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Subfunctors of Extension Functors

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Subfunctors of Extension Functors

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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ABSTRACT OF DISSERTATION

Subfunctors of Extension Functors

This dissertation examines subfunctors of Ext relative to covering (enveloping) classes and the theory of covering (enveloping) ideals. The notion of covers and envelopes by modules was introduced independently by Auslander-Smalø and Enochs and has proven to be beneficial for module theory as well as for representation theory. The first few chapters examine the subfunctors of Ext and their properties. It is showed how the class of precoverings give us subfunctors of Ext. Furthermore, the characterization of these subfunctors and some examples are given. In the latter chapters ideals, the subfunctors of Hom, are investigated. The definition of cover and envelope carry over to the ideals naturally. Classical conditions for existence theorems for covers led to similar approaches in the ideal case. Even though some theorems such as Salce's Lemma were proven to extend to ideals, most of the theorems do not directly apply to the new case. It is showed how Eklof & Trlifaj's result can partially be extended to the ideals generated by a set. In that case, one also obtains a significant result about the orthogonal complement of the ideal. We relate the existence theorems for covering ideals of morphisms by identifying the morphisms with objects in A_2 and obtain a sufficient condition for the existence of covering ideals in a more general setting. We finish with applying this result to the class of phantom morphisms.

KEYWORDS: Homological Algebra, Cover, Subfunctor of Ext, Covering ideal, Phantom morphism

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To my family for their love and endless support

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CONTENTS

Chapter 1 INTRODUCTION

The theory of covers and envelopes was first introduced by Enochs in [16] for classes of injective modules and flat modules. Auslander-Smalø had also introduced essentially the same notion in [4] for modules over an Artinian algebra. The study of these notions have proven to be beneficial for module theory as well as representation theory (see, for example, [2], [8], [11], [22], [30]).

With the help of these notions one can define resolutions for precovering (preenveloping) classes and then look at the derived functors of Hom relative to these classes. In particular one can prove that these derived functors (denoted $\mathrm{E}xt^1_{\mathcal{P}}$) give us a subfunctor of $\mathrm{E}xt^1$. The study of these relative derived functors have been considered in different settings (see [2], [12], [28], [45]) and have been useful for several areas such as tilting theory, finitistic dimension theory, representation theory of Artin algebras, etc. We will mostly focus on the subfunctors $\mathrm{E}xt^1_{\mathcal{P}}$ relative to a precovering (preenveloping) class $\mathcal{P}$ and formulate the characterization for the short exact sequences in $\mathrm{E}xt^1_{\mathcal{P}}$.

One of the recent trends in the theory of covers (envelopes) that attracted wide interest is the ideals. The concept of covers and envelopes carry over to the ideal case very naturally and hence leads to the statement of existence theorems for covers and envelopes associated to these ideals, similar to the ones obtained in the classical theory of covers and envelopes. Even though some theorems such as Salce's Lemma were proven to extend to ideals (see [32]), most of the theorems do not directly apply to the new case. In [37] we showed how Eklof & Trlifaj's result [15] can partially be extended to the ideals generated by a set. In that case, one also obtains a significant result about the orthogonal complement of the ideal. In collaboration with Prof. Estrada and Prof. A. Guil (see [21]), we also have studied the conditions under which existence of covering ideals were guaranteed in a more general setting. Our approach was to relate the existence theorems for covering ideals of morphisms by identifying the morphisms with objects in a Grothendieck category $\mathcal{A}_2$ of all representations by left R-modules of the quiver $\mathbb{A}_2 : \bullet \to \bullet$. Then one also obtains a new and elementary proof for the existence of phantom covers (cf. [31]).

The theory of ideal approximation has rapidly been developed and started to be a focus of interest in the last few years (see [21], [24], [25], [26], [31], [32], [37]). With the new tools in hand, the theory has the potential to be of long research interest for algebraist and will reveal examples of covers and envelopes that could not had been deduced from the classes of objects before.

We now give a more detailed description of the contents of the following chapters.

In chapter 2, we first give the definition of a (pre)covering $\mathcal{P}$. We see how a precovering class gives rise to a new kind of resolution, namely $\mathcal{P}$ -resolution, which are unique up to homotopy if the precovering class contains the projective modules. Then focusing on the precovering classes that contain projective modules, we see that one can obtain the derived functor $\mathrm{E}xt^n_{\mathcal{P}}$ with these new resolutions. Next we show that $\mathrm{E}xt^1_{\mathcal{P}}$ is a subfuntor of $\mathrm{E}xt^1$. We tweak the definition of a global dimension of R with respect to a precovering class and finish the chapter with a result about the global dimension of R with respect to $\mathcal{P}$.

In chapter 3, we first define absolutely pure modules (FP-injective modules). Then we recall that the class of absolutely pure modules form a preenveloping class. Next we see that this preenveloping class gives us an example for a subfunctor of $\mathrm{E}xt^1$ which is denoted by $\mathrm{A}xt$. Next, we examine some properties of absolutely pure modules and use these properties to get a result on a localization of $\mathrm{A}xt$.

In chapter 4, we define the concept of phantom morphism relative to a subfunctor $\mathrm{E}xt_{\mathcal{C}}$ where $\mathcal{C}$ is a preenveloping. Then we characterize these phantoms in terms of its special relation with the subfunctor $\mathrm{E}xt_{\mathcal{C}}$. Next we investigate the phantom morphism relative to $\mathrm{E}xt_{\mathcal{C}}$ when $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair and see that in that case direct sum of phantom morphisms relative to $\mathrm{E}xt_{\mathcal{C}}$ will not lose the property of being phantom. In the last section we examine a special case. We look at the phantom morphisms relative to $\mathrm{A}xt$ and apply the theorem and results from the previous section to this special case.

In chapter 5, we introduce the notion of an ideal and see how the definition of a cover and envelope carry over to the ideal case. We also define ideal cotorsion pairs in a similar way to the module case. A significant result of cotorsion theory proven by Eklof & Trlifaj is that if a pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is cogenerated by a set, then it is complete [15]. Motivated by the Eklof & Trlifaj argument, we prove a similar result for an ideal $\mathcal{I}$ when it is generated by a set of homomorphisms. We prove that if $\mathcal{I}$ is generated by a set then $\mathcal{I}^{\perp}$ is preenveloping and that $\mathcal{I}$ is precovering if it is closed under sums.

In the last chapter we give the results from [21], a joint work with Sergio Estrada and Pedro A. Guil. Our result is motivated by El Bashir's well-known theorem:

Theorem 1.0.1. *(El Bashir, [6]) Let $\mathcal{F}$ be a class of objects of a Grothendieck category $\mathcal{G}$ closed under coproducts and directed co-limits. If there exists a subset S of $\mathcal{F}$ such that each object in $\mathcal{F}$ is a directed co-limit of objects in S, then each object of $\mathcal{G}$ has an $\mathcal{F}$ -cover.*

We use this result to find a similar sufficient condition for an ideal to be covering. Then we prove that the class of phantom morphisms can easily be proven to be covering as a consequence. We conclude the chapter by showing that the kernel of a phantom cover is always pure injective.

We note that throughout the chapters all rings R are associative with identities and all modules are unitary. If for an R-module M there is no particular side mentioned, it is assumed to be a left R-module. Even though some of the results presented here can be extended to a more general setting we will focus on the category of R-modules.

Chapter 2 SUBFUNCTORS OF Ext

2.1 Preliminaries

In general it is a complicated task to describe all modules over an arbitrary ring. Since early 1960's injective envelopes and projective covers have been used to investigate properties of modules over an arbitrary ring. One can try to understand the structure of this type of classes and approximate arbitrary modules by the ones in these classes. So we start by giving the formal definition of such a class first.

Definition 2.1.1. *Let R be a ring and $\mathcal{P}$ be a class of R-modules. A morphism $\varphi : F \to M$ where $F \in \mathcal{P}$ is called a $\mathcal{P}$-precover of M if any morphism $G \to M$ where $G \in \mathcal{P}$ factors through φ. Moreover if every endomorphism of F satisfying $\varphi \circ f = \varphi$ is an automorphism, then φ is said to be a $\mathcal{P}$-cover of M. Then a class $\mathcal{P}$ is said to be (pre)covering if every module admits a $\mathcal{P}$-(pre)cover.*

In categorical sense, dual notion of a (pre)cover is (pre)envelope and defined by reversing the directions of arrows. Note that precovers are also known as left approximations due to Auslander and Reiten (see [2]).

Example 2.1.2. *The easiest example for a precovering is the class of all projective modules. For any module M there exists a surjective homomorphism $\varphi : F \to M$ where F is free. Then φ is a projective precover of M, since for any projective module P one can complete the following diagram,*

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow \\ F & \xrightarrow{\varphi} & M \end{array}$$

Moreover every $R-$module has a projective cover if and only if the ring R is perfect.

Example 2.1.3. *An R-module F is said to be flat if given any exact sequence $0 \to A \to B$ of R-modules, the tensored sequence $0 \to A \bigotimes_R F \to B \bigotimes_R F$ is exact. It was conjectured and then proven by Enochs that the class of flat modules is a covering [10].*

Example 2.1.4. *A short exact sequence in R-Mod $0 \longrightarrow X \xrightarrow{i} Y \longrightarrow Z \xrightarrow{p} 0$ is said to be pure exact if for every finitely presented R-module M, $Hom(M, Y) \to Hom(M, Z)$ is surjective. Then we say X is pure injective if every pure exact sequence with left term X is split exact. Warfield proved in [45] that every module has a pure injective envelope. Pure injective modules play a central role in the model theory (for a detailed exposition see [38]). Moreover pure injective modules are of interest for cotilting theory since it was proven recently that all cotilting modules are pure injective by Bazzoni in [7].*

One can generalize the definition of a free resolution to an arbitrary precovering $\mathcal{P}$. First we give the definition of a left $\mathcal{P}$ -resolution and then see under which condition this resolution will be exact. For the proof of existence of $\mathcal{P}$ -resolutions, we refer the reader to [22].

Proposition 2.1.5. *(Enochs, Jenda [22]) Let $\mathcal{P}$ be a precovering and M be an R-module, then there exist a complex*

$$... \ P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each $P_i \in \mathcal{P}$ for which

$$... \ Hom(P, P_1) \longrightarrow Hom(P, P_0) \longrightarrow Hom(P, M) \longrightarrow 0$$

is exact for any $P \in \mathcal{P}$. Such a complex is called a left $\mathcal{P}$ -resolution of M.

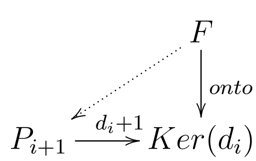
A right $\mathcal{C}$ -resolution for a preenveloping class $\mathcal{C}$ is defined similarly. Note that such a left $\mathcal{P}$ -resolution is not necessarily exact but the following proposition gives us a sufficient condition for such a resolution to be exact.

Proposition 2.1.6. *If $\mathcal{P}$ is a precovering which contains all projective modules then every R-module has an exact left $\mathcal{P}$ -resolution.*

Proof. Let M be a R-module and $P_0 \to M$ be a $\mathcal{P}$ -precover of M. Then choose a $\mathcal{P}$ -precover $P_1 \to Ker(P_0 \to M)$ for the kernel $Ker(P_0 \to M)$. Choosing precovers recursively, one obtains a $\mathcal{P}$ -resolution,

$$... \ P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

Now we will show that this complex is exact. Since $\mathcal{P}$ contains all projective modules, in particular it contains all free modules. For any $Ker(d_i)$ there exists a free module F and a surjective morphism $F \to Ker(d_i)$. But then the following can be completed to a commutative diagram,



Hence d_{i+1} is surjective. That is $Im(d_{i+1}) = Ker(d_i)$ and the $\mathcal{P}$ -resolution is exact. □

2.2 Subfunctors corresponding to a precovering class

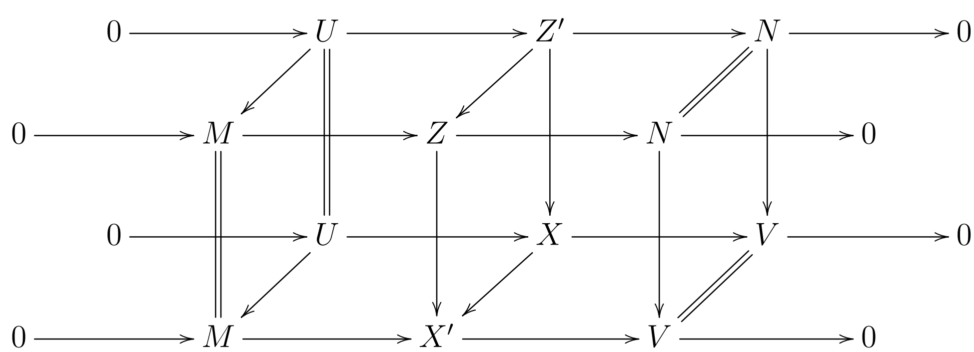
Extension of groups was first studied by Baer and observed to be an abelian group with the "Baer sum" (see [5]). It was Eilenberg and Steenrod who used free resolutions to compute $\mathrm{E}xt^1$ (see [13]). Following Baer's lead, we interpret $\mathrm{E}xt^1_R(U,V)$ as the quotient group of all the extensions of V by U module the homotopy equivalence, for details we refer reader to [40]. An extension of V by U denoted by an element $\xi \in \mathrm{E}xt(U,V)$ is a short exact sequence,

$$0 \longrightarrow U \longrightarrow X \longrightarrow V \longrightarrow 0$$

Two extensions are said to be equivalent if there is such a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & V & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & U & \longrightarrow & Y & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

where the middle map is an isomorphism. Then one can easily see that there is an equivalence relation in the set of all extensions of R-modules and with this viewpoint, one can interpret $\mathrm{E}xt^1$ as a bifunctor from $R\text{-}\mathrm{M}od^{op} \times R\text{-}\mathrm{M}od$ to the category Ab. That is $\mathrm{E}xt^1$ associates a pair (U,V) of modules to the group of all extension equivalence classes of V by U. Moreover if $f : N \to V$ and $g : U \to M$ are morphisms in $R\text{-}\mathrm{M}od$ then $Ext^1(f,g)$ maps an extension of V by U to an extension of N by M that is calculated by using a pushout along g followed by a pullback along f (or equivalently a pullback along f followed by a pushout along g) as following,



In this chapter we study the precovering classes that contain all the projective R-modules. We first show that for such a precovering class the left $\mathcal{P}$ -resolutions of modules are unique up to homotopy.

Proposition 2.2.1. *Let $\mathcal{P}$ be a precovering class that contains all the projective modules. If Q is a projective resolution of M and P is a $\mathcal{P}$ -resolution of M, then there exists a* *family of maps $(f_i)_{i\in\mathbb{N}}$ making the following diagram commutative,*

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\
\dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Moreover any such map is unique up to homotopy.

Proof. We will use induction. Since Q_0 is a projective module and $P_0 \to M$ is onto we conclude the existence of f_0.

To construct f_i assume that there exist f_{i-1} satisfying the desired conditions and consider the diagram,

$$\begin{array}{ccc}
Q_i & \xrightarrow{e_i} & Q_{i-1}\\
\vdots{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}}\\
P_i & \xrightarrow{d_i} & Im(d_i) = Ker(d_{i-1})
\end{array}$$

Note that $Im(f_{i-1}e_i) \subseteq Im(d_i) = Ker(d_{i-1})$, since $d_{i-1}f_{i-1}e_i = e_ie_{i-1}f_{i-2} = 0$. Then since Q_i is a projective module we can conclude the existence of f_i making the diagram commutative.

Now to prove uniqueness, assume that we have two such families of maps (f_i) and (g_i). Notice that $Im(f_0 - g_0) \subseteq Kerd_0 = Imd_1$ so,

$$\begin{array}{ccc}
 & & Q_0\\
 & \swarrow{\scriptstyle \exists s_0} & \downarrow{\scriptstyle f_0-g_0}\\
P_1 & \longrightarrow & Im(d_1)
\end{array}$$

In general since $Im(f_i - gi - s_{i-1}e_i) \subseteq Ker(d_i) = Im(d_{i+1})$ we get,

$$\begin{array}{ccc}
 & & Q_i\\
 & \swarrow{\scriptstyle \exists s_i} & \downarrow{\scriptstyle f_i-g_i-s_{i-1}e_i}\\
P_{i+1} & \longrightarrow & Im(d_{i+1})
\end{array}$$

So we conclude that $f_i - g_i = s_{i-1}e_i + d_{i+1}s_i$. That is $(f_i) \simeq (g_i)$. $\square$

Corollary 2.2.2. *Let $\mathcal{P}$ be a precovering class that contains all the projective modules. If $\bar{P}$ and P are two $\mathcal{P}$ -resolutions for M, then there exists a family of maps $(f_i)_{i\in\mathbb{N}}$ making the following diagram commutative,*

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \bar{P}_2 & \longrightarrow & \bar{P}_1 & \longrightarrow & \bar{P}_0 & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| & & \\
\dots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Moreover any such map is unique up to homotopy.

Proof. This follows in a similar way as Proposition 2.2.1. □

If $\mathcal{P}$ is a precovering containing all projective modules then using a left $\mathcal{P}$ -resolution one can compute the relative derived functor of Hom functor with respect to $\mathcal{P}$ denoted as $Ext^n_{\mathcal{P}}$ and it is well-defined by Corollary 2.2.2. Given a module M, lets compute the first degree E$xt^1_{\mathcal{P}}$ (M, N). We only need to look at the partial exact $\mathcal{P}$ -resolution,

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

Then E$xt^1_{\mathcal{P}}$ (M, N) consists of $f \in$ H$om(K, N)$ modulo I$m(Hom(P, N) \to$ H$om(K, N))$. Similar to the usual bijective correspondence one gets for the Ext^1 functor, we observe that there is a bijective correspondence between the morphisms $f \in$ H$om(K, N)$ modulo I$m(Hom(P, N) \to$ H$om(K, N))$ and the equivalence class of extensions of N by M that is obtained by a pushout from the above partial $\mathcal{P}$ -resolution as following,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & P' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

So we note that the equivalence class of the lower row is in E$xt^1_{\mathcal{P}}$ (M, N) if it is obtained by a pushout as above. We will use both of these interpretations whenever necessary.

Now we are ready to prove our main result which gives a nice relationship between E$xt^1_{\mathcal{P}}$ and Ext^1.

Theorem 2.2.3. *Let $\mathcal{P}$ be a precovering. Then $Ext^1_{\mathcal{P}}$ $(\_, N)$: R-$Mod^{op} \to Ab$ is a subfunctor of $Ext^1(\_, N)$: R-$Mod^{op} \to Ab$.*

Proof. Given a $\mathcal{P}$ -precover $Q_0 \to M$ for M with a surjection $P_0 \to Q_0$ where P_0 is a projective module. Let $S = Ker(P_0 \to Q_0)$ then $Q_0 \cong P_0/S$. So we have the following exact diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi|_{K_1}} & \circlearrowleft & \downarrow{\scriptstyle \pi} & \circlearrowleft & \| & & \\
0 & \longrightarrow & K_1/S & \longrightarrow & P_0/S & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

This yields to the morphism E$xt^1_{\mathcal{P}}$ $(M, N) \to$ E$xt^1(M, N)$. We need to show that:

- E$xt^1_{\mathcal{P}}$ $(M, N) \to$ E$xt^1(M, N)$ is an injection.

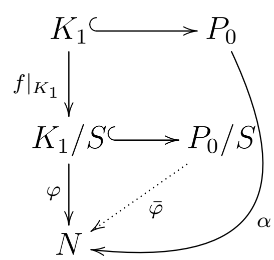
- For any map $M' \xrightarrow{f} M$, the restriction of the map $\mathrm{E}xt^1(M,N) \xrightarrow{\mathrm{E}xt^1(f,N)} \mathrm{E}xt^1(M',N)$ to $\mathrm{E}xt^1_{\mathcal{P}}\ (M,N)$ will be the morphism $\mathrm{E}xt^1_{\mathcal{P}}\ (f,N)$.

Lets first show the first map is an injection. From the commutative diagram above we get the following exact diagram,

$$\begin{array}{ccccccc}
\mathrm{H}om(P_0/S,N) & \longrightarrow & \mathrm{H}om(K_1/S,N) & \longrightarrow & \mathrm{E}xt^1_{\mathcal{P}}\ (M,N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{H}om(P_0,N) & \longrightarrow & \mathrm{H}om(K_1,N) & \longrightarrow & \mathrm{E}xt^1(M,N) & \longrightarrow & 0
\end{array}$$

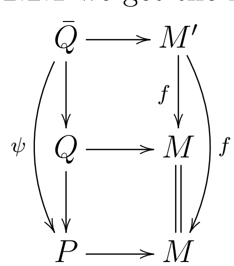
where $\mathrm{E}xt^1_{\mathcal{P}}\ (M,N) = \mathrm{H}om(K_1/S,N)/U$ st $U = \{g \in \mathrm{H}om(K_1/S,N) \mid \text{g can be extended to } P_0/S\}$. Moreover, $\mathrm{E}xt^1(M,N) = \mathrm{H}om(K_1,N)/V$ where $V = \{h \in \mathrm{H}om(K_1,N) \mid \text{h can be extended to } P_0\}$.

Given $\tilde{\varphi} = \varphi + U \in \mathrm{E}xt^1_{\mathcal{P}}\ (M,N)$ such that the map $\mathrm{E}xt^1_{\mathcal{P}}\ (M,N) \to \mathrm{E}xt^1(M,N)$ maps φ to $0+U$. That means, $\varphi f|_{K_1}$ can be extended to $\alpha : P_0 \to N$. Define $\bar{\varphi} : P_0/S \to N$ such that $\bar{\varphi}(x+S) = \alpha(x)$. So we have the following diagram,

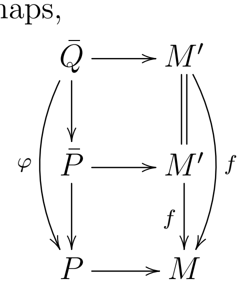


φ is well-defined. Moreover, $\bar{\varphi}|_{K_1/S} = \varphi$. That is $\bar{\varphi}$ is extension of φ to P_0/S. Hence $\varphi \in U$ in $\mathrm{E}xt^1_{\mathcal{P}}\ (M,N)$.

To prove the second part given $M' \xrightarrow{f} M$, and given projective resolution Q of M, projective resolution $\bar{Q}$ of M', $\mathcal{P}$ -resolution P of M, $\mathcal{P}$ -resolution $\bar{P}$ of M'. Then by Proposition 2.2.1 we get the following comparison maps,



and



By Proposition 2.2.1 we notice that φ and ψ are homotopy equivalent. They will still be homotopy equivalent if we take $\mathrm{H}om(\_,N)$ of each complex. Then the $\mathrm{H}om(\_,N)$ of the diagram of these complexes induce the same maps on homology groups. So we get the following commutative diagram of the homology groups,

$$\begin{array}{ccc} \mathrm{E}xt^1_{\mathcal{P}}\ (M,N) & \xrightarrow{\mathrm{E}xt^1_{\mathcal{P}}\ (f,N)} & \mathrm{E}xt^1_{\mathcal{P}}\ (M',N) \\ \downarrow & & \downarrow \\ \mathrm{E}xt^1(M,N) & \xrightarrow{\mathrm{E}xt^1(f,N)} & \mathrm{E}xt^1(M',N) \end{array}$$

which shows that $\mathrm{E}xt^1_{\mathcal{P}}\ (f,N)$ is just the restriction of $\mathrm{E}xt^1(f,N)$. □

Then one can easily obtain the following corollary.

Corollary 2.2.4. *Let $\mathcal{P}$ be a precovering. Then $Ext^1_{\mathcal{P}}\ (\_,\_) : R\text{-}Mod^{op} \times R\text{-}Mod \to Ab$ is a subfunctor of the bifunctor $Ext^1(\_,\_) : R\text{-}Mod^{op} \to Ab$.*

The proof above can easily be adapted to a preenveloping class. Now we give an example for such a subfunctor relative to the pure injective enveloping class.

Example 2.2.5. *Recall that in Example 2.1.4 we give the definition of a pure exact sequence and called a module X pure injective if every pure exact sequence with left term X is split exact. The class of pure injective class is an enveloping and gives rise to a subfunctor denoted as Pext which consists of all pure exact sequences. This subfunctor have been studied in details by Fuchs ([28]) and Warfield ([45]).*

2.3 Results on $\mathrm{E}xt_{\mathcal{P}}$

We discuss how some of the results concerning $\mathrm{E}xt$ can partially be carried over to $\mathrm{E}xt_{\mathcal{P}}$ and see what kind of conditions must be asserted for them to hold.

We say (left) global dimension of a ring R is n ($n \geq 0$) if $\mathrm{E}xt^{n+1}(M,N) = 0$ for all $M,N \in R$-M*od* and that $\mathrm{E}xt^n(M,N) \neq 0$ for some $M,N \in R$-M*od*. It is known that global dimension of R is 0 if one of the following equal condition is satisfied,

- Every submodule S of every R-module M is a direct summand of M. That is, every M is semisimple.
- All R-modules are injective(projective).
- R is a direct sum of simple R-modules.

Then one can extend the notion of global dimension for the functors $\mathrm{E}xt_{\mathcal{P}}$. We say global dimension of a ring with respect to a precovering $\mathcal{P}$ is n if $\mathrm{E}xt^{n+1}_{\mathcal{P}}(M,N) = 0$ for all $M,N \in R$-M*od* and that $\mathrm{E}xt^n_{\mathcal{P}}\ (M,N) \neq 0$ for some $M,N \in R$-M*od*. Now, we show that global dimension of a ring with respect to a precovering $\mathcal{P}$ is zero if it satisfies the following condition.

Theorem 2.3.1. *Let $\mathcal{P}$ be a precovering. The global dimension with respect to $\mathcal{P}$ is 0 if and only if for every $P \in \mathcal{P}$ and submodule $K \subset P$, K is a direct summand of P.*

Proof. ($\Rightarrow$) Assume that the global dimension with respect to $\mathcal{P}$ is 0. Now given $P \in \mathcal{P}$ and a submodule $K \subset P$, we have the following left $\mathcal{P}$-resolution of P/K

$$... \; P \longrightarrow P/K \longrightarrow 0$$

Then $\mathrm{E}xt^1_{\mathcal{P}} = \frac{\mathrm{H}om(K,K)}{Im(\mathrm{H}om(P,K) \to \mathrm{H}om(K,K))}$ which is 0 by assumption. That is $\mathrm{H}om(P,K) \to \mathrm{H}om(K,K)$ is surjective. Hence K is a retract of P, and a direct summand of P.

($\Leftarrow$) Assume that for every $P \in \mathcal{P}$ and submodule $K \subset P$, K is a direct summand of P. For any $n \geq 1$ to compute $\mathrm{E}xt^n_{\mathcal{P}}(M,N)$ consider a partial $\mathcal{P}$-resolution of M,

$$K_n \hookrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow ... \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Note that then $K_n \subset P_{n-1}$ is a direct summand of P_{n-1} and hence there is a retraction $q : P_{n-1} \to K_n$. So for any morphism $f : K_n \to N$ we have $f \circ q|_{K_n} = f$ hence $Ext^n_{\mathcal{P}} = 0$ for any $n \geq 1$. □

Now we prove that the first degree derived functor relative to precoverings containing projective modules are all isomorphic as explained in the given proposition.

Proposition 2.3.2. *Let $\mathcal{P}$ be a precovering. $\mathcal{P}$ contains the class of all projective modules if and only if $Hom(M,N) \cong Ext^0_{\mathcal{P}}(M,N)$ for any $M, N \in R$-Mod.*

Proof. Assume that $\mathcal{P}$ contains all projective modules. Given an R-module M and a $\mathcal{P}$-resolution of M,

$$... \; P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

which is exact by Remark 2.1.6. For an arbitrary R-module N, take $\mathrm{H}om(-,N)$ of the above complex,

$$0 \longrightarrow \mathrm{H}om(M,P) \longrightarrow \mathrm{H}om(P_0,P) \longrightarrow \mathrm{H}om(P_1,P) \; ...$$

Since the $\mathcal{P}$-resolution is exact, the complex above is exact as well. That is $\mathrm{E}xt^0_{\mathcal{P}}(M,N) = Ker(\mathrm{H}om(P_0,N) \to \mathrm{H}om(P_1,N)) \cong \mathrm{H}om(M,N)$.

To prove the other direction, assume that the given isomorphism holds for any $M, N \in R$-Mod. Let P be a projective module and

$$... \; P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P \longrightarrow 0$$

be a $\mathcal{P}$-resolution of P. Then

$$0 \longrightarrow \mathrm{H}om(P,N) \longrightarrow \mathrm{H}om(P_0,N) \longrightarrow \mathrm{H}om(P_1,N) \; ...$$

is exact for every module N by assumption. Then notice that $P_0 \xrightarrow{d_0} P$ is surjective since $0 \longrightarrow \mathrm{H}om(P, P/Im(d_0)) \longrightarrow \mathrm{H}om(P_0, P/Im(d_0))$ is exact. Then notice there exists a morphism f completing the following diagram,

$$\begin{array}{ccc} & & P \\ & \overset{f}{\swarrow} & \downarrow 1 \\ P_0 & \xrightarrow{d_0} & P \end{array}$$

Hence $d_0 \circ f = 1$ and P is a direct summand of P_0. Now we can conclude that all projective modules are in $\mathcal{P}$ since $\mathcal{P}$ is closed under direct summands. $\square$

2.4 Characterization of $\mathrm{E}xt^1_{\mathcal{P}}$

In this section we assume that $\mathcal{P}$ is closed under direct summands and contains all the projective modules. If $\mathcal{P}$ is not closed under direct summands, say $\bar{\mathcal{P}}$ is the closure of $\mathcal{P}$ under direct summands, then $\bar{\mathcal{P}}$ is also a precovering class. Moreover, if $P \to M$ is a $\mathcal{P}$ -precover then it will also be a $\bar{\mathcal{P}}$-precover. Then it is not difficult to see that one would get the same derived functor with respect to both classes.

Now we give a characterization for the short exact sequences in $\mathrm{E}xt^1_{\mathcal{P}}$ and also see how this characterization determines the elements of $\mathcal{P}$.

Proposition 2.4.1. $0 \longrightarrow N \longrightarrow U \longrightarrow M \longrightarrow 0$ *is in* $Ext^1_{\mathcal{P}}(M, N)$ *if and only if the s.e.s. has the following property:*
If $P' \in \mathcal{P}$ *and given any map* $P' \longrightarrow M$ *then*

$$\begin{array}{ccccccccc} & & & & & & P' & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

can be completed to a commutative diagram.

Proof. Suppose we have the given property. So the following can be completed to a commutative diagram,

$$\begin{array}{ccccccccc} & & & & P' & \longrightarrow & M & \longrightarrow & 0 \\ & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

But then,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P' & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Notice where $Coker(K \hookrightarrow P') \cong M \cong Coker(N \hookrightarrow U)$.Hence,

$$\begin{array}{ccc} K & \hookrightarrow & P' \\ \downarrow & & \downarrow \\ N & \hookrightarrow & U \end{array}$$

is a pushout.
That is, $0 \longrightarrow N \longrightarrow U \longrightarrow M \longrightarrow 0$ is in $\mathrm{E}xt^1_{\mathcal{P}}(M, N)$.

Now suppose $0 \longrightarrow N \longrightarrow U \longrightarrow M \longrightarrow 0$ is in $\mathrm{E}xt^1_{\mathcal{P}}(M, N)$. That is there exists a diagram:

$$\begin{array}{ccccccccc} & & & & & & P' & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Then since P' and $P \in \mathcal{P}$, there exist a map completing the diagram:

$$\begin{array}{ccc} & & P' \\ & \swarrow & \downarrow \\ P & \longrightarrow & M \end{array}$$

Hence there exist a map completing the commutative diagram,

$$\begin{array}{ccc} & & P' \\ & \swarrow & \downarrow \\ U & \longrightarrow & M \end{array}$$

□

Proposition 2.4.2. *Given a precovering class $\mathcal{P}$ which can be assumed to be closed under direct summands. An R-module L is in the precovering class $\mathcal{P}$ if and only if for any* *map $L \to M$ and s.e.s. $0 \longrightarrow N \longrightarrow U \longrightarrow M \longrightarrow 0$ in $Ext^1_{\mathcal{P}}(M, N)$ the diagram,*

$$\begin{array}{ccccccccc} & & & & & & L & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

can be lifted to a commutative diagram for any $N, M \in R - Mod$.

Proof. If $L \in \mathcal{P}$ then we know that lifting property holds.

To show the other direction, assume that lifting property holds for L. We know there exist a $\mathcal{P}$-precover $P \to L \to 0$ which yields to a exact sequence, $0 \longrightarrow K \hookrightarrow P \longrightarrow L \longrightarrow 0$ in $\mathrm{E}xt^1_{\mathcal{P}}(L, K)$. But then by assumption,

$$\begin{array}{ccccccccc} & & & & & & L & & \\ & & & & & \swarrow & \| & & \\ 0 & \longrightarrow & K & \hookrightarrow & P & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

can be completed. That is L is a direct summand of P. Since $P \in \mathcal{P}$ and $\mathcal{P}$ is closed under direct summands $L \in \mathcal{P}$ as well. □

We give yet another characterization of elements of a precovering class.

Proposition 2.4.3. *Let $\mathcal{P}$ be a precovering class containing all projective modules and closed under direct summands. Then a module M is in $\mathcal{P}$ if and only if $Ext^1_{\mathcal{P}}(M, N) = 0$ for all modules N.*

Proof. If M is in $\mathcal{P}$ then clearly $\mathrm{E}xt^1_{\mathcal{P}}(M, N) = 0$ for all modules N.

To prove other direction assume $\mathrm{E}xt^1_{\mathcal{P}}(M, N) = 0$ for all modules N. Let us consider the short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

where P is a $\mathcal{P}$-precover of M. Then by using the long-exact sequence theorem (Enochs-Jenda, Theorem 8.2.3, [22]) one applies $\mathrm{H}om(\_, K)$ to get the following exact sequence,

$$0 \longrightarrow \mathrm{H}om(M, K) \longrightarrow \mathrm{H}om(P, K) \longrightarrow \mathrm{H}om(K, K) \longrightarrow \mathrm{E}xt^1_{\mathcal{P}}(M, K) = 0 \ ...$$

That is $\mathrm{H}om(P, K) \to \mathrm{H}om(K, K)$ is surjective and thus M is a direct summand of P. Now we can conclude that M is in $\mathcal{P}$. □

We would now like to turn our attention to the lattice property of these subfunctors. In order to see the lattice property, we need to prove the following proposition first.

Proposition 2.4.4. *Suppose $Ext^1_{\mathcal{P}}$ and $Ext^1_{\bar{\mathcal{P}}}$ corresponds to the precovering classes $\mathcal{P}$ and $\bar{\mathcal{P}}$ respectively. Then the precovering class $\mathcal{P} \oplus \bar{\mathcal{P}}$ gives rise to the subfunctor $Ext^1_{\mathcal{P}} \cap Ext^1_{\bar{\mathcal{P}}}$.*

Proof. Suppose the class of s.e.s $\mathrm{E}xt^1_{\mathcal{P}} \cap \mathrm{E}xt^1_{\bar{\mathcal{P}}}$ corresponds to the precovering class Π . We claim that $\Pi = \mathcal{P} \oplus \bar{\mathcal{P}}$.

$\mathcal{P} \oplus \bar{\mathcal{P}} \subseteq \Pi$: Given $P \oplus \bar{P} \in \mathcal{P} \oplus \bar{\mathcal{P}}$ and a s.e.s in $\mathrm{E}xt^1_{\mathcal{P}} \cap \mathrm{E}xt^1_{\bar{\mathcal{P}}}$ with a map f:

$$\begin{array}{ccccccccc} & & & & & & \mathcal{P} \oplus \bar{\mathcal{P}} & & \\ & & & & & & \downarrow f & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \xrightarrow{\beta} & M & \longrightarrow & 0 \end{array}$$

Then there exist φ_1 and φ_2 completing the diagrams,

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \varphi_1 \swarrow & \downarrow \pi_1 & & \\ & & & & & & \mathcal{P} \oplus \bar{\mathcal{P}} & & \\ & & & & & & \downarrow f & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \xrightarrow{\beta} & M & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} & & & & & & \bar{P} & & \\ & & & & & \varphi_2 \swarrow & \downarrow \pi_2 & & \\ & & & & & & \mathcal{P} \oplus \bar{\mathcal{P}} & & \\ & & & & & & \downarrow f & & \\ 0 & \longrightarrow & N & \longrightarrow & U & \xrightarrow{\beta} & M & \longrightarrow & 0 \end{array}$$

Define φ such that $\varphi((x, y)) = \varphi_1(x) + \varphi_2(y)$. That is we get the following commutative diagram,

$$\begin{array}{ccc} & & P \oplus \bar{P} \\ & \varphi \swarrow & \downarrow f \\ U & \xrightarrow{\beta} & M \end{array}$$

So $\beta(\varphi(x, y)) = \beta(\varphi_1(x)) + \beta(\varphi_2(y)) = f(x, 0) + f(0, y) = f(x, y)$. That is $P \oplus \bar{P} \in \Pi$.

$\Pi \subseteq \mathcal{P} \oplus \bar{\mathcal{P}}$: Given $\tilde{P} \in \Pi$, there exist $\mathcal{P}$ and $\bar{\mathcal{P}}$-precoverings respectively,

$P \xrightarrow{f} \tilde{P} \longrightarrow 0$ and $\bar{P} \xrightarrow{g} \tilde{P} \longrightarrow 0$ which gives rise to exact sequences,

$$0 \longrightarrow Ker(f) \oplus Ker(g) \hookrightarrow P \xrightarrow{f} \tilde{P} \longrightarrow 0$$

in $\mathrm{E}xt^1_{\mathcal{P}}$ and

$$0 \longrightarrow Ker(f) \oplus Ker(g) \hookrightarrow \bar{P} \xrightarrow{g} \tilde{P} \longrightarrow 0$$

in $\mathrm{E}xt^1_{\bar{\mathcal{P}}}$ So we get the following exact sequence in $\mathrm{E}xt^1_{\mathcal{P}} \cap \mathrm{E}xt^1_{\bar{\mathcal{P}}}$,

$$0 \longrightarrow Ker(f) \oplus Ker(g) \hookrightarrow P \oplus \bar{P} \xrightarrow{(f,g)} \tilde{P} \longrightarrow 0$$

where $(f,g)(x,y) = f(x) + g(y)$. That is because for any $P_1 \in \mathcal{P}$ there exist φ_1 such that,

$$\begin{array}{ccccccccc} & & & & & & P_1 & & \\ & & & & & \overset{\varphi_1}{\swarrow} & \downarrow & & \\ 0 & \longrightarrow & Ker(f) \oplus Ker(g) & \hookrightarrow & P \oplus \bar{P} & \xrightarrow{(f,g)} & \tilde{P} & \longrightarrow & 0 \end{array}$$

where φ_1 is induced from φ,

$$\begin{array}{ccc} & & P_1 \\ & \overset{\varphi}{\swarrow} & \downarrow \\ P & \longrightarrow & \tilde{P} \end{array}$$

such that $\varphi_1(x) = (\varphi(x), 0)$.So $0 \longrightarrow Ker(f) \oplus Ker(g) \hookrightarrow P \oplus \bar{P} \longrightarrow \tilde{P} \longrightarrow 0$ is in $\mathrm{E}xt^1_{\mathcal{P}}$. The same way we can show that it is in $\mathrm{E}xt^1_{\bar{\mathcal{P}}}$.Hence,

$$\begin{array}{ccccccccc} & & & & & & \tilde{P} & & \\ & & & & & \swarrow & \downarrow id & & \\ 0 & \longrightarrow & Ker(f) \oplus Ker(g) & \hookrightarrow & P \oplus \bar{P} & \longrightarrow & \tilde{P} & \longrightarrow & 0 \end{array}$$

can be completed that is $\tilde{P}$ is a direct summand of $P \oplus \bar{P}$. Hence $\Pi = \mathcal{P} \oplus \bar{\mathcal{P}}$. □

Corollary 2.4.5. *Let the class of all subfunctors* $Ext^1_{\mathcal{P}}$ *that rises from precoverings be partially ordered by* $Ext^1_{\mathcal{P}} \leq Ext^1_{\bar{\mathcal{P}}} \Leftrightarrow Ext\,1_{\mathcal{P}} \subseteq Ext^1_{\bar{\mathcal{P}}}$. *Then the class of these subfunctors form a lattice.*

Proof. We need to show that any two subfunctors have a least upper bound and a greatest lower bound. We immediately notice that by Proposition 2.4.4 a greatest lower bound of the two subfunctors $\mathrm{E}xt^1_{\mathcal{P}}$, $\mathrm{E}xt^1_{\bar{\mathcal{P}}}$ is $\mathrm{E}xt^1_{\mathcal{P}} \cap \mathrm{E}xt^1_{\bar{\mathcal{P}}}$. One can easily observe that then the least upper bound of two such subfunctors is the intersection of all the subfunctors that rise from precoverings that contains union of those two subfunctors. □

Chapter 3 ABSOLUTELY PURE MODULES AND THE SUBFUNCTOR Axt

A is called an absolutely pure R-module if for every projective R-module P, every finitely generated submodule $S \subset P$ and every morphism $S \to A$, there is an extension to $P \to A$. In this section, we study some of the properties of the subfunctor Axt_R for a coherent ring R. A commutative ring R is coherent if every finitely generated submodule of a finitely presented module is finitely presented.

3.1 On absolutely pure modules

We gather some of the results on direct limit and localization of absolutely pure modules that will help us to get our main result in the upcoming section.

Lemma 3.1.1. *Given a directed set I and let $M_i \xrightarrow{\varphi_i} N_i$ be a set of onto maps. The map,*

$$\varinjlim M_i \xrightarrow{\varphi} \varinjlim N_i$$

defined as $\varphi([x]) = [\varphi_i(x)]$ where say $x \in M_i$ is onto as well.

Proof. By the universal property of direct limit, we see that the map φ is well-defined. But then it is clearly onto, since each φ_i is onto.

Lemma 3.1.2. *Given a family $(M_i)_{i\in I}$ of left R-modules where I is a directed set then,*

$$\varinjlim Hom(R^n, M_i) \cong Hom(R^n, \varinjlim M_i)$$

Proof. This follows since H$om(R^n, M_i) \cong M_i^n$ then notice that,

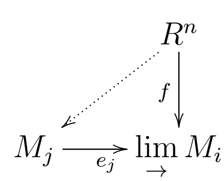
$$\varinjlim \mathrm{H}om(R^n, M_i) \cong \varinjlim M_i^n \cong (\varinjlim M_i)^n$$

On the other hand,

$$\mathrm{H}om(R^n, \varinjlim M_i) \cong (\varinjlim M_i)^n$$

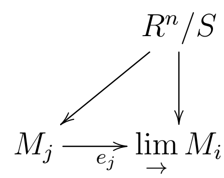
□

Lemma 3.1.3. *Given a family $(M_i)_{i\in I}$ of left R-modules where I is a directed set and given any linear $R^n \to \varinjlim M_i$ then we can find some $j \in I$ so that $f(R^n) \subset e_j(M_j)$. Then,*



can be completed to a commutative diagram.

Lemma 3.1.4. *Given a family $(M_i)_{i \in I}$ of left R-modules where I is a directed set and let $L = R^n/S$ where S is a finitely generated R-module (i.e. L is a finitely related R-module) then for any map $R^n/S \to \varinjlim M_i$ there is some $j \in I$ and a linear map $R^n/S \to M_j$ such that,*

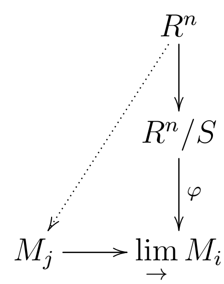


is commutative.

Proof. Given a map,

$$R^n/S \xrightarrow{\varphi} \varinjlim M_i$$

then by the Lemma 3.1.3 we know we can find $j \in I$ and a map h_j that completes the diagram,



What we want is to find such an $R^n \to M_j$ that has S in its kernel, so that we can get an induced map $R^n/S \to M_j$.

Now since

$$R^n \xrightarrow{h_j} M_j \xrightarrow{e_j} \varinjlim M_i$$

has S in its kernel.This means that, for $x \in S$ we get $e_j(h_j(x)) = 0$ in $\varinjlim M_i$. Using the definition of the equivalence relation that gives $\varinjlim M_i$ we see that this means that for some $k \in I$ with $j \leq k$ we have $f_{kj}(h_j(x)) = 0$.

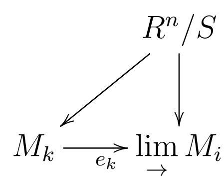
This means that

$$R^n \to M_j \to M_k$$

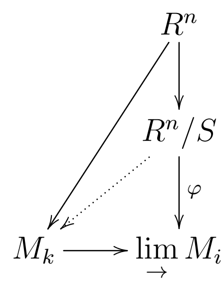
is 0 on S. But since S is finitely generated we can find one such k such that,

$$R^n \to M_j \to M_k$$

is 0 on all of S. So we conclude that,



can be induced from



$\square$

Fact 3.1.5. *Given a family $(M_i)_{i\in I}$ of left R-modules where there is a partial order on I and let $L = R^n/S$ where S is a finitely generated R-module (i.e. L is a finitely related R-module) then,*

$$\lim_{\rightarrow} Hom(R^n/S, M_i) \cong Hom(R^n/S, \lim_{\rightarrow} M_i)$$

In a noetherian ring we know that direct limit of injective modules is again injective. We have a similar relationship between the coherent rings and absolutely pure modules.

Theorem 3.1.6. *If R is left noetherian and if $((E_i), (f_{ji}))$ is a directed system of injective R-modules, then $\lim\limits_{\rightarrow} E_i$ is an injective R-module as well.*

Now we state the promised relationship between coherent rings and direct limit of absolutely pure modules. We will need this result to show the dependence between absolute purity of a module and its localization.

Theorem 3.1.7. *(Stenström, [42])R is a coherent ring if and only if the direct limit of absolutely pure R-modules is also absolutely pure.*

Now let's see how one can relate the direct limit to the localization for the following particular case.

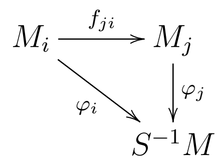
Theorem 3.1.8. *Let $S = \{1, s^2, s^3, ...\}$ be a multiplicative set in R and consider the system of modules where each $M_i = M$ and f_{ji} is multiplying by s^{j-i},*

$$M_0 \xrightarrow{s} M_1 \xrightarrow{s} M_2 \ldots$$

Then the direct limit obtained by the family $((M_i)_{i\in\mathbb{N}}, (f_{ji}))$ *has the following isomorphism,*

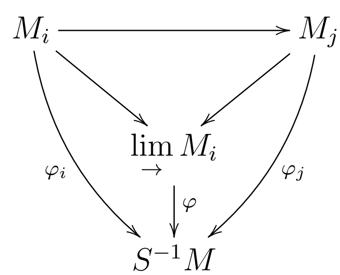
$$\lim_{\rightarrow} M_i \cong S^{-1}M$$

Proof. $f_{ji} : M_i \to M_j$ is defined by $f_{ji}(a) = s^{j-i}a$ now we define φ_i's ,



such that $\varphi_i(a) = \frac{a}{s^i}$ which clearly makes the diagram commutative.

Then by the universal property of the direct limit, there exists φ that makes the following diagram commutative,



So $\varphi([a]) = \frac{a}{s^i}$ for any $a \in M_i$. Then notice that φ is onto by definition.

To prove that φ is one-to-one, let $\varphi([a]) = \varphi([b])$ for some $a, b \in M$. That is $\frac{a}{s^i} = \frac{b}{s^j}$ for some i, j. Then $s^{t+j}a = s^{t+i}b$ for some t. Now we can notice that $[a] = [b]$ since $f_{ki}(a) = f_{kj}(b)$ for $k = j + i + t$. □

We prove the following isomorphism between localizations of a module with respect to different multiplicative sets and we will use this result to prove the upcoming theorem.

Lemma 3.1.9. *If* T *is the multiplicative set generated by* $\{r_1, r_2, ..., r_k\}$ *and* $S \subset T$ *is the multiplicative set generated by the single element* $r_1...r_k$ *then,*

$$S^{-1}M \cong T^{-1}M$$

for all R-modules M.

Proof. Define $S^{-1}M \xrightarrow{\varphi} T^{-1}M$ where φ acts as inclusion. Then clearly it is well-defined & one-to-one.

To show it is onto, given any $\frac{m}{r_1^{t_1}...r_k^{t_k}} \in T^{-1}M$ then if $t = max\{t_1, ..., t_k\}$ we get,

$$\varphi\left(\frac{r_1^{t-t_1}...r_k^{t-t_k}m}{(r_1...r_k)^t}\right) = \frac{m}{r_1^{t_1}...r_k^{t_k}}$$

Hence φ is onto. $\square$

Finally, we prove the relationship between the absolute purity of a module and its localization.

Theorem 3.1.10. *If R is coherent and if $T \subset R$ is any multiplicative set then $T^{-1}A$ is absolutely pure R-module whenever A is absolutely pure R-module.*

Proof. Note that $T \cong \lim\limits_{\rightarrow} S$ where the direct limit is over the finitely generated multiplicative subsets $S \subset T$.(Here the index set is partially ordered by inclusion, i.e. $i \leq j$ if $S_i \subset S_j$.)

Now if A is absolutely pure then by Theorem 3.1.7 $\lim\limits_{\rightarrow} A$ is absolutely pure as well. And by Theorem 3.1.8 and Fact 3.1.9 $\lim\limits_{\rightarrow} A \cong S^{-1}A$ for any finitely generated multiplicative set S. But since,

$$\lim_{\rightarrow}(S^{-1}A) \cong (\lim_{\rightarrow} S)^{-1}A$$

we can generalize the result that $\lim\limits_{\rightarrow} A \cong T^{-1}A$ holds for any multiplicative set T. Hence $T^{-1}A$ is absolutely pure as well. $\square$

With the help of this result, note that if $N \subset A$ is an absolutely pure preenvelope then $S^{-1}N \subset S^{-1}A$ is also an absolutely pure preenvelope in $S^{-1}R$. So an absolutely pure preenvelope of a module automatically gives us an absolutely pure preenvelope of the localization of the module.

3.2 The subfunctor Axt_R for coherent rings

D. Adams proved that the class of absolutely pure modules form a preenveloping in his thesis.

Theorem 3.2.1. *(Adams, [1]) Let R be a commutative ring and Abs be the class of all absolutely pure R-modules. Then Abs is preenveloping.*

Hence by 2.1.5 we conclude that every module has a *Abs*-resolution. With the help of the results given in the previous section, we get the following for Axt, the relative derived functor with respect to *Abs*.

Theorem 3.2.2. *If R is commutative, coherent and if M is finitely presented then,*

$$S^{-1}Axt^n_R(M,N) \cong Axt^n_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

for any module N.

Proof. First we need to prove that $S^{-1}\,\mathrm{H}om_R(M,A) \cong \mathrm{H}om_{S^{-1}R}(S^{-1}M, S^{-1}A)$ for finitely generated module M. Let,

$$F_1 \to F_0 \to M \to 0$$

be an exact sequence where F_1, F_0 are finitely generated free modules which exists since M is finitely presented. Then,

$$0 \to \mathrm{H}om(M,N) \to \mathrm{H}om(F_0,N) \to \mathrm{H}om(F_1,N)$$

is exact and hence we get the following exact sequence,

$$0 \to S^{-1}\,\mathrm{H}om(M,N) \to S^{-1}\,\mathrm{H}om(F_0,N) \to S^{-1}\,\mathrm{H}om(F_1,N)$$

So we get the following commutative diagram,

$$\begin{array}{ccccccc} 0 \longrightarrow & S^{-1}\,\mathrm{H}om(M,N) & \longrightarrow & S^{-1}\,\mathrm{H}om(F_0,N) & \longrightarrow & S^{-1}\,\mathrm{H}om(F_1,N) \\ & \downarrow \star & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & \mathrm{H}om_{S^{-1}R}(S^{-1}M,S^{-1}N) & \longrightarrow & \mathrm{H}om_{S^{-1}R}(S^{-1}F_0,S^{-1}N) & \longrightarrow & \mathrm{H}om_{S^{-1}R}(S^{-1}F_1,S^{-1}N) \end{array}$$

where $\star$ is also an isomorphism since the four-lemma gives us surjection and diagrams being commutative gives us injection.

Now let the following be an absolutely pure resolution of N,

$$0 \to N \to A_0 \to A_1 \to A_2 \to ...$$

Then we get the following isomorphisms,

$$\begin{aligned} S^{-1}Axt^n_R(M,N) &\cong S^{-1}\left(\frac{Ker(\mathrm{H}om(M,A_n) \to \mathrm{H}om(M,A_{n+1}))}{Im(\mathrm{H}om(M,A_{n-1}) \to \mathrm{H}om(M,A_n))}\right) \\ &\cong \frac{Ker(S^{-1}\,\mathrm{H}om(M,A_n) \to S^{-1}\,\mathrm{H}om(M,A_{n+1}))}{Im(S^{-1}(\mathrm{H}om(M,A_{n-1}) \to S^{-1}\,\mathrm{H}om(M,A_n))} \\ &\cong \frac{Ker(\mathrm{H}om_{S^{-1}R}(S^{-1}M,S^{-1}A_n) \to \mathrm{H}om_{S^{-1}R}(S^{-1}M,S^{-1}A_{n+1}))}{Im(\mathrm{H}om_{S^{-1}R}(S^{-1}M,S^{-1}A_{n-1}) \to S^{-1}\,\mathrm{H}om_{S^{-1}R}(S^{-1}M,S^{-1}A_n))} \\ &= Axt^n_{S^{-1}R}(S^{-1}M,S^{-1}N) \end{aligned}$$

Since we get the following absolutely pure resolution for $S^{-1}N$,

$$0 \to S^{-1}N \to S^{-1}A_0 \to S^{-1}A_1 \to S^{-1}A_2 \to ...$$

by Thm 3.1.10 and the fact that if $N \subset A$ is an absolutely pure preenvelope then $S^{-1}N \subset S^{-1}A$ is also an absolutely pure preenvelope in $S^{-1}R$. □

Chapter 4 PHANTOM MORPHISM RELATIVE TO A SUBFUNCTOR

Subfunctors of extensions also appeared in the study of ideals which we will investigate in the next two chapters. Herzog defined a phantom morphism relative to a sunfunctor in [32]. He showed that if $\mathcal{E}$ is a subfunctor of Ext with enough injectives then the class of $\mathcal{E}$ -phantom morphisms gives us a complete ideal cotorsion pair (see Chapter 6 for more details on ideal theory). So in this chapter we would like to focus on this special class of morphisms and give some proporties of them.

4.1 Properties of phantom morphisms

Suppose $\mathcal{C}$ is a preenveloping class which contains all the injective modules. Then the subfunctor corresponding to $\mathcal{C}$ will be denoted by $\mathrm{E}xt^1_{\mathcal{C}}$.

Definition 4.1.1. *Let $Ext^1_{\mathcal{C}}$ be a subfunctor of Ext^1. A morphism $f : M' \to M$ is called a phantom morphism for $Ext^1_{\mathcal{C}}$ (or $Ext^1_{\mathcal{C}}$ -phantom morphism) if the map*

$$Ext^1(f, N) : Ext^1(M, N) \to Ext^1(M', N)$$

has its image in $Ext^1_{\mathcal{C}}(M', N)$ for any module N. i.e Any pullback along f falls into $Ext^1_{\mathcal{C}}(M', N)$.

Definition 4.1.2. *Let $Ext^1_{\mathcal{C}}$ be a subfunctor of Ext^1. A R-module M is called a $Ext^1_{\mathcal{C}}$ -phantom object if id_M is a phantom morphism for $Ext^1_{\mathcal{C}}$.*

Proposition 4.1.3. *Let $\mathcal{C}$ be a preenveloping class that contains all the injective R-modules. Then M is a phantom object relative to $Ext^1_{\mathcal{C}}$ if and only if $Ext^1(M, C) = 0$ for any $C \in \mathcal{C}$.*

Proof. Assume that M is a phantom object relative to $\mathrm{E}xt^1_{\mathcal{C}}$. Then id_M is a phantom morphism relative to $\mathrm{E}xt^1_{\mathcal{C}}$. But this implies $\mathrm{E}xt^1(M, \_) = \mathrm{E}xt^1_{\mathcal{C}}(M, \_)$. Now notice that for any $C \in \mathcal{C}$, $\mathrm{E}xt^1_{\mathcal{C}}(M, C) = 0$ since, if the following short exact sequence is in $\mathrm{E}xt^1_{\mathcal{C}}(M, C)$

$$0 \longrightarrow A \longrightarrow U \longrightarrow M \longrightarrow 0$$

then we know there exist an extension making the following diagram commutative,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & \swarrow & & & & & \\ & & C & & & & & & \end{array}$$

Hence the sequence is split exact which gives us $\mathrm{E}xt^1_{\mathcal{C}}(M, C) = \mathrm{E}xt^1(M, C) = 0$.

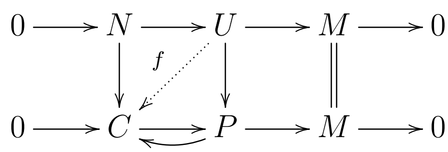
To prove the other way around, assume that $\mathrm{E}xt^1(M,C) = 0$ for any $C \in \mathcal{C}$. Given a short exact sequence in $Ext^1(M,N)$,

$$0 \longrightarrow N \longrightarrow U \longrightarrow M \longrightarrow 0$$

then for any morphism $N \to C$ where $C \in \mathcal{C}$ we get the following pushout,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & C & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Since $\mathrm{E}xt^1(M,C) = 0$ by assumption, the lower sequence is split exact. So we can define f as the composition of $U \to P \to C$,



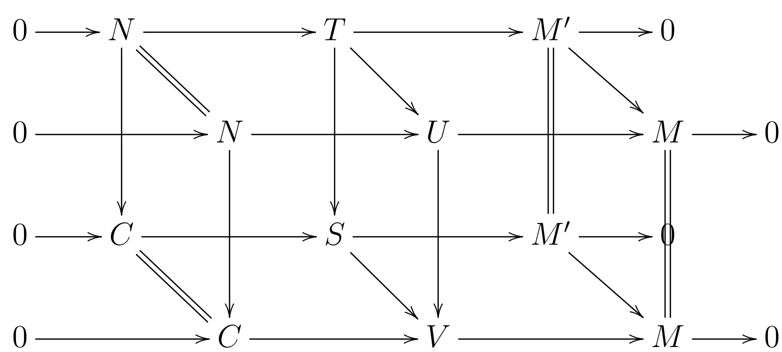
Then $0 \to N \to U \to M \to 0$ is in $\mathrm{E}xt^1_{\mathcal{C}}\ (M,N)$. We conclude that $\mathrm{E}xt^1(M,\_) = \mathrm{E}xt^1_{\mathcal{C}}\ (M,\_)$, i.e M is a phantom object relative to $\mathrm{E}xt^1_{\mathcal{C}}$. □

We find a sufficient and necessary condition for a morphism to be phantom for $\mathrm{E}xt^1_{\mathcal{C}}$.

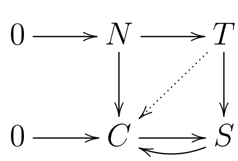
Proposition 4.1.4. *$f : M' \to M$ is a phantom morphism for $Ext^1_{\mathcal{C}}$ if and only if $Ext^1(f,C) = 0$ for any $C \in \mathcal{C}$.*

Proof. Assume that $\mathrm{E}xt^1(f,C) = 0$ for all $C \in \mathcal{C}$.

We want to show that $\mathrm{E}xt^1(f,N) : \mathrm{E}xt^1(M,N) \to \mathrm{E}xt^1(M',N)$ has its image in $\mathrm{E}xt^1_{\mathcal{C}}\ (M',N)$. Given R-modules and a morphism $N \to C$ where $C \in \mathcal{C}$, by pushout and pullbacks we get the following commutative diagram,



By assumption the far bottom row is split exact, so we get



Then we get the dotted map which proves that far top row in the previous diagram is in $\mathrm{E}xt^1_{\mathcal{C}}(M', N)$.

Assume that $f : M' \to M$ is a phantom morphism for $\mathrm{E}xt^1_{\mathcal{C}}$. Then for any module $C \in \mathcal{C}$ we have,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & T & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & \swarrow & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & C & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

since top row falls in $\mathrm{E}xt^1_{\mathcal{C}}(M', C)$ we have the dotted map. But then we see that the top row is split exact hence $\mathrm{E}xt^1(f, C) = 0$. □

We now turn our attention to the complete cotorsion pair $(\mathcal{F}, \mathcal{C})$ and see how a phantom morphism relative to $\mathrm{E}xt^1_{\mathcal{C}}$ can be related to the class $\mathcal{F}$. Recall that a pair of classes is cotorsion if $\mathcal{F}^{\perp} = \mathcal{C}$, $\mathcal{C}^{\perp} = \mathcal{F}$ and they have enough projectives and injectives respectively.

Theorem 4.1.5. *Suppose $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair. Then $f : M' \to M$ is a phantom morphism for $Ext^1_{\mathcal{C}}$ if and only if f can be factored through a R-module $F \in \mathcal{F}$.*

Proof. Given a morphism $f : M' \to M$, assume f factors through F for some $F \in \mathcal{F}$. For any $C \in \mathcal{C}$, we get the following pullbacks,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & T & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & S & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & U & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Since $F \in \mathcal{F} = ^{\perp}\mathcal{C}$ the middle row is split exact, hence also the top row. That is $\mathrm{E}xt^1(f, C) = 0$, i.e. f is phantom for $\mathrm{E}xt^1_{\mathcal{C}}$.

Now assume that $f : M' \to M$ is phantom for $\mathrm{E}xt^1_{\mathcal{C}}$. Since $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair there exists an exact sequence,

$$0 \longrightarrow C \longrightarrow F \longrightarrow M \longrightarrow 0$$

where $C \in \mathcal{C}$ and $F \in \mathcal{F}$ then looking at the pullback along f,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & U & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & C & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

notice that the top row is split exact since $\mathrm{E}xt^1(f, C) = 0$. But then f factors through F by the composition of the maps in the following diagram,

$$\begin{array}{ccc} U & \rightleftarrows & M' \\ \downarrow & & \downarrow f \\ F & \longrightarrow & M \end{array}$$

□

With the above characterization we now prove that phantom morphisms are in some sense closed under direct sums.

Corollary 4.1.6. *Suppose $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair.If $f_i : M'_i \to M_i$, $i \in I$, is phantom morphism for $Ext^1_{\mathcal{C}}$ then so is $\bigoplus_i f_i : \bigoplus_i M'_i \to \bigoplus_i M_i$.*

Proof. Since $f_i : M'_i \to M_i$ is $\mathrm{E}xt^1_{\mathcal{C}}$ -phantom for each $i \in I$, we conclude that they factor through some $P_i \in \mathcal{F}$,

$$M'_i \longrightarrow P_i \longrightarrow M_i$$

which gives us that $\bigoplus_i f_i$ factors through $\bigoplus_i P_i$ as following diagram suggests,

$$\bigoplus_i M'_i \longrightarrow \bigoplus_i P_i \longrightarrow \bigoplus_i M_i$$

Then we need to prove that $\bigoplus_i P_i \in \mathcal{F}$. Since ${}^{\perp}\mathcal{C} = \mathcal{F}$ it is enough to prove that if $\mathrm{E}xt^1(P_i, A) = 0$ for any $A \in \mathcal{C}$ then $\mathrm{E}xt^1(\bigoplus_i P_i, A) = 0$ for any $A \in \mathcal{C}$ as well.

Given projective resolutions for each P_i,

$$\ldots \longrightarrow Q_i^2 \longrightarrow Q_i^1 \longrightarrow Q_i^0 \longrightarrow P_i \longrightarrow 0$$

we get a projective resolution for $\bigoplus_i P_i$,

$$\ldots \longrightarrow \bigoplus_i Q_i^2 \longrightarrow \bigoplus_i Q_i^1 \longrightarrow \bigoplus_i Q_i^0 \longrightarrow \bigoplus_i P_i \longrightarrow 0$$

Now we notice that for any $A \in \mathcal{C}$, if

$$\mathrm{E}xt^1(P_i, A) = \frac{Ker(Hom(Q_i^1, A) \to Hom(Q_i^2, A))}{Im(Hom(Q_i^0, A) \to Hom(Q_i^1, A)))} = 0$$

then

$$\mathrm{E}xt^1(\bigoplus P_i, A) = \frac{Ker(Hom(\bigoplus Q_i^1, A) \to Hom(\bigoplus Q_i^2, A))}{Im(Hom(\bigoplus Q_i^0, A) \to Hom(\bigoplus Q_i^1, A)))} = 0$$

We can now conclude that $\bigoplus_i P_i \in \mathcal{F}$. Hence we get the result. □

4.2 Phantom morphisms relative to Axt

In this section we will denote the class of all absolutely pure modules by *Abs* and the representative set of all finitely presented modules by $\mathcal{P}$

Proposition 4.2.1. *[42]* $Abs = \mathcal{P}^{\perp}$.

Proof. $(Abs \supseteq \mathcal{P}^{\perp})$: Given $M \in \mathcal{P}^{\perp}$ then for any morphism $S \to M$ where $S \subset R^n$ a finitely generated submodule then we get the following diagram by the pushout,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & R^n & \longrightarrow & R^n/S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \rightleftarrows & U & \longrightarrow & R^n/S & \longrightarrow & 0 \end{array}$$

where the bottom row is split exact by assumption since $M \in \mathcal{P}^{\perp}$. Hence we conclude that $M \in Abs$.

$(Abs \subseteq \mathcal{P}^{\perp})$: Given $A \in Abs$, then for any finitely generated $S \subset R^n$ we get the short exact sequence,

$$0 \longrightarrow S \hookrightarrow R^n \longrightarrow R^n/S \longrightarrow 0$$

which yields to the long exact sequence and we get,

$$\mathrm{H}om(R^n, A) \longrightarrow \mathrm{H}om(S, A) \longrightarrow \mathrm{E}xt^1(R^n/S, A) \longrightarrow 0$$

But since A is absolutely pure module, $\mathrm{H}om(R^n, A) \to \mathrm{H}om(S, A)$ is onto and since the sequence is exact $\mathrm{H}om(S, A) \to \mathrm{E}xt^1(R^n/S, A)$ is onto as well. So we conclude that $\mathrm{E}xt^1(R^n/S, A) = 0$, that is $A \in \mathcal{P}^{\perp}$. □

Proposition 4.2.2. $({}^{\perp}Abs, Abs)$ *is a complete cotorsion pair.*

Proof. By Proposition 4.2.1, $Abs = \mathcal{P}^{\perp}$ hence we conclude that $Abs = ({}^{\perp}Abs)^{\perp}$ since $\mathcal{P}^{\perp} = ({}^{\perp}(\mathcal{P}^{\perp}))^{\perp}$. Moreover, due to Eklof and Trlifaj the pair $({}^{\perp}Abs, Abs)$ is complete since it is cogenerated by a set of modules $\mathcal{P}$. □p

Corollary 4.2.3. *$f : M' \to M$ is phantom for Axt^1 if and only if f can be factored through a module $P \in^{\perp} Abs$.*

Proof. By Proposition 4.2.2 $({}^{\perp}Abs, Abs)$ is a complete cotorsion pair, then by Theorem 4.1.5 we can conclude the result. □

Corollary 4.2.4. *If $f : M' \to M$ is phantom for Axt^1_R then so is $S^{-1}f : S^{-1}M' \to S^{-1}M$ for $Axt^1_{S^{-1}R}$.*

Proof. By corollary 4.2.3 we conclude that f factors through $M' \to P \to M$ for some $P \in^{\perp} Ab$. We want to show that $S^{-1}f : S^{-1}M' \to S^{-1}M$ factors as well. But notice that we have the following factorization,

$$S^{-1}M' \to S^{-1}P \to S^{-1}M$$

Now since $R \to S^{-1}R$ is a flat map we have the following isomorphism

$$\mathrm{E}xt^1_{S^{-1}R}(P \otimes S^{-1}R, A) \cong \mathrm{E}xt^1_{S^{-1}R}(S^{-1}P, A) \cong \mathrm{E}xt^1_R(P, A)$$

for any absolutely $S^{-1}R$-module A. But then A is also an absolutely R-module. Hence $\mathrm{E}xt^1_R(P, A) = 0$. We conclude that $S^{-1}P \in {}^{\perp}Abs$. Then by Corollary 4.2.3 we can conclude that $S^{-1}f$ is phantom for $Axt^1_{S^{-1}R}$. □

Corollary 4.2.5. *If $f_i : M_i' \to M_i$, $i \in I$, is phantom for Axt^1_R then so is $\bigoplus_i f_i : \bigoplus_i M_i' \to \bigoplus_i M_i$.*

Proof. It follows by the corollary 4.1.6, since $({}^{\perp}Abs, Abs)$ is a complete cotorsion pair. □

Chapter 5 IDEALS GENERATED BY A SET

Throughout this chapter we will focus on the ideals which are generated by a set of homomorphisms. First we examine how the elements of such ideals is characterized in Remark 5.2.2. This helps us to find sufficient conditions for a morphism to be in $\mathcal{I}^{\perp}$ (Lemma 5.2.3). We see that every homomorphism g in $\mathcal{I}^{\perp}$ has a small enough factorization (Lemma 5.2.6) which is motivated by Proposition(5.2.2) from Enochs and Jenda's pivotal book Relative Homological Algebra [22]. With all the tools in hand we prove that if an ideal $\mathcal{I}$ is generated by a set then $\mathcal{I}^{\perp}$ is a preenveloping class (Theorem 5.3.1).

We then give the definition for an ideal to be closed under sums and observe that being closed under sums is sufficient for an ideal to be precovering if it is generated by a set of morphisms (Theorem 5.4.3).

Finally we revise the definition of being generated by a set of homomorphisms. We see that if we allow infinite direct sums in the factorization of elements of an ideal, that luckily the results we have still hold.

5.1 Preliminaries

Following the lead of [24], we call an additive subfunctor of the bifunctor Hom : R-M$od^{op} \times$ R-M$od \to$ Ab an ideal $\mathcal{I}$ of R-Mod. This means that given R-homomorphisms f, g, h, t with $f, g \in \mathcal{I}$, then $f + g \in \mathcal{I}$ and $h \circ f \circ t \in \mathcal{I}$, whenever they are defined. The next definition is the natural extension of the usual notions of (pre)cover and (pre)envelope to ideals of morphisms (for a detailed exposition see [24]).

Definition 5.1.1. *Let $\mathcal{I} \subseteq Hom_R$ be an ideal in R-Mod and $M \in$ R-Mod. An $\mathcal{I}$-precover of M is a morphism $i : I \to M$ in $\mathcal{I}$ such that any morphism $i' : I' \to M$ in $\mathcal{I}$ factors through i. I.e., the following triangle can be completed to a commutative one,*

$$\begin{array}{ccc} I' & & \\ \vdots & \searrow^{i'} & \\ I & \xrightarrow{\ i\ } & M. \end{array}$$

An $\mathcal{I}$-precover $i : I \to M$ is said to be an $\mathcal{I}$-cover if every map j that completes the diagram

$$\begin{array}{ccc} I & & \\ {\scriptstyle j}\downarrow & \searrow^{i} & \\ I & \xrightarrow{\ i\ } & M \end{array}$$

is necessarily an automorphism. An ideal $\mathcal{I}$ is said to be (pre)covering if every R-module M admits an $\mathcal{I}$-(pre)cover.

$\mathcal{I}$-(pre)envelopes and (pre)enveloping ideals are defined dually.

Definition 5.1.2. *Given two ideals $\mathcal{I}$, $\mathcal{J} \subseteq Hom_R$ of R-modules define,*

$$\mathcal{I}^{\perp} = \{j |\, Ext^1(i,j) = 0 \text{ for all } i \in \mathcal{I}\}$$
$$^{\perp}\mathcal{J} = \{i |\, Ext^1(i,j) = 0 \text{ for all } j \in \mathcal{J}\}$$

An ideal cotorsion pair *in R-Mod is a pair $(\mathcal{I}, \mathcal{J})$ of ideals such that $\mathcal{I}^{\perp} = \mathcal{J}$ and $^{\perp}\mathcal{J} = \mathcal{I}$.*

5.2 Properties of an ideal generated by a set

In this section we observe how the elements of an ideal generated by a set can be factored through a certain kind of homomorphism. This observation helps us to identify the elements of $\mathcal{I}$ as well as of $\mathcal{I}^{\perp}$. We finish this section by proving Lemma 5.2.6 which will be the main tool to study when $\mathcal{I}^{\perp}$ is preenveloping.

Remark 5.2.1. *Let $\mathcal{I} = < f >$ where $f : M \to N$. Then $\varphi : U \to V$ is in I if and only if it has a factorization of the form,*

$$U \longrightarrow M^m \xrightarrow{f_{ji}} N^n \longrightarrow V$$

for some $1 \leq m, n$ where f_{ji} has entries either equal to f or 0.

Proof. Let $S = \{\varphi \,|\, \varphi \text{ has the desired factorization property}\}$. Clearly $f \in \mathcal{I}$ and $S \subseteq \mathcal{I}$, so it is enough to prove that S is an ideal. Let $g, g' \in S$ where,

$$g : U \xrightarrow{\alpha} M^m \xrightarrow{g_{ji}} N^n \xrightarrow{\beta} V$$

and

$$g' : U \xrightarrow{\tilde{\alpha}} M^{\tilde{m}} \xrightarrow{g'_{ji}} N^{\tilde{n}} \xrightarrow{\tilde{\beta}} V$$

then $g + g'$ has the following factorization,

$$U \xrightarrow{(\alpha,\tilde{\alpha})} M^{m+\tilde{m}} \xrightarrow{h} N^{n+\tilde{n}} \xrightarrow{(\beta,\tilde{\beta})} V$$

where

$$(\alpha, \tilde{\alpha})(u) = (\alpha(u), \tilde{\alpha}(u)),$$
$$(\beta, \tilde{\beta})(n, \tilde{n}) = \beta(n) + \tilde{\beta}(n)$$

and

$$h = \left[\begin{array}{c:c} (g_{ji}) & 0 \\ \hdashline 0 & (g'_{ji}) \end{array}\right]$$

for each $1 \leq j \leq n + \tilde{n}$ and $1 \leq i \leq m + \tilde{m}$. Given $u \in U$ say $\alpha(u) = (x_1, ..., x_m)$ and $\tilde{\alpha}(u) = (y_1, ..., y_{\tilde{m}})$ then,

$$u \xmapsto{(\alpha,\tilde{\alpha})} (\alpha(u), \tilde{\alpha(u)}) \xrightarrow{h} (g_{ji}(x_j), g'_{ji}(y_j)) \xrightarrow{(\beta,\tilde{\beta})} \beta(g_{ji}(x_j)) + \tilde{\beta}(g'_{ji}(y_j))$$

where we notice that,

$$g(u) + g'(u) = \beta(g_{ji}(x_j)) + \tilde{\beta}(g'_{ji}(y_j))$$

Hence we conclude that we have a factorization of $g + g'$. ☐

Remark 5.2.2. *Let $\mathcal{I} =< f^k >_{k\in K}$ be generated by a set of homomorphisms where $f^k : M_k \to N_k$. Then $\varphi : U \to V$ is in $\mathcal{I}$ if and only if it has the following factorization,*

$$U \longrightarrow M_{k_1}^{m_1} \oplus ... \oplus M_{k_t}^{m_t} \xrightarrow{(h_{ji})} N_{k_1}^{n_1} \times ... \times N_{k_t}^{n_t} \longrightarrow V$$

where $k_1, ..., k_t \in K$ and

$h_{ji} = f^{k_1}$ *or* 0 *for* $1 \leq j \leq n_1, 1 \leq i \leq m_1,$

.

.

.

$h_{ji} = f^{k_t}$ *or* 0 *for* $(n_1 + ... + n_{t-1}) \leq j \leq (n_1 + ... + n_t)$, $(m_1 + ... + m_{t-1}) \leq i \leq (m_1 + ... + m_t)$

and h_{ji} can be viewed as a matrix with entries,

$$\left[\begin{array}{c|ccc|c} f_{ji}^{k_1} & 0 & . & 0 & 0 \\ \hline 0 & . & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & . & 0 \\ \hline 0 & 0 & . & 0 & f_{ji}^{k_t} \end{array}\right]$$

Proof. Similar to the case where $\mathcal{I}$ is generated by a single homomorphism.

With the help of the above characterization, we find the following equivalent conditions for a morphism g to be in $\mathcal{I}^{\perp}$. So one can use either of these equivalent conditions to detect the elements of $\mathcal{I}^{\perp}$.

Lemma 5.2.3. *Let $\mathcal{I} =< f_k \,|\, k \in K >$ and $f_k : M_k \to N_k$, $g : U \to V$ be homomorphisms of R-modules then the following are equivalent,*

1. *g is in $\mathcal{I}^{\perp}$.*

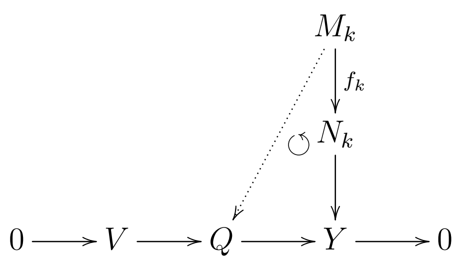
2. *Given any s.e.s.,*

$$0 \longrightarrow U \longrightarrow X \longrightarrow Y \longrightarrow 0$$

   the s.e.s. obtained by taking the pushout along g,

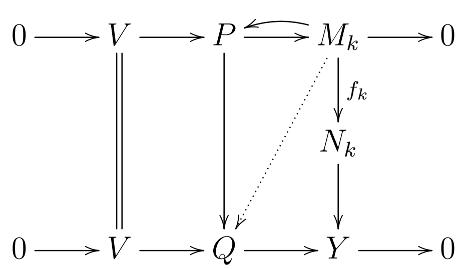
$$\begin{CD} 0 @>>> U @>>> X @>>> Y @>>> 0 \\ @. @VgVV @VVV @| @. \\ 0 @>>> V @>>> Q @>>> Y @>>> 0 \end{CD}$$

   can be completed to a commutative diagram for any $N_k \to Y$ where $k \in K$ as shown below,



3. *Let $0 \to V \to Q \to Q/V \to 0$ be the s.e.s. obtained by taking the pushout along g. Then for any $k \in K$ the composition Hom$(N_k, Q/V) \to$ Hom$(M_k, Q/V) \to$ Ext$^1(M_k, V)$ (obtained from $M_k \xrightarrow{f_k} N_k \longrightarrow Q/V$) is the zero map for any homomorphism $N_k \to Q/V$.*

Proof. ($1 \Rightarrow 2$) Assume $g \in \mathcal{I}^{\perp}$. Given a s.e.s. $\xi : 0 \longrightarrow U \longrightarrow X \longrightarrow Y \longrightarrow 0$ then E$xt^1(\tilde{f}, g)(\xi) = 0$ for any $\tilde{f} \in \mathcal{I}$. That is the resulting s.e.s we get by computing the pushout along g followed by the pullback along $\tilde{f}$ is split exact. Since for any $N_k \to Y$ the composition $M_k \to N_k \to Y$ is in $\mathcal{I}$, in particular the upper row in the following diagram,



is split exact. Hence we obtain the desired commutative triangle as shown above.

($2 \Rightarrow 1$) Assume that the second property holds for $g : U \to V$. We need to prove that g is in $\mathcal{I}^{\perp}$, that is E$xt^1(f, g)(\xi) = 0$ for any $f \in \mathcal{I}$ where ξ is any extension of codomain of f by U. One can easily observe that it is enough to show instead $\mathrm{E}xt^1(\tilde{f}, g)(\xi) = 0$ for $\tilde{f} \in \mathcal{I}$ where,

$$\tilde{f}: \; M_j \oplus M_i \xrightarrow{f} N_j \oplus N_i \longrightarrow Y$$

and where ξ is any s.e.s. $\xi : 0 \to U \to X \to Y \to 0$. By Remark 5.2.2, $f = \begin{pmatrix} f_j & 0 \\ 0 & f_i \end{pmatrix}$ where f_j, f_i are from the set of generators of $\mathcal{I}$. Given any $\varphi : N_j \oplus N_i \to Y$, we define $\varphi_j : N_j \to Y$ and $\varphi_i : N_i \to Y$ such that φ_j is the restriction of φ to $N_j \oplus 0$ and similarly φ_i is the restriction of φ to $0 \oplus N_i$. Then by assumption there exists α_j and α_i making the following diagrams commutative,

$$\begin{array}{ccccccccc} & & & & & & M_j & & \\ & & & & & \overset{\alpha_j}{\swarrow} & \big\downarrow{\scriptstyle f_j} & & \\ & & & & & \circlearrowleft & N_j & & \\ & & & & & & \big\downarrow{\scriptstyle \varphi_j} & & \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} & & & & & & M_i & & \\ & & & & & \overset{\alpha_i}{\swarrow} & \big\downarrow{\scriptstyle f_i} & & \\ & & & & & \circlearrowleft & N_i & & \\ & & & & & & \big\downarrow{\scriptstyle \varphi_i} & & \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

Then it is easy to see that the map $\alpha : M_j \oplus M_i \to Q$ defined as $\alpha(x_j, x_i) = \alpha_j(x_j) + \alpha_i(x_i)$ makes the following diagram commutative,

$$\begin{array}{ccccccccc} & & & & & & M_j \oplus M_i & & \\ & & & & & \overset{\alpha}{\swarrow} & \big\downarrow{\scriptstyle f} & & \\ & & & & & & N_j \oplus N_i & & \\ & & & & & & \big\downarrow{\scriptstyle \varphi} & & \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

If we compute the pullback along $\tilde{f}$ we get the following diagram,

$$\begin{array}{ccc} P & \longrightarrow & M_j \oplus M_i \\ \big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ Q & \longrightarrow & Y \end{array}$$

Now using the commutativity of the previous diagram, we get the following commutative diagram,

$$\begin{array}{ccccc} M_j \oplus M_i & & & & \\ & \overset{\psi}{\searrow} & & \overset{id}{\searrow} & \\ {\scriptstyle\alpha}\downarrow & & P & \longrightarrow & M_j \oplus M_i \\ & & \downarrow & & \downarrow \\ & \searrow & Q & \longrightarrow & Y \end{array}$$

So by the universal property of pullback diagrams we conclude that there exists a homomorphism ψ such that $M_j \oplus M_i \xrightarrow{\psi} P \longrightarrow M_j \oplus M_i$ is the identity homomorphism. Hence looking at the s.e.s. obtained by the pullback along $\tilde{f}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & P & \longrightarrow & M_j \oplus M_i & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \\ 0 & \longrightarrow & U & \longrightarrow & Q & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

we conclude that the upper row is split exact. That is $\mathrm{E}xt^1(\tilde{f}, g)$ maps ξ to a split exact sequence, i.e. $\mathrm{E}xt^1(\tilde{f}, g)(\xi) = 0$ for any s.e.s. ξ.

($2 \Leftrightarrow 3$) Assume that the second property holds. Given any s.e.s. $\xi : 0 \to U \to X \to Y \to 0$ by using the pushout along g we get,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow & & \| & & \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & Y \cong Q/V & \longrightarrow & 0 \end{array}$$

by assumption the lower row can be completed to a commutative diagram for any $N_k \to Q/V$ as shown below,

$$\begin{array}{ccccccccc} & & & & & & M_k & & \\ & & & & & \swarrow & \downarrow{\scriptstyle f_k} & & \\ & & & & & & N_k & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & Q/V & \longrightarrow & 0 \end{array}$$

So we get the diagram,

$$\begin{array}{ccccc} & & \mathrm{H}om(N_k, Q/V) & & \\ & & \downarrow & & \\ \mathrm{H}om(M_k, Q) & \longrightarrow & \mathrm{H}om(M_k, Q/V) & \longrightarrow & \mathrm{E}xt^1(M_k, V) \end{array}$$

with an exact row. But now our assumption holds if and only if the following composition,

$$\mathrm{H}om(N_k, Q/V) \longrightarrow \mathrm{H}om(M_k, Q/V) \longrightarrow \mathrm{E}xt^1(M_k, V)$$

is the zero map. $\square$

Corollary 5.2.4. *Let $\mathcal{I}$ be as in Lemma 5.2.3 and $g : U \to V$ be in $\mathcal{I}^\perp$. If $V' \subseteq V$ is a submodule such that $g(U) \subseteq V' \subseteq V$ and the map $Ext^1(M_k, V') \to Ext^1(M_k, V)$ is an injection for any $k \in K$ then $g : U \to V'$ is in $\mathcal{I}^\perp$ as well.*

Proof. Notice that by Lemma 5.2.3, $g : U \to V' \subseteq V$ is in $\mathcal{I}^\perp$ if and only if the following composition is 0 for any given $N_k \to Q'/V'$ and any given $k \in K$,

$$\mathrm{H}om(N_k, Q'/V') \longrightarrow \mathrm{H}om(M_k, Q'/V') \longrightarrow \mathrm{E}xt^1(M_k, V')$$

which is induced from the following s.e.s.,

$$0 \to V' \to Q' \to Q'/V' \to 0$$

Then the following diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & E(U) & \longrightarrow & E(U)/U & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & V' & \longrightarrow & Q' & \longrightarrow & Q'/V' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & V & \longrightarrow & Q & \longrightarrow & Q/V & \longrightarrow & 0 \end{array}$$

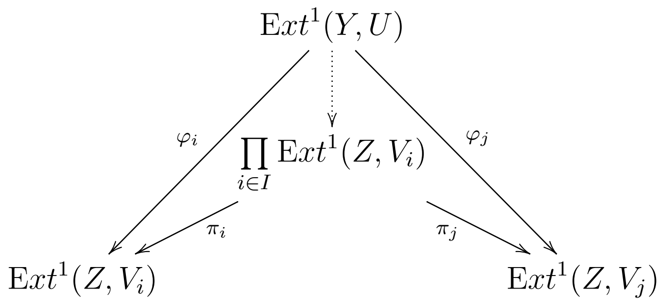
where $E(U)$ is the injective envelope of U, gives us,

$$\begin{array}{ccccc} \mathrm{H}om(N_k, Q'/V') & \longrightarrow & \mathrm{H}om(M_k, Q'/V') & \longrightarrow & \mathrm{E}xt^1(M_k, V') \\ \downarrow \cong & & \downarrow \cong & & \downarrow \\ \mathrm{H}om(N_k, Q/V) & \longrightarrow & \mathrm{H}om(M_k, Q/V) & \longrightarrow & \mathrm{E}xt^1(M_k, V) \end{array}$$

We notice that if $\mathrm{E}xt^1(M_k, V') \to \mathrm{E}xt^1(M_k, V)$ is an injection for every $k \in K$ then the composition on the top row is 0 for every $k \in K$. By Lemma 5.2.3 we conclude that $g : U \to V'$ is in $\mathcal{I}^\perp$. $\square$

Lemma 5.2.5. *Let $\mathcal{I}$ be as in Lemma 5.2.3. If each $g_i : U \to V_i$, $i \in I$ is in $\mathcal{I}^\perp$ then $g : U \to \prod_{i \in I} V_i$ where $g(x) = (g_i(x))_{i \in I}$ is in $\mathcal{I}^\perp$.*

Proof. Assume each $g_i : U \to V_i$, $i \in I$ is in $\mathcal{I}^{\perp}$. That is $\varphi_i = \mathrm{E}xt^1(\tilde{f}, g_i)$ is the zero map for any $\tilde{f} : Z \to Y$ in $\mathcal{I}$. Since $\mathrm{E}xt^1(Z, \prod_{i\in I} V_i) \cong \prod_{i\in I} \mathrm{E}xt^1(Z, V_i)$ we have the following diagram,



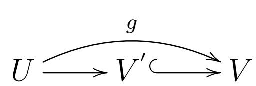
So there exists $\varphi : \mathrm{E}xt^1(Y, U) \to \prod_{i\in I} \mathrm{E}xt^1(Z, V_i)$ which makes the above diagram commutative. Now given any $\eta \in \mathrm{E}xt^1(Y, U)$ and say $\varphi(\eta) = (\xi_i)_{i\in I}$ then,

$$0 = \varphi_j(\eta) = \pi_j(\varphi(\eta)) = \pi_j((\xi_i)_{i\in I}) = \xi_j$$

That is, $\eta = 0$. Hence φ or $\mathrm{E}xt(\tilde{f}, g)$ is the zero map and $g \in \mathcal{I}^{\perp}$. □

Recall that our main goal is to prove that $\mathcal{I}^{\perp}$ is preenveloping when $\mathcal{I}$ is generated by a set. In order to prove that we need to show every $g \in \mathcal{I}^{\perp}$ has a "small" enough factorization. In the following lemma we show how one gets such a factorization.

Lemma 5.2.6. *Let $\mathcal{I}$ be as in Lemma 5.2.3 and $g : U \to V$ in $\mathcal{I}^{\perp}$. Then g can be factored through V' such that,*

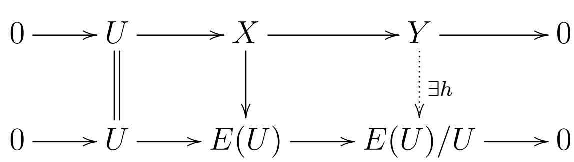


where the cardinality of V' is bounded by a cardinal number κ which depends only on $|U|$ and $\mathcal{I}$.

Proof. First we need to show that g is in $\mathcal{I}^{\perp}$ if and only if $\mathrm{E}xt(\tilde{f}, g)(\xi') = 0$ for the short exact sequence $\xi' : 0 \to U \to E(U) \to E(U)/U \to 0$ where $E(U)$ is the injective envelope of U. One direction is obvious. To show the other direction let,

$$\xi : 0 \to U \to X \to Y \to 0$$

be any short exact sequence. Since $E(U)$ is injective we get the following commutative diagram,



where $h : Y \to E(U)/U$ is induced from $X \to E(U)$. Now we look at $\mathrm{E}xt^1(h, id_U)(\xi^{'})$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow h & & \\
0 & \longrightarrow & U & \longrightarrow & E(U) & \longrightarrow & E(U)/U & \longrightarrow & 0 \\
 & & \downarrow id_U & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & E(U) & \longrightarrow & E(U)/U & \longrightarrow & 0
\end{array}$$

That is $\mathrm{E}xt^1(h, id_U)(\xi^{'}) = \xi$. Note that the upper row is a pullback along h since $X \to Y$ and $E(U) \to E(U)/U$ are epimorphisms and $U \xrightarrow{id_U} U$ is an isomorphism. Then,

$$\mathrm{E}xt^1(\tilde{f}, g)(\xi) = \mathrm{E}xt^1(\tilde{f}, g) \circ \mathrm{E}xt^1(h, id_U)(\xi^{'}) = \mathrm{E}xt^1(h \circ \tilde{f}, g)(\xi^{'}) = 0$$

Hence $g \in \mathcal{I}^{\perp}$.
Now we want to construct a "small" enough $V^{'}$ such that $g : U \to V^{'} \subset V$ is in $\mathcal{I}^{\perp}$. By Corollary 5.2.4, it is enough to show that $\mathrm{E}xt^1(M_k, V^{'}) \to \mathrm{E}xt^1(M_k, V)$ is an injection for any $k \in K$. But notice that this holds if the following map is an injection,

$$\prod_{k \in K} \mathrm{E}xt^1(M_k, V^{'}) \to \prod_{k \in K} \mathrm{E}xt^1(M_k, V)$$

But we have the following commutative diagram,

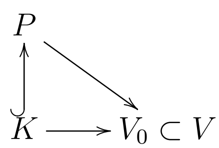
$$\begin{array}{ccc}
\mathrm{E}xt^1(\bigoplus_{k\in K} M_k, V^{'}) & \xrightarrow{\cong} & \prod_{k\in K} \mathrm{E}xt^1(M_k, V^{'}) \\
\downarrow & & \downarrow \\
\mathrm{E}xt^1(\bigoplus_{k\in K} M_k, V) & \xrightarrow{\cong} & \prod_{k\in K} \mathrm{E}xt^1(M_k, V)
\end{array}$$

So we conclude that in order to show $g : U \to V^{'}$ in $\mathcal{I}^{\perp}$, it is enough to show the left column is injective.

Now we will construct the desired $V^{'}$. Given an $i \in K$, let $0 \to K_i \to P_i \to M_i \to 0$ be a partial projective resolution of M_i. Then we obtain the following partial projective resolution for $\bigoplus_{i \in K} M_i$,

$$0 \longrightarrow K = \bigoplus_{i\in K} K_i \longrightarrow P = \bigoplus_{i\in K} P_i \longrightarrow \bigoplus_{i\in K} M_i \longrightarrow 0$$

We will construct an ascending chain of modules to obtain such a "small" $V^{'}$. Let $g(U) = V_0$ and for every $K \to V_0$ morphism that has an extension $P \to V$, choose one such extension $\alpha : P \to V$. So we have the following diagram,



Define $V_1 = \sum \alpha(P)$ where the sum is over all such chosen extensions $\alpha : P \to V$ for each $K \to V_0$. Then $V_0 \subset V_1$. Now we construct V_2 in a similar way. So we get an ascending chain of modules,

$$V_0 \subset V_1 \subset ... \subset V_\omega \subset V_{\omega+1} \subset ... \subset V_\beta$$

where β is the least cardinal number with $|K| < \beta$. Define $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ if $\lambda \leq \beta$ is a limit ordinal and let $V^{'} = V_\beta$.

Now we will show that $g : U \to V^{'}$ is in $\mathcal{I}^\perp$. By the previous observation all we need to show is that the homomorphism,

$$\mathrm{E}xt^1(\bigoplus_{k\in K} M_k, V^{'}) \to \mathrm{E}xt^1(\bigoplus_{k\in K} M_k, V)$$

is injective. Given a morphism $K \xrightarrow{\varphi} V^{'} \subset V$ that has an extension $P \xrightarrow{\Phi} V$, we want to show that then there is an extension $P \to V^{'}$. But now since $|K| < \beta$ we conclude that $Im(\varphi) \subseteq V_\alpha$ for some α such that $|\alpha| < \beta$; hence, by the construction of the ascending chain we can extend φ to $P \to V_{\alpha+1} \subset V^{'}$. This shows that

$$\mathrm{E}xt^1(\bigoplus_{k\in K} M_k, V^{'}) \to \mathrm{E}xt^1(\bigoplus_{k\in K} M_k, V)$$

is an injection. Now by Corollary 5.2.4 we conclude that $g : U \to V^{'} \subset V$ is in $\mathcal{I}^\perp$. Moreover, note that the cardinality of $V_{\alpha+1} = \sum \Phi(P)$ (where the sum is taken over all chosen extensions $\Phi : P \to V$ for each $K \to V_\alpha$) is bounded by,

$$|V_{\alpha+1}| \leq |P|^{|V_\alpha|^{|K|}}$$

Since,

$$|\,\mathrm{H}om(K, V_\alpha)| \leq |V_\alpha^K| = |V_\alpha|^K$$

and

$$|\Phi(P)| \leq |P|$$

We find a bound on $|V^{'}| \leq \sum_{\alpha<\beta} |V_\alpha|$. So we conclude that for any given U, if $g : U \to V$ is in $\mathcal{I}^\perp$ we can find a factorization $U \to V^{'} \to V$ where $|V^{'}| \leq \kappa$ for some cardinal number κ that depends on $|U|$, $|K|$ and $|P|$ and thus only on $|U|$ and $\mathcal{I}$. $\square$

5.3 Main result

We are now ready to prove the main theorem. The motivation for this result came from Eklof-Trlifaj's well-known theorem (Thm.10, [15]).

Theorem 5.3.1. *If $\mathcal{I}$ is generated by a set then $\mathcal{I}^{\perp}$ is a preenveloping class.*

Proof. Given an R-module U by Lemma 5.2.6 we find a cardinal number κ with the desired properties given in the lemma . We will use a similar argument to that of Rada-Saorín in [39]. Let $\{g_j\}_{j\in J}$ be the set of all the homomorphisms $g_j : U \to V_j$ in $\mathcal{I}^{\perp}$ (up to isomorphism) where $|V_j| \leq \kappa$. Then by lemma 5.2.5 the homomorphism,

$$U \xrightarrow{\prod g_j} \prod_{j\in J} V_j$$

is in $\mathcal{I}^{\perp}$. Moreover we claim that it is an $\mathcal{I}^{\perp}$ -preenvelope of U.
Given any $g : U \to V$ in $\mathcal{I}^{\perp}$ by Lemma 5.2.6 we get a factorization,

$$\overset{g}{\overbrace{U \longrightarrow V' \hookrightarrow V}}$$

where $|V'| \leq \kappa$. Then $g : U \to V' \subset V$ is isomorphic to g_j for some j. That is, there exists a map making the following diagram commutative,

$$\begin{array}{ccc} U & \xrightarrow{g_j} & V_j \\ {\scriptstyle g}\downarrow & \circlearrowleft & \swarrow \\ V' & & \end{array}$$

Now we get the following commutative diagram,

$$\begin{array}{ccc} U & \xrightarrow{g_j} & \prod_{j\in J} V_j \\ \downarrow & \swarrow & \\ V_j & & \\ \downarrow & & \\ V' & & \\ \downarrow & & \\ V & & \end{array}$$

We conclude that the following diagram is commutative where we use the composition of maps $\prod_{j\in J} V_j \to V_j \to V' \to V$,

$$\begin{array}{ccc} U & \xrightarrow{g_j} & \prod\limits_{j\in J} V_j \\ {\scriptstyle g}\big\downarrow & \circlearrowright & \swarrow \\ V & & \end{array}$$

That is $\mathcal{I}^{\perp}$ is a preenveloping class. □

5.4 Ideals closed under sums

We begin this section by the definition of being closed under sums for an ideal. Note that precovering ideals automatically satisfies this condition.

Definition 5.4.1. *The ideal $\mathcal{I}$ is said to be closed under sums if it satisfies one of the following equivalent conditions,*

- *If $(f_j)_{j\in J}$, $f_j : M_j \to N_j$ is any family of elements of $\mathcal{I}$ then $\bigoplus_{j\in J} f_j : \bigoplus_{j\in J} M_j \to \bigoplus_{j\in J} N_j$ is in $\mathcal{I}$.*
- *If $(g_j)_{j\in J}$, $g_j : M_j \to N$ is any family of elements of $\mathcal{I}$ then $g : \bigoplus_{j\in J} M_j \to N$ defined by $g((x_j)_{j\in J}) = \sum_{j\in J} g_j(x_j)$ is in $\mathcal{I}$.*

Now we draw our attention to ideals closed under direct sums. We will prove that being closed under sums is enough for an ideal generated by a set to be precovering. But first we consider an ideal that is generated by a single morphism to simplify the proof.

Theorem 5.4.2. *If $\mathcal{I}$ is the closure under direct sums of the ideal generated by a single homomorphism $f : M \to N$, then $\mathcal{I}$ is a precovering ideal.*

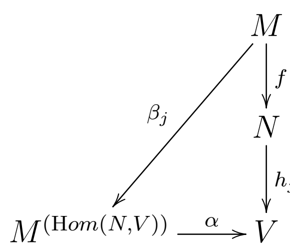
Proof. Given an arbitrary R-module V, we consider the homomorphism $\alpha : M^{(\mathrm{Hom}(N,V))} \to V$ defined by,

$$\alpha((x_g)_{g\in \mathrm{Hom}(N,V)}) = \sum_{g\in \mathrm{Hom}(N,V)} g(f(x_g))$$

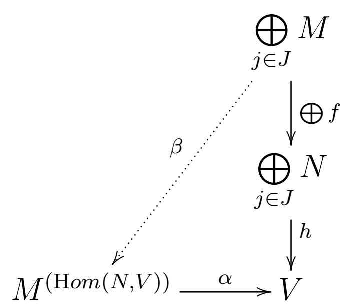
which is in $\mathcal{I}$ since it is closed under sums. Moreover we claim that it is an $\mathcal{I}$ -precover of V. Given a homomorphism,

$$\bigoplus_{j\in J} M \longrightarrow \bigoplus_{j\in J} N \xrightarrow{h} V$$

in $\mathcal{I}$. Define $h_j : N \to V$ such that $h = \sum_{j\in J} h_j$ and $\beta_j : M \to M^{(\mathrm{Hom}(N,V))}$ such that $\beta_j(x)$ is the element of $M^{(\mathrm{Hom}(N,V))}$ whose all entries are 0, except the one that corresponds to h_j, which is x. With the maps defined above the following diagram commutes,



Now we define β,



where $\beta((x_j)_{j\in J}) = \sum_{j\in J} \beta_j((x_j)$. Then notice that,

$$\begin{aligned}\alpha(\beta((x_j)_{j\in J})) &= \alpha(\sum_{j\in J} \beta_j(x_j)) = \sum_{j\in J} \alpha(\beta_j(x_j)) \\ &= \sum_{j\in J} h_j(f(x_j)) = h((f(x_j))_{j\in J}) \\ &= (h(\oplus f(x_j)_{j\in J}))\end{aligned}$$

So the above diagram is commutative and we conclude that V has an $\mathcal{I}$ -precover. That is $\mathcal{I}$ is a precovering ideal. □

Theorem 5.4.3. *If $\mathcal{I}$ is the closure under direct sums of the ideal generated by a set, then $\mathcal{I}$ is a precovering ideal.*

Proof. The proof follows very similarly to that of Theorem 5.4.2. □

In the following Proposition, we note that the orthogonal of an ideal generated by a set and the orthogonal of its closure should be the same.

Proposition 5.4.4. *Let $\mathcal{I} =< f_s >_{s\in S}$ generated by a set and $\mathcal{I}\,'$ be the smallest ideal that contains $\mathcal{I}$ and closed under sums. Then $\mathcal{I}^{\perp} = (\mathcal{I}\,')^{\perp}$.*

Proof. One way of the inclusion is easy. By definition $\mathcal{I} \subseteq \mathcal{I}'$ implies $(\mathcal{I}')^{\perp} \subseteq \mathcal{I}^{\perp}$.

To prove the other way of inclusion, let $g \in \mathcal{I}^{\perp}$ where $g : U \to V$. Then notice that $\mathrm{E}xt^1(f_s, g) = 0$ for all f_s. Now given any $T \subseteq S$ and a homomorphism,

$$\bigoplus M_t \xrightarrow{\oplus f_t} \bigoplus N_t$$

We want to prove that,

$$\mathrm{E}xt^1(\oplus f_t, g) : \mathrm{E}xt^1(\oplus N_t, U) \longrightarrow \mathrm{E}xt^1(\oplus M_t, V)$$

is the zero map. Let us observe the commutative diagram,

$$\begin{array}{ccc} \mathrm{E}xt^1(\bigoplus_{t\in T} N_t, U) & \xrightarrow{\cong} & \prod_{t\in T} \mathrm{E}xt^1(N_t, U) \\ \downarrow & & \downarrow \\ \mathrm{E}xt^1(\bigoplus_{t\in T} M_t, V) & \xrightarrow{\cong} & \prod_{t\in T} \mathrm{E}xt^1(M_t, V) \end{array}$$

Notice that the right column is the zero map since $\mathrm{E}xt^1(f_t, g) = 0$ for all $t \in T$. Hence we conclude that $\mathrm{E}xt^1(\oplus f_t, g) = 0$. Let us choose an arbitrary homomorphism in $\mathcal{I}'$,

$$U \xrightarrow{k} \bigoplus M_t \xrightarrow{\oplus f_t} \bigoplus N_t \xrightarrow{h} V$$

where $t \in T \subseteq S$. We have,

$$\mathrm{E}xt^1(h \circ \oplus f_t \circ k, g) = \mathrm{E}xt^1(k, id) \circ \mathrm{E}xt^1(\oplus f_t, g) \circ \mathrm{E}xt^1(h, id) = 0$$

Hence we conclude that $\mathcal{I}^{\perp} \subseteq (\mathcal{I}')^{\perp}$. □

5.5 Ideals generated by a set in the extended sense

We revise our definition of $\mathcal{I}$ being generated by a set.

Definition 5.5.1. *Let $(f_s)_{s\in S}$ be a set of homomorphisms where $f_s : M_s \to N_s$. $\mathcal{I}$ is said to be generated by $(f_s)_{s\in S}$ in the extended sense if every $\tilde{f} : U \to V$ in $\mathcal{I}$ has a factorization,*

$$U \longrightarrow \bigoplus_{s\in S} M_s^{\kappa_s} \xrightarrow{\bigoplus f_s} \bigoplus_{s\in S} N_s^{\kappa_s} \longrightarrow V$$

where κ_s is a cardinal number for each $s \in S$.

We note that if $\mathcal{I}$ is generated by a set in the extended sense, then it is closed under sums. So we can adapt the previous results to the new case of ideals generated by a set in the extended sense.

Corollary 5.5.2. *Let $\mathcal{I}$ be generated by a set of homomorphisms in the extended sense, then $\mathcal{I}^{\perp}$ is a preenveloping ideal.*

Proof. It follows from Theorem 5.3.1 and Proposition 5.4.4. □

Corollary 5.5.3. *Let $\mathcal{I}$ be generated by a set of homomorphisms in the extended sense, then $\mathcal{I}$ is a precovering ideal.*

Proof. Since $\mathcal{I}$ is closed under sums, we get the result by Theorem 5.4.3. □

There are a couple of further open questions that are worth pursuing and might be of interest about an ideal generated by a set. For example, whether the pair $(\mathcal{I}, \mathcal{I}^{\perp})$ is a cotorsion ideal pair if $\mathcal{I}$ is generated by a set. If this is proven to be correct, then the necessary conditions for completeness of such a cotorsion pair can be of interest.

Chapter 6 IDEALS CLOSED UNDER DIRECT LIMITS

The main motivation of this chapter is to introduce a new criterion for the existence of covers of modules associated to ideals of morphisms. Namely, we prove a sufficient condition to ensure that an ideal in R-M*od* is covering, in the sense that any R-module has an $\mathcal{I}$-cover (cf. [24]). And these sufficient conditions are satisfied by the ideal of phantom morphisms in R-M*od* introduced in [31]. In this chapter the key point is that one may identify ideals $\mathcal{I}$ of morphisms in R-M*od* with certain classes $\mathcal{I}(\mathcal{A}_2)$ of objects in the Grothendieck category $\mathcal{A}_2$ of all representations by left R-modules of the quiver $\mathbb{A}_2 : \bullet \to \bullet$. This allows to apply the general existence theorems developed in the literature for (pre)covers of modules with respect to a class of objects. The contents of this chapter are from [21], a joint work with Segio Estrada and Pedro A. Guil.

An important instance of ideal in R-M*od* is the ideal of phantom morphisms considered in [31]. A morphism $f : M \to N$ of left R-modules is called a *phantom morphism* if for each morphism $g : L \to M$, with L a finitely presented left R-module, the composition $f \circ g$ factors through a (finitely presented) projective module. Equivalently, for each left R-module A, the functor $\mathrm{E}xt^1(f, A)$ maps $\mathrm{E}xt^1(N, A)$ inside the subgroup $\mathrm{P}Ext^1(M, A)$ of $\mathrm{E}xt^1(M, A)$ consisting of all pure-exact sequences. It is straightforward to check that the class of all phantom morphisms forms an ideal, which we will denote by $\mathbf{Phant}(R)$.

6.1 The category $\mathcal{A}_2$

Let us denote by $\mathbb{A}_2$ the quiver with two vertices v_1, v_2 and an edge $a : v_1 \to v_2$. This may be thought as a small category. Let us consider the category $\mathcal{A}_2 = (\mathbb{A}_2, R\text{-M}od)$ of *a*ll representations of the quiver $\mathbb{A}_2$ by left R-modules. That is, the category of all covariant functors from $\mathbb{A}_2$ to R-M*od*. Note that an object $\overline{M}$ of $\mathcal{A}_2$ is just a morphism $\overline{M} \equiv M_1 \stackrel{f}{\to} M_2$ in R-M*od*. Whereas a morphism in $\mathcal{A}_2$ from $\overline{M} \equiv M_1 \stackrel{f}{\to} M_2$ to $\overline{N} \equiv N_1 \stackrel{g}{\to} N_2$ is a natural transformation; that is, a pair of morphisms (d, s) in R-M*od* for which the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle s} \\ N_1 & \xrightarrow[g]{} & N_2 \end{array}$$

is commutative. The category $\mathcal{A}_2$ is Grothendieck and the representations $R \stackrel{1_R}{\to} R$ and $0 \stackrel{0}{\to} R$ are projective representations that generate $\mathcal{A}_2$. It is also known that $\overline{P} \equiv P_1 \stackrel{f}{\to} P_2$ is a projective representation if, and only if, P_1 and P_2 are projective R-modules and f

is a splitting monomorphism. And $\overline{P}$ is flat provided that P_1 and P_2 are flat R-modules and f is a pure monomorphism (see [17, 23]).

We can associate to any ideal $\mathcal{I}$ in R-M*od*, the class of objects $\mathcal{I}(\mathcal{A}_2)$ in $\mathcal{A}_2$ consisting of those representations $\overline{M} \equiv M_1 \xrightarrow{f} M_2$ in $\mathcal{I}(\mathcal{A}_2)$ such that $f \in \mathcal{I}$. In particular, we will denote by $\mathbf{Phant}_R(\mathcal{A}_2)$ the class of all phantom morphisms in R-M*od*, considered as a class of representations in $\mathcal{A}_2$. Hence, $\overline{M} \equiv M_1 \xrightarrow{f} M_2$ belongs to $\mathbf{Phant}_R(\mathcal{A}_2)$ if and only if f is a phantom morphism in R-M*od*. It is clear, from the definition of phantom morphism, that flat representations of $\mathbb{A}_2$ belong to $\mathbf{Phant}_R(\mathcal{A}_2)$. This means that, if $\overline{F} \equiv F_1 \to F_2$ is a representation in which F_1 is a flat R-module, then $\overline{F} \in \mathbf{Phant}_R(\mathcal{A}_2)$. In particular, $\mathbf{Phant}_R(\mathcal{A}_2)$ contains a projective generator of $\mathcal{A}_2$.

As $\mathcal{A}_2$ is a Grothendieck category, we have the usual notions of *(pre)covers* and *(pre)envelopes* with respect to a class of representations $\mathcal{F}$ in $\mathcal{A}_2$. Namely, if $\mathcal{F}$ is a class of objects in $\mathcal{A}_2$ and $\overline{M} \in \mathcal{A}_2$, an $\mathcal{F}$-precover ($\mathcal{F}$-preenvelope) of $\overline{M}$ is a morphism $\overline{F} \xrightarrow{\varphi} \overline{M}$ (resp., $\overline{M} \xrightarrow{\varphi} \overline{F}$) with $\overline{F} \in \mathcal{F}$, such that $\mathrm{H}om(\overline{F'}, \overline{F}) \to \mathrm{H}om(\overline{F'}, \overline{M}) \to 0$ is exact (resp., $\mathrm{H}om(\overline{M}, \overline{F'}) \to \mathrm{H}om(\overline{F}, \overline{F'}) \to 0$ is exact), for every $\overline{F'} \in \mathcal{F}$. If, moreover, any $f : \overline{F} \to \overline{F}$ such that $\varphi \circ f = \varphi$ (resp. $f \circ \varphi = \varphi$) is an automorphism, then φ is called an $\mathcal{F}$-cover (resp., $\mathcal{F}$-envelope). We will say that a class $\mathcal{F}$ of representations in $\mathcal{A}_2$ is (pre)covering ((pre)enveloping)) if every $\overline{M} \in \mathcal{A}_2$ admits an $\mathcal{F}$-(pre)cover (resp., $\mathcal{F}$-(pre)envelope).

Throughout this chapter, all rings will be associative rings with identity element and all modules will be unitary left modules. We refer to [24, 22, 29, 46] for any undefined notion.

6.2 A Sufficient Condition for Covering Ideals

Let us begin this section by introducing the following definition which will be needed to state the main result of this section.

Definition 6.2.1. *Let $\mathcal{I}$ be an ideal in R-Mod. We will say that $\mathcal{I}$ is* closed under direct limits *if for any morphism $\{f_i : M_i \to N_i\}_I$ between directed systems of morphisms $\{g_{ij} : M_i \to M_j\}_{i \leq j}$ and $\{h_{ij} : N_i \to N_j\}_{i \leq j}$, satisfying that $f_i \in \mathcal{I}$ for every $i \in \mathcal{I}$, the induced morphism $\varinjlim f : \varinjlim M_i \to \varinjlim N_i$ also belongs to $\mathcal{I}$.*

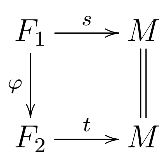
And we will say that the ideal $\mathcal{I}$ is the closure under direct limits *of a set of morphisms $\mathcal{I}_0 \subseteq \mathcal{I}$ if there exists a set $\mathcal{I}_0 \subseteq \mathcal{I}$ such that any morphism $f \in \mathcal{I}$ can be obtained in the above fashion from a morphism $\{f_i\}_I$ of direct systems with each $f_i \in \mathcal{I}_0$.*

We can now prove our promised criterion for the existence of covering ideals of morphisms.

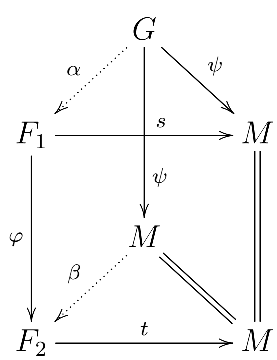
Theorem 6.2.2. *Let $\mathcal{I}$ be an ideal in R-Mod closed under direct limits. If $\mathcal{I}$ is the closure under direct limits of a set of morphisms $\mathcal{I}_0 \subseteq \mathcal{I}$, then $\mathcal{I}$ is a covering ideal of R-Mod.*

Proof. Note that our hypothesis implies that $\mathcal{I}(\mathcal{A}_2)$ is the closure under direct limits in the Grothendieck category $\mathcal{A}_2$ of the set $\mathcal{I}_0(\mathcal{A}_2)$. Hence $\mathcal{I}(\mathcal{A}_2)$ is a covering class in $\mathcal{A}_2$ by [6, Theorem 3.2].

Let us first show that this implies that $\mathcal{I}$ is a precovering ideal. Let M be an R-module and $\overline{F} \equiv F_1 \stackrel{\varphi}{\to} F_2$, an $\mathcal{I}(\mathcal{A}_2)$-cover of $\overline{1}_M \equiv M \stackrel{1_M}{\to} M$. Then there exist morphisms s, t such that the following diagram commutes,

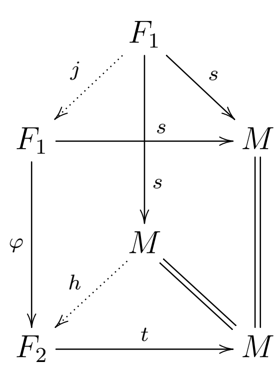


Now, given any $G \stackrel{\psi}{\to} M$ in $\mathcal{I}(\mathcal{A}_2)$, there exist morphisms α, β which make the following diagram commutative

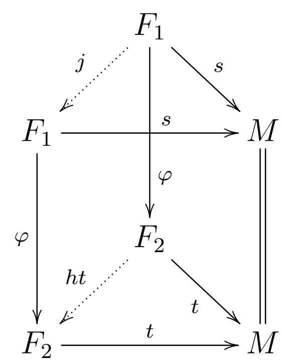


since $\overline{F}$ is an $\mathcal{I}(\mathcal{A}_2)$-precover. Note that this implies that $s = t \circ \varphi$ belongs to $\mathcal{I}$, since $\mathcal{I}$ is an ideal, and that the top triangle commutes. Hence, s is an $\mathcal{I}$-precover of M. This shows that $\mathcal{I}$ is a precovering ideal.

Moreover we claim that $F_1 \stackrel{s}{\to} M$ is not only an $\mathcal{I}$-precover of M but also an $\mathcal{I}(\mathcal{A}_2)$-cover of $M \stackrel{1_M}{\to} M$. To prove this claim it is enough to show that $F_1 \stackrel{s}{\to} M$ is isomorphic to $F_1 \stackrel{\varphi}{\to} F_2$. First, note that there exist morphisms j, h such that the following diagram commutes,



The commutativity of the bottom triangle implies $t \circ h = 1_M$, hence h is monic. We also conclude that $\varphi \circ j = h \circ s$ and $t \circ \varphi = s$ which gives us the following commutative diagram,



But since $F_1 \stackrel{\varphi}{\to} F_2$ is an $\mathcal{I}(\mathcal{A}_2)$-cover of $M \stackrel{1_M}{\to} M$, we conclude that j and $h \circ t$ are automorphisms. Then h is epic which implies that it is an isomorphism.

So we showed that $F_1 \stackrel{s}{\to} M$ is an $\mathcal{I}(\mathcal{A}_2)$-cover of $\overline{1}_M \equiv M \stackrel{1_M}{\to} M$. Then one can easily see that it is a $\mathcal{I}$-cover of M. This finishes the proof. □

As noted in [6, Theorem 3.3], the above arguments can be easily carried out to deduce the following stronger result under the assumption of Vopěnka's Principle (see e.g. [2, Chapter 6]).

Theorem 6.2.3. *(Assume Vopěnka's Principle) Any ideal $\mathcal{I}$ of R-Mod closed under direct limits is covering.*

Remark 6.2.4. *We would like to stress that the usual version of Wakamatsu's Lemma [46, Lemma 2.1.1] in $\mathcal{A}_2$ cannot be used to infer from Theorem 6.2.2 that the kernel of an $\mathcal{I}(\mathcal{A}_2)$-cover is an object in $\mathcal{I}(\mathcal{A}_2)^{\perp} = \{\overline{M} \in \mathcal{A}_2 |\ Ext^1_{\mathcal{A}_2}(\overline{F}, \overline{M}) = 0,\ \forall \overline{F} \in \mathcal{I}(\mathcal{A}_2)\}$. The reason is that the class of representations $\mathcal{I}(\mathcal{A}_2)$ of $\mathcal{A}_2$ which we have associated to any ideal $\mathcal{I}$ of R-Mod is not closed under extensions unless the ideal $\mathcal{I} = Hom$. Note that if $f : A \to B$ is a morphism in R-Mod such that $f \notin \mathcal{I}$ and we consider the following commutative diagram of splitting sequences,*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\tau_1} & A \oplus A & \xrightarrow{\pi_2} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle t_1 f \pi_2} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & B & \xrightarrow[t_1]{} & B \oplus B & \xrightarrow[p_2]{} & B & \longrightarrow & 0
\end{array}$$

then $0 : A \to B$ belongs to $\mathcal{I}$ but $t_1 \circ f \circ \pi_2 \notin \mathcal{I}$ since $f \notin \mathcal{I}$. This means that the representation $A \oplus A \stackrel{t_1 f \pi_2}{\longrightarrow} B \oplus B$ is an extension of the representation $0 : A \to B$ by itself which does not belong to $\mathcal{I}(\mathcal{A}_2)$.

Note that in the category of $\mathbb{A}_2$-representations, every morphism $f : M \to N$ is trivially an extension of 0 morphisms,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & M & = & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & N & = & N & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Suppose now that $f \in \mathcal{I}$, for some ideal $\mathcal{I}$. Both of the zero morphisms $0 : 0 \to N$ and $0 : M \to 0$ are then deconstructible in the sense of [14], by deconstructing N and M in the module category, and belong to $\mathcal{I}$. Thus $f \in \mathcal{I}$ will be also deconstructible. Hence any ideal $\mathcal{I}$ is deconstructible (in the sense that the class of representations $\mathcal{I}(\mathcal{A}_2)$ in $\mathcal{A}_2$ is deconstructible). However, in view of the above comment, the class $\mathcal{I}(\mathcal{A}_2)$ is never closed under $\mathcal{I}(\mathcal{A}_2)$-filtrations, for any ideal $\mathcal{I}$. This suggests that we may need a more sophisticated definition of deconstructibility in order to extend its usual applications in the classical approximation theory by objects (see, for instance [43, 44]) to this new framework.

6.3 Filtering Phantom Morphisms

Throughout this last section we will focus on the case in which the considered ideal $\mathcal{I}$ is the class $\mathbf{Phant}(R)$ of all phantom morphisms in R-M*od*. Our aim is to prove that each phantom morphism admits a nice filtration by "small" phantom morphism, and deduce from it that the ideal $\mathbf{Phant}(R)$ satisfies the hypothesis of Theorem 6.2.2.

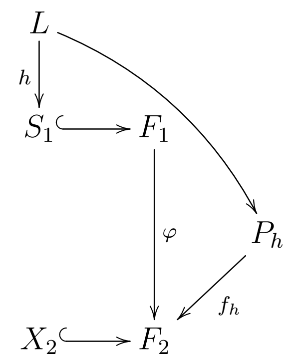
Let $\mathcal{A}$ be a Grothendieck category and λ, an ordinal number. Recall that a linearly ordered directed system of morphisms in $\mathcal{A}$, $\{f_{\alpha\beta} : A_\alpha \to A_\beta | \, \alpha \leq \beta < \lambda\}$, is called continuous if $A_\gamma = \varinjlim_{\alpha \leq \beta < \gamma} f_{\alpha\beta}$ for each ordinal limit $\gamma \leq \lambda$. Note that, in particular, this means that $A_0 = 0$.

A linearly ordered directed system of morphisms is called a *continuous directed union* if all morphisms in the system are monomorphisms.

Our goal in this section is to show in Corollary 6.3.4 that the class of phantom morphisms is the closure under directed unions of a set of phantom morphisms. To achieve this aim, we will start with the following technical lemma.

Lemma 6.3.1. *Let $\kappa \geq |R|$ be an infinite cardinal number, $\overline{F} \equiv F_1 \to F_2$, a representation in $\mathbf{Phant}_R(\mathcal{A}_2)$, and $X_1 \subseteq F_1, X_2 \subseteq F_2$ two subsets with $|X_1|, |X_2| \leq \kappa$. Then there is a phantom subrepresentation $\overline{S} \equiv S_1 \to S_2$ of $\overline{F}$ such that $|S_1|, |S_2| \leq \kappa$ and $X_1 \subseteq S_1$ and $X_2 \subseteq S_2$.*

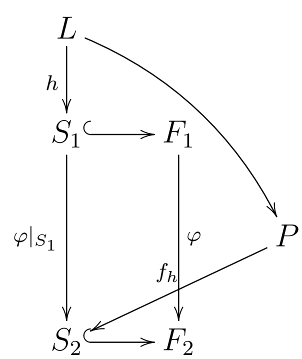
Proof. Let S_1 be the submodule of F_1 generated by X_1. Let L be any finitely presented R-module and $h : L \to S_1$, any homomorphism. As $\overline{F} \in \mathbf{Phant}_R(\mathcal{A}_2)$, there exists a finitely presented projective R-module P_h (whose cardinality is therefore, bounded by κ) such that $L \to F_2$ factors through it



Define

$$S_2 = X_2 + \sum \{\mathrm{I}m(f_h) |\, L \text{ is a finitely presented module and } h \in \mathrm{H}om(L, S_1)\}$$

and note that $|S_2| \leq \kappa$. Then, by construction, for any arbitrary finitely presented R-module L and any $h : L \to S_1$, there exists a projective R-module P such that the following diagram commutes



since $\mathrm{I}m(f_h) \subset S_2$. This means that $L \to S_2$ factors through P. And hence, $S_1 \xrightarrow{\varphi|_{S_1}} S_2$ belongs to $\mathbf{Phant}_R(\mathcal{A}_2)$. $\square$

Lemma 6.3.2. *The class* $\mathbf{Phant}_R(\mathcal{A}_2)$ *is closed under direct limits.*

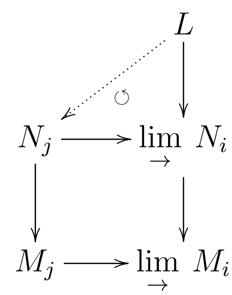
Proof. Let $\{f_i : N_i \to M_i\}_I$ be a morphism between two directed systems of morphisms $\{g_{ij} : N_i \to N_j\}_{i \leq j}$ and $\{h_{ij} : M_i \to M_j\}_{i \leq j}$ in R-Mod such that $f_i : N_i \to M_i$ is a phantom morphism, for each $i \in I$. And let $\varinjlim f_i : \varinjlim N_i \to \varinjlim M_i$ be the induced morphism in the direct limits. We must show that $\varinjlim f_i$ is also a phantom morphism.

Let L be any finitely presented left R-module. We have to prove that for any morphism $L \to \lim\limits_{\rightarrow} N_i$ the composition $L \to \lim\limits_{\rightarrow} N_i \to \lim\limits_{\rightarrow} M_i$ factors through a projective R-module.

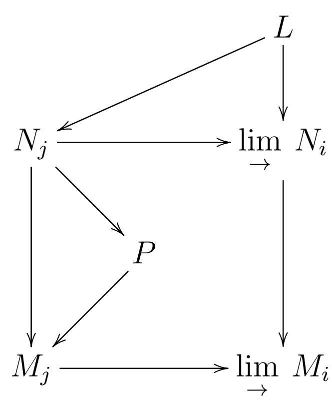
But, as L is finitely presented, we have an isomorphism

$$\mathrm{H}om(L, \varinjlim N_i) \cong \varinjlim \mathrm{H}om(L, N_i).$$

Hence, given a morphism $L \to \varinjlim N_i$, there is a $j \in I$ such that the following diagram can be completed



Now, as $f_j \in \mathbf{Phant}(R)$, we conclude that the composition $L \to N_j \to M_j$ factors through a projective R-module, say P



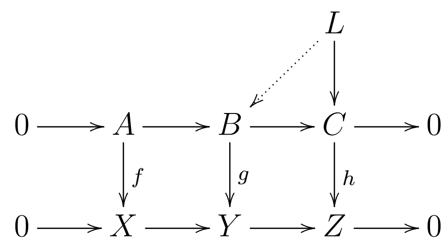
and then, by diagram chasing, we conclude that the morphism $L \to \varinjlim M_i$ factors through P as well. $\square$

Proposition 6.3.3. *Let us consider a morphism of exact sequences:*

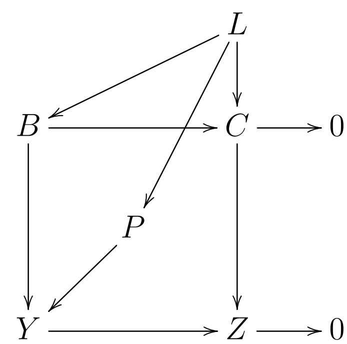
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

If g is a phantom morphism and the top sequence is pure exact, then $h : C \to Z$ is phantom.

Proof. Let L be a finitely presented R-module L and a morphism $L \to C$, there exists a morphism $L \to B$ which completes the following commutative diagram



Moreover, as $g \in \mathbf{Phant}(R)$, the composition $L \to B \to Y$ factors through a projective R-module, say P,



This shows that $L \to C \overset{h}{\to} Z$ also factors through P and hence, we deduce that $h \in \mathbf{Phant}(R)$.

□

Recall that a representation $\overline{M} \equiv M_1 \to M_2$ in $\mathcal{A}_2$ is of *type* κ (for κ an infinite cardinal number), if each of the R-modules M_1 and M_2 are generated at most by κ elements.

Corollary 6.3.4. *There exists an infinite cardinal κ such that every representation in $\mathcal{A}_2$ belongs to $\mathbf{Phant}_R(\mathcal{A}_2)$ if and only if it is the directed union of its subrepresentations in $\mathbf{Phant}_R(\mathcal{A}_2)$ of type κ.*

Proof. This follows from lemmas 6.3.1 and 6.3.2. □

Corollary 6.3.5. *Every module has a surjective phantom cover.*

Proof. It follows from Lemma 6.3.2 and Corollary 6.3.4 that the class $\mathbf{Phant}_R(\mathcal{A}_2)$ satisfies the conditions of Theorem 6.2.2. Finally, phantom covers are surjective because the class $\mathbf{Phant}_R(\mathcal{A}_2)$ contains a projective generator of $\mathcal{A}_2$. □

We are going to close this chapter by showing that the kernel of a phantom cover is always a pure injective module. First we need to prove the following lemma, which is of independent interest.

Lemma 6.3.6. *Let $\phi : M \to N$ be a phantom epimorphism, with kernel $u : K \to M$ and let $v : K \to K'$ be a pure monomorphism. Let us consider the pushout along u, v:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \xrightarrow{u} & M & \xrightarrow{\phi} & N & \longrightarrow & 0\\
 & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} & & \| & & \\
0 & \longrightarrow & K' & \xrightarrow[u']{} & X & \xrightarrow[\phi']{} & N & \longrightarrow & 0\\
 & & {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} & & & & \\
 & & K/K' & = & X/M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then ϕ' is also a phantom map.

Proof. Let F be a finitely presented module and consider a morphism $f : F \to X$. As the short exact sequence

$$0 \to K \xrightarrow{v} K' \xrightarrow{\pi} K/K' \to 0$$

is pure, there will exist a morphism $g : F \to K'$ such that $\pi \circ g = \pi' \circ f$. Then $\pi \circ g = \pi' \circ u' \circ g$ and thus, $\mathrm{I}m(f - u'g) \subseteq \mathrm{K}er(\pi') = M$. Therefore, there exists a unique $h : F \to M$ such that $v'h = f - u'g$. Now

$$\phi h = \phi' v' h = \phi' f - \phi' u' g = \phi' f,$$

where the last equality holds because $\phi' u' = 0$. Now, as ϕ is phantom, ϕh factors through a projective module. This shows that $\phi' f$ factors through a projective module and thus, ϕ' is phantom.

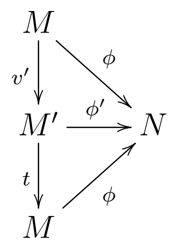
□

Proposition 6.3.7. *Let $\phi : M \to N$ be a phantom cover. Then $Ker(\phi)$ is a pure injective module.*

Proof. Let $K = \mathrm{K}er(\phi)$ and $u : K \to M$, the inclusion. We must show that K is pure injective. So let $u : K \to X$ be a pure monomorphism. We want to see that it admits a retract. As phantom covers are surjective, it follows from Lemma 6.3.6 that the pushout along u and v,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{u} & M & \xrightarrow{\phi} & N & \longrightarrow & 0\\
 & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} & & \| & & \\
0 & \longrightarrow & X & \xrightarrow[u']{} & M' & \xrightarrow[\phi']{} & N & \longrightarrow & 0
\end{array}$$

gives a phantom morphism $\phi' : M' \to N$. This leads to commutative triangles:



where $t : M' \to M$ comes from the fact that ϕ' is phantom and ϕ is a phantom (pre)cover. Let us denote by $w = t \restriction_X$ the restriction of t to $X \to K$. As ϕ is a cover, the morphism $tv' : M \to M$ is an automorphism. Hence, the restriction $w \circ v : K \to K$ is an automorphism of K and therefore, $r = (wv)^{-1}w$ is the desired retract of v. □

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AWARDS

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- *AMS Graduate Student Travel Grant*, Spring 2013 & 2014, Fall 2013
- *Department of Mathematics Fellowship*, Dept. of Mathematics, University of Kentucky, 201.
- *Edgar Enochs Scholarship in Algebra*, Dept. of Mathematics, University of Kentucky, 2011.
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