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THE h-VECTORS OF MATROIDS AND THE ARITHMETIC DEGREE OF SQUAREFREE STRONGLY STABLE IDEALS

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ABSTRACT OF DISSERTATION

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A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
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Director: Dr. Uwe Nagel, Professor of Mathematics
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Making use of algebraic and combinatorial techniques, we study two topics: the arithmetic degree of squarefree strongly stable ideals and the h-vectors of matroid complexes.

For a squarefree monomial ideal, I, the arithmetic degree of I is the number of facets of the simplicial complex which has I as its Stanley-Reisner ideal. We consider the case when I is squarefree strongly stable, in which case we give an exact formula for the arithmetic degree in terms of the minimal generators of I as well as a lower bound resembling that from the Multiplicity Conjecture. Using this, we can produce an upper bound on the number of minimal generators of any Cohen-Macaulay ideals with arbitrary codimension extending Dubreil's theorem for codimension 2.

A matroid complex is a pure complex such that every restriction is again pure. It is a long-standing open problem to classify all possible h-vectors of such complexes. In the case when the complex has dimension 1 we completely resolve this question and we give some partial results for higher dimensions. We also prove the 1-dimensional case of a conjecture of Stanley that all matroid h-vectors are pure $\mathcal{O}$-sequences. Finally, we completely characterize the Stanley-Reisner ideals of matroid complexes.

KEYWORDS: simplicial complex, matroid, h-vector, arithmetic degree, Stanley-Reisner ideal

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By
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Date: July 22, 2008

RULES FOR THE USE OF DISSERTATIONS

DISSERTATION

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Chapter 1 Introduction

Standing at the in the borderlands between combinatorics and commutative algebra is the field known as "combinatorial commutative algebra". Initiated by Stanley [21], combinatorial commutative algebra seeks to tie combinatorial objects (like simplicial complexes) together with algebraic ones (like ideals). The relationship goes in both directions — algebra can be used to answer combinatorial questions and combinatorics to answer algebraic ones. In our cases, we are concerned with two topics: the arithmetic degree of squarefree strongly stable ideals and the h-vectors of matroid complexes. We approach both of these problems using the techniques of combinatorial commutative algebra. In both cases, these algebraic seeming questions can be stated in a purely combinatorial manner. The first asks about the number of facets of certain simplicial complexes and the second about the number of faces in each dimension of certain other complexes (known as the *f-vector*).

Squarefree strongly stable ideals are a very special class of squarefree monomial ideals that correspond to the equally special class of shifted simplicial complexes. These are the complexes that arise as a result of "shifting" arbitrary complexes. The study of such shifting operators was initialed by Kalai [15] and allows one to reduce many general questions to questions about squarefree strongly stable ideals, which posses additional structure hopefully making the question more tractable. We do not study these shifting operators here, rather we look only at the results of shifting. However, the final result of Chapter 3 uses shifting and our results on squarefree strongly stable ideals to produce a general upper bound on the number of minimal generators of arbitrary Cohen-Macaulay ideals (Corollary 3.4.11).

The other potentially unfamiliar term in the title is "arithmetic degree", a term coined by Bayer and Mumford in [4] for an extension of the notion of the degree of an ideal giving a better measure of its computational complexity. The degree of an ideal and various generalizations have been studied for a very long time. Geometrically, the degree of a homogeneous ideal is a measure of the "bendiness" of its projective variety (or projective sub-scheme), V. However the ordinary degree only accounts for the irreducible components with the same codimension as V. It ignores everything else and does not account for multiplicities. For example, the variety consisting of 1 line and 100 points has degree 1, the same as the line alone. The arithmetic degree corrects this by considering every component. In our example of 1 line and 100 points, the arithmetic degree will be 101, reflecting the fact that this ideal is much more complex than that of a single line. We consider only the arithmetic degree of the squarefree strongly stable ideals discussed above. In this case, thanks to a result of Sturmfels-Trung-Vogel [24], the arithmetic degree measures the number of facets of a simplicial complex, making it a natural object of interest in combinatorics. We prove a lower bound for this quantity that depends only two basis invariants of the ideal (Theorem 3.3.1) and give an exact formula for computing the arithmetic degree directly from the minimal generators (Theorem 3.4.6).

Our final topic is the h-vectors of matroid complexes. An h-vector is a compact way to encode a Hilbert function; for simplicial complexes, knowing the h-vector is equivalent to knowing the f-vector. Matroids are a much studied topic in combinatorics. Starting with a set of vectors in some vector space, it is natural to ask which vectors of independent of which other vectors. This is, in some sense, a measure of how badly our set of vectors fails to be a basis. The collection of independent subsets of some set of vectors is knows as a matroid. Since a subset of an independent set is independent, we can naturally regard a matroid as a simplicial complex. The complexes that arise in this way are known as matroid complexes. When faced with a class of simplicial complexes, one of the most commonly asked questions it to characterize the h-vectors (or f-vectors) of the complexes in the class. For matroid complexes this remains unanswered. The two main results in this direction are due to Brown-Colbourn [7] and Chari [9]. The first says that the h-vector of a dimension d matroid complex satisfies

$$(-1)^j \sum_{i=0}^{j} (-\alpha)^i h_i \geq 0, \quad 0 \leq j \leq d+1$$

for any $\alpha \geq 1$ with equality if and only if $\alpha = 1$. In [9, Corollary 3], Chari resolved a conjecture of Hibi by showing that the h-vectors of matroids satisfy

$$\begin{aligned} &h_0 \leq h_1 \leq \cdots \leq h_{\lfloor (d+1)/2 \rfloor} \\ &h_i \geq h_{d-i+1}, \quad 0 \leq i \leq \left\lfloor \frac{d+1}{2} \right\rfloor \end{aligned}$$

These are the best results that the author is aware of. Additionally, there is the following 30 year old conjecture of Stanley [21].

Conjecture (Stanley)**.** *If* $\mathbf{h}$ *is the* h*-vector of a matroid complex then* $\mathbf{h}$ *is also the* h*-vector of a level monomial ideal.*

In Theorem 4.3.6 we prove this claim in the case that the complex has dimension 1 by explicitly constructing the claimed ideal.

The results of Chapter 4 fall into two categories: those that apply only to 1-dimensional complexes and those that work in higher dimensions. In Section 4.2 we completely classify all matroid complexes with dimension 1 in terms of partitions of the number of vertices (Theorem 4.2.20). By studying these partitions we compute the h-vector and describe the Stanley-Reisner ideal (Remark 4.2.21 and Theorem 4.2.9 respectively). This classification also allows us to resolve this case of the conjecture above. In Section 4.2 we describe some structure possessed by the set of 1-dimensional matroid h-vectors.

In Section 4.4 we try to, as far as possible, extend the results described above to higher dimensional complexes. Regarding h-vectors, if the initial degree of the Stanley-Reisner ideal is very large we give a description of the f-vector analogous the one for 1-dimensional h-vectors. (Theorem 4.4.19). We also completely describe the matroid complexes that are Gorenstein, that is, whose h-vector ends in 1 (Theorem 4.4.10). Our final, and most general, result is to give a complete description of the possible Stanley-Reisner ideals of matroid complexes (Theorem 4.4.31).

As an appendix, we include code written for the computer algebra system SAGE [22] that can perform many of the computations described in the preceding sections.

Chapter 2 Preliminary Results

2.1 Simplicial Complexes

The majority of this document deals with simplicial complexes and so we begin with some definitions related to this popular combinatorial object. First we need to fix a bit of notation. If X is any set we will write 2^X for its power set (the set of all subsets of X). We will write $\mathbb{N} = \{1, 2, \dots\}$ (the natural numbers), $\mathbb{Z}$ for the integers and $\mathbb{R}$ and $\mathbb{C}$ for the real and complex numbers respectively. The set of natural numbers including 0 is $\mathbb{N}_0$. If X is a finite set then we write $|X|$ to denote the number of elements in X.

Definition 2.1.1. Let X be a set and $\Delta \subseteq 2^X$. Then Δ is a *simplicial complex* if, whenever $F \in \Delta$ and $G \subseteq F$, $G \in \Delta$. We call X the *vertex set* of Δ. The elements of Δ are called its *faces* and we define $\dim F = |F| - 1$ and $\dim \Delta = \max\{\dim F \mid F \in \Delta\}$. If $F \in \Delta$ and $\dim F = d$ then we call F a *d-face* of Δ. The 0-faces are called *vertices* and the 1-faces are called *edges*.

Unless otherwise indicated we will always use Δ to refer to a simplicial complex. If $\Delta \neq \emptyset$ then $\emptyset \in \Delta$. This is the unique -1-dimensional face of Δ. It is at times important to distinguish between the complexes $\emptyset$ and $\{\emptyset\}$. The latter has dimension -1 while the former does not have a well-defined dimension.

We begin with an example of the simplest of all simplicial complexes, the simplex.

Example 2.1.2. Let X be a finite set with $|X| = n$ and $\Delta = 2^X$. Then, we say that Δ an $(n-1)$-simplex. It has n vertices, $\binom{n}{2}$ edges and, in general $\binom{n}{d}$ d-faces. Since the largest face of Δ is X itself, which has n elements, $\dim \Delta = n - 1$.

There are two standard ways to think about simplicial complexes. Each complex, Δ, is partially ordered by subset inclusion and so we can regard Δ as a poset and study it that way. Alternatively, we can regard Δ as a geometric object (explaining terms such as "vertex" "edge" and "dimension"). Observe Figure 2.1, depicting a 2-simplex with vertex set $\{1, 2, 3\}$ as a subset of $\mathbb{R}^2$.

The vertices of this figure correspond to the vertices of the complex with which they are labeled. The edge connecting the vertices with labels 1 and 2 corresponds the edge $\{1, 2\} \in \Delta$ and likewise for the other 2 edges. The 2-dimensional face $\{1, 2, 3\} \in \Delta$ is the 2-dimensional face containing vertices 1,2 and 3. One can construct such a subset by taking the convex hull of 3 properly chosen points. More generally, by taking the convex hull of $n + 1$ properly chosen points, on can construct a geometric subset corresponding to an n-simplex. Here "properly chosen" means that points must be in general position.

It is a standard fact that every simplicial complex with a finite vertex set can be thought of the union of n-simplices in $\mathbb{R}^N$ for some $N \gg 0$ such that the intersection of any two of them is a face of both simplices. one can choose N to be no larger than $2n + 1$. We will not prove this here (and in fact, technically need none of the

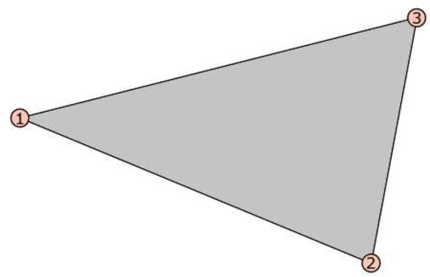


Figure 2.1: The 2-simplex

above discussion). The reader interested in this and other similar results can find them in many sources including [20]. We will use this primarily to generate pictures of simplicial complexes with small dimensions. The fact that the n-simplex has dimension n as a subset of R^N and $n+1$ vertices is the reason behind the definition of the dimension of a complex.

Remark 2.1.3. Suppose the vertex set of the complex Δ is $\{v_1, \ldots, v_n\}$. Naturally, one wants to call the v_i the vertices of Δ. There are 2 things that can go wrong here. First, it is not necessary that Δ contain all possible vertices. One can (and will) encounter such things as complexes with 4 vertices whose vertex set has 5 elements. Second, the elements v_i are not in the complex Δ. Rather, we have $\{v_i\} \in \Delta$. The distinction between an element and the set containing only that elements will not interest us and so we ignore it, happily referring to those v_i with $\{v_i\} \in \Delta$ as the vertices of Δ.

As in the above remark, it matters little what we call the vertices of Δ; relabeling them does not change the combinatorial structure of Δ (in the sense that the complexes are isomorphic, as in the next definition). With this in mind, we will almost universally use $[n] := \{1, 2, \ldots, n\}$ as our vertex set.

Definition 2.1.4. Let Δ and Γ be simplicial complexes and $\phi : \Delta \longrightarrow \Gamma$ a function. Then ϕ is an *isomorphism* if, whenever $F \subseteq G \in \Delta$, $\phi(G) \subseteq \phi(F) \in \Gamma$ and ϕ is bijective.

We summarize below some useful construction that produce new simplicial complexes from old ones.

Definition 2.1.5. Let Δ be a simplicial complex with vertex set X.

(a) The *k-skeleton* of Δ is $[\Delta]_k = \{F \in \Delta \mid \dim F \leq k\}$.

(b) If $W \subseteq X$ then the *restriction* of Δ to W is $\Delta|_W = \{F \in \Delta \mid F \subseteq W\}$. If $W = X - \{v\}$ then we will write $\Delta_{-v} = \Delta|_W$ and call Δ_{-v} the *deletion* of Δ with respect to v or the deletion of v from Δ.

(c) If $F \subseteq X$ then $\mathrm{link}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset, F \cup G \in \Delta\}$. We call this the *link* of Δ with respect to F.

(d) If $v \notin X$ then the cone over Δ is $C\Delta = \Delta \cup \{F \cup \{v\} \mid F \in \Delta\}$

That all of these are again simplicial complexes is easily checked using the definition.

Since if $G \in \Delta$ and $F \subseteq G$ then $F \in \Delta$, the complex Δ is determined completely by those faces that are not contained in any other face. We call these the *facets* of Δ. Typically, we will describe a simplicial complex by listing its facets. The above definitions are demonstrated in the next example.

Example 2.1.6. Consider the simplicial complex, Δ, depicted below in Figure 2.2. The facets of Δ are the faces (we drop the braces and commas for improved readabil-

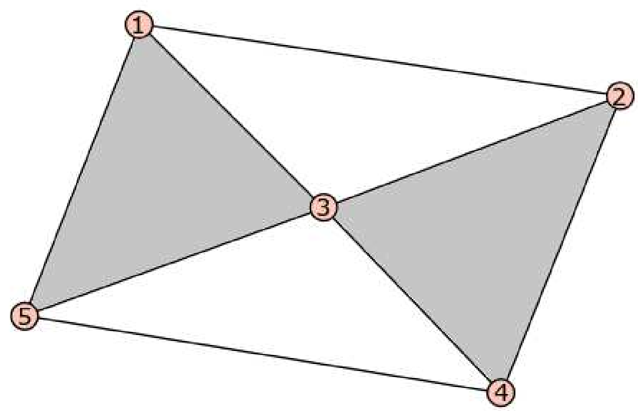


Figure 2.2: A simplicial complex

ity) 135, 234, 12 and 45. The dimension of Δ is the largest dimension of one of its facets, in this case 2.

The link of Δ with respect to the vertex 3 is the complex with facets 15 and 24, while the link with respect to the vertex 5 has facets 13 and 4. The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

In Chapter 4 we will be concerned with the particular class of simplicial complex defined below.

Definition 2.1.7. A simplicial complex Δ is said to be *pure* if all of its facets have the same dimension (which is necessarily $\dim \Delta$).

Having only a single facet, the complex in Figure 2.1 must be pure. The complex in Figure 2.2, however, is not pure as it has facets with dimensions 1 and 2.

One nice feature of pure simplicial complexes is that it is easy to detect when one is the cone of a smaller complex. If we know that Δ is a cone then we can delete one of its vertices and conclude many things by induction on the number of vertices in Δ.

Lemma 2.1.8. *Let Δ be a pure complex. The Δ is a cone if and only if there is some vertex $v \in \Delta$ such that* $\dim \Delta_{-v} < \dim \Delta$.

Proof. If Δ is the cone over another complex Γ then the facets of Δ are $F \cup \{v\}$ where F is a facet of Γ and $v \notin \Gamma$. Then clearly $\dim \Delta = \dim C\Gamma = 1 + \dim \Delta$, finishing one direction. For the converse, suppose $\dim \Delta_{-v} < \dim \Delta$. If F is a facet of Δ_{-v} then either F of $F \cup \{v\}$ is a facet of Δ. We have the former if and only if $v \in F$ and latter otherwise. If G is any facet of Δ then $\dim G = \dim \Delta$ by the purity of Δ. So the facets of Δ_{-v} must not be facets of Δ since Δ_{-v} has a smaller dimension. So every facet of Δ must contain v and $\Delta = C\Delta_{-v}$. QED

Remark 2.1.9. Applying the constructions in Definition 2.1.5 to a pure complex does not necessarily result in a pure complex. The exceptions are that the skeletons of pure complexes are pure, as are cones and links of pure complexes. Restrictions and deletions of pure complexes may not be pure. This is what make the results of Chapter 4 non-trivial. As a counterexample, consider the complex with facets 12, 23 and 34. Then deleting 2 gives the non-pure complex with facets 1, 3 and 34. The pure complexes whose restrictions are also pure are the matroid complexes discussed in Chapter 4.

After the dimension, the most basic bit of numerical information about Δ is its f-vector, which tells us the size of Δ is different dimension.

Definition 2.1.10. If Δ is a simplicial complex with $\dim \Delta = d$ then the f-vector of Δ is $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$ where $f_i = |\{F \in \Delta \mid \dim F = i\}|$.

It is implicitly assumed in the definition that $\Delta \neq \emptyset$. So Δ contains the unique -1-dimensional facet $\emptyset$ and $f_{-1} = 1$. The other entries tell us the number of faces in each dimension: f_0 is the number of vertices, f_1 the number of edges and so on. In Example 2.1.6, the complex depicted in Figure 2.2 has f-vector $(1, 5, 8, 2)$.

2.2 Stanley-Reisner Ideals and Commutative Algebra

The vast majority of our results in the coming chapters rely on the interplay between combinatorics and commutative algebra provided by the Stanley-Reisner ideal. First, we must fix some notation. Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K. Most of our results do not depend on the field chosen, but in general many things do. If $u \in S$ is a monomial then the *support* of u is $\operatorname{supp}(u) = \{i \mid x_i \text{ divides } u\}$. If $F \subseteq [n]$ then we can construct the unique squarefree monomial with support F, $x_F = \prod_{i \in F} x_i$. We define $x_\emptyset = 1$.

In this way, we have a bijection between the subsets of $[n]$ and the squarefree monomials in S. Since a simplicial complex on $[n]$ is a collection of subset of $[n]$, we can regard it instead as a collection of squarefree monomials in S. This motivates the next definition.

Definition 2.2.1. Let Δ be simplicial complex on $[n]$. The the *Stanley-Reisner ideal* of Δ is

$$I_\Delta = \langle x_F \mid F \notin \Delta \rangle \, .$$

The *Stanley-Reisner ring* of Δ is S/I_Δ.

By definition, the Stanley-Reisner ideal is a squarefree monomial ideal (that is, an ideal that can be generated by squarefree monomials). Compared to the number of monomial ideals (those generated by monomials) there are relatively few squarefree monomials. Nevertheless, this is an interesting and useful thing to study as many things can be related to simplicial complexes and thus to squarefree monomial ideals. Additionally, there are procedures that can turn arbitrary ideals is monomial ideals (the initial ideal) and monomial ideals into squarefree monomial ideals (polarization, see Remark 3.3.4).

Less formally, the Stanley-Reisner ideal of Δ is the ideal generated by the non-faces of Δ. In this way, the non-zero squarefree monomials in S/I_Δ (those outside of I_Δ) correspond to the faces of Δ. Hopefully, the algebraic properties of S/I_Δ reflect the properties of Δ. Some such results can be derived from the following lemma. If $F \subseteq [n]$ then we have an ideal $\mathfrak{m}_F = \langle x_i \mid i \in F \rangle$. It is well known that any monomial ideal can be written as the intersection of monomial ideals generated by powers of variables (its primary decomposition). For a squarefree monomial ideal, these powers are all 1. Below, we define $\overline{F} := [n] - F$ for any subset F of $[n]$.

Lemma 2.2.2. *Let Δ be a simplicial complex. Then the minimal primary decomposition of I_Δ is*

$$I_\Delta = \bigcap_{\substack{F \text{ a facet} \\ \text{of } \Delta}} \mathfrak{m}_{\overline{F}}.$$

Proof. Clearly all of the ideals above are prime and none is contained in any other by the maximality of the facets. Let u be a generator of I_Δ. Then, $\operatorname{supp}(u) \notin \Delta$ and so can not be contained in any facet. Thus, it must intersect $\overline{F}$ for every facet F. This means that u is in $\mathfrak{m}_{\overline{F}}$ for each F and thus in the intersection on the right-hand side of the decomposition.

Conversely, if u is a squarefree monomial in the intersection of the $\mathfrak{m}_{\overline{F}}$ it must have a non-empty intersection with each $\overline{F}$. Thus $\operatorname{supp}(u)$ is contained in no facet and is therefor not in Δ.

Finally, to see that our decomposition is minimal, suppose G is a facet of Δ and we intersect $\mathfrak{m}_{\overline{F}}$ for every facet except G. Since G is included in no face of Δ except itself, G has non-empty intersection with $\overline{F}$ whenever $F \neq G$. Thus x_G will be in this intersection. But $G \in \Delta$, which implies that $x_G \notin I_\Delta$. Thus, the stated decomposition must be minimal. QED

From this, we can get at some basic information about Δ. For example, $\dim I_\Delta = \dim S/I_\Delta = 1 + \dim \Delta$. To see this, note that $\dim I_\Delta$ is the smallest dimension of any of its primary components and $\dim \mathfrak{m}_{\overline{F}} = |F| = \dim F + 1$. Since $\dim \Delta$ is the largest dimension of one of its facets, we can immediately see that $\dim I_\Delta = \dim \Delta + 1$.

The Stanley-Reisner ideal is not just an ideal, it is a graded ideal and as such it has a Hilbert function: $h_{S/I_\Delta}(k) = \dim_K [S/I_\Delta]_k$, where $[S/I_\Delta]_k$ is the vector space of degree k homogeneous polynomial outside of I_Δ. As discussed above, the number of squarefree, degree k monomials outside of I_Δ is $f_k(\Delta)$ and so one might reasonably expect that there is some connection between the Hilbert function of I_Δ and the f-vector of Δ.

To see this we consider the generating function of the Hilbert function, the Hilbert series

$$H_{S/I}(t) = \sum_{i=1}^{\infty} h_{S/I}(i)t^i.$$

It is a well-know result of commutative algebra that the Hilbert function of a homogeneous ideal in S is eventually given by a polynomial and so the Hilbert series can be written as a rational function

$$H_{S/I}(t) = \frac{h_0 + h_1 t + \cdots h_d t^d}{(1-t)^d},$$

where $d = \dim I$. The sequence $(h_1, \ldots, h_d)$ is known as the h-vector of I. If $I = I_\Delta$ then we call this the h-vector of Δ and write $h(\Delta)$. While not equal to the f-vector, knowing $h(\Delta)$ is equivalent to knowing $f(\Delta)$. The relationship is most easily stated using the generating function:

$$\sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} = \sum_{i=1}^{d} h_i t^i.$$

We will not prove any of this here, instead directing the reader to [18] or [8] for proofs. Expanding this out, we get explicit formulas relating the h and f-vectors of a $(d-1)$-dimensional complex.

$$\begin{aligned} f_i &= \sum_{j=0}^{i} \binom{d-j}{i-j} h_j \\ h_i &= \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{i-1}. \end{aligned} \tag{2.1}$$

The last entry in the h-vector is $h_d = \sum_{j=0}^{d} (-1)^{d-j} f_{i-1} = 1 - (-1)^d \sum_{j=0}^{d} (-1)^i f_i = 1 - (-1)^d \mathcal{X}(\Delta)$, where $\mathcal{X}(\Delta)$ is the Euler characteristic of Δ. This value is known as the reduced Euler characteristic, written $\widetilde{\mathcal{X}}(\Delta)$.

The final piece of commutative algebra we will need is minimal free resolutions. Specifically, we will make use of multi-graded (or $\mathbb{Z}^n$-graded or fine-graded) free resolutions. If we have an exact sequence

$$F_\bullet : 0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} S \longrightarrow S/I \longrightarrow 0 \tag{2.2}$$

where each F_i is a free S-module then we call $F_\bullet$ a *free resolution* of S/I. Since the modules are all free, we can represent the maps ϕ_i as matrices with entries in S. If each such matrix contains no unit entries we say the resolution is *minimal.* For each ideal $I \subseteq S$, S/I has a unique (up to isomorphism of chain complexes) minimal free resolution. By the Hilbert Syzygy Theorem, every finitely generated S-module has a finite length resolution. In fact, the length (p in Equation (2.2)) is at most n. We call this length the *projective dimension* of S/I and write $\operatorname{pd}_S S/I$.

If I is homogeneous then the resolution may be taken to be homogeneous. In the sense that each free module can be written as

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{b_{ij}},$$

where $S(-j)$ is the module S with the degrees shifted so that $[S(-j)]_k = [S]_{k-j}$ (we call this S twisted (or shifted) by $-j$). The number b_{ij} are called the *graded Betti numbers* of S/I. We write $\beta^S_{ij} S/I = b_{ij}$ or simply $\beta_{ij} S/I$ if the ring is understood.

The first map in (2.2) $S \longrightarrow S/I$ is simply the canonical projection from S to one of its quotients. The kernel of this map is I and so $\operatorname{im}\phi_1 = I$ since the sequence is exact. So, we could remove the S from the front of $F_\bullet$ to get a minimal free resolution of I instead of S/I. We wish to point out that $\operatorname{pd}_S I = \operatorname{pd}_S S/I - 1$ for this reason.

The ideals we are working with are not simply graded; they are multi-graded (also known as fine-graded, or $\mathbb{Z}$-graded). To each monomial $\prod x_i^{a_i} \in S$ we can assign to it a *multi-degree* $\mathbf{a} = (a_1, \ldots, a_n)$. If I is a monomial ideal we can talk about the multi-graded components $[S/I]_{\mathbf{a}}$ and the multi-graded Hilbert function. More importantly, the resolution given in Equation (2.2) can be taken to multi-graded. So each free module can be written

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} S(-\mathbf{a})^{b_{i\mathbf{a}}}.$$

We then can define the *multi-graded Betti numbers* analogously to the graded case.

There are special types of resolutions with are particularly nice.

Definition 2.2.3. Let $I \subseteq S$ be homogeneous and $F_\bullet$ be a minimal free resolution of S/I.

(a) We say that I has *d-linear resolution* if $\beta_{i,i+j} I = 0$ whenever $i \neq d$.

(b) Let $p = \operatorname{pd}_S S/I$. Then S/I is *level* if $F_p = S(-D)$ for some D.

Said another way, the resolution $F_\bullet$ is linear if all of the non-zero entries the matrix representations of the maps ϕ_i are liner.

In light of part (a) of this definition, we define another numerical invariant that can be associated to the minimal free resolution: the regularity.

Definition 2.2.4. Let $I \subseteq S$ be a homogeneous ideal. Then the *Castelnouvo-Mumford regularity* (or simply the regularity) of S/I is

$$\operatorname{reg} S/I = \max\{j \mid \beta_{i,i+j} S/I \neq 0 \text{ for some } i\}$$

Of course, it also makes sense to talk about $\operatorname{reg} I$ instead of $\operatorname{reg} S/I$. Since $\beta_{ij} I = \beta_{i+1,j} S/I$ we immediately see that $\operatorname{reg} I = \operatorname{reg} S/I + 1$. If the projective dimension tells us the length of the resolution then the regularity might be said to tells us its width.

Remark 2.2.5. Knowing the regularity of I allows us to detect the linearity of its resolution. Since $\beta_{0,j} I \neq 0$ if and only if I has a generator with degree j. So $\operatorname{reg} I$ is at least D, where D is the largest degree of a minimal generator of I. In general it is much larger. If I has a d-linear resolution then I must be generated in degree d and $\operatorname{reg} I = d$, the smallest possible value.

Minimal free resolutions provide an important connection between the algebra of the ring S/I_Δ and the combinatorics and topology of the simplicial complex Δ. This connection is given by the formula of Hochster, whose proof may be found in [18, Corollary 5.12]. We will write $\widetilde{H}^i(\Delta; K)$ for the i-th reduced simplicial cohomology of Δ with coefficients in K and $\operatorname{supp}(\mathbf{a}) = \{i \mid a_i \neq 0\}$.

Theorem 2.2.6 (Hochster's Formula)**.** *Let $I = I_\Delta$. Then the multi-graded Betti numbers of I are $\beta_{i\mathbf{a}} = 0$ if $\mathbf{a} \notin \{0,1\}^n$ and*

$$\beta_{i\mathbf{a}} S/I = \dim_k \widetilde{H}^{|\mathbf{a}|-i-1}(\Delta|_{\operatorname{supp}(\mathbf{a})}; K)$$

We discussed pure complexes in Section 2.1. Related to this is the similar but more algebraic concept of Cohen-Macaulay complexes. We say that an ideal $I \subseteq S$ is *Cohen-Macaulay* if $\operatorname{codim} S/I = \operatorname{pd}_S S/I$. By the Auslander-Buchsbaum Theorem ([10, Theorem 19.9]), this is shortest possible resolution for any ideal with the given codimension.

If the Stanley-Reisner ideal of Δ is Cohen-Macaulay then we say that Δ is a Cohen-Macaulay complex over K. Some caution must be exercised here as it is possible for a complex to be Cohen-Macaulay over one field but not over another. Thus, being Cohen-Macaulay is not really a combinatorial property. Nevertheless, it has combinatorial consequences, as in the next Lemma.

Lemma 2.2.7. *Let Δ be a Cohen-Macaulay complex (over any field). Then Δ is pure.*

Proof. One of the nice properties posses by Cohen-Macaulay ideals is that they are unmixed, that is, all their associated primes have the same codimension. But the codimension of the associated primes of I_Δ determine the dimension of the facets of Δ by Lemma 2.2.2. In particular, I_Δ is unmixed if and only if Δ is pure. QED

Remark 2.2.8. The converse of this is false. For example, the complex with facets 12 and 34 is certainly pure, but its Stanley-Reisner ideal is not Cohen-Macaulay over any field. To see this, use Hochster's formula. If $\mathbf{a} = (1,0,1,0)$ then $\operatorname{supp}(\mathbf{a}) = \{1,3\}$. By Hochster's formula,

$$\begin{aligned}\beta_{3,\mathbf{a}} &= \dim_K \widetilde{H}^{3-2-1}(\Delta|_{13}; K)\\ &= \dim_K \widetilde{H}^{0}(\Delta|_{13}; K) = 1.\end{aligned}$$

The last equality is because $\Delta|_{13}$ consists of 2 vertices only, and is thus not connected (that is, its geometric realization is not connected in the topological sense); the 0-th reduced cohomology of any complex is 1 less than the number of connected components, in this case 2. So, $\operatorname{pd}_S S/I_\Delta \geq 3$, but $\dim \Delta = 1 = \dim S/I_\Delta - 1$, so $\operatorname{codim} S/I_\Delta = 2$ meaning that Δ is pure but not Cohen-Macaulay.

We used above (without proof) the fact that Cohen-Macaulay ideals are unmixed. The main algebraic fact about Cohen-Macaulay ideals that we need is that the lengths of their resolutions are determined by their codimensions, a fact that we have taken as the definition. This is quite a popular property and there are many other, equivalent, conditions for an ideal to be Cohen-Macaulay. Sections 13.4 and 13.5 of [18] list a number of these.

This is only a very brief overview of what is sometimes called combinatorial commutative algebra. The interested reader may find additional information about Stanley-Reisner ideals, simplicial complexes and many other topics in [18], [21] and [8].

Chapter 3 Arithmetic Degree

3.1 Introduction

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over some field K and $I \subset S$ a homogeneous ideal. One of the fundamental invariants of I is its *degree* and a natural refinement of the degree is the *arithmetic degree*, which includes the low dimensional and embedded components ignored by the degree. The utility of both concepts is clearly illustrated if I is squarefree monomial in which case it is the Stanley-Reisner ideal of some simplicial complex Δ. Then the degree of I is the number of faces of Δ with maximal dimension, while the arithmetic degree is the number of maximal faces (facets). This makes the arithmetic degree a natural object of study in the theory of Stanley-Reisner rings of non-pure simplicial complexes. As our main result (Theorem 3.4.6), we explicitly compute the arithmetic degree of Stanley-Reisner rings for the useful class of shifted complexes.

Suppose Δ is a simplicial complex on $\{1, \ldots, n\}$. Writing $x_\sigma = \prod_{i\in\sigma} x_i$ for each $\sigma \subseteq [n]$, the Stanley-Reisner ideal of Δ is $I_\Delta := \langle x_\sigma \mid \sigma \notin \Delta \rangle$. Now, consider $E = K\langle e_1, \ldots, e_n \rangle$, the exterior algebra of a vector space with basis $e_1, \ldots, e_n$. We make E into a graded K-algebra by setting $\deg e_i = 1$. If you want E to be multi-graded instead, we can do that to by setting the degree of e_i to be $(0, \ldots, 1, \ldots, 0)$ where the 1 is in the i-th position. Analogous to I_Δ we may define $J_\Delta = \langle e_\sigma \mid \sigma \notin \Delta \rangle \subseteq E$. We will call $K\{\Delta\} = E/J_\Delta$ the *exterior face ring* of Δ. We will make frequent, implicit use of the fact that there is a theory of Gröbner bases over E analogous to that over S. In particular, if $J \subseteq E$ it makes sense to talk about the squarefree monomial ideal $\operatorname{gin} J$. See [2] for information on these matters.

Let $I \subseteq S$ be a homogeneous ideal of codimension c and $F_\bullet$ a minimal $\mathbb{Z}$-graded free resolution of S/I. If we write $m_i(S/I)$ and $M_i(S/I)$ for the smallest and largest shifts respectively appearing in F_i then the Multiplicity Conjecture of Herzog-Huneke-Srinivasan claims that

$$\deg S/I \leq \frac{1}{c!} \prod_{i=1}^{c} M_i(S/I)$$

and, if I is Cohen-Macaulay,

$$\frac{1}{c!} \prod_{i=1}^{c} m_i(S/I) \leq \deg S/I.$$

A number of special cases of these conjectures have been known to be true for some time (see [13], [14], [17] for some of these). After this paper was written, the Cohen-Macaulay case of the Multiplicity conjecture was proven by [11] and the non-Cohen-Macaulay case by [6].

If $I = I_\Delta$ happens to be squarefree, we can consider the ideal $\hat{I} = J_\Delta \subseteq E$ and its graded minimal free resolution $\hat{F}_\bullet$. As before we write $\hat{m}_i(E/\hat{I})$ and $\hat{M}_i(E/\hat{I})$ for the smallest and largest twists in $\hat{F}_i$. The results of [1] imply that $\hat{m}_i(E/\hat{I}) \leq m_i(S/I)$

(and that this inequality is generally strict). In addition the arithmetic degree is at least the ordinary degree suggesting that, perhaps, if I is squarefree,

$$\frac{1}{c!}\prod_{i=1}^{c}\hat{m}_i(E/\hat{I}) \leq \operatorname{adeg} S/I.$$

This is false (see Example 3.3.3 for counterexamples). However, if we additionally assume that I is squarefree strongly stable (Definition 3.2.1), this becomes Corollary 3.3.2.

If I is any Cohen-Macaulay ideal with codimension 2 and initial degree d then Dubreil's theorem says that the number of minimal generators of I is at most $d+1$. As the final result of the Chapter, we give an extension of this to Cohen-Macaulay ideals with arbitrary codimension.

3.2 Preliminary Results

We are primarily interested in the special class of squarefree strongly stable ideals, defined below. If $u \in S$ is a monomial then we define the *support* of u to be $\operatorname{supp}(u) := \{i \mid x_i|u\}$. If v is a monomial in E we define $\operatorname{supp}(v)$ similarly.

Definition 3.2.1. Let $I \subseteq S$ be an ideal. Then we say that I is *strongly stable* if I is monomial and, for every monomial $u \in I$, $i \in \operatorname{supp}(u)$ and $j < i$, $\frac{ux_j}{x_i} \in I$. If I is squarefree and we only require this for $j \notin \operatorname{supp}(u)$ we say that I is *squarefree strongly stable*. An ideal $J \subseteq E$ is said to be squarefree strongly stable if the same condition holds for $\frac{ue_j}{e_i}$. A simplicial complex Δ such that I_Δ is squarefree strongly stable is said to be *shifted*.

Remark 3.2.2. Shifted complexes can be defined with referring to the Stanley-Reisner ideal. A complex is shifted if and only if, for every $\sigma \subseteq \Delta$, $j \in \Delta$ and $i > j$, $(\sigma - \{j\}) \cup \{i\} \in \Delta$. To see this, suppose that $(\sigma - \{j\}) \cup \{i\} \notin \Delta$. Then $u := \frac{x_\sigma x_i}{x_j} \in I_\Delta$. But $\frac{ux_j}{x_i} = x_\sigma \notin I_\Delta$. This can happen if and only if I_Δ is not squarefree strongly stable.

Remark 3.2.3. Why should one care about strongly stable or squarefree strongly stable ideals? The answer is that given any ideal $I \subseteq S$, the generic initial ideal (in the reverse-lexicographic order) $\operatorname{gin}_{rlex} I$ is strongly stable. If $J \subseteq E$ then $\operatorname{gin}_{rlex} J$ is squarefree strongly stable. Many properties of I are either preserved or well behaved when taking gin so it is natural to try to reduce to only proving the (squarefree) strongly stable case so that you can exploit the additional structure. This is how Corollary 3.4.11 works.

Let $u \in S$ be a squarefree monomial. Then there exists a unique monomial, $\hat{u} \in E$, with $\operatorname{supp}(\hat{u}) = \operatorname{supp}(u)$. If $I \subseteq S$ is a squarefree monomial ideal, then there is a corresponding monomial ideal $\hat{I} \subseteq E$,

$$\hat{I} := \langle \hat{u} \mid u \in G(I) \rangle$$

where $G(I)$ is the (unique) set of minimal monomial generators for I. It is clear from the definition that $\hat{I}$ is squarefree strongly stable if and only if I is. Given a squarefree ideal $I \subseteq S$ the ideal $I^{\mathrm{e}} \subseteq S$ corresponding to $\mathrm{gin}_{\mathrm{rlex}} \hat{I} \subseteq E$ is squarefree strongly stable. We will call I^{e} the *exterior shift* of I.

If u is a monomial then $\min(u) := \min(\mathrm{supp}(u))$ and $\max(u) := \max(\mathrm{supp}(u))$. We can easily read off several interesting invariants of I from its minimal generators as in the following lemma taken from [13, Lemma 4.1].

Lemma 3.2.4. *Let $I \subseteq S$ be a squarefree strongly stable ideal. Then*

(a) $\mathrm{codim}\, {}^{S}/_{I} = \max\{\min(u) \mid u \in I\} = \max\{\min(u) \mid u \in G(I)\}$

(b) $\mathrm{pd}_S\, {}^{S}/_{I} = \max\{\max(u) - \deg(u) + 1 \mid u \in G(I)\}$

Most of the coming arguments are built around induction on the number of minimal generators. To achieve this, we will need the next two lemmas.

Lemma 3.2.5. *Let $I = \langle u_1, \dots, u_s\rangle \subseteq S$ be a squarefree strongly stable ideal and $D := \max\{\deg(u) \mid u \in G(I)\}$. Then the ideal $I_1 := \langle u_2, \dots u_s\rangle$ (define $I_1 := \langle 0\rangle$ if I is principle) is squarefree strongly stable if either*

(a) u_1 is lexicographically smallest among the generators of degree D or

(b) u_1 has $\max u_1 - \deg u_1 + 1 = \mathrm{pd}_S\, S/I$ and has the largest degree among all minimal generators, u, with $\max u - \deg u + 1 = \mathrm{pd}_S\, S/I$; if there is more than one such generator, u_1 is the one that is lexicographically smallest.

Proof.

(a) Clearly it is enough to check stability on the minimal generators. Let $v \in G(I_1)$ and i and j be as required by Definition 3.2.1. If $\deg(v) < D = \deg(u_1)$ then it is clear that $\frac{vx_j}{x_i} \in I_1$. So assume that $\deg(v) = D$. Then, the only way stability could fail is if $\frac{vx_j}{x_i} = u_1$. By minimality of u_1 we have

$$u_1 <_{\mathrm{lex}} v <_{\mathrm{lex}} \frac{vx_j}{x_i}$$

and so $\frac{vx_j}{x_i} \neq u_1$ and I_1 is therefore squarefree strongly stable.

(b) Suppose that I_1 is not squarefree strongly stable. Then, there is a $v \in G(I_1)$ and $j < i$ such that $\frac{vx_j}{x_i} \notin I_1$. This means that $\frac{vx_j}{x_i} = mu_1$ for some monomial m and $\frac{vx_j}{x_i}$ is not a multiple of any other minimal generator. Since I_1 contains every minimal generator of I except u_1, we must have $\deg(v) \geq \deg(u_1)$. If u_1 and v have the same degree then $\max\left(\frac{vx_j}{x_i}\right) - \deg\left(\frac{vx_j}{x_i}\right) \leq \max(v) - \deg(v) \leq \max(u_1) - \deg(u_1)$. If one of these inequalities is strict then $\frac{vx_j}{x_i} \neq u_1$ and we are done. Otherwise the lexicographic minimality of u_1 shows that $u_1 \neq \frac{vx_j}{x_i}$.

So, we may assume that $\deg(m) \geq 1$. If $j \in \mathrm{supp}(m)$ then we could write $v = \frac{mx_i}{x_j}u_1$, which would mean that v is not a minimal generator. So, we must have $j \in \mathrm{supp}(u_1)$. We identify two cases

Case I: Assume $\max(v) \le \max(u_1)$ and write $v = m\frac{u_1 x_i}{x_j}$. Since $j \in \operatorname{supp}(u_1)$ and $j < i \le \max(v) \le \max(u_1)$, we have

$$\max(v) = \max\left(m\frac{u_1 x_i}{x_j}\right) \ge \max(mu_1) \ge \max(u) \ge \max(v,)$$

which implies that $\max(v) = \max(u_1)$. Now we can write

$$\frac{vx_j}{x_j} = \frac{mx_{\max(v)}}{x_{\min(m)}} \cdot \frac{u_1 x_{\min(m)}}{x_{\max(v)}}.$$

Since $\min(m) \le \max(v)$ the stability of I_1 implies that $\frac{u_1 x_{\min(m)}}{x_{\max(v)}} \in I_1$. This contradicts our assumption that $\frac{vx_j}{x_i}$ is not in I_1.

Case II: Assume $\max(v) > \max(u_1)$. If $\min(m) < \max(u_1)$ then

$$\frac{vx_j}{x_i} = \frac{mx_{\max(u_1)}}{x_{\min(m)}} \cdot \frac{u_1 x_{\min(m)}}{x_{\max(u_1)}},$$

where the second term is in I_1 by the first case. This again contradicts our assumption that $\frac{vx_j}{x_i} \notin I_1$. So, we must have $\min(m) > \max(u_1)$, which implies that $\deg(m) \le \max(v) - \max(u_1)$ (we can not have $\min(m) = \max(u_1)$ since $\frac{vx_j}{x_i} = mu_1$ is squarefree). By our choice of u_1, $\deg(v) - \deg(u_1) \ge \max(v) - \max(u_1)$ giving

$$\begin{aligned} \deg(v) = \deg(u_1) + \deg(m) &\le \deg(u_1) + (\max(v) - \max(u_1)) \\ &\le \deg(u_1) + \deg(v) - \deg(u_1) = \deg(v). \end{aligned}$$

So we must have $\max(v) - \max(u_1) = \deg(m)$. It follows that $m = x_{\max(u_1)+1} x_{\max(u_1)+2} \cdots x_{\max(v)}$, which implies that $\max(v) = \max(m) = \max\left(\frac{vx_j}{x_i}\right)$. So

$$\begin{aligned} \max(v) - \deg(v) &= \max(m) - (\deg(m) + \deg(u_1)) \\ &= (\max(m) - \deg(m)) - \deg(u_1) \\ &= \max(u_1) - \deg(u_1). \end{aligned}$$

So $\max(v) - \deg(v)$ is maximal, but v has a larger degree than u_1, contradicting our choice of u_1.

QED

Definition 3.2.6. Let $I \subseteq S$ be a homogeneous ideal. Then the *arithmetic degree* of S/I is

$$\operatorname{adeg} I = \operatorname{adeg}\left({}^{S}/_{I}\right) := \sum_{\mathfrak{p} \in \operatorname{Ass}(S/I)} \ell(H^0_{\mathfrak{p}}((S/I)_{\mathfrak{p}}) \cdot \deg(\mathfrak{p}),$$

where $\ell(M)$ denotes the length of M.

The results of [24] (in particular Lemma 2.2) imply that, if Δ is a simplicial complex on $[n] := \{1, \ldots, n\}$, $\operatorname{adeg}(S/I_\Delta)$ is the number of facets of Δ or, equivalently, the number of ideals in a minimal primary decomposition of I_Δ (each of which is generated by variables). By a standard abuse of notation, we will write $\operatorname{adeg}\Delta$ and $\operatorname{adeg} I_\Delta$ for $\operatorname{adeg}(S/I_\Delta)$. Note that Δ is pure if and only if $\operatorname{adeg}\Delta = \deg\Delta$.

We shall need to make some use of Alexander duality here and so let's define it and collect some basic results.

Definition 3.2.7. If Δ is a simplicial complex then the Alexander Dual of Δ is

$$\Delta^* := \{[n] - \sigma \mid \sigma \notin \Delta\}.$$

What does the Stanley-Reisner ideal of Δ^* look like? An minimal generator, x_σ, of I_{Δ^*} is a minimal non-face of Δ^*. So $[n] - \sigma \in \Delta$. Since σ is minimal among the non-faces of Δ^*, its complement is maximal among the faces of Δ. That is, σ is a facet of Δ. Reading the argument backwards gives us the converse. So, the number of facets of Δ is the same as the number of minimal generators of I_{Δ^*}. We will write $\mu(I)$ for the smallest size of a generating set of I.

The facets of Δ correspond to the components of the minimal primary decomposition of I_Δ. From above, we see that they also correspond to the minimal generators of I_{Δ^*}. Algebraically, the Alexander dual of a squarefree monomial ideal I (written I^*) interchanges the primary and the minimal generators. It interchanges a number of other things as well, as in the next lemma. Here, the initial degree of an ideal I, $\operatorname{init}(I)$ is the smallest degree of a minimal generator.

Proposition 3.2.8. *Let $I = I_\Delta$ be a squarefree monomial ideal and $I^* = I_{\Delta^*}$ be its Alexander dual. Then*

(a) $\operatorname{init}(I) = \operatorname{codim}(I^*)$

(b) (Terai) $\operatorname{pd}_S S/I = \operatorname{reg} I$

(c) (Eagon-Reiner) I is Cohen-Macaulay if and only if I^* has a linear free resolution.

Proof.

(a) This follows quickly from the definition. Let x_σ be a minimal generator of I with degree $\operatorname{init}(I)$. As discussed above, $\overline{\sigma} = [n] - \sigma$ is a facet of Δ^*. Since the degree of x_σ is minimal among the minimal generators of I, the dimension of $\overline{\sigma}$ is maximal among the facets of Δ^*; thus $\dim\Delta = |\overline{\sigma}| - 1$. So we have

$$\begin{aligned}\operatorname{codim}(I^*) &= n - \dim(I^*)\\ &= n - \dim\Delta - 1\\ &= n - |\overline{\sigma}|\\ &= |\sigma| = \operatorname{init}(I).\end{aligned}$$

(b) For the proof of this, we refer the reader to [25], which actually proves a generalization of this claim to squarefree modules.

(c) This follows immediately from (a) and (b).

QED

The proof of Theorem 3.3.1 will revolve around induction of the number of minimal generators. In order for this to work out, we need to know that, when we remove a minimal generator from a squarefree strongly stable ideal, the arithmetic degree decreases. We actually prove something more general about arbitrary squarefree ideals, but we will only need the squarefree strongly stable case.

Lemma 3.2.9. *Let Σ be a non-trivial simplicial complex on $[n]$ which is not a simplex and $I = I_\Sigma$ its Stanley-Reisner ideal. If $G(I) = \{u_1, \ldots, u_k\}$ and $I_1 = \langle u_2, \ldots, u_k \rangle$ then*

$$\operatorname{adeg}(S/I_1) \leq \operatorname{adeg}(S/I) + 1.$$

If Σ is shifted then this inequality is strict.

Proof. The minimal generators of I_Σ correspond to the facets of the Alexander dual, Σ^*, and vice-versa. By removing a minimal generator of I, we are removing a facet from Σ^* and examining the complex generated by the other facets. We wish to show that the arithmetic degree of I increases by at most 1 when we do this. Since the arithmetic degree of I is the number of minimal generators of the Stanley-Reisner ideal of the Alexander dual of I, if $\Delta := \Sigma^*$ and Δ_1 is the complex generated by all but one of the facets, our claim is equivalent to the claim the $\mu(I_{\Delta_1}) \leq \mu(I_\Delta) + 1$. Recall that the minimal generators of I_Δ are the minimal non-faces of Δ. We claim that this statement is true for any simplicial complex.

So, let Δ be a simplicial complex, Γ a facet of Δ and Δ' the complex generated by the other facets of Δ. Let $\gamma \notin \Delta$ be a minimal non-face. Since $\Delta' \subset \Delta$, γ is also a non-face of Δ'. We define a map from the minimal non-faces of Δ to those of Δ' as follows

$$\gamma \mapsto \begin{cases} \gamma & \text{if } \gamma \text{ is a minimal non-face of } \Delta' \\ \gamma \cap \Gamma & \text{otherwise} \end{cases}.$$

We must first show that this map is well defined. To this end, we show that, if γ is not a minimal non-face of Δ' then $\gamma \cap \Gamma$ is.

Suppose that $\tau \subset \gamma$ is a minimal non-face of Δ'. Since γ is minimal over Δ, we must have $\tau \in \Delta - \Delta'$, which implies that $\tau \subseteq \Gamma$ and τ is not contained in any other facet of Δ. Thus, $\tau \subseteq \gamma \cap \Gamma$, and $\gamma \cap \Gamma \notin \Delta'$. Note that any face of Δ that is not a face of Δ' must be contained in Γ and in no other facet of Δ. We claim that $\gamma \cap \Gamma$ is minimal with respect to this property.

First, it is clear that $\gamma \cap \Gamma \subseteq \Gamma$. If $\gamma \cap \Gamma$ were contained in some other facet of Δ then it would be a face of Δ', which we have already noted is false. So, $\gamma \cap \Gamma$ is contained in no facet of Δ except Γ. Next, suppose that $\rho \subseteq \gamma \cap \Gamma$ also has this property. Consider the set

$$\begin{aligned} F &:= \gamma - \big[(\gamma \cap \Gamma) - \rho\big] \\ &= \gamma \cap (\rho \cup \overline{\Gamma}). \end{aligned}$$

We claim that $F \notin \Delta$. First, we show that $F \not\subseteq \Gamma$ (we write $\overline{X}$ for $[n] - X$ when $X \subseteq [n]$):

$$\begin{aligned} F \cap \overline{\Gamma} &= \gamma \cap \left[(\rho \cap \overline{\Gamma}) \cup (\overline{\Gamma} \cap \overline{\Gamma})\right] \\ &= \gamma \cap \left[(\rho \cap \overline{\Gamma}) \cup \overline{\Gamma}\right] \\ &= \gamma \cap \overline{\Gamma}, \end{aligned}$$

where the last equality follows from $\rho \subseteq \Gamma$. In particular $F \cap \overline{\Gamma} \neq \emptyset$ and so F is not contained in Γ. Let Ω be another facet of Δ. As above we can write

$$\begin{aligned} F \cap \overline{\Omega} &= \gamma \cap \left[(\rho \cap \overline{\Omega}) \cup (\overline{\Gamma} \cap \overline{\Omega})\right] \\ &= (\gamma \cap \rho \cap \overline{\Omega}) \cup (\gamma \cap \overline{\Gamma} \cap \overline{\Omega}). \end{aligned}$$

Since we assumed that $\rho \subseteq \gamma \cap \Gamma \subsetneq \gamma$, $\rho \cap \gamma = \rho$. By assumption ρ is not contained in any facet except Γ and so $\rho \cap \overline{\Omega} \neq \emptyset$. So $F \cap \overline{\Omega}$ is non-empty and thus F is not contained in Ω. Therefore $F \notin \Delta$. But, $F \subseteq \gamma$ and γ is a minimal non-face of Δ, which means that $F = \gamma$, which could only happen if $\gamma \cap \Gamma - \rho = \emptyset$, or $\rho = \gamma \cap \Gamma$, giving us the desired minimality of $\gamma \cap \Gamma$. From before, we have $\tau \subseteq \gamma \cap \Gamma$ and τ is contained in no facet of Δ except Γ. So $\tau = \gamma \cap \Gamma$ is a minimal non-face of Δ', as desired.

Now we have a map from the minimal non-faces of Δ to the minimal non-faces of Δ'. Every minimal non-face of Δ' except possibly Γ is in the image of this map. Thus the number of minimal non-faces can increase by at most 1.

Now suppose that Δ is shifted and recall that a complex is shifted if and only if its Alexander dual is.

If Γ is not a minimal non-face of Δ' then we are done. So suppose otherwise. Since Γ is a minimal non-face of Δ' every proper subset $\gamma \subset \Gamma$, is contained in some facet of Δ other than Γ. Consider the set

$$A := \{i \notin \Gamma \mid \gamma \cup \{i\} \in \Delta \text{ for some } \gamma \subset \Gamma\}.$$

The set A is non-empty since otherwise each $\gamma \subset \Gamma$ would be a facet of Δ, contradicting that Γ is a facet of Δ.

Let $m = \max A$. Then, by shifting, $\gamma \cup \{m\} \in \Delta$ for every proper subset $\gamma \subset \Gamma$ and thus $\Gamma \cup \{m\}$ is a minimal non-face of Δ containing Γ. By the definition of our map the image of $\Gamma \cup \{m\}$ is Γ. Since we have already established that every facet except for Γ is in the image of our map this implies that the map is surjective and thus the number of minimal non-faces of Δ' is at most the number of minimal non-faces of Δ' As noted at the beginning of the proof, we can now apply Alexander duality to get the claimed inequality. QED

3.3 A Lower Bound

Using the above facts, we get the following lower bound for the arithmetic degree of a squarefree strongly stable ideal.

Theorem 3.3.1. *Let $I \subseteq S$ be a squarefree strongly stable ideal. Set $d := \min\{\deg(u) \mid u \in G(I)\}$ and $c := \operatorname{codim} S/I$. Then,*

$$\binom{d+c-1}{c} \leq \operatorname{adeg} I.$$

Proof. Let $D := \max\{\deg(u) \mid u \in G(I)\}$. We may, without loss of generality, assume that x_n divides some minimal generator, for otherwise we could pass to the ideal in $n-1$ variables $(I + \langle x_n \rangle)/\langle x_n \rangle$ and conclude by induction on n. Let u_1 be the lexicographically smallest minimal generator with degree D and I_1 the ideal generated by the other minimal generators. From Lemma 3.2.5 we know that I_1 is squarefree strongly stable. Let c_1 be the codimension of I_1; Lemma 3.2.4(a) implies that $c_1 \leq c$. We induct on the number of minimal generators. If $I = \langle 0 \rangle$ (which has arithmetic degree 1 and $d = c = 0$) the result is obvious. Otherwise, we can apply Lemma 3.2.9 to get

$$\binom{d+c_1-1}{c_1} \leq \operatorname{adeg} I_1 \leq \operatorname{adeg} I.$$

If $c = c_1$ we are done, so assume that $c_1 < c$. By Lemma 3.2.4(a) this must mean that $\min(u_1) = c$ and no other minimal generator has min c. Let u_2 be the minimal generator described in Lemma 3.2.5(b) and I_1' the squarefree strongly stable ideal generated by the other minimal generators. Suppose that $u_1 \neq u_2$. Since only u_1 has min c and $u_1 \in I_1'$ we must have $\operatorname{codim} I_1' = c$. Applying the above inequalities with the ideal I_1' in place of I_1 completes the argument in this case. It only remains to consider the case when $u_1 = u_2$. We claim that in this case I_1 is Cohen-Macaulay.

Since $c = \min(u_1)$ stability says that $x_c x_{c+1} \cdots x_{c+D-1} \in I$. The choice of u_1 implies that it has a larger min than any other minimal generator of I. So $x_c x_{c+1} \cdots x_{c+D-1}$ must be a minimal generator since it has a larger min than any minimal generator except for u_1. But then, $x_c x_{c+1} \cdots x_{c+D-1} = u_1$ because no minimal generator except for u_1 has min c. Our assumption that $u_1 = u_2$ means that $c = \max u_1 - \deg u_1 + 1 = \operatorname{pd}_S S/I$, showing that I is Cohen-Macaulay. So $\operatorname{adeg} S/I = \deg S/I$ and the claimed bound is the same as that in the Multiplicity Conjecture. The needed special case was first proven in [13, Theorem 4.7] but one can, of course, also apply the general proof for the Cohen-Macaulay case [11]. QED

We can regard Theorem 3.3.1 as a variation on the Multiplicity Conjecture as in the next corollary. In fact, the lower bound given is the smallest possible value of the lower bound in the Multiplicity Conjecture for a given initial degree and codimension.

Corollary 3.3.2. *Let $I \subseteq S$ be a squarefree strongly stable ideal with codimension c and $\hat{F}_\bullet$ a $\mathbb{Z}$-graded minimal free resolution of $\hat{I}$ over E. If $\hat{m}_i$ is the smallest degree shift in $\hat{F}_i$ then*

$$\frac{1}{c!}\prod_{i=1}^{c} \hat{m}_i = \binom{d+c-1}{c} \leq \operatorname{adeg} I,$$

where $d = \min\{\deg(u) \mid u \in G(I)\}$.

Proof. The inequality is Theorem 3.3.1 and the equality follows from the description of the minimal free resolution of squarefree strongly stable ideals in E given in [2]. QED

Example 3.3.3. Consider the ideal $I = \langle x_1x_3, x_2x_3, x_2x_4, x_1x_4\rangle = \langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle$. This has arithmetic degree 2, but the lower bound in Theorem 3.3.1 is 3; I is neither Cohen-Macaulay nor squarefree strongly stable. Using a computer system such as Macaulay 2 [12] to calculate the exterior shift of I, we get $I^e = \langle x_1x_2, x_2x_3, x_1x_3, x_2x_4\rangle$, which has the same initial degree and codimension but has arithmetic degree 3.

Remark 3.3.4. Alexander duality exchanges the arithmetic degree and the number of minimal generators as well as the initial degree and the codimension (Proposition 3.2.8). So, if I is squarefree strongly stable, we can apply Alexander duality to Theorem 3.3.1 to get that $\binom{d+c-1}{d} \leq \mu(I)$. If I is strongly stable then we can produce a squarefree strongly stable ideal by replacing each monomial $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ $(i_1 \leq i_2 \leq \cdots \leq i_d)$ with the squarefree monomial $u^{\mathrm{pol}} = x_{i_1}x_{i_2+1}\cdots x_{i_d+d-1}$. Applying this to each minimal generator of a strongly stable ideal I gives a squarefree strongly stable ideal I^{pol} called the *polarization* of I. Polarization fixes the number of minimal generators and the Hilbert function of strongly stable ideals. Thus the above lower bound on the number of minimal generators also holds for strongly stable ideals.

Example 3.3.5. Assume that the base field has characteristic 0. Let $I \subseteq S$ be an ideal with initial degree d, codimension c and $\mu(I) < \binom{d+c-1}{d}$. The ideal $J = \mathrm{gin}_{rlex}(I)$ is strongly stable and so we may apply Remark 3.3.4 to see that J must have more minimal generators than I in degrees larger than d. Since the graded Betti numbers of ideals with componentwise linear resolutions are fixed when taking the generic initial ideal ([3]), we see that I can not have a componentwise linear resolution (in particular I does not have a linear resolution).

3.4 An Upper Bound

We now establish an upper bound for the arithmetic degree of a squarefree strongly stable ideal. The idea behind this is to consider the ideal $J := J_\Delta \subseteq E$ and to remove a generator of degree D as is Lemma 3.2.5 to form an ideal J_1. Then we apply Alexander duality (which is an exact functor of E-modules) and the mapping cone to the exact sequence

$$0 \longrightarrow J_1 \hookrightarrow J \longrightarrow E/(J_1 : u)(-D) \longrightarrow 0$$

to examine the rate at which the number of minimal generators of the Alexander dual can grow as we build J one generator at a time.

Lemma 3.4.1. *Let $J \subseteq E$ be squarefree strongly stable. Let u be the minimal generator of J such that either:*

(a) u is lexicographically smallest among the generators of degree D or

(b) *u has $\max u - \deg u+$ is maximal and u has the largest degree among all minimal generators, v, with $\max v - \deg v + 1$ maximal; if there is more than one such generator, u is the one that is lexicographically smallest,*

as in Lemma 3.2.5. Let J_1 the ideal generated by the other minimal generators. Let $m := \max(u)$. Then $J_1 : u = \langle e_1, e_2, \ldots, e_m \rangle$.

Proof. Let $j \leq m$. If $j \in \operatorname{supp}(u)$ then $ue_j = 0 \in J_1$. Otherwise, stability tells us that $u' := \frac{ue_j}{e_m} \in J$. Since it is not a multiple of u it must be in J_1, which implies that $ue_j = u'e_m \in J_1$. Thus, $\langle e_1, \ldots, e_m \rangle \subseteq J_1 \colon u$. Now, we compare Hilbert functions.

The inclusion $J_1 \hookrightarrow J$ induces the short exact sequence

$$0 \longrightarrow J_1 \hookrightarrow J \longrightarrow E/(J_1 : u)(-D) \longrightarrow 0 \tag{3.1}$$

where D is the degree of u. The Hilbert function of $E/\langle e_1, \ldots, e_m \rangle$ in degree i is $\binom{n-m}{i}$ and the Hilbert function of $E/(J_1 : u)$ in degree i is $h_J(i + D) - h_{J_1}(i + D)$. The monomials whose support has a non-empty intersection with $[m]$ are 0 in $E/(J_1 : u)$, so $h_{E/J_1:u}(i) \leq \binom{n-m}{i}$. We will be done if we show that, whenever a is a degree i monomial with $\operatorname{supp}(a) \subseteq \{m + 1, \ldots, n\}$, then $au \notin J_1$. This would imply that $h_{E/(J_1:u)}(i) = h_J(i + D) - h_{J_1}(i + D) \geq \binom{n-m}{i}$, which would mean that $h_{E/(J_1:u)}(i) = \binom{n-m}{i}$ and so $J_1 : u = \langle e_1, \ldots, e_m \rangle$.

Suppose that $au \in J_1$. Then we must have $w|au$ for some $w \in G(J_1)$. If $\deg w \leq \deg u$ then, by stability, we can replace every $i \in \operatorname{supp}(w) \cap \operatorname{supp}(a)$ with a variable in the support of u to obtain a new monomial in J_1 which divides u. Since $u \notin J_1$ this can not happen. Thus $\deg w > \deg u$. If u is the monomial in part (a) of Lemma 3.2.5, which has maximal degree, we are done. So we may suppose that we are in the situation of part (b). The number of variables, e_i, with $i \leq \max(u)$ and $i \notin \operatorname{supp}(u)$ is $\max(u) - \deg(u)$, which is maximal among all minimal generators. Since w divides a multiple of u and $\min(a) > \max(u)$ every $i \leq \max(u)$ not in $\operatorname{supp}(u)$ is also not in $\operatorname{supp}(w)$. It follows that $\max(u) - \deg(u) \leq \max(w) - \deg(w)$, which is a contradiction since $\deg(w) > \deg(u)$. QED

For any E-module, M, we define $M^* := \operatorname{Hom}_E(M, E)$ to be the *Alexander dual* of M. It is straightforward to show that $(E/J_\Delta)^* \cong J_{\Delta^*}$, where Δ^* is the Alexander dual of Δ. We will write $J^* := J_{\Delta^*}$ Since E is an injective E-module, applying Alexander duality to (3.1) gives us a short exact sequence

$$0 \longrightarrow \langle e_1 \cdots e_m \rangle (D) \longrightarrow E/J^* \longrightarrow E/J_1^* \longrightarrow 0. \tag{3.2}$$

Applying the mapping cone to this sequence gives us the next lemma.

Lemma 3.4.2. *Let $J, J_1 \subseteq E$ and u be as in Lemma 3.4.1 and Δ, Δ_1 be the complexes corresponding to J, J_1 respectively. Then $\operatorname{adeg} \Delta - \operatorname{adeg} \Delta_1 = \max u - \max\left(\frac{e_1 \cdots e_{\max(u)}}{u}\right) - 1$, where we define $\max(1) = 0$.*

Proof. Let $D = \deg(u)$ and $m = \max(u)$. The arithmetic degree of J is the number of minimal generators of J^*, which is $\beta_1^E(E/J^*)$. We bound this using the following diagram, known as the mapping cone. Here, the left-most vertical sequences are the beginnings of minimal free resolutions and the horizontal maps are the ones induced by the map $\langle e_1 \cdots e_m\rangle(D) \longrightarrow E/J^*$. The right-most vertical sequence is a resolution but not necessarily minimal. We wish to examine which modules in the resolution must be removed to make it minimal (we say such modules *split*). The only splits occur between modules with the same twist in adjacent places in the resolution.

$$\begin{tikzcd}
0 \arrow[r] & \langle e_1 \cdots e_m \rangle (D) \arrow[r] & E/J^* \arrow[r] & E/J_1^* \arrow[r] & 0 \\
& E(D-m) \arrow[u] \arrow[r, dashed] & E \arrow[u] & E \arrow[u] & \\
& E(D-m-1)^m \arrow[u, "{[e_1 \cdots e_m]}"] \arrow[r, dashed] & F_1 \arrow[u] & E(D-m) \oplus F_1 \arrow[u] & \\
& \vdots \arrow[u] & F_2 \arrow[u] & E(D-m-1)^m \oplus F_2 \arrow[u] &
\end{tikzcd} \tag{3.3}$$

Note that $\operatorname{rk} F_1 = \operatorname{adeg} \Delta$. The only splits that could occur in the resolution of E/J_1^* occur between summands of F_1 and $E(D-m-1)^m$ since there is no map connecting $E(D-m)$ and F_2. If we consider the same resolutions multi-graded, then all splits must involve a summand of $E(D-m-1)^m$ with squarefree multi-degree since J^* is generated by squarefree monomials. Write $\mathbf{D}$ for the multi-degree of u and $\mathbf{m}$ for that of $e_1 \cdots e_m$.

Let $u' = \frac{e_1 \cdots e_m}{u}$. This has multi-degree $\mathbf{m} - \mathbf{D}$. From above, we see that this must be a minimal generator of J_1^* since its multi-degree is the same as the multi-graded twist in the first term in the resolution of $\langle e_1 \cdots e_m\rangle$. Thus, it appears in the second term of the resolution of E/J_1^* given by the mapping cone and it can not split since the only splits involve summands of $E(m-D-1)^m$. Every other minimal generator of J_1^* with (total) degree $m-D$ must be a minimal generator of J^*. Since J_1 and J are squarefree strongly stable, so are their Alexander duals. Thus, the description of the resolutions of squarefree strongly stable ideals in E given in [2] provides

$$\begin{aligned}
\beta^E_{1,(m-D+1)} J_1^* &= \sum_{m \in G(J_1^*)_{m-D}} \binom{\max(v)}{\max(v)-1} \\
&= \sum_{m \in G(J_1^*)_{m-D}} \max(v) \\
&= \beta^E_{1,(m-D+1)} J^* + \max\left(\frac{e_1 \cdots e_m}{u}\right).
\end{aligned}$$

The mapping cone above tells us that

$$\beta^E_{1,(m-D+1)}J_1^* = \beta^E_{1,(m-D+1)}J^* + m - (\#\text{ splits}).$$

Thus the number of splits is exactly $m - \max\left(\frac{e_1\cdots e_m}{u}\right)$. But the number of minimal generators of J_1^* is $\mu(J^*) + 1 - (\#\text{ of splits})$, which gives the result.

QED

Remark 3.4.3. The value, $\max u - \max\left(\frac{e_1\cdots e_{\max(u)}}{u}\right)$, given in Lemma 3.4.2 is always at least 1 unless $u = e_1$. So if J is squarefree strongly stable removing one of the minimal generators can never increase the arithmetic degree. Thus, in the shifted case, we can recover Lemma 3.2.9 from Lemma 3.4.2.

Remark 3.4.4. We can get slightly more information on the minimal generators of J^* using this method. Lemma 3.2.9 assures us that the arithmetic degree cannot increase, so one of the summands of F_1 *must* split and it must split with a summand of $E(D-m-1)^m$. So J^* must have a minimal generator of degree $m-D+1$ (equivalently Δ must have a facet of size $n - \max(u) + \deg(u) - 1$) whenever we can remove u without damaging stability. Thus Δ must contain a facet of size $n-p$ where p is the projective dimension of S/I_Δ.

To simplify notation we make a definition.

Definition 3.4.5. Let u be a squarefree monomial in S with $\max(u) = m$. Then we define

$$\operatorname{comax}(u) := \begin{cases} 0 & \text{if } u = x_1x_2\cdots x_d \\ \max\left(\frac{x_1x_2\cdots x_m}{u}\right) & \text{otherwise} \end{cases}$$

We can now compute the arithmetic degree of any squarefree strongly stable ideal.

Theorem 3.4.6. *Let $I = \langle u_1, u_2, \ldots, u_s\rangle \subseteq S$ be a squarefree strongly stable ideal with initial degree d. Then*

$$\operatorname{adeg} I = 1 + \sum_{i=1}^{s}(\max(u_i) - \operatorname{comax}(u_i) - 1)$$

Proof. Define $I_k = \langle u_1, \ldots, u_k\rangle$ where we have ordered the generators in such a way that each I_k is squarefree strongly stable. In particular $u_1 = x_1x_2\cdots x_d$, where d is the initial degree of I. Then $I_s = I$ and $I_1 = \langle u_1\rangle$, which has arithmetic degree d. If I is principle the claim is clear. Otherwise, by Lemma 3.4.2, we have

$$\operatorname{adeg} I_k - \operatorname{adeg} I_{k-1} = \max(u_k) - \operatorname{comax}(u_k) - 1.$$

Summing from $k = 2$ to $k = s$ we get

$$\operatorname{adeg} I - d = \sum_{k=2}^{s}(\max(u_k) - \operatorname{comax}(u_k) - 1).$$

Noting that $\max(u_1) - \operatorname{comax}(u_1) = d$, gives the claim. QED

Example 3.4.7. Let $I = I_d$ be the ideal generated by all squarefree monomials of degree d. This is squarefree strongly stable. We claim that $\operatorname{adeg} I_d = \binom{n}{d-1}$. To see this consider the ideals $I + x_n$ and $I\colon x_n$. Considering $(I + x_n)/\langle x_n\rangle$ as a squarefree strongly stable ideal in $n-1$ variables, induction on n gives that $\operatorname{adeg}(I+x_n) = \binom{n-1}{d-1}$. The ideal $I : x_n$ is also squarefree strongly stable. In fact $I : x_n = I_{d-1}$ as an ideal in $n-1$ variables. Again induction allows us to state that $\operatorname{adeg}(I : x_n) = \binom{n-1}{d-2}$. Since I is generated in a single degree, the minimal generating sets of $(I + x_n)/\langle x_n\rangle$ and $x_n(I : x_n)$ are disjoint and together give all the minimal generators of I. Now Theorem 3.4.6 tells us that $\operatorname{adeg} I = \operatorname{adeg}(I + x_n) + \operatorname{adeg}(x_n(I : x_n)) - 1$ Since $x_n(I : x_n) = \langle x_n\rangle \cap (I : x_n)$, we have $\operatorname{adeg}(I) = \operatorname{adeg}(I + x_n) + \operatorname{adeg}(I : x_n) = \binom{n-1}{d-1}+\binom{n-1}{d-2} = \binom{n-1}{d}$. More easily, one can note that the simplicial complex associated to I_d has every $(d-1)$-dimensional face as a facet. There are, of course, $\binom{n}{d-1}$ of these.

Note that the value given for the difference in the arithmetic degrees is bounded from above by $\deg(u) - 1$, giving us the next result.

Corollary 3.4.8. *Let $I \subseteq S$ be a squarefree strongly stable ideal with initial degree $d \geq 2$. Then, if I is not principle,*

$$\operatorname{adeg} I \leq \sum_{u\in G(I)} \deg(u) - \mu(I) - d + 2$$

Proof. The proof follows by induction on $\mu(I)$. Suppose $I = \langle u_1, u_2\rangle$ has 2 generators with degrees $d_1 \leq d_2$. Then we can write $I = I_1 + \langle u_2\rangle$ where I_1 is principle and squarefree strongly stable. Thus $I_1 = \langle x_1x_2 \cdots x_{d_1}\rangle$ and so $I_1^* = \langle x_1, \ldots, x_{d_1}\rangle$. Then Remark 3.4.4 tells us that $\max u_2 - \deg u_2 = 1$. By stability, if the degree of u_2 is larger than 1, $\min(u_2) = 1$

Then $u_2 = \frac{x_1x_2\cdots x_{d_2+1}}{x_i}$ for some $1 < i \leq d_2$. Since u_2 is not a multiple of u_1, $i \leq d_1$. By stability, $\frac{u_2x_i}{x_{d_1}} \in I$. If $i < d_1$ then this must be a multiple of u_1, which is a contradiction to $d_1 \in \operatorname{supp}(u_1)$. So $i = d_1$ and we can now compute a primary decomposition of I as

$$I = \bigcap_{j=1}^{d_1-1} \langle x_j\rangle \cap \langle x_{d_1}, x_{d_1+1}x_{d_1+2} \cdots x_{d_2+1}\rangle .$$

This last term can be written as the intersection of $\langle x_{d_1}, x_j\rangle$ where $d_1 < j \leq d_2 + 1$. There are then a total of $d_2 - d_1 + 1$ of these ideals. So we have an irredundant minimal primary decomposition of I with $(d_2 - d_1 + 1) + (d_1 - 1) = d_2$ ideals and so $\operatorname{adeg} \Delta = d_2$.

Now assume that I has at least 3 generators and $I = I_1 + \langle u\rangle$ with I_1 squarefree strongly stable and not principle. Taking care to make certain that u has maximal degree, we can be sure that I and I_1 have the same initial degree, d. By Lemma 3.4.2

and induction we have

$$\begin{aligned}\operatorname{adeg} I &\leq \operatorname{adeg} I_1 + \deg(u) - 1 \\ &\leq \sum_{v \in G(I_1)} \deg(v) - \mu(I_1) - d + 2 + \deg(u) - 1 \\ &\leq \sum_{v \in G(I)} \deg(v) - \mu(I) - d + 2,\end{aligned}$$

which completes the proof. QED

Remark 3.4.9. Theorem 3.4.8 is false for non-shifted complexes and for principle ideals. A principle ideal generated in degree d has arithmetic degree d rather than 1. Now, consider the ideal $I = \langle x_1x_4, x_2x_3\rangle$. This is the Stanley-Reisner ideal of a 4-gon, and thus has arithmetic degree 4. However, the predicted bound is 2. It does, however, hold for componentwise linear ideals since their graded Betti numbers (and in particular the degrees of their minimal generators) are stable under exterior shifting and $\operatorname{adeg} I \leq \operatorname{adeg} I^{\mathrm{e}}$. See [3, Theorem 2.1].

We now use the above results to produce an upper bound on the number of minimal generators of Cohen-Macaulay (CM) ideals of arbitrary codimension.

Lemma 3.4.10. *Let $J \subseteq S$ be a codimension c squarefree strongly stable ideal generated in degree d. Then the number of minimal generators of J with* $\max(u) - \operatorname{comax}(u) = s$ *is at most*

$$\binom{N(s)}{d-s+1} - \binom{N(s)-c}{d-s+1}$$

where $N(s) := \operatorname{pd} S/J + d - s - 1$. In addition

$$\operatorname{adeg} J \leq 1 + \sum_{s=1}^{d} (s-1)\left[\binom{N(s)}{d-s+1} - \binom{N(s)-c}{d-s+1}\right]$$

Proof. Let $p = \operatorname{pd} S/J$. If u is one of the minimal generators in question then, by Lemma 3.2.4, we must have $\min(u) \leq c$ and $\max(u) - d + 1 \leq p$. In addition, we can write u in the form $u = vx_{m-s+1}x_{m-s+2}\cdots x_m$, where $m = \max(u)$, $\max(v) \leq m-s-1$ and $\deg(v) = d - s$. If we depolarize u we get

$$u^{\mathrm{dep}} = u'x_{m-d+1}^{s}$$

where $u' = v^{\mathrm{dep}}$. The monomial u' must satisfy:

(1) $\deg(u') = d - s$

(2) $\operatorname{supp}(u') \subseteq \{1, 2, \ldots, m - d\}$

(3) $\min(u') = \min(v) \leq c$

We can count the number of possible u in S by instead counting the possible number of u'. Fix m. The number of monomials in S satisfying (1) and (2) is $\binom{m-s-1}{d-s}$ with the convention that $\binom{-1}{0} = \binom{-1}{-1} = 1$ to account for the case when $m = s = d$ (that is, $u = x_1x_2\cdots x_d$). Any monomial u' satisfying (1) and (2) but failing (3) must have $\operatorname{supp}(u') \subseteq \{c+1, \ldots, m-d\}$. There are $\binom{m-c-s-1}{d-s}$ of these (using the same convention as before). The max of u, m, may range over $d \leq m \leq p+d-1$. Summing over this range we get

$$\begin{aligned}
&\sum_{m=d}^{p+d-1} \binom{m-s-1}{d-s} - \sum_{m=d}^{p+d-1} \binom{m-c-s-1}{d-s} \\
&= \binom{d-s-1}{d-s} + \sum_{m=d+1}^{p+d-1} \binom{m-s-1}{d-s} - \sum_{m=d+c+1}^{p+d-1} \binom{m-c-s-1}{d-s} - \binom{-1}{d-s} \\
&= \sum_{k=d-s+1}^{N(s)} \binom{k-1}{d-s} - \sum_{k=d-s+1}^{N(s)-c} \binom{k-1}{d-s} \\
&= \sum_{k=N(s)-c+1}^{N(s)} \binom{k-1}{d-s}
\end{aligned}$$

Now, we may apply the identity $\sum_{j=0}^{N} \binom{j}{k} = \binom{N+1}{k+1}$ to get that

$$\sum_{k=N(s)-c+1}^{N(s)} \binom{k-1}{d-s} = \binom{N(s)}{d-s+1} - \binom{N(s)-c}{d-s+1}.$$

The statement about adeg follows by Theorem 3.4.6. QED

Corollary 3.4.11. *Let $I \subseteq S$ be CM with codimension c, initial degree d and $r = \operatorname{reg} I$. Let $N(s) = r + c - s - 1$ Then*

$$\mu(I) \leq 1 + \sum_{s=1}^{c} (s-1) \left[\binom{N(s)}{c-s+1} - \binom{N(s)-d}{c-s+1} \right]$$

where $\binom{-1}{-1} = \binom{-1}{0} = 1$.

Proof. Extending the field if necessary, we may assume that K is infinite. By [10, Theorem 15.20], $\operatorname{gin}_{\text{rlex}} I$ is strongly stable and, if we adjoin new variables to form a larger polynomial ring T, $J = (\operatorname{gin}(I))^{\text{pol}} \subseteq T$ is squarefree strongly stable. The ideal J has the same Hilbert function as I and thus the same codimension and initial degree. Since K is infinite, taking gin preserves the regularity by [5]. The Alexander dual of J, J^*, has codimension d, is generated in degree c and has projective dimension r. Since

$$\mu(I) \leq \mu(\operatorname{gin} I) = \mu(J) = \operatorname{adeg} J^*$$

the result follows from Lemma 3.4.10. QED

Remark 3.4.12. If $c = 2$ then the bound in Corollary 3.4.11 reduces to $\mu(I) \leq d+1$, Dubreil's theorem, which is more usually proven using the Hilbert-Burch theorem. In particular, in codimension 2, the bound does not depend on the regularity. However in codimension 3 the bound simplifies to $1+2d+d(r-d)+\frac{d(d-1)}{2}$, which does depend on the regularity.

Corollary 3.4.13. *Let I be squarefree strongly stable and Cohen-Macaulay of codimension 2. Then $\mu(I) = d+1$.*

Proof. Dubreil's theorem says that $\mu(I) \leq d+1$. As in Remark 3.3.4, we may combine Alexander Duality and Theorem 3.3.1 to get that $\mu(I) \geq d+1$. QED

Remark 3.4.14. Like Dubreil's theorem, the bound in Corollary 3.4.11 is sharp. Let J^* be the ideal generated by all monomials with degree c, min at most d and max at most $r-c-1$ (we assume that $r-c-1 \leq n$). The number of such monomials with $\max(u) - \operatorname{comax}(u) = s$ is exactly the bound given in Lemma 3.4.10. Clearly J^* is squarefree strongly stable. Thus, Lemma 3.4.10 gives exactly the arithmetic degree of J^*, which is the number of minimal generators of J. Since J^* has a c-linear minimal free resolution, J is CM of codimension c with the desired number of minimal generators.

Chapter 4 Matroid Complexes

4.1 Basics

Suppose that Δ is a d dimensional simplicial complex with n vertices and v some vertex of Δ. Then Δ_{-v} is a simplicial complex with $n-1$ vertices and $\text{link}_\Delta(v)$ has dimension $d-1$. Given information about the link and the deletion (which can be gotten, say, by induction) we can attempt to reconstruct information about the complex Δ. Of course, such an inductive procedure has no chance for success if Δ_{-v} and $\text{link}_\Delta(v)$ are not in the same class as Δ. So, we focus our attention on a particular class of complexes, a class with the property that most of its important subcomplexes are also within the class. This is the class of matroid complexes.

Definition 4.1.1. Let Δ be a simplicial complex on $[n]$. Then Δ is a *matroid complex* if one of the following equivalent conditions holds.

(i) For every $W \subseteq [n]$, $\Delta|_W$ is pure.

(ii) For every $W \subseteq [n]$, $\Delta|_W$ is Cohen-Macaulay.

(iii) For every $W \subseteq [n]$, $\Delta|_W$ is shellable.

It is always true that (iii) $\implies$ (ii) $\implies$ (i). We only need to show that if every subcomplex is pure then they are also shellable. This follows by induction, since a subcomplex of a matroid is again matroid. We will not do this proof here; the curious reader may refer to [21, Proposition 3.1].

Example 4.1.2. The complex depicted in Figure 4.1 is matroid while the one in Figure 4.2 is not. To see that the second is not, restrict to the subset $\{1, 3, 4\}$. Then we get $\{1\}$ and $\{3, 4\}$ as facets so the restriction is not pure. To see that Figure 4.1 is matroid, one can either use a brute force check of every subset of [6] or read ahead and use Lemma 4.1.10 or simply take the author's word for it. Note that Figure 4.2 depicts a Cohen-Macaulay (in fact shellable) complex.

In general, almost anything you do to a matroid complex will result in another matroid complex. We summarize some of the more useful constructions in the next proposition.

Proposition 4.1.3. *Let Δ be a matroid complex with vertex set $[n]$. Then the following complexes are also matroid.*

(a) $\Delta|_W$ *for every* $W \subseteq [n]$

(b) $C\Delta$, *the cone over* Δ

(c) $[\Delta]_k$, *the* k*-skeleton of* Δ

(d) $\text{link}_\Delta(F)$ *for every* $F \in \Delta$.

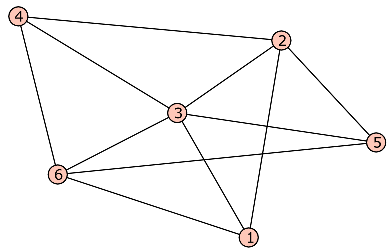


Figure 4.1: A matroid complex with 6 vertices

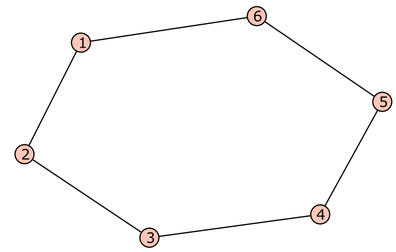


Figure 4.2: A non-matroid complex with 6 vertices

Proof.

(a) Since $(\Delta|_W)|_V = \Delta|_{W\cap V}$ and the left-hand side is, by definition pure this follows immediately from the definition.

(b) Let v be the vertex of the cone. Clearly the cone over a pure complex is pure, so let $W \subset [n]$. If $v \notin W$ then $(C\Delta)|_W = \Delta|_W$, which is pure because Δ is matroid. If $v \in W$ then $(C\Delta)|_W = C(\Delta|_W)$. By part (a) $\Delta|_W$ is matroid and so, by induction on the number of vertices, $C(\Delta|_W)$ is matroid and in particular pure.

(c) Note that $[\Delta|_W]_k = [\Delta]_k|_W$. As in part (b), if W is a proper subset of $[n]$ then this is matroid, and thus pure, by induction on the number of vertices. It only remains to check that $[\Delta]_k$ is itself pure. Suppose that $[\Delta]_k$ has a face F with $\dim F < k$. Since $F \in \Delta$ it must be contained in some facet with dimension $\dim \Delta \geq k$. It then follows that F must be contained in some k-dimensional face of Δ, which is then a face of $[\Delta]_k$. Thus F is not a facet of $[\Delta]_k$ and the k-skeleton is therefor pure.

(d) This time, we check that $\operatorname{link}_\Delta(F)|_W = \operatorname{link}_{\Delta|_W}(F)$, which will then be pure by induction. We then only need to know that $\operatorname{link}_\Delta(F)$ is pure. Suppose that $G \in \operatorname{link}_\Delta(F)$ is a facet. Then $G \cup F \in \Delta$ must be a facet of Δ. So $\dim(G \cup F) = \dim \Delta$ and then $\dim G = \dim \Delta - \dim F - 1 = \dim \operatorname{link}_\Delta(F)$. So the link is pure and thus matroid.

QED

Remark 4.1.4. All of the statements in Proposition 4.1.3 follow by the same sort of argument. First show that the desired construction commutes with restrictions. The proper restrictions will then be pure by induction on the number of vertices since restrictions of matroids are matroid. One then only has to check that the construction gives a pure complex. It is important to note that the purity of Δ does not follow from the purity of its restrictions. For an example, see Figure 4.3. In this complex, all of its proper restrictions are pure, while the complex itself is not.

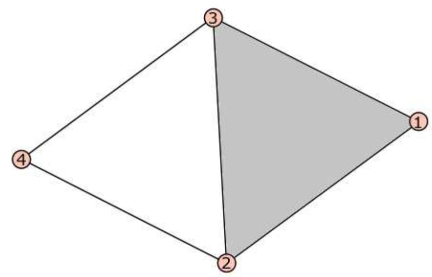


Figure 4.3: A non-pure complex whose proper restrictions are all pure

In addition to being Cohen-Macaulay, the Stanley-Reisner ideals of matroid complexes posses another, desirable property: they are level. This can be see by using Hochster's Forumula (Theorem 2.2.6) to compute the various degree components of the last term in the minimal free resolution of I_Δ. Taking the link and deletion we get, by standard results of simplicial homology, a long exact sequence. Since the links and deletions of matroid complexes are matroid, induction tells us that their Stanley-Reisner ideals are level. The long exact sequence and Hochster's formula then forces I_Δ to be level as well. For a complete proof of this fact, the reader may refer to Stanley's book [21].

In the spirit of Proposition 4.1.3 we give one more construction for producing new matroids from old ones.

Definition 4.1.5. Let Δ be a simplicial complex with vertex set V and v a vertex of Δ. Then we define

$$S_v\Delta = \{E \cup \{w\} \mid E \text{ a face of } \operatorname{link}_\Delta(v)\}$$

where w is a fixed vertex not in Δ. We consider $S_v\Delta$ as a simplicial complex with vertex set $V \cup \{w\}$. If $W = \{w_1, \ldots, w_k\}$ is a set of k vertices not in Δ then

$$S_v^W = S_v^k\Delta = \{E \cup \{w\} \mid E \text{ a face of } \operatorname{link}_\Delta(v), w \in W\}$$

is a simplicial complex on $V \cup W$. We call $S_v^W\Delta$ the *k-fold partial star avoiding* v.

See Figure 4.4 for an example of the result of this procedure. To get this, we start with a single 3-cycle (or a complete graph on 3 vertices if you prefer) and add 3 new vertices (labeled 4,5 and 6) connecting them by an edge to every vertex except vertex 1.

Figure 4.5 represents the results of applying this construction twice starting with a single edge between vertices 1 and 2 . First we add vertices 3 and 4 avoiding vertex 1 and then vertices 5 and 6 avoiding vertex 2. Note that the edges $\{3,5\}$ and $\{3,6\}$ are in the final complex.

The next lemma informs us that, if we start with a matroid complex, this construction will usually result in another matroid. In fact, we will later in Theorem 4.1.9

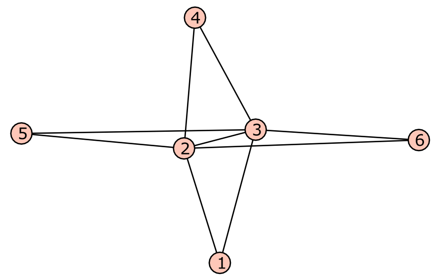


Figure 4.4: $S_1^3 K_3$

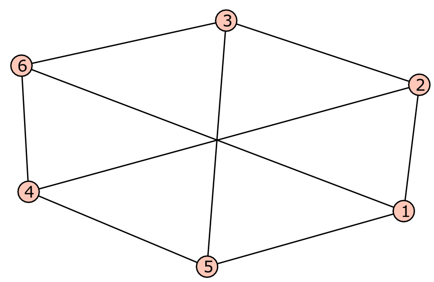


Figure 4.5: $S_2^2 S_1^2 K_2$

see that all matroid can be obtained from smaller matroids in this way. However, not every complex obtained by taking partial stars is matroid, even if we start with a "nice" complex (for example, paths can be obtained in this way). So, we must impose some additional conditions on this process. In particular, we need to choose the vertex we avoid properly, where the meaning of "properly" is given by the next definition.

Definition 4.1.6. Let Δ be a simplicial complex and v a vertex of Δ. Then we say that v is a *center* of Δ if $\mathrm{link}_\Delta(v)$ contains every other vertex of Δ.

Note that this definition depends only on the 1-skeleton of Δ; we are simply looking for vertices that are connected by an edge to every other vertex. More algebraically, if I_Δ is the Stanley-Reisner ideal of Δ, v is a center of Δ if and only if x_v does not appear in any degree 2 minimal generator of I_Δ.

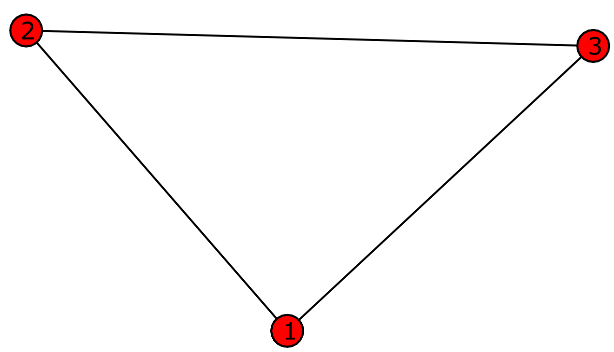
In Figure 4.4 vertex 2 and 3 are the only centers, while Figure 4.5 shows us a complex that has no centers. If Δ is a cone then the vertex of the cone is a center. The converse is false; to have a center it is only necessary that $[\Delta]_1$ be the 1-skeleton of a cone. This can be easily seen by looking at the Stanley-Reisner ideal and using the comment in the preceeding paragraph.

Lemma 4.1.7. *Let Δ be a matroid complex and $W = \{w_1, \ldots, w_k\}$ a set of vertices not in Δ. Then $S_v^W \Delta$ is matroid if and only if v is a center of Δ.*

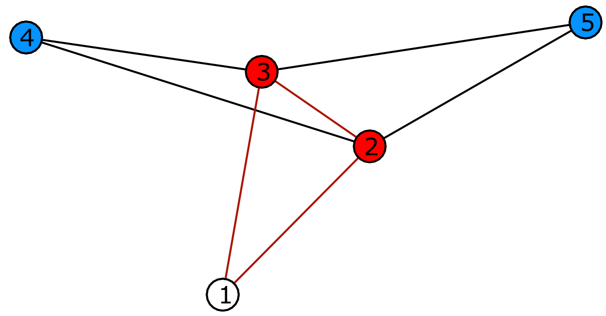
Proof. Assume that v shares an edge with every other vertex of Δ (i.e., v is a center of Δ). Let $\Gamma = S_v^W \Delta$. If X is a subset of vertices of Γ which contains only vertices of Δ then $\Gamma|_X = \Delta|_X$, which is pure. So suppose that $W \cap X \neq \emptyset$. If v is the only vertex of Δ contained in X then $\Gamma|_X$ is 0 dimensional and thus pure. So, assume that X intersects the other vertices of Δ. Let $X' = X \cap [\mathrm{link}_\Delta(v)]_0$. Since Δ is matroid, $\Gamma|_{X'} = \Delta|_{X'}$ is pure. A facet of $\Gamma|_X$ is of the form $\{v\} \cup E$ or $\{w_i\} \cup E$ where E is a facet of $\Gamma|_{X'}$. Since these all have the same size, $\Gamma|_X$ is pure and thus Γ is matroid.

Now, suppose that v is not a center and let $a \neq v$ be a vertex of Δ not in $\mathrm{link}_\Delta(v)$. If Δ is not pure then neither is $\Gamma = S_v^W \Delta$, so we may as well assume that Δ is pure. If Δ has dimension 0 then v will be a facet of $\Gamma = S_v \Delta$, which will have dimension 1. Suppose that $\dim \Delta > 1$. Then $\Delta_{\{a,v\} \cup W}$ has dimension 1 and has $\{v\}$ as a facet, thus is not pure so Γ is not matroid. QED

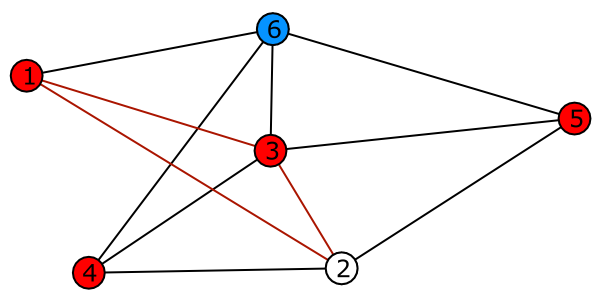
Example 4.1.8. Suppose we start with the complete graph on 3 vertices K_3, which is clearly matroid. Every vertex is a center so we may avoid any of them. Since the vertices of K_3 are all the same (in the sense that the link and deletion of any of the vertices in K_3 yield isomorphism pairs of complexes) it makes no difference which we choose. Let's pick 1.



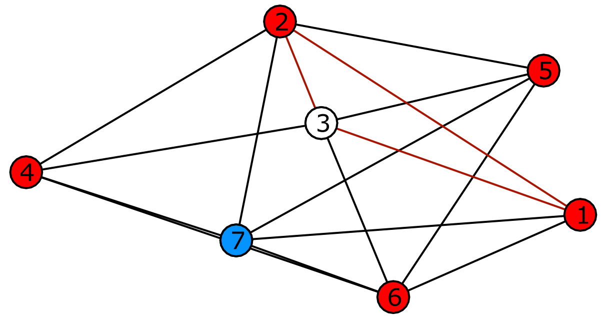
To form $S_1^2 K_3$ we add 2 new vertices (4 and 5) and draw an edge to every vertex of K_3 except 1. Since this complex is small, it is easily seen to be matroid, as guaranteed by Lemma 4.1.7.



Vertex 1 is now no longer a center (since $\{1,4\} \notin S_1^2 K_3$). Similarly, the new vertices we added are not centers either. So, if we wish to have matroid at the end we must choose on the vertices of the K_3 we started with. As before, the remaining vertices of K_3 (2 and 3) are identical, so we arbitrarily choose 2. Now we have a new center (2 this time) and can repeat our construction again, adding another vertex (6 this time) avoiding vertex 2 to obtain $S_2^1 S_1^2 K_3$.



As before, 2 is no longer a center and neither are the vertices that we added. The only center remaining is 3 so we add another vertex avoiding it to get $S_3^1 S_2^1 S_1^2 K_3$.



Being somewhat larger it is not as easy to check that this complex is matroid. The reader can either believe Lemma 4.1.7 or apply Lemma 4.1.10. At this point, there are no centers left so any attempt to continue this process will not result in a matroid.

The next Theorem allows us to, in many cases, reduce large matroid complexes to much smaller ones. This is particularly useful if in dimension 1 where it gives a constructive procedure that produces all matroid complexes.

Theorem 4.1.9. *Let Δ be a simplicial complex with dimension d. Then Δ is matroid if and only if $\Delta = S^{m_k}_{v_k} \cdots S^{m_1}_{v_1}\Gamma$ where Γ is a matroid such that $[\Gamma]_1$ is a complete graph and the v_i are distinct vertices of Γ. We allow for $[\Gamma]_1$ to be K_1, the complete graph on 1 vertex, i.e. a point.*

To prove this, we must first establish the special case when $\dim \Delta = 1$. Then, since the skeletons of matroids are themselves matroid, we can induct on the dimension and concern ourselves only with the facets of Δ. That Δ has facets in the correct places will be forced by purity. The next lemma is used to easily detect the matroid-ness of a 1-dimensional complex. This result amounts to saying that we can walk between any 2 vertices of a 1-dimensional matroid complex by taking at most 2 steps (assuming our steps are 1 edge long).

Lemma 4.1.10. *Let Δ be a 1-dimensional simplicial complex. Then Δ is matroid if and only if for every vertex v and every edge E, $\mathrm{link}_\Delta(v) \cap E \neq \emptyset$.*

Proof. Suppose there exists a vertex v and an edge E disjoint from the link of v. Let $L = [n] - \mathrm{link}_\Delta(v)$. Then Δ_L has $\{v\}$ and E as facets, and so is not matroid.

Conversely, suppose that there exists a subset $W \subseteq [n]$ such that Δ_W is not pure. So Δ_W must have a 0-dimensional facet, say $\{v\}$. Let $v \neq w \in W$. Since v is a facet of Δ_W we must have $\{v, w\} \notin \Delta$. Thus $W \cap \mathrm{link}_W(v) = \emptyset$ and so any edge, E, of Δ_W (there must be at least one since Δ_W is not pure) must also be disjoint from $\mathrm{link}_\Delta(v)$. Since E is also an edge of Δ the proof is complete. QED

Lemma 4.1.11. *Let Δ be a matroid with dimension 1. Then $\Delta = S_{v_k}^{m_k} \cdots S_{v_1}^{m_1} K_s$ where $k \leq s$ and the v_i are distinct vertices of K_s.*

Proof. Let v be a vertex of Δ and $n = f_0(\Delta)$. We define $\deg v = \deg_\Delta v = |\mathrm{link}_\Delta(v)|$. Choose, if possible, v so that $\deg v \neq n-1$. If there are no such vertices then $\Delta = K_n$ and we are done. Let W be the set of vertices of Δ_{-v} not in $\mathrm{link}_\Delta(v)$. If $E \in \Delta_W$ is an edge then $E \cap \mathrm{link}_\Delta(v) = \emptyset$, contradicting Lemma 4.1.10. So $\dim \Delta_W = 0$. If $w \in W$ and $\{v, w\} \in \Delta$ then $w \in \mathrm{link}_\Delta(v)$, a contradiction. Let Δ' be Δ with the vertices in W deleted. From above, we can see that any edge of Δ that is not in Δ' must be of the form $\{w, x\}$ where $w \in W$ and $x \in \mathrm{link}_\Delta(v)$. Thus, $\Delta = S_v^m \Delta'$ where $m = |W|$. Since Δ' has fewer vertices than Δ we can conclude by induction on n. QED

Since any simplicial complex of the form $S_{v_k}^{m_k} \cdots S_{v_1}^{m_1} K_s$ is matroid by Lemma 4.1.7, we now have a complete classification of 1 dimensional matroid complexes. Using this, we are now in a position to prove Theorem 4.1.9.

Proof. Assume that Δ is matroid. We induct on d, the dimension of Δ. If $d = 0$ then $\Delta = S_v^{n-1} K_1$, where v is the solitary vertex of K_1. So assume $d > 0$ and choose $v \in \Delta$ to be a vertex such that $\mathrm{link}_\Delta(v)$ does not contain every other vertex of Δ. If there are no such vertices then we may set $\Delta = \Gamma$ since the 1-skeleton of Δ must be complete. Let W be the set of vertices not in $\mathrm{link}_\Delta(v)$. Let Δ' be Δ with the vertices of W deleted. We need only show that $\Delta = S_v^W \Delta'$ since, by induction on the number of vertices, Δ' has the required form. By Lemma 4.1.11 we have $[\Delta]_1 = S_v^W [\Delta']_1$. In particular $[\Delta_W]_1$ has dimension 0. By definition, there are no edges (and thus no higher dimensional faces) of Δ in $\{v\} \cup W$. So, the only thing remaining to show is that, if E is a facet of $\mathrm{link}_\Delta(v)$ (which is matroid and thus pure) and $w \in W$ then $\{w\} \cup E \in \Delta$. By induction on d, if $F \in \mathrm{link}_\Delta(v)$ is not a facet, $\{w\} \cup F \in W$. Suppose that $E \cup \{w\} \notin \Delta$. Let $X = \{v, w\} \cup E$. For every $e \in E$, $(E - \{e\}) \cup \{w\} \in \Delta_X$ and since $E \cup \{w\} \notin \Delta_X$ these are all facets. By construction, $E \cup \{v\} \in \Delta_X$ contradicting the purity of Δ_X. Thus, $E \cup \{w\} \in \Delta$. Finally, we note that if $E \cup \{w\} \in W$ where $w \in W$ and $E \notin \mathrm{link}_\Delta(v)$ then an identical argument (interchanging v and w) shows that, again, $\Delta_{\{v,w\} \cup E}$ is not pure. Therefor, $\Delta = S_v^W \Delta'$ and we may conclude by induction on the number of vertices.

For the converse we simply note that, by Lemma 4.1.7, every complex of the form $S_{v_k}^{m_k} \cdots S_{v_1}^{m_1} \Gamma$ is matroid by provided that we choose the v_i so that they are centers of their respective complexes. Being a center depends only the 1-skeleton, which is, by assumption, complete. So we can simply choose the v_i to be distinct vertices of Γ and be assured that the partial star avoiding v_i is matroid. QED

4.2 Dimension 1

Our goal is to classify the h-vectors of all 1-dimensional matroid complexes. In fact, we do something stronger and classify all 1-dimensional matroid complexes up to isomorphism in terms of partitions.

Since we will be working exclusively with 1-dimensional complexes, it will be convenient to ignore the difference between the 0-dimensional complex $\mathrm{link}_\Delta(k)$ and the set of vertices of $\mathrm{link}_\Delta(k)$.

Lemma 4.1.11 provides us with a complete classification of 1 dimensional matroid complexes. It only remains to compute the possible h-vectors that this construction allows. Note that Lemma 4.1.7 allows us to form a new matroid complex $S^1_v\Delta$ whenever Δ has a center. If Δ has no center then we can easily give it a center by using the next easy lemma.

Lemma 4.2.1. *Let Δ be a 1-dimensional matroid complex with h-vector $(1, m-1, h_2)$ and $C_1\Delta$ the 1-skeleton of the cone over Δ. Then $C_1\Delta$ is matroid with h-vector $(1, m, h_2 + m)$.*

Proof. This can be easily shown either directly from the definition, or from Lemma 4.1.10. We apply Lemma 4.1.10; let v be a vertex of $C_1\Delta$ and E and edge. If both are contained in Δ they satisfy the condition in the lemma since Δ is matroid. This is also clear if v is the vertex of the cone since $\mathrm{link}_{C_1\Delta}(v)$ then contains every vertex of Δ and so the result is again clear. The only remaining case is if v is in Δ and E is not. But then E contains the vertex of the cone, which is clearly in $\mathrm{link}_{C_1\Delta}(v)$. Thus $C_1\Delta$ is matroid. The statement about h-vectors follows by noting that the f-vector of $C_1\Delta$ is $(1, m + 2, f_2 + m + 1)$ where f_2 is the number of edges of Δ. QED

While the title says "h-vector", we mostly work by computing f-vectors. It will thus be convenient to write the h-vector of a complex with f-vector $(1, f_0, f_1)$. This simply a special case of Equation (2.1).

$$h = (1, h_1 - 2, 1 - f_1 + f_2)$$

So, if Δ has h-vector $(1, m, h_2)$ then $S^i_v\Delta$ will have h-vector $(1, m+i, h_2+mi)$ since we are adding in exactly mi additional edges and i vertices. If we wish to stay in the class of matroid complexes then we must require that Δ have a center. If it doesn't then we first apply C_1. These two constructions in fact give every 1-dimensional matroid complex on n vertices and gives us a way to compute the h-vector. This will then give a complete classification of the h-vectors of 1-dimensional matroid complexes. Here the degree of a vertex v is the number of edges containing v. Note that v is a center of Δ if and only if v has degree $n - 1$.

Theorem 4.2.2. *Let $h = (1, m, h_2)$. Then h is the h-vector of a 1-dimensional matroid complex if and only if one of the following holds.*

1. *$h_2 = x(m - x)$ for some $\lfloor \frac{m}{2} \rfloor \le x \le m$.*
2. *$h_2 = h' + x(m - x + 1)$ where $\lfloor \frac{m}{2} \rfloor \le x \le m$ and $(1, x - 1, h')$ is the h-vector of a matroid complex.*

Proof. Suppose that Δ is a matroid complex with h-vector h. Let v be a vertex of Δ with degree $x + 1$ and $L = \mathrm{link}_\Delta(v) \cup \{v\}$. If $x = m = n - 2$ and Δ is not a cone then we may delete v to obtain a matroid complex Δ_{-v} with h-vector $(1, m - 1, h')$.

So the h-vector of Δ is $(1, m, h' + m)$, which satisfies condition 2 with $x = m$. If Δ is a cone then it satisfies condition 1 with $x = m$.

Assume $x \neq m$ and let $\Gamma = \Delta|_L$ If $w \in L - \{v\}$ and $a \notin L$ then $\{a, w\} \in \Delta$ by Lemma 4.1.10. So we may write $\Delta = S_v^{[n]-L}\Gamma$. By definition $|L| = x + 2$ and $h(\Gamma_{-v}) = (1, x - 1, h')$ for some h', as long as $\dim \Gamma_{-v} = 1$ (equivalently, as long as Γ is not a cone). Now simply note that $|[n]-L| = n-x-2 = m-x$ and that to form Δ from Γ_{-v} we must add edges $\{a, b\}$ for every $a \in L - \{v\}$ and $b \notin L - \{v\}$. There are a total of $x(m-x+1)$ such edges. Thus $h_2(\Delta) = h'+x(m-x+1)$. Now suppose that Γ is a cone so that $h(\Gamma) = (1, x, 0)$. Then from the comment just before the proof, we see that the h-vector of Δ is given by $(1, m, h_2)$ where $h_2 = 0+x|[n]-L| = x(m-x)$.

To get the inequalities, we simply take v to be a vertex with maximal degree. If $x < \lfloor \frac{m}{2} \rfloor$ then each vertex, $w \in L$, of $\Delta = S_v^{[n]-L}\Gamma$ has every vertex not in L in its link, by construction. There are $m - x \geq \lfloor \frac{m}{2} \rfloor$ such vertices meaning that w has a larger degree than v.

Conversely, if h satisfies one of the 2 conditions above, we must show that there is some matroid with h-vector h. There are naturally 2 cases.

Case 1. If $h = x(m - x)$ then the preceding paragraph tells us how to construct the matroid Δ. Let Γ be the cone over $x-1$ vertices with apex v and $\Delta = S_v^{m-x}\Gamma$. As noted above, $h(\Delta) = (1, m, x(m - x))$.

Case 2. Suppose $h = h' + x(m - x + 1)$ and there is some matroid, Γ with h-vector $(1, x - 1, h')$. Again, the needed construction is implicit in the preceding argument. We have that $C_1\Gamma$ is matroid with h-vector $(1, x, h' + x)$ and $\Delta = S_v^{m-x}C_1\Gamma$ has $x + 1 + 1 + (m - x) = m + 2$ vertices and $h_2(\Delta) = h' + x + x(m - x) = h' + x(m - x + 1)$. We choose the vertex v to be the new vertex added when forming $C_1\Gamma$ so that we may be assured that it is a center and that Δ is matroid (by Lemma 4.1.7).

QED

Shortly, we will give another form of the same classification the proves to be easier to work with in general. We will therefor have little need to refer to this result. However, when we discuss the diagram of 1-dimensional matroid h-vectors the construction of these conditions may make a bit more sense.

Remark 4.2.3. Using the above theorem, we can produce several easy examples of matroid h-vectors (assuming in each case that the final entry is positive): $(1, m, m)$, $(1, m, m - 1)$, $(1, m, 2(m - 1))$, $(1, m, 2(m - 2))$, $(1, m, 3m - 5)$. The last is produced using $x = m-1$ and $h' = m-3$, if $m \geq 3$ since $(1, m-2, m-3)$ is a matroid h-vector.

Remark 4.2.4. Theorem 4.2.2 provides us with a method for checking whether or not there is a matroid with the specified h-vector. In specific cases this can be somewhat tedious (although it is easily automated) however, we can eliminate certain small values immediately.

(i) There are no matroid h-vectors of the form $(1, m, h_2)$ where $0 < h_2 < m - 1$ because if there were then we would also have a matroid h-vector of the form $(1, x-1, x(m-x+1))$ for some x. However $x(m-x+1) > x(m-x) \geq m-1$ for all $1 \leq x < m$ (which excludes the first type of h-vectors as well).

(ii) Suppose $m \geq 6$ and $m < h_2 < 2(m-2)$. Then $(1, m, h_2)$ is not the h-vector of a matroid complex. To see this, note that the function $g(x) = x(m-x)$ only takes on values larger than $2(m-2)$ when $1 < x < m-1$ and $g(1) = g(m-1) = m-1$, excluding h-vectors of the first type. Similarly, the function $f(x) = x(m-x+1)$ takes on only values larger than $2(m-2)$ except for $f(1) = f(m) = m$. Thus, if $(1, m, h_2)$ is a matroid h-vector then there must be another matroid h-vector $(1, m-1, h_2 - m)$. But $0 < h_2 - m < (m-1) - 1$ and so by the above, there are no such h-vectors.

We now give a more closed form of Theorem 4.2.2. If Δ is a 1-dimensional matroid then, by Lemma 4.1.11 we know that we may write Δ in the form

$$\Delta = S_{v_1}^{W_1} \cdots S_{v_k}^{W_k} K_s.$$

From this it is straightforward to compute the f-vector and h-vector of Δ.

Theorem 4.2.5. *Let $h = (1, n-2, h_2)$, $h_2 \geq 0$. Then h is the $h-vector$ of a matroid if and only if there is a sequence of numbers $m_1, m_2, \ldots m_k$ such that $m_1 \geq 0$, $\sum_{i=1}^{k} m_i = n - k$ and*

$$h_2 = \binom{n-1}{2} - \sum_{i=1}^{k} \binom{m_i + 1}{2}$$

Proof. Assume h is the h-vector of some matroid, Δ. If Δ is not the complete graph on n vertices, K_n (for which the claim is obvious) then, we may write $\Delta = S_{v_1}^{W_1} \cdots S_{v_k}^{W_k} K_s$. Let $m_i = |W_i|$. By construction, $\dim \Delta_{W_i \cup \{v_i\}} = 0$ and, if X is not contained in any W_i, $\dim \Delta_X = 1$. Moreover, all of the $W_i \cup \{v_i\}$ are pairwise disjoint. Our construction guarantees that, if E is an edge not contained in any $W_i \cup \{v_i\}$ then $E \in \Delta$. It follows that $f_1(\Delta) = \binom{n}{2} - \sum \binom{m_i+1}{2}$. It is now easy to compute the h-vector of Δ and see that it is as claimed.

Conversely, if h has the form given in the Theorem, we may set $\Delta = S_{v_1}^{m_1} \cdots S_{v_k}^{m_k} K_s$, which, as we see above, is matroid and has the correct h-vector. QED

If $m \in \mathbb{N}_0^s$ and $\Delta = S_{v_s}^{m_s} \cdots S_{v_1}^{m_1} K_s$, where the v_i are distinct vertices of K_s then it is easily seen that if we choose the v_i in a different order we get isomorphic complexes (see Lemma 4.2.10). So without loss of generality, we will always assume that $v_i = i$ and suppress the notation.

Definition 4.2.6. If $m \in \mathbb{N}_0^s$ then we define $\Delta_{\mathbf{m}} = S^{m_s} \cdots S^{m_1} K_s$ where we agree that if $m_i = 0$ then $S^{m_i}\Gamma = \Gamma$ for any complex Γ.

Remark 4.2.7. So, what we have (from Lemma 4.1.11) is that every 1-dimensional matroid is isomorphic to one of the form $\Delta_{\mathbf{m}}$ for some sequence, $\mathbf{m}$, of non-negative integers with length s. However, we have chosen to construct $\Delta_{\mathbf{m}}$ in such a way that the first s vertices form a complete graph. Of course, we may always permute the vertices so that this occurs. The point is that, while Lemma 4.1.11 completely classifies all matroids with dimension 1, the notation $\Delta_{\mathbf{m}}$ does not since it implicitly assumes a particular ordering of the vertices. Since the author is incapable of distinguishing between isomorphic complexes, this may be considered only a minor notational annoyance.

If $\mathbf{m} = \mathbf{0}$ is the zero sequence then $\Delta_{\mathbf{0}} = K_s$ and if $\mathbf{m} = (m_1)$ then (with the understanding that K_1 is a single vertex) $\Delta_{(m_1)}$ is a 0-dimensional complex with $m_1 + 1$ vertices. Of course, all 0-dimensional complexes are matroid. In all other cases, $\dim \Delta_{\mathbf{m}} = 1$ as noted below in Proposition 4.2.22 (a).

The following is a restatement of Theorem 4.2.5 using this new notation.

Corollary 4.2.8. *If $m \in \mathbb{N}_0^s$ then h-vector of $\Delta_{\mathbf{m}}$ is $(1, h_1, h_2)$ where*

$$h_1 = s + \sum_{i=1}^{s} m_i$$
$$h_2 = \binom{n-1}{2} - \sum_{i=1}^{s} \binom{m_i + 1}{2}$$

Using an argument similar to that of Theorem 4.2.5 we can classify the possible Stanley-Reisner ideals of 1 dimensional matroid complexes. In fact, later we will see that this is a special case of a more general result (Lemma 4.4.3), which is itself a special case of a still more general result, Lemma 4.4.3, which follows immediately from Theorem 4.4.31.

Notation. If $\sigma \subseteq [n]$ and $\mathfrak{m}_\sigma = \langle x_i \mid i \in \sigma \rangle$ and $\hat{\mathfrak{m}}_\sigma^d$ is the ideal generated by all the squarefree monomials in $\mathfrak{m}^d$. If $|\sigma| < d$ then $\hat{\mathfrak{m}}_\sigma^d = \langle 0 \rangle$. We will write $\mathfrak{m} = \mathfrak{m}_{[n]}$.

Theorem 4.2.9. *Let $I \subseteq S$. Then I is the Stanley-Reisner ideal of a 1-dimensional matroid on $[n]$ with n vertices if and only if I has the form*

$$I = \sum_{i=1}^{k} \hat{\mathfrak{m}}_{\sigma_i}^2 + \hat{\mathfrak{m}}^3 \tag{4.1}$$

for some collection $\{\sigma_i\}$ of subsets of $[n]$ such that $\sigma_i \cap \sigma_j = \emptyset$ whenever $i \neq j$ and $n \geq \sum_{i=1}^{k} |\sigma_i|$.

Proof. Assume $I = I_\Delta$ for some 1-dimensional matroid Δ on $[n]$. Then we may write $\Delta = \Delta_{\mathbf{m}} = S_{v_k}^{W_k} \cdots S_{v_1}^{W_1} K_s$, where $|W_i| = m_i$. We consider $\mathbf{m} \in \mathbb{N}^s$, padding the end with 0 if needed. Then the number of vertices of Δ is $s + \sum m_i = n$. Let $\sigma_i := W_i \cup \{v_i\}$. The σ_i are all pairwise disjoint and $|\sigma_i| = m_i + 1$ We then get that $n = \sum_{i=1}^{k} |\sigma_i| + (s - k)$, where $s - k \geq 0$. By construction, any edge in σ_i is a non-face of Δ and thus any squarefree monomial in $\hat{\mathfrak{m}}_{\sigma_i}^d$ is in I. Again the construction assures us that these are the only non-edges of Δ. Since $\dim\Delta = 1$ every degree 3 squarefree monomial must also be in I. Thus I has the form given in equation (4.1).

Conversely, assume I is of the form given in equation (4.1). Then set $m_i = |\sigma_i| - 1$ and $s = n - \sum m_i$. Let $m = (m_1, \ldots, m_k, 0, \ldots, 0) \in \mathbb{N}^s$. The argument above shows us that I is the Stanley-Reisner ideal of $\Delta_{\mathbf{m}}$. QED

Using our knowledge of the Stanley-Reisner ideal, we show that $\Delta_{\mathbf{m}}$ is invariant up to isomorphism when the entries of $\mathbf{m}$ are permuted.

Lemma 4.2.10. *Let τ be a permutation on $[s]$ and $\tau\mathbf{m} = (m_{\tau 1}, \ldots, m_{\tau s})$. Then $\Delta_{\mathbf{m}} \cong \Delta_{\tau\mathbf{m}}$*

Proof. Let I and J be the Stanley-Reisner ideals of $\Delta_{\mathbf{m}}$ and $\Delta_{\tau\mathbf{m}}$ respectively. Write

$$I = \sum_{i=1}^{k} \hat{\mathfrak{m}}_{\sigma_i}^2 + \hat{\mathfrak{m}}^3$$

and

$$J = \sum_{i=1}^{k} \hat{\mathfrak{m}}_{\eta_i}^2 + \hat{\mathfrak{m}}^3.$$

Clearly, $|\sigma_{\tau i}| = |\eta_i|$ and so since they are all pairwise disjoint, $|\cup\sigma_i| = |\cup\eta_i|$. We may as well assume that they are equal to each other and equal to $[r]$ for some $r \leq n$. Select a bijection $\phi_i \colon \sigma_{\tau i} \to \eta_i$ for each i. Pasting these together (which is well defined only because the σ_i and η_j are pairwise disjoint) gives a permutation on $[n]$ (fixing everything not in $[r]$). This now induces an isomorphism $I \cong J$ which implies that $\Delta_{\mathbf{m}}$ and $\Delta_{\tau m}$ are isomorphic as well. QED

Remark 4.2.11. The h-vector does not uniquely determine the isomorphism class of the complex $\Delta_{\mathbf{m}}$. The matroids $\Delta_{(2,2)}$ and $\Delta_{(3,0,0)}$ both have h-vector $(1, 4, 4)$ but are not isomorphic since $\Delta_{(3,0,0)}$ has a vertex of degree 5 but all the vertices of $\Delta_{(2,2)}$ have degree 3. These complexes are depicted in Figures 4.4 and 4.5. On the other hand, the sequence $\mathbf{m}$, up to permutation, does uniquely determine the isomorphism class of $\Delta_{\mathbf{m}}$, as in the next result.

Lemma 4.2.12. *Suppose $\mathbf{m} \in N_0^s$ and $\mathbf{m}' \in N_0^{s'}$ and $\Delta_{\mathbf{m}} \cong \Delta_{\mathbf{m}'}$. Then $s = s'$ and there is a permutation σ of $[s]$ such that $\mathbf{m}' = \sigma\mathbf{m}$.*

Proof. By Lemma 4.2.10 we may assume that $m_1 \geq m_2 \geq \cdots \geq m_s$ and likewise for $\mathbf{m}'$. If $m_1 = 0$ then the claim is obvious since $\Delta_0 = K_s$. So assume that $m_1 > 0$. Write $\Delta = \Delta_{\mathbf{m}}$ and $\Delta' = \Delta_{\mathbf{m}'}$. Choose corresponding vertices $v \in \Delta$ and $v' \in \Delta'$ so that $\Delta_{-v} \cong \Delta'_{-v'}$. Then, if we write $\Delta_{-v} = \Delta_{\mathbf{a}}$ and $\Delta'_{-v'} = \Delta_{\mathbf{a}'}$ induction on the number of vertices tells us that $\mathbf{a}$ and $\mathbf{a}'$ differ only by a permutation, σ, of the indices. We may choose v so that $\mathbf{a} = (m_1, \ldots, m_{k-1}, m_k - 1, 0, \ldots, 0)$ where m_k is the last non-zero entry of $\mathbf{m}$. Since we are not deleting a vertex of degree $n-1$ the length of $\mathbf{a}$ is the same as that of $\mathbf{m}$ and likewise for $\mathbf{a}'$ and $\mathbf{m}'$. This gives us that $s = s'$. Now, we have $a_i' = m_{\sigma j}$. But, since we are only deleting a single vertex from Δ' we must have $a'_i = m'_i$ for all but 1 value of i. The unique m'_j that changes is simply reduced by 1. So, we may as well map that a'_j to $m_k - 1$. Then the permutation σ gives $\mathbf{m}' = \sigma\mathbf{m}$. QED

Similarly, the isomorphism class of a 1-dimensional matroid, Δ, is determined by the degree sequence of Δ. If Δ is a 1-dimensional complex (which we may regard as a graph) and v is a vertex of Δ the we define the *degree* of v, $\deg v = \deg_\Delta v$ to be the number of edges containing v, or equivalently, the number of vertices in its link. The *degree sequence* of Δ, $D(\Delta)$, is defined as the sequence $D_i = |\{v \in \Delta \mid \deg v = i\}|$.

Lemma 4.2.13. *If Δ and Δ' are 1-dimensional matroids on $[n]$ then $\Delta \cong \Delta'$ if and only if $D(\Delta) = D(\Delta')$.*

Proof. It is trivial that isomorphic complexes have equal degree sequences, so we only consider the other direction. Since the degree sequence determines $f_0(\Delta)$, we may assume that $f_0(\Delta) = f_0(\Delta') = n$. Let v be a vertex of Δ with minimal degree. If $\deg v = n-1$ then $\Delta = K_n$. But this is the only complex with degree sequence $D(K_n)$ ($D_{n-1}(K_n) = n$ and all other are 0). So assume that $\deg v < n-1$. Let v' be a vertex of Δ' with $\deg v = \deg v'$. Without loss of generality, we may write $\Delta = \Delta_{\mathbf{m}}$.

Since $\Delta_{\mathbf{m}}$ is invariant under permutations of the entries of $\mathbf{m}$ we may, still without loss of generality, assume that v is in the last group of vertices to be added and likewise for v' or they are the vertices being avoided. Then Δ_{-v} and $\Delta'_{-v'}$ have the same degree sequence (the degree of each vertex in the link of v (v') goes down by 1 and the others stay fixed). So, by induction on the number of vertices there is an isomorphism $\Delta_{-v} \to \Delta'_{-v'}$. since v and v' must both be "attached" to their deletions in the same manner (by the construction of $\Delta_{\mathbf{m}}$) this will lift to an isomorphism simply by mapping $v \mapsto v'$. QED

Remark 4.2.14. Since we can permute the entries of $\mathbf{m}$ as we like, we may as well assume that $m_1 \geq m_2 \geq \cdots \geq m_s$. Let $k = \max\{i \mid m_i \neq 0\}$. If $\Delta_{\mathbf{m}}$ has n vertices then $n = s + \sum_{i=1}^s m_i = s + \sum_{i=1}^k m_i$ or equivalently, $(m_1, \ldots, m_k)$ is a partition of $n - s$ with length $k \leq s$. By increasing each entry by 1, we can form a partition $\lambda = (m_1 + 1) + \cdots + (m_s + 1)$ of n. Thus, each 1 dimensional matroid complex corresponds to a partition, λ, of n and two matroids are isomorphism if and only if they have the same partition.

The partition, λ, is determined uniquely by the non-zero entries of $\mathbf{m}$ and n. It is often shorter to give λ this way, since n is usually understood.

Example 4.2.15. Let $n = 6$. Then, as in the above remark, the partitions we are concerned with are summarized in the following table (we allow $\emptyset$ as the unique partition of 0).

$n-s$	s	partitions of $n-s$	m
0	6	$\emptyset$	000000
1	5	1	10000
2	4	11	1100
		2	2000
		111	111
3	3	21	210
		3	300
		31	31
4	2	22	22
		4	40
5	1	5	5

The last is the 0-dimensional matroid with 6 vertices. This means we have a total of 10 matroids on 6 vertices (see Table 4.1). But there are only 8 distinct h-vectors of such complexes (those that end with 10, 9, 8, 7, 6, 4, 3 and 0), such there must be either 2 h-vectors each with 2 matroids or a single h-vector with 3 matroids. The matroids Δ_{22} and Δ_{300} both have h-vector $(1,4,4)$ and Δ_{2000} and Δ_{111} both have h-vector $(1,4,7)$. See Figure 4.6.

A similar computation with $n=7$ shows that there are a total of 14 1-dimensional matroids with 7 vertices but only 12 matroid h-vectors. As with $n=6$ there are two pairs of non-isomorphism matroids with the same h-vector. In this case it is Δ_{3000} and Δ_{2200} sharing the h-vector $(1,5,9)$ along with Δ_{1110} and Δ_{20000} having h-vector $(1,5,12)$.

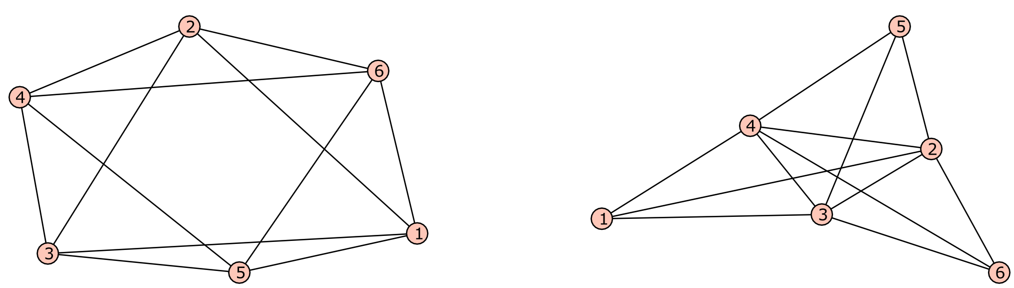


Figure 4.6: Δ_{111} and Δ_{2000} have the same h-vector but are not isomorphic

Extending the sequence $\mathbf{m}$ by appending 0 is equivalent to adjoining a new vertices to Δ along with every edge containing that vertex, that is, $C_1\Delta_{\mathbf{m}} = \Delta_{(m,0)}$. If $h(\Delta_{\mathbf{m_1}}) = h(\Delta_{\mathbf{m_2}})$ then $h(\Delta_{(\mathbf{m_1},0)}) = h(\Delta_{(\mathbf{m_2},0)})$.

Remark 4.2.16. As we see in the above example a partition of n does not necessarily produce of 1 dimensional complex. However the only exception is the trivial partition $\lambda = n$, which is the complex of n vertices. We can see from the definition of $\Delta_{\mathbf{m}}$ that as long as the length of $\mathbf{m}$ is at least 2, $\Delta_{\mathbf{m}}$ will contain a complete graph on at least 2 vertices and so $\dim \Delta_{\mathbf{m}} = 1$.

Notation. If λ is a partition of n then we will write $|\lambda| = n$ and $\ell(\lambda)$ for the length of λ. If $k > 1$ then $|\lambda|_k = \sum \binom{\lambda_i}{k}$ where we adopt the convention that $\binom{a}{b} = 0$ if $a < b$.

Definition 4.2.17.

(a) If λ is a partition of n then Δ_λ is the isomorphism class of the matroid defined by the sequence $(\lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$; $h(\lambda)$ is their common h-vector.

(b) If $\Delta \cong \Delta_{\mathbf{m}}$ is a matroid then λ_Δ is the partition $\sum_{i=1}^{s}(m_i + 1)$ of n. We will call λ_Δ the *partition associated to* Δ.

We have defined Δ_λ so that it is not a simplicial complex itself, but is rather a set of isomorphic simplicial complexes. We will, nonetheless, continue to write things like $\dim \Delta_\lambda$ and $h(\Delta_\lambda)$ to refer to any invariant of the class. We will also misuse such notation as $C\Delta_\lambda$ to refer to the class of complexes obtained from Δ_λ by, in this example, coning.

Remark 4.2.18. By the construction of $\Delta_{\mathbf{m}}$ we start with the complete graph on s vertices, where s is the length of $\mathbf{m}$. However, there may be complexes isomorphic to $\Delta_{\mathbf{m}}$ whose first s vertices do not define a complete graph. These are also elements of the isomorphism class Δ_λ. While these complexes are not strictly speaking accessible via the construction that the notation $\Delta_{\mathbf{m}}$ implies, we only need to be able to reach some element of their isomorphism class, Δ_λ since all the properties we are concerned with are invariant under isomorphism. This is all caused by the simplification in notation described just prior to Definition 4.2.6.

Example 4.2.19. Consider again Figure 4.6, which depicts the matroids corresponding to the sequences $(1, 1, 1)$ and $(2, 0, 0, 0)$ respectively. These are elements of the classes Δ_{2+2+2} and $\Delta_{3+1+1+1}$. Permuting the entries of the sequences will permute the vertices of the complexes. This will leave Δ_{111} unchanged. But, we can relabel the vertices so that $\{1, 2, 3\}$ does not defined a complete graph, which cannot be obtained by permuting the entries of $(1, 1, 1)$. This is an element of Δ_{2+2+2} that is not of the form $\Delta_{\mathbf{m}}$ for any sequence $\mathbf{m}$, which does not prevent it from being matroid.

Theorem 4.2.20. *There is a bijection between isomorphism classes of matroid complexes with dimension at most 1 and n vertices and partitions of n. In particular the number of isomorphism classes of 1 dimensional matroids with n vertices is $p(n) - 1$ where $p(n)$ is the number of partitions of n.*

Proof. The bijection is $\lambda \mapsto \Delta_\lambda$. The map $\Delta \mapsto \lambda_\Delta$ is its inverse. That every matroid with dimension 1 can be written as Δ_λ is essentially the content of Lemma 4.1.11. Proposition 4.2.22(a) takes care of the case when $\dim \Delta = 0$. That these maps are inverses to each other follows from their definitions. QED

Remark 4.2.21. If λ is a partition then $h(\lambda) = (1, n-2, h_2)$ where $n = |\lambda|$ and

$$h_2(\Delta_\lambda) = \binom{n-1}{2} - \sum_{i=1}^{\ell(\lambda)} \binom{\lambda_i}{2}$$
$$= \binom{n-1}{2} - |\lambda|_2.$$

So two partitions, λ and λ' determine the same h-vector if and only if $|\lambda| = |\lambda'|$ and $|\lambda|_2 = |\lambda|_2$. Examples of such pairs can be seen in Example 4.2.15. The first such pairs are $\lambda = 3+1+1$, $\lambda' = 2+2+2$ and $\lambda = 3+3$, $\lambda' = 4+1+1$.

Suppose there are k entries of λ equal to 1 and λ' is the partition of $n-k$ with these entries removed. Then $\sum \binom{\lambda_i}{2} = \sum \binom{\lambda'_i}{2}$. Then $\Delta_\lambda = C_1 \cdots C_1 \Delta_{\lambda'}$ (see the below Proposition) and $h(\lambda)$ is easily determined from $h(\lambda')$. So, in this sense, every matroid h-vector is induced from the h-vector of a smaller complex, one whose associated partition has no entry equal to 1.

The next proposition collects various facts about the relationship between 1-dimensional matroids and their associated partitions. If λ and λ' are partitions then we write $\lambda + \lambda'$ for the concatenation of λ and λ' as a partition of $|\lambda| + |\lambda'|$. Likewise, $\mathbf{m}, \mathbf{m}'$ is the concatenation of the sequences $\mathbf{m}$ and $\mathbf{m}'$.

Proposition 4.2.22.

(a) $\dim \Delta_\lambda = 0$ *if and only if* $\ell(\lambda) = 1$.

(b) $\Delta_{\mathbf{m},\mathbf{m}'} \cong [\Delta_{\mathbf{m}} * \Delta_{\mathbf{m}'}]_1$, *the one-skeleton of the join. Likewise for* $\Delta_{\lambda+\lambda'}$.

(c) $\Delta_{\mathbf{m},0} \cong C_1\Delta_{\mathbf{m}}$ *or equivalently* $\Delta_{\lambda+1} = C_1\Delta_\lambda$.

(d) Δ_λ *is a cone if and only if* $\lambda = (n-1)+1$

(e) *If* Δ *and* Δ' *are 1 dimensional matroids with* n *vertices then* $h(\Delta) = h(\Delta')$ *if and only if* $\binom{n-1}{2} - h_2(\lambda_\Delta) = \binom{n-1}{2} - h_2(\lambda_{\Delta'})$.

(f) *(Klivans) A 1 dimensional matroid,* $\Delta_{\mathbf{m}}$, *is isomorphic to a shifted complex if and only if* $\mathbf{m}$ *contains at most 1 non-zero entry.*

Proof.

(a) This follows immediately from the definition of Δ_λ, which is formed starting with a complete graph on $\ell(\lambda)$ vertices. Of course, this has dimension 0 if and only if $\ell(\lambda) = 1$.

(b) We induct on the length of $\mathbf{m}'$. If $\mathbf{m}$ has length 1 then $\dim \Delta_{\mathbf{m}} = 0$ and, by the definition $\Delta_{\mathbf{m}} * \Delta_{\mathbf{m}'} = \Delta_{\mathbf{m},\mathbf{m}'}$. Now, if $\mathbf{m}'$ has length $s > 1$ let $\mathbf{m}'' = (m_1, \ldots, m_{s-1})$. Then

$$\begin{aligned}\Delta_{\mathbf{m},\mathbf{m}'} \cong \Delta_{\mathbf{m},\mathbf{m}'',m_s} &\cong [\Delta_{\mathbf{m},\mathbf{m}''} * \Delta_{m_s}]_1 \\ &\cong [(\Delta_{\mathbf{m}} * \Delta_{\mathbf{m}''}) * \Delta_{m_s}]_1 \\ &\cong [\Delta_{\mathbf{m}} * \Delta_{\mathbf{m}'}]_1\end{aligned}$$

since the join is associative.

(c) $C_1\Delta$ is the 1-skeleton of $C\Delta$, the join of Δ and a single vertex. So this follows from part (b).

(d) From part (a), Δ_{m_1} has dimension 0 and from part (c) $\Delta_{\mathbf{m}} = C_1\Delta_{m_1} = C\Delta_{m_1}$ Conversely, if the sequence $\mathbf{m}$ has more than 1 non-zero entry $\Delta_{\mathbf{m}}$ can not be a cone, since it can then be written as the 1-skeleton of the join of two smaller complexes, at least one of which is not a cone. So $\ell(\lambda_\Delta) = 1$ and $\Delta_{\mathbf{m}} = C_1 \cdots C_1\Delta_{m_1}$, which is not a cone if there is more than one C_1.

(e) This follows immediately from Corollary 4.2.8 after noting that Δ_λ has $|\lambda|$ vertices.

(f) Let $\lambda = \lambda_\Delta$. By Theorem 4.2.9, we can write

$$I = \sum_{i=1}^{k} \hat{\mathfrak{m}}_{\sigma_i}^2 + \hat{\mathfrak{m}}^3$$

where $|\sigma_i| = \lambda_i$. We need to see that this ideal is squarefree strongly stable if and only if $\lambda_i = 1$ for all $i > 1$. One direction is easy; if only $lambda_1 = 1$ then it is clear that I_Δ will be squarefree strongly stable (after permuting the indices that so that σ_1 is the first $|\sigma_1|$ variables). Conversely, if $\lambda_2 \neq 1$ and $x_i x_j$ is the product of the two variables with the smallest indices in σ_2, $x_{i-1}x_j \notin I_\Delta$ since (by construction) $\{i-1, j\} \not\subseteq \sigma_2$ and it cannot be contained in any other σ_i since they are all pairwise disjoint. Thus I_Δ is not squarefree strongly stable whenever $\ell(\lambda) > 1$ (no matter how we permute the indices)

QED

Remark 4.2.23. Part (f) of Proposition 4.2.22 is in fact the same statement as Proposition 1 of [16] which states that dimension 1 (or rank 2) shifted matroids are exactly those obtained by starting with a dimension 0 complex and applying the C_1 operator repeatedly. This means precisely that our matroid is isomorphism to one of the form the form $\Delta_{(m_1,0,0,\ldots,0)}$. Equivalently, Δ_λ contains a shifted complex if and only if λ has only one entry not equal to 1.

If we regard the dimension 1 simplicial complex, Δ, as a graph one might want to ask about the size of the maximal cliques (that is, maximal subsets of vertices, W so that $\Delta|_W$ is a complete graph). If Δ is matroid then this is easy to determine from looking at the partition λ_Δ.

Lemma 4.2.24. *If* $\dim \Delta = 1$ *and* Δ *is matroid then, regarding* Δ *as a graph, all the maximal cliques of* Δ *have* $\ell(\lambda_\Delta)$ *vertices.*

Proof. Let $\lambda = \lambda_\Delta$. Of course, the sizes of the maximal cliques depends only on the isomorphism type of Δ, so we may assume that $\Delta = \Delta_{\mathbf{m}}$, where $m_i = \lambda_i - 1$. Let $s = \ell(\lambda)$. By definition, $\Delta|_{[s]}$ is a complete graph.

We may assume that λ is ordered so that $\lambda_1 \geq \lambda_2 \geq \cdots$. If Δ is itself a complete graph, then it is it's own unique maximal clique. This only happens if $\lambda = 1 + 1 + \cdots + 1$. Ignoring this single exception, we now assume that $\lambda_1 > 1$. Let v be one of the $\lambda_1 - 1$ vertices of $\Delta_{\mathbf{m}}$ added in the first batch (those avoiding vertex 1). Then $\lambda_{\Delta_{-v}} = (\lambda_1 - 1) + \lambda_2 + \cdots + \lambda_s$. This partition also has length s (since $\lambda_1 > 1$), so all its maximal cliques have size s, by induction on the number of vertices. If W is a maximal clique of Δ then, if $v \notin W$, W is also a maximal clique of Δ_{-v} and therefor $|W| = s$. Suppose that $v \in W$. Then $W - \{v\}$ is certainly a clique of Δ_{-v}, and we claim that it has size $s - 1$.

To see this, we first claim that $W' = (W - \{v\}) \cup \{1\}$ is a maximal cliques and so has s elements. Let $w \in W - \{v\}$ and suppose that $\{1, w\} \notin \Delta_{-v}$. Since W is a clique, $\{w, v\} \in \Delta$. Using Lemma 4.1.10, we see that $\mathrm{link}_\Delta(1) \cup \{w, v\}$ must be in Δ. Since $\{1, w\}$ is not in Δ we must have $\{1, v\} \in \Delta$. But, by our choice of v, $\{1, v\} \notin \Delta$ (v is attached to K_s avoiding 1). This contradiction indicates that our assumptions were wrong, so it must be that $\{1, w\} \in \Delta$, so that $W' = (W - \{v\}) \cup \{1\}$ is a clique.

Finally, we need to show that W' is a maximal clique. Suppose that V is a maximal clique with $W' \subseteq V$. Then V is also a clique of Δ and $W \subseteq (V - \{1\}) \cup \{v\}$. Let $w \in (V - \{1\})$. If we can show that $\{v, w\} \in \Delta$ then $(V - \{1\}) \cup \{v\}$ will be a clique of Δ and the maximality of W will imply that $W = (V - \{1\}) \cup \{v\}$, which is equivalent to $W' = V$. Since $1 \in V$ and V is a clique, $\{1, w\} \in \Delta$. Then, by Lemma 4.1.10, either $\{v, 1\} \in \Delta$ or $\{v, w\} \in \Delta$. Since the first is false by choice of v, the second must be true. So $W' = V$ and thus $|W| = |W'| = s$, as demanded.

QED

Remark 4.2.25. If we are given a 1 dimensional complex, Δ and we know that it is matroid, how can we find its associated partition, λ_Δ? The answer is to search for maximal subsets $\sigma_1, \ldots, \sigma_s \subseteq [n]$ so that $\dim \Delta_{\sigma_i} = 0$. Set $\lambda_i = |\sigma_i|$ to get the associated partition. This will be partition of n since the subsets σ_i must all be disjoint. Why? This is exactly the content of Theorem 4.2.9 describing the ideal of Δ and the subset σ_i are exactly the subsets that appear in that theorem.

A common problem in graph theory is to search for *cliques* of a graph, that is, subsets so that the restriction contains the maximal number of edges. We are doing the opposite and searching for "anti-cliques"— subsets whose restrictions contain the minimal number of edges, 0.

Within a fixed isomorphism class Δ_λ, we can ask how many different matroids does it contain? This can be answered by examining the Stanley-Reisner ideals associated to them.

Let λ be a fixed partition of n. Then we say a collection of disjoint subsets of $[n]$, Ω is a *set partition* subordinate to λ if $\Omega = \{\omega_1, \ldots, \omega_k\}$ where $|\omega_i| = \lambda_i$ for each $1 \leq i \leq \ell(\lambda) = k$.

Proposition 4.2.26. *Let λ be a partition of n and $s = \ell(\lambda)$. Then there is a bijection between the complexes in Δ_λ and the collection of set partitions subordinate to λ.*

Proof. Let $\Delta, \Delta' \in \Delta_\lambda$. Then, by Theorem 4.2.9 we may write

$$I_\Delta = \sum_{\omega \in \Omega} \hat{\mathfrak{m}}_\omega^2 + \hat{\mathfrak{m}}^3 \tag{4.2}$$

where Ω is a set partition subordinate to λ. Conversely, any set partition defines an ideal of the same form as (4.2). The two complexes, Δ and Δ', are *equal* if and only if $I_\Delta = I_{\Delta'}$. But $I_\Delta = I_{\Delta'}$ if and only if they are determined by the same set partition. QED

Example 4.2.27. Consider the partition $\lambda = 2 + 1 + 1$ of 4. The set partitions subordinate to λ are:

$$\begin{array}{c} 1|2|34 \\ 1|3|24 \\ 1|4|23 \\ 2|3|14 \\ 2|4|13 \\ 3|4|12 \end{array}$$

Among these only 1|2|34, 1|3|24, and 1|3|24 are of the form $\Delta_{\mathbf{m}}$ where $\mathbf{m}$ is one of the three permutation of $(1, 0, 0)$. The construction will always force us to add the vertex 4 avoiding either 1, 2 or 3. Thus $x_i x_4 \in I_{\Delta_{\mathbf{m}}}$ for exactly one of 1, 2 or 3. This implies that 1, 2 and 3 must be in different elements of the set partition.

Remark 4.2.28. The number of set partitions subordinate to λ is known to be given by the Faá di Bruno coefficients. Let $a_i = |\{j \mid \lambda_i = j\}|$ be the number of times that i appears in λ. Then the number of set partitions subordinate to λ is

$$\frac{n!}{a_1! a_2! \cdots a_k! 1!^{a_1} 2!^{a_2} \cdots k!^{a_k}}.$$

If $\lambda = 2 + 1 + 1$ then $a_1 = 2$ and $a_2 = 1$ and we get

$$\frac{4!}{2!1!1!^2 2!^1} = 6$$

set partitions. This is exactly the number we saw in Example 4.2.27.

4.2.1 The Set of Dimension 1 Matroid h-vectors

Now we consider the collection of matroid h-vectors and describe some structure this set possesses. To begin with, we give a table indicating which, out of all Cohen-Macaulay h-vectors, are matroid. The h-vector $(1, n-2, h_2)$ is a Cohen-Macaulay h-vector if and only if $h_2 \geq 0$. In fact, a 1-dimensional simplicial complex is Cohen-Macaulay if and only if it is connected, which is true if and only if $h_2 \geq 0$.

Table 4.1: Matroid h-vectors

n	h_2																												
2	**0**																												
3	**1**	**0**																											
4	**3**	**2**	**1**	**0**																									
5	**6**	**5**	**4**	**3**	**2**	1	**0**																						
6	**10**	**9**	**8**	**7**	**6**	5	**4**	**3**	2	1	**0**																		
7	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	7	**6**	**5**	**4**	3	2	1	**0**													
8	**21**	**20**	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	7	**6**	**5**	4	3	2	1	**0**							
9	**28**	**27**	**26**	**25**	**24**	**23**	**22**	**21**	**20**	**19**	**18**	**17**	**16**	**15**	14	**13**	**12**	11	**10**	9	8	**7**	**6**	5	4	3	2	1	**0**

In Table 4.1, the row number of each entry corresponds to the number of variables. The possible Cohen-Macaulay h-vectors are listed with the maximal values ($\binom{n-1}{2}$) being aligned on the left side. Those entries that are h_2 for a matroid with n vertices are shaded. The unshaded entries are not matroid h-vectors.

The first 2 rows of this table are automatic: there is only a single 1-dimensional complex with 2 vertices and only 2 pure complexes with 3 vertices. All of these are matroid and have h-vectors $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$. Moreover, for any n, there is a matroid with h-vector $(1, n-2, 0)$, namely, the cone over $n-1$ vertices. So we may shade the 0 on each row as well. From these, one may completely fill in the rest of the table. For notational convenience, we will write $m = n - 2$.

Recall that if $\dim \Delta = 1$ then $C_1\Delta$ is the 1-skeleton of the cone over Δ. By Lemma 4.2.1, whenever Δ is matroid so is $C_1\Delta$. On easily checks that, if $h(\Delta) = (1, m, h_2)$ then $h(C_1\Delta) = (1, m+1, h_2 + m + 1)$. Writing $h_2 = \binom{n-1}{2} - k$ for some k we see that $h_2 + m = \binom{n}{2} - k$. This is the entry in Table 4.1 directly below that of $h(\Delta)$. So if we have a shaded entry in Table 4.1, we may also shade each entry directly below it.

This still gives only a small portion of Table 4.1. To fill in the rest, we need another operation. Fortunately, we have one. Recall the definition of the partial star, S_v^k (Definition 4.1.5). If Δ is any matroid with a center (equivalently, $\Delta = C_1\Gamma$ for some other complex Γ) then $S_v^k\Delta$ is again matroid. We earlier computed the h-vector of $S_v^k\Delta$. If $k = 1$ then this gives us a "move" from $(1, m, h_2)$ to $(1, m+1, h_2 + m)$, which lies diagonally down and to the right. If $k = 2$ we first move down one and to the right one step and then down one step and to the right 2. Continue, moving an additional step to the right each time. This is illustrated below; the $\times$ indicates a matroid h-vector. Note that, since Δ must contain a center, we can only begin with an h-vector that has another h-vector directly above it, or with a 0.

$\times$						
$\times$						
$-$	$\times$					
$-$	$-$	$-$	$\times$			
$-$	$-$	$-$	$-$	$-$	$-$	$\times$

Thus, we may move straight down, or in parabolic arcs running parallel to the upper edge of Table 4.1. Everything we hit is guaranteed to be matroid by the results of the previous sections. That this gives all matroid h-vectors is the content of our classification of 1-dimensional matroids. Let's fill in the first few rows as an example.

Example 4.2.29. We will fill in the first 6 rows of the Table 4.1. We begin with all entries unshaded.

n	h_2									
2	0									
3	1	0								
4	3	2	1	0						
5	6	5	4	3	2	1	0			
6	10	9	8	7	6	5	4	3	2	1	0

We obtain the first row for free since the only 1-dimensional complex with 2 vertices is matroid. So, we first shade in the 0 in the first row. In fact, we may shade the 0 in each row, since they all are matroid h-vectors (as they are all cones).

n	h_2										
2	**0**										
3	1	**0**									
4	3	2	1	**0**							
5	6	5	4	3	2	1	**0**				
6	10	9	8	7	6	5	4	3	2	1	**0**

As in the discussion above, we may move directly down from any matroid h-vector and get another matroid h-vector. This gives us some additional shaded entries, which we put a box around to distinguish from those obtained in the previous step.

n	h_2										
2	**0**										
3	$\boxed{\mathbf{1}}$	**0**									
4	$\boxed{\mathbf{3}}$	$\boxed{\mathbf{2}}$	1	**0**							
5	$\boxed{\mathbf{6}}$	$\boxed{\mathbf{5}}$	4	$\boxed{\mathbf{3}}$	2	1	**0**				
6	$\boxed{\mathbf{10}}$	$\boxed{\mathbf{9}}$	8	$\boxed{\mathbf{7}}$	6	5	$\boxed{\mathbf{4}}$	3	2	1	**0**

Now we begin to move diagonally. The first entry at which we may begin this is the 1 on the second row. However, this gives us no new entries. The next choice is the 0 on the second row. From here we get more new entries.

n	h_2										
2	**0**										
3	**1**	**0**									
4	**3**	**2**	$\boxed{\mathbf{1}}$	**0**							
5	**6**	**5**	4	**3**	$\boxed{\mathbf{2}}$	1	**0**				
6	**10**	**9**	8	**7**	6	5	**4**	$\boxed{\mathbf{3}}$	2	1	**0**

We do this again with another entry. We can not use the 1 on the third row as it does not have a shaded entry above it. We may, however use the 2 on the third row.

n	h_2										
2	**0**										
3	**1**	**0**									
4	**3**	**2**	**1**	**0**							
5	**6**	**5**	$\boxed{\mathbf{4}}$	**3**	**2**	1	**0**				
6	**10**	**9**	8	**7**	$\boxed{\mathbf{6}}$	5	**4**	**3**	2	1	**0**

We may shade one more entry, the 8 on the last row as it lies directly below a matroid. This completes the table as every other valid move will land on an already shaded entry.

n	h_2										
2	**0**										
3	**1**	**0**									
4	**3**	**2**	**1**	**0**							
5	**6**	**5**	**4**	**3**	**2**	1	**0**				
6	**10**	**9**	$\boxed{8}$	**7**	**6**	5	**4**	**3**	2	1	**0**

Notice in Example 4.2.29 that many h-vectors can be reach in multiple ways. For example $(1, 3, 4)$ lies below $(1, 2, 1)$ and diagonally from $(1, 2, 2)$. It is often, but not always, true that different paths result in non-isomorphic matroids. In this case, there is only one matroid with h-vector $(1, 3, 4)$.

How are the moves described above reflected in the associated partitions? From Proposition 4.2.22 we see that a move directly down simply adds a 1 to the end of the partition. The diagonal moves are move complex. We may only apply these to partitions that contain a 1 (we will assume it is written last). Then, each time we move down a row, this 1 is increased by 1. That is we move from the partition $3+1+1$ to $3+1+2$ and then to $3+1+3$ and so on. At times, there will be multiple partitions in a given space. This will occur if and only if the matroids they produce have the same h-vector.

Below, we give a table indicating where one Table 4.1 the associated partitions are located as well as the number of matroid complexes with a specified h-vector. As a space saving measure, we will not use the + between the terms of the partitions and we will not list more than a single 1. We will use a subscript to indicate the number of times a value is repeated. For example, $321_2 = 3 + 2 + 1 + 1$.

Table 4.2: Associated partitions sorted by h-vector

n	h_2										
2	1_2										
3	1_3	21									
4	1_4	21_2	22	31							
5	1_5	21_3	221	31_2	32	–	41				
6	1_6	21_4	221_2	31_3 222	321	–	41_2 33	42	–	–	51

Note in Table 4.2 that there is only two spaces (and so only one corresponding matroid h-vector) containing more than one partition. This matches what we have seen before in Example 4.2.15 that there are only two matroid h-vectors that are the h-vectors of two different complexes. If we were to continue Table 4.2 to row 7 then we would get another pair of partitions with the same h-vectors, 2221 and

31_4 together with 41_3 and 331. These lie directly below the duplicated pairs on row number 6. This will occur again with 8 vertices and we also get a pair by moving diagonally since on row 7 there is a space all of whose partitions contain a 1. We also get another new pair 321_3 and 2222 from where a diagonal move an a vertical move happen to coincide.

Recall that for a partition, λ, $|\lambda|_k = \sum \binom{\lambda_i}{k}$. Two partitions generated the same h-vector if and only if $|\lambda|_k = |\gamma|_k$ for $k = 1, 2$. We may use this to define an equivalence relation $\sim$, on the set of partitions. Clearly, if $\lambda \sim \gamma$ then $\lambda + k \sim \gamma + k$ for any $k \in \mathbb{N}$. The converse is generally false. However, since $\binom{1}{k} = 0$ whenever $k > 1$, $\lambda + 1 \sim \gamma + 1$ does imply that $\lambda \sim \gamma$. In this sense, we want to talk about minimally equivalent partitions, that is, partitions so that no sub-partitions are equivalent.

For a given partition, λ of n, we may associate a monomial in n variables to it by raising x_i to the number of times i appears in λ. Of course, there is only one partition containing n, the trivial one. For our purpose we may largely ignore this since it gives a complex with dimension 0 rather than 1.

Definition 4.2.30. Let λ be a partition with $|\lambda| \leq n$. Then we define $x_\lambda \in K[x_1, \ldots x_n]$ by $x_\lambda = \prod x_i^{a_i}$ where $a_i = |\{j \mid \lambda_j = i\}|$.

Example 4.2.31. Consider $\lambda = 3+3+2+1+1+1$ as a partition of 11. There are 3 ones, 1 two and 2 threes, so $x_\lambda = x_1^3 x_2 x_3^2$. Notice that $\deg x_\lambda = 6 = \ell(\lambda)$

Remark 4.2.32. Every element of S can give a partition, but we do not get all partitions for numbers larger than n. In particular we do not get any partition containing a value larger than n. This can be corrected by using the polynomial ring, $K[x_1, x_2, \ldots]$, in countable many variables. This ring is, of course, not Noetherian and the author therefor does not wish to deal with it.

We consider all of these monomials as elements of a polynomial ring $S = K[x_1, \ldots, x_n]$. Since $x_\lambda = x_\gamma$ if and only if $\lambda = \gamma$ we can consider each partition of $\leq n$ to be an element of S.

Our results on matroids then allows us to think of S as a ring containing matroid complexes with dimension ≤ 1. We wish to make S into a ring containing, not matroid complexes, but matroid h-vectors. To this end, define an ideal

$$I = \langle x_\lambda - x_\gamma \mid h(\lambda) = h(\gamma) \rangle .$$

Unfortunately, this ideal is not homogeneous in the usual sense (for example $x_3^2 - x_1^2 x_2 \in I$ since $h(3+3) = h(4+1+1)$). We can correct this by changing what it means for an ideal to be homogeneous by changing the grading on S. In fact, we will want to consider 2 different gradings on S.

Notation. If $\mathbf{a} \in \mathbb{N}_0^n$ then $x^{\mathbf{a}} := \prod_{i=1}^n x_i^{a_i} \in S$. We will write $\mathbf{deg}\, x^{\mathbf{a}} = \mathbf{a}$ for the multi-degree of $x^{\mathbf{a}}$.

Definition 4.2.33. Let $\mathbf{w_1} = (1, 2, \ldots, n)$ and $\mathbf{w_2} = \left(\binom{1}{2}, \binom{2}{2} \ldots, \binom{n}{2}\right)$. Then, we make S into a bi-graded ring by defining, for any monomial $x^{\mathbf{a}} \in S$, $\mathbf{deg_w}\, x^{\mathbf{a}} = (\mathbf{w_1} \cdot \mathbf{a}, \mathbf{w_2} \cdot \mathbf{a})$, where $\mathbf{w_i} \cdot \mathbf{a}$ denotes the usual dot product of vectors. We will write $\deg_i x^{\mathbf{a}} = \mathbf{w_i} \cdot \mathbf{a}$.

Example 4.2.34. Consider again the monomial $x^{(3,1,2)} = x_1^3 x_2 x_3^2$. Using the above definition $\mathbf{deg_w}\, x^{(3,1,2)} = (11, 5)$. Recall from Example 4.2.31 that this is the monomial corresponding to the partition $\lambda = 3 + 3 + 2 + 1 + 1 + 1$. This partition has $|\lambda| = \deg_1 x_\lambda$ and $|\lambda|_2 = \deg_2 x_\lambda$.

A moment's thought will reveal that the situation in Example 4.2.34 is typical. That is, for any partition, λ, $\deg x_\lambda = (|\lambda|, |\lambda|_2)$. If $\ell(\lambda) > 1$ so that $\dim \Delta_\lambda > 1$ the bi-degree of x_λ tells us the h-vector of Δ_λ:

$$h(\Delta_\lambda) = \left(1, \deg_1 x_\lambda, \binom{\deg_1 x_\lambda - 1}{2} - \deg_2 x_\lambda\right).$$

Now, it is clear that using the bi-grading defined above makes the ideal I into a homogeneous ideal. Now we may make the following definition.

Definition 4.2.35. Let $S = K[x_1, \dots x_n]$ with the bi-grading from Definition 4.2.33 and

$$I = \langle x_\lambda - x_\gamma \mid \lambda, \gamma \text{ partitions of } \leq n, h(\Delta_\lambda) = h(\Delta_\gamma) \rangle .$$

Then we define the bi-graded ring $R_n := S/I$.

Each partition, λ, with $|\lambda| \leq n$ corresponds to an element of S and two partitions correspond to the same element in R_n if and only if they produce matroids with same h-vector. However, not every element of R_n corresponds to a matroid h-vector, only those whose $\mathbf{w_1}$-degree is at most n. One can correct this by using a polynomial ring with countable many variables, calling the resulting quotient R_∞. The idea I is not finitely generated in this case.

Remark 4.2.36. Each variable $x_i \in R_n$ (really we mean the image of x_i under the canonical projection) corresponds to the h-vector of the 0-dimensional complex with i vertices. A non-linear monomial whose $\mathbf{w_1}$-degree is at most n corresponds to a 1-dimensional matroid h-vector. Two monomials give the same element in R_n if and only if the matroids have the same h-vector.

Remark 4.2.37. What are the generators of the ideal I? Of course this depends on the choice of n. However increasing n only adds new minimal generators; it never alters the old ones. So we simply list out the first few generators (up to $n = 9$) that may be found in the ideal. Each generator appears for the first time the moment n becomes as large as its $\mathbf{w_1}$-degree.

$$\begin{aligned} I = \langle & x_1^3 x_3 - x_2^3, \\ & x_1^2 x_4 - x_3^2, \\ & x_1^5 x_5 - x_2 x_3 x_4, \\ & x_1 x_2 x_6 - x_4 x_5 \\ & x_2^2 x_5 - x_1 x_4 \\ & \dots \rangle \end{aligned}$$

Other generators can be found by brute force computation. One runs through every partition of n and finds a pair so with $|\lambda|_2 = |\gamma|_2$. The number of partitions grows rapidly as n does, but lists of partitions can still be obtained quickly by many computer algebra systems (in particular SAGE [22]).

Computing the generators of I one can then ask a computer algebra system to compute the Hilbert function of R_n, either bi-graded or with only the $\mathbf{w_1}$-grading. The way we have defined things, it is clear that the $\mathbf{w_1}$-graded Hilbert function of R_n in degree $i \leq n$ is the number of distinct h-vectors of matroid complexes with dimension at most 1. We list below the w_1-graded Hilbert function of R_9 up to degree 11 and the number of 1-dimensional matroid h-vectors.

Table 4.3: The number of at most 1-dimensional matroid h-vectors with at most 9 vertices

i	1	2	3	4	5	6	7	8	9	10	11
$\mathbf{w_1}$-Hilbert function	1	2	3	5	7	9	13	18	21	28	34
matroid h-vectors	1	2	3	5	7	9	13	18	21	26	33

Up to $i = 9$ this table gives the number of matroid h-vectors with dimension at most 1. This is, of course, simply one more than the number of matroid h-vectors with dimension 1. After this the Hilbert function and the number of matroid h-vectors no longer coincide.

4.3 A Conjecture of Stanley in Dimension 1

One of the long-standing conjectures on the h-vectors of matroid complexes is that they all occur as the Hilbert function of certain kinds of artinian monomial ideals. In dimension 1, we can positively resolve this conjecture. Throughout this section we write $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ for the maximal irrelevant ideal of S.

Definition 4.3.1. Let $I \subseteq S$ be a monomial ideal. Then we say that I is *pure* if the monomials outside of I that are maximal with respect to divisibility all have the same degree.

Example 4.3.2. The ideal $\mathfrak{m}^k$ is pure for any power of k. The ideal $I = \langle x_1^2, x_1, x_2, x_2^3 \rangle$ in $K[x_1, x_2]$ is not since $x_1, x_2^2 \notin I$ and are maximal under divisibility (that is, they do not divide any other monomials not in I) but have different degrees.

Conjecture 4.3.3 (Stanley)**.** *If* $\mathbf{h}$ *is the h-vector of a matroid complex then there is a pure monomial ideal with Hilbert function* $\mathbf{h}$.

Algebraically, pure ideals are level (Definition 2.2.3). An ideal is level if and only if $I\colon \mathfrak{m}$ (the *socle*) is generated in a single degree. This is the so-called *socle degree*, which is the same as the twist in the last module in the minimal free resolution (this fact is used in the proof of Theorem 4.3.6).

Lemma 4.3.4. *Let $I \subseteq S$ be a level monomial artinian ideal. Then I is pure.*

Proof. Since I is level, its socle, $I:\mathfrak{m}$, is generated in a single degree, say d. Let $u \notin I$ be a monomial maximal under divisibility. Then, by maximality $u\mathfrak{m} \subseteq I$ and so $u \in I:\mathfrak{m}$. Again, by the maximality of u, it must be a minimal generator of $I:\mathfrak{m}$ and thus has degree d. QED

If $\Delta \in \Delta_\lambda$ is a matroid with n vertices and $\dim \Delta = 0$ then we can consider the ideal J generated by all degree 3 monomials on $x_1, \ldots x_{n-1}$ in a polynomial ring with $n-1$ variables. Then $h(S/J) = (1, n-3) = h(\Delta)$. Clearly J is artinian. It is also strongly stable and the resolution of such ideals is given by the Eliahou-Kervaire resolution (see [18, Proposition 2.12]). This tells us, in particular, that J is level (in fact, it has a linear resolution).

Now, suppose that $\dim \Delta = 1$ but that Δ is a cone. Then Proposition 4.2.22 tells us that $\lambda = (n-1)+1$. Now, $h(\Delta) = (1, n-2, 0)$, which is the h-vector of the ideal, $J = \langle x_1, \ldots, x_{n-2} \rangle^3$ in a polynomial ring with $n-2$ variables. Again J has a linear resolution and is thus level. Since these two cases are easy to handle by hand, we can from here on out ignore them. Note that n and $(n-1)+1$ are the only partitions of n in which n or $n-1$ appear. So, we may assume that each entry of λ is at most $n-2$.

The following easy Lemma is a straightforward observation, but is critical in what follows.

Lemma 4.3.5. *Suppose that Δ is a matroid with $\dim \Delta = 1$. Then Δ is the 1-skeleton of a d-dimensional matroid if and only if $\ell(\lambda_\Delta) \geq d+1$.*

Proof. Let $\lambda = \lambda_\Delta$ and $s = \ell(\lambda)$. First, suppose that Δ is the 1-skeleton of a d-dimensional matroid, call it Γ. Then Γ contains a d-simplex, whose 1-skeleton is then a complete graph on $d+1$ vertices. Then, Lemma 4.2.24 says that $\ell(\lambda) \geq d+1$ QED

Theorem 4.3.6. *Let h be the h-vector of a matroid with dimension at most 1. Then there is an artinian, level monomial ideal with h-vector h and socle degree $n-2$ unless Δ is cone in which case it has socle degree $n-3$.*

Proof. Let $h = h(\Delta)$ for some matroid Δ. From the above comments, we can assume that $\dim \Delta = 1$ and that Δ is not a cone. Let $\lambda = \lambda_\Delta$ and $m = (\lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$. From Lemma 4.3.5 we know that $\ell(\lambda) \geq 2$ and so $\sum m_i \leq n-2$. Choose a set partition $\{\sigma_1 \ldots, \sigma_k\}$ where $|\sigma_i| = m_i$ (we ignore those m_j equal to 0). We can choose to do this so that σ_1 consists of the first m_1 numbers, σ_2 the next m_2 and so on. In this way we get a more canonical choice of set partition and we will consider everything to depend only on the partition, λ.

Let $R = K[x_1, \ldots, x_{n-2}]$ and $\mathfrak{n} \subseteq R$ be the maximal graded ideal. Define an ideal J_λ by

$$J_\lambda = \sum \mathfrak{n}_{\sigma_i}^2 + \mathfrak{n}^3 \subseteq R. \tag{4.3}$$

Note that this definition only make sense if $\sum m_i \leq n-2$, or equivalently if $\dim \Delta = 1$ and Δ is not a cone so that $\cup \sigma_i \subseteq [n-2]$. By construction J_λ is artinian so we need to show that $h(J_\lambda) = h(\lambda)$ and that J_λ is level.

We will work from the short exact sequence

$$0 \longrightarrow R/(J_\lambda : x_1)(-1) \xrightarrow{\cdot x_1} R/J_\lambda \longrightarrow R/(J_\lambda + \langle x_1 \rangle) \longrightarrow 0 \tag{4.4}$$

By our choice of $\{\sigma_i\}$, we can be assured that $1 \in \sigma_1$ and thus that x_1 properly divides a minimal generator of J_λ. That the sequence (4.4) is exact is then a standard algebraic fact. Since both ideals on the outside of the sequence are smaller that J_λ we may induct to assume that they have the proper Hilbert function. It is thus necessary to show that $J_\lambda + \langle x_1 \rangle$ and $J_\lambda : x_1$ are of the same form as J_λ.

First, consider $J_\lambda + \langle x_1 \rangle$. This is simply J_λ with every minimal generator that x_1 divides removed. That is,

$$J_\lambda + \langle x_1 \rangle = \sum_{i=2}^{s} \mathfrak{n}_{\sigma_i}^2 + \mathfrak{n}_{\sigma_1 - \{1\}}^2 + \mathfrak{n}^3$$

So $\overline{J_\lambda} = (J_\lambda + \langle x_1 \rangle)/\langle x_1 \rangle \subseteq \overline{R} = R/\langle x_1 \rangle$ is $J_{\overline{\lambda}}$ where $\overline{\lambda}$ is the partition $(\lambda_1 - 1) + \lambda_2 + \lambda_3 + \cdots$. By induction on $n = |\lambda|$ we get that $h(J_{\overline{\lambda}}) = h(\overline{\lambda}) = h(\Delta_{-v})$ for some vertex $v \in \Delta$. More precisely, if $\Delta = S^{W_s} \cdots S^{W_1} K_s$ then we may choose v to be any element of W_1. But $h(\overline{J_\lambda}) = h(J_\lambda + \langle x_1 \rangle)$ so the left side of (4.4) is what we need.

What about $J_\lambda \colon x_1$? By definition, J_λ contains every degree 3 monomial on $\{1, \ldots, n-2\}$, which implies that $J_\lambda \colon x_1$ contains every degree 2 monomial on $\{2, \ldots, n-2\}$. Additionally, $J_\lambda : x_1$ contains a degree 1 monomial for each element of σ_1 since $x_1 x_i \in J_\lambda$ for each $i \in \sigma_1$. So

$$J_\lambda \colon x_1 = \mathfrak{n}_{\sigma_1} + \mathfrak{n}^2 = \mathfrak{n}_{\sigma_1} + \mathfrak{n}_{[n-2]-\sigma_1}^2$$

and we easily see that $J_\lambda : x_1 = J_\gamma$ where γ is the partition $(n - 3 - \lambda_1) + 1$ of $n - 2 - \lambda_1$. One can see that $h(J_\gamma) = h(\gamma)$ be computing the h-vectors explicitly or by noting that $C\,\mathrm{link}_\Delta(v)$ is a cone and thus its h-vector, $h(\gamma)$, is determined by the 0 dimensional complex $\mathrm{link}_\Delta(v)$ and J_γ has the proper h-vector by induction on the dimension.

So the sequence (4.4) tells us that $h_i(J_\lambda) = h_{i-1}(\gamma) + h_i(\overline{\lambda})$. Corresponding to (4.4), we have another short exact sequence

$$0 \longrightarrow S/(I_\Delta \colon x_v)(-1) \xrightarrow{x_v} S/I_\Delta \longrightarrow S/I_\Delta + \langle x_v \rangle \longrightarrow 0$$

Since $I_\Delta \colon x_v$ is the Stalely-Reisner ideal of $C\,\mathrm{link}_\Delta(v)$ and $I_\Delta + \langle x_v \rangle$ that of Δ_{-v} which tells us that

$$\begin{aligned} h_i(\lambda) = h_i(\Delta) &= h_{i-1}(C\,\mathrm{link}_\Delta(v)) + h_i(\Delta_{-v}) \\ &= h_{i-1}(\gamma) + h_i(\overline{\lambda}) \\ &= h_i(J_\lambda) \end{aligned}$$

and so J_λ has the same h-vector as Δ_λ.

Now, we need to show that J_λ is level. To do this, we apply the mapping cone construction to sequence (4.4) using the fact that, by induction, $J_\lambda : x_1$ and $J_\lambda + \langle x_1 \rangle$

are level. Since all the ideals are artinian, they have projective dimension $n-2$. By induction on the number of variables we know the socle degree of $J_\lambda\colon x_1$ is $n-3$ since $C\,\mathrm{link}_\Delta(v)$ is a cone. We will also know that the socle degree of $J_\lambda + \langle x_1\rangle$ is $n-2$ provided that Δ_{-v} is not a cone. To see that this is true, recall that from Proposition 4.2.22(d), Δ_{-v} is a cone if and only if its associated partition has the form $(n-2)+1$, which could happen if $\lambda_1 = n-1$ since we assumed that λ_1 was the largest entry. This would then mean that Δ is itself a cone, a case we excluded at the beginning.

$$
\begin{tikzcd}
0 \arrow[r] & R/(J_\lambda : x_1)(-1) \arrow[r, "\cdot x_1"] & R/J_\lambda \arrow[r] & R/(J_\lambda + \langle x_1\rangle) \arrow[r] & 0 \\
 & R(-1) \arrow[u] \arrow[r] & R \arrow[u] & R \arrow[u] & \\
 & F_1(-1) \arrow[u] \arrow[r] & G_1 \arrow[u] & R(-1)\oplus G_1 \arrow[u] & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
 & F_{n-3}(-1) \arrow[u] \arrow[r] & G_{n-3} \arrow[u] & F_{n-4}(-1)\oplus G_{n-3} \arrow[u] & \\
 & R(-n+2)^a \arrow[u] \arrow[r] & G_{n-2} \arrow[u] & F_{n-3}(-1)\oplus G_{n-2} \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & R(-n+2)^a \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}
$$

We know that the final term in the rightmost row must split since otherwise the resolution would be longer than is allowed. Moreover, it must split with summands of G_{n-2}, which thus have twist $n-2$. No other summands of G_{n-2} can split. Any summand of G_{n-2} that does not split must also have twist $n-2$ since it is a summand of the final term in the resolution of $J_\lambda + \langle x_1\rangle$ which is $R(-n+2)^b$. Thus every summand of G_{n-2} has twist $n-2$ which means that J_λ is level. QED

Remark 4.3.7. Notice that in the proof of Theorem 4.3.6 we do not actually need to know which h-vectors can occur as the h-vectors of matroids. We only need to find a class of ideals indexed by matroids that in some sense respects links and deletions. Since the class of matroids is closed under both operations, induction and liberal use of the sequence (4.4) gets us both the h-vector and levelness.

We can extend part of the proof of Theorem 4.3.6 into its own lemma.

Lemma 4.3.8. *Let $I, J, K \subseteq S$ be homogeneous ideals, all with projective dimension p so that there is a short exact sequence*

$$0 \longrightarrow S/I(-d) \longrightarrow S/J \longrightarrow S/K \longrightarrow 0.$$

Assume that K is level with socle degree D and I is level with socle degree $D-d$. Then J is level with socle degree D.

Proof. As in the proof of Theorem 4.3.6, we use the mapping cone.

$$\begin{tikzcd}
0 \arrow[r] & S/I(-d) \arrow[r] & S/J \arrow[r] & S/K \arrow[r] & 0 \\
 & S(-d) \arrow[r] \arrow[u] & R \arrow[u] & R \arrow[u] & \\
 & F_1(-d) \arrow[r] \arrow[u] & G_1 \arrow[u] & S(-d) \oplus G_1 \arrow[u] & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
 & F_{p-1}(-d) \arrow[r] \arrow[u] & G_{p-1} \arrow[u] & F_{p-2}(-d) \oplus G_{p-1} \arrow[u] & \\
 & S(-D)^a \arrow[r] \arrow[u] & G_p \arrow[u] & F_{p-1}(-d) \oplus G_p \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & S(-D)^a \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

By assumption the minimal free resolution of S/K ends with $S(-D)^b$ for some b and has length p. So every $S(-D)^a$ summand in term $p+1$ of the right-hand column must split. Moreover it must split with a $S(-D)$ summand of G_p with twist $-D$. Any remaining summands of G_p will be in the minimal free resolution of S/K and so must have twist $-D$. So every summand in G_p, the final term of the minimal free resolution of S/J, has twist $-D$ meaning the J is level. QED

Remark 4.3.9. For our purposes the most useful case of Lemma 4.3.8 is when $I = I_\Delta$ is squarefree and we use the sequence

$$0 \longrightarrow S/(I\colon x_i)(-1) \longrightarrow S/I \longrightarrow S/(I + \langle x_i \rangle) \longrightarrow 0.$$

Since $I+\langle x_i\rangle$ is the Stanley-Reisner ideal of Δ_{-i} and we may choose i so that this is not a cone we may, by induction on whatever properties we are assuming our complexes have, assume that Δ_{-i} is matroid and thus level with socle degree n. Likewise, $I\colon x_i$ is the Stanley-Reisner ideal of $C\operatorname{link}_\Delta(i)$ it can also be assumed to be level. We only need to ensure that it has socle degree $n-1$. To see this, we need to ensure that $\operatorname{link}_\Delta(i)$ is not a cone. Provided that this is true we can apply Lemma 4.3.8 to see that I is level.

4.4 Higher dimensions

The following observation is elementary, but essential in what follows: there are very few pure complexes with large dimension and a small number of vertices.

Lemma 4.4.1. *Let $\Delta \subseteq 2^{[n]}$ be a pure complex with $\dim\Delta = n-2$ and $\operatorname{init}(I_\Delta) \geq n-1$. Then Δ is either the boundary of the n-simplex of the cone over the boundary of the $(n-1)$-simplex. In either case I_Δ is principle.*

Proof. We need to show that Δ has at most one non-$(n-2)$-face. Suppose $F, G \notin \Delta$, $F \neq G$ and $|F| = |G| = n-1$. Then $F\cup G = [n]$ and $F\cap G = n-2$. Since Δ contains no non-faces with $n-2$ vertices (since the initial degree of I_Δ at least $n-1$), $F\cap G \in \Delta$. Since Δ is pure, $F\cap G$ is not a facet. So it must be contained in a facet, which must be either F or G. So at least one must be in Δ. QED

Lemma 4.4.2. *Let $\Delta \subseteq 2^{[n]}$ be a matroid with $\dim\Delta = d$ and $\operatorname{init} I_\Delta = d+1$. If $F, G \notin \Delta$ and $|F| = |G| = d+1$ then either $|F\cap G| < d$ or every maximal, proper subset of $F\cup G$ is also a non-face.*

Proof. If $|F\cap G| = d$ then $|F\cup G| = d+2$. So $\Delta_{F\cup G}$ and $\dim\Delta_{F\cup G} \geq d-1$ since $\operatorname{init}(I_\Delta) = d+1$. satisfies the hypotheses of Lemma 4.4.1 if it has dimension d. If $\dim\Delta_{F\cup G} = d$ then, by Lemma 4.4.1 it can have at most on minimal non-face. But F and G are both minimal non-faces, a contradiction. So $\dim\Delta_{F\cup G} = d-1$, which means that no maximal, proper subset of $F\cup G$ (which all have $d+1$ vertices) can be a face of Δ. QED

Lemma 4.4.3. *Let Δ be a simplicial complex with initial degree d and $E = K < e_1, \ldots, e_n >$. Then, if Δ is matroid,*

$$[J_\Delta]_d = \sum_{\sigma_i} \hat{\mathfrak{m}}^d_{\sigma_i}$$

for some collection of subsets $\{\sigma_i\}$ of $[n]$ such that, if $i \neq j$, $|\sigma_i \cap \sigma_j| \leq d-2$.

Proof. Assume Δ is matroid. First, consider the case when $\operatorname{init}(J_\Delta) = d+1$. Let u be a monomial in $[J_\Delta]_d$. Let $F_1 = \operatorname{supp}(u)$ and let $F_2, \ldots, F_s$ be the supports of the other monomials in $[J_\Delta]_d$ such that $|F_i \cap F_1| = d-1$. Let $\sigma = \bigcup_i F_i$. Now, we can apply Lemma 4.4.2 to see that, if $v \in \hat{\mathfrak{m}}^d_\sigma$ is a squarefree monomial then $\operatorname{supp}(v)$ must intersect at least 2 of the F_i and so $v \in [J_\Delta]_d$. If the dimension, say k, is larger than $d+1$ then we may consider the $k-1$ skeleton of Δ and apply the above argument since $[J_{\Delta_{d-1}}]_d = [J_\Delta]_d$. QED

Remark 4.4.4. Lemma 4.4.3 is particularly useful if the initial degree is very large, by which we mean $\operatorname{init}(I_\Delta) = \dim I_\Delta = \dim \Delta + 1$ (the largest possible given the dimension). In this case, we have just completely described the structure of the Stanley-Reisner ideal in a way analogous to our 1-dimensional result, Theorem 4.2.9. In that result, we obtain disjoint subsets of vertices, σ_i so that $\dim \Delta|_{\sigma_i} = 0$. In larger dimensional cases, we obtain a collection of non-disjoint subsets W_i with the property that $\dim \Delta|_{W_i} = \dim \Delta - 1$. Since $|W_i \cap W_j| \leq \dim \Delta - 2$, $\Delta|_{W_i}$ and $\Delta|_{W_j}$ can have no facets in common. By purity, a facet of $\Delta|_{W_i}$ is a sub-facet of Δ. So, what we do is not to partition the vertices of Δ, but to partition the sub-facets of Δ. This is the idea behind Definition 4.4.11.

Theorem 4.4.5. *Let Δ be a matroid complex on $[n]$ with $f_0(\Delta) = n$ and $I = I_\Delta$ its Stanley-Reisner ideal. If $u \in I$ has degree $\operatorname{init} I$ and $i \in \operatorname{supp}(u)$ then $I' = I + \frac{u}{x_i}$ is the Stanley-Reisner ideal of a matroid complex.*

Proof. Let $F = \operatorname{supp} u - \{i\}$. Since u is a minimal generator of I, $\operatorname{supp}(u)$ is a minimal non-face of Δ and $I' = I + \frac{u}{x_i}$ is the ideal of the complex $\Delta' = \Delta_{-F}$. Thus, $F \in \Delta$. We must show that, if $W \subseteq [n]$, Δ_W is pure. If $F \cap W = \emptyset$ then $\Delta'_W = \Delta_W$, which is pure. Otherwise, if $\tau \in \Delta'_W$ then $\tau \in (\Delta_W)_{-(F\cap W)}$ and conversely. So $(\Delta_W)_{-(F\cap W)} = \Delta'_W$. Assume that $|W| \neq n$.

If $F \not\subseteq W$ then $(\Delta_{-F})_W = \Delta_W$, which is pure, so we may assume that $F \subseteq W$, in which case $(\Delta_{-F})_W = (\Delta_W)_{-F}$. If $\operatorname{supp}(u) \subseteq W$ then $(\Delta_{-F})_W = (\Delta_W)_{-F}$. So the ideal of Δ_W (which we consider in a polynomial ring in fewer variables) has u as a minimal generator and we get that $(\Delta_W)_{-F}$ is pure, since Δ_W is matroid. So, we reduce to the case when $F \subseteq W$, $i \notin W$. In this case $(\Delta_{-F})_W = ((\Delta_{W\cup\{i\}})_{-F})_{-i}$. The complex $((\Delta_{W\cup\{i\}})_{-F})$ is matroid since $\operatorname{supp}(u) \subseteq W \cup \{i\}$. Deleting a vertex from a matroid results in another matroid, so we are done provided that W is a proper subset of $[n]$.

So, we only need to show that Δ' is pure. Suppose Δ' is not pure. Let E be a facet of Δ' with non-maximal dimension. We claim that $E \subseteq F$. If there is any facet of Δ, G, such that $F \not\subseteq G$ then G is also a facet of Δ_{-F}. If $E \not\subseteq F$ then $(\Delta_{-F})_{G\cup E} = \Delta_{G\cup E}$, which is pure. But $(\Delta_{-F})_{G\cup E}$ has G and E as facets, a contradiction. So, if there is any facet of Δ that does not contain F, $E \subseteq F$. Now, if F is contained in every facet of Δ then, in particular, each vertex of F is contained in every facet of Δ. Thus Δ is a cone with vertex any element of F. But F divides a minimal generator of I_Δ, another contradiction. Thus any facet of Δ' with non-maximal dimension must be contained in F. Clearly it must then have $E = F - \{w\}$ for some $w \in F$. Then $E \cup \{i\} \subset \operatorname{supp}(u)$ and is thus in Δ. So, E must not be a facet. It follows that Δ' is pure, concluding the proof. QED

Let F be as in the above and $D = |F|$. We now have a short exact sequence of matroids

$$0 \longrightarrow {}^{S}\!/_{I_\Delta : x_F}(-D) \xrightarrow{\cdot x_F} {}^{S}\!/_{I_\Delta} \longrightarrow {}^{S}\!/_{I_\Delta + x_F} \longrightarrow 0. \tag{4.5}$$

Lemma 4.4.6. *Using the above notation,* $\operatorname{codim}(I_\Delta + x_F) = \operatorname{codim} I_\Delta$

Proof. If $|F| = 1$ then we simply need to note that no vertex of F can be the vertex of a cone. Using the sequence (4.5), see that, if $\operatorname{codim}(I_\Delta \colon x_F) > \operatorname{codim} I_\Delta$, we are done. So assume $\operatorname{codim}(I_\Delta \colon x_F) = \operatorname{codim} I_\Delta$. We must show that $\deg I_\Delta \neq \deg(I_\Delta \colon x_F)$. Let $\Gamma = \{\sigma \in \Delta \mid \sigma \cup F \in \Delta\}$ be the complex whose Stanley-Reisner ideal is $I_\Delta : x_F$. We have assumed that $\dim \Gamma = \dim \Delta$ and want to show that Γ has fewer faces with maximal dimension. This could only fail of $\Gamma = \Delta$. But, that would mean that F is contained in every facet of Δ, which means that Δ is a cone with vertex in F. We have already noted that this is impossible. QED

Remark 4.4.7. By construction the initial degree of $I_\Delta + x_F$ is $\deg x_F$. We can then apply Theorem 4.4.5 again to reduce the initial degree farther. Eventually, one will recover the fact that deleting a vertex from a matroid gives another matroid. We have just proven the following corollary.

Corollary 4.4.8. *Let $I = I_\Delta$ be the Stanley-Reisner ideal of a matroid complex and x_F a proper divisor of a minimal generator of I with degree* $\operatorname{init} I$*. Then $I + x_F$ is the Stanley-Reisner ideal of a matroid complex with* $\operatorname{codim} I = \operatorname{codim}(I + x_F)$.

4.4.1 Gorenstein Matroid Complexes

All matroid complexes are Cohen-Macaulay. A natural question is to wonder when they are something better than Cohen-Macaulay. For example, which matroid complexes are Cohen-Macaulay with linear resolutions or Gorenstein? We answer the second question. A S-module, M, is said to be *Gorenstein* if it is Cohen-Macaulay and the last module is its minimal free resolution has rank 1. Since matroids are level, the rank of the last module in its minimal free resolution is the same as the last non-zero entry in its h-vector. So, we are asking, for which matroid complexes does the h-vector end with 1 (those that end with 0 and cones and may be dealt with by passing to a lower dimensional complex).

The answer will turn out to be that the only Gorenstein matroid complexes are the complete intersections (all complete intersections are Gorenstein since they have Koszul resolutions). Identifying a squarefree monomial ideal as a complete intersection is easy. We simply need to check that the supports of all the minimal generators are pairwise disjoint. Moreover, all squarefree complete intersections are matroid. To see this, note that each complete intersection whose Stanley-Reisner ideal is not principle is the join of two smaller squarefree complete intersections. By induction both are matroid. Since the join of two matroids is matroid, we get the result after checking that all simplicial complexes whose Stanley-Reisner ideals are principle are matroid. We will use the following, rather technical, result to check that matroid complexes are complete intersections.

Lemma 4.4.9. *Let Δ be a matroid complex. Then Δ is a complete intersection if and only if Δ has a vertex w such that Δ_{-w} is a cone with vertex v over a complete intersection and x_w divides a unique minimal generator of I_Δ with degree* $\operatorname{init}(I_\Delta)$.

Proof. Assume I_Δ is a complete intersection. We induct on $\operatorname{init}(I_\Delta)$. If $\operatorname{init}(I_\Delta) = 1$ then we may pass to a smaller polynomial ring and conclude by induction on n. So, assume that $\operatorname{init}(I_\Delta) > 1$. Choose a face $F \in \Delta$ as dictated by Theorem 4.4.5 and set $\Delta' = \Delta_{-F}$. Since x_F properly divides a minimal generator, $\mu(I_{\Delta'}) \leq \mu(I_\Delta) = \operatorname{codim} I_\Delta = \operatorname{codim} I_{\Delta'}$. So $I_{\Delta'}$ is a complete intersection with the same codimension as I_Δ and a smaller initial degree. So by induction on the initial degree, either Δ' is a cone there is some vertex w such that Δ'_{-w} is a cone.

Suppose Δ' is a cone with vertex v. Since $F \notin \Delta'$, $v \notin F$. Since $I_{\Delta'}$ has no minimal generator divided by x_v and Δ is not a cone we must have $x_F x_v$ a minimal generator of I_Δ. Let $w \in F$. We claim that Δ_{-w} is a cone with vertex v. Let $G \in \Delta_{-w}$ with $v \notin G$. So $F \not\subset G$ and thus $G \in \Delta'$. Since Δ' is a cone, $G \cup \{v\} \in \Delta' \subseteq \Delta$. It follows that $G \cup \{v\} \in \Delta_{-w}$, which is then a cone.

Now, suppose that there is some vertex $w \in \Delta'$ such that Δ'_{-w} is a cone with vertex v. We know $\operatorname{init}(I_{\Delta'}) < \operatorname{init}(I_\Delta)$. By induction, we may take x_w to divide a minimal generator of $I_{\Delta'}$ with degree 2. But, by construction, there is only one such minimal generator, x_F. So $w \in F$ and $\Delta'_{-w} = \Delta_{-w}$ is then a cone. Note that x_w divides the minimal generator divided by x_F and in both cases the vertex w is contained in F. By repeatedly applying Theorem 4.4.5, we see that $I_\Delta + \langle x_w \rangle$ has the same codimension and number of minimal generators as I_Δ. This couldn't happen if x_w divided more than 1 minimal generator.

Conversely, assume that Δ has a vertex, w, dividing a unique degree $\operatorname{init}(I_\Delta)$ minimal degree generator of I_Δ such that Δ_{-w} is a cone over a complete intersection with vertex v. Choose a face F so that $w \in F$ and x_F properly divides the minimal generator divided by x_w. We will induct on the initial degree of I_Δ, starting with 2. If $\operatorname{init}(I_\Delta) = 2$ then $\Delta_{-F} = \Delta_{-w}$. By our assumptions, $\mu(I_{\Delta_{-w}}) = \mu(I_\Delta)$ and since Δ is not a cone, $\operatorname{codim} I_{\Delta_{-w}} = \operatorname{codim} I_\Delta$. Since Δ_{-w} is a cone over a complete intersection (and is thus itself a complete intersection), Δ must be a complete intersection.

So, assume $\operatorname{init} I_\Delta > 2$ so that $|F| \geq 2$. Then $(\Delta_{-F})_{-w} = \Delta_{-w}$ is a cone over a complete intersection and $I_\Delta + \langle x_F \rangle$ has a smaller initial degree than I_Δ. Note that, by the choice of w, I_Δ and $I_{\Delta_{-F}}$ have the same number of minimal generators and have the same codimension by Lemma 4.4.6. There are two cases. If Δ_{-F} is not a cone then, by induction, Δ_{-F} is a complete intersection with the same codimension as I_Δ. Thus, $\mu(I_\Delta) = \mu(I_\Delta + \langle x_F \rangle) = \operatorname{codim} I_\Delta$.

So, assume that Δ_{-F} is a cone with a vertex v. This means that x_v can divide no minimal generator of I_Δ except the one divided by x_w, $x_F x_v$. We can then pass to $I' = (I_{\Delta_{-F}} + \langle x_v \rangle)/\langle x_v \rangle$ contained in a polynomial ring with 1 fewer variables. This is then not the ideal of a cone and has the same number of minimal generators and the same codimension as I_Δ. Since $|F| \geq 2$ and our hypotheses are still satisfied, I' is a complete intersection by induction on the number of variables. As above, this then implies that I_Δ is also a complete intersection. QED

Note that a list of complete intersections will includes both the simplex (a cone) and the boundary of the simplex (deleting any vertex leaves behind a cone). In dimension 0 these are the only complete intersection, which is already clear. In dimension 1 we have the cones along with the 3 and 4 cycles as the only complete intersections for a total of 4 complete intersections. Note that, by our results on 1 dimensional matroids, these are also the only Gorenstein matroids since, if the complex is not a cone, it can be formed by adjoining vertices to a complete graph on k vertices. If $k \geq 3$ then $k = 3$ since all other complexes will have too much homology (the top homology group must have dimension 1 by Hochster's formula). So, we assume that $k = 2$ and we adjoin first $m_1 > 0$ and then $m_2 > 0$ vertices. Then

Theorem 4.4.10. *Let Δ be a matroid complex. Then Δ is Gorenstein if and only if Δ is a complete intersection.*

Proof. Every complete intersection is Gorenstein, so assume that Δ is Gorenstein. If Δ is a cone then we may delete a vertex and conclude by induction on the number of vertices. Let $d = \dim S/I_\Delta$. Choose a face F as required by Theorem 4.4.5 and set $\Delta' = \Delta_{-F}$. Let $I = I_\Delta$ and $I' = I_{\Delta'} = I + \langle x_F \rangle$. Consider the short exact sequence (4.5), where $D = |F|$

$$0 \longrightarrow {S}/{(I\colon x_F)}(-D) \xrightarrow{\cdot x_F} {S}/{I} \longrightarrow {S}/{I'} \longrightarrow 0. \tag{4.6}$$

Lemma 4.4.6 assures us that I and I' have the same codimension, c and since Δ' is matroid, I' is Cohen-Macaulay. We will induct on the initial degree (if $\operatorname{init}(I) = 1$ then we may induct on the number of variables). Since matroids are level, Δ' is Gorenstein if its Cohen-Macaulay type, $h_d(I')$, is 1. Note that I and I' have the same codimension, but have different degrees (since Δ and Δ' have different numbers of facets). Then, sequence (4.6) shows that $\operatorname{codim}(I\colon x_F) = \operatorname{codim} I$. Since Δ is Gorenstein, all of its links are Euler , that is $\widetilde{\mathcal{X}}(\operatorname{link}_\Delta(G)) = (-1)^{\dim \operatorname{link}_\Delta(G)}$. In particular, this means that $\operatorname{link}_\Delta(F)$ is Gorenstein since, being a matroid, it is already Cohen-Macaulay and level so that its Cohen-Macaulay type is its Euler characteristic. We want to show that I' is also Gorenstein. We do this by showing that, if $G \in \Delta'$, then $\operatorname{link}_{\Delta'}(G)$ is Gorenstein and in particular, Euler.

Let $G \in \Delta'$ and $\Gamma = \operatorname{link}_\Delta(G)$. Then it is easily seen that $\Gamma_{-F} = \operatorname{link}_{\Delta'}(G)$. We have already seen that Γ is Gorenstein and must have a smaller dimension than Δ. So, by induction on the dimension (the claim is trivial for dimension 0) Γ_{-F} is Gorenstein.

Now that we know Δ' is Gorenstein and $I_{\Delta'}$ has a smaller initial degree than Δ, induction tells us that $I_{\Delta'}$ is a complete intersection with codimension c. By Theorem 4.4.9 there is some vertex $w \in \Delta'$ such that Δ'_{-w} is a cone over a complete intersection and x_w divides a unique minimal generator of $I_{\Delta'}$ with degree $\operatorname{init} I_{\Delta'} = |F|$. There is only one such generator, F; so $w \in F$ and thus $\Delta_{-w} = \Delta'_{-w}$. Then, the only remaining thing to check is the uniqueness of the generator $x_F x_v$, where v is the vertex of Δ'_{-w}.

We separate off the case when $|F| = 1$, or equivalently $\operatorname{init}(I_\Delta) = 2$, and $\Delta' = \Delta_{-w}$. In this case we have already described that way that w (and by symmetry v) may be attached in Theorem 4.1.9. As in the proof of that result, if $G \in \Delta_{-w}$ and $v \in G$ then $(G - \{v\}) \cup \{w\} \in \Delta$. Let Ω be Δ with w and all of the vertices not in its link deleted. Since Δ_{-w} is a cone there is only one such vertex, v and $\dim \Omega < \dim \Delta$. So Δ must be a "double cone" with vertices v and w, which gives the needed uniqueness.

Now assume that $|F| > 1$. Clearly, we must have that if $u \neq x_F x_v$ is a minimal generator of I_Δ divided by x_w then $F \subseteq \mathrm{supp}(u)$. If $|F| \geq 2$ then we may repeat the above argument with $(F - \{x\}) \cup \{v\}$ in place of F (where $w \neq x \in F$) and get that $v \in \mathrm{supp}(u)$ as well. But then $x_F x_v$ divides u, a contradiction.

QED

As discussed just before Lemma 4.4.9, every squarefree complete intersection is both Cohen-Macaulay and Gorenstein. So, Theorem 4.4.10 says the set of matroid complexes and the set of Gorenstein complexes meet minimally.

4.4.2 The Sub-facet complex

In Section 4.2 we acquired a great deal of information about matroid complexes with dimension 1 by studying their associated partitions. Unfortunately, the author knows of no equivalent construction for larger dimensions. So, we compensate by replacing a large dimensional complexes with 1-dimensional complex any study its associated partition. We say that $G \in \Delta$ is a *sub-facet* of Δ is $G = F - \{v\}$ for some facet F and vertex v

Definition 4.4.11. Let Δ be any pure simplicial complex and $\mathscr{F}$ its set of sub-facets. We construct a 1-dimensional simplicial complex (a graph), $\mathscr{E}(\Delta)$, with vertex set $\mathscr{F}$. An edge $\{E, F\} \in \mathscr{E}(\Delta)$ if and only if $\dim \Delta|_{E \cup F} = \dim \Delta$. We call $\mathscr{E}(\Delta)$ the *sub-facet complex* of Δ. If $\dim \Delta = 2$ then we call $\mathscr{E}(\Delta)$ the *edge complex* of Δ.

Example 4.4.12. Let Δ be the octahedron. It has dimension 2 and f-vector $(1, 6, 12, 8)$. We will call the vertices $a, b \ldots, f$. Then, the Stanley-Reisner ideal of Δ is $I_\Delta = \langle ac, bc, ef \rangle$. The pair of edges $\{ab, ae\}$ is in $\mathscr{E}(\Delta)$ since their union contains the 2-face abe. On the other hand, $\{ae, af\} \notin \mathscr{E}(\Delta)$ since $aef \in I_\Delta$ and so the union of the edges does not contain a 2-face. Similarly, the restriction of Δ to $\{a, b, c, d\}$ has dimension 1 and so $\{ab, dc\} \in \mathscr{E}(\Delta)$. Repeating this for all $\binom{12}{2} = 66$ pairs of edges, we get the simplicial complex depicted in Figure 4.4.12. It has f-vector $(1, 12, 48)$. The dedicated reader may use Lemma 4.1.10 to show that $\mathscr{E}(\Delta)$ is in fact the matroid complex with associated partition $4 + 4 + 4$.

Remark 4.4.13. If $\dim \Delta = 1$ and Δ is pure then a sub-facet is simply a vertex and $\dim \Delta|_{\{v\} \cup \{w\}} = 1$ if and only if $\{v, w\} \in \Delta$. So $\mathscr{E}(\Delta) \cong \Delta$ (isomorphism, not equality because the vertices of $\mathscr{E}(\Delta)$ are of the form $\{v\}$ for some vertex v of Δ, a distinction of little importance).

The situation in Example 4.4.12 is the general one, that is, the sub-facet complex of any matroid is again a matroid.

Lemma 4.4.14. *If Δ is a matroid complex then $\mathscr{E}(\Delta)$ is also matroid.*

Proof. Let $d = \dim \Delta$. By definition, $\mathscr{E}(\Delta)$ has dimension 1, so we can check that it is matroid by applying Lemma 4.1.10. Suppose $\{E, F\} \in \mathscr{E}(\Delta)$ and G is a vertex of $\mathscr{E}(\Delta)$. We must show that $\mathrm{link}_{\mathscr{E}(\Delta)}(G) \cap \{E, F\} \neq \emptyset$. Assume that $\{F, G\} \notin \mathscr{E}(\Delta)$. Then we only need to show that $\dim \Delta|_{E \cup G} = d$, which implies that $\{E, G\} \in \mathscr{E}(\Delta)$.

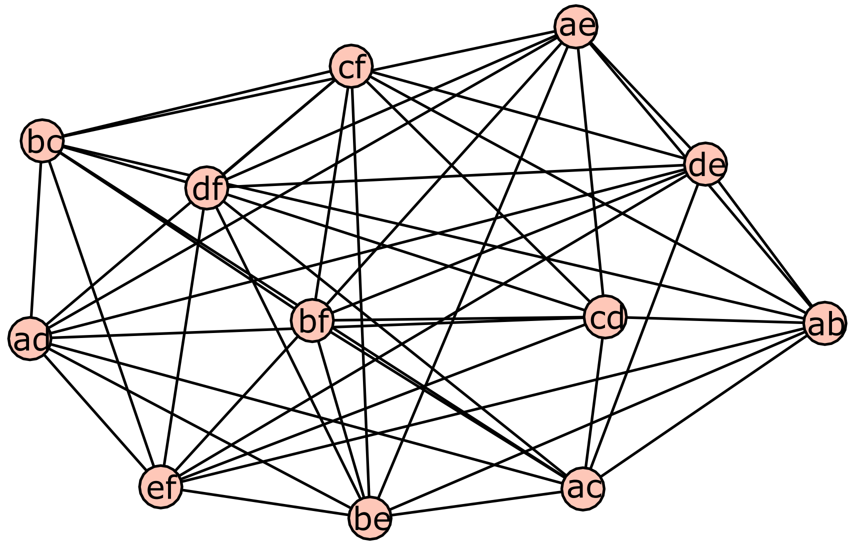


Figure 4.7: The edge complex of an octahedron

Let $W_1, \dots, W_k \subseteq [n]$ be the maximal subset such that $\dim \Delta|_{W_i} = d - 1$. Since $\dim \Delta|_E = \dim E = d - 1$, $E \subseteq W_i$ for some i.

We claim that, for any sub-facets, $A, B \in \Delta$, $\{A, B\} \notin \mathscr{E}(\Delta)$ if and only if $A, B \subseteq W_i$ for some i. On direction is clear since $\Delta|_{A\cup B} \subseteq \Delta|_{W_i}$ and $\dim \Delta|_{W_i} = d-1$. Since $A \in \Delta|_{A\cup B}$ we then get $d - 1 \leq \dim \Delta|_{A\cup B} \leq \dim \Delta|_{W_i} = d - 1$. The other direction is also easy: suppose that $\dim \Delta|_{A\cup B} = d-1$. Then $A \cup B$ (and thus A and B individually) must be contained in one of the maximal sets W_i.

Next, we claim that, if $\overline{W_i}$ is the set of sub-facets of Δ contained in W_i and $i \neq j$, $\overline{W_i} \cap \overline{W_j} = \emptyset$. Suppose to the contrary that E is a sub-facet contained in both W_i and W_j. If $W_i = E$ then $W_i \subseteq W_j$ and maximality implies that $W_i = W_j$. So, let $x \in W_i - E$. By the maximality of W_j, $\dim \Delta|_{W_j \cup \{x\}} = d$. Since Δ is matroid, the restriction is pure and so E must be contained in a d-face. But, $\Delta|_{W_j}$ has dimension $d - 1$ so the only d-face that could contain E is $E \cup \{x\}$. So $E \cup \{x\} \in \Delta$. But $E \cup \{x\} \subseteq W_i$ and $\dim \Delta|_{W_i} = d - 1$ so $E \cup \{x\} \notin \Delta$, a contradiction.

Returning to the main argument, since $\{F, G\} \notin \mathscr{E}(\Delta)$, $F, G \subseteq W_j$ for some j and since $\{E, F\} \in \mathscr{E}(\Delta)$, $E \not\subseteq W_j$. Suppose $E \subseteq W_i$ where $i \neq j$ (every sub-facet must be contained in some W_i since the sub-facet itself has dimension $d-1$ by the purity of Δ). Since G can not be contained in both W_i and W_j, we must have $G \not\subseteq W_i$, which means that $\{E, G\} \in \mathscr{E}(\Delta)$, which, by Lemma 4.1.10 is all we needed to show. QED

Remark 4.4.15. The proof of Lemma 4.4.14 tells us not only that $\mathscr{E}(\Delta)$ is a 1-dimensional matroid, it also gives us enough data to compute its associated partition.

Consider the sets of sub-facets $\overline{W_i}$, from the proof of Theorem 4.4.10. These are the maximal subsets of vertices of $\mathscr{E}(\Delta)$ such that the restriction has dimension 0. Moreover, they are all pairwise disjoint. Referring to Remark 4.4.4, we see that the associated partition of $\mathscr{E}(\Delta)$ is given by $|\overline{W_i}| = f_{d-1}(\Delta|_{W_i})$. We state this result as a corollary.

Corollary 4.4.16. *Let Δ be a matroid complex with $\dim \Delta = d$ and $W_1, \ldots, W_k$ be the maximal subsets of vertices such that $\dim \Delta|_{W_i} = d-1$. Then $W_i \cap W_j$ for $i \neq j$ does not contain any $(d-1)$-face of Δ and the associated partition of $\mathscr{E}(\Delta)$, $\lambda = \lambda_{\mathscr{E}(\Delta)}$, is given by $\lambda_i = f_{d-1}(\Delta|_{W_i})$. In particular, the length of $\lambda_{\mathscr{E}(\Delta)}$ is, k, the number of such subsets, and $|\lambda_{\mathscr{E}(\Delta)}| = f_{d-1}(\Delta)$.*

Proof. This is implicit in the proof of Lemma 4.4.14, as indicated in Remark 4.4.15. QED

Example 4.4.17. Consider the octahedron in Example 4.4.12. The maximal subsets so that the restriction is 1-dimensional are the 4 4-cyles. Since there are 4 edges in a 4-cycle, the partition associated to the edge complex of the octahedron is 4+4+4. The associated partition of the 1-skeleton of the octahedron is $2+2+2$ and from this we know that the first 3 entries in the f-vector are $(1, 6, 12)$. The partition associated to the edge complex tells us that we can divide our complex into 3 complexes each with 4 edges. There is only 1 matroid with 4 edges, the 4-cycle, which has 4 vertices. Each of these 4-cycles gives $\binom{4}{3} = 4$ non-2-faces, since they are 1-dimensional. Because the edge complex is associated to the partition $4+4+4$, there are no other missing non-2-faces. So $f_3 = \binom{6}{3} - 3\binom{4}{3} = 8$.

Remark 4.4.18. As we see in Example 4.4.12 (and will see more formally later in [ref]), for a 2-dimensional matroid complex, Δ, the partitions associated to $[\Delta]_1$ and $\mathscr{E}(\Delta)$ are enough information to determine the f-vector (and thus the h-vector). However, unlike the dimension 1 case, they do not determine the complex up to isomorphism. See Example 4.4.22.

We now consider again the special case of matroid complexes with large initial degree.

Theorem 4.4.19. *Let Δ be a simplicial complex on $[n]$ with $\dim \Delta = d$ and $\operatorname{init}(I_\Delta) = d+1$. Then $f_i(\Delta) = \binom{n}{i+1}$ for $-1 \leq i \leq d-1$ and*

$$f_d(\Delta) = \binom{n}{d+1} - \sum \binom{w_i}{d+1}$$

where $\lambda = \lambda_{\mathscr{E}(\Delta)}$ has the form $\lambda_i = \binom{w_i}{d}$.

Proof. Since $\operatorname{init}(I_\Delta) = d+1$, Δ must contain every subset of $[n]$ with at most d elements, which gives the claim about the small dimensional components of the f-vector. To calculate the final entry in $f(\Delta)$, let $W_1, \ldots, W_k$ be the maximal subsets $[n]$ such that $\dim \Delta|_{W_i} = d-1$. By Corollary 4.4.16, $\lambda_i = f_{d-1}(\Delta|_{W_i})$. Since Δ contains every $(d-1)$-face, we get that $\lambda_i = \binom{w_i}{d}$, where $w_i = |W_i|$. If F is any subset with $|F| = d+1$ and $F \not\subseteq W_i$ for any i then $F \in \Delta$ by the maximality of the W_i. By Corollary 4.4.16, there can be no such F in two distinct W_i. So, the $\sum \binom{w_i}{d+1}$ d-faces removed from the complex by the W_i are the only non-d-faces. The claim on the f-vector follow immediately because there are $\binom{n}{d+1}$ possible d-faces on $[n]$. QED

Remark 4.4.20. If $\dim \Delta = 1$ then $\mathscr{E}(\Delta) \cong \Delta$ and $\lambda_i = w_i$. Thus, Theorem 4.4.19 generalizes our results on the h-vectors of 1-dimensional complexes.

Definition 4.4.21. Let Δ be a matroid complex. Then we define $\lambda_\Delta^{(k)}$ to be the associated partition of $\mathscr{E}([\Delta]_k)$ for $1 \leq k \leq \dim \Delta$.

Example 4.4.22. Consider, once again, the octahedron. Its 1-skeleton has partition $2+2+2$. In Example 4.4.12 we computed its edge complex and found it to have partition $4+4+4$. So $\lambda^{(1)} = 2+2+2$ and $\lambda^{(2)} = 4+4+4$. Note that $|\lambda^{(1)}| = f_0 = 6$ and $|\lambda^{(2)}| = 12 = f_2$.

If $\dim \Delta = 1$ then $\lambda_\Delta^{(1)} = \lambda_\Delta$. From the results in Section 4.2 we know that this partition tells us both the number of vertices and the number of edges of Δ. In fact, it determines the complex Δ up to isomorphism. If $\dim \Delta > 1$ this is no longer true.

Example 4.4.23. Consider the ideals $I_1 = \langle x_1x_2x_3, x_4x_5x_6\rangle + \hat{\mathfrak{m}}^4$ and $I_2 = \langle x_1x_2x_3, x_3x_4x_5\rangle + \hat{\mathfrak{m}}^4$ in $K[x_1, \ldots, x_6]$. Let Δ_1 and Δ_2 be the corresponding simplicial complexes. These are, in fact, matroid complexes with dimension 2. Since the initial degree of each is 3, their 1-skeletons must be complete graphs and so $\lambda^{(1)} = 1+1+1+1+1+1$ and $f_1 = \binom{6}{2} = 15$. Computing the edge complexes (using the SAGE computer system [22] and the functions `edge_complex` and `find_partition` from Appendix 4.4.3) we see that $\lambda^{(2)} = 3+3+1+\cdots+1$. However, they are not isomorphic.

Some basic information about Δ can be easily read from the set of $\lambda_\Delta^{(k)}$.

Lemma 4.4.24. *Let Δ be a matroid complex.*

(a) $f_{k-1}(\Delta) = |\lambda_\Delta^{(k)}|$ *for* $1 \leq k < \dim \Delta$

(b) If $\operatorname{init} I_\Delta \neq 1$ *then* $\operatorname{init} I_\Delta - 1 = \min\{k \mid \lambda_\Delta^{(k)} \neq 1+\cdots+1\}$.

(c) If $\Delta = [\Gamma]_k$ *for some complex* Γ *with* $d = \dim \Gamma > k$ *then* $\ell(\lambda_\Delta^{(i)}) \geq \binom{d+1}{i}$ *for all* $1 \leq i \leq \dim k$.

Proof.

(a) By definition, the vertex set of $\mathscr{E}([\Delta]_k)$ is the number of sub-facets of $[\Delta]_k$. Since this is a pure complex, the sub-facets are simply the $(k-1)$-dimensional faces of Δ; there are $f_{k-1}(\Delta)$ of these. For any 1-dimensional matroid the sum of its associated partition is the size of its vertex set. The result immediately follows.

(b) The initial degree of I_Δ is the smallest degree of a minimal generator of I_Δ. Equivalently, it is the smallest size of a minimal non-face of Δ. So if $k-1 < \operatorname{init} I_\Delta$

then $[\Delta]_k$ will contain every k-dimensional face. Thus, if E, F are distinct sub-facets of $[\Delta]_k$ then $\Delta|_{E\cup F}$ must contain a k-face; by the definition of the sub-facet complex, $\{E, F\} \in \mathscr{E}([])_k \Delta$. The sub-facet complex is therefor a complete graph and thus has partition $1 + 1 + \cdots + 1$.

On the other hand, if $k = \operatorname{init} I_\Delta$ then there is some $(k-1)$ face that Δ does not contain; call is F. Let $E, G \subset F$ be proper subsets such that $E \cup G = F$ and $|E| = |G| = k-1$. Since F is a minimal non-face, $E, G \in \Delta$ and so $E, F \in \mathscr{E}([\Delta]_{k-1})$. However $[\Delta]_{k-1}|_{E\cup G} = \Delta|_F$ and $\dim \Delta|_F = \dim F - 1 = k-2$. So $\{E, F\} \notin \mathscr{E}([\Delta]_{k-1})$ and $\mathscr{E}([\Delta]_{k-1})$ is therefor not a complete graph. By the definition of the associated partition of a 1-dimensional complex, the only complex with partition $1 + 1 + \cdots + 1$ is the complete graph. Since $\mathscr{E}([\Delta]_{k-1})$ is not complete, it does not have partition $1 + 1 + \ldots + 1$.

(c) Suppose that Δ is the k-skeleton of some d-dimensional complex Γ. Since $\lambda_\Delta^{(i)}$ depends only on the i-skeleton of Δ and if $k = 1$ we have already seen this in Lemma 4.3.5 we induct on k. By Lemma 4.2.24 we only need to show that $\mathscr{E}(\Delta)$ contains a subset of $\binom{d+1}{k+1}$ vertices whose restriction is a complete graph. Let F be a dimension d face of Γ. Then all $\binom{d+1}{k}$ dimension $k-1$ subsets of F are in Δ. Let G_1 and G_2 be 2 distinct dimension $k-1$ subsets of F. Then, since $G_1 \cup G_2 \subseteq F$, $\Delta|_{G_1\cup G_2}$ must contain a k-face and therefor $\{G_1, G_2\} \in \mathscr{E}(\Delta)$. The restriction of $\mathscr{E}(\Delta)$ to these vertices is a complete graph and the claim about the length follows.

QED

4.4.3 The Stanley-Reisner Ideals of Matroid Complexes

Using the tools from the previous sections, we now discuss in full generality the Stanley-Reisner ideals of matroid complexes. The result we obtain states that in each degree, the Stanley-Reisner ideals of general matroids have a structure similar to that in the initial degree (Lemma 4.4.3).

What we will do is to simply ignore the small degree portions of the Stanley-Reisner ideal, forcing the initial degree up towards the case we have already studied. We will make use of the following notation.

Notation. Let $I \subseteq S$ be a monomial ideal. Then $G(I)$ is the unique set of monomial minimal generators of I. Then

$$G_d(I) = \{u \in G(I) \mid \deg u = d\}$$

$$G_{\geq d}(I) = \{u \in G(I) \mid \deg u \geq d\}.$$

Definition 4.4.25. Let Δ be a simplicial complex with dimension d and Stanley-Reisner ideal $I = I_\Delta$. Then we define $\overline{\Delta}$ to be the simplicial complex such that

$$I_{\overline{\Delta}} = \langle G_{d+1}(I) \rangle + \hat{\mathfrak{m}}^{d+2}.$$

We call $\overline{\Delta}$ the *completion* of Δ.

Example 4.4.26. Let Δ be, once again, the octahedron. It has Stanley-Reisner ideal $I_\Delta = \langle ac, bc, ef\rangle \subseteq K[a,b,c,d,e,f]$. Since I_Δ has no degree 3 or higher generators, $I_{\overline{\Delta}} = \hat{\mathfrak{m}}^4$. Thus the completion of the octahedron contains every 3-face.

Example 4.4.27. Let Δ be the double cone over a 3-cycle. One easily checks that Δ is matroid with $\dim \Delta = 2$. This has Stanley-Reisner ideal $I_\Delta = \langle abc, de\rangle \subseteq K[a,b,c,d,e]$. So $I_{\overline{\Delta}} = \langle abc\rangle + \hat{\mathfrak{m}}^4 = \langle abc, bcde, acde, abde\rangle$. This complex is again matroid and contains every 3-face except for $\{a,b,c\}$.

For any of this to make any use at all, it is necessary the $\overline{\Delta}$ be matroid whenever Δ is. Fortunately, this is true.

Lemma 4.4.28. *If Δ is a matroid complex then so is $\overline{\Delta}$.*

Proof. Let $d = \dim \Delta$ and $W \subset [n]$ be a proper subset. Then $\overline{\Delta}|_W = \overline{\Delta|_W}$ if $\dim \Delta|_W = \dim \Delta$. To see this, we examine the Stanley-Reisner ideals of each. To form $I_{\Delta|_W}$ we add the ideal $\langle x_i \mid i \notin W\rangle$. Thus,

$$I_{\Delta|_W} = \langle u \in G(I) \mid \operatorname{supp}(u) \subseteq W\rangle$$

and therefor, if $\dim \Delta|_W = \dim \Delta$,

$$\begin{aligned} I_{\overline{\Delta}|_W} &= \langle u \in G_{d+1}(I) \mid \operatorname{supp}(u) \subseteq W\rangle + \hat{m}_W^{d+2} \\ &= \big\langle u \in G_{d+1}(I_{\Delta|_W})\big\rangle + \hat{m}_W^{d+2} \\ &= I_{\overline{\Delta|_W}}. \end{aligned}$$

By induction on the number of vertices, $\overline{\Delta|_W}$ is matroid, and thus pure. Otherwise, if $\dim \Delta|_W < \dim \Delta$, $I_{\overline{\Delta}|_W} = I_{\overline{\Delta|_W}} + \hat{\mathfrak{m}}^{k+2}$ for some $k < d$. In this case $\overline{\Delta}|_W$ is the k-skeleton of $\overline{\Delta|_W}$, which is matroid. Skeletons of matroids are also matroid and so the restriction of $\overline{\Delta}$ is pure in this case as well.

So, it only remains to show that $\overline{\Delta}$ is itself pure. By definition $\overline{\Delta}$ contains every face with fewer than $d+1$ elements. So, we can concentrate on showing that, if $F \in \overline{\Delta}$ is a $(d-1)$-face it is contained in some d-face of $\overline{\Delta}$. If $F \in \Delta$ this is true because Δ is pure. Otherwise, $x_F \in I_\Delta$. If F is not contained in any d-face of $\overline{\Delta}$ then $x_F x_i \in I_{\overline{\Delta}}$ for every $i \notin F$. But $\deg(x_F x_i) = d+1$, and since $x_F \in I_\Delta$, none of these monomials can be minimal generators of I_Δ. Because $d+1$ is the initial degree of $I_{\overline{\Delta}}$ the only degree $d+1$ monomials in $I_{\overline{\Delta}}$ are its minimal generators, which are the degree $d+1$ minimal generators of I_Δ, which would contradict $x_F x_i \in I_{\overline{\Delta}}$ if we were to assume that. So $\overline{\Delta}$ is pure and is thus matroid. QED

Since the initial degree of $\overline{\Delta}$ is large, we can apply Theorem 4.4.19 to study the associated partition of $\mathscr{E}(\overline{\Delta})$ and use it to study $f_d(\overline{\Delta})$.

Corollary 4.4.29. *Let Δ be a matroid complex with $\dim \Delta = d$ then*

$$f(\overline{\Delta}) = \left(1, n, \binom{n}{2}, \ldots, \binom{n}{d}, \binom{n}{d+1} - \sum \binom{w_i}{d+1}\right),$$

where $\lambda_{\mathscr{E}(\overline{\Delta})} = \sum \binom{w_i}{d}$

Proof. Simply note that $\operatorname{init}(I_{\overline{\Delta}}) = d + 1$ and apply Theorem 4.4.19. QED

To study the structure of the Stanley-Reisner ideal of a matroid complex, we first show that we can remove rather large pieces of the ideal and still be left with the Stanley-Reisner of a matroid complex. In this way we will be able to study the minimal generating set of I_Δ degree by degree.

Lemma 4.4.30. *Let Δ be a matroid with $d = \dim \Delta$ and $I = I_\Delta$. If $I_k = \langle u \in G(I) \mid \deg u \neq k \rangle + \hat{\mathfrak{m}}^{d+2}$ is the Stanley-Reisner ideal of Δ_k then Δ_k is matroid for every $1 \leq k \leq d$*

Proof. As is our normal procedure, we first show that this commutes with restrictions and then show that Δ_k is pure for every k. First note since $\Delta \subseteq \Delta_k$ and I_k contains every squarefree degree $d+2$ monomial, $\dim \Delta_k = \dim \Delta$. To see that the restrictions behave properly, we examine the Stanley-Reisner ideal of the restriction to some $W \subseteq [n]$:

$$\begin{aligned} I_{\Delta_k|_W} &= I_k + \langle x_i \mid i \notin W \rangle \\ &= \langle u \in G(I) \mid \deg u \neq k, \operatorname{supp}(u) \not\subseteq W \rangle + \hat{m}_W^{d+2} \end{aligned}$$

If $\dim \Delta|_W = \dim \Delta$ then this ideal is equal to $I_{(\Delta|_W)_k}$. Otherwise, as in the proof of Lemma 4.4.28, $\Delta_k|_W$ is a skeleton of $(\Delta|_W)_k$. So, as long as W is a proper subset of $[n]$ we may thus conclude that the restriction of Δ_k is pure by induction on the number of vertices.

We now only need to prove that Δ_k is pure. Suppose to the contrary that $F \in \Delta_k$ is a facet with $\dim F \neq \dim \Delta_k = d$. By taking skeletons and inducting on the dimension, we may assume that $\dim F = d - 1$. If $F \in \Delta$ then, because $\Delta \subseteq \Delta_k$, it would be a facet of Δ contradicting the purity of Δ. So assume that $F \notin \Delta$, that is $x_F \in I$. Let $i \in [n]$. Since Δ is pure, $i \in G$ for some dimension d facet of Δ. This is then also a facet of Δ_k since $\Delta \subseteq \Delta_k$. So $\dim \operatorname{link}_\Delta(i) = d - 1$ for every $i \in [n]$. Moreover, as discussed above, we may assume that $(\Delta_k)_{-i}$ is pure. If $i \notin F$ then F is a facet of $(\Delta_k)_{-i}$ with dimension $d - 1$. If we can show that, for some $i \notin F$, $\dim(\Delta_k)_{-i} = d$ then we will have a contradiction since F is a facet of $(\Delta_k)_{-i}$.

Suppose, to the contrary, that $\dim(\Delta_k)_{-i} = d-1$ for every $i \notin \Delta$. Since $(\Delta_{-i})_k$ is a skeleton of $(\Delta_k)_{-i}$ we also have $\dim \Delta_{-i} = \dim(\Delta_{-i})_k < d$. But Δ is pure and so this implies that Δ must be a cone. Then, we may simply pass to a smaller polynomial ring and conclude by induction on n that Δ_k is pure. In particular, if $\Delta = C\Gamma$ then $\Delta_k = [C\Gamma_k]_d$. This is a contradiction, finishing the proof.

QED

Using this, and other results, we give a description of the structure of the Stanley-Reisner ideals of matroid complexes. This is a generalization of Theorem 4.2.9 and Lemma 4.4.3 to complexes with (almost) arbitrary initial degree.

Theorem 4.4.31. *Let Δ be simplicial complex that is not a cone with $1 \neq \operatorname{init} I_\Delta$ and $d = \dim \Delta$. Then Δ is matroid if and only if there are subsets $\sigma_{ij} \subseteq [n]$ such that*

$$I_\Delta = \sum_{i=1}^{d+1} \sum_j \hat{\mathfrak{m}}^i_{\sigma_{ij}} + \hat{\mathfrak{m}}^{d+2}$$

where $|\sigma_{ij}| = i$, $|\sigma_{ij_1} \cap \sigma_{ij_2}| \leq i-2$ *for all* $j_1 \neq j_2$ *and, if* $i_1 < i_2$ *then* $|\sigma_{i_1 j_1} \cap \sigma_{i_2 j_2}| \leq i_1 - 2$.

Proof. We know this is true if $d = 1$ from the results of Section 4.2. Since the Stanley-Reisner ideal of $[\Delta]_k$ is $I_\Delta + \hat{m}^{k+2}$ we know by induction on the dimension that I_Δ has the desired structure in every degree except $d+1$. The Stanley-Reisner ideal of the completion of Δ has the same degree $d+1$ minimal generators as I_Δ. Since init $I_{\overline{\Delta}} = d+1$ we may use Lemma 4.4.3 to get most of the result. It only remains to show that the supports of the degree $d+1$ minimal generators and the smaller degree minimal generators intersect as claimed.

Suppose otherwise, that there are minimal generators u, v with $d+1 = \deg u > \deg v$ and $|\mathrm{supp}(u) \cap \mathrm{supp}(v)| > \deg(v) - 2$. Since u is minimal, $|\mathrm{supp}(u) \cap \mathrm{supp}(v)| = \deg(v) - 1$.

If $\mathrm{supp}(u) \cup \mathrm{supp}(v) \neq [n]$ then we can restrict Δ to $\mathrm{supp}(u) \cup \mathrm{supp}(v)$ and conclude by induction on the number of vertices. So, we may assume that $\mathrm{supp}(u) \cup \mathrm{supp}(v) = [n]$. Moreover, by applying Lemma 4.4.31 repeatedly, we may assume that I has no minimal generators except in degrees $d+2$, $\deg(u) = d+1$ and $\deg(v)$. Let $i = \deg(v)$ and suppose that $\mathrm{supp}(v) \subseteq \sigma_{i,1}$; likewise, $\mathrm{supp}(u) \subseteq \sigma_{d+1,1}$. Since $|\mathrm{supp}(u) \cap \mathrm{supp}(v)| = i - 1$ we must have that there is some x_j such that $\frac{v}{x_j} | u$. If $\sigma_{i,1} \neq \mathrm{supp}(v)$ then some multiple of $\frac{v}{x_j}$ other than v would be in I, contradicting that u is a minimal generator.

We may as well assume that $v = x_1 x_2 \cdots x_i$ and $u = x_2 x_3 \cdots x_d$. Suppose that $\deg(v) > 2$ and consider $I : x_2$. This ideal contains $v' = \frac{v}{x_2}$ and $u' = \frac{u}{x_2}$ and $|\mathrm{supp}(v') \cap \mathrm{supp}(u')| = i - 3$. Since $\deg(v) = \mathrm{init}(I)$, v' is a minimal generator of $I : x_2$. If u' is also a minimal generator, we will have a contradiction since $I : x_2$ is the Stanley-Reisner ideal of a matroid with fewer vertices than Δ. Suppose otherwise, that there is some minimal generator $w \in I : x_2$ dividing u'. Clearly $\deg(w)$ is either i or $i-1$. If $\deg(w) = i - 1$ then wx_2 must be a minimal generator of I. But $w | u'$ implies that $wx_2 | u$, a contradiction. So $\deg(w) = 1$, or, equivalently, w is a minimal generator of I. Again, this is a contradiction since w divides u'. and thus $u = u' x_2$. So $u' \in G(I : x_2)$, which, as discussed above gives us a contradiction.

It now only remains to consider the case when $\mathrm{init}(I) = 2$. If $d = 1$ then there is nothing to show. So, assume that $d > 1$. In this case, we apply Theorem 4.1.9 to write $\Delta = S_{v_s}^{W_s} \cdots S_{v_1}^{W_1} \Gamma$ for some matroid complex Γ with $\mathrm{init}(I_\Gamma) > 2$ and $\dim \Gamma = d$. Any degree 2 generators of I except v must have support disjoint from $\mathrm{supp}(v)$. But, we have assumed that $\mathrm{supp}(v) \cup \mathrm{supp}(u) = [n]$ so this would contradict u being a minimal generator of I. So I has only 1 degree 2 generator. Since each time we apply S_v^W to a complex we gain a degree 2 generator $x_v x_i$ for each $i \in W$, this must mean that $\Delta = S_{v_1}^{W_1} \Gamma$, $\Gamma = \Delta_{-v_1}$ and $|W_1| = 1$.

For convenience, we again relabel the vertices so that $u = x_1 \ldots x_{d+1}$ and $v = x_{d+1} x_{d+2}$. The vertex v_1 is either $d+1$ or $d+2$. Since every variable except for x_{d+2} divides u, removing the degree 2 generators (as in Lemma 4.4.31) gives another matroid and so we have that u is the only degree $d+1$ generator of I. So u must also be a minimal generator of I_Γ. Now, we have that $\mathrm{init}(I_\Gamma) = d+1 = \dim \Gamma + 1$ and we can apply Lemma 4.4.1 to Γ and see that Γ is either the boundary of a simplex or the cone over the boundary of a simplex. Since, we have assumed that $f_0(\Gamma) = d+1$, Γ is not a cone. So Γ is the boundary of a $(d+1)$-simplex, and thus I_Γ is principle and generated in degree $d+2$. But this contradicts the assumed existence of u. With this final contradiction, we finish this case and conclude this direction of the proof.

For the other direction, we use are normal procedure and show that Stanley-Reisner ideal of a restriction of Δ has the same form as I. But this is clear since deleting a vertex v from Δ corresponds to removing every minimal generator that x_v divides from $G(I)$. So, we can simply replace each σ_{ij} with $\sigma_{ij} - \{v\}$. By induction on n, each restriction is matroid. It then only remains to show that Δ is pure.

Let F be a facet with $\dim F < d$. By taking skeletons and inducting on d, we may assume that $\dim F = d-1$ and that Δ has no facets with dimension less than $d-1$. Let $j \in F$ and consider $I\colon x_j$. This ideal also satisfy the conditions in the Theorem so $\operatorname{link}_\Delta(j)$ is matroid. As with the deletion, $F - \{j\}$ is a facet of $\operatorname{link}_\Delta(j)$. So, $\dim \operatorname{link}_\Delta(j) = d-2$. If $j \in G$ then for some d-face of Δ then $G - \{j\} \in \operatorname{link}_\Delta(j)$, a contradiction. Thus $F \cap G = \emptyset$. So every d-face is disjoint from every $(d-1)$-face. This implies that Δ is not connected, and so $[k]_{[}]\Delta$ is not connected either. But the Stanley-Reisner ideal of $[\Delta]_k$ satisfies Theorem 4.2.9 and so $[\Delta]_k$ is matroid and thus Cohen-Macaulay. Every Cohen-Macaulay complex with dimension at least 1 is connected. To see this, note that Hochster's formula requires that a Cohen-Macaulay complex have no reduced cohomology except possibly in homological degree d but a non-connected complex has non-zero reduced cohomology in degree 0, a contradiction. So Δ must be pure and therefor matroid. QED

Remark 4.4.32. Neither of the two restrictions (that $\operatorname{init}(I_\Delta) > 1$ and that Δ is not a cone) is very restrictive. If I_Δ has linear generators then we may simply remove them and look at everything in a smaller polynomial ring. It is present simply to avoid the silly statement $|\sigma_{1,j_1} \cap \sigma_{1,j_2}| \leq -1$. To remove the second restriction takes a bit more thought. This result gives not information at all about the generators with degree $d+2$. We only know that we need enough of them to have every squarefree degree $d+2$ monomial in I. By taking cones, we can raise the dimension without changing the generators of I, allowing us to break the result. For example, if $I = \langle x_1x_4, x_1x_2x_3, x_2x_3x_4\rangle$ is the ideal of a 1-dimensional matroid (the one with associated partition $2+1+1$) we have no problem with the degree 3 generators as our claim says nothing about them. However, if we take the cone we have a 2 dimensional complex and now are making claims about those same generators. The result would be false in this case.

Remark 4.4.33. The subsets σ_{ij} in Theorem 4.4.31 are the maximal subsets of $[n]$ such that $\dim \Delta|_{\sigma_{ij}} = i-1$ and $|\sigma_{ij}| \geq i$. By Corollary 4.4.16 we see that the non-one entries of $\lambda^{(i)}$ are $\lambda^{(i)}_j = f_i(\Delta|_{\sigma_{ij}})$. There are additionally enough entries equal to 1 to get $|\lambda^{(i)}| = f_i(\Delta)$.

Appendix: Computer Code for SAGE

In this appendix, we include some code written for the SAGE computer algebra system ([22]). SAGE is built as an extension for the Python language and is capable of acting as a common interface for many existing computer algebra systems. As prerequisites, it is assumed that the user has working installations of SAGE (version at least 2.5) and Macaulay 2 ([12]) (version at least 0.9.5). The first section defines a general SAGE class for simplicial complexes, the second a class that imports minimal free resolutions from Macaulay 2 to SAGE and the last special code for dealing with matroid complexes. Below long lines are wrapped to the line below. This is indicated with a / at the end of the line.

Simplicial Complexes

Given a list of facets or a squarefree monomial ideal in a polynomial ring, this class defines a simplicial complex have the specified data. One can then compute the f and h-vectors, the dimension, links and deletions, and the Alexander dual. Two pieces of data are required from the user: a multi-variate polynomial ring and either a list of facets or a squarefree monomial ideal in the given ring. The variables of the polynomial ring are used as the vertex set of the complex. Optionally, the user may give labels for the vertices. If not specified the vertices are labeled $1, 2, \ldots, n$. The facets may be given as any Python (or SAGE) iterable that having a method supporting the `in` keyword. Likewise for the labels themselves. For complexes with dimensions at most 1, a picture may be produced using SAGE's existing plotting abilities for graphs. This was used to generate virtually all of the figures in that you see above.

The large function at the end is a constructor, which sifts through the data provided by the user and returns instances of the class. Note that the complex $\emptyset$ must be handled as a special case. This is because the method used to compute the Stanley-Reisner ideal (by finding is primary components and intersecting them) will fail if there are no facets.

```
#from sage.all import SageObject
   import sage.misc.latex as latex
   import os#for system calls
   import re#regular expressions
   #attach "/home/erik/algebra/sage/resolution.sage"
   ##############################################################
   class Stanley_Reisner_ideal(SageObject):
       """documentation"""
       def __init__(self,ring,facets,vertices):
           self.__facets=facets
           self.__ring=ring
           self.__vertex_set=vertices#vertices as a list or string
```

```
 def _str_(self):
     return str(self.__facets)
 def _repr_(self):
     name='facet'
     if len(self.__facets)>1:
         name=name+'s'
     if len(self.__facets)<6:
         return "Simplicial complex with %s %s"%(name,self.__facets)
     else:
         return "Simplicial complex with %s vertices and %s /
facets"%(self.numvertices(), len(self.__facets))
 def __contains__(self,face):
     return self.is_face(face)
 def facets(self):
     try:
         return list(self.__facets)
     except AttributeError:
         raise NotImplementedError
 def dim(self):
     """Diminsion of the complex, not the ideal"""
     try:
         d=self.__dim
     except AttributeError:
         d=-1
         for face in self.__facets:
             facedim=len(face)-1
             if facedim>d:
                 d=facedim
     return d
 def ring(self):
     """The base ring"""
     return self.__ring
 def vertices(self):
     """A list of vertices of the complex, i.e. the zero-skeleton."""
     vertices=self.__vertex_set
     vlist=[i for i in range(len(vertices))]
     for v in self.__vertex_set:
         if not self.is_face([v]):
             vlist.remove(vertices.index(v))
     return [vertices[i] for i in vlist]
 def numvertices(self):
     """The number of vertices of the complex."""
     return len(self.vertices())
 def vertex_set(self):
     """The set on which the complex lives"""
     return list(self.__vertex_set)
 def ideal(self):
```

```
        """The Stanely-Reisner ideal of the complex, i.e., the ideal
        generated by the complex's non-faces."""
        try:
            return ideal(self.__ideal.gens())
        except AttributeError:
            R=self.__ring
            if len(self.facets())==1 and /
   len(self.facets()[0])==R.ngens():
                #special case if self is the simplex on its vertices.
                return ideal(R(0))
            x=R.gens()#the variables of R
            vertices=self.__vertex_set
            n=len(x)
            I=ideal(x)
            for face in self.__facets:
                comp=list(x)
                for vertex in face:
                    comp.remove(x[vertices.index(vertex)])
                #print comp
                I=I.intersection(ideal(comp))
                #print I
            self.__ideal=I
            return I
    def hvector(self,dim=None):
        """The h-vector of the complex.

       """
        d=self.dim()
        n=len(self.vertex_set())
        try:
            h=self.__hvector
        except AttributeError:
            h=self.ideal().hilbert_series().numerator().coefficients()
            if h[0]<0:#sage is not consistent with negative signs.
                h=[-i for i in h]
            while len(h)<=d+1:
                h.append(0)
        if dim==None:
            return h
        else:
            return h[dim]
        #old versions of sage don't have the hilbert_series function.
        #in that case, this code works using macaulay2
#        except AttributeError:
#            d=1+self.dim()
#            h=[1]
#            hilb=[1]
```

```
#            I=self.ideal()
#            for i in range(1,d+1):
#                hilbi=(macaulay2(str(i)).hilbertFunction(I))._sage_()
#                hilb.append(hilbi)
#            for i in range(1,d+1):
#                hi=sum([((-1)^j)*binomial(d,j)*hilb[i-j] for j in /
   range(i+1)])
#                h.append(hi)
#        self.__hvector=h
#       return h
    def fvector(self,dim=None):
        """The f-vector of the complex"""
        try:
            f=self.__fvector
        except AttributeError:
            h=self.hvector()
            f=[1]
            d=1+self.dim()
            for i in range(1,d+1):
                fi_1=sum([binomial(d-j,i-j)*h[j] for j in range(i+1)])
                f.append(fi_1)
        self.__fvector=f
        return f
    def link(self,F):
        """The complex of all faces G such that F\cap G=\emptyset and
        F\cup G is in the complex"""
        facets=self.facets()
        absF=len(F)
        link_facets=[]
        for G in facets:
            Gl=[x for x in G if not x in F]#Gl=G-F
            if len(Gl)==len(G)-absF:# true iff F\subset G
                link_facets.append(Gl)
        return simplicial_complex(self.ring(), link_facets, /
   self.vertex_set())
       #  I=self.ideal()
#         V=self.__vertex_set
#         var=(self.__ring).gens()
#         G=I.gens()
#         mF=prod([var[V.index(v)] for v in F])
#         Glink=[]
#         Fv=[V.index(v) for v in F]
#         for mon in G:
#             #I think this works by magic.
#             #print dict(map(None,Fv,[1 for i in F]))
#             Glink.append(mon.subs(dict(map(None,Fv,[1 for i in F]))))
#         J=ideal(Glink)+ideal(mF)
```

```
#         return simplicial_complex(self.__ring,J,self.__vertex_set)
    def delete(self,F):
        """Delete a set of vertices."""
        I=self.ideal()
        R=self.ring()
        if F==[]:
            return simplicial_complex(R, self.facets(), /
   self.__vertex_set)
        vertices=self.__vertex_set
        Fvert=[vertices.index(v) for v in F]
        Fvar=[R.gen(i) for i in Fvert]
        ## print Fvar
        return simplicial_complex(R,I+ideal(Fvar),vertices)
    def restrict(self,F):
        """Restrict to a subset of the vertices."""
        Fc=[]
        vertices=self.__vertex_set
        D=self
        for v in vertices:
            if not v in F:
                Fc.append(v)
        return self.delete(Fc)
    def is_face(self,F):
        """Returns true if F is a face and false otherwise"""
        if F==[]:
            return true
        var=(self.__ring).gens()
        vertices=self.__vertex_set
        m=prod(var[vertices.index(v)] for v in F)
        return not (m in self.ideal())
    def dual(self):
        """Return the alexander dual."""
        G=(self.ideal()).gens()
        I=ideal((self.__ring)(1))
        for gen in G:
            I=I.intersection(ideal(gen.variables()))
        return simplicial_complex(self.__ring, I, self.__vertex_set)
    def resolution(self):
        """The minimal free resolution of the
        Stanely-Reisner ideal.  This is computed using Macaulay2"""
        try:
            F=self._mfr
        except AttributeError:
            n=len(self.vertex_set())
            e=(identity_matrix(n)).rows()
            F=resolution(self.ideal(),degrees=e)
            self._mfr=F
```

```
        return F
    def betti(self):
        try:
            F=self._mfr
        except AttributeError:
            F=self.resolution()
            B=F.betti()
        return B
    def __adjacency_matrix(self):
        """Return the adjacency matrix of the complex if it has
        dimension \leq 1 and and error otherwise"""
        if self.dim()>1:
            raise NotImplementedError, 'Not implemented for complex /
with dimension larger than 1'
        vertices=self.vertices()
        n=len(vertices)
        A=matrix({(n-1,n-1):0})#make a nxn 0 matrix
        for a in vertices:
            for b in vertices:
                if self.is_face([a,b]):
                    A[vertices.index(a),vertices.index(b)]=1
        return A
    def plot(self,**kwds):
        """If you are a 1 (or 0) dimensional complex (a graph),
        plot yourself"""
        if self.dim()>1:
            raise NotImplementedError
        A=self.__adjacency_matrix()
        G=Graph(A)
        G.relabel(list(self.vertices()))
        return G.plot(**kwds)
    def homology(self,i):
        """return the vector space dimension of the i-th reduced
        simplicial cohomology group of self (this is the same dimension
        as the i-th reduced simplicial homology).

        This uses Hochster's forumula to compute
        $\dim\widetilde{H}^i(\Delta,K)$ as /
$\beta_{n-i-1,n}S/I_\Delta$"""
        N=len(self.vertex_set())
        sigma=self.vertices()#the 0-skeleton of self
        n=len(sigma)
        I=self.ideal()
        #if not N==n:
        #    raise NotImplementedError
        #compute the minimal free resolution of the
        #Stanley-Reisner ideal of self (self.ideal())
```

```
        d={(0,N-1):0}
        d.update(zip([(0,j-1) for j in sigma],[1 for j in sigma]))
        a=tuple(tuple(matrix(d))[0])# the vector whose support is sigma
        #e=(identity_matrix(N)).rows()#the multi-degrees of the /
   variables
        B=self.resolution()
        Hi=B[(n-i-1,a)]
        return Hi
    def is_CM(self):
        """True iff the complex is Cohen-Macaulay"""
        F=self.betti()
        return F.is_CM()
    def skeleton(self,k):
        """The k-skeleton of self"""
        def is_sf(g):
            return len(g.variables())==g.degree()
        d=self.dim()
        if k>d:
            return self#do nothing, should make a copy.
        if k==0:
            return self.vertices()
        I=self.ideal()
        R=self.ring()
        m=ideal(R.gens())
        tmp=(m^(k+2)).reduced_basis()
        msk=ideal([x for x in tmp if is_sf(x)])#the squarefree power /
   of m
        return simplicial_complex(R,I+msk,vertices=self.vertex_set())
class empty_complex(Stanley_Reisner_ideal):
    def __init__(self,ring,facets,vertices):
        self.__facets=facets
        self.__ring=ring
        self.__vertex_set=vertices
    def dim(self):
        raise AttributeError, "The empty set does not have a well /
   defined dimension."
    def ideal(self):
        R=self.__ring
        return ideal(R(1))
    def dual(self):
        """The Alexander dual of the empty set is the simplex."""
        return simplicial_complex(self.ring(), [self.vertex_set()], /
   self.vertex_set())
    def __adjacency_matrix(self):
        return matrix([])
    def is_CM(self):
        return True
```

```
    def skeleton(self,k):
        return self
    def homology(self,i):
        if i==-1:
            return 1
        else:
            return 0
#-------------------------------------------------------------------------
def simplicial_complex(ring,data,vertices=None):
    """Return a simplicial complex.  You may specifiy either the facets
    of the complex or its Stanely-Reisner ideal.
    USAGE:
             1. simplicial_complex(base_ring,ideal,vertices=None)
             2. simplicial_complex(base_ring,facets,vertices==None)
    INPUT:
             base_ring -- a polynomial ring over a field
             ideal -- a square-free monomial ideal in base_ring
             vertices -- a list or string.  If it is a string
                         each character will be treated as a
                         vertex.  If this is None, the vertices
                         will be 1,2, ... n, where n is the number of
                         variables of base_ring
             facets -- a list.  Its entries are lists whose entries
                        are in vertices.  If vertices is a string or
                        consists of length 1 strings, the entries of
                        facets may be strings."""
    if vertices==None:
        vertices=[i+1 for i in range(ring.ngens())]
    if sage.rings.ideal.is_Ideal(data):
        I=data#must be an ideal
        #check that the ideal  is squarefree monomial
        G=I.gens()
        for g in G:
            if not len(g.monomials())==1:
                raise TypeError, "Expected a monomial ideal"
        #compute the Stanely-Reisner ideal
        for g in G:
            if not len((g.monomials())[0].variables())==g.degree():
                raise TypeError, "Expected a squarefree monomial ideal"
            AssI=I.primary_decomposition()
            facets=[]
            x=list(ring.gens())
            n=len(x)
            for P in AssI:
                facet=[]
                for i in range(ring.ngens()):
                    if not(x[i] in P.gens()):
```

```
                        facet.append(vertices[i])
                #print facet
                facets.append(facet)
            return Stanley_Reisner_ideal(ring,facets,vertices)
    elif type(data)==list:
        if  data==[]:
            return empty_complex(ring,data,vertices)
        else:
            return Stanley_Reisner_ideal(ring,data,vertices)

    raise TypeError #Not a list or an ideal.  You did it wrong.
```

Resolutions

At this time SAGE does not offer support for free modules over anything except fields. Thus there is no way to actually have a minimal free resolution in SAGE. Instead we satisfy ourselves with importing the Macaulay 2 style Betti diagram giving the ranks of the different summands in the resolution. Both course and fine graded resolutions are supported. The user may assign degrees to the variables with minimal restrictions. In particular, the degree of each monomial must be positive.

Almost no math is being done by this code. Instead, all of the hard work is passed off to Macaulay 2 and the answer read back into SAGE. Thus, we only need to massage the data into a form that SAGE can understand using string substitutions and regular expressions. The data is stored in a Python dictionary (`dict`).

```
import sage.misc.latex as latex
   import os
   import re
   class resolution(SageObject,dict):
       """
       Use Macaulay 2 to get the MFR of an ideal.  Since SAGE doesn't
       support modules, we just return the Betti diagram.  The instances
       of this class behive like dictionaries except that they are
       displayed macaulay2 style and are immutable.

       USAGE: resolution(ideal)
              where ideal is an ideal in a (multivariate) polynomial ring.

       TODO: define a betti_diagram class so that self.betti() returns
       something that displays correctly.  Should it be immutable?

       TODO: currently, only rational fields are supported becaues
       thats all I can get into macaulay2
       """
       #Don't derive this from the dict class since, in principle, the
       #resolution is a complex not just the Betti diagram.
       #TODO: Try to get
       #the matrices out of macaulay2 and store them in a seperate
```

```
 #dictionary (remember the shifts)
 def __init__(self,I,degrees=None):
     n=(I.ring()).ngens()
     if degrees==None:
         degrees=[1 for i in range(n)]
     #error checking
     self.__degrees=degrees
     #print self.__degrees
     self.__ideal=I
     self.__get_betti()#use M2 to compute the MFR of I
 def __getitem__(self,key):
     """
     Return the Betti number corresponding to key.
     There are three cases depending on the type of key:
             1. key an integer: return the total Betti number
             2. key=(i,j) where i,j are integers: return
                the course-graded Betti number \beta_{ij}.
             3. key=(i,a) where i in an integer and a in a tuple:
                returns the fine-graded Betti number /
\beta_{i\vec{a}}.
     """
     B=self.dictionary()#maybe B=self instead?
     if type(key)==tuple:
         try:
             if key[1] in ZZ:
                 return self.__betti_dict_course[key]
             else:
                 return self.__betti_dict_fine[key]
         except KeyError:
             return 0
     elif key in ZZ:#this sould raise KeyError if key is not an /
integer
         s=0
         for entry in self.__betti_dict_course:
             if entry[0]==key:
                 s=s+self.__betti_dict_course[entry]#add up to get /
the total Betti number
         return s
     raise IndexError
 def __setitem__(self,key,item):
     raise TypeError
 def __contains__(self,key):
     """Allows one to use 'x in betti' style constructions"""
     return (key in self.__betti_dict_course) or key in /
self.__betti_dict_fine
 def _repr_(self):
     """Display things Macaulay2 style."""
```

```
        out={}
        n=((self.__ideal).ring()).ngens()
        total=[0 for i in range(n+1)]
        pd=0
        for key in self.__betti_dict_course:
            #print key
            i=key[0]
            j=key[1]#the last entry
            out[(i,j-i)]=(self.__betti_dict_course)[key]
            total[i]=total[i]+(self.__betti_dict_course)[key]
            if i>pd:
                pd=i
        self.__pd=pd
        M=matrix(out)
        Mstring=(M.transpose()).str()
        Mstring=Mstring.replace(' 0',' .')
        Mstring=Mstring.replace('[0','[.')
        lines=Mstring.splitlines(true)#true means keep the EOL /
characters
        for i in range(len(lines)):
            spacing="".join([' ' for x in range(5-len(str(i)))])
            lines[i]=spacing+str(i)+": "+lines[i]
        Mstring="".join(lines)
        Mstring='total: '+str(matrix(1,pd+1,total[:pd+1]))+'\n'+Mstring
        Mstring='       '+str(matrix(1,pd+1,range(pd+1)))+'\n'+Mstring
        Mstring=Mstring.replace('[','')
        Mstring=Mstring.replace(']','')
        return Mstring
    def _str_(self):
        return str(self.betti())
    def _latex_(self):
        """Return Latex tabular for the course-graded Macaulay2
        Betti diagram."""
        p=self.pd()
        format_string="".join(['r' for i in range(p+2)])
        betti_string=self._repr_()
        tab_start="\\begin{tabular}{%s}\n &"%format_string
        tab_end="\\end{tabular}\n"
        betti_string=self._repr_()
        #betti_string=betti_string.replace(' ','&')
        betti_string=re.sub(r"([:.\d]+)([ ]+)",r"\1 &\2",betti_string)
        betti_string=betti_string.replace('\n',"\\\\\n")
        out_string=tab_start+betti_string+"\n"+tab_end
        return out_string
    def __init_m2_field(self):
        """Create the field for use in macaulay2.
```

```
     This is pretty hacky, but SAGE doesn't offer a
     _macualay2_init method for fields.  Singular works OK.
     """
     field_dict={"ComplexField":"CC", "RealField":"RR", /
"RationalField": "QQ","FiniteField":None}
     R=(self.__ideal).ring()
     kk_str=str((R.base_ring()))#the string representation of the /
base ring
     name="".join((kk_str.split())[:2])#the first two words
     kk=field_dict[name]
     if kk==None:
         #the field is finite
         size=kk_str.pop()#its size
         size=size.replace('^',',')
         kk='GF(%s)'%size
     return kk
 def __init_m2_ideal(self):
     """Put the ideal of self into macaulay2 as a
      ideal. If the ideal is monomial this is multi-graded."""
     n=((self.__ideal).ring()).ngens()#the number of variables
     kk=self.__init_m2_field()
     if not kk=='QQ':
         raise NotImplementedError,'Only rational fields are /
supported'
     degrees_m2=str(self.__degrees)
     #change the python braces to macualay ones
     degrees_m2=degrees_m2.replace('[','{')
     degrees_m2=degrees_m2.replace(']','}')
     degrees_m2=degrees_m2.replace('(','{')
     degrees_m2=degrees_m2.replace(')','}')
     try:
         h=len((self.__degrees)[0])
     except TypeError:
         h=1
     heft_m2=str([1 for i in range(h)])
     heft_m2=heft_m2.replace('[','{')
     heft_m2=heft_m2.replace(']','}')
     R=macaulay2('%s[x_1..x_%s,Degrees=>%s,Heft=>%s]'%(kk,n, /
degrees_m2,heft_m2))
     J=(self.__ideal)._macaulay2_()#put the ideal into M2
     J=J.sub(R.vars())#make it graded
     return (J,degrees_m2,heft_m2)
 def __get_betti(self):
     """Fetch the Betti diagram from Macaulay2 and convert it
     to a python dictionary. Computes the fine and
     course resolutions"""
     #J=(self.__ideal)._macaulay2_()#put the ideal into M2
```

```
        J,deg_m2,heft_m2=self.__init_m2_ideal()
        B=(J.res()).betti('Weights=>%s'%heft_m2)
        Bs=macaulay2.toString(B)
        Bs=(Bs.str()).replace("new BettiTally from ",'')
        ## these string manipulations produce a string that looks
        ## like a python dict
        Bs=Bs.replace('-','')
        Bs=Bs.replace('\n','')
        Bs=re.sub(r"{([0-9, ]+)}",r"(\1)",Bs)
        Bs=re.sub(r"=>",":",Bs)
        #Bs=eval(Bs)
        betti=sage_eval(eval(Bs))
        Bs=re.sub(r"[(][0-9,]+[)],","",Bs)#delete the second term of /
the tuple
        self.__betti_dict=betti
        self.__get_betti_course()#compute the Z-graded MFR
        self.__get_betti_fine()
        return betti
    def __get_betti_course(self):
        """Return a dictionary containng the course Betti numbers"""
        betti=self.__betti_dict
        betti_course={}
        for key in betti:
            course_key=(key[0],key[2])
            if course_key in betti_course:
                /
betti_course[course_key]=betti_course[course_key]+betti[key]
            else:
                betti_course[course_key]=betti[key]
        self.__betti_dict_course=betti_course
        return betti_course
    def __get_betti_fine(self):
        """Return a dictionary containing the fine Betti numbers"""
        try:
            betti=self.__betti_dict
        except AttributeError:
            self.__get_betti()
            betti=self.__betti_dict
        betti_fine={}
        for key in betti:
            betti_fine[key[:2]]=betti[key]
        self.__betti_dict_fine=betti_fine
        return betti_fine
    def degrees(self):
        return self.__degrees
    def dictionary(self):
        """The values in the Betti diagram as a dictionary"""
```

```
        try:
            return self.__betti_dict
        except AttributeError:
            self.__get_betti()
            return self.__betti_dict
    def betti(self,grading='course'):
        """Return the Betti diagram of either the course of fine
        graded resolution as a dictionary.

        USAGE:
                F.betti('course')
                F.betti('fine')
        """
        if grading=='course':
            try:
                return dict(self.__betti_dict_course)
            except AttributeError:
                self.__get_betti_course()#compute the mfr
                return dict(self.__betti_dict_course)
        elif grading=='fine':
            try:
                return dict(self.__betti_dict_fine)
            except AttributeError:
                self.__get_betti_fine()
                return dict(self.__betti_dict_fine)
        else:
            raise TypeError, "grading must be either 'course' or 'fine'"
    def pd(self):
        """Projective dimension"""
        try:
            return self.__pd
        except AttributeError:
            B=self.dictionary()
            pd=0
            for key in B:
                if key[0]>pd:
                    pd=key[0]
            self.__pd=pd
        return self.__pd
    def regularity(self):
        """The regularity of the quotient by the ideal.  Careful about /
that."""
        try:
            return self.__regularity
        except AttributeError:
            B=self.__betti_dict_course
            reg=0
```

```
            for key in B:
                if key[1]-key[0]>reg:
                    reg=key[1]-key[0]
            self.__regularity=reg
        return self.__regularity
    def is_CM(self):
        """True is the ideal is Cohen-Macaulay, false otherwise."""
        I=self.__ideal
        R=I.ring()
        n=R.ngens()
        c=n-I.dimension()
        return self.pd()==c
    def is_gorenstein(self):
        """True iff the ideal is Gorenstein"""
        p=self.pd()
        return (self.is_CM() and self[p]==1)
    def ideal(self):
        return self.__ideal
########################################################################
```

Matroids

Finally, we have code that performs the computations and constructions described in Chapter 4. This is built on top of the simplicial complex code and returns instances of that class. Given any complex Δ this can return $S_v^W\Delta$. Given any partition λ it can also construct a complex in Δ_λ. For any 1-dimensional complex the edge complex can be constructed and finally, given a 1-dimensional matroid complex, Δ it can compute λ_Δ. This last takes a non-trivial amount of time (up to several minutes on a moderately powered machine). These functions all take simplicial complexes as input and do no checking to ensure that they are matroid. The user is advised to be cautious.

```
from sage.all import *
   import sage.misc.latex as latex
   #import os#for system calls
   #import re#regular expressions
   ##############################################################
   def matroid_attach(D,new,avoid):
       """
       attach a set of vertices, new, to the complex D avoiding vertex
       avoid.  It just iterates over the facets of the complex.
       """
       vert=D.vertex_set()
       facets=list(D.facets())
       newfacets=[]
       for F in facets:
           #facets.d(list(F))
           if avoid in F:
```

```
            #print "face=",F
            G=list(F)
            G.remove(avoid)
            for v in new:
                G1=list(G)#make a copy of G
                G1.append(v)
                facets.append(G1)
        #print facets
    return simplicial_complex(D.ring(),facets,vert)
def matroid_from_partition(R,L,vertices=None):
    """
    Makes a matroid from the given partition.  This just subtracts 1
    from each entry and feeds that into matroid_from_sequence.

    """
    m=[Li-1 for Li in L]
    return matroid_from_sequence(R,m,vertices)
def matroid_from_sequence(R,m, vertices=None, K=None, start=1, /
   avoid=1, s=None, depth=1):
    """
    Recursivly construct a matroid from a given sequence m starting
    with a complete graph on len(m) vertices by attaching m[0]
    vertices avoiding vertex 1, m[1] vertices avoiding vertex 2 and so
    on.
    """
    if len(m)+sum(m)>R.ngens():
        raise IndexError, "Not enough vertices."
    m.sort()
    m.reverse()#all the 0's at the end
    s=len(m)#given sequence
    n=s+sum(m)#number of vertices
    if vertices==None:
        vertices=list(range(1,R.ngens()+1))
    M=ideal(R.gens())
    if len(m)==1 and depth==1:#the complex has dimension 1
        return simplicial_complex(R, [[x] for x in vertices[:n]], /
   vertices)
    #form the complete graph on s vertices as the initial step
    if K==None:#only happends the first time through
        edges=[]
        for i in range(s):
            for j in range(i):
                edges.append([vertices[i],vertices[j]])
        K=simplicial_complex(R,edges,vertices)
        start=start+s
    #return the final complex, K, when m has no more non-zero entries /
   left.
```

```
    if m==[]:
        #print "done"
        return K
    if m[0]==0:
        #print "done"
        return K
    else:
        #print "adding",m[0],"vertices avoiding /
   vertex",vertices[avoid-1]
        #print "starting at",vertices[start-1]
        #print "new vertices=",vertices[start-1:start-1+m[0]]
        /
   K=matroid_attach(K,vertices[start-1:start-1+m[0]],vertices[avoid-1])
        K=matroid_from_sequence(R ,m[1:], vertices, K, start+m[0] /
   ,avoid+1, s, depth+1)
        return K
    return K
def step_by_step(R,part):
    """Makes a cool picture showing the steps to construct the matroid
    from the given partition.  The vertex being avoided is white, the
    new vertices are blue and the others are red.  The complete graph
    that you start with is highlighted in red.

    Returns a pair (D,P), D is a list of the complexes themselves and
    P is a list of plots.  Try

       sage: map(lambda x: x.plot(axes=False), step_by_step(R,part)[1])

    to display the plots.  A purely decorative function.
    """
    plots=[]
    #p=[[1,1,1],[3,1,1],[3,2,1]]
    s=len(part)
    p=[[1 for i in range(s)]]#list of partial partitions
    for i in range(1,s+1):
        p_t=[1 for t in range(s)]
        for j in range(i):
            #print i,j
            p_t[j]=part[j]
        p.append(p_t)
    print p
    D=[]
    #define some colors by their hex codes
    avoid='#ffffff'#white
    #old='#fec7b8'#pink. this breaks everything for some reason.
    old='#ff0000'#red
    new='#0593ff'#some kind of blue
```

```
    #edge='#37ff05'
    edge='#aa1500'#another red
    black='#000000'#black
    for l in p:
        m=[x-1 for x in l]
        #print m
        d=matroid_from_sequence(R,m)
        if p.index(l)==0:
            Ks=d.facets()
        D.append(d)
        t=text(str(l),(.5,0),axis_coords=True)
        v=p.index(l)
        new_edges=d.facets()
        map(new_edges.remove,Ks)
        edge_c={edge:Ks,black:new_edges}
        if v==0:#a special case for the first time through.  Changes /
   the colors
            v=1
            old_avoid=avoid
            avoid=old
            edge_c={black:d.facets()}
        end_slice=d.vertices()[-m[v-1]:]
        #print end_slice
        s=sum(l)
        #print edge_c
        plots.append(t+d.plot(edge_colors=edge_c, /
   vertex_colors={avoid:[v], old:d.vertices(), new:end_slice}))
        print avoid,old,new
        avoid=old_avoid
    return D,plots
def find_partition(cmplx):
    """
    Find the partition associated to a given 1-dimensional matroid.
    Does not check that the given complex is matroid or even that it
    has dimension 1.  Kind of slow.

    """
    D=cmplx
    V=D.vertices()
    #print V
    L=[]
    for v in V:
        if not v in D.vertices():
            continue#if v has already been deleted, skip to the next /
   step
        n=D.numvertices()# n=f_0(D)
        Dl=D.link([v]).vertices()
```

```
        delv=[i for i in V if (not i in Dl)]#the vertices not in link(v)
        k=(n-len(Dl))#the number of vertices not in link(v)
        D=D.delete(delv)#delete everything in
        L.append(k)
    L.sort()
    L.reverse()#from largest to smallest
    return L
def edge_complex(D):
    edges=["".join(map(str,F)) for F in D.skeleton(1).facets()]
    e=len(edges)
    faces=[]
    for i in range(e):
        for j in range(i):
            #print edges[i],edges[j]
            if D.restrict(edges[i]+edges[j]).dim()==2:
                faces.append([edges[i],edges[j]])
                #print edges[i],edges[j]
    #print faces
    vars=map(lambda edg: 'x_'+edg,edges)
    S=PolynomialRing(QQ,vars)
    return simplicial_complex(S,faces,vertices=edges)
def make_hvector_table(nmax,good_number=None,bad_number=None):
    if good_number==None:
        def good_number(n):
            return "\\textbf{%s}"%n
    if bad_number==None:
        def bad_number(n):
            return "%s"%n
    T="\\begin{tabular}{c|"+"".join(['c' for i in /
   range(binomial(nmax-1,2)+1)])+"}\n"
    T=T+"$n$&$h_2$\\\\\n\hline\\\\\n"
    for n in range(2,nmax+1):
        P=Partitions(n).list()[1:]
        S=set({})
        for p in P:
            S.add(binomial(n-1,2)-sum([binomial(x,2) for x in p]))
        T=T+"%s&"%n
        for h2 in range(binomial(n-1,2),0,-1):
            if h2 in S:
                T=T+good_number(h2)+"&"
            else:
                T=T+bad_number(h2)+"&"
        T=T+good_number(0)+"\\\\\n"
    T=T+"\end{tabular}"
    return T
```

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