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COMBINATORIAL ASPECTS OF EXCEDANCES AND THE FROBENIUS COMPLEX

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ABSTRACT OF DISSERTATION

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Eric Logan Clark
Lexington, Kentucky

Director: Dr. Richard Ehrenborg, Professor of Mathematics
Lexington, Kentucky 2011

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In this dissertation we study the excedance permutation statistic. We start by extending the classical excedance statistic of the symmetric group to the affine symmetric group $\widetilde{\mathfrak{S}}_n$ and determine the generating function of its distribution. The proof involves enumerating lattice points in a skew version of the root polytope of type A. Next we study the excedance set statistic on the symmetric group by defining a related algebra which we call the excedance algebra. A combinatorial interpretation of expansions from this algebra is provided. The second half of this dissertation deals with the topology of the Frobenius complex, that is the order complex of a poset whose definition was motivated by the classical Frobenius problem. We determine the homotopy type of the Frobenius complex in certain cases using discrete Morse theory. We end with an enumeration of Q-factorial posets. Open questions and directions for future research are located at the end of each chapter.

KEYWORDS: EXCEDANCE, AFFINE SYMMETRIC GROUP, ROOT POLYTOPE, DISCRETE MORSE THEORY, FROBENIUS COMPLEX

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Date: May 2, 2011

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Dedicated to my family
and
in memory of my grandmother,
Anna Logan Hayden

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Chapter 1 Introduction

Chapter 1 contains an overview of results that are found in this dissertation as well as a brief introduction to many of the mathematical ideas that will be contained throughout. References for these topics will be given for readers who are interested in a delving more deeply into these topics.

In Chapter 2 we extend the classical excedance statistic of the symmetric group to the affine symmetric group $\widetilde{\mathfrak{S}}_n$ and determine the generating function of its distribution. The proof involves enumerating lattice points in a skew version of the root polytope of type A. We also show that the left coset representatives of the quotient $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$ correspond to increasing juggling sequences and determine their Poincaré series.

A formal study of the excedance set statistic began with a paper by Ehrenborg and Steingrímsson [39]. Based on their work, Chapter 3 introduces the excedance algebra. We will provide a combinatorial interpretation for expansions in this algebra, and derive operators that make computing these expansions more efficient.

Motivated by the classical Frobenius problem, Chapter 4 introduces the Frobenius poset on the integers $\mathbb{Z}$, that is, for a sub-semigroup Λ of the non-negative integers $(\mathbb{N}, +)$, we define the order by $n \leq_\Lambda m$ if $m - n \in \Lambda$. When Λ is generated by two relatively prime integers a and b, we show that the order complex of an interval in the Frobenius poset is either contractible or homotopy equivalent to a sphere. We also show that when Λ is generated by the arithmetic sequence $\{a, a + d, a + 2d, \ldots, a + (a - 1)d\}$, the order complex is homotopy equivalent to a wedge of spheres.

In Chapter 5 we provide a method for enumerating Q-factorial posets. This answers a question and generalizes a result from work by Claesson and Linusson in [24]. The proof uses a q-enumeration that involves working with the polynomial part of certain Puiseux polynomials.

1.1 Dissertation results

Let $\mathfrak{S}_n$ denote the symmetric group, that is, the group of permutations on the elements $\{1, 2, \ldots, n\}$. A permutation statistic is a function from the symmetric group to the non-negative integers. Some of the most well-known permutations statistics are inversions, descents, excedances, and the major index. Given a permutation $\pi \in \mathfrak{S}_n$, we define the inversion set, descent set, and excedance set as follows:

$$
\begin{aligned}
\operatorname{Inv}(\pi) &= \{(i,j) \in [n] \times [n] \ : \ i < j, \pi(i) > \pi(j)\}, && (1.1)\\
\operatorname{Des}(\pi) &= \{i \in [n-1] \ : \ \pi(i) > \pi(i+1)\}, && (1.2)\\
\operatorname{Exc}(\pi) &= \{i \in [n] \ : \ \pi(i) > i\}, && (1.3)
\end{aligned}
$$

where $[n] = \{1, 2, \ldots, n\}$. The corresponding permutation statistics can then be written as $\operatorname{inv}(\pi) = |\operatorname{Inv}(\pi)|$, $\operatorname{des}(\pi) = |\operatorname{Des}(\pi)|$, and $\operatorname{exc}(\pi) = |\operatorname{Exc}(\pi)|$, while the major index is defined by

$$\operatorname{maj}(\pi) = \sum_{i \in \operatorname{Des}(\pi)} i. \tag{1.4}$$

Two classical results about permutation statistics are that excedances and descents, as well as inversions and the major index, are equidistributed. In particular,

$$\sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{exc}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{des}(\pi)} = \sum_{k=0}^{n-1} A(n, k+1) q^k,$$

where $A(n,k)$ represents the Eulerian numbers. The Eulerian number $A(n,k)$ is simply defined as the number of permutations from $\mathfrak{S}_n$ with $k-1$ descents. However, they can also be defined in the following ways:

$$A(n, k+1) = (k+1)A(n-1, k+1) + (n-k)A(n-1, k),$$

where $A(n, 0)$ is defined as $\delta_{n,0}$ and $A(n,k) = 0$ for $k \geq n+1$ and

$$x^n = \sum_{k=0}^{n} A(n,k) \binom{x+n-k}{n},$$

see [16, Chapter 1]. The symmetric group is a special case of a finite Weyl group of type A. Weyl groups are a family of Coxeter groups motivated by root systems, see Section 1.4. Every finite Weyl group has an associated affine Weyl group that is infinite and obtained by adding a single generator to the group. Lusztig [57] provided a permutation description of the affine group associated with the symmetric group, see Section 2.1. Combinatorial descriptions of other affine Weyl groups can be found in [42].

The wealth of combinatorics found in the symmetric group make it very natural to study the affine symmetric group, $\widetilde{\mathfrak{S}}_n$. Björner and Brenti [10] extended the idea of inversion to the affine symmetric group and found the generating function of its distribution. By virtue of the fact that Shi [65] showed the inversion statistic of an affine permutation is the same as its Coxeter length, this generating function is also the Poincaré series.

Theorem 1.1.1. *The Poincaré series for the affine Coxeter group $\widetilde{\mathfrak{S}}_n$ is given by*

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} = \frac{1-q^n}{(1-q)^n}.$$

This Poincaré series is a specific example of Theorem 1.4.2 due to Bott [56]. A combinatorial proof of Theorem 1.1.1 using the theory of juggling patterns was first given by Ehrenborg and Readdy in [38]. Björner and Brenti [10] also provided a combinatorial proof of this generating function identity by finding a bijection between $\widetilde{\mathfrak{S}}_n$ and $\mathbb{N}^n - \mathbb{P}^n$, that is, the collection of n-tuples with at least one zero.

In this dissertation, we first provide a different proof of this result using the Ehrenborg-Readdy juggling technique applied to representatives of left cosets in the quotient $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$. We then extend the excedance statistic to the affine symmetric group and find the generating function for its distribution (See Definition 2.3.1 and Theorem 2.6.5).

Definition 1.1.2. *Let $\pi \in \widetilde{\mathfrak{S}}_n$ be an affine permutation. The excedance statistic is given by*

$$\mathrm{exc}(\pi) = \sum_{i=1}^{n} \left| \left\lceil \frac{\pi(i)-i}{n} \right\rceil \right|.$$

Note that the symmetric group sits inside the affine symmetric group as a natural subgroup. In fact, it is a maximal parabolic subgroup. It is straightforward to check that Definition 1.1.2 agrees with the definition of excedance given in Equation (1.3) when restricted to $\mathfrak{S}_n$. The following identity is proved in Chapter 2.

Theorem 1.1.3. *The generating function for affine excedances is given by*

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\mathrm{exc}(\pi)} = \frac{1}{(1-q^2)^{n-1}} \sum_{k=0}^{n-1} A(n, k+1) \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1+k}{n-1-i} q^{2i+k}.$$

Observe that the Eulerian numbers appear in the generating function for the affine case as well. The proof of this fact relies on enumerating lattice points in dilations of a skew version of the root polytope.

In 2000 Ehrenborg and Steingrímsson [39] studied the excedance set statistic of a permutation. For every permutation $\pi \in \mathfrak{S}_n$ they encoded the excedance information into the *excedance word* $w(\pi) = w_1 w_2 \ldots w_{n-1}$ where $w_i = \mathbf{b}$ if i is an excedance of π and $w_i = \mathbf{a}$ if i is not an excedance of π. Given a word w of length $n-1$, denote by $[w]$ the number of permutations of $\mathfrak{S}_n$ whose excedance word is w. They proved the following four relations:

$$\begin{aligned} &[1] = 1, \\ &[\mathbf{a} \cdot u] = [u \cdot \mathbf{b}] = [u], \\ &[v \cdot \mathbf{ba} \cdot w] = [v \cdot \mathbf{ab} \cdot w] + [v \cdot \mathbf{a} \cdot w] + [v \cdot \mathbf{b} \cdot w]. \end{aligned}$$

They also provided an inclusion-exclusion formula for $[w]$. In an effort to find more explicit formulas, Clark and Ehrenborg [25] proved the following results.

Theorem 1.1.4. *For non-negative integers m and n, we have*

$$[\mathbf{b}^n \mathbf{a}^m] = \sum_{i \geq 0} S(n+1, i+1) \cdot S(m+1, i+1) \cdot i! \cdot (i+1)!,$$

and

$$[\mathbf{b}^n \mathbf{aba}^m] = \sum_{i \geq 0} S(n+2, i+2) \cdot S(m+2, i+2) \cdot (i+1) \cdot (i+1)!^2,$$

where S denotes the Stirling number of the second kind.

The expressions in Theorem 1.1.4 are particularly attractive because they are sign free, that is, each term is non-negative.

In this dissertation, based on the relations proved by Ehrenborg and Steingrímsson about excedance sets and excedance words, we define the excedance algebra to be the non-commutative algebra $k\langle \mathbf{a}, \mathbf{b}\rangle$ quotiented out with the ideal generated by the element $\mathbf{ba}-\mathbf{ab}-\mathbf{a}-\mathbf{b}$. A Ferrers shape can be generated from a given $\mathbf{ab}$-monomial, leading to the following result (see Theorem 3.2.1).

Theorem 1.1.5. *Let u be an $\mathbf{ab}$-monomial containing m copies of $\mathbf{a}$ and n copies of $\mathbf{b}$. Consider the expansion*

$$u = \sum_{i,j} c_{i,j} \cdot \mathbf{a}^{m-i} \cdot \mathbf{b}^{n-j}.$$

Then the coefficient $c_{i,j}$ enumerates the number of ways to place i copies of $\leftarrow$ and j copies of $\uparrow$ in the Ferrers shape $F(u)$ such that

(a) All the boxes to the west of an $\leftarrow$ must be empty.

(b) All the boxes to the north of an $\uparrow$ must be empty.

Theorem 1.1.5 appears without proof in [51]. A proof using permutation tableaux is given by Corteel and Williams [30], and the bijection between permutations tableaux and alternative tableaux can be found in [73]. We will provide formulas for specific families of the coefficients $c_{i,j}$.

We also define a linear map L from the excedance algebra to the polynomial ring $k[x, y]$ such that, given an $\mathbf{ab}$-monomial u, $L(u)|_{x=y=1}$ is the number of permutations whose excedance word is u. We then show that the difference operator Δ and the shift operator E can be used to compute $L(u)$ in the following way (see Theorem 3.3.4).

Theorem 1.1.6. *Let $u = u_1 u_2 \dots u_k$ be an $\mathbf{ab}$-monomial. Let U_i and V_i be the operators*

$$U_i = \begin{cases} \mathbf{x} & \text{if } u_i = \mathbf{a}, \\ \mathbf{y}E_x^1 + \mathbf{x}\Delta_x & \text{if } u_i = \mathbf{b}, \end{cases}$$

and

$$V_i = \begin{cases} \mathbf{x}E_y^1 + \mathbf{y}\Delta_y & \textit{if } u_i = \mathbf{a}, \\ \mathbf{y} & \textit{if } u_i = \mathbf{b}. \end{cases}$$

Then $L(u)$ is evaluated by applying the operators $U_1U_2\cdots U_k(1) = V_kV_{k-1}\cdots V_1(1)$.

The second part of this dissertation deals with topological combinatorics. Many times, combinatorial objects give rise naturally to topological spaces. These spaces tend to have a nice topology. In particular, many are homotopy equivalent to wedges of spheres. Examples include the independence complex of trees [36], the order complex of semi-modular lattices [13], the complex of disconnected graphs [72], and the neighborhood complex of the stable Kneser graph [12]. Forman [45] asked if there was some explanation for this behavior. In this dissertation, we will add to the list of examples of topological spaces that are homotopy equivalent to wedges of spheres.

The classical Frobenius problem asks for the largest integer that cannot be written as a non-negative integer combination of a collection of relatively prime positive integers, the so-called *Frobenius number*. It has received a great deal of attention over the years. Closed formulas for the Frobenius number exist when there are only two generators, when the generators form an arithmetic sequence [64], and when the generators form a geometric sequence [58]. See [5, Section 1.2] for a more in-depth introduction to this problem.

To define the Frobenius poset $P = (\mathbb{Z}, \leq_\Lambda)$, we first let Λ be the semigroup generated by a collection of relatively prime positive integers. We say $n \leq_\Lambda m$ if $m - n \in \Lambda$. We show that the order complex of intervals in the Frobenius poset is homotopy equivalent to a wedge of spheres or contractible in two cases (see Theorems 4.4.1 and 4.5.1).

Theorem 1.1.7. *Let the sub-semigroup Λ be generated by two relatively prime positive integers a and b with $1 < a < b$. The order complex of the associated Frobenius interval $[0, n]_\Lambda$, for $n \geq 1$, is homotopy equivalent to either a sphere or contractible,* *according to*

$$\Delta([0,n]_\Lambda) \simeq \begin{cases} \mathbb{S}^{2n/ab-2} & \textit{if } n \equiv 0 \bmod a \cdot b, \\ \mathbb{S}^{2(n-a)/ab-1} & \textit{if } n \equiv a \bmod a \cdot b, \\ \mathbb{S}^{2(n-b)/ab-1} & \textit{if } n \equiv b \bmod a \cdot b, \\ \mathbb{S}^{2(n-a-b)/ab} & \textit{if } n \equiv a+b \bmod a \cdot b, \\ \textit{point} & \textit{otherwise.} \end{cases}$$

Theorem 1.1.8. *Let Λ be the semigroup generated by the integers $\{a, a+d, a+2d, \ldots, a+(a-1)d\}$ where a and d are relatively prime. The order complex of the associated Frobenius interval $[0,n]_\Lambda$ is homotopy equivalent to a wedge of spheres where the ith Betti number satisfies*

$$\sum_{n \geq 0} \widetilde{\beta}_i q^n = q^{a+(i+1)(a+d)} \cdot [a]_{q^d} \cdot [a-1]_{q^d}^{i+1},$$

where $[a]_{q^d}$ represents the q^d-analogue, see Section 1.2.

For example, consider Λ generated by 3 and 4, that is, $\Lambda = \mathbb{N} - \{1,2,5\}$. The order complex of the interval $[0,15]_\Lambda$ contains the 32 chains

$$\begin{aligned} &\{\{3\},\{4\},\{6\},\{7\},\{8\},\{9\},\{11\},\{12\},\{3,6\},\{3,7\},\{3,9\},\{3,11\},\{3,12\}, \\ &\{4,7\},\{4,8\},\{4,11\},\{4,12\},\{6,9\},\{6,12\},\{7,11\},\{8,11\},\{8,12\},\{9,12\}, \\ &\{3,6,9\},\{3,6,12\},\{3,7,11\},\{3,9,12\},\{4,7,11\},\{4,8,11\},\{4,8,12\}, \\ &\{6,9,12\},\{3,6,9,12\}\}. \end{aligned}$$

The geometric representation of this simplicial complex is given in Figure 1.1. Observe that it is homotopy equivalent to a circle, $\mathbb{S}^1$, as predicted by Theorem 1.1.7.

These theorems are explicit examples in an area of mathematics where more general theory has been explored. For example, in d dimensions we can let Λ be a semigroup of $\mathbb{N}^d$ and define a partial order on $\mathbb{Z}^d$ by $\mu \leq_\Lambda \lambda$ if $\lambda - \mu \in \Lambda$. Hersh and Welker [49] give bounds on the vanishing homology groups of the order complex of intervals from this order. Additionally, the semigroup algebra $k[\Lambda] = \operatorname{span}\{x^\lambda =$

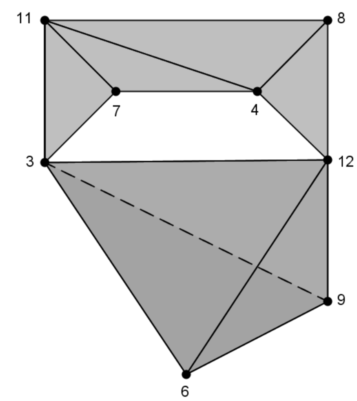

Figure 1.1: The geometric representation of the order complex $\Delta([0,15]_\Lambda)$ when Λ is generated by 3 and 4.

$x_1^{\lambda_1} \cdots x_d^{\lambda_d} : \lambda \in \Lambda\}$ is related to the homology of the order complex of the intervals of this order. See the work of Laudal and Sletsjøe [55] and Peeva, Reiner, and Sturmfels [61].

Lastly, we have a purely enumerative result about Q-factorial posets. Given a poset Q, a poset P is said to be Q-factorial if

1. $i <_P j$ implies $i <_Q j$,
2. $i <_Q j <_P k$ implies $i <_P k$.

See Figure 1.2 an example. Claesson and Linusson [24] showed that if Q is an n-chain, then there are $n!$ Q-factorial posets. Given an arbitrary poset Q, we provide a way to determine the number of Q-factorial posets. See Theorems 5.2.3 and 5.3.3.

1.2 Permutations, partitions, and rooks

A *permutation* is a bijection $\pi : [n] \longrightarrow [n]$. These bijections form a group under composition called the symmetric group, denoted $\mathfrak{S}_n$. Throughout this dissertation, we will denote permutations using one line notation $\pi = \pi_1\pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$.

Define a *board* B to be a finite subset of $\mathbb{Z}^2$. A *rook placement* on the board B is a finite subset C of the set B such that every two elements of C differ in both coordinates. That is, if C is the set of rooks, each pair of rooks is non-attacking.

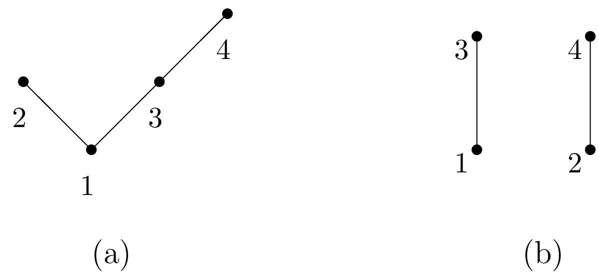

Figure 1.2: Let Q be a chain of 4 elements. Then P_1, (a), is a Q-factorial poset. P_2, (b), is not because $1 <_Q 2 <_{P_2} 4$ but $1 \not<_{P_2} 4$.

Let $r_k(B)$ be the number of ways to place k non-attacking rooks on the board B. It is straightforward to see there is a bijection between permutations from $\mathfrak{S}_n$ and rook placements using n rooks on a square board of size $n \times n$.

A *partition* of the set $[n]$ is a collection of sets (called *blocks*) $\{B_1, B_2, \ldots, B_k\}$ such that

- $B_i \neq \emptyset$,
- $B_i \cap B_j = \emptyset$ for $i \neq j$,
- $[n] = \bigcup_{i=1}^{k} B_i$.

Partitions of $[n]$ into k blocks are enumerated by the *Stirling numbers of the second kind*, denoted $S(n,k)$. These numbers follow the recursion

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k),$$

where $S(0,0) = 1$ and $S(n,0) = S(0,k) = 0$ for $n,k \geq 1$.

A *partition* of the non-negative integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_k \geq 1$ and $n = \sum_{i=1}^{k} \lambda_i$. A partition λ can be visualized as a *Young diagram* or *Ferrers board.* This is a left-justified array of squares where there are λ_i squares in row i. See Figure 1.3. Rook placements on Ferrers boards have been well studied. The next theorem can be found in Stanley's book [69, Corollary 2.4.2].

Theorem 1.2.1. *Let B be the "triangular board" associated with the partition $(n, n-1, \ldots, 2, 1)$. The rook number $r_k(B)$ is given by the Stirling number $S(n+1, n+1-k)$.*

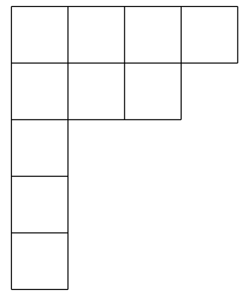

Figure 1.3: The Young diagram for the partition $(4, 3, 1, 1, 1)$ of the integer 10.

For positive integers n and d, the q^d-*analogue* of n is defined as follows:

$$[n]_{q^d} = 1 + q^d + (q^d)^2 + \cdots + (q^d)^{n-1}.$$

If it is clear from context that we are using q-analogues and $d = 1$, this will be written simply as $[n]$. Let $[n]! = [n] \cdot [n-1] \cdots [1]$, and define the q-binomial coefficient, or Gaussian polynomial, as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! \cdot [n-k]!}.$$

These q-analogues are powerful counting tools. For example, let $p(j, k, n)$ be the number of partitions of n using k parts where each part is less than or equal to j. Then we have the identity

$$\sum_{n \geq 0} p(j, k, n) \cdot q^n = \begin{bmatrix} j+k \\ j \end{bmatrix},$$

see [69, Proposition 1.3.19]. Also, consider the finite field $\mathrm{GF}(q)$ where q is a prime power, then $\begin{bmatrix} n \\ k \end{bmatrix}$ gives the number of k-dimensional subspaces of an n-dimensional vector space over $\mathrm{GF}(q)$.

Using these, we can clarify the earlier statement that inversions and the major index, recall Equations (1.2) and (1.4), are equidistributed. Thus, we have

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} = [n]! \tag{1.5}$$

Additionally, by setting $q = 1$ in the q-analogue $[n]$, we get n. Therefore, q-analogues are useful in refining enumeration. For example, $\binom{n}{k}$ counts the number of k element subsets of n elements, while $\begin{bmatrix} n \\ k \end{bmatrix}$ is the generating function for the number of k element subsets of n elements weighted by the sum of their elements, see [16, Theorem 2.25]. That is, we have

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} q^{i_1 + i_2 + \cdots + i_k - 1 - 2 - \cdots - k}.$$

1.3 Posets

A partially ordered set, or *poset*, P is a set of elements along with an order relation denoted $\leq_P$ (or $\leq$ if P is understood) that satisfies reflexivity, antisymmetry, and transitivity. We denote by $\widehat{0}$ and $\widehat{1}$ the unique minimal and maximal elements of the poset, respectively, if they exist. A *chain* of length n in the poset is any set of $n+1$ comparable elements $\{x_0 < x_1 < \cdots < x_n\}$ from P. We say that y *covers* x in P, denoted $x \prec y$, if there is no $z \in P$ such that $x < z < y$. An *interval* $[x, y]$ in the poset P is the collection of elements $z \in P$ such that $x \leq z \leq y$. A chain $\{x_0 < x_1 < \cdots < x_n\}$ is said to be *maximal* (or *saturated*) in the interval $[x, y]$ if $x = x_0 \prec x_1 \prec \cdots \prec x_n = y$. An *upper bound* of the two elements x and y is an element $z \in P$ such that $z \geq x$ and $z \geq y$. The element z is the *least upper bound* of x and y if for all upper bounds w of x and y, we have $z \leq w$. The least upper bound of x and y is denoted $x \vee y$ and is read "x join y." The *greatest lower bound* of two elements can be dually defined, and is denoted as $x \wedge y$, or "x meet y." A *lattice* is a poset for which every pair of elements has a least upper bound and a greatest lower bound. The *Hasse diagram* of a poset is a graph whose vertices are the elements of the poset and whose edges represent cover relations. See Figure 1.4 (a).

A poset is said to be $(2+2)$-*free* if it does not contain an induced sub-poset that is isomorphic to the union of two disjoint 2-chains. Fishburn [43] showed that a poset is $(2+2)$-free when it is isomorphic to an interval order. Bogart [15] also proved a poset is $(2+2)$-free if and only if the set of strict downsets of P can be linearly ordered by inclusion. See Section 5.1. The following result is due to Bousquet-Mélou, Claesson, Dukes, and Kitaev [17].

Theorem 1.3.1 (Bousquet-Mélou, et. al)**.** *The generating function of unlabel* $(2+2)$-

free posets is

$$P(t) = \sum_{n\geq 0} \prod_{i=1}^{n} (1-(1-t)^i).$$

Similarly defined are the $(3+1)$-free posets. Interestingly, it is well-known that the number of posets that are both $(2+2)$-free and $(3+1)$-free is given by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, see [70, Exercise 6.19 ddd].

The *Möbius function* μ on intervals of a poset has many important uses in combinatorics. For $x < y$, it is defined recursively on the interval $[x, y]$ with

$$\mu(x, y) = -\sum_{x\leq z<y} \mu(x, z),$$

and the base case $\mu(x, x) = 1$. When applied to the divisor lattice, D_n, we see this is a generalization of the Möbius function from number theory. The Möbius function also, through the Möbius inversion theorem, generalizes the Principle of Inclusion-Exclusion, see the second paragraph of [69, Chapter 3].

Among other topics, this dissertation deals with poset topology. As such, we need a way to obtain a topological space from a given poset. This space will be a simplicial complex. A *simplicial complex* Δ on a vertex set V is a collection of subsets of V that is closed under containment, that is, if $S, T \subseteq V$, $S \subseteq T \subseteq V$, and $T \in \Delta$, then $S \in \Delta$. The elements of Δ are called *faces* of the simplicial complex. Maximal faces are called *facets*. The *order complex* of a poset P with a minimal and maximal element, denoted $\Delta(P)$, is a simplicial complex where the faces of dimension k are given by chains of length k in the poset $P - \{\widehat{0}, \widehat{1}\}$. See Figure 1.4 for the Boolean algebra on three elements and its order complex. We remove the minimal and maximal element because, otherwise, the order complex will always be contractible.

One important tool in the study of the topology of a simplicial complex Δ is the *Euler characteristic*, $\chi(\Delta)$. It is defined as the alternating sum $f_0 - f_1 + f_2 - \cdots$ of the number of i-dimensional faces of the complex. It is sometimes convenient to use the *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$. This comes from the alternating sum of the faces of the complex including the empty face. That is, the reduced Euler characteristic is

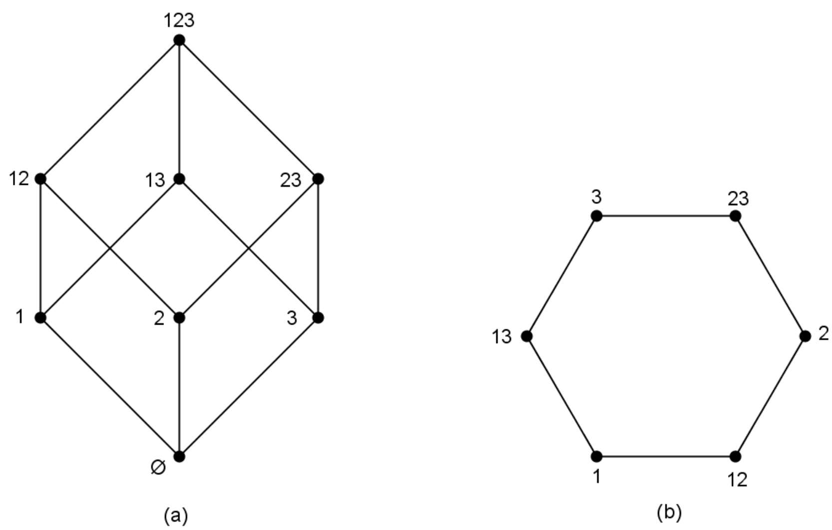

Figure 1.4: The Hasse diagram of the Boolean algebra on three elements and its order complex.

given by

$$\widetilde{\chi}(\Delta) = -f_{-1} + f_0 - f_1 + f_2 - \cdots = \chi(\Delta) - 1.$$

Therefore, if c_i denotes the chains of length i in the order complex $\Delta(P)$, we have

$$\widetilde{\chi}(\Delta(P)) = c_0 - c_1 + c_2 - \cdots = \mu(P),$$

where the second equality is a classic theorem due to Philip Hall [47].

Another important poset is the *face poset* of a simplicial complex Δ, denoted $\mathcal{F}(\Delta)$. Its vertex set is the set of faces of the complex with order defined by inclusion. Note that order complexes and face posets are related. Given a simplicial complex Ψ, the order complex of the face poset of Ψ, $\Delta(\mathcal{F}(\Psi))$, produces the barycentric subdivision of the original complex Ψ.

The theory of posets covers a great deal of topics. For the early work in poset and lattice theory, see the work of Birkhoff [8]. For a contemporary treatment of posets, including theory related to the Möbius function and additional references, see [69, Chapter 3].

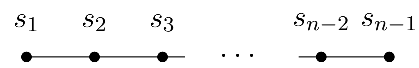

Figure 1.5: The Coxeter graph of the symmetric group $\mathfrak{S}_n$.

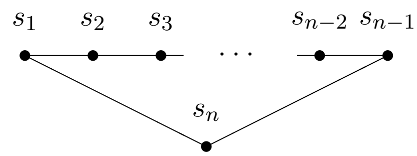

Figure 1.6: The Coxeter graph of the affine symmetric group $\widetilde{\mathfrak{S}}_n$.

1.4 Coxeter groups

A group W is a *Coxeter group* if it has a set of generators S whose only relations are of the form $(ss')^{m(s,s')} = 1$ for all $s, s' \in S$ where $m(s, s')$ is a (possibly infinite) positive integer and $m(s, s) = 1$. The group along with the generating set is known as the *Coxeter system*, (W, S).

A Coxeter group is frequently represented by its *Coxeter graph* (or *Dynkin diagram*) whose vertices are the elements of the generating set S and whose edges are the pairs $\{s, s'\}$ such that $m(s, s') \geq 3$. If $m(s, s') \geq 4$ or $m(s, s') = \infty$, the edge in the graph is labeled by $m(s, s')$. A Coxeter system is said to be *irreducible* if the Coxeter graph is connected.

The standard example of a Coxeter group is the Coxeter group of type A_{n-1}. This is isomorphic to the symmetric group $\mathfrak{S}_n$. Its generating set is $S = \{s_1, s_2, \ldots, s_{n-1}\}$ where s_i is the adjacent transposition $(i, i+1)$. For example, the permutation 2314 is obtained by $s_2 s_1(1234)$. We see that $s_i s_j = s_j s_i$ for $|i - j| \geq 2$. That is, $m(s_i, s_j) = 2$ for $|i - j| \geq 2$. However, we see that $m(i, i+1) = 3$. It remains to show that there are no relations among the generators other than the Coxeter relations. The Coxeter graph of $\mathfrak{S}_n$ is given in Figure 1.5.

The *length* of an element w in a Coxeter group, denoted $\ell(w)$, is the minimum number of generators required to express w as a product of generators. The *Poincaré series* (or polynomial if W is finite) of a Coxeter group is the length generating function, that is, $W(q) = \sum_{w \in W} q^{\ell(w)}$. The following theorem, found in [11, Chapter 7], gives a formula for the Poincaré series of finite Coxeter groups.

Theorem 1.4.1. *Let* (W, S) *be a finite irreducible Coxeter system, and let* $n = |S|$. *Then there exists positive integers* $e_1, e_2, \ldots, e_n$*, called the* exponents of (W, S)*, such that the Poincaré series* $W(q)$ *is given by the product*

$$W(q) = \prod_{i=1}^{n} [e_i + 1]_q.$$

For the symmetric group $\mathfrak{S}_n$, the exponents are $1, 2, \ldots, n-1$. Therefore, since the inversion statistic on the symmetric group is equal to the length, see [11, Proposition 1.5.2], Theorem 1.4.1 implies Equation (1.5).

Every finite Coxeter group can be viewed as a reflection group, that is, a group generated by reflections in certain hyperplanes in Euclidean space. For the symmetric group, these hyperplanes are $x_i = x_{i+1}$ for $1 \leq i \leq n-1$ in $\mathbb{R}^n$. A special class of these reflection groups are the Weyl groups of root systems. These groups arise naturally in the study of Lie algebras. A finite subset Φ of $\mathbb{R}^n - \{0\}$ is called a *crystallographic root system* if it spans $\mathbb{R}^n$ and for all elements α and β of Φ, the following hold.

1. The only integral multiple of α in Φ are α and $-\alpha$.

2. Φ is closed under reflection through the hyperplane orthogonal to α.

3. The reflection of β through the hyperplane orthogonal to α can be computed by adding an integral multiple of α to β.

See [11, Chapter 1] and [50] for more about root systems. Given a root system Φ, the *Weyl group* is the group generated by reflections about the hyperplanes orthogonal to α for all α in Φ. As an example, consider $\Phi = \{\mathbf{e}_i - \mathbf{e}_j \;:\; 1 \leq i, j \leq n, i \neq j\}$, the root system of type A_n. Most of the finite, irreducible Coxeter groups are also Weyl groups. The only exceptions are H_3, H_4, and $I_2(m)$. These root systems also give rise to a family of polytopes. The *root polytope* of a root system Φ is given by the convex hull of the elements of Φ. For an example of a root polytope, see Figure 2.3.

To each of these finite Weyl groups is an associated affine Weyl group that results from adding a reflection in an affine hyperplane. This makes the group infinite. The hyperplane added to create the affine symmetric group, denoted $\widetilde{A}_n$ or $\widetilde{\mathfrak{S}}_n$, is $x_1 = x_n + 1$. The Coxeter graph of the affine symmetric group is given in Figure 1.6.

The Poincaré series for affine Weyl groups have been completely determined by Bott. The following theorem is due to him, and can be found in [11, 50, 56].

Theorem 1.4.2 (Bott)**.** *Let (W, S) be an affine Weyl group, and let $e_1, e_2, \ldots, e_n$ be the exponents of the corresponding finite group. Then the Poincaré series of W is given by the rational function*

$$W(q) = \prod_{i=1}^{n} \frac{[e_i + 1]_q}{1 - q^{e_i}}.$$

Using the exponents for the symmetric group $\mathfrak{S}_n$, $1, 2, \ldots, n-1$, this implies Theorem 1.1.1.

The affine symmetric group also has a more combinatorial description given by Lusztig [57]. A bijection $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}$ is an element of $\widetilde{\mathfrak{S}}_n$ if and only if it satisfies the following two properties:

1. $\pi(i + n) = \pi(i) + n$ for all integers i.
2. $\sum_{i=1}^{n} (\pi(i) - i) = 0$.

Thus, we can think of these bijections as "infinite," but periodic, permutations.

The symmetry group of regular polytopes is another example of a Coxeter group and were studied long before the formal definition of Coxeter group was given. However, the formal study of Coxeter groups began with Coxeter [31] and Witt [75], where the finite irreducible Coxeter groups were classified. Coxeter groups show up in a many different contexts, including combinatorics, algebra, and geometry. For an introduction to Coxeter and reflection groups in general, see [50]. The combinatorial properties of Coxeter groups can be found in [11].

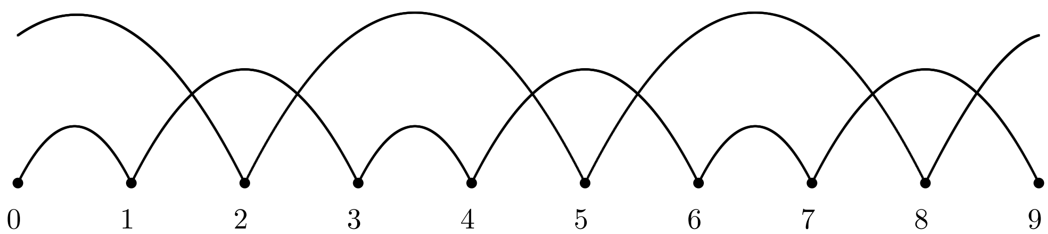

Figure 1.7: The two ball juggling pattern $\mathbf{a}$ = (1,2,3).

1.5 Juggling

The mathematics of juggling began to receive serious attention around 1994 when the paper by Buhler, Eisenbud, Graham, and Wright [18] was published. Since then, it has generated quite a following. See, for example, [19, 23, 62, 66]. Most of the terminology from this section can be found in [38].

Juggling patterns can be viewed geometrically, see Figure 1.7. The horizontal axis represents time, and at each point one juggling ball is caught and thrown. Such a juggling pattern is called *simple*. The arcs of this figure represent the paths of the juggling balls. We can see that the balls thrown at time steps 0, 3, 6, etc. are thrown high enough to land one time unit later. Likewise, the balls thrown at time steps 1, 4, 7, etc. and 2, 5, 8, etc. are thrown high enough to land two and three times units later, respectively. This is a periodic pattern with period $d = 3$. Since there are two infinite paths, this is a juggling pattern with two balls. This pattern is described by the vector $\mathbf{a} = (1, 2, 3)$. That is, for $0 \leq i \leq d-1$, the ball thrown at time $i \mod d$ is thrown so that it lands a_i time units later.

Buhler, Eisenbud, Graham, and Wright [18] showed that the number of simple juggling patterns of period d and at most n balls is equal to n^d. Ehrenborg and Readdy [38] generalized this result in two ways. They included juggling patterns where the juggler is allowed to catch and thrown more than one ball at any given time step, called a *multiplex*, and provided a q-analogue of the result. We see that the paths of the juggling balls in Figure 1.7 cross between time steps 1 and 2; this is called a *crossing*. Another crossing occurs between time steps 2 and 3. Due to the periodic nature of the juggling pattern, crossing occur later in the pattern as well. However, in a single period of this pattern, there are two crossings. Using q to the number of crossings as a weight, we get the following q-analogue result.

Theorem 1.5.1 (Ehrenborg, Readdy)**.** *The sum of the weight of simple juggling sequences, with period d and at most n balls is equal to*

$$[n]^d = (1 + q + q^2 + \cdots + q^{n-1})^d.$$

Using this result, they were then able to give an elementary computation of the Poincaré series of the affine Weyl group $\widetilde{A}_{d-1}$.

1.6 Topological tools

Topology is the study of topological spaces and invariants of these spaces. The most well-known invariants are the homology and cohomology groups of a space which assign an algebraic structure to each space. Homology will be briefly defined later in this section. However, there are spaces with the same homology groups that are not homeomorphic, that is, homology can not distinguish between these spaces. This motivates the notion of homotopy equivalence; it is a finer invariant than homology. That is, if two topological spaces are homotopy equivalent then they have the same homology groups, but the reverse is not true.

A standard example of a topological space is the n-dimensional sphere, $\mathbb{S}^n$, which is the collection of points of distance one from the origin in the $(n+1)$-dimensional Euclidean space. That is, the sphere is given by

$$\mathbb{S}^n = \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \ : \ \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

The *wedge sum* of two connected topological spaces X and Y is a way to create a new space. Let x and y be elements of X and Y, respectively. Then the wedge

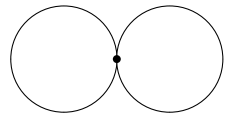

Figure 1.8: The wedge of spheres $\mathbb{S}^1 \vee \mathbb{S}^1$.

$X \vee Y$ is the space created by identifying the points x and y to the same point. See Figure 1.8. In Chapter 4, we extend this notion so that the wedge sum with the sphere $\mathbb{S}^0$ is defined. That is, given a space X, the wedge $X \vee \mathbb{S}^0$ is obtained by adding an isolated point to the space X. For more about wedge sums, see [48, Chapter 0].

A *CW complex* or *cell complex* is an example of a topological space. It is a collection of *cells*, or topological spaces homeomorphic to a closed ball, along with maps describing how the cells are glued together. The technical definition of a *CW* complex along with many of their properties can be found in [48, Appendix].

Two continuous functions $f, g : X \to Y$ are *homotopic* if there exists a continuous function $h : X \times [0,1] \to Y$ such that for $x \in X$, $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Two topological spaces X and Y are homotopic if there exists two continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity on Y and $g \circ f$ is homotopic to the identity on X. Here, f and g are called *homotopy equivalences.*

Slightly weaker than homotopy is the idea of homology. Given an n-dimensional simplicial complex Δ, define A_k as the Abelian group generated by the k-dimensional faces of Δ. Let $\partial_k : A_k \to A_{k-1}$ be the map that sends a face to its boundary, that is,

$$\partial_k(\{x_1, \ldots, x_{k+1}\}) = \sum_{i=1}^{k+1} (-1)^i \{x_1, \ldots, \widehat{x}_i, \ldots, x_{k+1}\},$$

where $\widehat{x}_i$ means that the vertex x_i is deleted. Observe that applying the boundary map twice gives us zero, that is, $\partial_i \circ \partial_{i+1} = 0$. Using this collection of sets and maps, we can form a *chain complex*,

$$0 \longrightarrow A_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} A_0 \longrightarrow 0.$$

Since $\partial_i \circ \partial_{i+1} = 0$ we have $\mathrm{im}(\partial_{i+1}) \subseteq \ker(\partial_i)$. Using this containment, the *kth homology group* is defined as

$$H_k(\Delta) = \frac{\ker(\partial_i)}{\mathrm{im}(\partial_{i+1})}.$$

The *Betti numbers*, denoted β_k, of a space Δ are defined as the rank of the homology groups, that is, β_k is the rank of $H_k(\Delta)$.

Closely related to homology is *reduced homology.* For reduced homology, the group $A_{-1} \cong \mathbb{Z}$ corresponding to the span of the empty set is added to the chain complex, giving

$$\cdots A_0 \xrightarrow{\partial_0} A_{-1} \longrightarrow 0.$$

Therefore, the reduced homology groups $\widetilde{H}_k$ of a space are equal to the homology groups for all $k \neq 0$. However, H_0 and $\widetilde{H}_0$ differ by a rank of one. The *reduced Betti numbers*, $\widetilde{\beta}_k$ are then the Betti numbers associated to the reduced homology. Intuitively, the reduced Betti numbers of a space counts the number of "holes." That is, for a sphere $\mathbb{S}^n$, we have $\widetilde{\beta}_n = 1$ and $\widetilde{\beta}_k = 0$ for $0 \leq k \leq n-1$. More generally, we have the following proposition.

Proposition 1.6.1. *Let X be a wedge of spheres containing α_k spheres of dimension k. Then the reduced Betti number $\widetilde{\beta}_k$ of X is given by α_k.*

The Betti numbers are also related to the Euler characteristic defined in Section 1.3. Namely, for a simplicial complex Δ, we have

$$\chi(\Delta) = \beta_0 - \beta_1 + \beta_2 - \cdots ,$$

and

$$\widetilde{\chi}(\Delta) = \widetilde{\beta}_0 - \widetilde{\beta}_1 + \widetilde{\beta}_2 - \cdots .$$

These relations show that the Euler characteristic is invariant under homotopy equivalences.

1.7 Discrete Morse theory

Discrete Morse theory is a technique developed by R. Forman used to simplify the topology of a simplicial complex while preserving its homotopy type. This is done by describing the given simplicial complex as a CW complex. Discrete Morse theory is a combinatorial extension of Morse theory, a technique used to analyze the topology of a manifold. Since its development in the late nineties, discrete Morse theory has been used to determine the topology of simplicial complexes that arise from combinatorial objects.

For example, the *independence complex* of a graph G, $\mathrm{Ind}(G)$ is the set of independent sets of vertices of G, that is, sets I such that for $u, v \in I$, (u, v) is not an edge of G. Discrete Morse theory can be used to show the following for chains on n elements, L_n, and n-cycles, C_n, see [53].

Theorem 1.7.1 (Kozlov)**.** *The independence complex of a path, respectively a cycle, is given by*

$$\mathrm{Ind}(L_n) \simeq \begin{cases} \mathbb{S}^{k-1} & \text{if } n = 3k, \\ \text{point} & \text{if } n = 3k+1, \\ \mathbb{S}^{k} & \text{if } n = 3k+2, \end{cases}$$

$$\mathrm{Ind}(C_n) \simeq \begin{cases} \mathbb{S}^{k-1} \vee \mathbb{S}^{k-1} & \text{if } n = 3k, \\ \mathbb{S}^{k-1} & \text{if } n = 3k \pm 1. \end{cases}$$

Discrete Morse theory can also be used to study the order complex of posets. The partition lattice Π_n is the poset whose elements are partitions of the set $\{1, 2, \ldots, n\}$ ordered by refinement. Björner [9] proved the following.

Theorem 1.7.2 (Björner)**.** *The simplicial complex* $\Delta(\Pi_n)$ *is homotopy equivalent to a wedge of* $(n-1)!$ *copies of* $\mathbb{S}^{n-3}$.

This result has been generalized several times, most recently by Ehrenborg and Jung in [37].

We now consider the original formulation of discrete Morse theory. We begin with definitions and theorems given by Forman [44, 45]. Given a simplicial complex Δ,

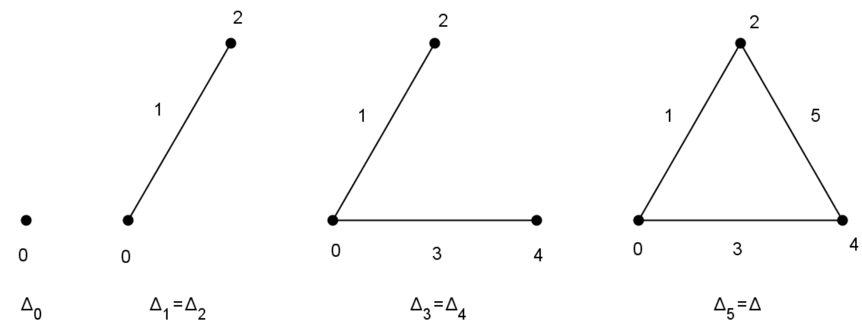

Figure 1.9: A discrete Morse function on the complex Δ along with subcomplexes obtained from the order prescribed by the function.

let $\beta \in \Delta$ be a simplex with $p+1$ vertices. Since β is a p-dimensional simplex, we denote it $\beta^{(p)}$.

Definition 1.7.3. *A function* $f : \Delta - \{\emptyset\} \longrightarrow \mathbb{R}$ *is a* discrete Morse function *if for every* $\beta^{(p)} \in \Delta$ *the following two conditions hold.*

1. $|\{\gamma^{(p+1)} \supset \beta \;:\; f(\gamma) \leq f(\beta)\}| \leq 1,$

2. $|\{\alpha^{(p-1)} \subset \beta \;:\; f(\alpha) \geq f(\beta)\}| \leq 1.$

If both of these are zero, the simplex β *is said to be* critical*.*

For an example of a discrete Morse function, see the complex Δ in Figure 1.9.

Thus, we get the following theorem.

Theorem 1.7.4 (Forman)**.** *Suppose* Δ *is a simplicial complex with a discrete Morse function. Then* Δ *is homotopy equivalent to a CW complex with exactly one cell of dimension* p *for each critical simplex of dimension* p*.*

Essentially, a discrete Morse function gives an order to build the complex. See Figure 1.9. Following this order, note non-critical simplices will be added in pairs. For example, in Figure 1.9, in order to add the line segment $f^{-1}(1)$ the point $f^{-1}(2)$ must also be added. Going backwards, removing these pairs of non-critical simplices is known as a simplicial collapse. A *simplicial collapse* is the removal of two simplices

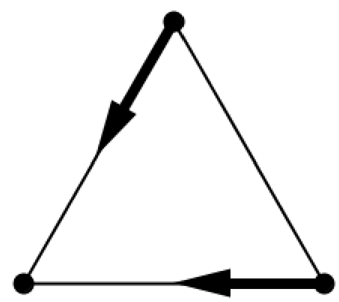

Figure 1.10: The gradient vector field on Δ associated to the discrete Morse function given in Figure 1.9.

$\alpha^{(p-1)} \subset \beta^{(p)}$ such that β is a facet of Δ and β is the only simplex containing α. It is straightforward to see that performing a simplicial collapse does not change the homotopy type of the complex. Therefore, the homotopy type of the complex only changes when a critical simplex is added.

In general, finding a discrete Morse function can be tedious. However, based on the discussion in the previous paragraph, we are simply looking for pairs of simplices. Therefore, Forman gave the following definitions.

Definition 1.7.5. *A* gradient vector field V *on* Δ *is a collection of pairs of simplices of* Δ, $\{\alpha^{(p)} \subset \beta^{(p+1)}\}$, *such that each simplex is in at most one pair of* V. *A* V-path *is a sequence of simplices*

$$\alpha_0^{(p)} \subset \beta_0^{(p+1)} \supset \alpha_1^{(p)} \subset \cdots \subset \beta_r^{(p+1)} \supset \alpha_{r+1}^{(p)},$$

such that, for all $i = 0, 1, \ldots, r$, *the pair* $\{\alpha_i, \beta_i\}$ *is an element of* V *and* $\alpha_i \neq \alpha_{i+1}$. *The* V*-path is said to be a* non-trivial closed path *if* $r \geq 1$ *and* $\alpha_0 = \alpha_{r+1}$.

See Figure 1.10. Finally, we get the following theorem which relates gradient vector fields to discrete Morse functions.

Theorem 1.7.6 (Forman)**.** *A discrete vector field* V *is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed* V*-paths.*

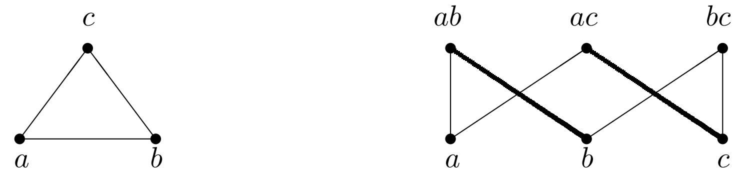

Figure 1.11: An empty triangle and an acyclic matching on its face poset. The critical 0-cell and 1-cell indicate that the triangle is homotopy equivalent to a 1-sphere.

There is a more combinatorial way to view discrete Morse theory. The following formulation is due to Chari [22] and is the way discrete Morse theory will be used in this dissertation. Further details on this formulation can be found in [53].

A *partial matching* in a poset P is a partial matching in the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ such that $(x, y) \in M$ implies $x \prec y$ and each $x \in P$ belongs to at most one element of M. For $(x, y) \in M$ we write $x = d(y)$ and $y = u(x)$, where d and u stand for down and up, respectively. A partial matching M on P is *acyclic* if there does not exist a cycle

$$z_1 \succ d(z_1) \prec z_2 \succ d(z_2) \prec \cdots \prec z_n \succ d(z_n) \prec z_1,$$

in P with $n \geq 2$ and all z_i, $i = 1, \ldots, n$ distinct. Given a partial matching, the unmatched elements are called *critical*. If there are no critical elements, the acyclic matching is *perfect*.

We now state the main result from discrete Morse theory. For a simplicial complex Δ, let $\mathcal{F}(\Delta)$ denote the face poset of Δ.

Theorem 1.7.7 (Chari)**.** *Let Δ be a simplicial complex. If M is an acyclic matching on $\mathcal{F}(\Delta) - \{\widehat{0}\}$ and k_i denotes the number of critical i-dimensional cells of Δ then the complex Δ is homotopy equivalent to a CW complex Δ_k which has k_i cells of dimension i.*

Given an acyclic matching, it is straightforward to construct a discrete Morse function by induction. At each step, select either a critical cell $\alpha^{(p)}$ or a matched pair $\alpha^{(p)} \subset \beta^{(p+1)}$ all of whose proper faces have been mapped previously by the discrete

Morse function, and assign values larger than those already used to the selected face(s). Therefore, we see that Theorem 1.7.7 is equivalent to Theorem 1.7.4.

In some cases it is possible to know exactly the homotopy type of Δ_k. For example, if Δ_k has one 0-cell and n i-cells then Δ_k is homotopy equivalent to a wedge of n i-dimensional spheres. See Figure 1.11. Additionally, when the critical cells are facets of the original complex, we get the following theorem (see Theorem 4.2.7).

Theorem 1.7.8. *Let M be a Morse matching on $\mathcal{F}(\Delta)$ such that all k_i critical cells of dimension i are facets of Δ. Then the complex Δ is homotopic equivalent to a wedge of spheres, that is,*

$$\Delta \simeq \bigvee_{i} \bigvee_{j=1}^{k_i} \mathbb{S}^i.$$

However, given a set of critical cells having different dimensions, in general it is impossible to conclude from this data that the CW complex Δ_k is homotopy equivalent to a wedge of spheres. For example, a torus has a 0-cell, two 1-cells, and a 2-cell as its cellular decomposition and this is certainly not homotopy equivalent to a wedge of 1- and 2-dimensional spheres.

Some work has been done to determine when the exact homotopy type can be determined. For example, Kozlov [54] gives a more general sufficient condition on an acyclic Morse matching for the complex to be homotopy equivalent to a wedge of spheres enumerated by the critical cells.

In order to use discrete Morse theory, it is necessary to determine when a matching is acyclic. The Patchwork theorem [53, Theorem 11.10] gives one way to construct an acyclic matching from smaller acyclic matchings.

Theorem 1.7.9. *Assume that $\varphi : P \to Q$ is an order-preserving poset map, and assume that there are acyclic matchings on the fibers $\varphi^{-1}(q)$ for all $q \in Q$. Then the union of these matchings is itself an acyclic matching on P.*

Chapter 2 Affine excedances

2.1 Introduction

The symmetric group $\mathfrak{S}_n$ has many interesting permutation statistics. The most well-known statistics are inversions, descents, excedances, and the major index. The two most classical results are the descent statistic and the excedance statistic are equidistributed, and the inversion statistic and the major index are equidistributed. The symmetric group $\mathfrak{S}_n$ is also a finite Weyl group, which is a special case of Coxeter groups. In this terminology the group is known as A_{n-1} and it is viewed as the group generated by reflections in the hyperplanes $x_i = x_{i+1}$, $1 \leq i \leq n-1$. To every finite Weyl group W there is the associated affine Weyl group $\widetilde{W}$. Geometrically this is obtained by adding one more generator to the group corresponding to a reflection in an affine hyperplane which makes the group infinite. In the case of the symmetric group the affine hyperplane is $x_1 = x_n + 1$ and the group is denoted by $\widetilde{A}_{n-1}$.

Lusztig [57] found the following combinatorial description of the affine Weyl group $\widetilde{A}_{n-1}$. Define an affine permutation π to be a bijection $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying the following two conditions:

$$\pi(i+n) = \pi(i) + n \quad \text{for all } i, \quad \text{and} \quad \sum_{i=1}^{n} (\pi(i) - i) = 0. \tag{2.1}$$

Let $\widetilde{\mathfrak{S}}_n$ be the set of all affine permutations. It is straightforward to see that $\widetilde{\mathfrak{S}}_n$ forms a group under composition. Lusztig then asserts that the group of affine permutations $\widetilde{\mathfrak{S}}_n$ and the affine Weyl group $\widetilde{A}_{n-1}$ are isomorphic. Viewing $\widetilde{\mathfrak{S}}_n$ as a Coxeter group, it has n generators $s_1, \ldots, s_n$ given by

$$s_i(j) = \begin{cases} j+1 & \text{if } j \equiv i \bmod n, \\ j-1 & \text{if } j \equiv i+1 \bmod n, \\ j & \text{otherwise.} \end{cases}$$

Furthermore, the Coxeter relations when $n \geq 3$ are

$$s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1, \quad \text{and} \quad (s_i s_j)^2 = 1 \text{ for } i - j \not\equiv -1, 0, 1 \bmod n,$$

where we view the indices modulo n. For $n = 2$ the relations are $s_1^2 = s_2^2 = 1$ and there is no relation between s_1 and s_2. Observe that the symmetric group $\mathfrak{S}_n$ is embedded in the group of affine permutations. We can view the symmetric group as generated by the reflections $s_1, \ldots, s_{n-1}$.

Shi [65] and Björner and Brenti [10] were the first to study the group of affine permutations $\widetilde{\mathfrak{S}}_n$. Björner and Brenti extended the inversion statistic from the symmetric group $\mathfrak{S}_n$ to the group of affine permutations $\widetilde{\mathfrak{S}}_n$ by defining

$$\operatorname{inv}(\pi) = \sum_{1 \leq i < j \leq n} \left| \left\lfloor \frac{\pi(j) - \pi(i)}{n} \right\rfloor \right|, \text{ for } \pi \in \widetilde{\mathfrak{S}}_n. \tag{2.2}$$

Recall that the length of an element in a Coxeter group is given by the minimum number of generators required to express the group element as a product of generators. Shi showed that the inversion number is equal to the length of the affine permutation [65].

The next step is to look at the distribution of the inversions statistic, i.e., the length. Bott's formula, Theorem 1.4.2, solves this for any affine Weyl group in terms of the exponents of the finite group [56]. In the A_{n-1} case one has

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} = \frac{1 - q^n}{(1 - q)^n}.$$

Björner and Brenti gave a combinatorial proof of this generating function identity by finding a bijection between $\widetilde{\mathfrak{S}}_n$ and $\mathbb{N}^n - \mathbb{P}^n$, that is, the collection of n-tuples with at least one zero. An earlier combinatorial proof was given by Ehrenborg and Readdy [38] using juggling sequences.

In this paper we will refine the Ehrenborg–Readdy juggling approach to give a proof of the length distribution of the coset representatives of the parabolic subgroup. We then extend the excedance statistic from the symmetric group $\mathfrak{S}_n$ to affine permutations. It is well-known that the generating polynomial for the excedance statistic is given by the Eulerian polynomial, that is,

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{exc}(\pi)} = \sum_{k=0}^{n-1} A(n, k+1) q^k,$$

where $A(n,k)$ is the Eulerian number. For affine permutations we determine the associated generating function for the excedance statistic. In doing so we reformulate the problem to instead counting lattice points on the hyperplane $x_1 + \cdots + x_n = 0$ which are certain distances in the L^1-norm from the point $(-1, \ldots, -1, 0, \ldots, 0)$. The proof involves working with the $(n-1)$-dimensional root polytope R_{n-1} of type A defined as the convex hull of the vectors $\mathbf{v}_{i,j} = \mathbf{e}_i - \mathbf{e}_j$ for $1 \leq i, j \leq n$. The Ehrhart series for the root polytope is given by

$$\mathrm{Ehr}(R_{n-1}, t) = \frac{\sum_{i=0}^{n-1} \binom{n-1}{i}^2 t^i}{(1-t)^n}; \tag{2.3}$$

see [2, 21, 59, 60].

Recently Ardila, Beck, Hosten, Pfeifle, and Seashore gave a natural triangulation of the root polytope and a combinatorial description of this triangulation [1]. Unfortunately the excedance statistic requires us to work with a skew version of the root polytope, but the triangulation still applies to the skew root polytope.

The faces of the Ardila–Beck–Hosten–Pfeifle–Seashore triangulation are in a natural correspondence with a combinatorial structure they named staircases. For a quick example of a staircase, see Figure 2.2. Ardila et al. enumerated the number of staircases and used this to give a combinatorial proof of the Ehrhart series of the root polytope. In order to count lattice points inside the skew root polytope we use an additional parameter ℓ corresponding to an extra additive term in the dilation of the associated simplex in the triangulation of the skew root polytope; compare Figures 2.4 and 2.5. This extra condition requires the associated staircase to have ℓ of the first k columns non-empty. Completing this enumeration allows us to count the lattice points inside the skew root polytope and determine the generating function for the affine excedance set statistics.

This chapter also appears in [26].

2.2 Coset representatives and increasing juggling patterns

The affine symmetric group $\widetilde{\mathfrak{S}}_n$ consists of all bijections satisfying the relations in equation (2.1). This is a group under composition. From the first condition observe that the affine permutation π is uniquely determined by the entries $\pi(1), \pi(2), \ldots, \pi(n)$. Moreover, since π is a bijection, it permutes the congruency classes modulo n. Hence we can write

$$\pi(i) = nr_i + \sigma(i),$$

where r_i is an integer such that $\sum_{i=1}^{n} r_i = 0$ and $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation in $\mathfrak{S}_n$. Following Björner and Brenti [10] we write this as

$$\pi = (r_1, \ldots, r_n|\sigma) = (\mathbf{r}|\sigma).$$

Observe that the embedding of the symmetric group $\mathfrak{S}_n$ in the group of affine permutations is exactly the map which sends the permutation σ to the affine permutation $(\mathbf{0}|\sigma)$.

Consider a left coset D in the quotient $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$. To pick a coset representative it is natural to choose the element π of least length in the coset D. This element π satisfies the inequalities $\pi(1) < \pi(2) < \cdots < \pi(n)$. We will study these coset representatives by considering their associated juggling sequences.

We refer the reader to the papers [18, 35, 38] and the book [62] for more on the mathematics of juggling. Here we give a brief introduction. A *juggling sequence* of period n is a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of positive integers such that $a_i + i$ are distinct modulo n. This can be viewed as a directed graph where there is an edge from t to $t + a_{t \bmod n}$ for all integers t. This symbolizes that a ball thrown at time t is caught at time $t + a_{t \bmod n}$. At each vertex of this graph the indegree and outdegree are each 1. This directed graph decomposes into connected components and each component is an infinite path. A path corresponds to a ball in the time-space continuum. The number of balls of is given by the mean value $(a_1 + \cdots + a_n)/n$; see [18].

A crossing is two directed edges $i \longrightarrow j$ and $k \longrightarrow \ell$ such that $i < k < j < \ell$; see [38]. The number of crossings $\text{cross}(\mathbf{a})$ of a juggling sequence $\mathbf{a}$ is the number of crossings such that $1 \leq i \leq n$. This extra condition implies that number of crossings is finite and we are not counting crossings that are equivalent by a shift of a multiple of the period n.

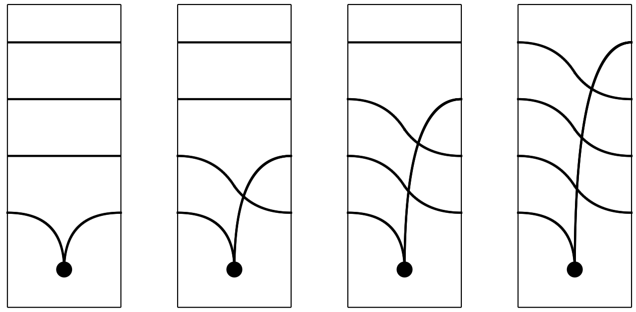

Figure 2.1: The four juggling cards C_0^*, C_1^*, C_2^* and C_3^*.

We call a juggling sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *increasing* if $a_1 \leq a_2 \leq \cdots \leq a_n$. In juggling terms this states that a ball thrown at time i is caught before the ball thrown at time j, for $1 \leq i < j \leq n$.

Theorem 2.2.1. *The sum of $q^{\mathrm{cross}(\boldsymbol{a})}$ over all increasing juggling sequences $\boldsymbol{a}$ of period n having at most m balls is given by the Gaussian coefficient*

$$\sum_{\boldsymbol{a}} q^{\mathrm{cross}(\boldsymbol{a})} = \begin{bmatrix} m+n-1 \\ n \end{bmatrix}.$$

Similarly, the sum of $q^{\mathrm{cross}(\boldsymbol{a})}$ over all increasing juggling sequences $\boldsymbol{a}$ of period n having exactly m balls is given by the Gaussian coefficient

$$\sum_{\boldsymbol{a}} q^{\mathrm{cross}(\boldsymbol{a})} = q^{m-1} \begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}.$$

Proof. We prove this using juggling cards. See Figure 2.1. Note that these juggling cards are the mirror images of the cards introduced in [38]. As in [38] by taking n juggling cards $C_{i_1}^*, C_{i_2}^*, \ldots, C_{i_n}^*$ and repeating them we construct a juggling pattern of period n having $\max(i_1, i_2, \ldots, i_n) + 1$ number of balls. However, note that with these cards it is always the ball in the lowest orbit that lands first. That is, the balls land according to their height.

If we use the cards $C_{i_1}^*, C_{i_2}^*, \ldots, C_{i_n}^*$, where $i_1 \leq i_2 \leq \cdots \leq i_n$, the ball thrown at time j will be in an orbit lower than the ball thrown at time $j+1$. Hence the ball thrown at time j will land before the next ball thrown. Hence the juggling pattern will be increasing.

Also observe that if $i_j > i_{j+1}$ then the jth ball would land after the $(j+1)$st ball and the pattern would not be increasing. Thus all increasing juggling sequences are in bijective correspondence with weakly increasing lists of indices.

Since the card C_i^* has i crossings, the sought after sum is given by

$$\sum_{\mathbf{a}} q^{\mathrm{cross}(\mathbf{a})} = \sum_{0 \le i_1 \le i_2 \le \cdots \le i_n \le m-1} q^{i_1+i_2+\cdots+i_n},$$

which is one of the combinatorial expressions for the Gaussian coefficient $\begin{bmatrix} m+n-1 \\ n \end{bmatrix}$.

To obtain the number of increasing juggling patterns having exactly m balls, we require the last card to be C_{m-1}^*. Thus the sum is restricted by the condition $i_n = m-1$, giving the factor q^{m-1}. The sum is now over $0 \le i_1 \le i_2 \le \cdots \le i_{n-1} \le m-1$ which produces the desired Gaussian coefficient. □

There is a natural bijection between juggling patterns $\mathbf{a}$ having exactly m balls and affine permutations π such that $i - \pi(i) < m$ for all i. Namely, given an affine permutation π, define the juggling pattern $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ by $a_i = m - i + \pi(i)$ for $i = 1, \ldots, n$. This states that the ball thrown at time i is caught at time $\pi(i) + m$. Furthermore, Theorem 4.2 in [38] states that the length of the affine permutation π and the number of crossings of the juggling pattern are related by

$$\ell(\pi) + \mathrm{cross}(\mathbf{a}) = n \cdot (m-1). \tag{2.4}$$

Theorem 2.2.2. *The sum $q^{\ell(\pi)}$ over all affine permutations $\pi \in \widetilde{\mathfrak{S}}_n$ such that $\pi(1) < \pi(2) < \cdots < \pi(n)$ is given by*

$$\sum_{\pi} q^{\ell(\pi)} = \frac{1}{(1-q)^{n-1}[n-1]!}.$$

Proof. Consider a coset representative π with the extra condition that $i - \pi(i) < m$ for all i. The condition $\pi(1) < \cdots < \pi(n)$ implies that π corresponds to an increasing juggling pattern having exactly m balls. By equation (2.4) and Theorem 2.2.1, we have

$$\sum_{\substack{\pi(1)<\cdots<\pi(n)\\ i-\pi(i)<m}} q^{n(m-1)-\ell(\pi)} = q^{m-1}\begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}.$$

By dividing by $q^{n(m-1)}$, substituting $q \longmapsto q^{-1}$ and using the fact that Gaussian coefficients are symmetric, we obtain

$$\sum_{\substack{\pi(1)<\cdots<\pi(n)\\ i-\pi(i)<m}} q^{\ell(\pi)} = \begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix} = \frac{[m+n-2][m+n-3]\cdots[m]}{[n-1]!}.$$

Finally by letting m tend to infinity the result follows. $\square$

Observe that this gives another evaluation of the Poincaré series of the group of affine permutations.

Corollary 2.2.3. *The Poincaré series for $\widetilde{\mathfrak{S}}_n$ is given by*

$$\sum_{\pi\in\widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} = \frac{1-q^n}{(1-q)^n}.$$

Proof. We have that

$$\begin{aligned}
\sum_{\pi\in\widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} &= \left(\sum_{\pi\in\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n} q^{\ell(\pi)}\right)\left(\sum_{\sigma\in\mathfrak{S}_n} q^{\ell(\sigma)}\right) \\
&= \frac{[n]!}{(1-q)^{n-1}[n-1]!} \\
&= \frac{1-q^n}{(1-q)^n}.
\end{aligned}$$

$\square$

The approach in Theorems 2.2.1 and 2.2.2 presents a bijection between the coset representatives and partitions λ of length at most $n-1$. Such a bijection was given in [10, Theorem 4.4]. Given a partition $\lambda = (\lambda_1 \leq \cdots \leq \lambda_{n-1})$, let $(a_1, a_2, \ldots, a_n)$ be the increasing juggling sequence defined using the juggling cards $C_0^*, C_{\lambda_1}^*, \ldots, C_{\lambda_{n-1}}^*$. Note that $a_1 = 1$, that is, this juggling sequence begins with a 1 throw. Hence we cannot subtract a positive integer from each entry to make another juggling sequence.

Let m be the number of balls of this juggling pattern, that is, $m = \lambda_{n-1} + 1$. Now construct the affine permutation by $\pi(i) = a_i + i - m$ for $1 \leq i \leq m$. The inverse of this bijection is given by letting $m = 2 - \pi(1)$ and $a_i = \pi(i) - i + m$. Then the partition is obtained by determining which juggling cards are used to create the juggling sequence $(a_1, \ldots, a_n)$.

This bijection differs from the one given by Björner and Brenti [10]. Their bijection has the extra advantage that the entries of the partition record inversions of the affine permutation.

2.3 Affine excedances

Recall that for a permutation $\sigma \in \mathfrak{S}_n$, an excedance of σ is an index i such that $i < \sigma(i)$. The *excedance statistic* of σ is the number of excedances, that is,

$$\operatorname{exc}(\sigma) = |\{i \in [n] \ : \ i < \sigma(i)\}|.$$

Observe that a permutation has at most $n-1$ excedances. The number of permutations in $\mathfrak{S}_n$ with k excedances is given by the Eulerian number $A(n, k+1)$.

Definition 2.3.1. *For an affine permutation $\pi \in \widetilde{\mathfrak{S}}_n$ define the excedance statistic by*

$$\operatorname{exc}(\pi) = \sum_{i=1}^{n} \left| \left\lceil \frac{\pi(i) - i}{n} \right\rceil \right|.$$

Observe that this definition of excedances agrees with the classical definition on the symmetric group, that is, for a permutation σ we have that $\operatorname{exc}((\mathbf{0}|\sigma)) = \operatorname{exc}(\sigma)$.

In order to analyze the distribution of the excedance statistic, we need to introduce a few notions. The L^1-norm of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is given by

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

The n-dimensional crosspolytope is given by $\lozenge_n = \operatorname{conv}\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\}$, where the vector $\mathbf{e}_i$ is the ith standard unit vector in $\mathbb{R}^n$. Note that $\partial\lozenge_n$ is the unit sphere in the L^1-norm. Let H_n be the hyperplane in $\mathbb{R}^n$ defined by

$$H_n = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \;:\; \sum_{i=1}^{n} x_i = 0 \right\}.$$

and let L_n be the lattice $L_n = H_n \cap \mathbb{Z}^n$.

We now reformulate the excedance statistic of an affine permutation.

Proposition 2.3.2. *For $\sigma \in \mathfrak{S}_n$, define the vector $\boldsymbol{p}_\sigma \in \{-1, 0\}^n$ by $p_\sigma(i) = -1$ if i is an excedance of σ and 0 otherwise. Then for an affine permutation $\pi \in \widetilde{\mathfrak{S}}_n$ with $\pi = (r_1, \ldots, r_n|\sigma) = (\boldsymbol{r}|\sigma)$, the excedance statistic is given by*

$$\mathrm{exc}(\pi) = \|\boldsymbol{r} - \boldsymbol{p}_\sigma\|_1.$$

Proof. For $1 \le i \le n$ we have that

$$\begin{aligned} \left| \left\lceil \frac{\pi(i) - i}{n} \right\rceil \right| &= \left| \left\lceil \frac{nr_i + \sigma(i) - i}{n} \right\rceil \right| \\ &= \begin{cases} |r_i| & \text{if } i \ge \sigma(i), \\ |r_i + 1| & \text{if } i < \sigma(i). \end{cases} \end{aligned}$$

That is, we get this "+1" wherever we have an excedance in the permutation σ. The result follows by summing over all i. $\square$

The next lemma expresses the generating function of affine excedances in terms of Eulerian numbers and generating functions of distances.

Lemma 2.3.3. *Let $\boldsymbol{p}_k$ be the lattice point $(\underbrace{-1, \ldots, -1}_{k}, 0, \ldots, 0)$ in $\mathbb{R}^n$. Then the following identity holds:*

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\mathrm{exc}(\pi)} = \sum_{k=0}^{n-1} A(n, k+1) \sum_{\boldsymbol{r} \in L_n} q^{\|\boldsymbol{r} - \boldsymbol{p}_k\|_1}.$$

Proof. Let σ be a permutation with k excedances. By permuting the coordinates of the vector $\mathbf{r}$, we have that

$$\sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r} - \mathbf{p}_\sigma\|_1} = \sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r} - \mathbf{p}_k\|_1}.$$

Since there are $A(n, k+1)$ permutations with k excedances, the lemma follows. $\square$

This lemma reduces the problem of determining the number of affine permutations with i excedances to computing the number of points in the lattice L_n at distance i from the points $\mathbf{p}_0, \dots, \mathbf{p}_{n-1}$. We begin by noting the following lemma.

Lemma 2.3.4. *For $\boldsymbol{r} \in L_n$ and $0 \le k \le n-1$ we have that $\|\boldsymbol{r} - \boldsymbol{p}_k\|_1 \ge k$ and $\|\boldsymbol{r} - \boldsymbol{p}_k\|_1 \equiv k \bmod 2$.*

Proof. For an integer x we have that $|x| \ge x$, hence

$$\|\mathbf{r} - \mathbf{p}_k\|_1 = \sum_{i=1}^{n} |r_i - p_i| \ge \sum_{i=1}^{n} (r_i - p_i) = k.$$

Also observe that $|x|$ and x have the same parity. That is $|x| \equiv x \bmod 2$. Therefore,

$$\|\mathbf{r} - \mathbf{p}_k\|_1 = \sum_{i=1}^{n} |r_i - p_i| \equiv \sum_{i=1}^{n} (r_i - p_i) = k \bmod 2.$$

□

This lemma tells us that for $0 \le k \le n-1$, the boundary of the crosspolytope centered at $\mathbf{p}_k$ will not intersect lattice points in L_n until its kth dilation and then only every other integer dilation after that. Thus we are interested in lattice points $\mathbf{r}$ at distance $2t + k$ from $\mathbf{p}_k$, where t is a non-negative integer. Therefore, we define the following polytope which will be the focus of our study.

Definition 2.3.5. *For non-negative integers t and k, we define $P_{t,k}$ to be the set*

$$P_{t,k} = ((2t + k)\Diamond_n + \boldsymbol{p}_k) \cap H_n.$$

That is, $\boldsymbol{x} \in P_{t,k}$ if and only if $\|\boldsymbol{x} - \boldsymbol{p}_k\|_1 \le 2t + k$ and $\boldsymbol{x} \in H_n$.

In the case $t = k = 0$ we have that $P_{0,0}$ is a point. In the other cases, $P_{t,k}$ is obtained by cutting a dilated crosspolytope with a hyperplane which is parallel to two facets of the crosspolytope. For $k \ge 1$ and $t = 0$ the hyperplane is the affine span of a facet of the crosspolytope and hence the set $P_{0,k}$ is an $(n-1)$-dimensional simplex. Finally, for $t > 0$ the hyperplane cuts the interior of crosspolytope. Hence the combinatorial type of $P_{t,k}$ in this case does not depend on the parameters t and k.

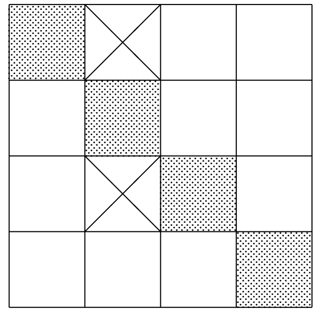

Figure 2.2: A visualization of the staircase $I = ((1,2),(3,2))$. Observe that this staircase is only contained in two other staircases, namely by adding either $(4,2)$ or $(3,4)$.

2.4 The root polytope

We begin to study the case $t = 1$ and $k = 0$. That is, we are intersecting the crosspolytope $2\lozenge_n$ with the hyperplane H_n. This is the root polytope and its structure will be used in developing the other cases. We first verify that $P_{1,0}$ is the root polytope.

Proposition 2.4.1. *The polytope $P_{1,0}$ is the $(n-1)$-dimensional root polytope R_{n-1}, that is, its vertices are $\boldsymbol{v}_{i,j} = \boldsymbol{e}_i - \boldsymbol{e}_j$ for $1 \le i, j \le n$ and $i \neq j$.*

Proof. The vertices of the crosspolytope $2\lozenge_n$ are partitioned into the two sets $\{2\mathbf{e}_i\}_{1\le i\le n}$ and $\{-2\mathbf{e}_i\}_{1\le i\le n}$ by the hyperplane H_n. Hence the edges of the crosspolytope that are cut by H_n are of the form $[2\mathbf{e}_i, -2\mathbf{e}_j]$ for $i \neq j$. The midpoint of these edges are $\mathbf{e}_i - \mathbf{e}_j = \mathbf{v}_{i,j}$, which are precisely the vertices of $P_{1,0}$. □

We introduce now the work of Ardila, Beck, Hosten, Pfeifle, and Seashore [1] who have studied the combinatorial structure of the root polytope in depth. The next definition and theorem are due to them.

Definition 2.4.2 (Ardila, et al.)**.** *We call the list $I = ((i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m))$ a* staircase *of size m in an n by n array if*

1. $1 \le i_1 \le i_2 \le \cdots \le i_m \le n$ *and* $1 \le j_1 \le j_2 \le \cdots \le j_m \le n$,

2. $(i_s, j_s) \neq (i_t, j_t)$ *for* $s \neq t$, *and*

3. $i_s \neq j_t, 1 \leq s, t \leq m$.

Let $\boldsymbol{v}_I$ *denote the set* $\boldsymbol{v}_I = \{\boldsymbol{v}_{i_1,j_1}, \boldsymbol{v}_{i_2,j_2}, \ldots, \boldsymbol{v}_{i_m,j_m}\}$.

Note that the third condition above is the essential condition. One of its implications is that the diagonal element (i, i) is not part of any staircase. In pictures we always shade these diagonal elements. Also note that if we remove a pair from a staircase, the resulting list is also a staircase. That is, the collection of staircases forms a simplicial complex. This simplicial complex is in fact spherical.

Theorem 2.4.3 (Ardila, et al.)**.** *The collection* $\{\text{conv}(\boldsymbol{v}_I)\}_I$*, where* I *ranges over all staircases in an* n *by* n *array, is a triangulation of the boundary the root polytope* R_{n-1}*, that is,* ∂R_{n-1}.

The three dimensional root polytope R_3 is the cuboctahedron, which consists of 8 triangles and 6 squares. However, in the triangulation of its boundary, each square is cut into 2 triangles. Hence the triangulation in this case has $8 + 2 \cdot 6 = 20$ facets. The staircase in Figure 2.2 corresponds to an edge of the cuboctahedron. This edge lies in two facets: the facet $\text{conv}(\mathbf{v}_{1,2}, \mathbf{v}_{3,2}, \mathbf{v}_{4,2})$, which is a triangular face of the cuboctahedron, and the facet $\text{conv}(\mathbf{v}_{1,2}, \mathbf{v}_{3,2}, \mathbf{v}_{3,4})$, which is one-half of the square face $\text{conv}(\mathbf{v}_{1,2}, \mathbf{v}_{3,2}, \mathbf{v}_{3,4}, \mathbf{v}_{1,4})$.

Definition 2.4.4. *For a staircase* $I = ((i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m))$ *let* Γ_I *be the* m*-dimensional simplex* $\text{conv}(\{\mathbf{0}\} \cup \boldsymbol{v}_I)$*. Furthermore, let* C_I *be the simplicial cone generated by the set* $\boldsymbol{v}_I$*, that is,*

$$C_I = \left\{ \sum_{s=1}^{m} \lambda_s \boldsymbol{v}_{i_s,j_s} \;\; : \;\; \lambda_s \geq 0 \right\}.$$

Theorem 2.4.3 implies that $\{C_I \; : \; I \text{ is a staircase}\}$ is a complete simplicial fan. In particular, we know that the hyperplane H_n is the disjoint union of the relative interiors of the cones C_I, that is,

$$H_n = \biguplus_I \text{relint}(C_I),$$

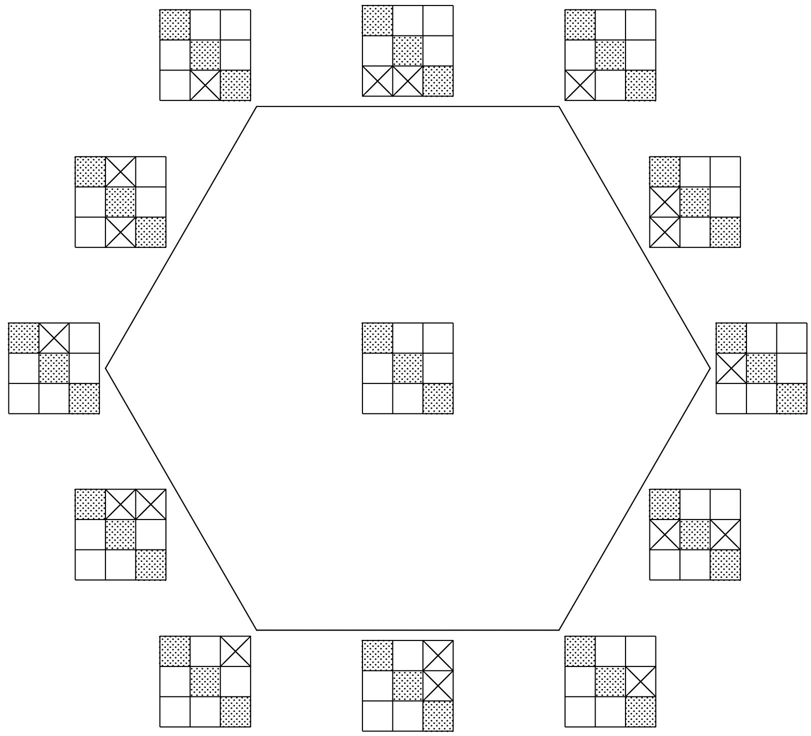

Figure 2.3: The root polytope $P_{1,0}$ (hexagon) with faces labeled with the associated staircases.

where I ranges over all staircases. Thus, for every lattice point $\mathbf{w} \in L_n$ we know that $\mathbf{w}$ is contained in the relative interior of one cone C_I for exactly one staircase $I = ((i_1, j_1), \ldots, (i_m, j_m))$. In that case we may write

$$\mathbf{w} = \sum_{s=1}^{m} \lambda_s \mathbf{v}_{i_s, j_s},$$

where each λ_s is a positive integer.

2.5 The skew root polytope

We now investigate the number of lattice points contained in the polytope $P_{t,k}$.

Proposition 2.5.1. *Let $\boldsymbol{w}$ be a lattice point in the relative interior of the cone C_I, that is, $\boldsymbol{w} \in L_n \cap \operatorname{relint}(C_I)$, where I is the staircase $((i_1, j_1), \ldots, (i_m, j_m))$. In other words, we can write $\boldsymbol{w}$ as the positive linear combination $\boldsymbol{w} = \sum_{s=1}^{m} \lambda_s \boldsymbol{v}_{i_s, j_s}$. Then* *the L^1-norm between the two points $\boldsymbol{w}$ and $\boldsymbol{p}_k$ is given by*

$$\|\boldsymbol{w}-\boldsymbol{p}_k\|_1 = k - 2\ell + 2\sum_{s=1}^{m}\lambda_s,$$

where $\ell = |[k]\cap\{j_1, j_2, \ldots, j_m\}|$, that is, ℓ is the number of non-empty columns among the first k columns of the staircase I.

Proof. Consider the difference

$$\begin{aligned}\mathbf{w}-\mathbf{p}_k &= \sum_{s=1}^{m}\lambda_s(\mathbf{e}_{i_s}-\mathbf{e}_{j_s}) + \sum_{i=1}^{k}\mathbf{e}_i \\ &= \sum_{r=1}^{k}\left(\begin{cases}\lambda_s+1 & \text{if } r=i_s,\\ -\lambda_s+1 & \text{if } r=j_s,\\ 1 & \text{otherwise}\end{cases}\right)\cdot\mathbf{e}_r + \sum_{r=k+1}^{n}\left(\begin{cases}\lambda_s & \text{if } r=i_s,\\ -\lambda_s & \text{if } r=j_s,\\ 0 & \text{otherwise}\end{cases}\right)\cdot\mathbf{e}_r.\end{aligned}$$

Using that $\lambda_s \geq 1$ we have that the L^1-norm is given by

$$\|\mathbf{w}-\mathbf{p}_k\|_1 = \sum_{r=1}^{k}\begin{cases}\lambda_s+1 & \text{if } r=i_s,\\ \lambda_s-1 & \text{if } r=j_s,\\ 1 & \text{otherwise}\end{cases} + \sum_{r=k+1}^{n}\begin{cases}\lambda_s & \text{if } r=i_s,\\ \lambda_s & \text{if } r=j_s,\\ 0 & \text{otherwise.}\end{cases}$$

Each λ_s appears twice in this sum. Furthermore, there are $k-\ell$ ones and ℓ negative ones also in the sum. This proves the proposition. □

Definition 2.5.2. *For a staircase $I = ((i_1, j_1), \ldots, (i_m, j_m))$ define the apex $\boldsymbol{a}_I$ to be the sum $\boldsymbol{a}_I = \sum_{s=1}^{m}\boldsymbol{v}_{i_s,j_s}$. Furthermore, let $\Delta_{t,k}(I)$ be the simplex*

$$\Delta_{t,k}(I) = \boldsymbol{a}_I + (t-m+\ell)\Gamma_I,$$

where ℓ is defined as in Proposition 2.5.1.

Theorem 2.5.3. *The collection of simplices $\{\Delta_{t,k}(I)\}_I$, where I ranges over all staircases, partitions the lattice points of $P_{t,k}$ into disjoint sets.*

Proof. Observe that the simplex $\Delta_{t,k}(I)$ is contained in the relative interior of C_I and that the relative interiors are pairwise disjoint. Hence the simplices $\{\Delta_{t,k}(I)\}_I$ are

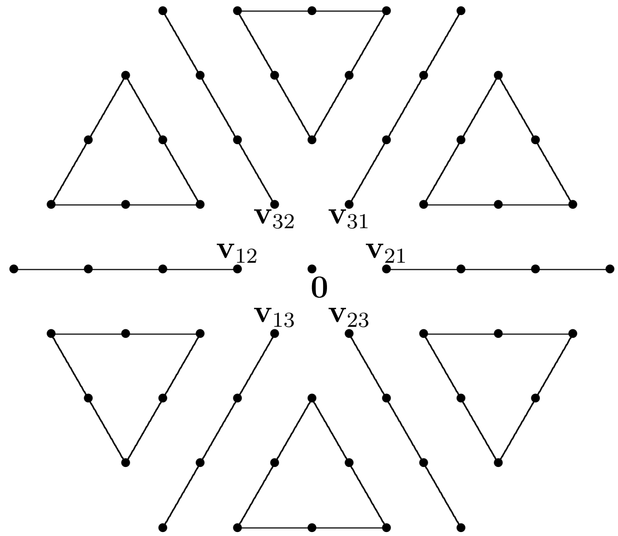

Figure 2.4: Partitioning the lattice points of $P_{t,0}$. The origin corresponds to the empty face, the six lines to the vertices, and the six triangles to the edges. Hence the number of lattice points in $P_{t,0}$ is $\binom{t}{0} + 6\binom{t}{1} + 6\binom{t}{2}$.

pairwise disjoint. Now assume that $\mathbf{w}$ is a lattice point inside the polytope $P_{t,k}$. Then there is exactly one staircase $I = ((i_1, j_1), \ldots, (i_m, j_m))$ such that $\mathbf{w} = \sum_{s=1}^{m} \lambda_s \mathbf{v}_{i_s,j_s}$, where each λ_s is a positive integer. By Proposition 2.5.1 and the definition of the polytope $P_{t,k}$, we have that

$$k - 2\ell + 2\sum_{s=1}^{m} \lambda_s = \|\mathbf{w} - \mathbf{p}_k\|_1 \leq 2t + k.$$

By cancelling k on both sides, dividing by 2 and subtracting m, we have

$$\sum_{s=1}^{m} (\lambda_s - 1) \leq t + \ell - m. \tag{2.5}$$

We can now write $\mathbf{w}$ as

$$\mathbf{w} = \sum_{s=1}^{m} \lambda_s \mathbf{v}_{i_s,j_s} = \mathbf{a}_I + \sum_{s=1}^{m} (\lambda_s - 1)\mathbf{v}_{i_s,j_s}.$$

Now by the inequality (2.5) we have $\mathbf{w} \in \mathbf{a}_I + (t + \ell - m)\Gamma_I = \Delta_{t,k}(I)$, proving that each lattice point is inside at least one simplex $\Delta_{t,k}(I)$. $\square$

Proposition 2.5.4. *Let I be a staircase of size m and let ℓ be the number of non-empty columns among the first k columns. Then the number of lattice points con-*

tained in the simplex $\Delta_{t,k}(I)$ and the number of lattice points in the simplex $\Delta_{t,k}(I)$ intersected with the boundary of $P_{t,k}$ are given by

$$\begin{aligned} |\Delta_{t,k}(I) \cap L_n| &= \binom{t+\ell}{m}, \\ |\Delta_{t,k}(I) \cap \partial P_{t,k} \cap L_n| &= \binom{t+\ell-1}{m-1}. \end{aligned}$$

Proof. We know by definition that $\Delta_{t,k}(I)$ is the $(t-m+\ell)$-dilation of a standard m-simplex. It is well-known that the number of lattice points contained in this dilation is $\binom{m+(t-m+\ell)}{m} = \binom{t+\ell}{m}$. Furthermore, for a lattice point $\mathbf{w}$ to be on the boundary of $P_{t,k}$, there is equality in inequality (2.5). Hence $\Delta_{t,k}(I) \cap \partial P_{t,k}$ is the $(t-m+\ell)$-dilation of a standard $(m-1)$-simplex and the result follows. $\square$

We are using the convention that $\binom{n}{-1} = \delta_{n,-1}$, so that Proposition 2.5.4 also holds for the empty staircase. Note that this convention agrees with the formal power series $\sum_{t \geq 0} \binom{t+m}{m} x^t = \frac{1}{(1-x)^{m+1}}$.

Combining Theorem 2.5.3 and Proposition 2.5.4 we have the following result.

Proposition 2.5.5. *The number of lattice points in the polytope $P_{t,k}$ and on its boundary $\partial P_{t,k}$ are given by*

$$\sum_{I} \binom{t+\ell}{m}, \qquad \textit{respectively,} \qquad \sum_{I} \binom{t+\ell-1}{m-1},$$

where I ranges over all staircases in an n by n array, m is the size of the staircase I and ℓ is the number of non-empty columns among the first k columns of I.

Example 2.5.6. We can visualize Proposition 2.5.5 as follows. Consider the case when $n = 3$, that is, the associated root polytope is a hexagon; see Figure 2.3. First we view the case $k = 0$. The partitioning of the lattice points in the polytope $P_{t,0}$ is shown in Figure 2.4. In this case each simplex $\Delta_{t,0}(I)$ contains $\binom{t}{m}$ lattice points. Next consider the case $k = 2$; see Figure 2.5. Going from the simplices $\Delta_{t,0}(I)$ to the simplices $\Delta_{t,2}(I)$ observe that some of them have been stretched by an additive

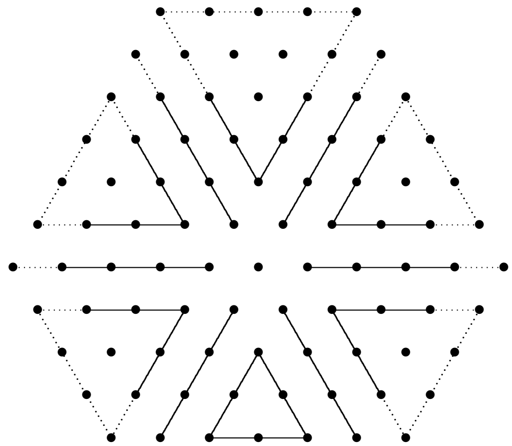

Figure 2.5: Partitioning the lattice points of $P_{t,2}$. Observe that the number of lattice points is $\binom{t}{0} + 2 \cdot \binom{t}{1} + 4 \cdot \binom{t+1}{1} + \binom{t}{2} + 4 \cdot \binom{t+1}{2} + \binom{t+2}{2}$.

term of ℓ. By comparing the stretching factor with the labels in Figure 2.3, we note that this additive term ℓ is exactly the number of non-empty columns among the first $k = 2$ columns of the staircase diagram.

2.6 Enumerating staircases

Proposition 2.6.1. *The number of staircases of size m in an n by n array with ℓ of the first k columns nonempty is given by*

$$\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}.$$

Proof. We begin by assuming a staircase will go through exactly a rows and b columns in the n by n array. We must first choose the ℓ of the first k columns that will be used in $\binom{k}{\ell}$ ways. We pick the remaining columns in $\binom{n-k}{a-\ell}$ ways. Now picking the disjoint rows can be done in $\binom{n-a}{b}$ ways. We have to get a path from the upper left to the lower right of this a by b subarray that goes through every row and column of the subarray. We must do this in $m-1$ steps where the steps are horizontal $(1,0)$, vertical $(0,1)$ and diagonal $(1,1)$. Note that we have to cover a vertical distance of $a-1$ and a horizontal distance of $b-1$. This can only be done with $m-a$ horizontal steps, $m-b$ vertical steps and $a+b-m-1$ diagonal steps. Hence the number of possibilities is given by the trinomial coefficient $\binom{m-1}{m-a,m-b,a+b-m-1}$. Thus, the number of paths is given by

$$\sum_{a,b} \binom{k}{\ell}\binom{n-k}{a-\ell}\binom{n-a}{b}\binom{m-1}{m-a, m-b, a+b-m-1}.$$

To evaluate this sum, consider

$$\begin{aligned}
&\binom{n-1}{\ell-1}\binom{n-\ell}{n-k}\sum_{a,b}\binom{n-k}{a-\ell}\binom{n-a}{b}\binom{m-1}{m-a, m-b, a+b-m-1} \\
= \quad &\binom{n-1}{m}\binom{m-1}{\ell-1}\sum_{a}\binom{m-\ell}{a-\ell}\binom{n-a}{k-\ell}\sum_{b}\binom{m}{b}\binom{n-m-1}{n-a-b} \\
= \quad &\binom{n-1}{m}\binom{m-1}{\ell-1}\sum_{a}\binom{m-\ell}{a-\ell}\binom{n-a}{k-\ell}\binom{n-1}{n-a} \\
= \quad &\binom{n-1}{m}\binom{m-1}{\ell-1}\binom{n-1}{k-\ell}\sum_{a}\binom{m-\ell}{a-\ell}\binom{n+\ell-k-1}{n-a-k+\ell} \\
= \quad &\binom{n-1}{m}\binom{m-1}{\ell-1}\binom{n-1}{k-\ell}\binom{m+n-k-1}{n-k} \\
= \quad &\binom{n-1}{\ell-1}\binom{n-\ell}{n-k}\binom{n-1}{m}\binom{m+n-k-1}{m-\ell}.
\end{aligned}$$

The Vandermonde identity was used in second and fourth steps. The other three steps are a veritable orgy of expressing the binomial coefficients in terms of factorials and shuffling the factorials around. The result now follows by multiplying by $\binom{k}{\ell}$ and dividing by $\binom{n-1}{\ell-1}$ and $\binom{n-\ell}{n-k}$. □

Now, combining Propositions 2.5.5 and 2.6.1, we immediately obtain the following theorem.

Theorem 2.6.2. *The number of lattice points contained in the polytope $P_{t,k}$ and on its boundary $\partial P_{t,k}$ is*

$$|P_{t,k} \cap L_n| = \sum_{m=\ell}^{n-1}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}\binom{t+\ell}{m},$$

respectively,

$$|\partial P_{t,k} \cap L_n| = \sum_{m=\ell}^{n-1}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}\binom{t+\ell-1}{m-1}.$$

Lemma 2.6.3. *For non-negative integers n and k,*

$$\sum_{\ell=0}^{k}\sum_{m=\ell}^{n-1}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}x^{m-\ell}(1-x)^{n-m-1}$$
$$=\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1+k}{n-1-i}x^{i}.$$

Proof. We start with

$$\begin{aligned}
&\sum_{\ell=0}^{k}\sum_{m=\ell}^{n-1}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}x^{m-\ell}(1-x)^{n-m-1}\\
=\;&\sum_{\ell=0}^{k}\sum_{p=0}^{n-1-\ell}\binom{k}{\ell}\binom{n-1}{p}\binom{2n-2-k-p}{n-1-k+\ell}x^{n-1-\ell-p}(1-x)^{p}\\
=\;&\sum_{\ell=0}^{k}\sum_{p=0}^{n-1-\ell}\binom{k}{\ell}\binom{n-1}{p}\\
&\quad\cdot\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1-p}{n-1-k+\ell-i}x^{n-1-\ell-p}(1-x)^{p}\\
=\;&\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}x^{n-1-k-i}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-1}{i+k-\ell}\\
&\quad\cdot\sum_{p=0}^{i+k-\ell}\binom{i+k-\ell}{p}(1-x)^{p}x^{i+k-\ell-p}\\
=\;&\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1+k}{i+k}x^{n-1-k-i}\\
=\;&\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1+k}{n-1-i}x^{i}.
\end{aligned}$$

In the first equality we make the substitution $p=n-m-1$. The second equality comes from expanding the term $\binom{2n-2-k-p}{n-1-k+\ell}$ using the classical Vandermonde identity.

The third equality is the trinomial coefficient identity

$$\begin{aligned}
\binom{n-1}{p}\binom{n-1-p}{n-1-k+\ell-i} &= \binom{n-1}{p,\, n-1-k+\ell-i,\, -p+k-\ell+i} \\
&= \binom{n-1}{n-1-k+\ell-i}\binom{k-\ell+i}{p} \\
&= \binom{n-1}{i+k-\ell}\binom{i+k-\ell}{p}.
\end{aligned}$$

Also note that that the last binomial coefficient is zero for $i+k-\ell < p \leq n-1-\ell$. The fourth equality is the binomial theorem applied to $((1-x)+x)^{i+k-\ell} = 1$ followed by collapsing the sum over ℓ using the Vandermonde identity. The last step is by the substitution $i \longmapsto n-1-k-i$ and by the symmetry of the binomial coefficients. $\square$

Proposition 2.6.4. *For any* $0 \leq k \leq n-1$,

$$\sum_{\boldsymbol{r} \in L_n} q^{\|\boldsymbol{r}-\boldsymbol{p}_k\|} = \frac{1}{(1-q^2)^{n-1}} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i+k}.$$

Proof. First observe that by the substitution $t = s+m-\ell$,

$$\begin{aligned}
\sum_{t \geq 0} \binom{t+\ell-1}{m-1} q^{2t+k} &= q^{2(m-\ell)+k} \sum_{s \geq \ell-m} \binom{s+m-1}{m-1} q^{2s} \\
&= \frac{q^k}{(1-q^2)^{n-1}} q^{2(m-\ell)}(1-q^2)^{n-1-m}. \qquad (2.6)
\end{aligned}$$

Hence we have that

$$\begin{aligned}
\sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r}-\mathbf{p}_k\|} &= \sum_{t \geq 0} |\partial P_{t,k} \cap L_n| q^{2t+k} \\
&= \sum_{t \geq 0} \sum_{m=\ell}^{n-1} \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}\binom{t+\ell-1}{m-1} q^{2t+k} \\
&= \frac{q^k}{(1-q^2)^{n-1}} \sum_{m=\ell}^{n-1} \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{n-1}{m} \\
&\qquad \cdot \binom{n+m-k-1}{m-\ell} q^{2(m-\ell)}(1-q^2)^{n-1-m} \\
&= \frac{q^k}{(1-q^2)^{n-1}} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i},
\end{aligned}$$

where the third step is by equation (2.6) and the last step is Lemma 2.6.3. $\square$

Observe that Proposition 2.6.4 reduces to the Ehrhart series of the root polytope R_{n-1} in the case $k = 0$; see equation (2.3). The difference in the power of $1 - t = 1 - q^2$ comes from Proposition 2.6.4 counting lattice points on the boundary, while the Ehrhart series counts lattice points in the polytope.

Finally, combining Lemma 2.3.3 with Proposition 2.6.4, we obtain the generating function associated with the excedance statistic of affine permutations.

Theorem 2.6.5. *The generating function for affine excedances is given by*

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\mathrm{exc}(\pi)} = \frac{1}{(1-q^2)^{n-1}} \sum_{k=0}^{n-1} A(n, k+1) \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1+k}{n-1-i} q^{2i+k}.$$

2.7 Concluding remarks

We end with some open problems and directions for continued research.

Question 2.7.1. There is not much known about how classic permutation statistics generalize to affine permutations. In analogy with the definitions of the inversion and excedance statistics (equation (2.2) and Definition 2.3.1), it is natural to consider the expression

$$f(\pi) = \sum_{i=1}^{n} \left| \left\lfloor \frac{\pi(i+1) - \pi(i)}{n} \right\rfloor \right| \tag{2.7}$$

as an affine analogue of the descent statistic though it does not exactly generalize the descent statistic. However, there are only a finite number of affine permutations with a given value of the f statistic. Hence the generating function

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{f(\pi)}$$

is well-defined. Does this have a nice generating function?

Question 2.7.2. The numerator in Proposition 2.6.4 has a nice combinatorial interpretation. The coefficient of q^{2i+k} counts the number of lattice paths from $(0, 0)$ to $(n-1, n-1)$ which go through the point $(i, n-1-k-i)$. In particular, the sum of these coefficients is $\binom{2n-2}{n-1}$. Is there a more bijective reason for this interpretation?

Question 2.7.3. Observe that the numerator in Proposition 2.6.4 is symmetric. Recall that the Ehrhart series of reflexive polytopes have this property and the root polytope is reflexive. Are there other reflexive polytopes that have a skew version with a symmetric numerator?

Question 2.7.4. Proposition 2.6.1 enumerates staircases where ℓ of the first k columns are non-empty. The result is a product of three binomial coefficients. Is there a more bijective proof, which avoids all the binomial coefficient manipulations?

Question 2.7.5. Since simple juggling patterns are so closely related with the affine Weyl group $\widetilde{A}_{n-1}$, it is natural to ask for juggling interpretations for the other Weyl groups. See the paper [42] for permutation interpretations for these groups.

Question 2.7.6. Is there an excedance statistic for finite Coxeter groups in general? Bagno and Garber [3] have extended the excedance statistic to the infinite classes B_n and D_n. Furthermore, could the statistic be extended to the associated affine groups? A first step in this direction would be to consider the B_n case, that is, the group of signed permutations, and see if their excedance statistic can be extended to the affine group $\widetilde{B}_n$. Would the associated calculations involve a skew version of the root polytope of type B?

Chapter 3 The excedance algebra

3.1 Introduction

The excedance set of a permutation $\pi = \pi_1 \cdots \pi_{n+1}$ in the symmetric group $\mathfrak{S}_{n+1}$ is the set $\{i \,:\, \pi_i > i\}$. Note that this set is a subset of the set $\{1, \ldots, n\}$. In order to study the number of permutations with a given excedance set, Ehrenborg and Steingrímsson [39] introduced the equivalent notion of the excedance word, that is, $u(\pi) = u_1 \cdots u_n$ where $u_i = \mathbf{a}$ if $\pi_i \leq i$ and $u_i = \mathbf{b}$ otherwise. They denoted the number of permutations with a given excedance word u by $[u]$ and proved the following four relations:

$$\begin{aligned}
&[1] = 1,\\
&[\mathbf{a} \cdot u] = [u \cdot \mathbf{b}] = [u],\\
&[v \cdot \mathbf{ba} \cdot w] = [v \cdot \mathbf{ab} \cdot w] + [v \cdot \mathbf{a} \cdot w] + [v \cdot \mathbf{b} \cdot w].
\end{aligned}$$

It is straightforward to see that these relations allow us to compute the excedance set statistic. Inspired by the last relation, we have the following definition.

Definition 3.1.1. *Let the* excedance algebra $\mathcal{E}$ *be the non-commutative algebra* $k\langle \mathbf{a}, \mathbf{b} \rangle$ *quotiented out with the ideal generated by the element* $\mathbf{ba} - \mathbf{ab} - \mathbf{a} - \mathbf{b}$.

Notice that this algebra has a linear basis of monomials of the form $\mathbf{a}^m \mathbf{b}^n$. Thus, we will be interested in the expansion of an arbitrary monomial into this standard basis, that is, given an $\mathbf{ab}$-monomial u with m copies of $\mathbf{a}$ and n copies of $\mathbf{b}$,

$$u = \sum_{i,j} c_{i,j}(u) \cdot \mathbf{a}^{m-i} \mathbf{b}^{n-j}.$$

We will write just $c_{i,j}$ if the monomial u is clear from the context.

Other references to the excedance set statistic are [25], where an explicit expression for $[\mathbf{b}^n \mathbf{a}^m]$ and $[\mathbf{b}^n \mathbf{aba}^m]$ are given, [41], where the excedance set statistic is related to the linear coefficient of the chromatic polynomial of Ferrers graphs, and [74] where one of the many conjectures by Ehrenborg and Steingrímsson is proved. The excedance set is also explored in [71, Exercise 3.40e].

3.2 Expansion

Given an **ab**-monomial $u = u_1 \cdots u_n$, define the associated Ferrers shape $F(u)$ by the following procedure. First create a lattice walk in the plane where the ith step is a unit north step if $u_i = \mathbf{a}$ and a unit east step if $u_i = \mathbf{b}$. Complete this walk to a Ferrers shape by adding the walk where we first take all the north steps followed by all the east steps. This Ferrers shape of a word is essentially described in [70, Exercise 7.59].

Note that if $u = \mathbf{b}^n \mathbf{a}^m$ then the shape $F(u)$ is a rectangle, whereas in the other extreme $u = \mathbf{a}^m \mathbf{b}^n$, $F(u)$ is the empty shape. Furthermore, the number of boxes in the Ferrers shape $F(u)$ is the number of inversions in the word u, that is, the number of pairs of $\mathbf{b}$ and $\mathbf{a}$ where the $\mathbf{b}$ is before the $\mathbf{a}$.

Theorem 3.2.1. *Let u be an* **ab***-monomial containing m copies of* $\mathbf{a}$ *and n copies of* $\mathbf{b}$*. Consider the expansion*

$$u = \sum_{i,j} c_{i,j} \cdot \mathbf{a}^{m-i} \cdot \mathbf{b}^{n-j}.$$

Then the coefficient $c_{i,j}$ enumerates the number of ways to place i copies of $\leftarrow$ and j copies of $\uparrow$ in the Ferrers shape $F(u)$ such that

(a) All the boxes to the west of a $\leftarrow$ must be empty.

(b) All the boxes to the north of a $\uparrow$ must be empty.

Theorem 3.2.1 appears without proof in [51]. A proof using permutation tableaux is given by Corteel and Williams [30], and the bijection between permutations tableaux and alternative tableaux can be found in [73]. For completeness, we provide a direct proof of Theorem 3.2.1 below.

Note that $c_{0,0} = 1$ because of the empty placement. Moreover $c_{m,n} = 0$ since with m copies of $\leftarrow$ there is no position to place a $\uparrow$ in the first column.

Before proving this combinatorial expansion we digress with a combinatorial interpretation of the analogous result in the Weyl algebra.

Theorem 3.2.2. *The Weyl algebra is defined by* $k\langle \mathbf{a}, \mathbf{b}\rangle/(\mathbf{ba} - \mathbf{ab} - 1)$. *Assume that the* **ab***-monomial* u *contains* m *copies of* **a** *and* n *copies of* **b**. *Then the following expansion holds*

$$u = \sum_i r_i \cdot \mathbf{a}^{m-i} \cdot \mathbf{b}^{n-i},$$

where r_i *is the number of rook placements of* i *rooks on the Ferrers board* $F(u)$.

This result is implicit in [68, Theorem 3.7]. The proof of Theorem 3.2.2 follows the same outline as the proof of Theorem 3.2.1, but it is easier.

Proof of Theorem 3.2.1: The proof is by induction on the number of inversions of the word u. When u has no inversions, we have that $u = \mathbf{a}^m\mathbf{b}^n$ and there is nothing to prove.

Consider the case when $u = v \cdot \mathbf{ba} \cdot w$. Then we have $u = v \cdot \mathbf{ab} \cdot w + v \cdot \mathbf{b} \cdot w + v \cdot \mathbf{a} \cdot w$. For each of these terms we have a case.

(**ab**) Consider a placement of i copies of $\leftarrow$ and j copies of $\uparrow$ on the board $F(v\mathbf{ab}w)$. By adding an empty square we obtain the board $F(v\mathbf{ba}w)$.

(**b**) Consider a placement of $i - 1$ copies of $\leftarrow$ and j copies of $\uparrow$ on the Ferrers board $F(v\mathbf{b}w)$. By adding a row we obtain the board $F(v\mathbf{ba}w)$. Furthermore, place a $\leftarrow$ in the last entry of this new row.

(**a**) This case is similar to the previous case with the change that we add a column and place a $\uparrow$ in its last entry.

In each case we obtain a legal placement on the board $F(v\mathbf{ba}w)$ with i copies of $\leftarrow$ and j copies of $\uparrow$. Furthermore each legal placement on this board is obtained by one of these three cases, proving the equality. □

Example 3.2.3. Consider the monomial **baba** which has the expansion

$$\mathbf{baba} = \mathbf{b} + 2\mathbf{bb} + \mathbf{a} + 4\mathbf{ab} + 3\mathbf{abb} + 2\mathbf{aa} + 3\mathbf{aab} + \mathbf{aabb}.$$

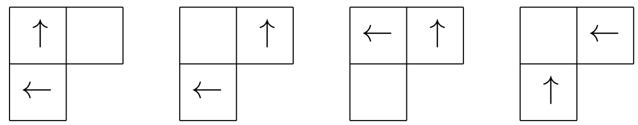

Figure 3.1: The four valid placements of one $\leftarrow$ and one $\uparrow$ in the Ferrers shape $F(\mathbf{baba})$.

For the monomial **ab** in this expansion, we must place $2-1=1$ copy of $\leftarrow$ and $2-1=1$ copy of $\uparrow$ in the Ferrers shape, $F(\mathbf{baba})$. There are four such placements, giving the monomial **ab** a 4 as its coefficient. See Figure 3.1.

Corollary 3.2.4. *The coefficient $c_{i,0}$ is given by the ith elementary symmetric function $e_i(\lambda_1,\ldots,\lambda_m)$, where $\lambda=(\lambda_1,\ldots,\lambda_m)$ is the partition associated to the shape $F(u)$. Similarly, the dual statement is $c_{0,j}=e_j(\lambda_1^*,\ldots,\lambda_n^*)$, where $\lambda^*=(\lambda_1^*,\ldots,\lambda_n^*)$ is the dual partition of λ.*

Proof. When $j=0$ we only have to place $\leftarrow$ in the Ferrers shape and the condition of Theorem 3.2.1 reduces to at most one $\leftarrow$ in each row. Since the partition λ contains the length of each row, the enumeration is given by the ith elementary symmetric function. The dual result follows by only placing $\uparrow$ and using the column lengths, which are given by the dual partition. $\square$

In analogue with the equality $\sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}(u)=[u]$, we have the next corollary.

Corollary 3.2.5. *Let u be an* **ab**-*monomial containing m copies of* **a** *and n copies of* **b**. *Then the following three statements are true.*

(a) If $u=v\mathbf{a}$ then $\sum_{i=0}^{m} c_{i,n}(u)=[v]$.

(b) If $u=\mathbf{b}v$ then $\sum_{j=0}^{n} c_{m,j}(u)=[v]$.

(ba) If $u=\mathbf{b}v\mathbf{a}$ then $c_{m-1,n}(u)=[v]=c_{m,n-1}(u)$.

Proof. When $u = v\mathbf{a}$ observe that the Ferrers shape $F(v)$ is obtained from $F(u)$ by removing the first row. Hence, given a legal placement using $\leftarrow$ and $\uparrow$ on $F(v)$, we obtain a legal placement on $F(u)$ by adding another row on top and inserting copies of $\uparrow$ in the top row in the columns that are missing a $\uparrow$. This placement has exactly n copies of $\uparrow$ and the statement (a) follows.

Note that statements (a) and (b) are equivalent under the involution $u \longmapsto \overline{u}^*$ where $*$ denotes reading the monomial backwards and the bar exchanges the **a**s and **b**s.

When $u = \mathbf{b}v\mathbf{a}$ observe that the Ferrers shape $F(v)$ is obtained by removing the first row and the first column from $F(u)$. Given a legal placement on $F(v)$, first add back the first column and insert copies of $\leftarrow$ in this column in the rows that are missing a $\leftarrow$. Next add back the top row and insert copies of $\uparrow$ in the top row in every column that are missing a $\uparrow$. Observe that we obtain a legal placement on $F(u)$ that has $m-1$ copies of $\leftarrow$ and n copies of $\uparrow$. This proves the first equality in (ba). The second follows by first adding the column and then the row. □

A quick algebraic observation also gives the following relations among the coefficients.

Lemma 3.2.6. *For a non-negative integer k we have the identity*

$$\sum_{i+j=k} (-1)^j \cdot c_{i,j} = \delta_{k,0}.$$

Proof. Consider the map $\phi : \mathcal{E} \longrightarrow k[x]$ defined by $\phi(\mathbf{a}) = x$ and $\phi(\mathbf{b}) = -x$. Observe that ϕ is a well-defined ring homomorphism since $\phi(\mathbf{ba} - \mathbf{ab} - \mathbf{a} - \mathbf{b}) = 0$. Directly we have $\phi(u) = (-1)^n \cdot x^{m+n}$ but also

$$\phi(u) = \phi\left(\sum_{i,j} c_{i,j} \cdot \mathbf{a}^{m-i}\mathbf{b}^{n-j}\right) = \sum_{i,j} c_{i,j} \cdot (-1)^{n-j} \cdot x^{m+n-i-j}.$$

The result follows by comparing the coefficients of x^{m+n-k}. □

Example 3.2.7. As example we have that

$$c_{1,1} = c_{2,0} + c_{0,2} = e_2(\lambda) + e_2(\lambda^*).$$

This can also be seen by Theorem 3.2.1 and using the identity

$$\begin{aligned}(\lambda_1 + \cdots + \lambda_m) \cdot (\lambda_1 + \cdots + \lambda_m - 1) \quad &- \\ \left(\binom{\lambda_1}{2} + \cdots + \binom{\lambda_m}{2}\right) \quad &- \quad \left(\binom{\lambda_1^*}{2} + \cdots + \binom{\lambda_n^*}{2}\right) = e_2(\lambda) + e_2(\lambda^*).\end{aligned}$$

3.3 The operators E and Δ

Lemma 3.3.1. *Let p be a polynomial in one variable. In the excedance algebra, the following two identities hold.*

1. $\mathbf{b} \cdot p(\mathbf{a}) = p(\mathbf{a}+1) \cdot \mathbf{b} + \mathbf{a} \cdot (p(\mathbf{a}+1) - p(\mathbf{a}))$,
2. $p(\mathbf{b}) \cdot \mathbf{a} = \mathbf{a} \cdot p(\mathbf{b}+1) + \mathbf{b} \cdot (p(\mathbf{b}+1) - p(\mathbf{b}))$.

Proof. We start by proving the first identity. Expand the **ab**-monomial $\mathbf{b} \cdot \mathbf{a}^k$. As we expand, the $\mathbf{b}$ will move further right through the word. There are two cases. The $\mathbf{b}$ either reaches the end of the word, killing i copies of $\mathbf{a}$ in the process, or the $\mathbf{b}$ kills $i-1$ copies of $\mathbf{a}$ and is then itself killed by the ith $\mathbf{a}$. In both cases, there are $\binom{k}{i}$ ways of choosing the **a**s. Hence, we have

$$\begin{aligned}\mathbf{b} \cdot \mathbf{a}^k &= \sum_{i=0}^{k} \binom{k}{i} \mathbf{a}^{k-i} \cdot \mathbf{b} + \sum_{i=1}^{k} \binom{k}{i} \mathbf{a}^{k-i+1} \\ &= (\mathbf{a}+1)^k \cdot \mathbf{b} + \mathbf{a} \cdot \left((\mathbf{a}+1)^k - \mathbf{a}^k\right).\end{aligned}$$

This completes the proof for $p(\mathbf{a}) = \mathbf{a}^k$. The result for a general polynomial follows by linearity. The second identity follows by the symmetry $u \longmapsto \overline{u}^*$. $\square$

Let $p(x)$ be a polynomial in x and define E_x^c to be the shift operator

$$E_x^c(p(x)) = p(x+c).$$

Also, let Δ_x represent the difference operator

$$\Delta_x(p(x)) = p(x+1) - p(x).$$

Using these, the result of Lemma 3.3.1 can be rewritten as follows.

Lemma 3.3.2. *Let p be a polynomial in one variable. In the excedance algebra, the following two identities hold.*

1. $\mathbf{b} \cdot p(\mathbf{a}) = E^1_{\mathbf{a}}(p(\mathbf{a})) \cdot \mathbf{b} + \mathbf{a} \cdot \Delta_{\mathbf{a}}(p(\mathbf{a}))$,

2. $p(\mathbf{b}) \cdot \mathbf{a} = \mathbf{a} \cdot E^1_{\mathbf{b}}(p(\mathbf{b})) + \mathbf{b} \cdot \Delta_b(p(\mathbf{b}))$.

Define the linear map $L : \mathcal{E} \longrightarrow k[x, y]$ by $L(\mathbf{a}^m\mathbf{b}^n) = x^m y^n$. Note that L is not an algebra homomorphism, but it is a vector space isomorphism. Also, denote by $\mathbf{x}$ and $\mathbf{y}$ the two operators on $k[x, y]$ defined by multiplication by x and y, respectively. It is also clear from the definition of L that for $\mathbf{ab}$-monomial u, $[u] = L(u)|_{x=y=1}$.

Lemma 3.3.3. *Let u be an* $\mathbf{ab}$*-polynomial. Then we have*

1. $L(\mathbf{a} \cdot u) = \mathbf{x}L(u)$,

2. $L(\mathbf{b} \cdot u) = (\mathbf{y}E^1_x + \mathbf{x}\Delta_x)(L(u))$,

3. $L(u \cdot \mathbf{b}) = \mathbf{y}L(u)$,

4. $L(u \cdot \mathbf{a}) = \left(\mathbf{x}E^1_y + \mathbf{y}\Delta_y\right)(L(u))$.

Proof. Since every term in the expansion of the polynomial $\mathbf{a} \cdot u$ starts with an $\mathbf{a}$, part one follows. For part two, we note that by linearity it is enough to consider $u = \mathbf{a}^m\mathbf{b}^n$. Then, using Equation 1 from Lemma 3.3.2, we have

$$\begin{aligned} L(\mathbf{b} \cdot \mathbf{a}^m\mathbf{b}^n) &= L\left(E^1_{\mathbf{a}}(\mathbf{a}^m)\mathbf{b}^{n+1} + \mathbf{a}\Delta(\mathbf{a}^m)\mathbf{b}^n\right) \\ &= E^1_x(x^m)y^{n+1} + x\Delta(x^m)y^n \\ &= \left(E^1_x\mathbf{y} + \mathbf{x}\Delta_x\right)(x^m y^n) \\ &= \left(\mathbf{y}E^1_x + \mathbf{x}\Delta_x\right)(L(\mathbf{a}^m\mathbf{b}^n)). \end{aligned}$$

Parts three and four are exactly analogous, using Equation 2 from Lemma 3.3.2. □

Using this lemma, we are able to evaluate the map L for an arbitrary $\mathbf{ab}$-monomial u.

Theorem 3.3.4. *Let $u = u_1 u_2 \ldots u_k$ be an* **ab***-monomial. Let U_i and V_i be the operators*

$$U_i = \begin{cases} \mathbf{x} & \text{if } u_i = \mathbf{a}, \\ \mathbf{y}E_x^1 + \mathbf{x}\Delta_x & \text{if } u_i = \mathbf{b}, \end{cases}$$

and

$$V_i = \begin{cases} \mathbf{x}E_y^1 + \mathbf{y}\Delta_y & \text{if } u_i = \mathbf{a}, \\ \mathbf{y} & \text{if } u_i = \mathbf{b}. \end{cases}$$

Then $L(u)$ is evaluated by applying the operators $U_1 U_2 \cdots U_k(1) = V_k V_{k-1} \cdots V_1(1)$.

Proof. This is a straightforward proof by induction on the length of the monomial u. □

Example 3.3.5. Applying this theorem to the monomial **baba**, we get

$$\begin{aligned} L(\mathbf{baba}) &= U_1 U_2 U_3 U_4(1) \\ &= (\mathbf{y}E_x^1 + \mathbf{x}\Delta_x)\mathbf{x}(\mathbf{y}E_x^1 + \mathbf{x}\Delta_x)\mathbf{x}(1) \\ &= (\mathbf{y}E_x^1 + \mathbf{x}\Delta_x)\mathbf{x}(y(x+1) + x) \\ &= (\mathbf{y}E_x^1 + \mathbf{x})\Delta_x(x^2y + xy + x^2) \\ &= (x+1)^2y^2 + (x+1)y^2 + (x+1)^2y + x(2x+1)y + x \cdot 1 \cdot y + x(2x+1) \\ &= y + 2y^2 + x + 4xy + 3xy^2 + 2x^2 + 3x^2y + x^2y^2. \end{aligned}$$

Notice that with the substitution $x \longmapsto \mathbf{a}$ and $y \longmapsto \mathbf{b}$, this expansion coincides with the expansion of **baba** in Example 3.2.3.

There are several corollaries to Theorem 3.3.4 which we now discuss.

Corollary 3.3.6. *Let u be the* **ab***-monomial $u = \mathbf{a}^{p_n}\mathbf{b}\mathbf{a}^{p_{n-1}}\mathbf{b}\cdots\mathbf{b}\mathbf{a}^{p_0}$. Then we have*

$$L(u)|_{y=0} = \mathbf{x}^{p_n+1}\Delta_x\mathbf{x}^{p_{n-1}+1}\Delta_x \cdots \Delta_x\mathbf{x}^{p_1+1}\Delta_x x^{p_0}.$$

Likewise, for $u = \mathbf{b}^{q_m}\mathbf{a}\mathbf{b}^{q_{m-1}}\mathbf{a}\cdots\mathbf{a}\mathbf{b}^{q_0}$, we have

$$L(u)|_{x=0} = \mathbf{y}^{q_0+1}\Delta_y\mathbf{y}^{q_1+1}\Delta_y \cdots \Delta_y\mathbf{y}^{q_{m-1}+1}\Delta_y y^{q_m}.$$

Corollary 3.3.7. *Consider the* $\mathbf{ab}$*-monomials* $(\mathbf{ba})^n$ *and* $\mathbf{b}^n\mathbf{a}^m$*. Then*

1. $L((\mathbf{ba})^n)\big|_{y=0} = (\mathbf{x}\Delta_x\mathbf{x})^n(1)$,

2. $L(\mathbf{b}^n\mathbf{a}^m)\big|_{y=0} = (\mathbf{x}\Delta_x)^n(x^m)$.

Corollary 3.3.8. *The following operator identities hold.*

1. $(\mathbf{y}E_x^1 + \mathbf{x}\Delta_x)^n x^m = (\mathbf{x}E_y^1 + \mathbf{y}\Delta_y)^m y^n$.

2. $(\mathbf{y}E_x^1\mathbf{x} + \mathbf{x}\Delta_x\mathbf{x})^n = (\mathbf{x}E_y^1\mathbf{y} + \mathbf{y}\Delta_y\mathbf{y})^n$.

Proof. The first identity holds by equating the expansions of $L(\mathbf{b}^n\mathbf{a}^m)$ using U_i and V_i. The second identity comes from equating the different expansions of $L((\mathbf{ba})^n)$. □

3.4 Gandhi polynomials and Genocchi numbers

Permutations that have an alternating excedance set, that is, permutations with excedance word $(\mathbf{ba})^n$, have been well studied. They are enumerated by the *Genocchi numbers*, G_n. Specifically, we have $[(\mathbf{ba})^n] = G_{2(n+2)}$. These numbers can be defined by the generating function

$$\sum_{i\geq 1} G_{2n}\frac{x^{2n}}{(2n)!} = x\cdot\tan\left(\frac{x}{2}\right).$$

In 1970, Gandhi [46] conjectured that the following family of polynomials provided a formula for the Genocchi numbers:

$$A_n(x) = x^2\cdot A_{n-1}(x+1) - (x-1)^2\cdot A_{n-1}(x),$$

where $A_0(x) = 1$. This conjecture was independently proven a few years later by Carlitz [20] and Riordan and Stein [63]. Dumont [32] also provided a combinatorial interpretation for Gandhi's conjecture. He also introduced a related family of polynomials, $B_n(x) = x^2\cdot A_{n-1}(x+1)$, and showed that this family follows the recursion

$$\begin{aligned} B_n(x) &= x^2(B_{n-1}(x+1) - B_{n-1}(x)),\\ B_1(x) &= x^2. \end{aligned}$$

It is straightforward from the definition of $B_n(x)$ that the coefficient of x^2 is equal to $A_{n-1}(1)$, that is, $B_{n,2} = A_{n-1}(1) = G_{2n}$. For more information about Genocchi numbers and some of their generalizations, see [33, 34, 40].

From Corollary 3.3.7, we know that

$$L((\mathbf{ba})^n)|_{y=0} = (\mathbf{x}\Delta_x\mathbf{x})^n(1).$$

By comparing this with the recursion defining the polynomial B_n, it is straightforward to see that they differ only by a factor of x. Therefore, we get the following theorem.

Theorem 3.4.1. *Consider the expansion of the* **ab***-monomial* $(\mathbf{ba})^n$*. Then we have*

$$c_{i,n} = [x^{n+1-i}]B_n(x),$$

where $[x^{n+1-i}]$ *indicates the coefficient of* x^{n+1-i}*.*

3.5 Concluding remarks

We are left with some open questions.

Question 3.5.1. The placements of $\leftarrow$ and $\uparrow$ in the Ferrers shape $F(u)$ are examples of *alternative tableaux*, introduced by X. Viennot in [73]. Alternative tableaux of a given shape can be enumerated using a method called *matrix Ansatz*. Can this method be used to find explicit expressions for any of the coefficients $c_{i,j}$ other than $c_{i,0}$ and $c_{0,j}$? See [29, 51] for more information about matrix Ansatz.

Question 3.5.2. Do the coefficients of expansions of other specific families of monomials, such as $\mathbf{b}^n\mathbf{a}^m$, have other combinatorial interpretations like the coefficients of $(\mathbf{ba})^n$? For more about the monomial $\mathbf{b}^n\mathbf{a}^m$, see [25].

Question 3.5.3. For positive integers n and k, let $\Phi_{n,k}$ be the collection of surjections $\phi : [-n] \cup [n] \longrightarrow [n]$ such that $\phi(m) \leq |m|$ for all $m \in [-n] \cup [n]$ and the cardinality of $\phi^{-1}(1)$ is k. Dumont showed in [32] that the coefficient of x^k in $B_n(x)$ is equal to the cardinality of $\Phi_{n,k}$, that is $B_{n,k} = |\Phi_{n,k}|$. Using this, is there a combinatorial proof of Theorem 3.4.1?

Question 3.5.4. What can be said about the excedance algebra itself? What algebraic properties does it have?

Chapter 4 The Frobenius complex

4.1 Introduction

The classical Frobenius problem is to find the largest integer for which change cannot be made using coins with the relatively prime denominations $a_1, a_2, \ldots, a_d$; see for instance [5, Section 1.2]. We will reformulate this question by introducing the following poset.

Let Λ be a sub-semigroup of the non-negative integers $\mathbb{N}$, that is, Λ is closed under addition and the element 0 lies in Λ. We define the Frobenius poset $P = (\mathbb{Z}, \leq_\Lambda)$ on the integers $\mathbb{Z}$ by the order relation $n \leq_\Lambda m$ if $m - n \in \Lambda$. We denote by $[n, m]_\Lambda$ the interval from n to m in the Frobenius poset, that is,

$$[n, m]_\Lambda = \{i \in [n, m] \ : \ i - n, m - i \in \Lambda\}.$$

Observe that the interval $[n, m]_\Lambda$ in the Frobenius poset is isomorphic to the interval $[n+i, m+i]_\Lambda$, that is, the interval $[n, m]_\Lambda$ only depends on the difference $m - n$. Also note that each interval is self-dual by sending i in $[0, n]_\Lambda$ to $n - i$.

In this form, the original Frobenius problem would be to find the largest integer n that is not comparable to zero in the Frobenius poset when Λ is generated by $\{a_1, a_2, \ldots, a_d\}$. The largest such integer is known as the Frobenius number. In general, calculating the Frobenius number is difficult. However, in the case where the semigroup is generated by two relatively prime integers a and b, it is well known that the Frobenius number is given by $ab - a - b$. Also, when the semigroup is generated by the arithmetic sequence $\{a, a + d, \ldots, a + (a - 1)d\}$, the Frobenius number was shown by Roberts [64] to be $(a - 1) \cdot d$. We study the topology of the order complex of intervals of this poset in these two cases.

The technique we use is discrete Morse theory which was developed by Forman, see [44, 45]. Thus we construct an acyclic partial matching on the face poset of the order complex by using the Patchwork Theorem. We then identify the unmatched (critical) cells. These cells tell us the number and dimension of cells in a CW complex

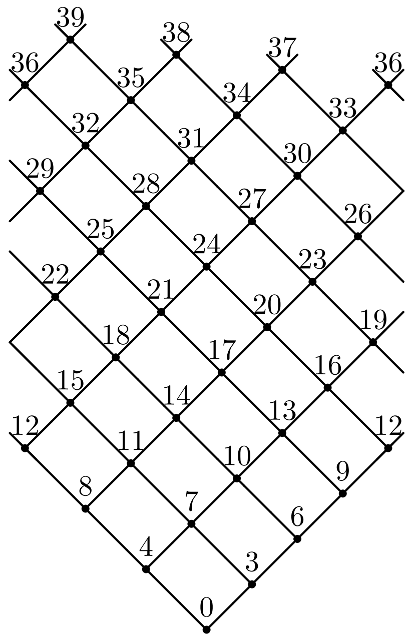

Figure 4.1: The filter generated by 0 in the Frobenius poset corresponding to the semigroup Λ generated by $a = 3$ and $b = 4$, that is, $\Lambda = \mathbb{N} - \{1, 2, 5\}$. Note that you get a better picture by rolling the page into a cylinder.

to which our order complex is homotopy equivalent. Using extra structure about the critical cells, we can determine exactly what the homotopy type is.

For the semigroup Λ generated by two relatively prime integers a and b with $1 < a < b$, our first result is that the order complex of an interval $[0, n]_\Lambda$ in the Frobenius poset is either contractible or homotopy equivalent to a sphere. The exact statement depends on the congruence class of n modulo $a \cdot b$; see Theorem 4.4.1. The second result handles the case when the semigroup is generated by the arithmetic sequence $\{a, a + d, \ldots, a + (a - 1)d\}$. In this case the order complex is homotopy equivalent to a wedge of spheres of different dimensions. The generating function of the ith reduced Betti number is a polynomial that factors into explicit terms; see Theorem 4.5.1.

This chapter also appears in [27].

4.2 Discrete Morse theory

Discrete Morse theory, developed by Forman, is a tool used to study the topology of simplicial complexes. We recall the following definitions and theorems from discrete Morse theory. See [44, 45] for further details.

For a simplicial complex Λ, let $\beta^{(p)}$ denote a p-dimensional faces of Λ.

Definition 4.2.1. *A function* $f : \Lambda \longrightarrow \mathbb{R}$ *is a* discrete Morse function *if for every* $\beta^{(p)} \in \Lambda$ *we have*

1. $|\{\gamma^{(p+1)} \supset \beta \ : \ f(\gamma) \leq f(\beta)\}| \leq 1$,

2. $|\{\alpha^{(p-1)} \subset \beta \ : \ f(\alpha) \geq f(\beta)\}| \leq 1$.

If both of these are zero, the simplex β *is said to be* critical*.*

We now state Forman's original formulation of discrete Morse theory; see [45].

Theorem 4.2.2 (Forman)**.** *Suppose* Λ *is a simplicial complex with a discrete Morse function. Then* Λ *is homotopy equivalent to a CW complex with exactly one cell of dimension* p *for each critical simplex of dimension* p*.*

Chari [22] provided a combinatorial reformulation of discrete Morse theory. It is this reformulation that will be used in this chapter, so we now discuss it. See [53] for more details.

Definition 4.2.3. *A* partial matching *in a poset* P *is a partial matching in the underlying graph of the Hasse diagram of* P*, that is, a subset* $M \subseteq P \times P$ *such that* $(x, y) \in M$ *implies* $x \prec y$ *and each* $x \in P$ *belongs to at most one element of* M*. For* $(x, y) \in M$ *we write* $x = d(y)$ *and* $y = u(x)$*, where* d *and* u *stand for down and up, respectively.*

Definition 4.2.4. *A partial matching* M *on* P *is* acyclic *if there does not exist a cycle*

$$z_1 \succ d(z_1) \prec z_2 \succ d(z_2) \prec \cdots \prec z_n \succ d(z_n) \prec z_1,$$

in P with $n \geq 2$, and all $z_i \in P$ distinct. Given a partial matching, the unmatched elements are called critical. *If there are no critical elements, the acyclic matching is* perfect.

We now state the reformulated main result from discrete Morse theory. For a simplicial complex Δ, let $\mathcal{F}(\Delta)$ denote the poset of faces of Δ ordered by inclusion. The following can be found in [53, Theorem 11.13].

Theorem 4.2.5 (Chari)**.** *Let Δ be a simplicial complex. If M is an acyclic matching on $\mathcal{F}(\Delta) - \{\widehat{0}\}$ and k_i denotes the number of critical i-dimensional cells of Δ, then the complex Δ is homotopy equivalent to a CW complex Δ_k which has k_i cells of dimension i.*

For us it will be convenient to work with the reduced discrete Morse theory, that is, we include the empty set.

Corollary 4.2.6. *Let Δ be a simplicial complex and let M be an acyclic matching on $\mathcal{F}(\Delta)$. Then the complex Δ is homotopy equivalent to a CW complex Δ_k which has $k_0 + 1$ cells of dimension 0 and k_i cells of dimension i for $i > 0$.*

In particular, if the matching from Corollary 4.2.6 is perfect, then Δ_k is contractible. Also, if the matching has exactly one critical cell then Δ_k is a combinatorial d-sphere where d is the dimension of the cell.

Given a set of critical cells of differing dimension, in general it is impossible to conclude that the CW complex Δ_k is homotopy equivalent to a wedge of spheres. See Kozlov [54, Section 3] for an example. However, in certain cases, this is possible.

Theorem 4.2.7. *Let M be a Morse matching on $\mathcal{F}(\Delta)$ such that all k_i critical cells of dimension i are facets of Δ. Then the complex Δ is homotopic equivalent to a wedge of spheres, that is,*

$$\Delta \simeq \bigvee_{i} \bigvee_{j=1}^{k_i} \mathbb{S}^i.$$

Proof. By the above statement, the complex Δ without the critical cells is contractible. In particular, the boundary of each of the critical cells contract to a point. Since all of the critical cells are maximal, they can be independently added back into the complex. $\square$

Kozlov [54] gives a more general sufficient condition on an acyclic Morse matching for the complex to be homotopy equivalent to a wedge of spheres enumerated by the critical cells.

We are interested in finding an acyclic matching on the face poset of the Frobenius complex. The Patchwork Theorem [53, Theorem 11.10] gives us a way of constructing one.

Theorem 4.2.8 (The Patchwork Theorem)**.** *Assume that $\varphi : P \to Q$ is an order-preserving poset map, and assume that there are acyclic matchings on the fibers $\varphi^{-1}(q)$ for all $q \in Q$. Then the union of these matchings is itself an acyclic matching on P.*

4.3 Generating functions

Recall that the *order complex* $\Delta(P)$ of a bounded poset P is the collection of chains in P, that is,

$$\Delta(P) = \{\{x_1, x_2, \ldots, x_k\} : \widehat{0} < x_1 < x_2 < \cdots < x_k < \widehat{1}\}$$

ordered by inclusion. Also, the reduced Euler characteristic of the order complex $\Delta(P)$ is given by the Möbius function of P. We call the order complex of the face poset of a Frobenius interval the *Frobenius complex.*

Consider the interval $[0, n]_\Lambda$. Observe that if n does not belong to the semigroup Λ then we consider the order complex $\Delta([0, n]_\Lambda)$ to be the empty set which we view as contractible. This is distinct from the case when n equals one of the generators, that is, when the order complex $\Delta([0, n]_\Lambda)$ only contains the empty set. In this case, we view this as a sphere of dimension -1.

We begin our study of the topology of the Frobenius complex by finding the reduced Euler characteristic. Let $c_k(n)$ denote the number of chains of length k in the Frobenius interval $[0, n]_\Lambda$.

Theorem 4.3.1. *The generating function for the f-polynomial of an order complex is given by*

$$\sum_{n \geq 0} f(\Delta(n), x) \cdot q^n = \frac{1}{1 - x \sum_{n \geq 1} c_1(n) \cdot q^n}.$$

Proof.

$$\begin{aligned}
\sum_{n \geq 0} f(\Delta(n), x) \cdot q^n &= \sum_{n \geq 0} \sum_{k \geq 0} f_{k-1}(\Delta(n)) \cdot x^k \cdot q^n \\
&= \sum_{k \geq 0} \sum_{n \geq k} c_k(n) \cdot q^n \cdot x^k \\
&= \sum_{k \geq 0} \left(\sum_{n \geq 1} c_1(n) \cdot q^n \right)^k \cdot x^k \\
&= \frac{1}{1 - x \sum_{n \geq 1} c_1(n) \cdot q^n}
\end{aligned}$$

□

Corollary 4.3.2. *Let $\mu(n)$ denote the Möbius function of the interval $[0, n]_\Lambda$. Then*

$$\sum_{n \geq 0} \mu(n) \cdot q^n = \frac{1}{1 + \sum_{n \geq 1} c_1(n) \cdot q^n}.$$

Proof. By setting $x = -1$ in Theorem 4.3.1 and using Philip Hall's expression for the Möbius function, the result follows. □

Now assuming that Λ is generated by two relatively prime positive integers a and b, we have that

$$1 + \sum_{n \geq 1} c_1(n) \cdot q^n = \frac{1 - q^{ab}}{(1 - q^a) \cdot (1 - q^b)};$$

see [4, Exercise VIII.1.5]. Hence the Möbius function is given by

$$\begin{aligned}\sum_{n\geq 0}\mu(n)\cdot q^n &= \frac{(1-q^a)\cdot(1-q^b)}{1-q^{ab}}\\ &= 1-q^a-q^b+q^{a+b}+q^{ab}-q^{ab+a}-q^{ab+b}+q^{ab+a+b}+\cdots.\end{aligned}$$

Since the reduced Euler characteristic of the Frobenius complex takes on the values $+1$, -1, or 0, we are lead to conjecture Theorem 4.4.1.

Now assume that Λ is generated by $\{a, a+d, a+2d, \ldots, a+(a-1)d\}$ where a and d are relatively prime.

Lemma 4.3.3. *Any non-zero element of Λ can be written uniquely as a sum of one generator and a multiple of the positive integer a.*

Proof. Assume that n is an integer that can be written as a sum of k generators where $k > 1$. That is,

$$\begin{aligned} n &= (a+s_1d)+(a+s_2d)+\cdots+(a+s_kd)\\ &= ka+(s_1+s_2+\cdots+s_k)d.\end{aligned}$$

Using the division algorithm, we write $s_1+s_2+\cdots+s_k = qa+r$ where $0 \leq r \leq a-1$. Therefore, we get $n = (k+q-1)a+(a+rd)$. □

From this lemma, it is straightforward to see that

$$\begin{aligned}1+\sum_{n\geq 1}c_1(n)\cdot q^n &= \frac{1}{1-q^a}\left(1+\sum_{i=1}^{a-1}q^{a+id}\right)\\ &= \frac{1-q^d+q^{a+d}-q^{a+ad}}{(1-q^a)\cdot(1-q^d)}\end{aligned}$$

Hence, using Corollary 4.3.2, the Möbius function is given by

$$\sum_{n\geq 0}\mu(n)\cdot q^n = \frac{(1-q^a)\cdot(1-q^d)}{1-q^d+q^{a+d}-q^{a+ad}}.$$

4.4 Two generators

When the semigroup Λ is generated by two generators a and b which are not relatively prime, the case can be reduced by dividing by their greatest common divisor. Hence we assume that they are relatively prime, that is, $\gcd(a, b) = 1$. Furthermore, we also assume that $1 < a < b$.

With two generators, the associated Frobenius poset can be embedded on a cylinder. By Bezout's identity there are two integers p and q such that $p \cdot a + q \cdot b = 1$. Define a group morphism $\gamma : \mathbb{Z} \longrightarrow \mathbb{Z}_{2ab} \times \mathbb{Z}$ by $\gamma(x) = ((p \cdot a - q \cdot b) \cdot x, x)$, that is, the first coordinate is modulo $2 \cdot a \cdot b$ which corresponds to encircling the cylinder. Observe that $\gamma(a) = ((p \cdot a - q \cdot b) \cdot a, a) = ((p \cdot a + q \cdot b) \cdot a, a) = (a, a)$ and $\gamma(b) = ((p \cdot a - q \cdot b) \cdot b, b) = ((-p \cdot a - q \cdot b) \cdot b, b) = (-b, b)$. Hence the two cover relations $x \prec x + a$ and $x \prec x + b$ in the Frobenius poset translates to $\gamma(x) + (a, a) = \gamma(x + a)$ and $\gamma(x) + (-b, b) = \gamma(x + b)$. In other words, to take an a step we make the step (a, a) on the cylinder and a b step corresponds to the step $(-b, b)$. As an example, see Figure 4.1 where $a = 3$ and $b = 4$.

In general, the Frobenius poset is not a lattice. We have the four relations $a <_\Lambda a + b$, $b <_\Lambda a + b$, $a <_\Lambda ab$, and $b <_\Lambda ab$. However, since $ab - a - b$ is the Frobenius number we have $a + b \not\leq_\Lambda ab$, showing that the poset is not a lattice. In Figure 4.1, we see that 3 and 4 are both lower bounds for 7 and 12.

In the case where the two generators are 2 and 3, the semigroup is $\mathbb{N} - \{1\}$ and the order complex $\Delta([0, n]_\Lambda)$ consists of all subsets of the interval $[2, n - 2]$ that do not contain two consecutive integers. This is known as the complex of sparse subsets. Its homotopy type was first determined by Kozlov [52, Proposition 4.6]. See also [36, Corollary 6.3] where it appears as the independence complex of a path. Billera and Myers [7, Corollary 2] showed this complex is non-pure shellable.

Theorem 4.4.1. *Let the sub-semigroup Λ be generated by two relatively prime positive integers a and b with $1 < a < b$. The order complex of the associated Frobenius interval $[0, n]_\Lambda$, for $n \geq 1$, is homotopy equivalent to either a sphere or contractible,* *according to*

$$\Delta([0,n]_\Lambda) \simeq \begin{cases} \mathbb{S}^{2n/ab-2} & \text{if } n \equiv 0 \bmod a\cdot b, \\ \mathbb{S}^{2(n-a)/ab-1} & \text{if } n \equiv a \bmod a\cdot b, \\ \mathbb{S}^{2(n-b)/ab-1} & \text{if } n \equiv b \bmod a\cdot b, \\ \mathbb{S}^{2(n-a-b)/ab} & \text{if } n \equiv a+b \bmod a\cdot b, \\ \text{point} & \text{otherwise.} \end{cases}$$

We now turn our attention to the proof of this result. Consider the Frobenius interval $[0,n]_\Lambda$. Define the three sets B_ℓ, C_ℓ and D_ℓ as follows:

$$\begin{aligned} B_\ell &= \{\ell ab + 2b, \ell ab + 3b, \ldots, \ell ab + (a-1)b\}, \\ C_\ell &= \{b, ab, ab+b, 2ab, 2ab+b, 3ab, \ldots, (\ell-1)ab + b, \ell ab\}, \\ D_\ell &= C_\ell \cup \{\ell ab + b\}. \end{aligned}$$

Note that $C_0 = \emptyset$, $D_0 = \{b\}$, and $C_{\ell+1} = D_\ell \cup \{(\ell+1)ab\}$.

Example 4.4.2. (Part 1) For $a = 3$, $b = 4$ and $n = 20$, the Frobenius complex consists of 216 chains (faces) and has f-vector $(f_0, f_1, \ldots, f_5) = (1, 13, 51, 80, 56, 15)$.

Let Q be the infinite chain $\{a < a+b < ab+a < ab+a+b < 2ab+a < \cdots\}$ adjoined with a new maximal element $\widehat{1}_Q$, that is,

$$Q = \{m \in \mathbb{N} \;:\; m \equiv a, a+b \bmod ab\} \cup \{\widehat{1}_Q\}.$$

We now define a map φ from the face poset of the order complex $\Delta([0,n]_\Lambda)$ to the poset Q. We will later show that φ is an order-preserving poset map with natural matchings on the fibers. Let φ be defined by

$$\varphi(x) = \begin{cases} \ell ab + a & \text{if } \ell ab + a <_\Lambda n, \\ & C_\ell \subseteq x, \\ & B_t \cap x = \emptyset \text{ for } 0 \leq t \leq \ell, \\ & \text{and } \ell ab + b \notin x; \\ \ell ab + a + b & \text{if } \ell ab + a + b <_\Lambda n, \\ & D_\ell \subseteq x, \\ & B_t \cap x = \emptyset \text{ for } 0 \leq t \leq \ell, \\ & \text{and } \ell ab + ab \notin x; \\ \widehat{1}_Q & \text{otherwise.} \end{cases}$$

In order to make acyclic pairings on the fibers of φ, it will be useful to have a description of the chains that are mapped to the maximal element $\widehat{1}_Q$ and their structure. Let Γ denote this collection of chains in the Frobenius poset, that is, $\Gamma = \varphi^{-1}(\widehat{1}_Q)$.

Lemma 4.4.3. *The collection Γ consists of the chains x that satisfy one of the following four conditions:*

1. *There exists a non-negative integer λ such that $C_\lambda \subseteq x$, $\lambda ab + b \notin x$, $B_\lambda \cap x \neq \emptyset$, and $B_t \cap x = \emptyset$ for $0 \leq t \leq \lambda - 1$.*

2. *There exists a non-negative integer λ such that $D_\lambda \subseteq x$, $B_\lambda \cap x \neq \emptyset$, and $B_t \cap x = \emptyset$ for $0 \leq t \leq \lambda - 1$.*

3. *There exists a non-negative integer λ such that $x = C_\lambda$ and $\lambda ab + a \not<_\Lambda n$.*

4. *There exists a non-negative integer λ such that $x = D_\lambda$ and $\lambda ab + a + b \not<_\Lambda n$.*

We will refer to the condition met by a chain as its *type* and the associated λ as its *parameter*. For example, any chain containing both b and $2b$ would be a chain of type 2 with parameter 0.

Proof of Lemma 4.4.3. It is straightforward to see that any element of type $1, 2, 3$, or 4 would indeed be mapped to $\widehat{1}_Q$ by the map φ.

Suppose x is a chain belonging to Γ but is not of type $1, 2, 3$, or 4. We claim that if $C_\ell \subseteq x$ then $D_\ell \subseteq x$. Suppose not, that is, suppose $\ell ab + b \notin x$. Since x is not mapped to $\ell ab + a$, we must have either $B_t \cap x \neq \emptyset$ for some $0 \leq t \leq \ell$ or $\ell ab + a \not\leq_\Lambda n$ by definition of φ. Let t be the smallest integer such that $B_t \cap x \neq \emptyset$. If $t = \ell$, then x is a chain of type 1 with parameter ℓ. If $0 \leq t \leq \ell - 1$ then x is a chain of type 2 with parameter t. This is a contradiction. Therefore, we must have $B_t \cap x = \emptyset$ for all $0 \leq t \leq \ell$ and $\ell ab + a \not\leq_\Lambda n$.

We claim this forces x to be a chain of type 3 with parameter ℓ, that is, $x = C_\ell$. We already know $C_\ell \subseteq x$ so we must show that $x \subseteq C_\ell$. Note that we can write all elements of C_ℓ in the form $tab + ia + jb$ where $0 \leq t \leq \ell$, $ia + jb < ab$, and $(i, j) = (0, 0)$ or $(0, 1)$. Note, however, that the elements 0 and $\ell ab + b$ are of this form but are in neither C_ℓ nor x. Thus, we must show that all elements that are not of this form are not elements of x.

Suppose $0 \leq t \leq \ell - 1$. If $i = 0$ and $2 \leq j \leq a - 1$, we know that $tab + jb \notin x$ because $B_t \cap x = \emptyset$. Now assume $1 \leq i \leq (b - 1)$ and $j = 0$. Then $tab + ia$ cannot be in x since it is not comparable to $tab + b \in x$. Similarly, if $i, j \geq 1$, we know that $tab + ia + jb$ cannot be in x since it is not comparable to $(t + 1)ab \in x$.

We now let $t = \ell$. Again, we cannot have $i = 0$ and $2 \leq j \leq a - 1$ because $B_\ell \cap x = \emptyset$. If $i \geq 1$, since $\ell ab + a \not\leq_\Lambda n$, we see that $\ell ab + ia + jb$ is not comparable to n and cannot be in x.

Finally, we suppose $t \geq \ell + 1$. Using the Frobenius number, we know that

$$\begin{aligned} n &\leq \ell ab + a + (ab - a - b) \\ &= (\ell + 1)ab - b \end{aligned}$$

since $\ell ab + a$ would be comparable to any larger integer. This means that $tab + ia + jb \not\leq_\Lambda n$ for $t \geq \ell + 1$ and thus cannot be an element of x. Therefore, $x = C_\ell$ and x is a chain of type 3. This is also a contradiction, so we must have $D_\ell \subseteq x$.

By an analogous argument, using the fact that x is not mapped by φ to $\ell ab + a + b$, we see that $D_\ell \subseteq x$ implies $C_{\ell+1} \subseteq x$. The only difference occurs when we are assuming $B_t \cap x = \emptyset$ for $0 \leq t \leq \ell$ and $\ell ab + a + b \not\leq_\Lambda n$ and are trying to prove

$\ell ab + ia + jb$ is not an element of x for $i \geq 1$. Here we must separate the cases $j = 0$ and $j \geq 1$. If $j = 0$, the sum $\ell ab + ia$ is not in x since it is not comparable to $\ell ab + b \in x$. If $j \geq 1$, the sum $\ell ab + ia + jb$ is not comparable to n because $\ell ab + a + b \not\leq_\Lambda n$.

Since $C_0 = \emptyset \subseteq x$, these two claims together show that $C_0 \subseteq D_0 \subseteq C_1 \subseteq D_1 \subseteq \cdots \subseteq x$. This cannot happen since x is a finite chain. Therefore, every element in Γ must be one of the above four types. $\square$

Continuation of Example 4.4.2. (Part 2) The fiber Γ consists of the following 24 chains given by type.

- Type 1: $\{8\}, \{8, 11\}, \{8, 12\}, \{8, 14\}, \{8, 16\}, \{8, 17\}, \{8, 11, 14\}, \{8, 11, 17\}, \{8, 12, 16\}, \{8, 14, 17\}, \{8, 11, 14, 17\}$. These chains all have parameter $\lambda = 0$.

- The chains of type 2 are obtained by adjoining the element 4 to every chain of type 1. Each of these has parameter $\lambda = 0$. This is an example of (i) and (ii) of the next lemma.

- Type 3: There is one chain $\{4, 12\}$ and it has parameter $\lambda = 1$.

- Type 4: There is one chain $\{4, 12, 16\}$ and it has parameter $\lambda = 1$.

The structure of Γ is given in the following lemma.

Lemma 4.4.4. *The following four conditions hold for the collection Γ.*

(i) Let x be a chain of type 1 with parameter λ in Γ. Then $x \cup \{\lambda ab + b\}$ is a chain in Γ of type 2 with the same parameter λ.

(ii) Let y be a chain of type 2 with parameter λ in Γ. Then $y - \{\lambda ab + b\}$ is a chain in Γ of type 1 with the same parameter λ.

(iii) Let x be a chain of type 1 with parameter λ and y be a chain of type 2 with parameter μ such that $x \prec y$. Then $\lambda \geq \mu$ holds with equality if and only if $y = x \cup \{\lambda ab + b\}$.

(iv) If z is an element of type 4, then z does not cover any element of type 1 or 2.

Proof. Suppose x is a chain of type 1 with parameter λ. In order to prove the first statement, we must ensure that $x \cup \{\lambda ab + b\}$ is a valid chain. By definition we know that λab and $\lambda ab + jb$ are elements of x for some $2 \leq j \leq (a-1)$. It is clear that $\lambda ab + b$ lies in the interval $[\lambda ab, \lambda ab + jb]_\Lambda$ because

$$\lambda ab + b - \lambda ab = b \text{ and } \lambda ab + jb - (\lambda ab + b) = (j-1)b$$

which are both elements of the semigroup Λ. Thus, $x \cup \{\lambda ab + b\}$ is a valid chain in Λ. Since D_λ is a subset of $x \cup \{\lambda ab + b\}$ and the other requirements are met by the elements from x, we see that $x \cup \{\lambda ab + b\}$ is a chain of type 2 with parameter λ.

To prove the second statement, we simply note that if y is a chain of type 2 with parameter λ, removing the element $\lambda ab + b$ from y gives a chain of type 1 with parameter λ.

Let x and y be chains as described in the third statement. If we assume that $y = x \cup \{\lambda ab + b\}$ then it is clear from the definition of the types in Lemma 4.4.3 that $\lambda = \mu$. Now suppose that y is not $x \cup \{\lambda ab + b\}$. Since $\lambda ab + b$ is not an element of y, we know that D_t for $t \geq \lambda$ is not contained in y. Therefore we cannot have $\lambda \leq \mu$.

The fourth statement can be proven by noting every chain of type 1 or 2 has a non-empty intersection with B_λ for some λ. However, any chain z of type 4 does not intersect B_λ for any λ. Therefore z cannot cover any element of type 1 or 2. $\square$

We now turn our attention to the map φ.

Lemma 4.4.5. *The map $\varphi : \mathcal{F}(\Delta([0, n]_\Lambda)) \longrightarrow Q$ is an order-preserving poset map.*

Proof. To show that φ is order-preserving, let x and y be elements of the Frobenius complex such that $x \subseteq y$.

Suppose $\varphi(x) = \ell ab + a$. This means we have $C_\ell \subseteq x \subseteq y$. Since $tab + b \in y$ for $0 \leq t \leq \ell - 1$, y cannot be mapped to $tab + a$ by the definition of φ. Likewise, since

$tab + ab \in y$, y cannot be mapped to $tab + a + b$. Therefore, $\varphi(x) \leq_Q \varphi(y)$. The argument is similar if $\varphi(x) = \ell ab + a + b$.

Suppose $\varphi(x) = \widehat{1}_Q$ and x is a chain of type 1 or type 2 with parameter λ. Then $C_\lambda \subseteq x \subseteq y$. Again, since $tab + b$ and $tab + ab$ are elements of y for $0 \leq t \leq \lambda - 1$, y cannot be mapped by φ to $tab + a$ or $tab + a + b$, respectively. Also, since $B_\lambda \cap y \neq \emptyset$, we cannot have y mapped to $tab + a$ or $tab + b$ for $t \geq \lambda$. Therefore $\varphi(y) = \widehat{1}_Q$.

Suppose x is of type 3 with parameter λ. As before, since $C_\lambda \subset y$, the element y cannot be mapped to $tab + a$ or $tab + a + b$ for $0 \leq t \leq \lambda - 1$. Since $\lambda ab + a \not<_\Lambda n$, y cannot be mapped to $\lambda ab + a$ or $\lambda ab + a + b$. Finally, using the Frobenius number as we did in Lemma 4.4.3, we know that $n \leq (\lambda + 1)ab - b$. Thus y cannot be mapped to $tab + a$ or $tab + a + b$ for $t \geq \lambda + 1$ since they are not comparable with n. Thus $\varphi(y) = \widehat{1}_Q$. Using similar reasoning, we see this holds if x is of type 4 as well. Therefore, φ is order-preserving. □

Lemma 4.4.6. *For $m <_Q \widehat{1}_Q$, the collection $\{(x, x \cup \{m\}) : m \notin x \in \varphi^{-1}(m)\}$ is a perfect acyclic matching on the fiber $\varphi^{-1}(m)$.*

Proof. Let $x \in \varphi^{-1}(\ell ab + a)$ and suppose $\ell ab + a \in x$. It is clear that $d(x) = x - \{\ell ab + a\}$ belongs to the Frobenius complex $\Delta([0, n]_\Lambda)$ since we simply remove the element $\ell ab + a$. Also, since $\ell ab + a <_\Lambda n$, we see from the definition of φ that removing the element $\ell ab + a$ will not affect where x is mapped. Thus, we have $\varphi(d(x)) = \ell ab + a$.

Suppose $\ell ab + a \notin x$. We need to argue that $u(x) = x \cup \{\ell ab + a\}$ is an element of the Frobenius complex and that it is mapped to $\ell ab + a$. By definition of φ we know that $\ell ab \in x$. Thus we only need to ensure that elements which are comparable to ℓab but not to $\ell ab + a$ are not contained in x. The only such elements are those that can only be written as $\ell ab + jb$ for $1 \leq j \leq a - 1$. However, these are elements of $B_\ell \cup \{\ell ab + b\}$ and are therefore not in x. Again, since the element $\ell ab + a$ does not affect where x is mapped, we conclude $\varphi(u(x)) = \ell ab + a$.

Now suppose that $x \in \varphi^{-1}(\ell ab + a + b)$ with $\ell ab + a + b \in x$. By similar reasoning as above, we see that $d(x) = x - \{\ell ab + a + b\}$ is an element of the Frobenius complex that is also mapped to $\ell ab + a + b$. Suppose $\ell ab + a + b \notin x$. To see that $u(x) = x \cup \{\ell ab + a + b\}$ is an element of the Frobenius complex that is mapped to $\ell ab + a + b$, we note that $\ell ab + b \in x$ by definition. The only elements that are comparable to $\ell ab + b$ but not to $\ell ab + a + b$ are elements that can only be written as $\ell ab + jb$ where $2 \leq j \leq a$. Since these are elements of $B_\ell \cup \{\ell ab + ab\}$, they are not allowed to be in x. Therefore, $\varphi(u(x)) = \ell ab + a + b$ and this matching is perfect.

Finally, the matching on each fiber is clearly acyclic since the same element is either added or removed from a chain. $\square$

Continuation of Example 4.4.2. (Part 3) In our example, 152 chains are mapped to the element 3 in Q, 40 chains are mapped to 7 in Q, and none are mapped to 15 and 19. All of these chains are matched perfectly in our Morse matching.

Thus we have reduced the problem to finding an acyclic matching on the fiber $\Gamma = \varphi^{-1}(\widehat{1}_Q)$.

Lemma 4.4.7. *The collection* $\{(x, x \cup \{\lambda ab + b\})$: x *is a chain of type* 1 *with parameter* $\lambda\}$ *is an acyclic matching on* Γ *where the critical cells are the chains of type* 3 *and* 4.

Proof. We have seen from parts (i) and (ii) of Lemma 4.4.4 that to every element x of type 1 there exists a corresponding element y of type 2 with the same parameter and vice-versa. In other words, this is a perfect matching on chains of type 1 and 2. Chains of type 3 and 4 are left unmatched.

We must now show that this matching is acyclic, that is, a directed cycle of the form described in Definition 4.2.4 cannot exist. Let z_1 be a chain of type 2 with parameter λ. Then $d(z_1) = z_1 - \{\lambda ab + b\}$ is an element of type 1 with the same λ. Part (iii) of Lemma 4.4.4 tells us that any z_2 different from z_1 will have a smaller parameter. Therefore, we cannot return to z_1 using our matching. Hence the matching is acyclic. $\square$

Continuation of Example 4.4.2. (Part 4) All type 1 chains are matched with chains of type 2. We are only left with the two chains $\{4, 12\}$ and $\{4, 12, 16\}$ of type 3 and 4, respectively. As we will see in the proof of Theorem 4.4.1, these two can be matched together without creating any cycles in the matching. Hence, the complex is contractible.

Lemma 4.4.8. *Let $n = kab + r$ for $0 \leq r < ab$. If $r = 0, a, b,$ or $a + b$, then the matching given in Lemma 4.4.7 has exactly one critical cell. If $r = jb$ for $2 \leq j \leq a - 1$, there are exactly two unmatched chains of Γ. Otherwise, there are no critical cells in Γ. More precisely, the critical cells of Γ are given by*

$$\begin{cases} \{D_{k-1}\} & \text{if } n = kab, \\ \{C_k\} & \text{if } n = kab + a, \\ \{C_k\} & \text{if } n = kab + b, \\ \{D_k\} & \text{if } n = kab + a + b, \\ \{C_k, D_k\} & \text{if } n = kab + ib, 2 \leq i \leq a - 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. The only elements of Γ that were not matched are those of type 3 and 4 in Lemma 4.4.3. Thus, we need to determine the number of type 3 and 4 elements in Γ, that is, the number of integers λ such that $\lambda ab <_\Lambda n$ and $\lambda ab + a \not<_\Lambda$ (type 3) and integers λ such that $\lambda ab + b <_\Lambda n$ and $\lambda ab + a + b \not<_\Lambda n$ (type 4).

Using the Frobenius number, we know that every integer smaller than

$$n - (ab - a - b) = (k - 1)ab + a + b + r$$

is comparable with n with respect to the order $<_\Lambda$. We do not need to check $tab + a$ or $tab + a + b$ for $0 \leq t < k$ because these numbers are always comparable to n (unless $r = 0$ when we must check $(k - 1)ab + a + b$). We also do not need to consider $tab + a$ or $tab + a + b$ for $t \geq k + 1$ because we would have $tab + a, tab, tab + a + b$, and $tab + b$ all not contained in $[0, n]_\Lambda$. Thus, we only need to check $(k - 1)ab + a + b$ (if $r = 0$), $kab + a$, and $kab + a + b$

There are nine cases to consider.

– $r \notin \Lambda$. Then we have both

$$kab + a \not<_\Lambda n \text{ and } kab \not<_\Lambda n$$

and

$$kab + a + b \not<_\Lambda \text{ and } kab + b \not<_\Lambda n.$$

Therefore, there are no critical cells.

Otherwise, r belongs to the semigroup Λ and we can write $r = ia + jb$, where i and j are unique non-negative integers.

– $(i, j) = (0, 0)$. We see that $kab + a \not<_\Lambda n$, but also $kab \not<_\Lambda n$. Similarly, we have $kab + a + b \not<_\Lambda n$ and $kab + b \not<_\Lambda n$. Finally, we check and see that

$$(k-1)ab + a + b \not<_\Lambda n$$

since $kab - ((k-1)ab + a + b) = ab - a - b$ which is the Frobenius number. Also

$$(k-1)ab + b <_\Lambda n$$

because $kab - ((k-1)ab + b) = ab - b = (a-1)b \in \Lambda$. Thus we have one critical cell D_{k-1}.

– $(i, j) = (1, 0)$. We can easily see that $kab + a + b \not<_\Lambda n$ and $kab + b \not<_\Lambda n$. However, we have $kab + a \not<_\Lambda n$ while $kab <_\Lambda n$. Therefore, we have one critical cell C_k.

– $(i, j) = (0, 1)$. In this case we again see that $kab + a + b \not<_\Lambda n$ and $kab + b \not<_\Lambda n$. However, we still have $kab + a \not<_\Lambda n$, while $kab <_\Lambda n$. Thus we have one critical cell C_k.

– $(i, j) = (1, 1)$. First we note that $kab + a <_\Lambda n$. Thus we only check to see that $kab + a + b \not<_\Lambda n$ and $kab + b <_\Lambda n$. This is easily true, so there is one critical cell D_k.

– $i = 0, 2 \leq j \leq a - 1$. Clearly $kab + a \not<_\Lambda n$ while $kab <_\Lambda n$. Also, we see that $kab + a + b \not<_\Lambda n$ while $kab + b <_\Lambda n$. Thus the unmatched cells are C_k and D_k.

– $i \geq 1, j \geq 2$. Both $kab+a$ and $kab+a+b$ are both comparable with n. Therefore there are no critical cells.

– $i \geq 2, j = 0$. We see that $kab + a$ is comparable with n. Also, both $kab + a + b$ and $kab + b$ are not comparable with n. Therefore there are no critical cells.

– $i \geq 2, j = 1$. Then $kab + a$ and $kab + a + b$ are both comparable with n. Therefore there are no critical cells.

□

Proof of Theorem 4.4.1. By applying the Patchwork Theorem to the function φ we see that the homotopy type of $\Delta([0,n]_\Lambda)$ depends only on the fiber $\varphi^{-1}(\widehat{1}_Q) = \Gamma$. Applying Lemmas 4.4.7 and 4.4.8, there is only one critical cell when $n \equiv 0, a, b, a+b \mod ab$ and no critical cells in every other case except when $i = 0$ and $2 \leq j \leq a-1$. However, we claim in this last case we can add the pair (C_k, D_k) to the matching on Γ and still be left with an acyclic matching.

Lemma 4.4.4 (iv) shows that a chain of type 4 does not cover any chain of type 1. Hence, when adding the edge (C_k, D_k) to the Morse matching of Γ, it will not create any directed cycles through the chain D_k. Hence the matching is still acyclic and there are no critical cells in this case.

The critical cells for $n \equiv 0, a, b, a+b \mod ab$ can be easily seen to be of dimension $2n/ab-2, 2(n-a)/ab-1, 2(n-b)/ab-1$, and $2(n-a-b)/ab$, respectively. Therefore, applying the main theorem of reduced discrete Morse theory, Corollary 4.2.6, proves the result. □

4.5 Generators in the arithmetic sequence $\{a, a+d, \ldots, a+(a-1)d\}$

Recall that the q^d-analogue of a non-negative integer a is defined as follows:

$$[a]_{q^d} = 1 + q^d + (q^d)^2 + \cdots + (q^d)^{a-1}.$$

Theorem 4.5.1. *Let Λ be the semigroup generated by the integers $\{a, a+d, a+2d, \ldots, a+(a-1)d\}$ where a and d are relatively prime. The order complex of the* *associated Frobenius interval* $[0,n]_\Lambda$ *is homotopy equivalent to a wedge of spheres where the* i*th Betti number satisfies*

$$\sum_{n\geq 0} \widetilde{\beta}_i q^n = q^{a+(i+1)(a+d)} \cdot [a]_{q^d} \cdot [a-1]_{q^d}^{i+1}.$$

Example 4.5.2. For the generators $\{4,5,6,7\}$, that is $a=4$ and $d=1$, we have

$$\begin{aligned}\sum_{n\geq 0} \widetilde{\beta}_1 q^n &= q^{14}\cdot[4]\cdot[3]^2 = q^{14}+\cdots+3q^{20}+q^{21},\\ \sum_{n\geq 0} \widetilde{\beta}_2 q^n &= q^{19}\cdot[4]\cdot[3]^3 = q^{19}+4q^{20}+\cdots+q^{28},\end{aligned}$$

and no other generating polynomial contains the q^{20} term. Hence the Frobenius complex $\Delta([0,20]_\Lambda)$ is homotopy equivalent to a wedge of three circles and four 2-spheres.

Roberts showed in [64] that the Frobenius number of the arithmetic sequence $\{a, a+d, a+2d, \ldots, a+sd\}$ is given by

$$\left(\left\lfloor \frac{a-2}{s} \right\rfloor + 1\right)\cdot a + (d-1)\cdot(a-1) - 1.$$

Therefore, for the generators $\{a, a+d, a+2d, \ldots, a+(a-1)d\}$, we have the Frobenius number

$$\left(\left\lfloor \frac{a-2}{a-1} \right\rfloor + 1\right)\cdot a + (d-1)(a-1) - 1 \quad = \quad (a-1)d.$$

We will proceed as before and use discrete Morse theory and the Patchwork Theorem. Define the set A as follows:

$$A = \{a+d, a+2d, \ldots, a+(a-1)d\}.$$

Definition 4.5.3. *Given* n*, let* R *be the chain* $\{1,2,3,4,\ldots,n-a\}$ *with a maximal element* $\widehat{1}_R$ *adjoined. That is,*

$$R = \{1,2,3,\ldots,n-a\} \cup \{\widehat{1}_R\}.$$

If $x = \{x_1, x_2, \ldots, x_k\} \in \Delta([0,n]_\Lambda)$ and we define $x_0 = 0$, let $\psi : \mathcal{F}(\Delta([0,n]_\Lambda)) \to R$ be a map defined by

$$\psi(x) = \begin{cases} x_{i-1} + a, & x_i - x_{i-1} \notin A, \\ & x_j - x_{j-1} \in A \\ & \text{for } 1 \leq j \leq i-1, \\ x_k + a, & n - x_k \notin \{a\} \cup A, \\ & x_j - x_{j-1} \in A \\ & \text{for } 1 \leq j \leq k, \\ \widehat{1}_R, & \text{otherwise.} \end{cases}$$

Example 4.5.4. Suppose $a = 3$, $d = 1$, and $n = 19$. Then $A = \{4, 5\}$.

- $\psi(\{3, 7, 12\}) = 3$ since $x_1 - x_0 = 3$.
- $\psi(\{4, 8, 15\}) = 11$ since $x_3 - x_2 = 7$.
- $\psi(\{5, 9, 13\}) = 16$ since $n - x_3 = 6$.
- $\psi(\{5, 9, 14\}) = \widehat{1}_R$.

Lemma 4.5.5. *The element $m \cdot d$ is not contained in the semigroup Λ for any $1 \leq m \leq a - 1$.*

Proof. Suppose $m \cdot d \in \Lambda$. Then, by Lemma 4.3.3 we know $m \cdot d = (a + i \cdot d) + j \cdot a$. Thus, we see that $(m - i) \cdot d = (j + 1) \cdot a$. Since a and d are relatively prime, it must be that $m - i \geq a$, that is, $m \geq a + i \geq a$. Since $m \leq a - 1$, the result follows. □

Lemma 4.5.6. *Let x_i and x_j be elements of a chain x such that $x_i - x_j \in \{a\} \cup A$. Then the open interval $(x_i, x_j)_\Lambda$ is empty.*

Proof. Suppose $x_j - x_i = a + md \in \{a\} \cup A$ and $e = x_i + (a + \ell d) \in (x_i, x_j)$, that is, $0 \leq \ell < m \leq (a - 1)$. Then $x_j - e = (m - \ell)d$. By the previous lemma, we know that x_j and e are not comparable. Therefore, e cannot exist. □

The following lemma is an immediate consequence of Lemma 4.5.6 and the definition of the function ψ.

Lemma 4.5.7. *If x and y are chains such that $x \subseteq y$ and $\psi(x) = x_{i-1} + a$ then $x_j = y_j$ for $1 \leq j \leq i-1$. In particular, if $\psi(x) = \widehat{1}_R$ then $x = y$.*

We can finally give a few properties of the map ψ.

Lemma 4.5.8. *The map $\psi : \mathcal{F}(\Delta([0,n]_\Lambda)) \to R$ is an order preserving poset map.*

Proof. Suppose x and y are chains such that $x \subseteq y$ with $\psi(x) = x_{i-1} + a$. We wish to show that $\psi(y) \geq x_{i-1} + a$. The previous lemma says that $x_j = y_j$ for $1 \leq j \leq i-1$. In particular, $y_j - y_{j-1} \in A$. Therefore, by definition of the function ψ, we have $\psi(y) \geq y_{i-1} + a = x_{i-1} + a$. $\square$

Lemma 4.5.9. *For $m <_R \widehat{1}_R$, the collection $\{(x, x \cup \{m\}) : m \notin x \in \psi^{-1}(m)\}$ is a perfect acyclic matching on the fiber $\psi^{-1}(m)$.*

Proof. Suppose $\psi(x) = x_{i-1} + a$ and $x_{i-1} + a \in x$. That is, $x_i = x_{i-1} + a$. It is clear that $d(x) = x - \{x_i\}$ is a valid chain in the Frobenius complex since we are simply removing an element. We need to check that $\psi(d(x)) = x_{i-1} + a$. We know that $d(x)_j - d(x)_{j-1} = x_j - x_{j-1} \in A$ for $1 \leq j \leq (i-1)$. Suppose $d(x)_i - d(x)_{i-1} = x_{i+1} - x_{i-1} \in A$. Then, by Lemma 4.5.6, $(x_{i-1}, x_{i+1})_\Lambda$ would have to be empty. This contradicts the fact that $x_i \in (x_{i-1}, x_{i+1})_\Lambda$ in the chain x. Since $d(x)_i - d(x)_{i-1} \notin A$ and $d(x)_j - d(x)_{j-1} \in A$ for $1 \leq j \leq (i-1)$, we have $\psi(d(x)) = d(x)_{i-1} + a = x_{i-1} + a$.

Now suppose that $\psi(x) = x_{i-1} + a$ and $x_{i-1} + a \notin x$. It is clear that $u(x) = x \cup \{x_{i-1} + a\}$ would be mapped to $x_{i-1} + a$. Thus, it must be shown that $u(x)$ is a valid chain, that is, $x_{i-1} + a$ is comparable to x_i. We know that $x_i - x_{i-1} \notin A$. Suppose

$$x_i - x_{i-1} = (a + s_1 d) + (a + s_2 d) + \cdots + (a + s_k d)$$

where $s_1 \leq s_2 \leq \cdots \leq s_k \leq a - 1$ and $k \geq 2$. Then

$$x_i - (x_{i-1} + a) = (a + (s_1 + s_2)d) + (a + s_3 d) + \cdots + (a + s_k d).$$

If $s_1 + s_2 \leq a - 1$, then we have written this difference as a sum of generators. Therefore, x_i and $x_{i-1} + a$ are comparable.

If $s_1+s_2 > a-1$, then the difference is larger than $(a-1)d$, which is the Frobenius number of the generators. Thus, x_i and $x_{i-1}+a$ are comparable. Therefore, $u(x)$ is a valid chain.

Finally, the matching on the fiber is clearly acyclic since the same element is either added or removed from a chain. □

Using the Patchwork Theorem, we have an acyclic matching on $\mathcal{F}(\Delta([0,n]_\Lambda))$ whose only critical cells are the elements of the fiber $\psi^{-1}(\widehat{1}_R)$. Note that Lemma 4.5.7 says that each of these cells are maximal. Therefore, due to Theorem 4.2.7, we will have that $\Delta([0,n]_\Lambda)$ is homotopy equivalent to a wedge of spheres whose number and dimension corresponds to the number and dimension of the critical cells.

Thus, we are interested in counting the number of chains that are mapped to $\widehat{1}_R$. The following lemma is straightforward from the definition of the function ψ.

Lemma 4.5.10. *The fiber $\psi^{-1}(\widehat{1}_R)$ consists of elements $x = \{x_1, x_2, \ldots, x_k\}$ where $x_i - x_{i-1} \in A$ for $1 \leq i \leq k$ and $n - x_k \in \{a\} \cup A$.*

Proof of Theorem 4.5.1. We know from Lemma 4.5.10 that the critical cells are in bijection with compositions of n where the last part belongs to the set $\{a\} \cup A$ and the remaining parts belong to the set A. Furthermore, if such a composition has $i+2$ parts, it will contribute to the i-dimensional homology. Hence, fixing i, we obtain the generating function

$$\begin{aligned}
\sum_{n\geq 0} \widetilde{\beta}_i q^n &= \left(\sum_{k=0}^{a-1} q^{a+kd}\right) \cdot \left(\sum_{\ell=1}^{a-1} q^{a+\ell d}\right)^{i+1} \\
&= q^{a+(i+1)(a+d)} \cdot \left(\sum_{k=0}^{a-1} q^{kd}\right) \cdot \left(\sum_{\ell=0}^{a-2} q^{\ell d}\right)^{i+1} \\
&= q^{a+(i+1)(a+d)} \cdot [a]_{q^d} \cdot [a-1]_{q^d}^{i+1}.
\end{aligned}$$

□

4.6 Concluding remarks

A more general situation is to consider a sub-semigroup Λ of $\mathbb{N}^d$ and define a partial order on $\mathbb{Z}^d$ by $\mu \leq_\Lambda \lambda$ if $\lambda - \mu \in \Lambda$. Define the semigroup algebra $k[\Lambda]$ as the linear span of the monomials whose powers belong to Λ, that is, $k[\Lambda] = \mathrm{span}\{x^\lambda = x_1^{\lambda_1} \cdots x_d^{\lambda_d} : \lambda \in \Lambda\}$. Laudal and Sletsjøe [55] and Peeva, Reiner, and Sturmfels [61] make the connection between the homology of the order complex of intervals in this partial order and the semigroup algebra $k[\Lambda]$.

Theorem 4.6.1 (Laudal–Sletsjøe and Peeva–Reiner–Sturmfels)**.** *For Λ a sub-semigroup of $\mathbb{N}^d$ with the associated monoid Λ, the following two equalities hold*

$$\dim_k \mathrm{Tor}_i^{k[\Lambda]}(k,k)_\lambda = \dim_k \widetilde{H}_{i-2}(\Delta([0,\lambda]_\Lambda), k),$$

$$\dim_k \mathrm{Ext}^i_{k[\Lambda]}(k,k)_\lambda = \dim_k \widetilde{H}^{i-2}(\Delta([0,\lambda]_\Lambda), k),$$

for all $\lambda \in \Lambda$ and $i \geq 0$.

For more information, see the dissertation of Stamate [67].

The papers [14, 49, 61] continue to study the topology of the intervals in this partial order. Hersh and Welker [49] give bounds on the indices of the non-vanishing homology groups of the order complex of the intervals. Peeva, Reiner, and Sturmfels [61] show that the semigroup ring $k[\Lambda]$ is Koszul if and only if each interval in Λ is Cohen-Macaulay.

As a corollary, we obtain

Corollary 4.6.2. *Let a and b be relatively prime integers such that $1 < a < b$. Let R denote the ring $k[y,z]/(y^b - z^a)$. Then the multigraded Poincaré series*

$$\begin{aligned} P_k^R(t,q) &= \sum_{n\in\Lambda}\sum_{i\geq 0} \dim_k \left(\mathrm{Tor}_i^R(k,k)_n\right) t^i q^n \\ &= \sum_{n\in\Lambda}\sum_{i\geq 0} \dim_k \left(\mathrm{Ext}_R^i(k,k)_n\right) t^i q^n \end{aligned}$$

is given by the rational function

$$\frac{1 + tq^a + tq^b + t^2 q^{a+b}}{1 - t^2 q^{ab}}.$$

Proof. Let Λ be the semigroup generated by a and b. Observe that the ring R is isomorphic to the semigroup ring $k[\Lambda]$. By combining Theorems 4.4.1 and 4.6.1 the multigraded Poincaré series is given by

$$P_k^R(t,q) = 1 + tq^a + tq^b + t^2q^{a+b} + t^2q^{ab} + t^3q^{ab+a} + t^3q^{ab+b} + t^4q^{ab+a+b} + \cdots ,$$

which is the sought-after rational generating function. $\square$

We now highlight three open questions.

Question 4.6.3. The Frobenius poset generated by two relatively prime integers can be embedded on a cylinder. There are many results (see, for example, [6, 28]) on posets that can be embedded in the plane. Can any of these results be extended to cylindrical posets?

Question 4.6.4. There are other classes of generators, such as a geometric sequence, that have closed formulas for the Frobenius number, see [58]. Does the Frobenius complex have a nice topological representation in this case?

Question 4.6.5. More generally, all computational evidence suggests that the Frobenius complex – even for randomly selected generators – has a relatively simple topology, that is, it is torsion-free. Is there a set of generators that creates torsion in the associated Frobenius complex?

Chapter 5 Enumerating Q-factorial posets

5.1 Introduction

A poset is said to be $(2+2)$-*free* if it does not contain an induced sub-poset that is isomorphic to the union of two disjoint 2-chains. These $(2+2)$-free posets have been completely categorized by Bogart [15]. Given a poset P, he defined the *strict downset* of $x \in P$ by

$$D(x) = \{y : y < x\}$$

and proved the poset is $(2+2)$-free if and only if the set of strict downsets of P can be linearly ordered by inclusion. Fishburn [43] also showed that a poset is $(2+2)$-free when it is isomorphic to an interval order. Bousquet–Mélou, Claesson, Dukes, and Kitaev [17] found bijections between the following four sets: unlabeled $(2+2)$-free posets, a class of involutions, a family of permutations, and ascent sequences. Claesson and Linusson [24] then constructed a family of $(2+2)$-free posets they called factorial posets. They were then able to show that there were $n!$ such posets with n vertices. In the following chapter, we will extend the construction used by Claesson and Linusson to create and enumerate a new family of posets. We will end with a few open questions and directions for further research.

5.2 Q-factorial posets

Definition 5.2.1. *Let P and Q be labeled posets on the set $[n] = \{1, 2, \ldots, n\}$. The poset P is said to be Q*-factorial *if*

1. $i <_P j$ *implies* $i <_Q j$,

2. $i <_Q j <_P k$ *implies* $i <_P k$.

Let $\mathcal{F}(Q)$ denote the collection of Q-factorial posets.

An example was given in Figure 1.2. Given an arbitrary poset Q, we would like to know the number of Q-factorial posets, that is, the cardinality $|\mathcal{F}(Q)|$.

For a poset Q, we will denote by $I(Q)$ the set of all strict inequalities in Q. For example, $I(B_2) = \{1 < 2, 1 < 3, 1 < 4, 2 < 4, 3 < 4\}$, see Figure 5.1. If P is a Q-factorial poset, it is clear from the first condition that the set of strict inequalities of P is a subset of the set of strict inequalities of Q, that is $I(P) \subseteq I(Q)$. In other words, we can create P by removing inequalities from Q. However, care must be taken when removing relations because removing one might force the removal of others.

Lemma 5.2.2. *Suppose P is a Q-factorial poset, $i <_Q j <_Q k$, and $i \not<_P k$. Then it must be that $j \not<_P k$.*

Proof. Suppose $j <_P k$. Then we have $i <_Q j <_P k$. The second condition from Definition 5.2.1 forces $i <_P k$ which is a contradiction. $\square$

When removing relations we must also be aware of transitivity. That is, we must have $i <_P k$ if $i <_P j$ and $j <_P k$. However, the forced removal of elements in Lemma 5.2.2 breaks this chain of transitivity. Other than this, the relations can be removed independently of each other. More specifically, the removal of $h <_Q i$ does not force the removal of $j <_Q k$ for any $k \neq i$.

For every poset Q define the polynomial $\mathcal{R}(Q)$ by

$$\mathcal{R}(Q) = \sum_{P \in \mathcal{F}(Q)} q^{|I(Q)-I(P)|}.$$

Note that we can compute $|\mathcal{F}(Q)|$ by setting $q = 1$ in $\mathcal{R}(Q)$.

Let k be an element of Q. Define $\mathcal{F}_k(Q) \subseteq \mathcal{F}(Q)$ as the collection of Q-factorial posets P that can only be created by removing inequalities of the form $i \leq_Q k$. That is,

$$\mathcal{F}_k(Q) = \{P \in \mathcal{F}(Q) \ : \ a <_Q b \text{ but } a \not<_P b \text{ implies } b = k\}.$$

Likewise, let $\mathcal{R}_k(Q)$ be the polynomial defined by

$$\mathcal{R}_k(Q) = \sum_{P \in \mathcal{F}_k(Q)} q^{|I(Q)-I(P)|}.$$

Theorem 5.2.3. *For a poset Q, the polynomial $\mathcal{R}(Q)$ is given by the product*

$$\mathcal{R}(Q) = \prod_{k \in Q} \mathcal{R}_k(Q).$$

Proof. On the left hand side, the coefficient of q^m is the number of posets $P \in \mathcal{F}(Q)$ formed by removing m inequalities from $I(Q)$. Since inequalities of the form $i <_Q k$ for different values of k can be removed independently, on the right hand side we get a contribution to q^m when the number of inequalities of the form $i <_Q k$ removed for every $k \in Q$ sum to m. This gives precisely a poset $P \in \mathcal{F}(Q)$. □

Thus, it remains to compute $\mathcal{R}_k(Q)$.

5.3 Computing $\mathcal{R}_k(Q)$

Definition 5.3.1. *A* Puiseux polynomial f *on the variables $x_1, x_2, \ldots, x_n$ is a linear combination of the terms $x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where the exponents $p_1, p_2, \ldots, p_n$ are rational numbers. The* polynomial part *of a Puiseux polynomial is the sum of all the terms that have only non-negative integer exponents, denoted* $\mathrm{poly}(f)$.

Example 5.3.2. Consider the Puiseux polynomial $f(x_1, x_2, x_3) = 1 + 2x_1x_2 + x_2^{1/2}x_3 + x_1^2x_2x_3 + x_3^{5/3}$. Then we have the polynomial part

$$\mathrm{poly}(f) = 1 + 2x_1x_2 + x_1^2x_2x_3.$$

For each $v \in Q$, let x_v be a variable associated with the vertex v. For $k \in Q$ and v an element of the sub-poset $Q_{\le k}$, let $C(v)$ be the number of maximal chains of $Q_{\le k}$ that contain v. Now define the weight

$$\ell(v) = \begin{cases} 1 & \text{if } v = k, \\ x_v^{1/C(v)} & \text{otherwise.} \end{cases}$$

Theorem 5.3.3. *The polynomial $\mathcal{R}_k(Q)$ is given by*

$$\mathcal{R}_k(Q) = \mathrm{poly}\left(\prod_c \sum_{i=1}^{m} \prod_{j=1}^{i} \ell(v_j) \right)\Bigg|_{x_v = q \ \forall \ v \le_Q k}$$

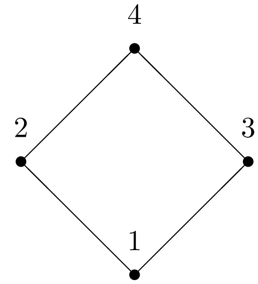

Figure 5.1: The Hasse diagram of the Boolean algebra on two elements, B_2. $\mathcal{R}(B_2) = 1 \cdot (1+q)^2 \cdot (1+2q+q^2+q^3) = 1+4q+6q^2+5q^3+3q^4+q^5$.

where the first product is over all maximal chains $c = \{k = v_1 > v_2 > \cdots > v_m\}$ *in* $Q_{\leq k}$.

Proof. We consider removing inequalities from one maximal chain at a time. Let $c = \{k = v_1 > v_2 > \cdots > v_m\}$ be a maximal chain. Suppose the relation $v_i <_Q k$ is removed from this maximal chain. Then $v_i <_Q k$ must also be removed from every maximal chain that contains v_i. Since the exponent of x_{v_i} is $1/C(v_i)$, the only way to get integral exponent on x_{v_i} is if $v_i <_Q k$ is not removed from any maximal chain containing v_i or is removed from all maximal chains containing v_i.

By removing $v_i <_Q k$, Lemma 5.2.2 states that we must also remove the relation $v_j <_Q k$ for $1 \leq j < i$. Hence, for this maximal chain we get the sum of the products $\prod_{j=1}^{i} \ell(v_j)$.

The terms of this product with a fractional exponent indicate that a relation was removed from some but not all of the maximal chains containing it. Since this is not possible, those terms are discarded. What remains, the polynomial part, indicate exactly which relations can be removed. By setting all of the variables equal to q, we get exactly how many relations were removed. $\square$

Example 5.3.4. Consider the poset in Figure 5.1. We first need to compute $\mathcal{R}_4(B_2)$. The weights for each element are:

$$\ell(1) = x_1^{1/2}, \;\; \ell(2) = x_2, \;\; \ell(3) = x_3, \;\; \ell(4) = 1$$

Since there are two maximal chains, we have the product

$$\left(1 + x_2 + x_2 \cdot x_1^{1/2}\right) \cdot \left(1 + x_3 + x_3 \cdot x_1^{1/2}\right).$$

The polynomial part of this product is

$$1 + x_2 + x_3 + x_2 \cdot x_3 + x_2 \cdot x_3 \cdot x_1.$$

This means that for inequalities of the form $i <_{B_2} 4$ we could remove none of them, we could remove only the inequality where $i = 2$, only the inequality where $i = 3$, both where $i = 2$ and $i = 3$, or all three where $i = 2$, $i = 3$ and $i = 1$. However, in order to find $\mathcal{R}_4$, we set all variables equal to q. That is

$$\mathcal{R}_4(B_2) = 1 + 2q + q^2 + q^3.$$

Following the same process, we get $\mathcal{R}_2(B_2) = \mathcal{R}_3(B_2) = 1 + q$ and $\mathcal{R}_1(B_2) = 1$. Therefore, by Theorem 5.2.3,

$$\begin{aligned} \mathcal{R}(B_2) &= 1 \cdot (1+q)^2 \cdot (1 + 2q + q^2 + q^3) \\ &= 1 + 4q + 6q^2 + 5q^3 + 3q^4 + q^5. \end{aligned}$$

In particular, setting $q = 1$, there are 20 B_2-factorial posets.

Corollary 5.3.5. *Assume that the poset Q is a tree, that is, Q has a minimal element $\widehat{0}$ and every other element covers exactly one other element. Then $R_k(Q)$ is given by*

$$\mathcal{R}_k(Q) = 1 + q + q^2 + \cdots + q^{|[\widehat{0},k]|-1}.$$

Proof. Assume the one maximal chain in $Q_{\leq k}$ has m elements. Since there is only one, the first product in Theorem 5.3.3 will only have one term. Also, the exponent on every variable will be one eliminating the possibility of getting fractional exponents. Therefore

$$\begin{aligned} \mathcal{R}_k(Q) &= 1 + \sum_{i=2}^{m} \prod_{j=2}^{i} x_{v_j} \Bigg|_{x_v = q \ \forall \ v \leq k} \\ &= 1 + q + \cdots + q^{|[\widehat{0},k]|-1}. \end{aligned}$$

□

By letting Q be an n-chain and using this corollary, the result by Claesson and Linusson that $\mathcal{F}(Q) = n!$ is straightforward.

5.4 Concluding remarks

We end this chapter with a few open questions.

Question 5.4.1. Claesson and Linusson [24] showed that when Q is an n-chain, the number of posets P satisfying the three constraints $x <_P y$ implies $x <_Q y$, $x <_Q y <_P z$ implies $x <_P z$, and $x <_P y <_Q z$ implies $x <_P z$ is given by C_n, the nth Catalan number. Does this result extend to the other posets Q?

Question 5.4.2. Are there any posets Q other than trees where the number of Q-factorial posets can be calculated? If so, can the finer statistic $\mathcal{R}(Q)$ also be determined?

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Vita

Eric L. Clark

- Background
  - Born: April 25, 1983
  - Hometown: Owensboro, Kentucky
- Education
  - 2011: Ph.D. University of Kentucky, expected
  - 2007: M.A. University of Kentucky
  - 2005: B.A. Transylvania University
- Professional Experience
  - Teaching Assistant, University of Kentucky, 2005–2011.
  - Adjunct Professor, Transylvania University, 2010.
  - Algebra Cubed Fellow, University of Kentucky, 2007–2008.
- Publications
  - "The Frobenius complex," to appear in the *Annals of Combinatorics*, with Richard Ehrenborg.
  - "Excedances of affine permutations," *Advances in Applied Mathematics* **46** (2011), 175–191, with Richard Ehrenborg.
  - "Explicit expressions for extremal excedance set statistics," *European Journal of Combinatorics* **31** (2010), 270–279, with Richard Ehrenborg.
  - "Bluma's method: A different way to solve quadratics," *Georgia Council of Teachers of Mathematics: Reflections*, Vol. 53, No. 3 (2009), 19–21, with Richard Millman.
- Selected Presentations
  - "The Frobenius complex," poster presentation, Formal Power Series and Algebraic Combinatorics, San Francisco State University, August 2, 2010.
  - "An introduction to discrete Morse theory," Graduate Student Colloquium, University of Kentucky, April 14, 2010.
  - "The Frobenius poset," (invited) AMS Southeastern Sectional Meeting, University of Kentucky, March 27, 2010.
  - "The mathematics of juggling," with Richard Ehrenborg, Geek Week presentation, University of Kentucky, April 8, 2009.

- "An introduction to the affine symmetric group," Discrete CATS Seminar, University of Kentucky, February 16, 2009.
- "Bluma's method," AMS/MAA Joint Meeting, Washington, D.C., January 7, 2009.
- "Computing the bracket of excedance sets," Graduate Student Combinatorics Conference, University of California – Davis, April 13, 2008.
- Algebra Cubed poster presentation (with five other Algebra Cubed Fellows), NSF GK-12 National Meeting, Washington, D.C., March 2008.

• Awards

- Steckler Fellowship, Fall 2009 – Spring 2010, University of Kentucky.
- Research Assistantship, Summer 2009, University of Kentucky.
- College of Arts and Sciences Certificate for Outstanding Teaching, April 2009, University of Kentucky.
- Van Meter Fellowship, Fall 2007 – Spring 2008, University of Kentucky.
- William T. Young Scholar, 2001 – 2005, Transylvania University.
- Hollian Society Inductee, Spring 2005, Transylvania University.