e^{2i\lambda_0 y} \varphi(y)\, dy$$

and

$$|C_0| \leqslant \left\| \frac{d}{d\lambda}\left(\lambda \breve{\beta}(\lambda)\right) \right\|_{L^\infty}.$$

From the compactness of the potential q and the asymptotic condition of $\alpha(\lambda)$ we know that α has finitely many zeros in $\mathbb{C}^- \cup \mathbb{R}$. We will prove:

Proposition 3.3.5. *Suppose that $R > 0$ and $q \in C_0^\infty([-R, R])$. Let $\alpha(\lambda)$ be the $(1,1)$ entry of the transition matrix for q. For $\varphi \in C_0^\infty(\mathbb{R})$, let $\alpha(\lambda, \mu)$ be the $(1,1)$ entry for the transition matrix of $q + \mu\varphi$, so that $\alpha(\lambda, 0) = \alpha(\lambda)$.*

(i) Suppose that $\{\lambda_i\}_{i=1}^m$ are the isolated zeros of $\alpha(\lambda)$ in $\mathbb{C}^- \cup \mathbb{R}$ and $\lambda_i \neq 0$ is one of the zeros of $\alpha(\lambda)$ of multiplicity $n \geqslant 2$, i.e. $\alpha(\lambda) = (\lambda - \lambda_i)^n g(\lambda)$ for some analytic function g with $g(\lambda_i) \neq 0$. Then, for some $\varphi \in C_0^\infty(\mathbb{R})$ and all sufficiently small $\mu \neq 0$, $\alpha(\lambda, \mu)$ has n simple zeros in the disc $D_{r_i}(\lambda_i)$.

(ii) Suppose that after the perturbation in part (i), Λ_j is a simple zero of $\alpha(\lambda, \mu)$ on the real axis, $\Lambda_j \neq 0$. Then, there is a function $\varphi \in C_0^\infty(\mathbb{R})$ so that, for all real, nonzero, and sufficiently small μ', $\alpha(\lambda, \mu')$ has no zeros on the real axis near ω_j.

In each case, we may choose φ to have support in $(-2R, -R) \cup (R, 2R)$.

Proof. (i) We first claim that there exists a function $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geqslant 0$ such that

$$\widehat{\varphi}(\lambda_i) = \int_{-\infty}^{\infty} e^{2i\lambda_i x} \varphi(x)\, dx \neq 0$$

for all i.

Indeed, let $2\lambda_i = \xi_i + i\eta_i$, then we have

$$\widehat{\varphi}(\lambda_i) = \int_{-\infty}^{\infty} (\cos \xi_i x + i \sin \xi_i x) e^{\eta_i x} \varphi(x)\, dx$$

with $e^{\eta_i x}\varphi(x) \geqslant 0$ for all x.

If we let $\xi = \max\{\xi_1, ...\xi_i, ...\xi_m\}$, $r = \pi/2\xi$ and make $|\mathrm{supp}(\varphi)| \leqslant r$, then at least one of $\cos \xi_i x$ and $\sin \xi_i x$ does not change sign on $\mathrm{supp}(\varphi)$ for all i. So $\widehat{\varphi}(\lambda_i) \neq 0$ for all i.

Using the Taylor expansion of $\widehat{\varphi}(\lambda)$ and $\lambda\breve{\beta}(\lambda)$ we can write (3.3.7) as

$$\alpha(\lambda,\mu) = (\lambda-\lambda_i)^n g(\lambda) + \mu c_{\varphi_i}\lambda_i\breve{\beta}(\lambda_i) + C_0(\lambda-\lambda_i)\mu + \mathcal{O}\left(\mu^2\right). \tag{3.3.8}$$

If we can establish the following inequalities

$$|\lambda\breve{\beta}(\lambda)|_{L^\infty} \lesssim_q 1 \tag{3.3.9}$$

and

$$|(\lambda\breve{\beta}(\lambda))'|_{L^\infty} \lesssim_q 1 \tag{3.3.10}$$

where $\lambda \in \overline{D(\lambda_i, r_i)}$ then it is clear that

$$\begin{aligned}|\alpha(\lambda,\mu) - (\lambda-\lambda_i)^n g(\lambda)| &= |\mu c_{\varphi_i}\lambda_i\breve{\beta}(\lambda_i) + C_0(\lambda-\lambda_i)\mu + \mathcal{O}\left(\mu^2\right)| \\ &\leqslant |\lambda-\lambda_i|^n|g(\lambda)|\end{aligned}$$

for μ sufficiently small and $\lambda \in \partial D(\lambda_i, r_i)$. Rouché's Theorem shows that the number of zeros of $\alpha(\lambda,\mu)$ and $\alpha(\lambda,0)$ agree (with multiplicities) in $D(\lambda_i, r_i)$. That is, $\alpha(\lambda,\mu)$ has exactly n zeros there.

To prove estimates (3.3.9) and (3.3.10), we use the boundary condition of the Jost functions

$$\begin{aligned}\lambda\breve{\beta}(\lambda) &= \int_{-R}^{R} e^{-2i\lambda y}\left\{\langle -\lambda\overline{q(y)}, p_2(y)\rangle \cdot n^-_{(1)}(y,\lambda)\right\}\, dy \\ &= \int_{-R}^{R} e^{-2i\lambda y}\left(-\lambda\overline{q(y)}n^-_{11}(y,\lambda) + p_2(y)n^-_{21}(y,\lambda)\right) dy.\end{aligned} \tag{3.3.11}$$

From direct computation its derivative is

$$\begin{aligned}\frac{d}{d\lambda}\left(\lambda\breve{\beta}(\lambda)\right) &= -2i\int_{-R}^{R} e^{-2i\lambda y} y\left(-\lambda\overline{q(y)}n^-_{11}(y,\lambda) + p_2(y)n^-_{21}(y,\lambda)\right) dy \\ &+ \int_{-R}^{R} e^{-2i\lambda y}\left(-\overline{q(y)}n^-_{11}(y,\lambda) + p_2(y)n^-_{21}(y,\lambda)_\lambda\right) dy \\ &+ \int_{-R}^{R} e^{-2i\lambda y}\left(-\lambda\overline{q(y)}n^-_{11}(y,\lambda)_\lambda + p_2(y)n^-_{21}(y,\lambda)_\lambda\right) dy.\end{aligned} \tag{3.3.12}$$

Inequalities (3.3.9) and (3.3.10) follow from these expressions and Lemma 3.2.3.

Now we want to show that the zeros of $\alpha(\lambda,\mu)$ are simple. For $0 \leqslant k \leqslant n-1$, consider the disc around the k^{th} root of unity of $-\gamma_i$

$$D_k := D\left(|\gamma_i|^{\frac{1}{n}} e^{i(\phi+2\pi k)/n} + \lambda_i,\ \rho|\gamma_i|^{\frac{1}{n}}\right) \tag{3.3.13}$$

where

$$\gamma_i = \frac{\mu c_{\varphi_i}\lambda_i\breve{\beta}(\lambda_i)}{g(\lambda_i)},\quad \phi = \arg\gamma_i + \pi$$

Notice that $\rho < \pi/n \Rightarrow D_k \cap D_l$ is empty for $k \neq l$. We now expand $g(\lambda)$ at $\lambda = \lambda_i$ and get

$$\alpha(\lambda, \mu) = (\lambda - \lambda_i)^n g(\lambda_i) + \mathcal{O}(\lambda - \lambda_i)^{n+1}.$$

For $\lambda \in \partial D_k$,

$$\left|(\lambda - \lambda_i)^n g(\lambda_i) + \mu c_{\varphi_i} \lambda_i \breve{\beta}(\lambda_i) - \alpha(\lambda, \mu)\right| \leqslant C_0 |\gamma_i|^{1+\frac{1}{n}}.$$

On the other hand, if we choose $\rho > 2C_0|\gamma_i|^{\frac{1}{n}}$ then for $\lambda \in \partial D_k$,

$$\begin{aligned}
|(\lambda - \lambda_i)^n g(\lambda_i) + \gamma_i g(\lambda_i)| &= |\gamma_i| \rho \left(1 + \mathcal{O}\left(\rho^2\right)\right) \\
&\geqslant C_0 |\gamma_i|^{1+\frac{1}{n}} \\
&\geqslant |(\lambda - \lambda_i)^n g(\lambda_i) + \mu c_{\varphi_i} \lambda_i \breve{\beta}(\lambda_i) - \alpha(\lambda, \mu)|.
\end{aligned}$$

Since the discs D_k are disjoint, Rouché's theorem now shows that there is exactly one zero of $\alpha(\lambda, \mu)$ in each D_k. This shows that all n zeros are simple.

(ii) After the first step of perturbation in (i), $\alpha(\lambda, \mu)$ has simple zeros $\{\Lambda_j\}_{j=1}^l$. Suppose Λ_j is a zero of $\alpha(\lambda, \mu)$ on the real axis. We make another small perturbation of the potential as above and formulate

$$\alpha(\lambda, \mu') = (\lambda - \Lambda_j) h(\lambda, \mu) + \mu' c_{\psi_j} \Lambda_j \breve{\beta}(\Lambda_j, \mu) + C_0'(\lambda - \Lambda_j)\mu' + \mathcal{O}\left(\mu'^2\right) \tag{3.3.14}$$

where

$$(\lambda - \Lambda_j) h(\lambda, \mu) = \alpha(\lambda, \mu)$$

and we define

$$D_j := D\left(\Gamma_j + \Lambda_j,\ \rho' |\Gamma_j|\right) \tag{3.3.15}$$

where

$$\Gamma_j = \frac{\mu' c_{\psi_j} \Lambda_j \breve{\beta}(\Lambda_j, \mu)}{h(\Lambda_j, \mu)}.$$

Given $\Lambda_j \in \mathbb{R}$, we can make appropriate choices of small parameter μ' and $\psi \in C_0^\infty(\mathbb{R})$ such that $\Im(\Gamma_j + \Lambda_j)$ is strictly nonzero and $D_j \cap \mathbb{R}$ is empty. Since there are only finitely many zeros, we can choose μ which works for all $j = 1, 2, ..., l$. Repeating the argument in (i) we get the desired conclusion.

□

Proposition 3.3.5 shows that there is a dense subset of $q \in H^{2,2}(\mathbb{R})$ for which $\alpha(\lambda; q)$ has at most finitely many simple zeros in $\mathbb{C}^-$ and no zeros on $\mathbb{R}$. Owing to the continuity of α in q, the fact that α is analytic in $\mathbb{C}^-$, and the continuity of the map $q \mapsto \alpha(\,\cdot\,, q)$ imply that this set is also open.

Proposition 3.3.6. *Suppose that $q_0 \in H^{2,2}(\mathbb{R})$ and that $\alpha(\,\cdot\,; q_0)$ has exactly n simple zeros in $\mathbb{C}^-$ and no zeros on $\mathbb{R}$. There is a neighborhood $\mathcal{N}$ of q_0 in $H^{2,2}(\mathbb{R})$ so that all $q \in \mathcal{N}$ have these same properties.*

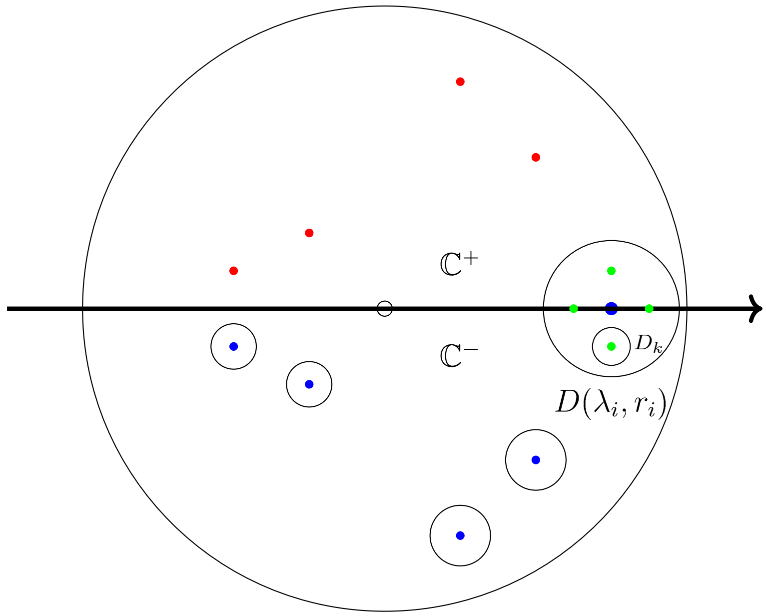


Figure 3.1: Zeros of $\breve{\alpha}$ and α in the λ plane

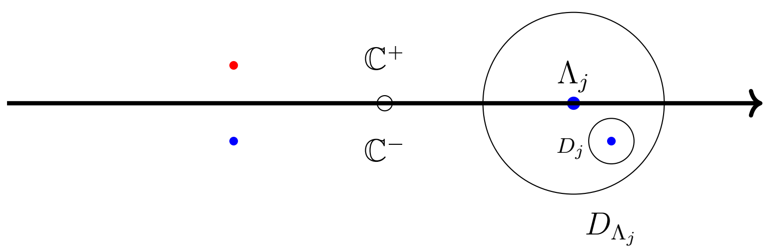


Figure 3.2: Simple zero of $\alpha(\lambda, \mu)$ on $\mathbb{R}$

Proof. Since $|\alpha(\lambda; q_0)|$ does not vanish on $\mathbb{R}$, we have $|\alpha(\lambda; q_0)| \geqslant c$ for some $c > 0$. It follows from Lipschitz continuity of $q \mapsto \alpha(\,\cdot\,; q)$ in $H^1(\mathbb{R})$ that there is an $r_0 > 0$ so that $|\alpha(\lambda; q)| \geqslant c/2$ for all q with $\|q - q_0\|_{H^{2,2}} < r_0$. Next, let $\eta_1 = \inf_{j \neq k} |\lambda_j - \lambda_k|$, $\eta_2 = \inf_k |\operatorname{Im} \lambda_k|$, and $\eta = \frac{1}{2} \inf(\eta_1, \eta_2)$. By Proposition 3.2.4 and analyticity there is an $r_1 > 0$ so that the n simple zeros of α remain simple and move a distance no more than η for $q \in H^{2,2}(\mathbb{R})$ with $\|q - q_0\|_{H^{2,2}} < r_1$. Take $\mathcal{N} = B(q_0, r)$ where $r < \inf(r_1, r_2)$. $\square$

3.4 Lipschitz Continuity of Spectral Data for Generic Potentials

Finally we prove that, for the open subset of generic $H^{2,2}$ potentials, the zeros of $\breve{\alpha}$ and the associated norming constants are continuous functions of q. We order the zeros by modulus and, given two zeros with the same modulus, order by increasing phase in $(0, \pi)$. We recall that, if $\breve{\alpha}(\lambda_k) = 0$, there is a constant b_k with the property

$$e^{-2i\lambda_k x} \begin{pmatrix} n_{11}^{-}(x,\lambda_k) \\ \lambda n_{21}^{-}(x,\lambda_k) \end{pmatrix} = b_k e^{2i\lambda_k x} \begin{pmatrix} \lambda^{-1} n_{12}^{+}(x,\lambda_k) \\ n_{22}^{+}(x,\lambda_k) \end{pmatrix} \tag{3.4.1}$$

If $\breve{\alpha}'(\lambda_k) \neq 0$, one defines the norming constant as $C_k = \frac{b_k}{\zeta_k \breve{\alpha}'(\lambda_k)}$ where $\lambda_k = (\pm\zeta_k)^2$. The discrete scattering data are composed of the pairs (λ_k, C_k).

Proposition 3.4.1. *Suppose that q_0 is a generic potential with n simple zeros of $\breve{\alpha}$ in $\mathbb{C}^+$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a listing of the zeros of $\breve{\alpha}$ with the ordering as described above, and set*

$$d_\Lambda = \min\left(\min_{1 \leqslant j \neq k \leqslant N} |\lambda_j(q_0) - \lambda_k(q_0)|,\ \min(\operatorname{Im} \lambda_k) \right).$$

There is a neighborhood $\mathcal{N}$ of q_0 so that:

(i) For any $q \in \mathcal{N}$, $\breve{\alpha}(\lambda; q)$ has exactly n simple zeros in $\mathbb{C}^+$, no zeros on $\mathbb{R}$, and $|\lambda_j(q) - \lambda_j(q_0)| \leqslant \frac{1}{2} d_\Lambda$.

(ii) The estimate $|\lambda_j(q) - \lambda_j(q_0)| \leqslant C \|q - q_0\|_{H^{2,2}}$ holds for C uniform in $q \in \mathcal{N}$.

(iii) The estimate $|b_j(q) - b_j(q_0)| \leqslant C \|q - q_0\|_{H^{2,2}}$ holds for C uniform in $q \in \mathcal{N}$.

(iv) The estimate $|C_j(q) - C_j(q_0)| \leqslant C \|q - q_0\|_{H^{2,2}}$ holds for C uniform in $q \in \mathcal{N}$.

Proof. (i) From Proposition 3.3.6 we immediately conclude that there is a neighborhood $\mathcal{N}$ of q_0 for which $q \in \mathcal{N}$ has exactly n simple zeros in $\mathbb{C}^+$ with no singularities on the real axis. We can establish continuity of the simple zeros as a function of q (and hence both the estimate $|\lambda_j(q) - \lambda_j(q_0)| \leqslant \frac{1}{2} d_\Lambda$ and assertion (ii)) by exploiting simplicity of the zeros and the implicit function theorem for Banach spaces applied to the equation $\alpha(\lambda_j(q); q) = 0$ regarding α as a function on $\mathbb{C}^- \times H^{2,2}(\mathbb{R})$. This function is analytic in $\lambda \in \mathbb{C}^-$ by Proposition 3.2.4 and differentiable in q because the functions occurring in the Wronskian formula (3.1.5) may be computed by convergent Volterra series which are analytic in q. Since $\lambda_j(q_0)$ is a simple zero, we have $\alpha'(\lambda_j(q_0), q_0) \neq 0$ which is the differential condition for the implicit function to be applicable.

(ii) The implicit function theorem also guarantees that the function $\lambda_j(q)$ will be C^1 as a function of q, and hence Lipschitz continuous. See [20].

(iii) Uniqueness for the equation (3.1.1) guarantees that at least one of $n_{12}^{+}(0, \lambda_j)$ and $n_{22}^{+}(0, \lambda_j)$ is nonzero at $q = q_0$. Suppose that $n_{12}^{+}(0, \lambda_j(q_0)) \neq 0$. By shrinking the neighborhood if needed we may assume that $n_{12}^{+}(0, \lambda_j(q)) > 0$ strictly for all $q \in \mathcal{N}$. We may then compute from (3.4.1) that $b_j(q) = \lambda_j(q) n_{21}^{-}(0, \lambda_j(q)) / n_{22}^{+}(0, \lambda_j(q))$ which, as a product and quotient of Lipschitz continuous functions of q, is itself Lipschitz continuous in q.

(iv) Finally, $\alpha'(\lambda_k)$ can easily be expressed in terms of α through a Cauchy integral over a small circle around λ_k due to the analyticity of α in $\mathbb{C}^-$, and the Lipschitz continuity of b_j and $\alpha(\lambda_j)$ in q extends to the norming constants C_j. $\square$

Chapter 4 Beals-Coifman Solutions

In this chapter we construct the Beals-Coifman solutions for (1.1.7). It follows from (1.1.6) and the discussion in the Introduction that the scattering data is given by

$$m^+(x,\zeta) = m^-(x,\zeta)e^{-ix\zeta^2\,\mathrm{ad}(\sigma)}\begin{pmatrix} a(\zeta) & \breve{b}(\zeta) \\ b(\zeta) & \breve{a}(\zeta)\end{pmatrix} \tag{4.0.1}$$

where the symmetries (1.2.5) hold. In order to elucidate properties of the scattering data we recall the integral equations for $m^\pm$.

Assuming that $q \in L^1 \cap L^2$ (so that both Q and P are L^1 matrix-valued functions), the Jost solutions $m^\pm$ are solutions of the integral equations

$$m^+(x,\zeta) = I - \int_x^\infty e^{-i(x-y)\zeta^2\,\mathrm{ad}(\sigma)}\left[(\zeta Q(y) + P(y))\, m(y,\zeta)\right] dy$$

$$m^-(x,\zeta) = I + \int_{-\infty}^x e^{-i(x-y)\zeta^2\,\mathrm{ad}(\sigma)}\left[(\zeta Q(y) + P(y))\, m(y,\zeta)\right] dy$$

with $\det m^+(x) = \det m^-(x) = 1$.

Observe that

$$\begin{pmatrix} m_{11}^+(x,\zeta) \\ m_{21}^+(x,\zeta)\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \int_x^\infty \begin{pmatrix} \zeta q m_{21}^+ + p_1 m_{11}^+ \\ e^{2i\zeta^2(x-y)}\left[-\overline{q} m_{11}^+ + p_2 m_{21}^+\right]\end{pmatrix} dy \tag{4.0.2}$$

$$\begin{pmatrix} m_{12}^+(x,\zeta) \\ m_{22}^+(x,\zeta)\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_x^\infty \begin{pmatrix} e^{-2i\zeta^2(x-y)}\left[\zeta q m_{22+} + p_1 m_{12}^+\right] \\ \zeta q m_{12}^+ + p_2 m_{22}^+\end{pmatrix} dy \tag{4.0.3}$$

$$\begin{pmatrix} m_{11}^-(x,\zeta) \\ m_{21}^-(x,\zeta)\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^x \begin{pmatrix} \zeta q m_{21}^- + p_1 m_{11}^- \\ e^{2i\zeta^2(x-y)}\left[-\overline{q} m_{11}^- + p_2 m_{21}^-\right]\end{pmatrix} dy \tag{4.0.4}$$

$$\begin{pmatrix} m_{12}^-(x,\zeta) \\ m_{22}^-(x,\zeta)\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^x \begin{pmatrix} e^{-2i\zeta^2(x-y)}\left[\zeta q m_{22}^- + p_1 m_{12}^-\right] \\ \zeta q m_{12}^- + p_2 m_{22}^-\end{pmatrix} dy \tag{4.0.5}$$

Write $m_{(1)}^+ = (m_{11}^+, m_{21}^+)^T$, $m_{(2)}^+ = (m_{12}^+, m_{22}^+)^T$, and similarly for $m_{(1)}^-$ and $m_{(2)}^-$. Using the fact that $\det m^+ = \det m^- = 1$, it is easy to deduce that

$$\breve{a}(\zeta) = \begin{vmatrix} m_{11}^- & m_{12}^+ \\ m_{21}^- & m_{22}^+ \end{vmatrix} = W\left(m_{(1)}^-, m_{(2)}^+\right) \tag{4.0.6}$$

$$a(\zeta) = \begin{vmatrix} m_{11}^+ & m_{12}^- \\ m_{21}^+ & m_{22}^- \end{vmatrix} = W\left(m_{(1)}^+, m_{(2)}^-\right) \tag{4.0.7}$$

It follows from (4.0.1), the first line of (4.0.2), and the second line of (4.0.3) that

$$a(\zeta) = 1 - \int_{-\infty}^{\infty} \left(\zeta q m_{21}^{+} + p_1 m_{11}^{+}\right) dy, \tag{4.0.8}$$

$$\breve{a}(\zeta) = 1 - \int_{-\infty}^{\infty} \left(\zeta q m_{12}^{+} + p_2 m_{22}^{+}\right) dy. \tag{4.0.9}$$

(4.0.10)

Using (1.1.9), (4.0.1), the first line of (4.0.4), and the second line of (4.0.5), we also have

$$a(\zeta) = 1 + \int_{-\infty}^{\infty} \left(\zeta q m_{12}^{-} + p_1 m_{22}^{-}\right) dy \tag{4.0.11}$$

$$\breve{a}(\zeta) = 1 + \int_{-\infty}^{\infty} \left(\zeta q m_{21}^{-} + p_1 m_{11}^{-}\right) dy \tag{4.0.12}$$

From Lemma 3.2.1 and change of variable formula (1.2.7) we have

- $m_{(1)}^{+}$ has a bounded analytic continuation to $\mathrm{Im}(\zeta^2) < 0$
- $m_{(2)}^{+}$ has a bounded analytic continuation to $\mathrm{Im}(\zeta^2) > 0$
- $m_{(1)}^{-}$ has a bounded analytic continuation to $\mathrm{Im}(\zeta^2) > 0$
- $m_{(2)}^{-}$ has a bounded analytic continuation to $\mathrm{Im}(\zeta^2) < 0$

It follows from these observations, (4.0.6), and (4.0.7) that

- $a(\zeta)$ has an analytic extension to $\mathrm{Im}(\zeta^2) < 0$
- $\breve{a}(\zeta)$ has an analytic extension to $\mathrm{Im}(\zeta^2) > 0$

To construct the Beals-Coifman solutions, we will need the asymptotic behavior of $m_{(1)}^{\pm}$ as $x \to \mp\infty$ and $m_{(2)}^{\pm}$ as $x \to \mp\infty$. An argument with the dominated convergence theorem, exploiting the decay of the exponential $\exp(\pm i(x-y)\zeta^2 \,\mathrm{ad}(\sigma))$, shows that

$$\begin{aligned}
\lim_{x\to-\infty} m_{21}^{+}(x,\zeta) &= 0, \quad \mathrm{Im}(\zeta^2) < 0 \\
\lim_{x\to-\infty} m_{12}^{+}(x,\zeta) &= 0, \quad \mathrm{Im}(\zeta^2) > 0 \\
\lim_{x\to+\infty} m_{12}^{-}(x,\zeta) &= 0, \quad \mathrm{Im}(\zeta^2) < 0 \\
\lim_{x\to+\infty} m_{21}^{-}(x,\zeta) &= 0, \quad \mathrm{Im}(\zeta^2) > 0
\end{aligned}$$

It now follows from these relations, the integral equations (4.0.2), (4.0.3), (4.0.4), (4.0.5), and the integral identities (4.0.8), (4.0.9), (4.0.11), and (4.0.12) that

$$\lim_{x\to-\infty} m_{(1)}^{+}(x,\zeta) = \begin{pmatrix} a(\zeta) \\ 0 \end{pmatrix} \quad \mathrm{Im}(\zeta^2) < 0 \tag{4.0.13}$$

$$\lim_{x\to-\infty} m^{+}_{(2)}(x,\zeta) = \begin{pmatrix} 0 \\ \breve{a}(\zeta) \end{pmatrix} \quad \operatorname{Im}(\zeta^2) > 0 \tag{4.0.14}$$

$$\lim_{x\to+\infty} m^{-}_{(1)}(x,\zeta) = \begin{pmatrix} \breve{a}(\zeta) \\ 0 \end{pmatrix} \quad \operatorname{Im}(\zeta^2) > 0 \tag{4.0.15}$$

$$\lim_{x\to+\infty} m^{-}_{(2)}(x,\zeta) = \begin{pmatrix} 0 \\ a(\zeta) \end{pmatrix} \quad \operatorname{Im}(\zeta^2) < 0. \tag{4.0.16}$$

4.1 Construction of Beals-Coifman Solutions

We now define the right-hand Beals-Coifman solutions by

$$M_r(x,\zeta) = \begin{cases} \left(\dfrac{m^{-}_{(1)}(x,\zeta)}{\breve{a}(\zeta)},\, m^{+}_{(2)}(x,\zeta) \right), & \operatorname{Im}(\zeta^2) > 0, \\[2ex] \left(m^{+}_{(1)}(x,\zeta),\, \dfrac{m^{-}_{(2)}(x,\zeta)}{a(\zeta)} \right), & \operatorname{Im}(\zeta^2) < 0. \end{cases}$$

These solutions are piecewise analytic, and bounded as $x \to -\infty$ by the boundedness of the normalized Jost solutions and the functions $a(\zeta)$ and $\breve{a}(\zeta)$ (so long as $a(\zeta)$ and $\breve{a}(\zeta)$ have no zeros). By (4.0.15) and (4.0.14), they are normalized so that

$$\lim_{x\to\infty} M_r(x,\zeta) = I, \quad \operatorname{Im}\zeta^2 \neq 0,$$

and are bounded as $x \to -\infty$.

Similarly, the left-hand Beals-Coifman solutions, given by

$$M_\ell(x,\zeta) = \begin{cases} \left(m^{-}_{(1)}(x,\zeta),\, \dfrac{m^{+}_{(2)}(x,\zeta)}{\breve{a}(\zeta)} \right), & \operatorname{Im}(\zeta^2) > 0, \\[2ex] \left(\dfrac{m^{+}_{(1)}(x,\zeta)}{a(\zeta)},\, m^{-}_{(2)}(x,\zeta) \right), & \operatorname{Im}(\zeta^2) < 0 \end{cases}$$

are piecewise analytic, bounded as $x \to +\infty$, and normalized so that

$$\lim_{x\to-\infty} M_\ell(x,\zeta) = I, \quad \operatorname{Im}\zeta^2 \neq 0.$$

Both M_r and M_ℓ have boundary values as $\pm\Im\zeta^2 \downarrow 0$. We denote these respectively by $M_r^{\pm}$ and $M_\ell^{\pm}$. We now compute the jump relations for these boundary values. In what follows, we write

$$f(x) \underset{x\to\pm\infty}{\sim} g(x)$$

if $\lim_{x\to\pm\infty} |f(x) - g(x)| = 0$.

From (4.0.1) it is easy to see that, for $\Im\zeta^2 = 0$,

$$m^+_{(1)}(x,\zeta) \underset{x\to-\infty}{\sim} \begin{pmatrix} a(\zeta) \\ e^{2ix\zeta^2} b(\zeta) \end{pmatrix}$$

$$m^+_{(2)}(x,\zeta) \underset{x\to-\infty}{\sim} \begin{pmatrix} \breve{b}(\zeta) \\ e^{2ix\zeta^2} \breve{a}(\zeta) \end{pmatrix}$$

$$m^-_{(1)}(x,\zeta) \underset{x\to+\infty}{\sim} \begin{pmatrix} \breve{a}(\zeta) \\ -e^{2ix\zeta^2} b(\zeta) \end{pmatrix}$$

$$m^-_{(2)}(x,\zeta) \underset{x\to+\infty}{\sim} \begin{pmatrix} a(\zeta) \\ -e^{2ix\zeta^2} \breve{b}(\zeta) \end{pmatrix}$$

It follows from these relations that

$$M^+_\ell(x,\zeta) \underset{x\to+\infty}{\sim} e^{-ix\zeta^2\,\mathrm{ad}(\sigma)} \begin{pmatrix} \breve{a}(\zeta) & 0 \\ -b(\zeta) & \dfrac{1}{\breve{a}(\zeta)} \end{pmatrix} \tag{4.1.1}$$

$$M^-_\ell(x,\zeta) \underset{x\to+\infty}{\sim} e^{-ix\zeta^2\,\mathrm{ad}(\sigma)} \begin{pmatrix} \dfrac{1}{a(\zeta)} & -\breve{b}(\zeta) \\ 0 & a(\zeta) \end{pmatrix} \tag{4.1.2}$$

and

$$M^+_r(x,\zeta) \underset{x\to-\infty}{\sim} e^{-ix\zeta^2\,\mathrm{ad}(\sigma)} \begin{pmatrix} \dfrac{1}{\breve{a}(\zeta)} & \breve{b}(\zeta) \\ 0 & \breve{a}(\zeta) \end{pmatrix} \tag{4.1.3}$$

$$M^-_r(x,\zeta) \underset{x\to-\infty}{\sim} e^{-ix\zeta^2\,\mathrm{ad}(\sigma)} \begin{pmatrix} a(\zeta) & 0 \\ b(\zeta) & \dfrac{1}{a(\zeta)} \end{pmatrix} \tag{4.1.4}$$

From (4.1.1) and (4.1.2), we compute that

$$M^+_\ell(x,\zeta) = M^-_\ell(x,\zeta) e^{-ix\zeta^2\,\mathrm{ad}(\sigma)} v_\ell(\zeta), \quad v_\ell(\zeta) = \begin{pmatrix} 1 & \dfrac{\breve{b}(\zeta)}{\breve{a}(\zeta)} \\ -\dfrac{b(\zeta)}{a(\zeta)} & 1 - \dfrac{\breve{b}(\zeta) b(\zeta)}{\breve{a}(\zeta) a(\zeta)} \end{pmatrix} \tag{4.1.5}$$

while, from (4.1.3) and (4.1.4), we see that

$$M_r^+(x,\zeta) = M_r^-(x,\zeta)e^{-ix\zeta^2\,\mathrm{ad}(\sigma)}v_r(\zeta), \quad v_r(\zeta) = \begin{pmatrix} 1 - \dfrac{\breve{b}(\zeta)b(\zeta)}{a(\zeta)\breve{a}(\zeta)} & \dfrac{\breve{b}(\zeta)}{a(\zeta)} \\ \\ -\dfrac{b(\zeta)}{\breve{a}(\zeta)} & 1 \end{pmatrix} \tag{4.1.6}$$

4.1.1 Residue conditions

The functions $\breve{a}(\zeta)$ and $a(\zeta)$ admit analytic continuations respectively to Ω^+ and Ω^- where

$$\Omega^\pm = \{z \in \mathbb{C} : \pm \mathrm{Im}\ z^2 > 0\}.$$

The contour Σ bounds the regions $\Omega^\pm$. We denote by Ω^{++} the first quadrant

$$\Omega^{++} = \{\zeta \in \mathbb{C} : \mathrm{Re}\ \zeta > 0,\ \mathrm{Im}\ \zeta > 0\}$$

and by $\mathbb{C}^\times$ the set $\{z \in \mathbb{C} : z \neq 0\}$.

We have shown in the direct scattering problem that there is an open and dense subset U of $H^{2,2}(\mathbb{R})$ with the following properties: $a(z)$ has only finitely many simple zeros in $\mathbb{C}^-$, and $a(z)$ has no zeros on the real line. We will assume that the initial condition is in this subset U.

Due to symmetries, $\breve{a}(\zeta)$ has a finite number of simple zeros $\zeta_i \in \Omega^{++}$, $i = 1, ...n$:

$$\breve{a}(\zeta_i) = 0, \quad \breve{a}'(\zeta_i) \neq 0.$$

$\breve{a}$ also has zeros at $-\zeta_1, ..., -\zeta_n$, while a has zeros at $(\pm\overline{\zeta_1}, \ldots \pm \overline{\zeta_n})$. We denote by $\mathcal{Z}$ the set

$$\{\pm\zeta_1, \ldots, \pm\zeta_n, \pm\overline{\zeta_1}, \ldots, \pm\overline{\zeta_n})\}$$

and

$$\mathcal{Z}_\pm = \mathcal{Z} \cap \Omega^\pm.$$

For each $\zeta_i \in \Omega^{++}$ there are four resonances: $\zeta_i, -\zeta_i \in \Omega^+$ and $\overline{\zeta}_i, -\overline{\zeta}_i \in \Omega^-$. We assume that $a(\zeta)$ and $\breve{a}(\zeta)$ do not vanish on Σ, ruling out algebraic solitons.

At $\zeta = \pm\zeta_i$, using the symmetry given in (1.2.3) we have the linear dependence relation

$$\begin{bmatrix} m_{11}^-(x, \pm\zeta_i) \\ m_{21}^-(x, \pm\zeta_i) \end{bmatrix} = \pm b_i \begin{bmatrix} m_{12}^+(x, \pm\zeta_i) \\ m_{22}^+(x, \pm\zeta_i) \end{bmatrix} e^{2i\zeta_i x} \tag{4.1.7}$$

Similarly, at $\zeta = \overline{\zeta}_i$,

$$\begin{bmatrix} m_{12}^-(x, \pm\overline{\zeta}_i) \\ m_{22}^-(x, \pm\overline{\zeta}_i) \end{bmatrix} = \mp \overline{b}_i \begin{bmatrix} m_{11}^+(x, \pm\overline{\zeta}_i) \\ m_{21}^+(x, \pm\overline{\zeta}_i) \end{bmatrix} e^{-2i\overline{\zeta}_i^{\,2} x} \tag{4.1.8}$$

Finally, we define the norming constants c_i, for $k = 1, ...n$

$$c_i = c_{\pm\zeta_i} = \frac{b_i}{\breve{a}'(\zeta_i)}. \tag{4.1.9}$$

Due to symmetry reduction, $c_{\pm\overline{\zeta_i}} = -\overline{c}_i$.

Fix index i, for $\zeta \in \{\pm\zeta_i, \pm\overline{\zeta}_i\}$,

$$\mathrm{Res}_{z=\zeta} M_r(x,z) = \lim_{z\to\zeta} M_r(x,z)V_x(\zeta)$$

with $V_x(\zeta)$ given as follows:

$$V_x(\pm\zeta_i) = \begin{pmatrix} 0 & 0 \\ c_i e^{-2ix\zeta^2} & 0 \end{pmatrix}, \quad V_x(\pm\overline{\zeta_i}) = \begin{pmatrix} 0 & -\overline{c}_i e^{2ix\overline{\zeta}^2} \\ 0 & 0 \end{pmatrix}.$$

4.2 The Riemann-Hilbert Problems in the λ Variables

We can recast the left and right RHP's (4.1.5) and (4.1.6) in terms of the dependent variables $N^\pm$ and the spectral variable $z = \zeta^2$. The new RHP is an RHP with contour $\mathbb{R}$. Applying the automorphism

$$\varphi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \zeta^{-1}b \\ \zeta c & d \end{pmatrix}$$

to the Jost and Beals-Coifman solutions and exploiting the odd symmetry of off-diagonal components, and even symmetry of diagonal components, with respect to the reflection $\zeta \mapsto -\zeta$, we may define first

$$n^\pm(x,\lambda) = \varphi(m^\pm(x,\zeta)), \quad \lambda = \zeta^2$$

and then

$$N_*(x,\lambda) = \varphi(N_*(x,\zeta^2)), \quad \lambda = \zeta^2$$

where $* = \ell, r$. To get the correct normalization as $x \to \infty$, in the remaining part of the dissertation we work with the first row of $\varphi(M_*(x,\zeta))$. More explicitly, for $* = r$ we define

$$N_r^+(x,\lambda) = \left(\frac{n_{11}^-(x,\lambda)}{\breve{\alpha}(\lambda)}, n_{12}^+(x,\lambda)\right) \tag{4.2.1}$$

$$N_r^-(x,\lambda) = \left(n_{11}^+(x,\lambda), \frac{n_{12}^-(x,\lambda)}{\alpha(\lambda)}\right) \tag{4.2.2}$$

Set

$$\alpha(\lambda) = a(\zeta^2), \quad \beta(\lambda) = \zeta^{-1}\breve{b}(\zeta), \quad \breve{\alpha}(\lambda) = \breve{a}(\zeta^2), \quad \breve{\beta}(\lambda) = \zeta^{-1}b(\zeta).$$

From the symmetries (1.2.5), it follows that

$$\breve{\alpha}(\lambda) = \overline{\alpha(\lambda)}, \quad \breve{\beta}(\lambda) = \overline{\beta(\lambda)}. \tag{4.2.3}$$

We can now compute jump relations for the pairs (N_ℓ^+, N_ℓ^-) and (N_r^+, N_r^-). It follows from (4.1.5), (4.1.6), and the definitions above that

$$N_\ell^+(x,\lambda) = N_\ell^-(x,\lambda)e^{-i\lambda x \,\mathrm{ad}\,\sigma} \begin{pmatrix} 1 & \dfrac{\beta(\lambda)}{\breve{\alpha}(\lambda)} \\ -\lambda\dfrac{\breve{\beta}(\lambda)}{\alpha(\lambda)} & 1 - \lambda\dfrac{\beta(\lambda)\breve{\beta}(\lambda)}{\breve{\alpha}(\lambda)\alpha(\lambda)} \end{pmatrix} \tag{4.2.4}$$

$$N_r^+(x,\lambda) = N_r^-(x,\lambda)e^{-i\lambda x \,\mathrm{ad}\,\sigma}\begin{pmatrix} 1-\lambda\dfrac{\beta(\lambda)\breve{\beta}(\lambda)}{\breve{\alpha}(\lambda)\alpha(\lambda)} & \dfrac{\beta(\lambda)}{\alpha(\lambda)} \\ & \\ -\lambda\dfrac{\breve{\beta}(\lambda)}{\breve{\alpha}(\lambda)} & 1 \end{pmatrix} \tag{4.2.5}$$

Setting $\tilde{\rho} = \beta/\breve{\alpha}$ and $\rho = \beta/\alpha$ respectively in (4.2.4) and (4.2.5) and using the symmetries (4.2.3), we conclude that

$$N_\ell^+(x,\lambda) = N_\ell^-(x,\lambda)e^{-ix\lambda \,\mathrm{ad}\,\sigma}\begin{pmatrix} 1 & \tilde{\rho}(\lambda) \\ & \\ \lambda\overline{\tilde{\rho}(\lambda)} & 1+\lambda|\breve{\rho}(\lambda)|^2 \end{pmatrix} \tag{4.2.6}$$

$$N_r^+(x,\lambda) = N_r^-(x,\lambda)e^{-ix\lambda \,\mathrm{ad}\,\sigma}\begin{pmatrix} 1+\lambda|\rho(\lambda)|^2 & \rho(\lambda) \\ & \\ \lambda\overline{\rho(\lambda)} & 1 \end{pmatrix} \tag{4.2.7}$$

For $N = N_r$ and $\lambda = \zeta^2$ where $\zeta \in \{\pm\zeta_i, \pm\overline{\zeta}_i\}$,

$$\mathrm{Res}_{z=\lambda} N_r(x,z) = \lim_{z\to\lambda} N_r(x,z)J_x^r(z)$$

with $J_x^r(\lambda)$ given as follows.

$$J_x^r(\lambda) = \begin{pmatrix} 0 & 0 \\ C_i\lambda_i e^{2i\lambda_i x} & 0 \end{pmatrix}, \qquad J_x^r(\overline{\lambda}) = \begin{pmatrix} 0 & -\overline{C_i}e^{-2i\overline{\lambda_i}x} \\ 0 & 0 \end{pmatrix}.$$

where $C_\lambda = 2c_\zeta$.

Similarly,

$$\mathrm{Res}_{z=\lambda} N_l(x,z) = \lim_{z\to\lambda} N_l(x,z)J_x^l(z)$$

with $J_x^l(\lambda)$ given as follows:

$$J_x^l(\lambda) = \begin{pmatrix} 0 & \tilde{C}_i\lambda_i^{-1}e^{-2i\lambda_i x} \\ 0 & 0 \end{pmatrix}, \qquad J_x^r(\overline{\lambda}) = \begin{pmatrix} 0 & 0 \\ -\overline{\tilde{C}_i}e^{2i\overline{\lambda_i}x} & 0 \end{pmatrix}.$$

where $\tilde{C}_\lambda$ is constructed in (6.3.12). We will build the left RHP through conjugation in Section 6.3.

Chapter 5 Two Riemann-Hilbert Problems and their equivalence

In this chapter we study the RHPs that defines the inverse scattering map. We only discuss the 'right' RHP problem since the discussion for the 'left' RHP is similar.

We begin by formulating precisely the RHPs $(\mathbb{R}, J)$ (see Problem 5.1.1) and (Σ, v) (see Problem 5.1.4). Next, we prove that these two problems are equivalent through change of variable . We then prove that the RHP (Σ, v) has a unique solution. We use these facts to show that the Beals-Coifman integral equation associated to the RHP for $(\mathbb{R}, J)$ has a unique solution provided that $\rho \in H^{2,2}(\mathbb{R})$. Finally, we show that the solution $M_\pm$ of Problem 5.1.4 obeys (1.1.7) as a function of x, and obtain reconstruction formulas for $Q(x)$ and $P(x)$ in terms of the solution μ of the Beals-Coifman integral equation for Problem 5.1.4. Using the equivalence of Problems 5.1.1 and 5.1.4, we obtain the reconstruction formula (1.2.20) that will be used in the next section to analyze the inverse scattering map.

In this chapter, for each pole $\lambda_i \in \mathbb{C}^+$, let Γ_i be a circle centered at λ_i of sufficiently small radius to be lie in the open upper half-plane and to be disjoint from all other circles. By doing so we replace the residue condition of the Riemann-Hilbert problem with Schwarz invariant jump conditions across closed contours. The equivalence of this new RHP on augmented contours with the original one is a well-established result (see [26] Sec 6). The purpose of this replacement is to make use of

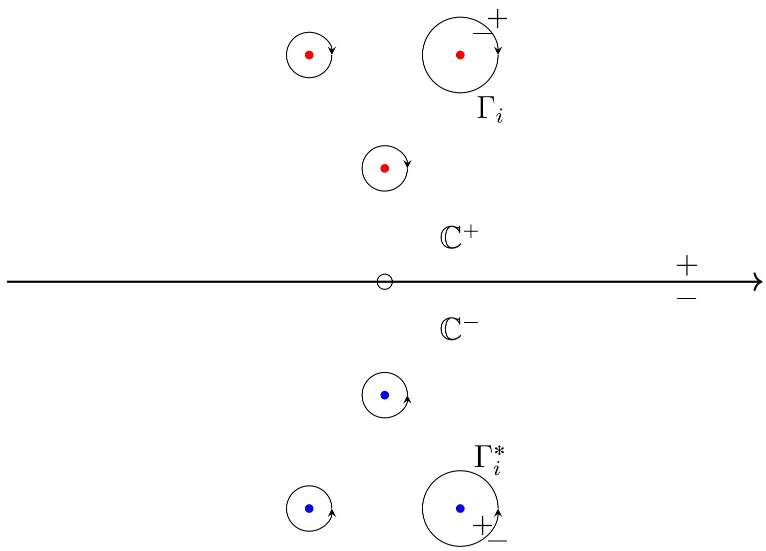
1. Vanishing lemma of homogeneous RHPs from [26] Theorem 9.3.

2. The Plemelj formula (2.3.3) over closed contours.

5.1 Two RHPs

We formulate precisely the RHPs on two types of contours Λ and Σ' . In the next section we prove their equivalence.

Figure 5.1: The Augmented Contour Λ



Problem 5.1.1. Fix $x \in \mathbb{R}$ and let $(\rho, \{C_i, \lambda_i\}_{i=1}^n)$ be a set of scattering data such that $\rho \in H^{2,2}(\mathbb{R})$ and $\lambda_i \in \mathbb{C}^+$, $C_i \in \mathbb{C}_\times$. Find a vector-valued function $\mathbf{N}(x, \cdot)$ with the following properties:

(i) (Analyticity) $\mathbf{N}(x, z)$ is a row vector-valued analytic function of z for $z \in \mathbb{C} \backslash \Lambda$ where

$$\Lambda = \mathbb{R} \bigcup \{\Gamma_1, ..., \Gamma_n, \Gamma_1^*, ..., \Gamma_n^*\}$$

(ii) (Normalization) There are two types of normalization. Either,

A. $\mathbf{N}(x, z) = (1, 0) + \mathcal{O}(z^{-1})$ as $z \to \infty$, or

B. $\mathbf{N}(x, z) = \mathcal{O}(z^{-1})$ as $z \to \infty$.

(iii) (Jump condition) For each $\lambda \in \Lambda$, $\mathbf{N}$ has continuous boundary values $\mathbf{N}_\pm(\lambda)$ as $z \to \lambda$ from $\mathbb{C}^\pm$. Moreover, the jump relation

$$\mathbf{N}_+(x, \lambda) = \mathbf{N}_-(x, \lambda) J_x(\lambda)$$

holds, where for $\lambda \in \mathbb{R}$

$$J_x(\lambda) = e^{-i\lambda x \operatorname{ad} \sigma} \begin{pmatrix} 1 + \lambda |\rho(\lambda)|^2 & \rho(\lambda) \\ \lambda \overline{\rho(\lambda)} & 1 \end{pmatrix}$$

and for $\lambda \in \Gamma_i \cup \Gamma_i^*$

$$J_x(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \dfrac{C_i\,\lambda_i e^{2i\lambda x}}{\lambda - \lambda_i} & 1 \end{pmatrix} & \lambda \in \Gamma_i, \\ \\ \begin{pmatrix} 1 & \dfrac{\overline{C_i}\,e^{-2ix\lambda}}{\lambda - \overline{\lambda_i}} \\ 0 & 1 \end{pmatrix} & \lambda \in \Gamma_i^* \end{cases}$$

Remark 5.1.2. Problem 5.1.1 has two types of normalization at infinity. Type A which has an inhomogeneous boundary condition at infinity is needed for the reconstruction of the potential q. Type B, the homogeneous one is suitable for proving the existence and uniqueness of the solution.

Now we derive the Beals-Coifman integral equation for Problem 5.1.1. The unique solvability of Problem 5.1.1 is equivalent to unique solvability of this integral equation.

We set

$$\nu = \mathbf{N}^+(1 + W_x^+)^{-1} = \mathbf{N}^-(1 - W_x^-)^{-1} \tag{5.1.1}$$

where

$$W_x^+ = \begin{pmatrix} 0 & 0 \\ \lambda\overline{\rho(\lambda)}e^{2i\lambda x} & 0 \end{pmatrix}, \quad W_x^- = \begin{pmatrix} 0 & \rho(\lambda)e^{-2i\lambda x} \\ 0 & 0 \end{pmatrix} \quad \lambda \in \mathbb{R}$$

$$W_x^+ = \begin{pmatrix} 0 & 0 \\ \dfrac{C_i\,\lambda_i e^{2i\lambda x}}{\lambda - \lambda_i} & 0 \end{pmatrix}, \quad W_x^- = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda \in \Gamma_i$$

$$W_x^+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad W_x^- = \begin{pmatrix} 0 & \dfrac{\overline{C_i}\,e^{-2ix\lambda}}{\lambda - \overline{\lambda_i}} \\ 0 & 0 \end{pmatrix} \quad \lambda \in \Gamma_i^*$$

Using (4.2.1)-(4.2.2) we write down ν explicitly: for $\lambda \in \mathbb{R}$

$$\nu(x,\lambda) = \begin{cases} \begin{pmatrix} \dfrac{n_{11}^-(x,\lambda)}{\breve{a}(\lambda)} & n_{12}^+(x,\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{2i\lambda x}\lambda\overline{\rho(\lambda)} & 1 \end{pmatrix} \\ \\ \begin{pmatrix} n_{11}^+(x,\lambda) & \dfrac{n_{12}^-(x,\lambda)}{\alpha(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & \rho(\lambda)e^{-2i\lambda x} \\ 0 & 1 \end{pmatrix} \end{cases}$$

for $\lambda \in \Gamma_i$

$$\nu(x,\lambda) = \begin{cases} \begin{pmatrix} \dfrac{n_{11}^{-}(x,\lambda)}{\breve{a}(\lambda)} & n_{12}^{+}(x,\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{C_i\,\lambda_i e^{2i\lambda x}}{\lambda-\lambda_i} & 1 \end{pmatrix} \\ \begin{pmatrix} \dfrac{n_{11}^{-}(x,\lambda)}{\breve{a}(\lambda)} - \dfrac{C_i\,\lambda_i e^{2i\lambda x} n_{12}^{+}(x,\lambda)}{\lambda-\lambda_i} & n_{12}^{+}(x,\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

and for $\lambda \in \Gamma_i^*$

$$\nu(x,\lambda) = \begin{cases} \begin{pmatrix} n_{11}^{+}(x,\lambda) & \dfrac{n_{12}^{-}(x,\lambda)}{\alpha(\lambda)} + \dfrac{\overline{C_i}\, e^{-2ix\lambda} n_{11}^{+}(x,\lambda)}{\lambda-\overline{\lambda_i}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} n_{11}^{+}(x,\lambda) & \dfrac{n_{12}^{-}(x,\lambda)}{\alpha(\lambda)} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\overline{C_i}\, e^{-2ix\lambda}}{\lambda-\overline{\lambda_i}} \\ 0 & 1 \end{pmatrix} \end{cases}$$

From (5.1.1) we have

$$\mathbf{N}^{+} - \mathbf{N}^{-} = \nu\left(W_x^{+} + W_x^{-}\right).$$

The Plemelj formula (2.3.3) and Type A normalization together give the following Beals-Coifman integral equation:

$$\nu = (1,0) + \mathcal{C}_W \nu = (1,0) + C_\Lambda^{+}(\nu W_x^{-}) + C_\Lambda^{-}(\nu W_x^{+}) \tag{5.1.2}$$

Similarly, for Type B normalization we have that

$$\nu = \mathcal{C}_W \nu = C_\Lambda^{+}(\nu W_x^{-}) + C_\Lambda^{-}(\nu W_x^{+}) \tag{5.1.3}$$

For $\lambda \in \mathbb{R}$ and Type A normalization we have

$$\nu_{11}(x,\lambda) = 1 + \int_{-\infty}^{\infty} \frac{\nu_{12}(x,s) s\overline{\rho(s)} e^{2isx}}{s-\lambda+i0} \frac{ds}{2\pi i} + \sum_{i=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_i} \frac{\nu_{12}(x,s) C_i\,\lambda_i e^{2isx}}{(s-\lambda)(s-\lambda_i)} ds \tag{5.1.4}$$

and

$$\nu_{12}(x,\lambda) = \int_{-\infty}^{\infty} \frac{\nu_{11}(x,s)\rho(s) e^{-2isx}}{s-\lambda-i0} \frac{ds}{2\pi i} + \sum_{i=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_i^*} \frac{\nu_{11}(x,s)\overline{C_i}\, e^{-2isx}}{(s-\lambda)(s-\overline{\lambda_i})} ds \tag{5.1.5}$$

An application of Cauchy's integral formula on (5.1.4) and (5.1.5) gives the following integro-algebraic equations:

$$\nu_{11}(x,\lambda) = 1 + \int_{-\infty}^{\infty} \frac{\nu_{12}(x,s) s\overline{\rho(s)} e^{2isx}}{s-\lambda+i0} \frac{ds}{2\pi i} + \sum_{i=1}^{n} \frac{\nu_{12}(x,\lambda_i) C_i\,\lambda_i e^{2i\lambda_i x}}{\lambda-\lambda_i} \tag{5.1.6}$$

and

$$\nu_{12}(x,\lambda) = \int_{-\infty}^{\infty} \frac{\nu_{11}(x,s)\rho(s)e^{-2isx}}{s-\lambda-i0}\frac{ds}{2\pi i} - \sum_{i=1}^{n} \frac{\nu_{11}(x,\overline{\lambda_i})\overline{C_i}\, e^{-2i\overline{\lambda_i}x}}{\lambda-\overline{\lambda_i}}. \tag{5.1.7}$$

To close the system above, we evaluate (5.1.4) and (5.1.5) at the eigenvalues to get

$$\nu_i^+ = \nu_{12}(x,\lambda_i) = \int_{-\infty}^{\infty} \frac{\nu_{11}(x,s)\rho(s)e^{-2isx}}{s-\lambda_i}\frac{ds}{2\pi i} - \sum_{k=1}^{n} \frac{\nu_{11}(x,\overline{\lambda_k})\overline{C_k}e^{-2i\overline{\lambda_k}x}}{\lambda_i-\overline{\lambda_k}} \tag{5.1.8}$$

$$\nu_i^- = \nu_{11}(x,\overline{\lambda_i}) = 1 + \int_{-\infty}^{\infty} \frac{\nu_{12}(x,s)s\overline{\rho(s)}e^{2isx}}{s-\overline{\lambda_i}}\frac{ds}{2\pi i} + \sum_{k=1}^{n} \frac{\nu_{12}(x,\lambda_k)C_k\lambda_k e^{2i\lambda_k x}}{\overline{\lambda_i}-\lambda_k} \tag{5.1.9}$$

To write down the integral equation (5.1.3) explicitly, we just remove the "1" term from the RHS of equation (5.1.4) and equation (5.1.9).

The solution to Problem 5.1.1 with Type A normalization, in terms of

$$\nu(x,s) = (\nu_{11},\nu_{12})$$

should then be

$$\mathbf{N}(x,z) = (1,0) + \frac{1}{2\pi i}\int_{\Gamma} \frac{\nu(x,s)(W_x^+(s)+W_x^-(s))}{s-z}\, ds \tag{5.1.10}$$

$$= (1,0) + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\nu(x,s)(W_x^+(s)+W_x^-(s))}{s-z}\, ds \tag{5.1.11}$$

$$+ \left(\sum_{k=1}^{n} \frac{\nu_{12}(x,\lambda_k)C_k\lambda_k e^{2i\lambda_k x}}{z-\lambda_k}, \sum_{k=1}^{n} \frac{-\nu_{11}(x,\overline{\lambda_k})\overline{C_k}e^{-2i\overline{\lambda_k}x}}{z-\overline{\lambda_k}} \right)$$

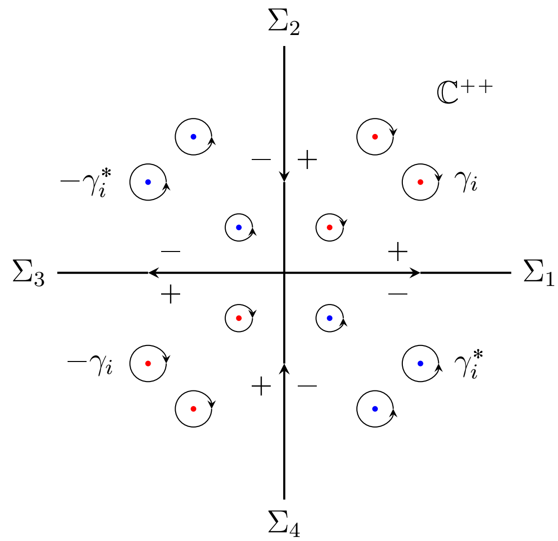
Using (1.2.20) we obtain

$$q(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \nu_{11}(x,s)\rho(s)e^{-2isx}ds - \sum_{k=1}^{n} 2i\nu_{11}(x,\overline{\lambda_k})\overline{C_k}e^{-2i\overline{\lambda_k}x} \tag{5.1.12}$$

Remark 5.1.3. Note that (5.1.10) which is a direct consequence of the Beals-Coifman integral equation involves the value of ν on $\mathbb{R}$ and $\{\nu_{11}\restriction\Gamma_i^*, \nu_{12}\restriction\Gamma_i\}$ while for (5.1.11) we only need the value of ν on $\mathbb{R}$ and $\{\nu_{11}(\overline{\lambda}_i), \nu_{12}(\lambda_i)\}_{i=1}^n$. We use the second form to re construct the potential and study the Lipschitz continuity of the inverse scattering map. We mention that the Beals-Coifman integral equation (5.1.2) can be derived from the integro-algebraic equations (5.1.6) and (5.1.7). To do this, we extend (5.1.6) and (5.1.7) to Γ_i^* and Γ_i respectively to obtain $\{\nu_{11}\restriction\Gamma_i^*, \nu_{12}\restriction\Gamma_i\}$.

In the next section we study the existence and uniqueness of the solution to Problem 5.1.1. To make use of the symmetry relations of the jump conditions, we need the following Riemann-Hilbert problem with jump contour Σ' and its equivalence with Problem 5.1.1

Figure 5.2: The Augmented Contour Σ'



Problem 5.1.4. Given functions $r(\zeta)$, $\breve{r}(\zeta)$, $\{\zeta_i\}_{i=1}^N \subset \mathbb{C}^{++}$, $\{c_i\}_{i=1}^N \subset \mathbb{C}^\times$, where

$$\zeta = \sqrt{\lambda}, \quad \zeta_i = \sqrt{\lambda_i}, \quad c_i = \frac{1}{2}C_i, \quad r(\zeta) = \zeta\rho(\zeta^2), \quad \breve{r}(\zeta) = -\zeta\overline{\rho(\zeta^2)} \tag{5.1.13}$$

and the Schwarz-invariant contour Σ' given by Figure 5.2 where

$$\Sigma = \bigcup_{j=1}^{4} \Sigma_j$$

$$\Sigma' = \Sigma \bigcup \{\pm\gamma_i\}_{i=1}^n \bigcup \{\pm\gamma_i^*\}_{i=1}^n$$

find a matrix-valued meromorphic function $M(x, \zeta)$ with the following properties:

(i) (Analyticity) $M(x, z)$ is analytic in $\mathbb{C}\backslash\Sigma'$ and has continuous boundary values $M_\pm$ on Σ' and $M_\pm$ satisfy

$$M_+(x, \zeta) = M_-(x, \zeta)v_x(\zeta)$$

where on Σ

$$v_x(\zeta) = \begin{pmatrix} 1 - r(\zeta)\breve{r}(\zeta) & e^{-2ix\zeta^2}r(\zeta) \\ -e^{2ix\zeta^2}\breve{r}(\zeta) & 1 \end{pmatrix}$$

with

$$\breve{r}(\zeta) = -\overline{r(\overline{\zeta})} \tag{5.1.14}$$

and

$$v_x(\zeta) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \dfrac{c_i\, e^{2ix\zeta^2}}{\zeta \mp \zeta_i} & 1 \end{pmatrix} & \zeta \in \pm\gamma_i, \\[2ex] \begin{pmatrix} 1 & \dfrac{\overline{c_i}\, e^{-2ix\zeta^2}}{\zeta \mp \overline{\zeta_i}} \\ 0 & 1 \end{pmatrix} & \zeta \in \pm\gamma_i^* \end{cases}$$

(ii) (Normalization)

A. $M(x,\zeta) = I + \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

B. $M(x,\zeta) = \mathcal{O}(\zeta^{-1})$ as $\zeta \to \infty$.

Following the same approach in dealing with Problem 5.1.1, we can also obtain the following Beals-Coifman solution to Problem 5.1.4:

$$\mu = M^+(1 + w_x^+)^{-1} = M^-(1 - w_x^-)^{-1}. \tag{5.1.15}$$

where for $\zeta \in \Sigma$

$$\mu(x,\zeta) = \begin{cases} \begin{pmatrix} \dfrac{m_{11}^-(x,\zeta)}{\breve{a}(\zeta)} & m_{12}^+(x,\zeta) \\[2ex] \dfrac{m_{21}^-(x,\zeta)}{\breve{a}(\zeta)} & m_{22}^+(x,\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\[2ex] e^{2i\zeta^2 x}\breve{r}(\zeta) & 1 \end{pmatrix} \\[6ex] \begin{pmatrix} m_{11}^+(x,\zeta) & \dfrac{m_{12}^-(x,\zeta)}{a(\zeta)} \\[2ex] m_{21}^+(x,\zeta) & \dfrac{m_{22}^-(x,\zeta)}{a(\zeta)} \end{pmatrix} \begin{pmatrix} 1 & e^{-2i\zeta^2 x} r(\zeta) \\[2ex] 0 & 1 \end{pmatrix} \end{cases} \tag{5.1.16}$$

and for $\zeta \in \pm\gamma_i$

$$\mu(x,\zeta) = \begin{cases} \begin{pmatrix} \dfrac{m_{11}^-(x,\zeta)}{\breve{a}(\zeta)} & m_{12}^+(x,\zeta) \\[2ex] \dfrac{m_{21}^-(x,\zeta)}{\breve{a}(\zeta)} & m_{22}^+(x,\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\[2ex] -\dfrac{c_i\, e^{2ix\zeta^2}}{\zeta \mp \zeta_i} & 1 \end{pmatrix} \\[6ex] \begin{pmatrix} \dfrac{m_{11}^-(x,\zeta)}{\breve{a}(\zeta)} - \dfrac{c_i\, e^{2ix\zeta^2} m_{12}^+(x,\zeta)}{\zeta \mp \zeta_i} & m_{12}^+(x,\zeta) \\[2ex] \dfrac{m_{21}^-(x,\zeta)}{\breve{a}(\zeta)} - \dfrac{c_i\, e^{2ix\zeta^2} m_{22}^+(x,\zeta)}{\zeta \mp \zeta_i} & m_{22}^+(x,\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\[2ex] 0 & 1 \end{pmatrix} \end{cases}$$

and for $\zeta \in \pm\gamma_{i*}$

$$\mu(x,\zeta) = \begin{cases} \begin{pmatrix} m_{11}^+(x,\zeta) & \dfrac{m_{12}^-(x,\zeta)}{a(\zeta)} + \dfrac{\overline{c_i}\, e^{-2ix\zeta^2} m_{11}^+(x,\zeta)}{\zeta \mp \overline{\zeta_i}} \\ m_{21}^+(x,\zeta) & \dfrac{m_{22}^-(x,\zeta)}{a(\zeta)} + \dfrac{\overline{c_i}\, e^{-2ix\zeta^2} m_{21}^+(x,\zeta)}{\zeta \mp \overline{\zeta_i}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} m_{11}^+(x,\zeta) & \dfrac{m_{12}^-(x,\zeta)}{a(\zeta)} \\ m_{21}^+(x,\zeta) & \dfrac{m_{22}^-(x,\zeta)}{a(\zeta)} \end{pmatrix} \begin{pmatrix} 1 & \dfrac{\overline{c_i}\, e^{-2ix\zeta^2}}{\zeta \mp \overline{\zeta_i}} \\ 0 & 1 \end{pmatrix} \end{cases}$$

In analogy to Problem 5.1.1 we can deduce the the following Beals-Coifman integral equation for Problem 5.1.4. For Type A normalization:

$$\mu = I + \mathcal{C}_w \mu = I + C_{\Sigma'}^+(\mu w_x^-) + C_{\Sigma'}^-(\mu w_x^+) \tag{5.1.17}$$

where I is the 2×2 identity matrix. And for Type B normalization:

$$\mu = \mathcal{C}_w \mu = C_{\Sigma'}^+(\mu w_x^-) + C_{\Sigma'}^-(\mu w_x^+) \tag{5.1.18}$$

In this case, for $\zeta \in \Sigma$

$$\begin{aligned} \mu_{11}(x,\zeta) = &- C_\Sigma^- \left[\mu_{12}(x,\cdot)\breve{r}(\cdot) e^{2ix(\cdot)^2} \right](\zeta) \\ &+ \sum_{i=1}^n \left(\frac{\mu_{12}(x,\zeta_i) c_i\ e^{2ix\zeta_i^2}}{\zeta - \zeta_i} + \frac{\mu_{12}(x,-\zeta_i) c_i\ e^{2ix\zeta_i^2}}{\zeta + \zeta_i} \right) \end{aligned} \tag{5.1.19}$$

$$\begin{aligned} \mu_{12}(x,\zeta) = &\ C_\Sigma^+ \left[\mu_{11}(x,\cdot) r(\cdot) e^{-2ix(\cdot)^2} \right](\zeta) \\ &- \sum_{i=1}^n \left(\frac{\mu_{11}(x,\overline{\zeta_i}) \overline{c_i}\ e^{-2ix\overline{\zeta_i}^2}}{\zeta - \overline{\zeta_i}} + \frac{\mu_{11}(x,-\overline{\zeta_i}) \overline{c_i}\ e^{-2ix\overline{\zeta_i}^2}}{\zeta + \overline{\zeta_i}} \right) \end{aligned} \tag{5.1.20}$$

$$\begin{aligned} \mu_{21}(x,\zeta) = &- C_\Sigma^- \left[\mu_{22}(x,\cdot)\breve{r}(\cdot) e^{2ix(\cdot)^2} \right](\zeta) \\ &+ \sum_{i=1}^n \left(\frac{\mu_{22}(x,\zeta_i) c_i\ e^{2ix\zeta_i^2}}{\zeta - \zeta_i} + \frac{\mu_{22}(x,-\zeta_i) c_i\ e^{2ix\zeta_i^2}}{\zeta + \zeta_i} \right) \end{aligned} \tag{5.1.21}$$

$$\begin{aligned} \mu_{22}(x,\zeta) = &\ C_\Sigma^+ \left[\mu_{21}(x,\cdot) r(\cdot) e^{-2ix(\cdot)^2} \right](\zeta) \\ &- \sum_{i=1}^n \left(\frac{\mu_{21}(x,\overline{\zeta_i}) \overline{c_i}\ e^{-2ix\overline{\zeta_i}^2}}{\zeta - \overline{\zeta_i}} + \frac{\mu_{21}(x,-\overline{\zeta_i}) \overline{c_i}\ e^{-2ix\overline{\zeta_i}^2}}{\zeta + \overline{\zeta_i}} \right) \end{aligned} \tag{5.1.22}$$

and in order to close the system, we have

$$\mu_{11}(x,\pm\overline{\zeta_i}) = -\int_\Sigma \frac{\mu_{12}(x,s)\breve{r}(s)e^{2is^2x}}{s\mp\overline{\zeta_i}}\frac{ds}{2\pi i} + \sum_{k=1}^n\left(\frac{\mu_{12}(x,\zeta_k)c_ke^{2i\zeta_k^2x}}{\pm\overline{\zeta_i}-\zeta_k}+\frac{\mu_{12}(x,-\zeta_k)c_ke^{2i\zeta_k^2x}}{\pm\overline{\zeta_i}+\zeta_k}\right) \tag{5.1.23}$$

$$\mu_{12}(x,\pm\zeta_i) = \int_\Sigma \frac{\mu_{11}(x,s)r(s)e^{-2is^2x}}{s\mp\zeta_i}\frac{ds}{2\pi i} - \sum_{k=1}^n\left(\frac{\mu_{11}(x,\overline{\zeta_k})\overline{c_k}e^{-2i\overline{\zeta_k}^2x}}{\pm\zeta_i-\overline{\zeta_k}}+\frac{\mu_{11}(x,-\overline{\zeta_k})\overline{c_k}e^{-2i\overline{\zeta_k}^2x}}{\pm\zeta_i+\overline{\zeta_k}}\right) \tag{5.1.24}$$

$$\mu_{21}(x,\pm\overline{\zeta_i}) = -\int_\Sigma \frac{\mu_{22}(x,s)\breve{r}(s)e^{2is^2x}}{s\mp\overline{\zeta_i}}\frac{ds}{2\pi i} + \sum_{k=1}^n\left(\frac{\mu_{22}(x,\zeta_k)c_ke^{2i\zeta_k^2x}}{\pm\overline{\zeta_i}-\zeta_k}+\frac{\mu_{22}(x,-\zeta_k)c_ke^{2i\zeta_k^2x}}{\pm\overline{\zeta_i}+\zeta_k}\right) \tag{5.1.25}$$

$$\mu_{22}(x,\pm\zeta_i) = \int_\Sigma \frac{\mu_{21}(x,s)r(s)e^{-2is^2x}}{s\mp\zeta_i}\frac{ds}{2\pi i} - \sum_{k=1}^n\left(\frac{\mu_{21}(x,\overline{\zeta_k})\overline{c_k}e^{-2i\overline{\zeta_k}^2x}}{\pm\zeta_i-\overline{\zeta_k}}+\frac{\mu_{21}(x,-\overline{\zeta_k})\overline{c_k}e^{-2i\overline{\zeta_k}^2x}}{\pm\zeta_i+\overline{\zeta_k}}\right) \tag{5.1.26}$$

Again, to write down the integral equation (5.1.17) explicitly, we just add 1 to the RHS of equation (5.1.19) (5.1.22) (5.1.23) and (5.1.26).

5.2 Equivalence of two RHPs

Definition 5.2.1. For fixed x and $\lambda \in \mathbb{R}$ and $\{\lambda_i\}_{i=1}^n \subset \mathbb{C}^+$, we define the following two $n+1$ dimensional vectors:

$$\nu_1 = (\nu_{11}(x,\lambda), \oplus_i\{\nu_{11}(x,\lambda_i)\})$$

$$\nu_2 = (\nu_{12}(x,\lambda), \oplus_i\{\nu_{12}(x,\overline{\lambda}_i)\})$$

We also define a Hilbert space $X = L^2_\lambda(\mathbb{R}) \oplus \mathbb{C}^n$.

We begin with the following change of variable formulas: for $\zeta \in \Sigma$

$$\mu_{11}(x,\zeta) = \nu_{11}(x,\zeta^2), \quad \mu_{12}(x,\zeta) = \zeta\nu_{12}(x,\zeta^2). \tag{5.2.1}$$

and for $\zeta_i \in \mathbb{C}^{++}$

$$\mu_{11}(x, \overline{\zeta}_i) = \nu_{11}(x, \overline{\zeta}_i^2), \quad \mu_{12}(x, \zeta_i) = \zeta_i \nu_{12}(x, \zeta_i^2). \tag{5.2.2}$$

It is easy to see that

$$\mu_{11}(x, -\zeta) = \mu_{11}(x, \zeta), \quad \mu_{12}(x, -\zeta) = -\mu_{12}(x, \zeta).$$

Lemma 5.2.2. *For $\nu = (\nu_1, \nu_2)$ a solution of the homogeneous Beals-Coifman equation (5.1.3) i.e.*

$$\nu(x, \lambda) = [C_W \nu](x, \lambda),$$

then $\nu_{11}(x, \lambda), \nu_{12}(x, \lambda) \in L^2(\mathbb{R})$ implies that $\mu_{11} r, \mu_{12} \breve{r}$ given by (5.2.1) belong to $L^2(\Sigma) \cap L^1(\Sigma)$.

Proof. We first notice that

$$\begin{aligned}
\int_\Sigma |\mu_{11}(x, \zeta) r(\zeta)|^2 \, |d\zeta| &= \int_0^\infty |\mu_{11}(x, t) r(t)|^2 + |\mu_{11}(x, -t) r(t)|^2 \\
&\quad + |\mu_{11}(x, it) r(it)|^2 + |\mu_{11}(x, -it) r(-it)|^2 \, dt \\
&= 2 \int_0^\infty \left|\nu_{11}(x, t^2) t \rho(t^2)\right|^2 + \left|\nu_{11}(x, -t^2) t \rho(-t^2)\right|^2 dt \\
&= \int_{-\infty}^\infty |\nu_{11}(x, u) \rho(u)|^2 \sqrt{|u|} du
\end{aligned} \tag{5.2.3}$$

The improper integral above is convergent given $\nu_{11} \in L^2(\mathbb{R})$ and $\rho \in H^{2,2}(\mathbb{R})$.

Now we want to show that

$$\int_\Sigma |\mu_{12}(x, \zeta) \breve{r}(\zeta)|^2 \, |d\zeta| < \infty. \tag{5.2.4}$$

In this case we have

$$\begin{aligned}
\int_\Sigma |\mu_{12}(x, \zeta) \breve{r}(\zeta)|^2 \, |d\zeta| &= \int_\Sigma |\zeta^2 \, \nu_{12}(x, \zeta^2) \rho(\zeta^2)|^2 \, |d\zeta| \\
&= 2 \int_0^\infty \left|t^2 \, \nu_{12}(x, t^2) \rho(t^2)\right|^2 + \left|t^2 \, \nu_{12}(x, -t^2) \rho(-t^2)\right|^2 dt \\
&= \int_{-\infty}^\infty |\nu_{12}(x, u) \rho(u)|^2 |u|^{3/2} du
\end{aligned}$$

Again, the improper integral above is convergent given $\nu_{12} \in L^2(\mathbb{R})$ and $\rho \in H^{2,2}(\mathbb{R})$. The L^1 boundedness is an easy consequence of the Cauchy-Schwarz inequality. $\square$

Lemma 5.2.3. *For $\nu = (\nu_1, \nu_2)$ a solution of the homogeneous Beals-Coifman equation (5.1.3), define*

$$\mu(x, \zeta) = \begin{pmatrix} \mu_{11}(x, \zeta) & \mu_{12}(x, \zeta) \\ \mu_{21}(x, \zeta) & \mu_{22}(x, \zeta) \end{pmatrix} = \begin{pmatrix} \nu_{11}(x, \zeta^2) & \zeta \nu_{12}(x, \zeta^2) \\ -\zeta \overline{\nu_{12}(x, \zeta^2)} & \overline{\nu_{11}(x, \zeta^2)} \end{pmatrix} \tag{5.2.5}$$

and for $i = 1, ..., n$

$$\begin{pmatrix} \mu_{11}(x, \pm\overline{\zeta}_i) & \mu_{12}(x, \pm\zeta_i) \\ \\ \mu_{21}(x, \pm\overline{\zeta}_i) & \mu_{22}(x, \pm\zeta_i) \end{pmatrix} = \begin{pmatrix} \nu_{11}(x, \overline{\zeta}_i^2) & \pm\zeta_i \nu_{12}(x, \zeta_i^2) \\ \\ \mp\overline{\zeta}_i \overline{\nu_{12}(x, \zeta_i^2)} & \overline{\nu_{11}(x, \overline{\zeta}_i^2)} \end{pmatrix} \tag{5.2.6}$$

then μ *solves integral equation* (5.1.18).

Proof. We first build equations (5.1.19)-(5.1.22) from the homogeneous form of Equation (5.1.4)-(5.1.9) using the change of variable given in (5.1.13). We first deal with the discrete part of equations (5.1.4)-(5.1.9). Using change of variable formulas from (5.1.13), simple computation gives us

$$\frac{\nu_{11}(x, \overline{\lambda}_i)\overline{C_i}e^{-2i\overline{\lambda}_i x}}{\lambda - \overline{\lambda}_i} = \frac{1}{\zeta}\left(\frac{\mu_{11}(x, \overline{\zeta}_i)\overline{c_i}e^{-2i\overline{\zeta}_i^2 x}}{\zeta - \overline{\zeta}_i} + \frac{\mu_{11}(x, -\overline{\zeta}_i)\overline{c_i}e^{-2i\overline{\zeta}_i^2 x}}{\zeta + \overline{\zeta}_i}\right)$$

and

$$\frac{\nu_{12}(x, \lambda_i)C_i\,\lambda_i e^{2i\lambda_i x}}{\lambda - \lambda_i} = \frac{\mu_{12}(x, \zeta_i)c_i\,e^{2i\zeta_i^2 x}}{\zeta - \zeta_i} + \frac{\mu_{12}(x, -\zeta_i)c_i\,e^{2i\zeta_i^2 x}}{\zeta + \zeta_i}$$

So we get the discrete part of formula (5.1.19)-(5.1.22).

We can use the conclusion of the previous lemma and the change of variable formulas (2.4.3) and (2.4.4) to deduce that

$$\begin{aligned} C_{\mathbb{R}}^{-}\left(\nu_{12}(x, \cdot\,)(\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2ix(\,\cdot\,)}\right)(\lambda) &= C_{\Sigma}^{-}\left[\mu_{12}(x, \diamond)(\diamond)\overline{\rho((\diamond)^2)}e^{2ix(\diamond)^2}\right](\zeta) \\ &= -C_{\Sigma}^{-}\left[\mu_{12}(x, \diamond)\breve{r}(\diamond)e^{2ix(\diamond)^2}\right](\zeta) \end{aligned} \tag{5.2.7}$$

and

$$\begin{aligned} \zeta C_{\mathbb{R}}^{+}\left[\nu_{11}(x, \diamond)\rho(\diamond)e^{-2ix(\diamond)}\right](\zeta^2) &= C_{\Sigma}^{+}\left[\nu_{11}(x, (\diamond)^2)r(\diamond)e^{-2ix(\diamond)^2}\right](\zeta) \\ &= C_{\Sigma}^{+}\left[\mu_{11}(x, \diamond)r(\diamond)e^{-2ix(\diamond)^2}\right](\zeta) \end{aligned} \tag{5.2.8}$$

We use the fact that $\overline{C_{\mathbb{R}}^{+}(f)} = -C_{\mathbb{R}}^{-}(\overline{f})$ and the change of variable formula (5.1.13) to deduce the following integral equations from (5.1.4) and (5.1.5)

$$\begin{aligned} \mu_{22}(x, \zeta) &= \overline{\nu_{11}(x, \zeta^2)} \\ &= -C_{\mathbb{R}}^{+}\left[\overline{\nu_{12}(x, \cdot\,)}(\,\cdot\,)\rho(\,\cdot\,)e^{-2ix(\,\cdot\,)}\right](\zeta^2) + \sum_{i=1}^{n}\frac{\overline{\nu_{12}(x, \lambda_i)}\overline{C_i}\,\overline{\lambda}_i e^{-2i\overline{\lambda}_i x}}{\lambda - \overline{\lambda}_i} \\ &= C_{\Sigma}^{+}\left[\mu_{21}(x, \cdot\,)r(\,\cdot\,)e^{-2ix(\,\cdot\,)^2}\right](\zeta) \\ &\quad - \sum_{i=1}^{n}\left(\frac{\mu_{21}(x, \overline{\zeta}_i)\overline{c_i}\ e^{-2ix\overline{\zeta}_i^2}}{\zeta - \overline{\zeta}_i} + \frac{\mu_{21}(x, -\overline{\zeta}_i)\overline{c_i}\ e^{-2ix\overline{\zeta}_i^2}}{\zeta + \overline{\zeta}_i}\right) \end{aligned}$$

and

$$
\begin{aligned}
\mu_{21}(x,\zeta) &= -\zeta\overline{\nu_{12}(x,\zeta^2)} \\
&= \zeta C_{\mathbb{R}}^{-}\left[\overline{\nu_{11}(x,\,\cdot\,)\rho(\,\cdot\,)}e^{2ix(\,\cdot\,)}\right](\zeta^2) + \zeta\sum_{i=1}^{n}\frac{\overline{\nu_{11}(x,\overline{\lambda_i})}C_i e^{2i\lambda_i x}}{\lambda-\lambda_i} \\
&= -C_{\Sigma}^{-}\left[\mu_{22}(x,\,\cdot\,)\breve{r}(\,\cdot\,)e^{2ix(\,\cdot\,)^2}\right](\zeta) \\
&\quad + \sum_{i=1}^{n}\left(\frac{\mu_{22}(x,\zeta_i)c_i\ e^{2ix\zeta_i^2}}{\zeta-\zeta_i} + \frac{\mu_{22}(x,-\zeta_i)c_i\ e^{2ix\zeta_i^2}}{\zeta+\zeta_i}\right)
\end{aligned}
$$

Equations (5.1.23)-(5.1.26) can be derived by following the same approach. ☐

Lemma 5.2.4. *Suppose $\nu_1,\nu_2 \in L^2_\lambda(\mathbb{R})\oplus\mathbb{C}^n$ solve Equation (5.1.3) then we can construct solution $M(x,\zeta)$ to Problem 5.1.4 with Type B normalization such that $M_\pm(x,\,\cdot\,)\in\partial C_\Sigma(L^2)$.*

Proof. We construct the solution to Problem 5.1.4 in terms of μ_{11} and μ_{12}. By (5.2.5) the 2×2 matrix μ in (5.1.15) has the following symmetry condition:

$$
\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} = \begin{pmatrix} \mu_{11}(x,\zeta) & \mu_{12}(x,\zeta) \\ -\overline{\mu_{12}(x,\overline{\zeta})} & \overline{\mu_{11}(x,\overline{\zeta})} \end{pmatrix} \quad \zeta\in\Sigma \tag{5.2.9}
$$

As a consequence of the Plemelj formula, the solution $M(x,z)$ to Problem 5.1.4 is given by

$$
M(x,z) = \int_{\Sigma'}\frac{\mu(x,s)\,(w_x^+(s)+w_x^-(s))}{s-z}\frac{ds}{2\pi i} \tag{5.2.10}
$$

Combining the symmetry condition given by (5.2.9) with (5.1.16) we get the following boundary condition:

$$
M^+(x,\zeta) = \begin{pmatrix} \mu_{11}(x,\zeta)-\mu_{12}(x,\zeta)e^{2ix\zeta^2}\breve{r}(\zeta) & \mu_{12}(x,\zeta) \\ -\overline{\mu_{12}(x,\zeta)}-\overline{\mu_{11}(x,\zeta)}e^{2ix\zeta^2}\breve{r}(\zeta) & \overline{\mu_{11}(x,\zeta)} \end{pmatrix} \quad \zeta\in\Sigma \tag{5.2.11}
$$

$$
M^-(x,\zeta) = \begin{pmatrix} \mu_{11}(x,\zeta) & \mu_{12}(x,\zeta)-\mu_{11}(x,\zeta)e^{-2ix\zeta^2}r(\zeta) \\ -\overline{\mu_{12}(x,\zeta)} & \overline{\mu_{11}(x,\zeta)}+\overline{\mu_{12}(x,\zeta)}e^{-2ix\zeta^2}r(\zeta) \end{pmatrix} \quad \zeta\in\Sigma \tag{5.2.12}
$$

Now we can appeal to Lemma 5.2.2 and the L^2 boundedness of the Cauchy projection to conclude that $M_\pm(x,\,\cdot\,)\in\partial C_\Sigma(L^2)$. ☐

Using the same reasoning in Lemma 5.2.3 and Lemma 5.2.4 we can also obtain the following lemma:

Lemma 5.2.5. *Suppose $\nu_1-1,\nu_2\in L^2(\mathbb{R})\oplus\mathbb{C}^n$, then we can construct solution $M(x,\zeta)$ to Problem 5.1.4 with Type A normalization and $M_\pm(x,\,\cdot\,)-I\in\partial C_\Sigma(L^2)$.*

Remark 5.2.6. The following proposition is an application of the *Vanishing Lemma* from [26]. We give the details here for completeness.

Proposition 5.2.7. *The solution to Problem 5.1.4 with Type B normalization given by* (5.2.10) *is identically zero.*

Proof. We first recall that the symmetry reduction condition for the entries of the transition matrix is given as follows:

$$\breve{r}(\zeta) = -\overline{r(\overline{\zeta})}.$$

Thus for $\zeta \in i\mathbb{R}$ we compute

$$r(\overline{\zeta}) = -\overline{\breve{r}(\zeta)}, \quad -\breve{r}(\overline{\zeta}) = \overline{r(\zeta)}$$

Taking complex conjugate of both terms above we get:

$$\overline{r(\overline{\zeta})} = -\breve{r}(\zeta), \quad \overline{-\breve{r}(\overline{\zeta})} = r(\zeta)$$

So we conclude that on $i\mathbb{R}$

$$v(\zeta) = v(\overline{\zeta})^{\dagger} \tag{5.2.13}$$

where † denotes complex conjugation and transpose of a given matrix. It is trivial that the same equality holds on $\{\pm\gamma_i \bigcup \pm\gamma_i^*\}_{i=1}^n$. So we conclude that (5.2.13) holds on $\Sigma'\backslash\mathbb{R}$.

Now we formulate the matrix-valued function: $F(\zeta) = M(\zeta)M(\overline{\zeta})^{\dagger}$ and we want to show that

$$\int_{-\infty}^{\infty} F_{\pm}(\zeta)d\zeta = 0. \tag{5.2.14}$$

It is clear that $F(\zeta)$ is analytic in $\mathbb{C}\backslash\Sigma$ by the Schwarz reflection principle. Now suppose that $\zeta \in \mathbb{C}\backslash\mathbb{R}$, then we have

$$\begin{aligned}
F_+(\zeta) &= M_+(\zeta)M_-(\overline{\zeta})^{\dagger} \\
&= M_-(\zeta)v(\zeta)(v(\overline{\zeta})^{-1})^{\dagger}M_+(\overline{\zeta})^{\dagger} \\
&= M_-(\zeta)M_+(\overline{\zeta})^{\dagger} \\
&= F_-(\zeta)
\end{aligned}$$

where the third equality above comes from (5.2.13).

By Morera's theorem $F(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash\mathbb{R}$. Since $M_{\pm}(x,\cdot)$ is L^2 on $\mathbb{R}$, $F(\zeta)$ is integrable. Also we can write

$$\begin{aligned}
M(x,z) &= \int_{\Sigma'} \frac{\mu(x,s)\left(w_x^+(s) + w_x^-(s)\right)}{s-z}\frac{ds}{2\pi i} \\
&= \frac{1}{z}\int_{\Sigma'} \frac{s}{s-z}\mu(x,s)\left(w_x^+(s) + w_x^-(s)\right)\frac{ds}{2\pi i} \\
&\quad - \frac{1}{z}\int_{\Sigma'} \mu(x,s)\left(w_x^+(s) + w_x^-(s)\right)\frac{ds}{2\pi i}
\end{aligned}$$

so we have

$$M(x,z) \sim \mathcal{O}\left(\frac{1}{z}\right), \quad z \in \mathbb{C}\backslash\Sigma$$

(5.2.14) follows from Cauchy's theorem.

For $\zeta \in \mathbb{R}$ we have that

$$F_+(\zeta) = M_+(\zeta)M_-(\zeta)^\dagger = M_-(\zeta)v(\zeta)M_-(\zeta)^\dagger$$

and

$$F_-(\zeta) = M_-(\zeta)M_+(\zeta)^\dagger = M_-(\zeta)v(\zeta)^\dagger M_-(\zeta)^\dagger$$

and (5.2.14) we get

$$\int_{-\infty}^{\infty} M_-(\zeta)\left(v(\zeta) + v(\zeta)^\dagger\right) M_-(\zeta)^\dagger = 0. \tag{5.2.15}$$

where for $\zeta \in \mathbb{R}$

$$v(\zeta) + v(\zeta)^\dagger = 2\begin{pmatrix} 1 + |r(\zeta)|^2 & e^{-2ix\zeta^2} r(\zeta) \\ e^{2ix\zeta^2}\overline{r(\zeta)} & 1 \end{pmatrix}.$$

$v(\zeta) + v(\zeta)^\dagger$ is positive definite since it is Hermitian and has positive eigenvalues. From (5.2.15) we conclude that $M_-(\zeta) = M_+(\zeta) = 0$ on $\mathbb{R}$. From Morera's theorem we conclude that $M(\zeta)$ is analytic in a neighborhood of every point on $\mathbb{R}$. Since $M(\zeta) = 0$ for $\zeta \in \mathbb{R}$, analytic continuation gives us that $M(\zeta) = 0$ holds all the way up to the first complex part of Σ. Applying the jump condition on this part shows that $M_\pm(\zeta)$ agree and and vanish. We can apply the same argument to the remaining parts of Σ' and conclude that $M(\zeta) \equiv 0$ on the entire complex plane. This completes the proof. $\square$

Corollary 5.2.8. *The homogeneous Beal-Coifman integral equation given by* (5.1.3) *has only the trivial solution.*

Proof. An immediate consequence of change of variable formula (5.2.1) and Proposition 5.2.7. $\square$

The following lemma is a standard result from [26]:

Lemma 5.2.9. *If M is a solution to Problem 5.1.4 with $M_\pm - \mathbf{1} \in \partial C_\Sigma(L^2)$, then M is unique.*

5.3 Reconstruction of the Potential

In this section we show that the solution to Problem 5.1.4 with Type A normalization solves a differential equation of the form (1.1.7) and obtain explicit formulas for $Q(x)$ and $P(x)$ having the correct structure (see Remark 5.3.2).

Proposition 5.3.1. *The functions $M_\pm$ obey the differential equation (1.1.7) where M, P and Q are constructed from the solution μ of (5.1.17) as follows:*

$$M(x,\zeta) = I + \int_{\Sigma'} \frac{\mu(x,s)\,(w_x^+(s) + w_x^-(s))}{s-\zeta}\frac{ds}{2\pi i} \tag{5.3.1}$$

$$Q(x) = -\frac{1}{2\pi}\operatorname{ad}\sigma\left(\int_{\Sigma'} \mu(x,\zeta)\left(w_x^+(\zeta) + w_x^-(\zeta)\right)\,d\zeta\right) \tag{5.3.2}$$

$$P(x) = Q(x)i(\operatorname{ad}\sigma)^{-1}Q(x) \tag{5.3.3}$$

Remark 5.3.2. We set

$$f(x,\zeta) = \mu(x,\zeta)\left(w_x^+(\zeta) + w_x^-(\zeta)\right).$$

More explicitly

$$f(x,\zeta) = \begin{pmatrix} -\mu_{12}(x,\zeta)\breve{r}(\zeta)e^{2i\zeta^2 x} & \mu_{11}(x,\zeta)r(\zeta)e^{-2i\zeta^2 x} \\ -\mu_{22}(x,\zeta)\breve{r}(\zeta)e^{2i\zeta^2 x} & \mu_{21}(x,\zeta)r(\zeta)e^{-2i\zeta^2 x} \end{pmatrix} \quad \zeta \in \Sigma \tag{5.3.4}$$

$$f(x,\zeta) = \begin{pmatrix} \mu_{12}(x,\zeta)\dfrac{c_i\, e^{2ix\zeta^2}}{\zeta \mp \zeta_i} & 0 \\ \mu_{22}(x,\zeta)\dfrac{c_i\, e^{2ix\zeta^2}}{\zeta \mp \zeta_i} & 0 \end{pmatrix} \quad \zeta \in \pm\gamma_i \tag{5.3.5}$$

$$f(x,\zeta) = \begin{pmatrix} 0 & \mu_{11}(x,\zeta)\dfrac{\overline{c_i}\, e^{-2ix\zeta^2}}{\zeta \mp \overline{\zeta_i}} \\ 0 & \mu_{21}(x,\zeta)\dfrac{\overline{c_i}\, e^{-2ix\zeta^2}}{\zeta \mp \overline{\zeta_i}} \end{pmatrix} \quad \zeta \in \pm\gamma_{i^*} \tag{5.3.6}$$

which, together with the formula (5.3.2), shows that

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ -\overline{q(x)} & 0 \end{pmatrix},$$

as required. Here

$$q(x) = -\frac{1}{\pi}\int_\Sigma e^{-2ix\zeta^2} r(\zeta)\mu_{11}(x,\zeta)\,d\zeta - \sum_{i=1}^n 4i\mu_{11}(x,\overline{\zeta_i})\overline{c_i}e^{-2i\overline{\zeta}_i^2 x} \tag{5.3.7}$$

We begin the proof with the following lemma:

Lemma 5.3.3.

$$\int_{\Sigma'} s^n f(x,s)ds = \int_\Sigma s^n f(x,s)ds + \int_{\Sigma'\setminus\Sigma} s^n f(x,s)ds \tag{5.3.8}$$

is diagonal when n is odd and off-diagonal when n is even.

Proof. For the first term on the RHS of (5.3.8) note that when n is even (odd) , $s^n f(x,s)$ is odd (even) off-diagonal and even (odd) on-diagonal. We can deduce these relations from the evenness (oddness) of the diagonal (off-diagonal) entries of $\mu(x,\zeta)$. Given the orientation of the contour Σ' as shown in Figure 5.2, the even (odd) terms integrate to zero while the odd (even) terms persist. To deal with the second term, we can now use Cauchy integral formula and the fact that μ_{11} , μ_{22} are even and that μ_{12} , μ_{21} are odd to arrive at the desired conclusion. $\square$

Proof of Proposition 5.3.1. For $\zeta \in \Sigma$, let

$$v_x(\zeta) = e^{-ix\zeta^2 \operatorname{ad}\sigma} v(\zeta).$$

Differentiating the jump relation

$$M_+(x,\zeta) = M_-(x,\zeta) v_x(\zeta)$$

with respect to x and using the fact that $\operatorname{ad}\sigma$ is a derivation, we compute

$$\begin{aligned}
\frac{dM_+}{dx} &= \frac{dM_-}{dx} v_x + M_- \left(-i\zeta^2 \operatorname{ad}\sigma(v_x)\right) \\
&= \frac{dM_-}{dx} v_x + M_- \left(-i\zeta^2 \operatorname{ad}\sigma\left((M_-)^{-1} M_+\right)\right) \\
&= \frac{dM_-}{dx} v_x + i\zeta^2 \operatorname{ad}\sigma(M_-) v_x - i\zeta^2 \operatorname{ad}\sigma(M_+)
\end{aligned}$$

We conclude that

$$\frac{d}{dx} M_+(x,\zeta) + i\zeta^2 \operatorname{ad}\sigma(M_+) = \left(\frac{d}{dx} M_-(x,\zeta) + i\zeta^2 \operatorname{ad}\sigma(M_-)\right) v_x. \tag{5.3.9}$$

Using the fact that

$$M_\pm(x,\zeta) - \mathbf{1} = C^\pm \left[\mu(x,\,\cdot\,)\left(w_x^-(\,\cdot\,) + w_x^+(\,\cdot\,)\right)\right] \tag{5.3.10}$$

and Lemma 2.3.1(ii), we conclude that

$$\begin{aligned}
i\zeta^2 \operatorname{ad}\sigma(M_\pm)(x,\zeta) = i \operatorname{ad}\sigma\Big[& C^\pm \left((\,\cdot\,)^2 f(x,\,\cdot\,)\right) \\
& - \frac{\zeta}{2\pi i}\int_{\Sigma'} f(x,s)\,ds - \frac{1}{2\pi i}\int_{\Sigma'} s f(x,s)\,ds\Big]
\end{aligned}$$

where $f(x,\zeta)$ is given by (5.3.4). It follows from Lemma 5.3.3 and (5.3.4) that the matrix-valued integral

$$\int_\Sigma \zeta f(x,\zeta)\,d\zeta$$

is a diagonal matrix. Hence,

$$\operatorname{ad}\sigma\left(\int_{\Sigma'} s f(x,s)\,ds\right) = 0.$$

Hence defining $Q(x)$ by (5.3.2), we have

$$i\zeta^2 \operatorname{ad}\sigma(M_\pm)(x,\zeta) = i \operatorname{ad}\sigma\left[C^\pm\left((\,\cdot\,)^2 f(x,\,\cdot\,)\right)\right] + \zeta Q(x). \tag{5.3.11}$$

Note that Cauchy projection is bounded on L^2 and $\nu_{11} - 1 \in L^2$ for each fixed x. Through change of variable and following the same argument given in Lemma 5.2.2 we can show that the first term defines an L^2 function of ζ for each x. Next, observe that, by (5.3.10) and Lemma 2.3.1(i),

$$\zeta Q(x)(M_\pm - \mathbf{1}) = Q(x)C^\pm\left[(\,\cdot\,)f(x,\,\cdot\,)\right] - Q(x)R(x) \tag{5.3.12}$$

where $R(x)$ is given by

$$\begin{aligned} R(x) &= \frac{1}{2\pi i}\int_{\Sigma'} \mu(x,\zeta)(w_x^+(\zeta) + w_x^-(\zeta))\, d\zeta \\ &= i(\operatorname{ad}\sigma)^{-1}Q(x) \end{aligned}$$

and the first right-hand term of (5.3.12) is an L^2 function of ζ for each x.

Now define

$$W_\pm = \frac{dM_\pm}{dx} + i\zeta^2 \operatorname{ad}\sigma(M_\pm) - \zeta Q(x)M_\pm(x) - Q(x)R(x)M_\pm(x).$$

By (5.3.10), (5.3.11), (5.3.12), and the identity

$$\begin{aligned} -\zeta Q(x)M_\pm(x) - Q(x)R(x)M_\pm(x) = &-\zeta Q(x) - \zeta Q(x)(M_\pm - \mathbf{1}) \\ &- Q(x)R(x) - Q(x)R(x)(M_\pm - \mathbf{1}), \end{aligned}$$

it now follows that $(W_+, W_-) \in \partial C_\Sigma(L^2)$ for each fixed x. More explicitly,

$$\begin{aligned} W_\pm = &\frac{d}{dx}C^\pm[f(x,\cdot)] + i \operatorname{ad}\sigma\left[C^\pm\left((\,\cdot\,)^2 f(x,\,\cdot\,)\right)\right] - Q(x)\left[C^\pm\left((\,\cdot\,)f(x,\,\cdot\,)\right)\right] \\ &- Q(x)R(x)\left[C^\pm\left((\,\cdot\,)f(x,\,\cdot\,)\right)\right] \end{aligned}$$

Also

$$W_+ = W_- v_x,$$

and we can check that $W(x,z)$ has the same residue condition in the complex plane as Problem 5.1.4. It follows from Proposition 5.2.7 that $W_+ = W_- = 0$.

□

Recalling that we construct μ from $\nu = (\nu_{11}, \nu_{12})$ as in Proposition 5.2.3, we may use the reconstruction formula (5.3.7), the change of variables formula (2.3.2), and the odd symmetry of the integrand in (5.3.7) to conclude that

$$q(x) = -\frac{1}{\pi}\int_{\mathbb{R}} e^{-2ix\lambda}\rho(\lambda)\nu_{11}(x,\lambda)\, d\lambda - \sum_{i=1}^{n} 2i\nu_{11}(x,\overline{\lambda_i})\overline{C_i}e^{-2i\overline{\lambda_i}x}. \tag{5.3.13}$$

We conclude this chapter with the derivation of the 1-soliton solution from Problem 5.1.1.

5.4 1-soliton solution

Suppose $\rho(\lambda) = 0$ for all $\lambda \in \mathbb{R}$ and $\breve{\alpha}$ has only one zero λ_1 in $\mathbb{C}^+$. Then (5.1.4) and (5.1.5) become :

$$\nu_{11}(x,\lambda) = 1 + \frac{\nu_{12}(x,\lambda_1)\lambda_1 e^{2i\lambda_1 x} C_1}{\lambda - \lambda_1} \tag{5.4.1}$$

and

$$\nu_{12}(x,\lambda) = -\frac{\nu_{11}(x,\overline{\lambda}_1) e^{-2i\overline{\lambda}_1 x}\overline{C}_1}{\lambda - \overline{\lambda}_1} \tag{5.4.2}$$

Since ν_{12} and ν_{11} are analytic in $\mathbb{C}^{\pm}$ respectively, to close the system formed by (5.4.1) and (5.4.2), we set $\lambda = \overline{\lambda}_1$ and $\lambda = \lambda_1$ for ν_{11} and ν_{12} respectively to get

$$\nu_{11}(x,\overline{\lambda}_1) = 1 + \frac{\nu_{12}(x,\lambda_1)\lambda_1 e^{2i\lambda_1 x} C_1}{\overline{\lambda}_1 - \lambda_1} \tag{5.4.3}$$

and

$$\nu_{12}(x,\lambda_1) = -\frac{\nu_{11}(x,\overline{\lambda}_1) e^{-2i\overline{\lambda}_1 x}\overline{C}_1}{\lambda_1 - \overline{\lambda}_1} \tag{5.4.4}$$

Now we can first solve for $\nu_{11}(x,\overline{\lambda}_1)$ to get $\nu_{12}(x,\lambda)$ given in (5.4.2) and use the reconstruction formula (5.3.13) to get the 1-soliton solution to equation (1.1.3):

$$q(x) = \frac{8i\eta^2 e^{-2i\overline{\lambda}_1 x}\overline{C}_1}{4\eta^2 + e^{-4\eta x}|C_1|^2\lambda_1} \tag{5.4.5}$$

where

$$\lambda_1 = \xi + i\eta$$

Using the notations of [12] , we set

$$\lambda_1 = i\Delta^2 e^{-i(\pi/2-\gamma)} = \Delta^2\cos\gamma + i\Delta^2\sin\gamma, \ (0 < \gamma < \pi)$$

and

$$C_1 = 2\frac{\eta}{\Delta} e^{2i\sigma_0} e^{2\eta x_0}.$$

we obtain

$$q(x) = 4\frac{i\eta}{\Delta}\frac{e^{-2i\sigma}e^{2\theta}}{e^{4\theta} + e^{i\gamma}} \tag{5.4.6}$$

where

$$\theta = \eta(x - x_0)$$

$$\sigma = \xi x + \sigma_0$$

Using (33a) and (33b) of [12] we get

$$e^{-2i\mu^+} = \exp\left(i\int_x^{+\infty}|q|^2 dy\right) = \left(\frac{e^{4\theta} + e^{i\gamma}}{e^{4\theta} + e^{-i\gamma}}\right)^2$$

Now we invert the gauge transformation (1.1.2) to get the 1-soliton solution to Equation (1.1.1)

$$\begin{aligned}
u(x) &= q(x)\ \exp\left(-i\int_{-\infty}^{x} |q|^2 dy\right) \qquad (5.4.7)\\
&= q(x)\ \exp\left(i\int_{x}^{\infty} |q|^2 dy\right) \exp\left(-i\int_{-\infty}^{\infty} |q|^2 dy\right)\\
&= \frac{4i\eta}{\Delta}\frac{e^{-2i(\sigma+2\gamma)}e^{2\theta}}{e^{4\theta}+e^{-i\gamma}}e^{-i\mu^+}\\
&= 4i\Delta\sin\gamma\ \frac{e^{2\theta}\ e^{-2i(\sigma+2\gamma)}}{e^{4\theta}\ +\ e^{-i\gamma}}\ e^{\frac{i}{2}\mu^+}\ e^{-\frac{3}{2}i\mu^+}\\
&= 4i\Delta\sin\gamma\frac{e^{2\theta}\ e^{-2i(\sigma+2\gamma)}}{[(e^{4\theta}\ +\ e^{i\gamma})(e^{4\theta}\ +\ e^{-i\gamma})]^{1/2}}\ e^{-\frac{3}{2}i\mu^+}
\end{aligned}$$

The denominator of this expression simplifies to

$$e^{2\theta}\sqrt{e^{4\theta}+e^{-4\theta}+e^{i\gamma}+e^{-i\gamma}} = \sqrt{2}e^{2\theta}\sqrt{\cosh(4\theta)+\cos\gamma}.$$

which leads to

$$\begin{aligned}
u(x) &= 2\sqrt{2}i\Delta\sin\gamma\ \frac{e^{-2i(\sigma+2\gamma)}}{\sqrt{\cosh(4\theta)+\cos\gamma}}\ e^{-\frac{3}{2}i\mu^+} \qquad (5.4.8)\\
&= 2\sqrt{2}i\Delta\sin\gamma\ \frac{e^{-2i(\sigma+2\gamma)}}{\sqrt{\cosh(4\theta)+\cos\gamma}}\ \exp\left(\frac{3i}{4}\int_{x}^{+\infty} |q|^2 dy\right)
\end{aligned}$$

Remark 5.4.1. We notice that (5.4.6) and (5.4.7) differ from (31) and (33c) of [12] by a factor of i. This is legitimate since we can multiply both sides of equations by i. (1.1.1) and (1.1.3).

5.4.1 Time Evolution

Recall the time dependence of the scattering data

$$\rho_t = -4i\lambda^2\rho \qquad (5.4.9)$$

and for $k = 1, 2, ..., N$

$$\frac{d\,C_k}{dt} = -4i\lambda_k^2 C_k \qquad (5.4.10)$$

For one-soliton solution (5.4.6) we can write the time dependent parameters θ and σ as

$$\theta = \eta(x-x_0) = \Delta^2\sin(\gamma)\left(x-\widetilde{x_0}+4\Delta^2\cos(\gamma)t\right) \qquad (5.4.11)$$

and

$$\sigma = \xi x+\sigma_0 = \Delta^2\cos(\gamma)x+\widetilde{\sigma_0}+2\Delta^4\cos(2\gamma)t \qquad (5.4.12)$$

where

$$C_1|_{t=0} = 2\frac{\eta}{\Delta}e^{2i\widetilde{\sigma_0}}e^{2\eta\widetilde{x_0}}$$

We now write the soliton solution in the form of hyperbolic function. We first introduce two parameters c and ω with $|c| < 2\sqrt{\omega}$. Denote $c/2\sqrt{\omega} = -\cos\gamma$, $\Delta = \omega^{1/4}/\sqrt{2}$ then

$$\sqrt{4\omega - c^2} = 2\sqrt{\omega}\sin\gamma = 4\Delta^2\sin(\gamma)$$

$$\left(\frac{4\omega - c^2}{\sqrt{\omega}}\right)^{1/2} = 2\omega^{1/4}\sin\gamma = 2\sqrt{2}\Delta\sin\gamma \tag{5.4.13}$$

and

$$c = -4\Delta^2\cos\gamma$$

This gives

$$\cosh(4\theta) + \cos\gamma = \cosh\left(\sqrt{4\omega - c^2}(x - \widetilde{x}_0 - ct)\right) - \frac{c}{2\sqrt{\omega}} \tag{5.4.14}$$

and

$$\begin{aligned} e^{-2i\sigma} &= \exp\left(2i\Delta^2(\cos\gamma)x - 2i\widetilde{\sigma}_0 - 4i\Delta^4(\cos 2\gamma)t\right) \\ &= \exp i\left(\omega t - 2\widetilde{\sigma}_0 + \frac{c}{2}(x - ct)\right) \end{aligned} \tag{5.4.15}$$

Now we have the following sech-form soliton:

$$u(x,t) = iR(x - \widetilde{x}_0 - ct)\exp\left(i\omega t + i\frac{c}{2}(x - ct) - 2i\widetilde{\sigma}_0 - i\gamma - \frac{3i}{4}\int_{-\infty}^{x - \widetilde{x}_0 - ct}|R(y)|^2 dy\right) \tag{5.4.16}$$

with

$$R(x) = \left(\frac{4\omega - c^2}{\sqrt{\omega}}\right)^{1/2}\frac{1}{\sqrt{\cosh(\sqrt{4\omega - c^2}x) - \frac{c}{2\sqrt{\omega}}}}. \tag{5.4.17}$$

Also using gauge transformation (1.1.2) we have

$$q(x,t) = iR(x - \widetilde{x}_0 - ct)\exp\left(i\omega t + i\frac{c}{2}(x - ct) - 2i\widetilde{\sigma}_0 - i\gamma + \frac{i}{4}\int_{-\infty}^{x - \widetilde{x}_0 - ct}|R(y)|^2 dy\right) \tag{5.4.18}$$

We can see from (5.4.16) and (5.4.18) that both solitons have velocity -4ξ and amplitude $2\sqrt{2}\eta/|\lambda_1|^{1/2}$.

In this chapter we prove Theorem 1.3.8 by studying the RHP given by Problem 5.1.1 to reconstruct q on $(-a, \infty)$ for any $a > 0$, and studying the RHP given by Problem 6.3.1 to reconstruct q on $(-\infty, a)$ for any such a. Recall the construction formula given by (1.2.20). Let us write $q(x) = q_1(x) + q_2(x)$ where

$$\begin{aligned} q_1(x) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \nu_{11}(x,s)\rho(s)e^{-2isx}ds \\ q_2(x) &= -\sum_i 2i\nu_i^-(x)\overline{C_i}e^{-2i\overline{\lambda_i}x} \end{aligned}$$

Direct computation gives

$$(q_1)_{xx} = \frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}4\lambda^2\rho(1+\nu^\flat) - \frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\rho(\lambda)\left(-4i\lambda\nu_x^\flat + \nu_{xx}^\flat\right) \tag{6.0.1}$$

$$x^2 q_1(x) = -\frac{1}{4\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\rho''(1+\nu^\flat) - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\left(2\rho'\nu_\lambda^\flat + \rho\nu_{\lambda\lambda}^\flat\right) \tag{6.0.2}$$

and

$$\begin{aligned} (q_2)_{xx} &= -\sum_i 2i\left(\nu_i^-(x)\right)_{xx}\overline{C_i}e^{-2i\overline{\lambda_i}x} - \sum_i 4\overline{\lambda}_i\left(\nu_i^-(x)\right)_x\overline{C_i}e^{-2i\overline{\lambda_i}x} \\ &\quad + \sum_i 8i\overline{\lambda}_i^2\nu_i^-(x)\overline{C_i}e^{-2i\overline{\lambda_i}x} \end{aligned} \tag{6.0.3}$$

$$x^2 q_2(x) = -x^2\sum_i 2i\nu_i^-(x)\overline{C_i}e^{-2i\overline{\lambda_i}x} \tag{6.0.4}$$

where for (6.0.2) we used $(-2i)^{-1}(d/d\lambda)e^{-2i\lambda x} = xe^{-2i\lambda x}$ and integrated by parts. From these formulas and the mapping properties of the Fourier transform, we are going to obtain the sufficient conditions for Lipschitz continuity of the map $\{\rho, \{\lambda_i, C_i\}_{i=1}^n\} \to q$.

We'll make repeated use of the operator $\mathscr{S}$ given by the following formulas

$$\begin{aligned} \mathscr{S}[h](x,\lambda) &= C_{\mathbb{R}}^-\left[C_{\mathbb{R}}^+\left(\rho(\,\cdot\,)h(\,\cdot\,)e^{-2ix(\,\cdot\,)}\right)(\diamond)\left((\diamond)\overline{\rho(\diamond)}e^{2ix(\diamond)}\right)\right](\lambda) \\ &= \frac{1}{2i\pi^2}\int_0^{-\infty} e^{2i\lambda\xi}\int_x^\infty (\widehat{\rho} * \widehat{h})(\xi')\,\widehat{\overline{\rho}}\,'(\xi-\xi')\,d\xi'\,d\xi, \quad h \in L^2 \end{aligned} \tag{6.0.5}$$

$$\begin{aligned} \mathscr{S}[1](x,\lambda) &= C_{\mathbb{R}}^-\left\{C_{\mathbb{R}}^+\left[\rho(\diamond)e^{-2i(\diamond)x}\right](\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2i(\,\cdot\,)x}\right\}(\lambda) \\ &= \frac{1}{2i\pi^2}\int_0^{-\infty} e^{2i\lambda\xi}\int_x^\infty \widehat{\rho}(\xi')\,\widehat{\overline{\rho}}\,'(\xi-\xi')\,d\xi'\,d\xi. \end{aligned} \tag{6.0.6}$$

These formulas follow from elementary properties of the Fourier transform and the fact that $C^\pm$ act in Fourier representation as multiplication by the characteristic functions of $\mathbb{R}^\pm$.

Remark 6.0.1. The operator $\mathscr{S}$ is denoted $A_{00}B_{00}$ in Lemma 6.1.5 and the first term of $\mathcal{K}_{00}$ in (6.2.12). Different notations are used in different contexts.

For Problem 5.1.1 , we solve

$$\nu_1^\sharp = \mathbf{f}^\sharp + \mathcal{K}[\nu_1^\sharp] \tag{6.0.7}$$

for $\nu_1^\sharp(x,\,\cdot\,) \in L^2(\mathbb{R}) \oplus \mathbb{C}^n$ and $\mathbf{f}^\sharp$ given by (6.2.8). Similarly, for Problem 6.3.1, we solve

$$\tilde{\nu}_1^\sharp = \tilde{\mathbf{f}}^\sharp + \tilde{\mathcal{K}}[\tilde{\nu}_1^\sharp] \tag{6.0.8}$$

for $\tilde{\nu}^\sharp(x,\,\cdot\,) \in L^2(\mathbb{R}) \oplus \mathbb{C}^n$. Let

$$V_n \subset S \times (\mathbb{C}_\times \times \mathbb{C}^+)^n$$

be the bounded open subset given in Definition 1.3.5. We will first prove:

Proposition 6.0.2. *For any $a > 0$, the map*

$$(\rho, \{C_i, \lambda_i\}_{i=1}^n) \mapsto q \tag{6.0.9}$$

defined by Problem 5.1.1 and the formula (5.3.13) on any bounded subset of V_n is a locally Lipschitz continuous map from V to $H^{2,2}(-a,\infty)$.

We give the proof of Proposition 6.0.2 in Section 6.2. By essentially identical arguments, we may prove:

Proposition 6.0.3. *For any $a > 0$, the map*

$$(\tilde{\rho}, \{\widetilde{C}_i, \lambda_i\}_{i=1}^n) \mapsto \tilde{q} \tag{6.0.10}$$

defined by Problem 6.3.1 and the formula (6.3.17) on any bounded subset of V_n is a locally Lipschitz continuous map from V to $H^{2,2}(-\infty,a)$.

Finally, we will prove:

Proposition 6.0.4. *For any $a > 0$, we have $q(x) = \tilde{q}(x)$ for $x \in (-a,a)$, so the maps (6.0.9)–(6.0.10) together define a locally Lipschitz mapping $\mathcal{I}$ on any bounded subset of V_n*

$$(\rho, \{C_i, \lambda_i\}_{i=1}^n) \mapsto q$$

with the property that $\mathcal{I}(\mathcal{R}(q)) = q$ on U, an open neighborhood of 0 in $H^{2,2}(\mathbb{R})$, and $\mathcal{R}\circ\mathcal{I} = I$ on V.

Before proving the propositions above, we first establish the mapping properties of the operator $\mathscr{S}$ and estimates on the resolvent operator $(I-\mathscr{S})^{-1}$ which will be extensively used in the next section. We will need the Banach space

$$W_0 = \left\{ f \in L^2(\mathbb{R}) : \widehat{f},\ \widehat{f'} \in L^1(\mathbb{R}) \right\}. \tag{6.0.11}$$

in which $H^{2,2}(\mathbb{R})$ is compactly embedded.

Lemma 6.0.5. *Suppose that $\rho \in H^{2,2}(\mathbb{R})$ then*

(i) $\mathscr{S}$, $\mathscr{S}_x$, *and* $\mathscr{S}_{xx}$ *are bounded operators on* L^2 *with norm* $\leqslant C\left\|\rho\right\|^2_{H^{2,2}}$ *where* C *is independent of* x*.*

(ii) The operator $\mathscr{S}$ *is Hilbert-Schmidt with* $\left\|\mathscr{S}\right\|_{\mathrm{HS}} \leqslant C\left\|\rho\right\|^2_{H^{2,2}}$ *and*

$$\lim_{x\to\infty}\left\|\mathscr{S}\right\|_{\mathrm{HS}} = 0$$

uniformly in bounded subsets of $H^{2,2}(\mathbb{R})$*.*

(iii) For any $a \geqslant 0$*, the map*

$$\begin{aligned}(-a,\infty)\times W_0 &\longrightarrow \mathcal{B}(L^2)\\ (x,\rho) &\mapsto \mathscr{S}\end{aligned}$$

is continuous.

Proof. (i) Let $\mathcal{F}$ denote the Fourier transform. From (6.0.5) we see that the operator $\mathcal{F}\mathscr{S}\mathcal{F}^{-1}$ has integral kernel

$$K(\xi,\xi'';x) = \int_x^\infty \widehat{\bar{\rho}}\,'(\xi-\xi')\widehat{\rho}(\xi'-\xi'')\,d\xi', \quad \xi\leqslant 0 \tag{6.0.12}$$

up to trivial constants, so that

$$\left\|\mathcal{F}(\mathscr{S}\check{h})\right\|_{L^2} \leqslant C\left\|\widehat{\rho}\right\|_{L^1}\left\|\widehat{\rho}'\right\|_{L^1}\left\|h\right\|_{L^2}. \tag{6.0.13}$$

The estimate (6.0.13) shows that $\left\|\mathscr{S}\right\|_{\mathcal{B}(L^2)}$ is bounded by $C\left\|\rho\right\|^2_{H^{2,2}}$ and from (6.0.11), we also have $\left\|S\right\|_{\mathcal{B}(L^2)} \leqslant C\left\|\rho\right\|^2_{W_0}$. Differentiating (6.0.12) with respect to x we have

$$K_x(\xi,\xi'';x) = -\widehat{\bar{\rho}}\,'(\xi-x)\widehat{\rho}(x-\xi'') \tag{6.0.14}$$

so that

$$\left\|\mathscr{S}_x\right\|_{\mathcal{B}(L^2)} \leqslant \left\|\mathscr{S}_x\right\|_{\mathrm{HS}} \leqslant \left\|\widehat{\rho}\,'\right\|_{L^2}\left\|\widehat{\rho}\right\|_{L^2}.$$

Differentiating again we find

$$\left\|\mathscr{S}_{xx}\right\|_{\mathcal{B}(L^2)} \leqslant \left\|\mathscr{S}_{xx}\right\|_{\mathrm{HS}} \leqslant \left\|\widehat{\rho}\,''\right\|_{L^2}\left\|\widehat{\rho}\right\|_{L^2} + \left\|\widehat{\rho}\,'\right\|^2_{L^2}.$$

(ii) From (6.0.12) and integration by parts we have

$$\begin{aligned}K(\xi,\xi'';x) &= -\widehat{\bar{\rho}}(\xi-x)\widehat{\rho}(x-\xi'') - \int_x^\infty \widehat{\bar{\rho}}(\xi-\xi')\widehat{\rho}'(\xi'-\xi'')\,d\xi'\\ &= K_1(\xi,\xi'';x) + K_2(\xi,\xi'';x)\end{aligned}$$

where $\xi < 0$ and $\xi' \geqslant x \geqslant -a$. Clearly

$$\left\|K_1\right\|_{L^2(\mathbb{R}^-\times\mathbb{R})} \leqslant \left(\int_x^\infty \left|\widehat{\rho}(t)\right|^2\,dt\right)^{1/2}\left\|\widehat{\rho}\right\|_{L^2} \leqslant C_a(1+|x|)^{-1/2}\left\|\widehat{\rho}\right\|_{L^{2,1}}\left\|\widehat{\rho}\right\|_{L^2}. \tag{6.0.15}$$

Since $|\xi' - \xi| \geqslant x$ and $|\xi - \xi'| \geqslant \xi'$, we have

$$|\widehat{\rho}(\xi - \xi')| \leqslant C_a(1+|x|)^{-1/2}(1+|\xi'|)^{-3/2}(1+|\xi-\xi'|)^2\,|\widehat{\rho}(\xi-\xi')|$$

It follows that

$$|K_2(\xi,\xi'')| \leqslant C_a(1+|x|)^{-1/2}\,\|\widehat{\rho}\|_{L^{2,2}}\,\|\widehat{\rho}'\|_{L^2} \tag{6.0.16}$$

The estimates (6.0.15)–(6.0.16) show that

$$\|\mathscr{S}\|_{\mathrm{HS}} \leqslant C_a(1+|x|)^{-1/2}\,\|\widehat{\rho}\|_{L^{2,2}}\,\|\widehat{\rho}\|_{H^1}$$

which proves (ii).

(iii) Write $\mathscr{S} = \mathscr{S}_{x,\rho}$. Using the technique that proved (6.0.13) we have

$$\|\mathscr{S}_{x,\rho_1} - \mathscr{S}_{x,\rho_2}\|_{\mathcal{B}(L^2)} \leqslant C_a\left(\|\widehat{\rho}_1 - \widehat{\rho}_2\|_{L^1}\,\|\widehat{\rho}_1'\|_{L^1} + \|\widehat{\rho}_2\|_{L^1}\,\|\widehat{\rho}_1' - \widehat{\rho}_2'\|_{L^1}\right) \tag{6.0.17}$$

uniformly in $x \geqslant -a$. On the other hand, by (6.0.14) and Young's inequality,

$$\|\partial\mathscr{S}/\partial x\|_{\mathcal{B}(L^2)} \leqslant C\,\|\widehat{\rho}'\|_{L^1}\,\|\widehat{\rho}\|_{L^1}$$

so that

$$\|\mathscr{S}_{x,\rho} - \mathscr{S}_{y,\rho}\|_{\mathcal{B}(L^2)} \leqslant C|x-y|^{1/2}\,\|\widehat{\rho}'\|_{L^1}\,\|\widehat{\rho}\|_{L^1}\,. \tag{6.0.18}$$

Combining (6.0.17) and (6.0.18) we obtain the claimed continuity. □

Remark 6.0.6. Since all estimates in the proof of Lemma 6.0.5 are bilinear in ρ, it follows that $\rho \mapsto \mathscr{S}$, $\rho \mapsto \mathscr{S}_x$, and $\rho \mapsto \mathscr{S}_{xx}$ are locally Lipschitz maps from $H^{2,2}$ to the bounded operators on L^2.

We can now construct the resolvent $(I-\mathscr{S})^{-1}$ as a bounded operator on L^2. Although we do not obtain the kind of explicit integral representation we obtained for resolvents in the direct problem, we are able to extend the resolvent family to a bounded operator on the space $L^2((-a,\infty)\times\mathbb{R})$ by the uniformity of the estimate (6.0.19) with respect to x.

Lemma 6.0.7. *Suppose that $\rho \in H^{2,2}(\mathbb{R})$ and $a > 0$. The resolvent $(I-\mathscr{S})^{-1}$ exists as a bounded operator on L^2 and*

$$\sup_{x\geqslant -a}\left\|(I-\mathscr{S})^{-1}\right\|_{\mathcal{B}(L^2)} \leqslant C \tag{6.0.19}$$

with C uniform in ρ in a fixed bounded subset of $H^{2,2}(\mathbb{R})$. Moreover

$$\sup_{x\geqslant -a}\left\|(I-\mathscr{S}_{x,\rho})^{-1} - (I-\mathscr{S}_{x,\sigma})^{-1}\right\|_{\mathcal{B}(L^2)} \leqslant C\,\|\rho-\sigma\|_{H^{2,2}} \tag{6.0.20}$$

with C uniform in ρ, σ in a fixed bounded subset of $H^{2,2}(\mathbb{R})$.

Proof. The estimate (6.0.20) follows from (6.0.19) and the second resolvent identity. It suffices to prove (6.0.19).

By Proposition 5.2.8, Lemma 6.0.5(ii), and Fredholm theory, the resolvent $(I - \mathscr{S}_{x,\rho})^{-1}$ exists for all $(x, \rho) \in \mathbb{R} \times H^{2,2}(\mathbb{R})$.

To bound the norm of the resolvent, fix a bounded subset B of $H^{2,2}(\mathbb{R})$. By Lemma 6.0.5(ii), there is an R so that $\|\mathscr{S}_{x,\rho}\|_{\mathcal{B}(L^2)} < 1/2$ for $x \geqslant R$ and all $\rho \in B$. Thus $(I - \mathscr{S}_{x,\rho})^{-1}$ exists for all such (x, ρ) and $\|(I - \mathscr{S}_{x,\rho})^{-1}\|_{\mathcal{B}(L^2)} \leqslant 2$. To control the resolvent for $(x, \rho) \in [-a, R] \times B$, we note that $[-a, R] \times B$ is a compact subset of $\mathbb{R} \times X$ By Lemma 6.0.5(iii) and the second resolvent formula, the map $(x, \rho) \to (I - \mathscr{S}_{x,\rho})^{-1}$ is a continuous map from $\mathbb{R} \times W_0$ to $\mathcal{B}(L^2)$. The continuous image of the set $[-a, R] \times B$ is compact, hence bounded, in $\mathcal{B}(L^2)$. $\square$

Remark 6.0.8. (i) The family of operators $(I - \mathscr{S}_{x,\rho})^{-1}$ for $x \geqslant -a$ defines a bounded operator R_ρ from $L^2((-a, \infty) \times \mathbb{R})$ to itself by the formula

$$(R_\rho f)(x, \cdot) = (I - \mathscr{S}_{x,\rho})^{-1} f(x, \cdot).$$

By Lemma 6.0.7 the map $\rho \to R_\rho$ is locally bounded and locally Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $\mathcal{B}(L^2((-a, \infty) \times \mathbb{R}))$. (ii) Lemma 6.0.5(i) shows that $\mathscr{S}$, $\mathscr{S}_x$, and $\mathscr{S}_{xx}$ extend in the same way to bounded operators on $L^2((-a, \infty) \times \mathbb{R})$ to itself, Lipschitz continuous in ρ.

6.1 Uniform Resolvent Bound

The reconstruction procedure can be reduced in each case to solving an $n+1$ vector-valued integral equation . We will show that, by iteration, we can decouple the system (5.1.4)-(5.1.9) into integral equations for $\left(\nu_{11}, \left\{\nu_i^- - 1\right\}\right)$ and $\left(\nu_{12}, \left\{\nu_i^+\right\}\right)$. Let

$$X = L^2(\mathbb{R}) \oplus \mathbb{C}^n.$$

We expect that the vectors

$$\begin{aligned} \nu_1^\sharp &= \left(\nu_{11} - 1, \{\nu_i^- - 1\}\right) \\ \nu_2^\sharp &= \left(\nu_{12}, \{\nu_i^+\}\right) \end{aligned}$$

each belong to X. Now set $Y = X \oplus X$ and

$$\nu^\sharp = \left(\nu_1^\sharp, \nu_2^\sharp\right).$$

We expect that $\nu^\sharp \in Y$. With respect to the direct sum decomposition $Y = X \oplus X$, the system of equations (5.1.4)– (5.1.9) takes the form

$$\nu^\sharp = f + K\nu^\sharp \tag{6.1.1}$$

where

$$K = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \tag{6.1.2}$$

and the operators $A, B : X \to X$ are defined as follows. With respect to the decomposition $L^2(\mathbb{R}) \oplus \mathbb{C}^n$, write

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}, \quad B = \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$$

Denote by h a generic function in $L^2(\mathbb{R})$ and by $\{h_j\}_{j=1}^n$ a generic n-tuple in $\mathbb{C}^n$. Then

$$A_{00}[h](\lambda) = C_- \left(h(\,\cdot\,)(\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2i(\,\cdot\,)x} \right)(\lambda) \tag{6.1.3}$$

$$A_{01}[h_1, \ldots, h_n](\lambda) = \sum_{k=1}^{n} \frac{h_k C_k \lambda_k e^{2i\lambda_k x}}{\lambda - \lambda_k} \tag{6.1.4}$$

$$A_{10}[h] = \left\{ \int_{-\infty}^{\infty} \frac{h(x,s) s\overline{\rho(s)} e^{2isx}}{s - \overline{\lambda_j}} \frac{ds}{2\pi i} \right\}_{j=1}^{n} \tag{6.1.5}$$

$$A_{11}[h_1, \ldots, h_n] = \left\{ \sum_{k=1}^{n} \frac{h_k C_k \lambda_k e^{2i\lambda_k x}}{\overline{\lambda_j} - \lambda_k} \right\}_{j=1}^{n} \tag{6.1.6}$$

and

$$B_{00}[h](\lambda) = C_+ \left(h(\,\cdot\,)\rho(\,\cdot\,)e^{-2i(\,\cdot\,)x} \right)(\lambda)(\lambda) \tag{6.1.7}$$

$$B_{01}[h_1, \ldots, h_n](\lambda) = -\sum_{k=1}^{n} \frac{h_k \overline{C_k} e^{-2i\overline{\lambda_k}x}}{\lambda - \overline{\lambda_k}} \tag{6.1.8}$$

$$B_{10}[h] = \left\{ \int_{-\infty}^{\infty} \frac{h(x,s)\rho(s) e^{-2isx}}{s - \lambda_j} \frac{ds}{2\pi i} \right\}_{j=1}^{n} \tag{6.1.9}$$

$$B_{11}[h_1, \ldots, h_n] = -\left\{ \sum_{k=1}^{n} \frac{h_k \overline{C_k} e^{-2i\overline{\lambda_k}x}}{\lambda_j - \overline{\lambda_k}} \right\} \tag{6.1.10}$$

Boundedness of these operators on their respective spaces follows from the facts that ρ and $(\,\cdot\,)\rho(\,\cdot\,)$ belong to $L^\infty(\mathbb{R})$, that the Cauchy projectors $C^\pm$ are bounded operators on L^2, and the explicit formulas.

The inhomogeneous term f is given by $0 \oplus f_2$ where

$$f_2 = \begin{pmatrix} C_+ \left(\rho(\,\cdot\,)e^{-2i(\,\cdot\,)x} \right) - \sum_{k=1}^{n} \dfrac{\overline{C_k} e^{-2i\overline{\lambda_k}x}}{\lambda - \overline{\lambda_k}} \\ \\ \displaystyle\int_{-\infty}^{\infty} \frac{\rho(s) e^{-2isx}}{s - \lambda_1} \frac{ds}{2\pi i} - \sum_{k=1}^{n} \frac{\overline{C_k} e^{-2i\overline{\lambda_k}x}}{\lambda_1 - \overline{\lambda_k}} \\ \vdots \\ \displaystyle\int_{-\infty}^{\infty} \frac{\rho(s) e^{-2isx}}{s - \lambda_n} \frac{ds}{2\pi i} - \sum_{k=1}^{n} \frac{\overline{C_k} e^{-2i\overline{\lambda_k}x}}{\lambda_n - \overline{\lambda_k}} \end{pmatrix}. \tag{6.1.11}$$

It will be useful to iterate (6.1.1) to obtain

$$\nu^\sharp = f + Kf + K^2 \nu^\sharp \tag{6.1.12}$$

since

$$K^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} \tag{6.1.13}$$

so that the system decouples into separate equations for $\nu_1^\sharp$ and $\nu_2^\sharp$. This means that it suffices to solve the iterated equation for $\nu_1^\sharp$, which yields functions ν_{11} and $\{\nu_i^- - 1\}$ for the reconstruction formula (5.3.13). We will prove that there exists a unique solution to (6.1.1) using Fredholm theory and its connection to Problem 5.1.1; we will carry out estimates using (6.1.12). Owing to the special structure of the problem, the resolvent $(I-K)^{-1}$ needed to solve (6.1.1) exists if and only if the resolvent $(I-K^2)^{-1}$ needed to solve (6.1.12) exists.

Lemma 6.1.1. *Suppose that X is a Banach space and $A, B \in \mathcal{B}(X)$ are operators with the property that AB and BA are compact. Let K be given by (6.1.2). Then $\ker(I-K)$ is trivial if and only if $\ker(I-K^2)$ is trivial.*

Proof. First, recall that if A and B are bounded operators, then $(I-AB)^{-1}$ exists if and only if $(I-BA)^{-1}$ exists. Since AB and BA are compact it follows that $\ker(I-AB)$ is trivial if and only if $\ker(I-BA)$ is trivial. Hence, $\ker(I-K^2)$ is trivial if and only if $\ker(I-AB)$ is trivial. Finally, a simple computation shows that the map

$$\psi \mapsto (\psi, B\psi)$$

is an isomorphism from $\ker(I-AB)$ onto $\ker(I-K)$, so $\ker(I-K)$ is trivial if and only if $\ker(I-AB)$ is trivial. □

We prove the existence of $(I-K^2)^{-1}$ in two steps. First, we show that $\ker(I-K)$ is trivial by exploiting the uniqueness of solutions for the Problem 5.1.1.

Lemma 6.1.2. $\ker(I-K)$ *is trivial.*

Proof. See Corollary 5.2.8. □

Lemma 6.1.3. *The operator $I-K^2$ is Fredholm on X.*

Proof. It suffices to show that K^2 is compact on X, or equivalently that AB and BA are compact on Y. We give the argument for AB since the argument for BA is similar. In Lemma 6.2.8, we will show that $A_{00}B_{00}$ is compact. Since all of the operators A_{ij}, B_{ij} with $(i,j) \neq (0,0)$ are either finite-rank or bounded, and their products are all compositions of bounded (finite rank) operators with finite rank (bounded) operators, thus compactness of K^2 is immediate. □

Now we prove a uniform resolvent bound needed for the Lipschitz continuity of the inverse scattering map. We will prove uniformity over sets of the following form.

Proposition 6.1.4. *Fix $a \in \mathbb{R}$ and a bounded subset of V_n as in Definition 1.3.5. Then,*

$$\sup_{x>-a} \left\| (I-K^2)^{-1} \right\|_{\mathcal{B}(Y)} \lesssim_{a,V_n} 1.$$

The first step is to obtain a large-x bound on the resolvent.

Lemma 6.1.5. *Fix $a \in \mathbb{R}$ and a set V_n as in Definition 1.3.5. Then, there is a number $R > 0$ depending on V_n so that*

$$\sup_{x \geqslant R} \left\| (I - K^2)^{-1} \right\|_{\mathcal{B}(Y)} \lesssim_{a,V_n} 1$$

Proof. Our goal is to show that $\|AB\| \to 0$ as $x \to \infty$. Since

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A$$

and $\|A\|_{\mathcal{B}(X)}$, $\|B\|_{\mathcal{B}(X)}$ are bounded uniformly in x for scattering data in S, it suffices to show that $(I - AB)^{-1}$ is uniformly bounded for large x. But

$$I - AB = \begin{pmatrix} I - C_{00} & -C_{01} \\ \\ -C_{10} & I - C_{11} \end{pmatrix}$$

where

$$C_{00} = A_{00}B_{00} + A_{01}B_{10} \qquad\qquad C_{01} = A_{00}B_{01} + A_{01}B_{11}$$
$$C_{10} = A_{10}B_{00} + A_{11}B_{10} \qquad\qquad C_{11} = A_{10}B_{01} + A_{11}B_{11}.$$

The explicit form of C_{00}, C_{01}, C_{10} and C_{11} is given by (6.2.13)–(6.2.16) respectively. It is shown in Lemma 6.0.5 that

$$\lim_{x \to +\infty} \|A_{00}B_{00}\|_{\mathcal{B}(L^2(\mathbb{R}))} = 0.$$

uniformly in ρ in a bounded subset of $H^{2,2}(\mathbb{R})$ From the explicit formulas, it is easy to see that

$$\|A_{01}\|_{\mathcal{L}(\mathbb{C}^n, L^2(\mathbb{R}))}, \quad \|A_{11}\|_{\mathcal{L}(\mathbb{C}^n)}, \quad \|B_{01}\|_{\mathcal{L}(\mathbb{C}^n, L^2(\mathbb{R}))}, \quad \|B_{11}\|_{\mathcal{L}(\mathbb{C}^n)}$$

vanish as $x \to \infty$ owing to the exponential decay in x of factors $e^{2i\lambda_k x}$. On the other hand,

$$\|A_{00}\|_{\mathcal{B}(L^2(\mathbb{R}))}, \quad \|B_{00}\|_{\mathcal{B}(L^2(\mathbb{R}))}, \quad \|A_{10}\|_{\mathcal{L}(L^2(\mathbb{R}), \mathbb{C}^n)}, \quad \|B_{10}\|_{\mathcal{L}(L^2(\mathbb{R}), \mathbb{C}^n)}$$

are all uniformly bounded for ρ in a bounded subset of $H^{2,2}$ and $\{\lambda\}_{i=1}^n$ in a bounded subset of $\mathbb{C}^+$. Thus it remains only to estimate $\|A_{10}B_{00}\|_{\mathcal{L}(L^2(\mathbb{R}), \mathbb{C}^n)}$. A single entry is given by

$$[(A_{10}B_{00}h)]_j = \int_{\mathbb{R}} \frac{C^+ \left[h(\cdot)\rho(\cdot)e^{-2ix(\cdot)} \right](s)}{s - \overline{\lambda}_j} s\overline{\rho(s)} e^{2isx} \frac{ds}{2\pi i}$$

For $h \in L^2(\mathbb{R})$ and $z \in \mathbb{C}^-$, we have

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{s - z} h(s) ds = -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1}{s - z} \left(C^- h \right)(s) ds$$

since $I = C^+ - C^-$, C^+h is the boundary value of a function analytic in the upper half-plane, and we may close the contour to show that the contribution from the C^+ term is zero. It follows that

$$[(A_{10}B_{00}h)]_j = \mathcal{C}_{\mathbb{R}}[(A_{00}B_{00})h](\overline{\lambda}_j)$$

so that

$$|[(A_{10}B_{00}h)]_j| \leqslant c(\mathrm{Im}\lambda_j) \left\| (A_{00}B_{00})h \right\|_{L^2(\mathbb{R})}$$

which goes to zero as $x \to \infty$ uniformly in ρ in a bounded subset of $H^{2,2}(\mathbb{R})$ and $h \in L^2(\mathbb{R})$ by Lemma 6.0.5. □

Next, we bound

$$\sup_{-a\leqslant x\leqslant R}\left\|(I-K^2)^{-1}\right\|_{\mathcal{B}(X)}$$

using a continuity-compactness argument. We will embed V_n in a compact subset of the space $Z_0 = W_0 \times (\mathbb{C}_\times)^n \times (\mathbb{C}^+)^n$ where W_0 is given by (6.0.11). The proof of Lemma 6.0.7 can easily be adapted to show that the map

$$\begin{aligned}[][-a,\infty)\times Z_0 &\longrightarrow \mathcal{B}(X)\\ (x,(\rho,\{C_k\}_{k=1}^n,\{\lambda_k\}_{k=1}^n)) &\mapsto K\end{aligned}$$

is continuous. By mimicking the arguments given there we can prove:

Lemma 6.1.6. *Fix $a\in\mathbb{R}$, $R>0$, and a bounded subset of V_n as in Defintion 1.3.5. Then,*

$$\sup_{-a\leqslant x\leqslant R}\left\|(I-K^2)^{-1}\right\|_{\mathcal{B}(X)} \lesssim_{a,R,V_n} 1.$$

Proof of Proposition 6.1.4. An immediate consequence of Lemmas 6.1.5 and 6.1.6. □

6.1.1 Lipschitz Continuity of the Resolvent

The resolvent $(I-K^2)^{-1}$ depends on x as parameter; we now use the notation $(I-K_x^2)^{-1}$ to emphasize this dependence. The operator K lifts to an operator $K^\sharp$ on the space

$$X^\sharp = L^2([-a,\infty),X) \tag{6.1.14}$$

via the formula

$$\left[K^\sharp F\right](x,\lambda) = \left[K_x F(x,\,\cdot\,)\right](\lambda).$$

Immediately from Proposition 6.1.4, we have

Proposition 6.1.7. *Fix $a\in\mathbb{R}$ and a set V_n as in Definition 1.3.5. Then*

$$\left\|(I-(K^\sharp)^2)^{-1}\right\|_{B(X^\sharp)} \lesssim_{a,V_n} 1$$

Denote by v_n and v_n' elements of the set V_n. We write $K^\sharp = K^\sharp(v_n)$ to emphasize the dependence of $K^\sharp$ on the scattering data. For $v_n = (\rho,\{C_k,\lambda_k\}_{k=1}^n)$ define

$$\|v_n\|_S = \|\rho\|_{H^{2,2}} + \sup_{1\leqslant k\leqslant n}|C_k| + \sup_{1\leqslant k\leqslant n}|\lambda_k|.$$

Lemma 6.1.8. *The estimate*

$$\left\|K^\sharp(v_n)-K^\sharp(v_n')\right\|_{\mathcal{B}(X^\sharp)} \lesssim_{a,v_n} \|v_n-v_n'\|_{V_n}$$

holds.

We omit the proof, which is an easy consequence of the explicit formulae (6.1.3) – (6.1.10). It is clear that a similar estimate holds for $\left\|K^\sharp(s)^2-K^\sharp(s')^2\right\|_{\mathcal{B}(X^\sharp)}$.

From Lemma 6.1.8, Proposition 6.1.7, and the second resolvent formula, we immediately obtain:

Proposition 6.1.9. *Fix $a\in\mathbb{R}$, $c>0$ and a set S as in Definition 1.3.4. Then, for any $s,s'\in S$,*

$$\left\|(I-K^\sharp(s)^2)^{-1}-(I-K^\sharp(s')^2)^{-1}\right\|_{\mathcal{B}(X^\sharp)} \lesssim_{a,c,S} \|s-s'\|_S\,.$$

6.2 Mapping Properties

From (6.1.12) we obtain an integral equation for $\nu^\flat(x,\lambda) := \nu_{11}(x,\lambda) - 1$

$$\nu_1^\sharp = \mathbf{f}^\sharp + \mathcal{K}[\nu_1^\sharp] \tag{6.2.1}$$

In this case the operator $\mathcal{K} = AB$ where AB is given by (6.1.13) and

$$\nu_1^\sharp = \big(\nu_{11} - 1, \{\nu_i^- - 1\}\big) = \big(\nu^\flat(x,\lambda),\ \nu^\flat(x,\overline{\lambda}_1)\ \ldots\ \nu^\flat(x,\overline{\lambda}_n)\big)^T$$

We now use the preceding analysis to show that the reconstruction formula (1.2.20) defines a continuous map from $S \times (\mathbb{C}^+ \times \mathbb{C}_\times)^n$ to $H^{2,2}((-a,\infty))$.

We want to obtain the following two lemmas:

Lemma 6.2.1. *Suppose that the maps*

$$(\rho, \{\lambda_k, C_k\}_{k=1}^n) \to \nu_i^- \tag{6.2.2}$$

defined on a bounded subset of V_n are Lipschitz continuous. Then

$$(\rho, \{\lambda_k, C_k\}_{k=1}^n) \longmapsto q_2$$

is Lipschitz continuous.

Lemma 6.2.2. *Suppose that the following functions*

$$\begin{gathered}\nu_{11}(x,\lambda) - 1, \quad \partial\nu_{11}(x,\lambda)/\partial x, \quad \partial^2\nu_{11}/\partial x^2, \\ \partial\nu_{11}(x,\lambda)/\partial\lambda, \quad \langle\lambda\rangle^{-1}\partial^2\nu_{11}(x,\lambda)/\partial\lambda^2\end{gathered} \tag{6.2.3}$$

are all Lipschitiz continuous maps from a bounded subset of V_n to $L^2((-a,\infty)\times\mathbb{R})$. Then the map

$$(\rho, \{\lambda_k, C_k\}_{k=1}^n) \longmapsto q_1$$

is Lipschitz continuous.

The proof of Lemma 6.2.1 and 6.2.2 follows directly from the expressions given in (6.0.1)–(6.0.4).

To get the Lipschitz continuous maps (6.2.2) and (6.2.3) listed in Lemma 6.2.1 and Lemma 6.2.2 we study the mapping properties of the integral equation (6.2.1) and also the following equations involves derivatives in λ and x:

$$(\nu_1^\sharp)_\lambda = \mathbf{f}_\lambda^\sharp + (\mathcal{K}[\nu_1^\sharp])_\lambda \tag{6.2.4}$$

$$\langle\lambda\rangle^{-1}(\nu_1^\sharp)_{\lambda\lambda} = \langle\lambda\rangle^{-1}\mathbf{f}_{\lambda\lambda}^\sharp + \langle\lambda\rangle^{-1}(\mathcal{K}[\nu_1^\sharp])_{\lambda\lambda} \tag{6.2.5}$$

$$(\nu_1^\sharp)_x = \mathbf{f}_x^\sharp + \mathcal{K}_x[\nu_1^\sharp] + \mathcal{K}[(\nu_1^\sharp)_x] \tag{6.2.6}$$

$$(\nu_1^\sharp)_{xx} = \mathbf{f}^\sharp_{xx} + 2\mathcal{K}_x[(\nu_1^\sharp)_x] + \mathcal{K}_{xx}[\nu_1^\sharp] + \mathcal{K}[(\nu_1^\sharp)_{xx}] \tag{6.2.7}$$

First, we show that the inhomogeneous term

$$\mathbf{f}^\sharp = A[f_2] = \left(f^\sharp(x,\lambda)\ f^\sharp(x,\overline{\lambda}_1)\ \dots\ f^\sharp(x,\overline{\lambda}_n)\right)^T$$

has the required properties. For scattering data

$$(\rho, \{\lambda_k, C_k\}_{k=1}^n) \in S \times (\mathbb{C}^+ \times \mathbb{C}_\times)^n$$

we give the explicit formulas:

$$f^\sharp(x,\lambda) = f_1^\sharp(x,\lambda) + f_2^\sharp(x,\lambda) + f_3^\sharp(x,\lambda) + f_4^\sharp(x,\lambda) \tag{6.2.8}$$

where

$$\begin{aligned}
f_1^\sharp(x,\lambda) &= \sum_k \frac{1}{\lambda-\lambda_k}\left(-\sum_j \frac{\overline{C}_j e^{-2i\overline{\lambda}_j x}}{\overline{\lambda}_j - \lambda_k} C_k \lambda_k e^{2i\lambda_k x}\right)\\
f_2^\sharp(x,\lambda) &= \sum_k \frac{1}{\lambda-\lambda_k}\left(\int_{\mathbb{R}} \frac{\rho(s)e^{-2ixs}}{s-\lambda_k}\frac{ds}{2\pi i} C_k\lambda_k e^{2i\lambda_k x}\right)\\
f_3^\sharp(x,\lambda) &= -\sum_k \overline{C}_k e^{-2i\overline{\lambda}_k x} C^- \left[\frac{(\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2ix(\,\cdot\,)}}{(\,\cdot\,)-\overline{\lambda}_k}\right](\lambda)\\
f_4^\sharp(x,\lambda) &= C^- \left\{C^+\left[\rho(\diamond)e^{-2i(\diamond)x}\right](\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2i(\,\cdot\,)x}\right\}(\lambda)
\end{aligned}$$

We can get $f^\sharp(x,\overline{\lambda}_j)$, $1 \leqslant j \leqslant n$, by substituting $\overline{\lambda}_j$ for $\lambda \in \mathbb{R}$ and changing the corresponding Cauchy projection C_- to a Cauchy integral over the real line.

Lemma 6.2.3. *For any fixed $x \in (-a,\infty)$, $f^\sharp$ given by (6.2.8) and indices $i = 1,2,3,4$ and $1 \leqslant j \leqslant n$ we have that*

$$f_i^\sharp(x,\overline{\lambda}_j), \quad \partial f_i^\sharp(x,\overline{\lambda}_j)/\partial x, \quad \partial^2 f_i^\sharp(x,\overline{\lambda}_j)/\partial x^2$$

are all contained in some bounded set of $\mathbb{C}$.

Lemma 6.2.4. *For $f^\sharp$ given by (6.2.8) and indices $i = 1,2,3$ and $1 \leqslant j \leqslant n$ we have that*

$$f_i^\sharp(x,\overline{\lambda}_j), \quad \partial f_i^\sharp(x,\overline{\lambda}_j)/\partial x, \quad \partial^2 f_i^\sharp(x,\overline{\lambda}_j)/\partial x^2$$

all belong to $L^2([a,+\infty)$ and

$$\begin{gathered}
f_i^\sharp(x,\lambda), \quad \partial f_i^\sharp(x,\lambda)/\partial x, \quad \partial^2 f_i^\sharp/\partial x^2,\\
\partial f_i^\sharp(x,\lambda)/\partial\lambda, \quad \langle\lambda\rangle^{-1}\partial^2 f_i^\sharp(x,\lambda)/\partial\lambda^2
\end{gathered}$$

all belong to $L^2([-a,+\infty)\times\mathbb{R})$*. For* f_4 *we have the following estimates:*

$$\left\| f_4^\sharp(x,\,\cdot\,)\right\|_{L^2_\lambda} \lesssim (1+|x|)^{-1}\,\|\rho\|_{H^{2,0}}\,\|\rho\|_{L^2}\,, \tag{6.2.9a}$$

$$\left\| (f_4^\sharp)_x\right\|_{L^2((-a,\infty)\times\mathbb{R})} \leqslant \|\rho\|_{L^{2,1}}\,\|\rho\|_{L^2} \tag{6.2.9b}$$

$$\left\| (f_4^\sharp)_{xx}\right\|_{L^2((-a,\infty)\times\mathbb{R})} \leqslant \|\rho\|_{L^{2,2}}\,\|\rho\|_{L^{2,1}} \tag{6.2.9c}$$

$$\left\| (f_4^\sharp)_\lambda(x,\,\cdot\,)\right\|_{L^2_\lambda} \lesssim (1+|x|)^{-1}\,\|\rho\|_{H^2}\,\|\rho\|_{H^{2,2}} \tag{6.2.9d}$$

$$\left\| \langle\,\cdot\,\rangle^{-1}(f_4^\sharp)_{\lambda\lambda}(x,\,\cdot\,)\right\|_{L^2_\lambda} \lesssim (1+|x|)^{-1}\,\|\rho\|^2_{H^{2,2}} \tag{6.2.9e}$$

Remark 6.2.5. Since $\mathbf{f}^\sharp$ and its derivatives are bilinear in the scattering data $(\rho,\{\lambda_k\},\{C_k\})$, the estimates used to prove the lemma above can easily be adapted to show that $\mathbf{f}^\sharp$ and its derivatives are Lipschitz continuous as a function of $(\rho,\{\lambda_k,C_k\}_{k=1}^n)\in S\times(\mathbb{C}^+\times\mathbb{C}_\times)^n$.

Proof. We establish the inequalities (6.2.9a)-(6.2.9e) and the other part of the lemma is trivial. It follows from (6.0.6) that, Fourier transforming in λ,

$$\widehat{f_4^\sharp}(x,\xi) = \chi_-\frac{1}{2\pi^2 i}\int_x^\infty \widehat{\overline{\rho}}\,'(\xi-\xi')\,\widehat{\rho(\xi')}\,d\xi'. \tag{6.2.10}$$

Note that, in (6.2.10), $\xi\leqslant 0$ and $\xi'\geqslant x\geqslant -a$. We'll sketch the proofs assuming $a=0$ and when $a>0$, we simply write

$$\int_x^\infty = \int_x^0 + \int_0^\infty$$

and use the fact that $|\xi-a|>|\xi|$. The same conclusions follow.

Using (6.2.10) and the inequality $(1+|x|)\leqslant(1+|\xi-\xi'|)$, we easily recover (6.2.9a) using $\|\widehat{\rho}'\|_{L^1}\leqslant C\|\widehat{\rho}'\|_{L^{2,1}}$, Young's inequality, and Plancherel's theorem. Estimates (6.2.9b) and (6.2.9c) follow by differentiating (6.2.10).

To prove (6.2.9d) it suffices to estimate $\|\widehat{f_4^\sharp}\|_{L^{2,1}}$. Since $0\leqslant x\leqslant|\xi'|$ and $|\xi|\leqslant|\xi'|\leqslant|\xi-\xi'|$ we may estimate

$$(1+|\xi|)(1+|x|)\left|\widehat{f_{4\lambda}^\sharp}(x,\xi)\right| \leqslant \frac{1}{2\pi^2}\int_x^\infty (1+|\xi-\xi'|)\left|\widehat{\overline{\rho}}\,'(\xi-\xi')\right|(1+|\xi'|)\,|\widehat{\rho}(\xi')|\,d\xi'.$$

and use Young's inequality together with the estimate $\|xf'(x)\|_{L^2}\leqslant\|f\|_{H^{2,2}}$.

To prove (6.2.9e), we compute

$$\begin{aligned}(f_4^\sharp)_{\lambda\lambda} &= -\frac{2i}{\pi^2}\int_{-\infty}^0 e^{2i\lambda\xi}\xi^2\int_x^\infty \widehat{\rho}(\xi')\widehat{\overline{\rho}}\,'(\xi-\xi')\,d\xi'\,d\xi \\ &= -\frac{2i}{\pi^2}\int_{-\infty}^0 e^{2i\lambda\xi}\xi^2\frac{d}{d\xi}\left(\int_x^\infty \widehat{\rho}(\xi')\widehat{\overline{\rho}}(\xi-\xi')\right)d\xi'\,d\xi \\ &= \frac{2i}{\pi^2}\int_{-\infty}^0 e^{2i\lambda\xi}\left(2i\lambda\xi^2+2\xi\right)\int_x^\infty \widehat{\rho}(\xi')\widehat{\overline{\rho}}(\xi-\xi')\,d\xi'\,d\xi\end{aligned} \tag{6.2.11}$$

Following the pattern of the previous arguments, it is easy to see that

$$\left\| \langle\lambda\rangle^{-1} (f_4^\sharp)_{\lambda\lambda} \right\|_{L^2_\lambda} \lesssim (1+|x|)^{-1} \|\rho\|^2_{H^{2,2}}$$

$\square$

We now analyze the operator $\mathcal{K}$, an $(n+1)\times(n+1)$ matrix form operator:

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{00} & \mathcal{K}_{01} \\ \mathcal{K}_{10} & \mathcal{K}_{11} \end{pmatrix} \tag{6.2.12}$$

where for fixed x

$$\begin{aligned} &\mathcal{K}_{00} : L^2_\lambda(\mathbb{R}) \to L^2_\lambda(\mathbb{R}) \\ &\mathcal{K}_{01} : \mathbb{C}^n \to L^2_\lambda(\mathbb{R}) \\ &\mathcal{K}_{10} : L^2_\lambda(\mathbb{R}) \to \mathbb{C}^n \\ &\mathcal{K}_{11} : \mathbb{C}^n \to \mathbb{C}^n \end{aligned}$$

The operators $\mathcal{K}_{ij}$ are as follows:

$$\begin{aligned} \mathcal{K}_{00}(\nu^\flat) = {} & C^- \left[\left\{ C_+ \left[\nu^\flat(x,\diamond)\rho(\diamond)e^{-2i(\diamond)x} \right](\,\cdot\,) \right\} (\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2i(\,\cdot\,)x} \right](\lambda) \\ & + \sum_{k=1}^n \frac{\displaystyle\int_{\mathbb{R}} \frac{\nu^\flat(x,s)\rho(s)e^{-2isx}}{s-\lambda_k}\frac{ds}{2\pi i}}{\lambda-\lambda_k} \lambda_k C_k e^{2i\lambda_k x} \end{aligned} \tag{6.2.13}$$

$$\begin{aligned} \mathcal{K}_{01}\left(\nu^\flat(x,\overline{\lambda}_1)\dots\nu^\flat(x,\overline{\lambda}_n)\right) = {} & -\sum_{k=1}^n C^- \left[\frac{(\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2i(\,\cdot\,)x}}{(\,\cdot\,)-\overline{\lambda}_k} \right](\lambda)\nu^\flat(x,\overline{\lambda}_k)\overline{C}_k e^{-2i\overline{\lambda}_k x} \\ & - \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\lambda_k C_k e^{2i\lambda_k x}}{(\lambda-\lambda_k)(\lambda_k-\overline{\lambda}_j)} \right) \nu^\flat(x,\overline{\lambda}_j)\overline{C}_j e^{-2i\overline{\lambda}_j x} \end{aligned} \tag{6.2.14}$$

$\mathcal{K}_{10}(\nu^\flat)$ is an n dimensional vector where the i-th entry is given by

$$\int_{\mathbb{R}} \frac{C^+\left[\nu^\flat(x,\cdot)\rho(\cdot)e^{-2ix(\cdot)}\right](s)}{s-\overline{\lambda}_i} s\overline{\rho(s)}e^{2isx}\frac{ds}{2\pi i} + \sum_{k=1}^n \frac{\displaystyle\int_{\mathbb{R}} \frac{\nu^\flat(x,s)\rho(s)e^{-2isx}}{s-\lambda_k}\frac{ds}{2\pi i}}{\overline{\lambda}_i-\lambda_k}\lambda_k C_k e^{2i\lambda_k x} \tag{6.2.15}$$

$\mathcal{K}_{11}\left(\nu^\flat(x,\overline{\lambda}_1)\dots\nu^\flat(x,\overline{\lambda}_n)\right)$ is an n dimensional vector where the i-th entry is given by

$$-\sum_{k=1}^n \int \frac{s\overline{\rho(s)}e^{2isx}}{(s-\overline{\lambda}_k)(s-\overline{\lambda}_i)}\frac{ds}{2\pi i}\nu^\flat(x,\overline{\lambda}_k)\overline{C}_k e^{-2i\overline{\lambda}_k x} - \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\lambda_k C_k e^{2i\lambda_k x}}{(\overline{\lambda}_i-\lambda_k)(\lambda_k-\overline{\lambda}_j)} \right) \nu^\flat(x,\overline{\lambda}_j)\overline{C}_j e^{-2i\overline{\lambda}_j x} \tag{6.2.16}$$

Since
$$(I-K^2)^{-1} = \begin{pmatrix} (I-AB)^{-1} & 0 \\ 0 & (I-BA)^{-1} \end{pmatrix}$$
and $\mathcal{K} = AB$, the following lemma is an immediate consequence of Proposition 6.1.9.

Lemma 6.2.6. *For fixed x, $(I-\mathcal{K})^{-1}$ exists as a bounded operator on the space $X = L^2_\lambda(\mathbb{R}) \oplus \mathbb{C}^n$ to itself by the formula*
$$(R_s f)(x,\cdot) = (I-\mathcal{K})^{-1} f(x,\cdot)$$
and
$$\sup_{x\in(-a,\infty)} \|(I-\mathcal{K})^{-1}\|_{\mathcal{B}(X)} \lesssim_{\rho,\lambda_k,C_k} 1 \tag{6.2.17}$$
such that
$$\|\nu_1^\sharp(x,\,\cdot\,)\|_{L^2(\mathbb{R})\oplus\mathbb{C}^n} \leqslant C\|f^\sharp(x,)\|_{L^2(\mathbb{R})\oplus\mathbb{C}^n} \tag{6.2.18}$$
for a fixed C uniform in $x \in [-a,\infty)$. Moreover, we can estimate the mixed norm on the space $X^\sharp$ given by (6.1.14)
$$\|\nu_1^\sharp\|^2_{X^\sharp} \leqslant C_S \int_{-a}^\infty \|f^\sharp(x,\,\cdot\,)\|^2_{L^2(\mathbb{R})\oplus\mathbb{C}^n} dx \tag{6.2.19}$$

Remark 6.2.7. By Lemma 6.2.6 the map $(\rho, \{\lambda_k, C_k\}_{k=1}^n) \mapsto (I-\mathcal{K})^{-1}$ is locally bounded and locally Lipschitz continuous from $S \times (\mathbb{C}^+ \times \mathbb{C}_\times)^n$ to $\mathcal{B}(X)$

Lemma 6.2.8. *Fix $a \geqslant 0$. The maps $\rho \to \nu^\flat_\lambda$ and $\rho \to \langle\lambda\rangle^{-1}\nu^\flat_{\lambda\lambda}$ are Lipschitz continuous from $H^{2,2}(\mathbb{R})$ to $L^2((-a,\infty)\times\mathbb{R})$.*

Proof. By the integral equation (6.2.1) and the estimates (6.2.9a)–(6.2.9e), it suffices to show that the maps $\rho \to (\mathscr{S}\nu^\flat)_\lambda$ and $\rho \to (\mathscr{S}\nu^\flat)_{\lambda\lambda}$ are Lipschitz continuous. From the identity (6.0.5), we compute (up to trivial constants)
$$\begin{aligned} \mathcal{F}(\mathscr{S}[h]_\lambda)(x,\xi) &= \xi \int_x^\infty \left(\widehat{\rho} * \widehat{h}\right)(\xi')\widehat{\overline{\rho}}\,'(\xi-\xi'), d\xi' \\ &= \xi(\widehat{\rho} * \widehat{h})(x)\widehat{\overline{\rho}}(\xi - x) - \xi \int_x^\infty (\widehat{\rho}\,' * \widehat{h})(\xi')\,\widehat{\overline{\rho}}(\xi-\xi')\,d\xi' \end{aligned} \tag{6.2.20}$$

where in the second step we integrated by parts. The first right-hand term in (6.2.20) has $L^2((-a,\infty)\times\mathbb{R})$-norm estimated by $\|\widehat{\rho}\|_{L^1}\,\|h\|_{L^2}\,\|\widehat{\rho}\,'\|_{L^{2,1}}$. To estimate the second right-hand term in (6.2.20) we note that
$$\left|\xi \int_x^\infty (\widehat{\rho}\,' * \widehat{h})(\xi')\widehat{\overline{\rho}}(\xi-\xi')\,d\xi'\right| \leqslant \int_x^\infty |(\widehat{\rho}\,' * \widehat{h})(\xi')|\,|\xi-\xi'|\,|\widehat{\overline{\rho}}(\xi-\xi')|\,d\xi \tag{6.2.21}$$

since ξ and ξ' have opposite sign. By Young's inequality, the right-hand side of (6.2.21) has $L^2((-a,\infty)\times\mathbb{R})$-norm estimated by $\|\widehat{\rho}\,'\|_{L^1}\,\|\widehat{\rho}\|_{L^{2,1}}\,\|h\|_2$. Hence
$$\left\|\frac{\partial\mathscr{S}[h]}{\partial\lambda}\right\|_{L^2((-a,\infty)\times\mathbb{R})} \leqslant C\,\|\widehat{\rho}\,'\|_{L^1}\,\|\widehat{\rho}\|_{H^{2,2}}\,\|h\|_{L^2}\,.$$

Applying this estimate with $h(\lambda) = \nu^\flat$ we see that $(\partial \mathscr{S}[\nu^\flat]/\partial\lambda \in L^2((-a,\infty)\times\mathbb{R})$. The Lipschitz continuity of $\nu^\flat$ in ρ together with the bilinear estimates above show that $\rho \to \partial\mathscr{S}[\nu^\flat]/\partial\lambda$ is Lipschitz.

To study $\langle\lambda\rangle^{-1}\partial^2\mathscr{S}[\nu^\flat]/\partial\lambda$, we use the same integration by parts trick used in (6.2.11) to conclude that

$$\left(\frac{\partial^2(\mathscr{S}[\nu^\flat])}{\partial\lambda^2}\right)(x,\lambda) = \frac{2i}{\pi}\int_{-\infty}^{0} e^{2i\lambda\xi}(2i\lambda\xi^2+2\xi)\int_x^\infty (\widehat{\rho}\,'*\nu^\flat)(\xi')\,\widehat{\overline{\rho}}(\xi-\xi')\,d\xi'\,d\xi.$$

By the Plancherel theorem, it suffices to bound the $L^2((-a,\infty)\times\mathbb{R})$-norm of the function

$$G(x,\xi) = (1+|\xi|)^2\int_x^\infty (\widehat{\rho}\,'*\nu^\flat)(\xi')\,\widehat{\overline{\rho}}(\xi-\xi')\,d\xi'.$$

As $\xi<0$ and $\xi'\geqslant x$ we may estimate

$$|G(x,\xi)| \leqslant \int_x^\infty |(\widehat{\rho}\,'*\widehat{\nu^\flat})(\xi')|\,(1+|\xi-\xi'|)^2)|\widehat{\overline{\rho}}(\xi-\xi')|\,d\xi'.$$

By Young's inequality we get

$$\|G(x,\,\cdot\,)\|_{L^2(\mathbb{R})} \leqslant \|\widehat{\rho}\,'\|_{L^1}\left\|\widehat{\nu^\flat}\right\|_{L^1}\|\widehat{\rho}\|_{L^{2,2}}$$

where

$$\left\|\widehat{\nu^\flat}\right\|_{L^1} \leqslant \|\langle\xi\rangle\widehat{\nu^\flat}\|_{L^2} \leqslant \|\widehat{\nu^\flat}\|_{L^2} + \left\|\frac{\partial\nu^\flat}{\partial\lambda}\right\|_{L^2}$$

which gives the desired estimate. $\square$

Lemma 6.2.9. *The maps $(\rho,\{\lambda_k,C_k\}_{k=1}^n)\mapsto(\nu_1^\sharp)_\lambda$ and $(\rho,\{\lambda_k,C_k\}_{k=1}^n)\mapsto\langle\lambda\rangle^{-1}(\nu_1^\sharp)_{\lambda\lambda}$ are Lipschitz continuous from $S\times(\mathbb{C}^+\times\mathbb{C}_\times)^n$ to $X^\sharp$.*

Proof. For integral equations (2) and (3) by computing the derivatives in λ we get

$$\nu^\flat(x,\lambda)_\lambda = f(x,\lambda)_\lambda + \mathcal{K}_{00}(\nu^\flat)_\lambda + \mathcal{K}_{01}\left(\nu^\flat(x,\overline{\lambda}_1)\dots\nu^\flat(x,\overline{\lambda}_n)\right)_\lambda$$

$$\begin{aligned}\langle\lambda\rangle^{-1}\nu^\flat(x,\lambda)_{\lambda\lambda} &= \langle\lambda\rangle^{-1}f(x,\lambda)_{\lambda\lambda} + \langle\lambda\rangle^{-1}\mathcal{K}_{00}(\nu^\flat)_{\lambda\lambda}\\ &\quad+\langle\lambda\rangle^{-1}\mathcal{K}_{01}\left(\nu^\flat(x,\overline{\lambda}_1)\dots\nu^\flat(x,\overline{\lambda}_n)\right)_{\lambda\lambda}\end{aligned}$$

We notice that the operator $\mathcal{K}_{01}$ contains the following:

$$C^-\left[\frac{(\,\cdot\,)\overline{\rho(\,\cdot\,)}e^{2i(\,\cdot\,)x}}{(\,\cdot\,)-\overline{\lambda}_k}\right](\lambda) = \frac{1}{\pi}\int_0^{-\infty} e^{2i\lambda\zeta}\widehat{h}(\zeta-x)d\zeta$$

where

$$h(s) = \frac{s\,\overline{\rho(s)}}{s-\overline{\lambda}_k}$$

Taking the second derivative with respect to λ we get

$$-\frac{4}{\pi}\int_0^{-\infty} e^{2i\lambda\zeta}\zeta^2\,\widehat{h}(\zeta-x)d\zeta$$

By Plancheral's theorem, to evaluate the L^2 norm we look at

$$\|(\,\cdot\,)^2\widehat{h}(\,\cdot\,-x)\chi_-\|_{L^2}$$

If $x\geqslant 0$, then we have

$$\|(\,\cdot\,)^2\widehat{h}(\,\cdot\,-x)\chi_-\|_{L^2}\leqslant\|(\,\cdot\,-x)^2\widehat{h}(\,\cdot\,-x)\chi_-\|_{L^2}\lesssim\|\rho\|_{H^{2,2}}$$

If $-a<x<0$, we use the inequality:

$$(|\zeta-x|+|a|)^2\geqslant|\zeta+(a-x)|^2\geqslant|\zeta|^2$$

Using Lemma 6.2.4, the conclusion $\nu^\flat\in L^2((-a,+\infty)\times\mathbb{R})$ from Lemma 6.2.6 and Lemma 6.2.8 we conclude that $\nu^\flat(x,\lambda)_\lambda$ and $\langle\lambda\rangle^{-1}\nu^\flat(x,\lambda)_{\lambda\lambda}$ both belong to $L^2([a,+\infty)\times\mathbb{R})$. The derivatives of the remaining components of $\nu_1^\sharp$ with respect to λ are all zero. □

Now we turn to equation (6.2.6) and equation (6.2.7). We only need to show that the inhomogenous terms belong to $L^2_\lambda\oplus\mathbb{C}^n$.

Proposition 6.2.10. *Suppose that $\rho\in S$ and $\{C_k\}_{k=1}^n$ and $\{\lambda_k\}_{k=1}^n$ in some bounded set of $\mathbb{C}$. Then for any $x\in(-a,\infty)$, $\mathcal{K}_x$ and $\mathcal{K}_{xx}$ are bounded operators on $L^2_\lambda\oplus\mathbb{C}^n$ and the operator norm is uniform in x.*

Proof. The study of the boundedness of the derivatives in x variables of those operators given by (6.2.13)-(6.2.16) only requires the analysis of the mapping properties of Cauchy projection C_+ from (6.2.15). The analysis of other parts of the operators is either trivial or has been done in Paper I. For fixed x and some $h\in L^2(\mathbb{R})$, we have

$$C^+\left[h(\,\cdot\,)\rho(\cdot)e^{-2ix(\cdot)}\right](s)=\frac{1}{\pi}\int_0^\infty\left[\hat{h}(\,\cdot\,)*\hat{\rho}(\,\cdot\,+x)\right](\xi)e^{2i\xi s}d\xi \tag{6.2.22}$$

Computing the first and second order derivative of x, we get

$$C^+\left[h(\,\cdot\,)\rho(\cdot)e^{-2ix(\cdot)}\right]_x(s)=\frac{1}{\pi}\int_0^\infty\left[\hat{h}(\,\cdot\,)*\hat{\rho}'(\,\cdot\,+x)\right](\xi)e^{2i\xi s}d\xi \tag{6.2.23}$$

$$C^+\left[h(\,\cdot\,)\rho(\cdot)e^{-2ix(\cdot)}\right]_{xx}(s)=\frac{1}{\pi}\int_0^\infty\left[\hat{h}(\,\cdot\,)*\hat{\rho}''(\,\cdot\,+x)\right](\xi)e^{2i\xi s}d\xi \tag{6.2.24}$$

Using Plancherel's identity we have

$$\left\|C^+\left[h(\,\cdot\,)\rho(\cdot)e^{-2ix(\cdot)}\right]_x\right\|_{L^2}\lesssim\|\hat{h}\|_{L^2}\|\hat{\rho}'\|_{L^1}.$$

and

$$\left\|C^+\left[h(\,\cdot\,)\rho(\cdot)e^{-2ix(\cdot)}\right]_{xx}\right\|_{L^2}\lesssim\|\hat{h}\|_{L^1}\|\hat{\rho}''\|_{L^2}.$$

In our case, we have $h = \nu^\flat$ and we know from Lemma 6.2.8 that

$$\|\widehat{\nu^\flat}\|_{L^1} \leqslant \|\langle\xi\rangle\widehat{\nu^\flat}\|_{L^2} \leqslant \left\|\frac{\partial \nu^\flat}{\partial \lambda}\right\|_{L^2} + \|\widehat{\nu^\flat}\|_{L^2}$$

and the two terms on the right hand side both belong to $X^\sharp$ by Lemma 6.2.9. ☐

Remark 6.2.11. Since all estimates in the proof of Proposition 6.2.10 are bilinear in the scattering data $(\rho, \{\lambda_k, C_k\}_{k=1}^n)$, it follows that $(\rho, \{\lambda_k, C_k\}_{k=1}^n) \mapsto \mathcal{K}_x$ and $(\rho, \{\lambda_k, C_k\}_{k=1}^n) \mapsto \mathcal{K}_{xx}$ are locally Lipschitz maps from $S \times (\mathbb{C}^+ \times \mathbb{C}_\times)^n$ to the bounded operators on $L^2_\lambda \oplus \mathbb{C}^n$.

Lemma 6.2.12. *The maps $(\rho, \{\lambda_k, C_k\}_{k=1}^n) \mapsto (\nu_1^\sharp)_x$ and $(\rho, \{\lambda_k, C_k\}_{k=1}^n) \mapsto (\nu_1^\sharp)_{xx}$ are Lipschitz continuous from $S \times (\mathbb{C}^+ \times \mathbb{C}_\times)^n$ to $X^\sharp$.*

Proof. The conclusion follows from Lemma 6.2.6 and Lemma 6.2.10. ☐

6.3 Proof of the Inverse Scattering Map

To reconstruct the potential q on the left, we use the standard trick of conjugating to a new left RHP that gives good estimates on the inverse map for $x < a$. This RHP yields solutions normalized so that $\lim_{x\to-\infty} \tilde{N}(x,z) = (1,0)$, and gives a stable reconstruction of q on any interval $(-\infty, a)$. In this case, the jump matrix J_x across the real line is replaced by

$$J_x^\ell(\lambda) = e^{-ix\lambda\,\mathrm{ad}\,\sigma}\begin{pmatrix} 1 & \tilde{\rho}(\lambda) \\ \lambda\overline{\tilde{\rho}(\lambda)} & 1 + \lambda|\tilde{\rho}(\lambda)|^2 \end{pmatrix},$$

where

$$\tilde{\rho}(\lambda) = \tilde{\rho}(\zeta^2) = \zeta^{-1}\breve{b}(\zeta)/\breve{a}(\zeta) \tag{6.3.1}$$

and the jump matrices across the circles are replaced by

$$J_x^\ell(\lambda_k) = \begin{pmatrix} 0 & \tilde{C}_k e^{-2i\lambda_k x} \\ 0 & 0 \end{pmatrix}, \quad J_x^\ell(\overline{\lambda_k}) = \begin{pmatrix} 0 & 0 \\ -\overline{\tilde{C}_k}\overline{\lambda}_k e^{2i\overline{\lambda}_k x} & 0 \end{pmatrix} \tag{6.3.2}$$

To construct(6.3.1)-(6.3.2), we need to write $\breve{\alpha}$ and α in terms of the reflection coefficient ρ. Here we have to take into consideration the zeros of $\breve{a}(\zeta) = \breve{\alpha}(\lambda)$. Since $\breve{\alpha}(\lambda)$ and $\alpha(\lambda)$ have the simple zeros $\{\lambda_k : \Im(\lambda_k) > 0\}_{k=1}^N$ and $\{\overline{\lambda}_k : \Im(\overline{\lambda}_k) < 0\}_{k=1}^N$ respectively. we define:

$$\breve{\gamma}(\lambda) = \prod_{k=1}^N \frac{\lambda - \overline{\lambda}_k}{\lambda - \lambda_k}\breve{\alpha}(\lambda), \quad \gamma(\lambda) = \prod_{k=1}^N \frac{\lambda - \lambda_k}{\lambda - \overline{\lambda}_k}\alpha(\lambda). \tag{6.3.3}$$

$\breve{\gamma}(\lambda)$ is analytic in the upper half plane where it has no zeros, while γ is analytic in the lower half plane where it has no zeros. Also $\breve{\gamma}$ and $\gamma \to 1$ as $|\lambda| \to \infty$ in the respective half planes. And both $\breve{\gamma}$ and γ have no zeros on the real line.

Therefore we have

$$\log\breve{\gamma}(\lambda) = \int_{-\infty}^{+\infty} \frac{\log\breve{\gamma}(\xi)}{\xi - \lambda}\frac{d\xi}{2\pi i}, \qquad \int_{-\infty}^{+\infty} \frac{\log\gamma(\xi)}{\xi - \lambda}\frac{d\xi}{2\pi i} = 0 \quad \Im(\lambda) > 0$$

and

$$\log \gamma(\lambda) = -\int_{-\infty}^{+\infty} \frac{\log \gamma(\xi)}{\xi - \lambda} \frac{d\xi}{2\pi i}, \qquad \int_{-\infty}^{+\infty} \frac{\log \breve{\gamma}(\xi)}{\xi - \lambda} \frac{d\xi}{2\pi i} = 0 \quad \Im(\lambda) < 0.$$

Subtracting these equations from one another and using (6.3.3), we obtain

$$\log \breve{\alpha}(\lambda) = \sum_{k=1}^{N} \log \left(\frac{\lambda - \lambda_k}{\lambda - \overline{\lambda}_k} \right) + \int_{-\infty}^{+\infty} \frac{\log (\gamma(\xi)\breve{\gamma}(\xi))}{\xi - \lambda} \frac{d\xi}{2\pi i}, \quad \Im(\lambda) > 0 \tag{6.3.4}$$

and

$$\log \alpha(\lambda) = \sum_{k=1}^{N} \log \left(\frac{\lambda - \overline{\lambda}_k}{\lambda - \lambda_k} \right) - \int_{-\infty}^{+\infty} \frac{\log (\gamma(\xi)\breve{\gamma}(\xi))}{\xi - \lambda} \frac{d\xi}{2\pi i}, \quad \Im(\lambda) < 0 \tag{6.3.5}$$

From $\breve{\alpha}(\xi)\alpha(\xi) = \breve{\gamma}(\xi)\gamma(\xi)$ and the fact that that $\breve{\alpha}(\xi)\alpha(\xi) = \left(1 + \xi|\rho(\xi)|^2\right)^{-1}$, (6.3.4) and (6.3.5) can be written as

$$\log \breve{\alpha}(\lambda) = \sum_{k=1}^{N} \log \left(\frac{\lambda - \lambda_k}{\lambda - \overline{\lambda}_k} \right) - \int_{-\infty}^{+\infty} \frac{\log(1 + \xi|\rho(\xi)|^2)}{\xi - \lambda} \frac{d\xi}{2\pi i}, \quad \Im(\lambda) > 0, \tag{6.3.6}$$

$$\log \alpha(\lambda) = \sum_{k=1}^{N} \log \left(\frac{\lambda - \overline{\lambda}_k}{\lambda - \lambda_k} \right) + \int_{-\infty}^{+\infty} \frac{\log(1 + \xi|\rho(\xi)|^2)}{\xi - \lambda} \frac{d\xi}{2\pi i}, \quad \Im(\lambda) > 0. \tag{6.3.7}$$

Taking exponential on both sides we obtain:

$$\breve{\alpha}(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{\lambda - \overline{\lambda}_k} \exp \left(-\int_{-\infty}^{+\infty} \frac{\log(1 + \xi|\rho(\xi)|^2)}{\xi - \lambda} \frac{d\xi}{2\pi i} \right) \tag{6.3.8}$$

$$\alpha(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \overline{\lambda}_k}{\lambda - \lambda_k} \exp \left(\int_{-\infty}^{+\infty} \frac{\log(1 + \xi|\rho(\xi)|^2)}{\xi - \lambda} \frac{d\xi}{2\pi i} \right) \tag{6.3.9}$$

A straightforward computation shows that the jump matrices (4.2.4) and (4.2.5) are related by

$$J_x^{\ell}(\lambda) = \begin{pmatrix} \alpha(\lambda) & 0 \\ 0 & \alpha(\lambda)^{-1} \end{pmatrix} J_x^{r}(\lambda) \begin{pmatrix} \breve{\alpha}(\lambda) & 0 \\ 0 & \breve{\alpha}(\lambda)^{-1} \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

We now derive $\breve{C}_k$ in (6.3.2) from the set of scattering data $\{\rho, \lambda_k, C_k\}$ of the right RHP. Recall that for the right RHP,

$$N_r^{+}(x, \lambda) = \left(\frac{n_{11}^{-}(x, \lambda)}{\breve{\alpha}(\lambda)}, n_{12}^{+}(x, \lambda) \right)$$

$$N_r^{-}(x, \lambda) = \left(n_{11}^{+}(x, \lambda), \frac{n_{12}^{-}(x, \lambda)}{\alpha(\lambda)} \right)$$

If $\breve{\alpha}(\lambda_k) = 0$, then

$$n_{11}^{-}(x, \lambda_k) = B_k \lambda_k n_{12}^{+}(x, \lambda_k) e^{2i\lambda_k x} \tag{6.3.10}$$

$$n_{12}^{-}(x, \overline{\lambda}_k) = -\overline{B_k} n_{11}^{+}(x, \overline{\lambda}_k) e^{-2i\overline{\lambda}_k x} \tag{6.3.11}$$

and we have the norming constant

$$C_k = \frac{B_k}{\breve{\alpha}'(\lambda_k)}$$

For the left RHP, we have

$$\begin{aligned}
N_l^+(x,\lambda) &= \left(n_{11}^-(x,\lambda), \frac{n_{12}^+(x,\lambda)}{\breve{\alpha}(\lambda)}\right) \\
N_l^-(x,\lambda) &= \left(\frac{n_{11}^+(x,\lambda)}{\alpha(\lambda)}, n_{12}^-(x,\lambda)\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\operatorname{Res}_{\lambda=\lambda_k} N_l^+(x,\lambda) &= \frac{1}{\breve{\alpha}'(\lambda_k)} \left(0, n_{12}^+(x,\lambda_k)\right) \\
&= \frac{e^{-2i\lambda_k x}}{B_k \lambda_k \breve{\alpha}'(\lambda_k)} \left(0, n_{11}^-(x,\lambda_k)\right)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Res}_{\lambda=\overline{\lambda}_k} N_l^-(x,\lambda) &= \frac{1}{\alpha'(\overline{\lambda}_k)} \left(n_{11}^+(x,\overline{\lambda}_k), 0\right) \\
&= -\frac{e^{2i\overline{\lambda}_k x}}{\overline{B_k} \alpha'(\overline{\lambda}_k)} \left(n_{12}^-(x,\overline{\lambda}_k), 0\right)
\end{aligned}$$

We now define

$$\tilde{C}_k = \frac{1}{B_k \breve{\alpha}'(\lambda_k)} = \frac{1}{C_k \left(\breve{\alpha}'(\lambda_k)\right)^2} \tag{6.3.12}$$

Now we arrive at the following left Riemann-Hilbert problem:

Problem 6.3.1. Fix $x \in \mathbb{R}$ and let $(\tilde{\rho}, \{\lambda_k, \tilde{C}_k\}_{k=1}^n)$ such that $\tilde{\rho} \in S$ and for $i = 1, ...n$, $\lambda_i \in \mathbb{C}^+$, $\tilde{C}_i \in \mathbb{C}_\times$. Find a vector-valued function $\tilde{\mathbf{N}}(x, \cdot)$ with the following properties:

(i) (Analyticity) $\tilde{N}(x,z)$ is an analytic function of z for $z \in \mathbb{C} \backslash \Lambda$ where

$$\Lambda = \mathbb{R} \cup \{\Gamma_1, ..., \Gamma_n, \Gamma_1^*, ..., \Gamma_n^*\}$$

(ii) (Normalization) $\tilde{\mathbf{N}}(x,z) = (1,0) + \mathcal{O}(z^{-1})$ as $z \to \infty$.

(iii) (Jump condition) For each $\lambda \in \Lambda$, $\tilde{\mathbf{N}}$ has continuous boundary values $\tilde{\mathbf{N}}_\pm(\lambda)$ as $z \to \lambda$ in $\mathbb{C}^\pm$. Moreover, the jump relation

$$\tilde{\mathbf{N}}_+(x,\lambda) = \tilde{\mathbf{N}}_-(x,\lambda) J_x^l(\lambda)$$

holds, where for $\lambda \in \mathbb{R}$

$$J_x^l(\lambda) = e^{-i\lambda x \operatorname{ad}\sigma} \begin{pmatrix} 1 & \tilde{\rho}(\lambda) \\ \lambda \overline{\tilde{\rho}(\lambda)} & 1 + \lambda |\tilde{\rho}(\lambda)|^2 \end{pmatrix}$$

and for $\lambda \in \Gamma_i \cup \Gamma_i^*$

$$J_x^l(\lambda) = \begin{cases} \begin{pmatrix} 1 & \dfrac{\widetilde{C}_i e^{-2i\lambda x}}{\lambda_i(\lambda - \lambda_i)} \\ 0 & 1 \end{pmatrix} & \lambda \in \Gamma_i, \\ \begin{pmatrix} 1 & 0 \\ \dfrac{\overline{\widetilde{C}_i}\, e^{2ix\lambda}}{\lambda - \overline{\lambda_i}} & 1 \end{pmatrix} & \lambda \in \Gamma_i^* \end{cases}$$

We mention that solvability of Problem 6.3.1 is equivalent to the solvability of the following integral equation:

$$\tilde{\nu} = (1,0) + \mathcal{C}_{\tilde{w}}\tilde{\nu} = (1,0) + C_\Lambda^+(\tilde{\nu}\tilde{w}_x^-) + C_\Lambda^-(\tilde{\nu}\tilde{w}_x^+) \tag{6.3.13}$$

Suppose now that $N(x,z)$ solves Problem 5.1.1 and for $z \in \mathbb{C}\backslash\Lambda$ let

$$\widetilde{\mathbf{N}}(x,z) = \mathbf{N}(x,z) \begin{pmatrix} \delta(z) & 0 \\ 0 & \delta(z)^{-1} \end{pmatrix}. \tag{6.3.14}$$

where

$$\delta(z) = \begin{cases} \breve{\alpha}(z) & \mathrm{Im} z > 0 \\ \alpha(z) & \mathrm{Im} z < 0 \end{cases} \tag{6.3.15}$$

Then $\widetilde{\mathbf{N}}$ solves Problem 6.3.1. We recover $\tilde{q}(x)$ from the formula

$$\tilde{q}(x) = \lim_{z\to\infty} 2iz\widetilde{\mathbf{N}}_{12}(x,z) \tag{6.3.16}$$

where the limit is taken in a direction not tangential to $\mathbb{R}$. In terms of the Beals-Coifman solution,

$$\tilde{q}(x) = -\frac{1}{\pi}\int_{\mathbb{R}} e^{-2ix\lambda}\tilde{\rho}(\lambda)\tilde{\nu}_{11}(x,\lambda)\,d\lambda - \sum_i 2i\tilde{\nu}_{11}(x,\overline{\lambda_i})\overline{\widetilde{C}_i}e^{-2i\overline{\lambda_i}x}. \tag{6.3.17}$$

Lemma 6.3.2. *For any $x \in \mathbb{R}$, $q(x) = \tilde{q}(x)$.*

Proof. The solutions $\mathbf{N}(x,z)$ and $\widetilde{\mathbf{N}}(x,z)$ have large-z asymptotic expansions in $\mathbb{C}\backslash\mathbb{R}$ of the form

$$\mathbf{N}(x,z) = \mathbf{e}_1 + \frac{\mathbf{N}_{-1}(x)}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \quad \widetilde{\mathbf{N}}(x,z) = \mathbf{e}_1 + \frac{\widetilde{\mathbf{N}}_{-1}(x)}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)$$

while the function $\delta(z)$ given by (6.3.15) satisfies

$$\delta(z) = 1 + \mathcal{O}\left(\frac{1}{z}\right).$$

From these formulae, it is easy to see that

$$\mathbf{N}_{12}(x,z) = \widetilde{\mathbf{N}}_{12}(x,z) + \mathcal{O}\left(\frac{1}{z^2}\right).$$

We now use the fact that the reconstruction formulas (5.3.13) for q and (6.3.17) for $\tilde{q}$ are equivalent to the formulae

$$\begin{aligned} q(x) &= \lim_{z\to\infty} 2iz\mathbf{N}_{12}(x,z) \\ \tilde{q}(x) &= \lim_{z\to\infty} 2iz\widetilde{\mathbf{N}}_{12}(x,z) \end{aligned}$$

to conclude that $q = \tilde{q}$. $\square$

Define

$$\mathcal{I} = q(x)\chi(x) + \tilde{q}(x)(1-\chi(x))$$

The map $\mathcal{I}$ is Lipschitz continuous from a bounded subset of V_n to $H^{2,2}$.

Proposition 6.3.3. *Suppose that $q \in U_n$.*

(i) There exists at most one matrix-valued solution $M(x,z)$, meromorphic for $z \in \mathbb{C}\backslash\Sigma$, of the problem

$$\begin{aligned} \frac{d}{dx}M(x,z) &= -iz^2 \operatorname{ad}\sigma(M) + zQ(x)M + P(x)M, \\ \lim_{x\to+\infty} M(x,z) &= \mathbf{1}, \\ M(x,z) & \text{ is bounded as } x \to -\infty. \end{aligned}$$

(ii) There exists at most one matrix-valued solution $M(x,z)$, analytic for $z \in \mathbb{C}\backslash\Sigma$, of the problem

$$\begin{aligned} \frac{d}{dx}M(x,z) &= -iz^2 \operatorname{ad}\sigma(M) + zQ(x)M + P(x)M, \\ \lim_{x\to-\infty} M(x,z) &= \mathbf{1}, \\ M(x,z) & \text{ is bounded as } x \to +\infty. \end{aligned}$$

Proof. We prove (i) since the proof of (ii) is similar. Suppose that M_1 and M_2 are two such solutions. It is easy to prove that there is a matrix $A(z)$ with $\det A(z) = 1$ and

$$M_1(x,z) = M_2(x,z)e^{-iz^2x\operatorname{ad}\sigma}A(z) = M_2(x,z)\begin{pmatrix} A_{11}(z) & A_{12}(z)e^{-2iz^2x} \\ A_{21}(z)e^{2iz^2x} & A_{22}(z) \end{pmatrix}$$

Using the exponential blow-up of the factors $e^{\pm iz^2x}$ as $x \to \infty$ together with the asymptotic conditions it is easy to see that $A_{12}(z) = A_{21}(z) = 0$ for $z \in \mathbb{C}\backslash\Sigma$. We can then use the asymptotic condition as $x \to +\infty$ to show that $A_{11}(z) = A_{22}(z) = 1$. Hence $M_1 = M_2$. $\square$

Proposition 6.3.4. *For any v_n belonging to a bounded subset of V_n, $\mathcal{I}(v_n) \in U_n$ and $\mathcal{R} \circ \mathcal{I}$ is the identity map. Moreover, the map $\mathcal{R}$ is one-to-one from U onto V.*

Proof. For given $v_n \in V_n$ we solve the Beals-Coifman integral equations for Problem 5.1.1 and Problem 6.3.1 obtaining solutions ν and $\tilde{\nu}$. By Lemma 5.2.3 of Chapter 5 we can construct solutions μ and $\breve{\mu}$ to the Beals-Coifman integral equations for the corresponding RHP's on Σ. Now define

$$\begin{aligned} M(x,z) &= \mathbf{1} + \int_{\Sigma'} \mu(x,\zeta)\left(w_x^+(\zeta) + w_x^-(\zeta)\right)\frac{1}{\zeta - z}\frac{d\zeta}{2\pi i}, \\ \tilde{M}(x,z) &= \mathbf{1} + \int_{\Sigma'} \tilde{\mu}(x,\zeta)\left(\tilde{w}_x^+(\zeta) + \tilde{w}_x^-(\zeta)\right)\frac{1}{\zeta - z}\frac{d\zeta}{2\pi i}. \end{aligned}$$

The functions M and $\tilde{M}$ solve (1.1.7) and are analytic in $z \in \mathbb{C}\backslash\Sigma'$. Using the estimate (6.2.9), the boundedness of $(I - \mathcal{K})^{-1}$,and the construction of μ from ν in (5.2.1)-(5.2.2), it is easy to see that $\|\mu - \mathbf{1}\|_{L^2}$ is bounded uniformly for $x > -a$. Using this estimate, the Riemann-Lebesgue lemma, and the formula

$$\begin{aligned} M(x,z) = \mathbf{1} &+ \int_{\Sigma'} \left(\mu(x,\zeta) - \mathbf{1}\right)\left(w_x^+ + w_x^-\right)\frac{1}{\zeta - z}\frac{d\zeta}{2\pi i} \\ &+ \int_{\Sigma'} \left(w_x^+ + w_x^-\right)\frac{1}{\zeta - z}\frac{d\zeta}{2\pi i} \end{aligned}$$

we see that $M(x,z) \to \mathbf{1}$ as $x \to +\infty$. A similar argument shows that $\tilde{M}(x,z) \to \mathbf{1}$ as $x \to -\infty$. It now follows from (6.3.14) that $M(x,z)$ is bounded as $x \to -\infty$. Hence, $M(x,z)$ is the unique "right" Beals-Coifman solution for $q = \mathcal{I}(v_n)$.

Let $r(\zeta) = \zeta\rho(\zeta^2)$, then $M_\pm(x,\zeta)$ satisfy the jump relation

$$M_+(x,\zeta) = M_-(x,\zeta)v_x(\zeta), \quad v_x(\zeta) = e^{-ix\zeta^2 \operatorname{ad}\sigma}\begin{pmatrix} 1 - r(\zeta)\breve{r}(\zeta) & r(\zeta) \\ -\breve{r}(\zeta) & 1 \end{pmatrix},$$

This is equivalent to the statement that $\mathcal{R} \circ \mathcal{I} = \mathrm{Id}$ since the scattering data for the unique Beals-Coifman solutions corresponds exactly to the input $(\rho, \{C_k, \lambda_k\}_{k=1}^n)$.

Next, we prove that $\mathcal{R}$ is one-to-one. Suppose that q_1 and q_2 are potentials with $\mathcal{R}(q_1) = \mathcal{R}(q_2)$. We can construct "right" Beals-Coifman solutions M_1 and M_2 which both satisfy the *same* jump relation. It follows that $(M_1 - M_2)_+ = (M_1 - M_2)_- v_x$ and $(M_1 - M_2)_\pm \in \partial C(L^2)$ so that $M_1 = M_2$ by Lemma 5.2.7. But then the reconstruction formulas show that $q_1 = q_2$, so $\mathcal{R}$ is one-to-one. □

Proposition 6.3.5. *For any $a > 0$, we have $q(x) = \tilde{q}(x)$ for $x \in (-a,a)$, so the maps (6.0.9)–(6.0.10) together define a locally Lipschitz mapping $\mathcal{I}$ on any bounded subset of V_n*

$$(\rho, \{C_k, \lambda_k\}_{k=1}^n) \mapsto q$$

with the property that $\mathcal{I}(\mathcal{R}(q)) = q$ on U, an open neighborhood of 0 in $H^{2,2}(\mathbb{R})$, and $\mathcal{R} \circ \mathcal{I}$ is the identity map.

Proof of Theorem 1.3.8. By Proposition 6.3.5, $\mathcal{R} \circ \mathcal{I}$ extends to the identity map on V, and $\mathcal{R}$ is one-to-one from U to V. Theorem 1.3.8 now follows. $\square$

Chapter 7 Inverse Scattering Solution to DNLS

The purpose of this chapter is to prove that the function

$$q(x,t) = \mathcal{I}\left(\Phi_t \circ (\mathcal{R}q_0)(\,\cdot\,)\right)(x) \tag{7.0.1}$$

gives a classical solution to (1.1.3) if $q_0 \in \mathcal{S}(\mathbb{R}) \cap U$ where U is the spectrally determined set in Theorem 1.3.2. Equation (8.1.3) from Section 8.1 is the zero-curvature representation of (1.1.3) where $q = q(x,t)$:

$$\psi_x = -i\zeta^2\psi + \zeta Q(x,t)\psi + P(x,t)\psi, \tag{7.0.2}$$

$$\begin{aligned} \psi_t = &-i\zeta^4\psi + 2\zeta^3 Q(x,t)\psi + i\zeta^2 \begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix}\psi \\ &+ i\zeta \begin{pmatrix} 0 & q_x \\ \overline{q}_x & 0 \end{pmatrix}\psi + \frac{i}{4}\begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix}\psi \\ &+ \frac{1}{2}\begin{pmatrix} -q_x\overline{q} + q\overline{q}_x & 0 \\ 0 & -q\overline{q}_x + q_x\overline{q} \end{pmatrix}\psi. \end{aligned} \tag{7.0.3}$$

Recall that a *fundamental solution* to this system is an invertible matrix-valued solution $\psi(x,t,\zeta)$. The computations in Section 8.1 imply the following criterion which is the key to our analysis.

Lemma 7.0.1. *Suppose that $q_0 \in \mathcal{S}(\mathbb{R}) \cap U$. Then $q \equiv q(x,t) \in C^\infty(\mathbb{R}\times\mathbb{R})$,*

$$Q(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\overline{q(x,t)} & 0 \end{pmatrix}, \quad P(x,t) = \frac{i}{2}\begin{pmatrix} |q(x,t)|^2 & 0 \\ 0 & -|q(x,t)|^2 \end{pmatrix},$$

and suppose that there exists a fundamental solution $\psi(x,t,\zeta)$ of the system (7.0.2)–(7.0.3) for each $\zeta \in \Sigma$. Then $q(x,t)$ is a classical solution of (1.1.3).

Our main result is:

Theorem 7.0.2. *Suppose that $q_0 \in \mathcal{S}(\mathbb{R}) \cap U$. Then the function (7.0.1) is a classical solution of (1.1.3).*

Remark 7.0.3. From the continuity of the solution map (1.3.4), it follows that if $q_0 \in \mathcal{S}(\mathbb{R}) \cap U$, then the function (7.0.1) is a strong solution of (1.1.3).

By Lemma 7.0.1, it suffices to construct a fundamental solution for (7.0.2)–(7.0.3). We will do so using the RHP.

Suppose that $M_\pm$ solve the RHP

$$\begin{aligned} M_+(x,t,\zeta) &= M_-(x,t,\zeta)e^{it\theta\,\mathrm{ad}(\sigma)}v(\zeta) \\ M_\pm - I &\in \partial C_\Sigma(L^2) \end{aligned} \tag{7.0.4}$$

where

$$v(\zeta) = \begin{pmatrix} 1 - r(\zeta)\breve{r}(\zeta) & r(\zeta) \\ \\ -\breve{r}(\zeta) & 1 \end{pmatrix}, \quad \zeta \in \Sigma$$

$$v(\zeta) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \dfrac{c_i}{\zeta \mp \zeta_i} & 1 \end{pmatrix} & \zeta \in \pm\gamma_i, \\ \\ \begin{pmatrix} 1 & \dfrac{\overline{c_i}}{\zeta \mp \overline{\zeta_i}} \\ 0 & 1 \end{pmatrix} & \zeta \in \pm\gamma_i^* \end{cases}$$

$$\theta(x, t, \zeta) = -\left(\zeta^2 \frac{x}{t} + 2\zeta^4\right)$$

In terms of the transition matrix

$$T(\zeta) = \begin{pmatrix} a(\zeta) & \breve{b}(\zeta) \\ b(\zeta) & \breve{a}(\zeta) \end{pmatrix}$$

we have $r(\zeta) = \breve{b}(\zeta)/a(\zeta)$, and we have used the symmetries

$$\breve{a}(\zeta) = \overline{a(\overline{\zeta})}, \quad \breve{b}(\zeta) = -\overline{b(\overline{\zeta})}.$$

We also have $a(-\zeta) = a(\zeta)$, $b(-\zeta) = -b(\zeta)$ so that $r(\zeta)$ is odd. We have the factorization $v = (I - w_-)^{-1}(I + w_+)$ where

$$w_- = \begin{pmatrix} 0 & r(\zeta) \\ 0 & 0 \end{pmatrix}, \quad w_+ = \begin{pmatrix} 0 & 0 \\ -\breve{r}(\zeta) & 0 \end{pmatrix} \tag{7.0.5}$$

We define

$$w^{\pm}_{x,t}(\zeta) = e^{it\theta \operatorname{ad}\sigma} w^{\pm}(\zeta), \tag{7.0.6}$$

so that

$$\frac{\partial w^{\pm}_{x,t}}{\partial x} = -i\zeta^2 \operatorname{ad}\sigma(w^{\pm}_{x,t}), \quad \frac{\partial w^{\pm}_{x,t}}{\partial t} = -i\zeta^4 \operatorname{ad}\sigma w^{\pm}_{x,t}. \tag{7.0.7}$$

Proposition 7.0.4. *Suppose that $M_\pm$ solve the RHP (7.0.4). Let*

$$Q(x,t) = -\frac{1}{2\pi} \operatorname{ad}\sigma \left[\int_{\Sigma'} \mu \left(w^+_{x,t} + w^-_{x,t} \right) \right]$$

$$P(x,t) = iQ(x,t)(\operatorname{ad}\sigma)^{-1}Q(x,t) = \frac{i}{2} \begin{pmatrix} |q|^2 & 0 \\ \\ 0 & -|q|^2 \end{pmatrix}.$$

Then $M_\pm$ are fundamental solutions for the Lax equations

$$\frac{\partial M_\pm}{\partial x} = -i\zeta^2 \operatorname{ad}\sigma(M_\pm) + \zeta Q(x) M_\pm + P(x) M_\pm, \tag{7.0.8}$$

$$\frac{\partial M_\pm}{\partial t}(x,t,\zeta) = -2i\zeta^4 \operatorname{ad}\sigma(M_\pm) + A(x,t,\zeta)M_\pm(x,t,\zeta) \tag{7.0.9}$$

where

$$\begin{aligned} A(x,t,\zeta) = 2\zeta^3 \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix} + i\zeta^2 \begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix} + i\zeta \begin{pmatrix} 0 & q_x \\ \overline{q}_x & 0 \end{pmatrix} \\ + \frac{i}{4}\begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -q_x\overline{q} + q\overline{q}_x & 0 \\ 0 & -q\overline{q}_x + q_x\overline{q} \end{pmatrix} \end{aligned} \tag{7.0.10}$$

Proof. We have already shown in Proposition 5.3.1 that, for each fixed t, $M_\pm$ obeys (7.0.8). In Section 8.2, we show that (7.0.9) also holds. Straightforward computation then shows that the functions $\psi_\pm = M_\pm e^{it\theta\sigma}$ obey (7.0.2)–(7.0.3). □

The proof of Theorem 7.0.2 is an immediate consequence of Proposition 7.0.4 and Lemma 7.0.1.

8.1 Gauge Equivalence

In this section, we provide details about the correspondence between the gauge transformation relating solutions u and q of DNLS equations (1.1.1) and (1.1.3) ($\varepsilon = -1$) and a matrix gauge transformation relating their respective Lax pairs (L, A) and (L', A').

We write $v = \overline{u}$ and $r = \overline{q}$. We know from the original paper of Kaup and Newell [12] that the DNLS equation (1.1.1) is equivalent to the zero-curvature condition [1]

$$L_t - A_x + [L, A] = 0 \tag{8.1.1}$$

where the operators L and A are given by

$$L = -i\zeta^2\sigma + \zeta U(x),$$
$$A = (-i)\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & -A_{11} \end{pmatrix}.$$

Here $U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}$, while

$$\begin{aligned} A_{11} &= 2\zeta^4 + \zeta^2 uv \\ A_{12} &= 2i\zeta^3 u - \zeta u_x + i\zeta u^2 v \\ A_{21} &= 2i\zeta^3 v + \zeta v_x + i\zeta uv^2. \end{aligned}$$

The zero-curvature representation (8.1.1)

$$L_t - A_x + [L, A] = \zeta \begin{pmatrix} 0 & u_t - iu_{xx} - (u^2v)_x \\ v_t + iv_{xx} - (v^2u)_x & 0 \end{pmatrix} = 0$$

gives the evolution equations

$$\begin{aligned} u_t &= iu_{xx} + (u^2v)_x, \\ v_t &= -iv_{xx} - (v^2u)_x. \end{aligned} \tag{8.1.2}$$

Proposition 8.1.1. *The zero-curvature representation associated to (1.1.3) has the form*

$$L'_t - A'_x + [L', A'] = 0 \tag{8.1.3}$$

where the Lax pair (L', A') is equivalent to (L, A) through a matrix gauge transformation and

$$\begin{aligned} L' &= -i\zeta^2\sigma + \zeta Q(x) + P(x) \\ A' &= (-i)\begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & -A'_{11} \end{pmatrix} \end{aligned} \tag{8.1.4}$$

[1]This terminology refers to a geometrical interpretation of (1.1.4) where the matrix operators L and A are seen as connection coefficients.

with

$$
\begin{aligned}
A'_{11} &= 2\zeta^4 + \zeta^2 qr - \frac{1}{4}q^2r^2 + \frac{i}{2}\left(q_x r - q r_x\right) \\
A'_{12} &= 2i\zeta^3 q - \zeta q_x \\
A'_{21} &= 2i\zeta^3 r + \zeta r_x \ .
\end{aligned}
\tag{8.1.5}
$$

Proof. A 2×2 matrix-valued function $G(x,t)$ defines a gauge transformation to a new Lax pair

$$L' = GLG^{-1} + G_x G^{-1} \tag{8.1.6}$$

$$A' = GAG^{-1} + G_t G^{-1}. \tag{8.1.7}$$

Indeed, if $\psi_x = L\psi$ and $\psi_t = A\psi$, then the function $\Psi = G\psi$ satisfies $\Psi_x = L'\Psi$ and $\Psi_t = A'\Psi$. We seek a gauge transformation in the matrix form

$$G(x,t) = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \tag{8.1.8}$$

A simple computation shows that

$$L' = -i\zeta^2\sigma + \zeta Q + P$$

where, setting

$$q = e^{-2i\varphi}u, \quad r = e^{2i\varphi}v,$$

we have

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad P = \begin{pmatrix} -i\varphi_x & 0 \\ 0 & i\varphi_x \end{pmatrix}.$$

We wish to choose φ so that[2]

$$P = iQ(\operatorname{ad}\sigma)^{-1}Q = \begin{pmatrix} -\frac{i}{2}qr & 0 \\ 0 & \frac{i}{2}qr \end{pmatrix}.$$

It follows that

$$\varphi(x,t) = \frac{1}{2}\int_{-\infty}^{x} qr\,dy = \frac{1}{2}\int_{-\infty}^{x} uv\,dy. \tag{8.1.9}$$

We get A' in the form $A' = (-i)\begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & -A'_{11} \end{pmatrix}$ with

$$
\begin{aligned}
A'_{11} &= 2\zeta^4 + \zeta^2 qr - \varphi_t \\
A'_{12} &= 2i\zeta^3 q - \zeta u_x e^{2i\varphi} + i\zeta q^2 r \\
A'_{21} &= 2i\zeta^3 r + \zeta v_x e^{-2i\varphi} + i\zeta q r^2 \ .
\end{aligned}
$$

We can express $u_x e^{-2i\varphi}$ and $v_x e^{2i\varphi}$ in terms of q and r by differentiating the identities $u = e^{2i\varphi}q$ and $v = e^{-2i\varphi}r$ to obtain

$$u_x e^{-2i\varphi} = q_x + iq^2 r, \quad v_x e^{2i\varphi} = r_x - iqr^2. \tag{8.1.10}$$

[2]This condition insures that the RHP associated to the inverse scattering map will be properly normalized for large scattering parameter.

We compute φ_t using that u and v obey the equations (8.1.2)

$$\varphi_t = \left(\frac{1}{2}\int_{-\infty}^{x} uv\, dy\right)_t = \frac{i}{2}\left(u_x v - u v_x\right) + \frac{3}{4}(u^2 v^2) = -\frac{1}{4}q^2 r^2 + \frac{i}{2}\left(q_x r - q r_x\right)$$

Finally, a short computation shows that the condition $L'_t - A'_x + [L', A'] = 0$ gives the following equations (in the order (11), (12), (21), (22) of entries in the matrices):

$$\begin{aligned} -\frac{i}{2}\left(q_t r + q r_t\right) - \frac{i}{2}\left(r^2 q q_x + q^2 r r_x\right) - \frac{1}{2}\left(r q_{xx} - q r_{xx}\right) &= 0 \\ q_t - i q_{xx} + q^2 r_x - \frac{i}{2} r^2 q^3 &= 0 \\ r_t + i r_{xx} + r^2 q_x + \frac{i}{2} r^3 q^2 &= 0 \\ \frac{i}{2}\left(q_t r + q r_t\right) + \frac{i}{2}\left(r^2 q q_x + q^2 r r_x\right) + \frac{1}{2}\left(r q_{xx} - q r_{xx}\right) &= 0 \end{aligned}$$

In particular, the (12) and (21) equations hold, the (11) and (22) equations are vacuous. It shows that (8.1.3) give a zero-curvature representation of (1.1.3), and that the transformation (1.1.2) indeed maps solutions of (1.1.1) to solutions of (1.1.3). $\square$

8.2 Time-Evolution of Solutions to the RHP

We prove that the solutions $M_\pm$ of the RHP (7.0.4) solve equation (7.0.9), completing the proof of Proposition 7.0.4. The computation is similar to that presented in the proof of Proposition 5.3.1 except that now, we take into account the time evolution. We write

$$g_\pm \doteq h_\pm$$

if

$$g_+ - h_+ = C^+ k, \quad g_- - h_- = C^- k \qquad \text{for the same function } k \in L^2(\Sigma).$$

Since the RHP (7.0.4) has a unique solution, we have:

Lemma 8.2.1. *Suppose that $G_\pm \doteq 0$ and $G_+ = G_- v_{x,t}$. Then $G_+ = G_- = 0$.*

We will differentiate the jump relation in (7.0.4) and use a commutator formula to show that the function

$$G_\pm = \frac{\partial M_\pm}{\partial t} - A(x,t,\zeta) M_\pm - M_\pm (2i\zeta^4 \sigma)$$

obeys $G_\pm \doteq 0$ and $G_+ = G_- v_{x,t}$, proving Proposition 7.0.4. A computation similar to that leading to (5.3.9) gives

$$\frac{\partial M_+}{\partial t} + i\zeta^4 \operatorname{ad}\sigma(M_+) = \left(\frac{\partial M_-}{\partial t} + i\zeta^4 \operatorname{ad}\sigma M_-\right) v_{x,t}. \tag{8.2.1}$$

We need to evaluate $i\zeta^4 \operatorname{ad}\sigma(M_\pm)$ modulo terms $\doteq 0$. To this end, we use the commutator formula (2.3.6)

$$\zeta^m C^\pm [f](\zeta) \doteq -\sum_{j=0}^{m-1} \zeta^{m-1-j} \frac{f_j}{2\pi i} \tag{8.2.2}$$

The function

$$f(x,t,\zeta)=\mu(x,t,\zeta)\left(w_{x,t}^{+}(\zeta)+w_{x,t}^{-}(\zeta)\right)=\left(\begin{array}{cc}-\breve{r}(\zeta)\mu_{12}(x,t,\zeta) & r(\zeta)\mu_{11}(x,t,\zeta)\\ & \\ -\breve{r}(\zeta)\mu_{22}(x,t,\zeta) & r(\zeta)\mu_{21}(x,t,\zeta)\end{array}\right)$$

and its moments

$$f_j(x,t)=\int_{\Sigma'}\zeta^j f(x,t,\zeta)\,d\zeta,\quad j=0,1,2,3.$$

play a crucial role in the the computations since the solution of the RHP has the large-z asymptotic expansion

$$M(x,z)\sim\mathbf{1}-\frac{1}{z}\frac{f_0}{2\pi i}-\frac{1}{z^2}\frac{f_1}{2\pi i}+\ldots. \tag{8.2.3}$$

By symmetry, f_0 and f_2 vanish on the diagonal, while f_1 and f_3 vanish off the diagonal, so that in particular

$$\operatorname{ad}\sigma(f_1)=\operatorname{ad}\sigma(f_3)=0. \tag{8.2.4}$$

We recall the reconstruction formula

$$\begin{aligned}\left(\begin{array}{cc}0 & q\\ -\overline{q} & 0\end{array}\right)&=-\frac{1}{2\pi}\operatorname{ad}\sigma(f_0) \qquad (8.2.5)\\ &=-\frac{1}{\pi}\int_{\Sigma'}\left(\begin{array}{cc}0 & \mu_{11}(x,\zeta)r(\zeta)e^{-2is^2x}\\ -\mu_{22}(x,\zeta)\breve{r}(\zeta)e^{2is^2x} & 0\end{array}\right)d\zeta\end{aligned}$$

which implies

$$f_0=\pi\left(\begin{array}{cc}0 & -q\\ -\overline{q} & 0\end{array}\right). \tag{8.2.6}$$

The formula

$$M_\pm=\mathbf{1}+C^\pm f \tag{8.2.7}$$

implies that

$$\zeta(\mathbf{1}-M_\pm)\doteq\frac{1}{2i}\left(\begin{array}{cc}0 & -q\\ -\overline{q} & 0\end{array}\right)\doteq\frac{1}{2i}\left(\begin{array}{cc}0 & -q\\ -\overline{q} & 0\end{array}\right)M_\pm \tag{8.2.8}$$

where the last step follows from the fact that $M_\pm\doteq\mathbf{1}$. In the course of the computations, we will need to evaluate the off-diagonal matrix f_2. It follows from (7.0.8) and the identity $f=M_+-M_-$ that

$$\frac{\partial f}{\partial x}=-i\zeta^2\operatorname{ad}\sigma(f)+\zeta Q(x)f+P(x)f.$$

Hence

$$\begin{aligned}\frac{d}{dx}f_0&=\int_{\Sigma'}\left(-i\zeta^2\operatorname{ad}\sigma(f)+\zeta Q(x)f+P(x)f\right)d\zeta\\ &=-i\operatorname{ad}\sigma(f_2)+Q(x)f_1+P(x)f_0. \qquad (8.2.9)\end{aligned}$$

Lemma 8.2.2. *The identity*

$$\frac{\partial M_\pm}{\partial t}\doteq 0. \tag{8.2.10}$$

holds.

Proof. Recall that $M_\pm = \mu(\mathbf{1} \pm w^\pm_{x,t})$ where $w_{x,t}$ is given by (7.0.6). Assume that $r \in \mathcal{S}(\Sigma)$. We claim that $(\partial\mu/\partial t)(x,t,\,\cdot\,) \in L^2(\Sigma)$. If so, then (8.2.10) follows by differentiating (8.2.7).

We prove that $(\partial\mu/\partial t)(x,t,\,\cdot\,) \in L^2(\Sigma)$ using the Beals-Coifman integral equation satisfied by μ

$$\mu = \mathbf{1} + C^+(\mu w^-_{x,t}) + C^-(\mu w^+_{x,t}) = \mathbf{1} + \mathcal{C}_w\mu.$$

We obtain

$$\frac{\partial\mu}{\partial t}(x,t,\zeta) = g(x,t,\zeta) + \mathcal{C}_w\left(\frac{\partial\mu}{\partial t}\right) \tag{8.2.11}$$

where

$$g(x,t,\zeta) = C^-\left(\mu\frac{\partial w^+_{x,t}}{\partial t}\right) + C^+\left(\mu\frac{\partial w^-_{x,t}}{\partial t}\right).$$

Since $\mu - \mathbf{1} \in L^2(\Sigma)$ and $\partial w^\pm_{x,t}/\partial t \in L^\infty(\Sigma) \cap L^2(\Sigma)$, it follows that $g(x,t,\,\cdot\,) \in L^2(\Sigma)$ for each (x,t), and eq. (8.2.11) for $\partial\mu/\partial t$ can be solved in $L^2(\Sigma)$. $\square$

From the commutator formula (8.2.2) (applying it successively for $j = 4$ and $j = 3$) and the identity (8.2.4), we have

$$\begin{aligned}
2i\zeta^4 \operatorname{ad}\sigma(M_\pm) &\doteq -\zeta^3\frac{1}{\pi}\operatorname{ad}\sigma\,[f_0] - \zeta\frac{1}{\pi}\operatorname{ad}\sigma\,[f_2] \qquad (8.2.12)\\
&\doteq 2\zeta^3 Q(x)M_\pm + 2\zeta^3 Q(x)(I - M_\pm) - \frac{\zeta}{\pi}\operatorname{ad}\sigma\,[f_2]\\
&\doteq 2\zeta^3 Q(x)M_\pm + \frac{\zeta^2}{\pi i}Q(x)f_0 + \frac{\zeta}{\pi i}Q(x)f_1 + \frac{Q(x)}{\pi i}f_2 - \frac{\zeta}{\pi}\operatorname{ad}\sigma\,[f_2]
\end{aligned}$$

We wish to re-write the last four terms on the right-hand side of (8.2.12) as coefficients times $M_\pm$, modulo the equivalence relation. We will see that terms involving f_1 cancel so we will keep separate track of these. We have (using again the identity matrix $\mathbf{1} = M_\pm + (\mathbf{1} - M_\pm)$)

$$\begin{aligned}
\frac{\zeta^2}{\pi i}Q(x)f_0 &= i\zeta^2\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}M_\pm + i\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}\left[\zeta^2(\mathbf{1} - M_\pm)\right]\\
\text{(by (8.2.2), } m = 2) \quad &\doteq i\zeta^2\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}M_\pm(x)\\
&\quad + i\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}\left[\zeta\frac{f_0}{2\pi i} + \frac{f_1}{2\pi i}\right]\\
&\doteq i\zeta^2\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}M_\pm + \frac{\zeta}{2}\begin{pmatrix}0 & |q|^2 q\\ -|q|^2\overline{q} & 0\end{pmatrix}M_\pm\\
&\quad + \frac{\zeta}{2}\begin{pmatrix}0 & |q|^2 q\\ -|q|^2\overline{q} & 0\end{pmatrix}(\mathbf{1} - M_\pm) + \frac{1}{2\pi}\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}f_1\\
\text{(by (8.2.2), } m = 1) \quad &\doteq i\zeta^2\begin{pmatrix}|q|^2 & 0\\ 0 & -|q|^2\end{pmatrix}M_\pm + \frac{\zeta}{2}\begin{pmatrix}0 & |q|^2 q\\ -|q|^2\overline{q} & 0\end{pmatrix}M_\pm
\end{aligned}$$

$$-\frac{i}{4}\begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} M_\pm + \frac{1}{2\pi}\begin{pmatrix} |q|^2 & 0 \\ 0 & -|q|^2 \end{pmatrix} f_1.$$

Collecting all the terms, we get

$$\begin{aligned} &2i\zeta^4 \operatorname{ad}\sigma(M_\pm) - 2\zeta^3 Q(x) M_\pm \\ &\quad + \begin{pmatrix} -i\zeta^2|q|^2 & 0 \\ 0 & i\zeta^2|q|^2 \end{pmatrix} M_\pm - \frac{1}{2}\begin{pmatrix} 0 & \zeta|q|^2 q \\ -\zeta|q|^2\overline{q} & 0 \end{pmatrix} M_\pm \\ &\doteq \frac{1}{4}\begin{pmatrix} -i|q|^4 & 0 \\ 0 & i|q|^4 \end{pmatrix} M_\pm + \frac{1}{\pi} Q(x)(\operatorname{ad}\sigma)^{-1} Q(x) f_1 + \frac{1}{\pi i} Q(x)\zeta f_1 \\ &\quad + \frac{1}{\pi i} Q(x) f_2 - \frac{\zeta}{\pi}\operatorname{ad}(\sigma)\,[f_2] \end{aligned} \tag{8.2.13}$$

We are now able to simplify the right hand side of (8.2.13) using (8.2.9):

$$\begin{aligned} &\frac{1}{\pi} Q(x)(\operatorname{ad}\sigma)^{-1} Q(x) f_1 + \frac{1}{\pi i} Q(x) f_2 \\ &\quad \doteq -\frac{1}{2}\begin{pmatrix} -q\overline{q}_x & 0 \\ 0 & -\overline{q} q_x \end{pmatrix} M_\pm + \frac{i}{4}\begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} M_\pm \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\pi i} Q(x)\zeta f_1 - \frac{\zeta}{\pi}\operatorname{ad}\sigma\,[f_2] &= \begin{pmatrix} 0 & i\zeta q_x \\ i\zeta\overline{q}_x & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \zeta|q|^2 q \\ -\zeta|q|^2\overline{q} & 0 \end{pmatrix} \\ \doteq \begin{pmatrix} 0 & i\zeta q_x \\ i\zeta\overline{q}_x & 0 \end{pmatrix} M_\pm &+ \frac{1}{2}\begin{pmatrix} -q_x\overline{q} & 0 \\ 0 & -\overline{q}_x q \end{pmatrix} M_\pm \\ -\frac{1}{2}\begin{pmatrix} 0 & \zeta|q|^2 q \\ -\zeta|q|^2\overline{q} & 0 \end{pmatrix} M_\pm &+ \frac{i}{4}\begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} M_\pm. \end{aligned}$$

Eq. (8.2.13) now becomes the equivalence relation:

$$\begin{aligned} &2i\zeta^4 \operatorname{ad}\sigma(M_\pm) - 2\zeta^3 Q(x) M_\pm \\ &+ \begin{pmatrix} -i\zeta^2|q|^2 & 0 \\ 0 & i\zeta^2|q|^2 \end{pmatrix} M_\pm - \frac{i}{4}\begin{pmatrix} |q|^4 & 0 \\ 0 & -|q|^4 \end{pmatrix} M_\pm \\ &- \begin{pmatrix} 0 & i\zeta q_x \\ i\zeta\overline{q}_x & 0 \end{pmatrix} M_\pm + \frac{1}{2}\begin{pmatrix} -q\overline{q}_x + q_x\overline{q} & 0 \\ 0 & -\overline{q} q_x + \overline{q}_x q \end{pmatrix} M_\pm \doteq 0\ . \end{aligned} \tag{8.2.14}$$

We combine (8.2.14) with Lemma 8.2.2 and Lemma 8.2.1 to obtain eq. (7.0.9) and conclude the proof of Proposition 7.0.4.

Bibliography

[1] Ablowitz, M., Kaup, D. J., Newell, A., Segur, H. (1974). The inverse scattering transform-Fourier analysis for nonlinear problems, *Studies in Appl. Math.* 53:249–315.

[2] Beals, R., Coifman, R. R. (1984). Scattering and inverse scattering for first order systems. *Comm. Pure Appl. Math.* 37:39–90.

[3] Beals, R., Deift, P., Tomei, C. (1988). Direct and inverse scattering on the line. *Mathematical Surveys and Monographs*, 28. American Mathematical Society, Providence, RI.

[4] Borghese, M., Jenkins, R., McLaughlin, K. T.-R. Long-time asymptotic behavior of the focusing nonlinear Schrodinger equation. Preprint, arXiv:1604.07436, 2016.

[5] Champeaux, S., Laveder, D. Passot, T., Sulem, P.-L. (1999). Remarks on the parallel propagation of small-amplitude Alfvén waves. *Nonlin. Processes in Geophys.* 6:169–178.

[6] Deift, P., Venakides, S., Zhou, X. (1997). New results in small dispersion KdV by an extension of the steepest descent method of Riemann-Hilbert problems. *Intern. Math. Res. Not.* 6:285–299.

[7] Deift, P., Zhou, X. (1993). A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation. *Ann. of Math.* (2) 137:295–368.

[8] Deift, P., Zhou, X. (2003). Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. Dedicated to the memory of Jürgen K. Moser. *Comm. Pure Appl. Math.* 56:1029–1077.

[9] Dyatlov, S., Zworski, M. Mathematical theory of scattering resonances, Version 0.1 (March 2017). http://math.mit.edu/dyatlov/res/res 20170323.pdf.

[10] Hayashi, N., Ozawa, T. (1992). On the derivative nonlinear Schrödinger equation. *Physica D* 55:14–36.

[11] Jenkins, R., Liu, J., Perry, P., Sulem, S. (2017). Global well-posedness and large-time asymptotics for the derivative non-linear Schrödinger equation. Preprint. arXiv:1706.06252

[12] Kaup, D. J., Newell, A. C. (1978). An exact solution for a derivative nonlinear Schrödinger equation. *J. Mathematical Phys.* 19:798–801.

[13] Lee, J.-H. (1983). Analytic properties of Zakharov-Shabat inverse scattering problem with a polynomial spectral dependence of degree 1 in the potential. Thesis (Ph. D.), Yale University.

[14] Lee, J.-H. (1989). Global solvability of the derivative nonlinear Schrödinger equation. *Trans. Amer. Math. Soc.* **314**:107–118.

[15] Liu, J., Perry, P., Sulem, C. (2015). Global Existence for the Derivative Nonlinear Schrdinger Equation by the Method of Inverse Scattering. Comm. Partial Differential Equations 41 (2016), no. 11, 1692-1760

[16] Liu, J., Perry, P., Sulem, C. (2016). Large-time behavior of soliton-free solutions of the derivative nonlinear Schrödinger equation, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, in press. http://doi.org/10.1016/j.anihpc.2017.04.002

[17] Liu, Xiao (2013). Analytical and numerical results for some classes of nonlinear Schrödinger equations. Thesis (Ph.D.), University of Toronto.

[18] Mjolhus, E. (1976). On the modulational instability of hydromagnetic waves parallel to the magnetic field. *Journal of Plasma Physics* 16:321–334.

[19] Pelinovsky, D.E., Shimabukoro, Y. (2015). Global existence in the derivative NLS equation with the inverse scattering transform method. Int. Math. Res. Not. https://doi.org/10.1093/imrn/rnx051

[20] Pöschel, J., Trubowitz, E, Inverse spectral theory. Pure and Applied Mathematics, 130. Academic Press, Inc., Boston, MA, 1987. x+192 pp. ISBN: 0-12-563040-9

[21] Shimabukuro, Y., Saalmann, A., Pelinovsky, D.(2017) The derivative NLS equation: global existence with solitons. arXiv:1703.05277.

[22] Tovbis, A., Venakides, S. (2000) The eigenvalue problem for the focusing nonlinear Schrodinger equation: new solvable cases. Physica D, 146, 2000, 250264.

[23] Trogdon, T., Olver, S. Riemann-Hilbert problems, their numerical solution, and the computation of nonlinear special functions. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016.

[24] Wu, Y. (2014). Global well-posedness on the derivative nonlinear Schrdinger equation. Anal. PDE 8 (2015), no. 5, 11011112.

[25] Zakharov, V.E., Shabat, A.B. (1971). Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **34**:62–69; Z. Èksper. Teoret. Fiz. 61, no. 1, 118–134

[26] Zhou, X. (1989). The Riemann-Hilbert problem and inverse scattering. *SIAM J. Math. Anal.* 20:966–986.

[27] Zhou, X. (1998). L^2-Sobolev space bijectivity of the scattering and inverse scattering transforms. *Comm. Pure Appl. Math.* 51:697–731.

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