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New Perspectives of Quantum Analogues

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NEW PERSPECTIVES OF QUANTUM ANALOGUES

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Yue Cai
Lexington, Kentucky

Director: Dr. Margaret A. Readdy, Professor of Mathematics
Lexington, Kentucky 2016

ABSTRACT OF DISSERTATION

NEW PERSPECTIVES OF QUANTUM ANALOGUES

In this dissertation we discuss three problems. We first show the classical q-Stirling numbers of the second kind can be expressed more compactly as a pair of statistics on a subset of restricted growth words. We extend this enumerative result via a decomposition of a new poset which we call the Stirling poset of the second kind. The Stirling poset of the second kind supports an algebraic complex and a basis for integer homology is determined. A parallel enumerative, poset theoretic and homological study for the q-Stirling numbers of the first kind is done. We also give a bijective argument showing the (q, t)-Stirling numbers of the first and second kind are orthogonal. In the second part we give combinatorial proofs of q-Stirling identities via restricted growth words. This includes new proofs of the generating function of q-Stirling numbers of the second kind, the q-Vandermonde convolution for Stirling numbers and the q-Frobenius identity. A poset theoretic proof of Carlitz's identity is also included. In the last part we discuss a new expression for q-binomial coefficients based on the weighting of certain 01-permutations via a new bistatistic related to the major index. We also show that the bistatistics between the inversion number and major index are equidistributed. We generalize this idea to q-multinomial coefficients evaluated at negative q values. An instance of the cyclic sieving phenomenon related to flags of unitary spaces is also studied.

KEYWORDS: q-Stirling numbers, poset topology, cyclic sieving phenomenon, symmetric functions, major index

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1.1 Introduction to q-analogues

This dissertation is focused on extending the classical theory of quantum analogues, also known as q-analogues to one which reveals enumerative, poset theoretic and topological structure. The study of q-analogues can be traced back to the 1700's in work of Euler on theta functions as well as q-series for the pentagonal numbers [23, Chapter 16]. In 1916 Major Percy Alexander MacMahon ushered in the modern era of q-analogues. He enumerated mathematical objects by keeping track of their inherent structure using the indeterminate q raised to some statistic.

For example, let $\mathfrak{S}_n$ denote the set of all permutations of the n-element set $\{1, 2, \ldots, n\}$. For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, the *inversion number* of π, denoted $\mathrm{inv}(\pi)$ is

$$\mathrm{inv}(\pi) = |\{(i, j) : i < j \text{ and } \pi_i > \pi_j\}|.$$

In 1916 MacMahon [47, Page 318] gave the q-enumeration of the symmetric group $\mathfrak{S}_n$ using the inversion statistic.

Theorem 1.1.1 (MacMahon, 1916) *For any nonnegative integer n, the following identity holds.*

$$\begin{aligned}\sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{inv}(\pi)} &= 1 \cdot (1 + q) \cdot (1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) \\ &= [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q \\ &= [n]_q!.\end{aligned}$$

Here for any nonnegative integer n, $[n]_q = 1+q+\cdots+q^{n-1}$ denotes the q-analogue of n and $[n]_q!$ denotes the q-analogue of $n! = n \cdot (n-1)\cdots 2 \cdot 1$. Observe $\lim_{q\to 1}[n]_q = n$, and $\lim_{q\to 1}[n]_q! = n!$. Thus one can view Theorem 1.1.1 as extending the usual enumeration of the symmetric group $\mathfrak{S}_n$ from $|\mathfrak{S}_n| = n!$ to the q-enumeration $[n]_q!$.

The *Gaussian polynomial* or *q-binomial* is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

This appears to be a naive extension of the usual definition of the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}$. However, MacMahon again found a combinatorial interpretation for the q-binomial, that is, a mathematical object with a natural q-weighting which gives the q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Let $\mathfrak{S}(0^{n-k}, 1^k)$ be the set of all 0-1 bit strings consisting of $n-k$ zeros and k ones. For $w = w_1w_2\cdots w_n \in \mathfrak{S}(0^{n-k}, 1^k)$ define the *descent set* $D(w)$ to be

$$D(w) = \{i : w_i > w_{i+1}\} \subseteq \{1, 2, \ldots, n-1\},$$

while the *major index* of w is defined to be the sum of all the elements of the descent set:

$$\mathrm{maj}(w) = \sum_{i \in D(w)} i.$$

MacMahon's combinatorial interpretation of the q-binomial is as follows.

Theorem 1.1.2 (MacMahon, 1916) *For any integers $n \geq k \geq 0$, the following identity holds.*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \mathfrak{S}(0^{n-k},1^k)} q^{\mathrm{inv}(w)} = \sum_{w \in \mathfrak{S}(0^{n-k},1^k)} q^{\mathrm{maj}(w)}.$$

As a corollary from this result, the statistics $\mathrm{inv}(\cdot)$ and $\mathrm{maj}(\cdot)$ are equidistributed over the symmetric group $\mathfrak{S}_n$. Foata [24] gave the first bijective proof of this result. An algebraic interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is that this is the number of k-dimensional subspaces of the n-dimensional vector space over the field $\mathbb{F}_q$ of q elements. See Dempsey's dissertation for vector space proofs of q-identities [19].

MacMahon's theorem foreshadowed the idea of Mahonian statistics in combinatorics, that is, a statistic on a set of words giving the same distribution as the inversion number or major index as well as joint distributions of pairs of statistics. See Zeilberger and Bressoud [82] for their proof of the q-Dysson conjecture, and Galovich and White's characterization of integer Mahonian statistics [29, 30]. For a sampling of research on joint distributions, we refer the reader to Rawlings' work on Worpitzky identities [59], Gessel and Reutenauer's work connecting enumerating permutations by cycle type and descent with characters from the representation theory of the symmetric group [32], Babson and Steingrímsson's work on Mahonian statistics for generalized permutation patterns [2], and Shareshian and Wachs' work on q-Eulerian polynomials [66].

The *q-multinomial coefficient* is defined as

$$\begin{bmatrix} n \\ \alpha_1, \alpha_2, \ldots, \alpha_\ell \end{bmatrix}_q = \frac{[n]_q!}{[\alpha_1]_q! \cdot [\alpha_2]_q! \cdots [\alpha_\ell]_q!},$$

where α_i are positive integers such that $n = \alpha_1 + \alpha_2 + \cdots \alpha_\ell$. Let

$$M = \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, \ell^{\alpha_\ell}\}$$

be a multiset of cardinality n, and let $\mathfrak{S}_M$ be the set of all permutations on the elements of M. MacMahon [47, Page 317] gave the following combinatorial interpretation of the q-multinomial.

Theorem 1.1.3 (MacMahon, 1916) *The q-multinomial satisfies the following identity.*

$$\begin{bmatrix} n \\ \alpha_1, \alpha_2, \ldots, \alpha_\ell \end{bmatrix}_q = \sum_{w \in \mathfrak{S}_M} q^{\mathrm{inv}(w)}.$$

In particular, the q-binomial is a special case of the q-multinomial with the relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k, n-k \end{bmatrix}_q.$$

1	1	1	1	1	1	1	1	1	etc
	1	3	7	15	31	63	127	255	etc
		1	6	25	90	301	966	3025	etc
			1	10	65	350	1701	7770	etc
				1	15	140	1050	6951	etc
					1	21	266	2646	etc
						1	28	461	etc
							1	36	etc
								1	etc
									etc

Table 1.1: Stirling's original table of expanding z^n.

1.2 Stirling numbers

The Stirling numbers of the first and second kind are named after James Stirling who was interested in Newton series of a function, that is, expansions of the form

$$f(z) = \sum_{k\geq 0} a_k \cdot z \cdot (z-1) \cdots (z-k+1).$$

Stirling found that the coefficients are nonnegative integers in the case $f(z) = z^n$:

$$z^n = \sum_{k=0}^{n} A_k^n \cdot z \cdot (z-1) \cdots (z-k+1).$$

Table 1.1 is Stirling's original computation of the expansion of z^n up to $n = 9$ [71, Page 8]. Reading the table vertically one obtains the equations in modern language as follows:

$$\begin{aligned}
z &= z, \\
z^2 &= z + z(z-1), \\
z^3 &= z + 3z(z-1) + z(z-1)(z-2), \\
z^4 &= z + 7z(z-1) + 6z(z-1)(z-2) + z(z-1)(z-2)(z-3), \\
z^5 &= z + 15z(z-1) + 25z(z-1)(z-2) + 10z(z-1)(z-2)(z-3), \\
&\quad + z(z-1)(z-2)(z-3)(z-4), \\
\cdots
\end{aligned}$$

See also [76, Page 24] for Stirling's techique for computing this table.

Later in his book Stirling gave a table for the coefficients $\sigma(m+k, m)$.

$$\frac{1}{z^{m+1}} = \sum_{k\geq 0} \frac{\sigma(m+k, m)}{z \cdot (z+1) \cdots (z+m+k)}.$$

k	$\pi \in \Pi_{4,k}$			$S(4,k)$
1	1234			1
2	1/234	134/2	124/3	7
	123/4	12/34	13/24	
	14/23			
3	1/2/34	1/24/3	1/23/4	6
	12/3/4	13/2/4	14/2/3	
4	1/2/3/4			1

Table 1.2: All set partitions of $\{1, 2, 3, 4\}$.

Again these coefficients are nonnegative integers. The values A_k^n and $\sigma(m+k, m)$ were named the *Stirling numbers of the second kind* and the *Stirling number of the first kind* respectively by the Danish mathematician Niels Nielsen [55, Page 68].

Using modern notation,

$$z^n = \sum_{k \geq 0} S(n,k) \cdot z \cdot (z-1) \cdots (z-k+1), \tag{1.2.1}$$

and

$$\frac{1}{z^{m+1}} = \sum_{k \geq 0} \frac{c(m+k, m)}{z \cdot (z+1) \cdots (z+m+k)}. \tag{1.2.2}$$

Stirling realized these two sequences are orthogonal. Again, in modern notation

$$\sum_{k \geq 0} S(n,k) \cdot s(k,m) = \delta_{m,n}$$

where $\delta_{m,n}$ denotes the Kronecker delta and $s(k,m) = (-1)^{k-m} \cdot c(k,m)$ is the *signed Stirling number of the first kind.* By Möbius inversion, Equation (1.2.2) becomes

$$\sum_{k=0}^{n} s(n,k) \cdot z^k = z \cdot (z-1) \cdots (z-n+1). \tag{1.2.3}$$

A combinatorial interpretation for the Stirling number of the second kind $S(n,k)$ is it equals the number of set partitions of n elements into k disjoint blocks. A *set partition* of $\{1, 2, \ldots, n\}$ is written $\pi = B_1/B_2/\cdots/B_k$ where the blocks B_i are nonempty and pairwise disjoint with $\bigcup_{1 \leq i \leq k} B_i = \{1, 2, \ldots, n\}$. See Figure 1.2 for an example. In contrast, the Stirling number of the first kind counts the number of permutations in $\mathfrak{S}_n$ having k cycles. See Figure 1.3 for an example.

Another combinatorial interpretation of the Stirling numbers is the special case of the valuation on Motzkin paths and Facard paths; see Viennot [78]. We briefly

k	$\pi \in \mathfrak{S}_4$			$c(4,k)$
1	(1234)	(1243)	(1324)	6
	(1342)	(1423)	(1432)	
2	(1)(234)	(1)(243)	(12)(34)	11
	(13)(24)	(14)(23)	(134)(2)	
	(143)(2)	(124)(3)	(142)(3)	
	(123)(4)	(132)(4)		
3	(1)(2)(34)	(1)(24)(3)	(1)(23)(4)	6
	(12)(3)(4)	(13)(2)(4)	(14)(2)(3)	
4	(1)(2)(3)(4)			1

Table 1.3: All permutations in $\mathfrak{S}_4$ in cycle notation, the sum of the Stirling numbers of the first kind gives $4! = 24$.

describe the construction here. A *path* is a sequence $(s_0, s_1, \ldots, s_n)$ of elements in $S = \mathbb{N} \times \mathbb{N}$. The s_i are the *vertices* of the path and the pair (s_{i-1}, s_i) is the *i-th elementary step* of the path. A *Motzkin path* is a path such that $s_0 = (0,0)$ and each elementary step (s_{i-1}, s_i) is one of the three types: $s_i - s_{i-1} = (1,1)$, $s_i - s_{i-1} = (1,0)$, or $s_i - s_{i-1} = (1,-1)$. The coordinate y_i of the vertex $s_i = (x_i, y_i)$ is called the *level* of s_i. For any elementary step (s_{i-1}, s_i), if the vertex s_{i-1} is at level k, define the *weight* or *valuation* $v(s_{i-1}, s_i)$ of this step as

$$v(s_{i-1}, s_i) = \begin{cases} 1 & \text{if } s_i - s_{i-1} = (1,1), \\ k+1 & \text{if } s_i - s_{i-1} = (1,0), \\ 0 & \text{if } s_i - s_{i-1} = (1,-1). \end{cases}$$

The weight of a Motzking path p is then given by the product

$$v(p) = v(s_0, s_1) \cdot v(s_1, s_2) \cdots v(s_{n-1}, s_n).$$

Theorem 1.2.1 (Viennot, 1984) *The Stirling number of the second kind $S(n,k)$ is given by*

$$S(n,k) = \sum_p v(p),$$

where the sum is over all Motzkin paths of length $n-1$ starting at level 0 and ending at level $k-1$.

In contrast, a *Favard path* is a path such that $s_0 = (0,0)$ and each elementary step (s_{i-1}, s_i) is one of the three types: $s_i - s_{i-1} = (1,1)$, $s_i - s_{i-1} = (0,1)$, or $s_i - s_{i-1} = (0,2)$. For any elementary step (s_{i-1}, s_i), if the vertex s_{i-1} is at level k,

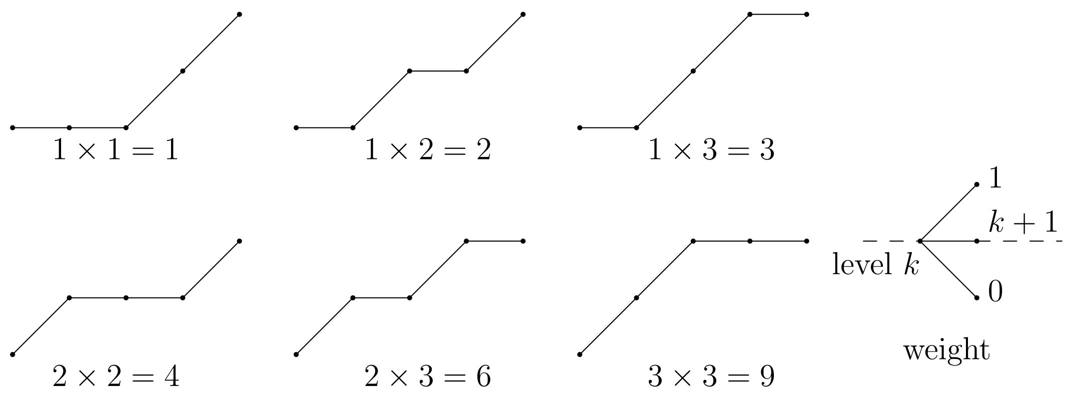


Figure 1.1: The non-zero weighted Motzkin paths of length 4 starting at $(0,0)$ and ending at level 2. Summing over the weights gives $S(5,3) = 25$.

define the *weight* of this step, $v(s_{i-1}, s_i)$ as

$$v(s_{i-1}, s_i) = \begin{cases} 1 & \text{if } s_i - s_{i-1} = (1,1), \\ k+1 & \text{if } s_i - s_{i-1} = (0,1), \\ 0 & \text{if } s_i - s_{i-1} = (0,2). \end{cases}$$

Similarly we have the following theorem.

Theorem 1.2.2 (Viennot, 1984) *The Stirling number of the first kind $c(n,k)$ is given by*

$$c(n,k) = \sum_p v(p),$$

where the sum is over all Favard paths starting at level 0 and ending at level $n-1$ with $k-1$ $(1,1)$-steps.

Figures 1.1 and 1.2 give examples of determining the Stirling numbers $S(5,3)$ and $c(5,3)$ using Motzkin paths and Favard paths respectively.

The *q-Stirling number of the second kind* is given by Cigler [15] using the recurrence formula

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q \cdot S_q[n-1,k], \text{ for } 1 \le k \le n,$$

with boundary conditions $S_q[n,0] = \delta_{n,0}$ and $S_q[0,k] = \delta_{0,k}$. In the same paper, Cigler also gave recurrence formula for the *q-Stirling number of the first kind.*

$$c_q[n,k] = c_q[n-1,k-1] + [n-1]_q \cdot c_q[n-1,k], \text{ for } 1 \le k \le n,$$

where $c_q[n,0] = \delta_{n,0}$ and $c_q[0,k] = \delta_{0,k}$. and the *signed q-Stirling number of the first kind* is $s_q[n,k] = (-1)^{n-k} \cdot c_q[n,k]$.

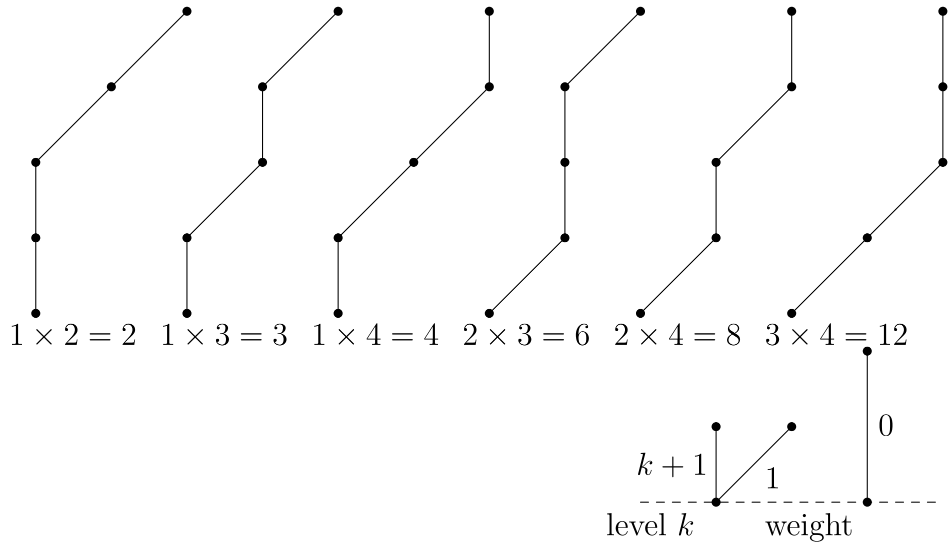


Figure 1.2: The non-zero weighted Farvard paths starting at $(0, 0)$ and ending at level 4 with two $(1, 1)$-steps. Summing over the weights gives $c(5, 3) = 35$.

The q-Stirling numbers have been studied extensively. Milne [50, 51, 52] considered q-identities using restricted growth functions, while Wachs and White [79] found different statistics on *RG*-words to express q-Stirling numbers $S_q[n, k]$. Garsia and Remmel [31] as well as de Médicis and Leroux [17] studied the rook placements and 01-matrices interpretation for both $S_q[n, k]$ and $c_q[n, k]$. Ehrenborg and Readdy [21] gave a juggling pattern interpretation of the q-Stirling number of the second kind. We now give an overview of these formulations.

Recall a *set partition* of the elements $\{1, 2, \ldots, n\}$ is a decomposition of this set into mutually disjoint nonempty sets called blocks. Let $\Pi_{n,k}$ denote the set of all partitions of $\{1, 2, \cdots, n\}$ into k disjoint blocks. A partition $\pi \in \Pi_{n,k}$ will be represented using the standard notation, $\pi = B_1/B_2/\cdots/B_k$, where the blocks are ordered so that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$.

Given a partition $\pi \in \Pi_{n,k}$, we can encode it using a *restricted growth word* or *RG*-word $w = w(\pi) = w_1 w_2 \cdots w_n$, where $w_i = j$ if the element i occurs in the jth block B_j of π. For example, the partition $\pi = 14/236/57$ has *RG*-word $w = w(\pi) = 1221323$. Let $\mathcal{R}(n, k)$ denote the set of all *RG*-words of length n with maximal letter k.

Restricted growth words are also known as *restricted growth functions.* A restricted growth function $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, k\}$ is a surjective map which satisfies $f(1) = 1$ and $f(i) \leq \max(f(1), f(2), \ldots, f(i-1)) + 1$ for $i = 2, 3, \ldots, n$. They have been studied by Hutchinson [37] and Milne [50, 51].

Wachs and White [79] studied four statistics on *RG*-words that enumerate the q-Stirling numbers. For an *RG*-word $w \in \mathcal{R}(n, k)$, define

$$L(w) = \{i : w_i \text{ is the leftmost occurrence of } w_i\}$$

w	$lb(w)$	$ls(w)$	$rb(w)$	$rs(w)$	$\mathrm{maj}(w)$	$\widehat{\mathrm{maj}}(w)$
1123	0	4	5	0	0	3
1213	1	3	4	1	1	3
1231	2	3	3	2	1	3
1223	0	4	4	0	0	4
1232	1	4	3	1	2	4
1233	0	5	3	0	0	5

Table 1.4: Six statistics on RG-words in $\mathcal{R}(4,3)$ which give $S_q[4,3] = q^2 + 2q + 3$.

and

$$R(w) = \{i : w_i \text{ is the rightmost occurrence of } w_i\}.$$

For any $1 \leq i \leq n$, let

$$\begin{aligned} lb_i(w) &= |\{j \in L(w) : j < i \text{ and } w_j > w_i\}|, \\ ls_i(w) &= |\{j \in L(w) : j < i \text{ and } w_j < w_i\}|, \\ rb_i(w) &= |\{j \in R(w) : j > i \text{ and } w_j > w_i\}|, \\ rs_i(w) &= |\{j \in R(w) : j > i \text{ and } w_j < w_i\}|. \end{aligned}$$

The letters l, r, b, s stand for "left", "right", "bigger" and "smaller" respectively. For an RG-word $w \in \mathcal{R}(n,k)$, define

$$\begin{aligned} lb(w) &= \sum_{i=1}^{n} lb_i(w), \qquad ls(w) = \sum_{i=1}^{n} ls_i(w), \\ rb(w) &= \sum_{i=1}^{n} rb_i(w), \qquad rs(w) = \sum_{i=1}^{n} rs_i(w). \end{aligned}$$

Relations between the lb, ls statistics and the q-Stirling numbers of the second kind were first proved by Milne [52, Theorem 4.6]. Wachs and White gave similar results using the rb and rs statistics [79, Corollary 5.4].

Theorem 1.2.3 (Milne, 1982; Wachs–White, 1991) *The lb, ls, rb, rs statistics satisfy the following identities.*

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} q^{lb(w)} = \sum_{w \in \mathcal{R}(n,k)} q^{rs(w)},$$

and

$$q^{\binom{k}{2}} \cdot S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} q^{ls(w)} = \sum_{w \in \mathcal{R}(n,k)} q^{rb(w)}.$$

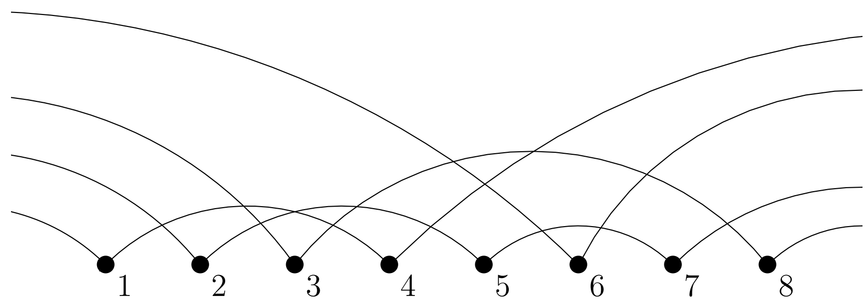


Figure 1.3: The juggling pattern associated to the set partition 14/257/38/6.

Sagan [64] introduced an analogue of the major index on RG-words. For an RG-word $w \in \mathcal{R}(n,k)$, the *major index* and the *dual major index* are

$$\mathrm{maj}(w) = \sum_{lb_i(w)>0} w_i, \quad \text{and} \quad \widehat{\mathrm{maj}}(w) = \sum_{ls_i(w)>0} (w_i - 1).$$

Theorem 1.2.4 (Sagan, 1991) *The q-Stirling numbers of the second kind satisfy the following relations.*

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} q^{\mathrm{maj}(w)}, \quad \textit{and} \quad q^{\binom{k}{2}} \cdot S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} q^{\widehat{\mathrm{maj}}(w)}.$$

Table 1.4 illustrates the above six statistics on the set of RG-words $\mathcal{R}(4,3)$. As a remark, it is only a coincidence in the table that $lb(w) = rs(w)$. This is not true in general. For example, $lb(123112) = 5$ but $rs(123112) = 4$.

White [80] introduced classes of statistics to interpolate between the above statistics on RG-words and gave sufficient conditions for the classes to be q-Stirling distributed.

For a partition $\pi \in \Pi_{n,k}$, associate it with a *juggling pattern* consisting of k paths with each path corresponding to a block of the partition. The ith path joins the vertices corresponding to the elements in block B_i. See Figure 1.3 for an example. Let $\mathrm{cross}(\pi)$ denote the number of crossings in the juggling pattern that is associated with the partition π. In [21] Ehrenborg and Readdy gave the following relation between juggling patterns and the q-Stirling numbers of the second kind.

Theorem 1.2.5 (Ehrenborg–Readdy, 1996) *The q-Stirling number of the second kind satisfies the following relation.*

$$q^{\binom{k}{2}} \cdot S_q[n,k] = \sum_{\pi \in \Pi_{n,k}} q^{\mathrm{cross}(\pi)}.$$

Another combinatorial structure that is used to study Stirling numbers of the first kind and the second kind is rook placements. A *staircase chessboard* of length m is

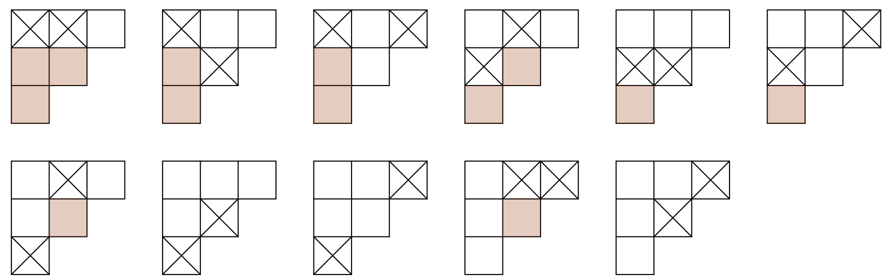


Figure 1.4: All rook placements in the set $P(4,2)$ where the shaded red squares indicate $\mathrm{s}(T)$. The total weight is $c_q[4,2] = q^3 + 3q^2 + 4q + 3$.

a board with $m - i$ squares in the ith row for $i = 1, 2, \ldots, m-1$ and each row of squares is left-justified. A *rook placement* is a way to place rooks on a chessboard that satisfies certain conditions.

Definition 1.2.6 *Let $P(m,n)$ be the set of all ways to place n rooks onto a staircase chessboard of length m so that no two rooks are in the same column. Let $Pd(m,n) \subseteq P(m,n)$ be the subset with no two rooks in the same row.*

For any rook placement T, denote by $\mathrm{s}(T)$ the number of squares to the south of the rooks in T. Let $\mathrm{d}(T)$ denote the number of squares to the south of a rook with no other rooks lying to its east. In [31, Page 247] Garsia and Remmel gave combinatorial interpretations of q-Stirling numbers via rook placements.

Theorem 1.2.7 (Garsia–Remmel, 1986) *The q-Stirling numbers of the first kind and the second kind are given by*

$$c_q[n,k] = \sum_{T \in P(n,n-k)} q^{\mathrm{s}(T)},$$

and

$$S_q[n,k] = \sum_{T \in Pd(n,n-k)} q^{\mathrm{d}(T)}.$$

See Figures 1.4 and 1.5 for examples. A similar construction using 0-1 tableaux was given by de Médicis and Leroux [17, Page 90].

The q-Stirling numbers have the generating functions

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k] \cdot x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_q[n,k] \cdot (x)_{k,q},$$

where $(x)_{k,q} = \displaystyle\prod_{m=0}^{k-1} (x - [m]_q)$.

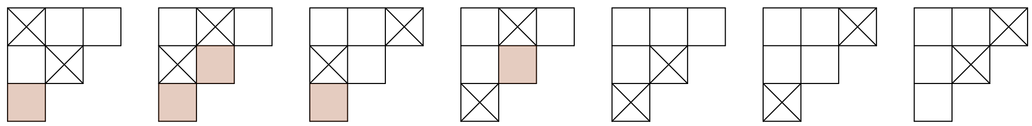


Figure 1.5: All rook placements in the set $Pd(4, 2)$ where the shaded red squares indicate $\mathrm{d}(T)$. The total weight is $S_q[4, 2] = q^2 + 3q + 3$.

In [78] Viennot has some beautiful results in which he gave combinatorial constructions for orthogonal polynomials and their moment generating functions. One well-known relation between the ordinary signed Stirling numbers of the first kind and Stirling numbers of the second kind is their orthogonality. A bijective proof of the orthogonality of their q-analogues via 0-1 tableaux was given by de Médicis and Leroux [17, Proposition 3.1].

Theorem 1.2.8 (de Médicis–Leroux, 1993) *The q-Stirling numbers are orthogonal, that is, for $m \leq n$*

$$\sum_{k=m}^{n} s_q[n, k] \cdot S_q[k, m] = \delta_{m,n}$$

and

$$\sum_{k=m}^{n} S_q[n, k] \cdot s_q[k, m] = \delta_{m,n}.$$

Moreover, there is a combinatorial proof of this result.

1.3 Poset preliminaries

In this section we give some basic background on posets. For further details, we refer the reader to Stanley's treatise [69].

A *partially ordered set* or *poset* is a finite set P of elements with a binary relation denoted $<_P$ or $<$ that is reflexive, antisymmetric and transitive. The unique minimal element in a poset P is denoted by $\widehat{0}$ and the unique maximal element is denoted by $\widehat{1}$. A *chain* of length k in a poset P, denoted C_k, is a string of comparable elements $x_0 < x_1 < \cdots < x_k$. We say the element y *covers* x or x *is covered by* y, denoted $x \prec y$ or $y \succ x$, if there is no element $z \in P$ such that $x < z < y$. A *maximal* or *saturated chain* in P is of the form $\widehat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = \widehat{1}$. In this case we say the chain has length n. For elements $x, y \in P$ the *interval* $[x, y]$ is the set of all elements $z \in P$ such that $x \leq z \leq y$.

A poset P is *graded* if all of its maximal chains have the same length. The *rank* of an element x in a graded poset P, denoted $\rho(x)$, is the length of any maximal chain from the unique minimal element $\widehat{0}$ to x. The poset is graded of rank n if $\rho(\widehat{1}) = n$.

Figure 1.6: The chain of length three. Here $F(P, x) = 1 + x + x^2 + x^3$.

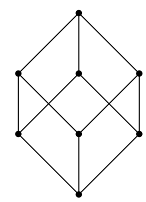


Figure 1.7: The Boolean algebra B_3 on three elements. Here $F(P, x) = 1+3x+3x^2+x^3 = (1 + x)^3$.

The *rank generating function* of a graded poset is given by the polynomial

$$F(P, x) = \sum_{a \in P} x^{\rho(a)}.$$

The *Cartesian product* or *direct product* of two posets P and Q is the poset $P \times Q$ on the set $\{(s, t) : s \in P \text{ and } t \in Q\}$ such that $(s, t) \leq (s', t')$ in $P \times Q$ if $s \leq s'$ in P and $t \leq t'$ in Q. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P and the edges are the cover relations. For example, the *Boolean algebra* B_n is the poset with elements all subsets of the set $\{1, 2, \ldots, n\}$ such that $S \leq T$ in B_n if $S \subseteq T$ as sets, that is, B_n consists of the subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. It is straightforward to check that $\widehat{0}$ is the empty set and $\widehat{1}$ is the whole set $\{1, 2, \ldots, n\}$. Figure 1.6 and Figure 1.7 are examples of the Hasse diagrams of a chain and a Boolean algebra. Both are graded posets.

1.4 Discrete Morse Theory

In general, "spaces" are studied in topology and "functions" are studied in analysis. However, there are deep relations between functions defined on a space and the structure of the space. The study of these relations is the goal of *Morse theory*, which was introduced by Marston Morse [53]. In particular, by understanding the critical points of a smooth function on the manifold, called a Morse function, one can recover the topology of the space.

There are many classical applications and different approaches of Morse theory. For example, the Morse-Bott theory, introduced by Bott [7, 8] considered smooth functions on a manifold whose critical set is a closed submanifold, Kühnel studied the Morse theory of piecewise linear functions [44], and Goresky and MacPherson [33] developed stratified Morse theory. Among these developments, *discrete Morse theory*, adapted by Robin Forman [27], gives combinatorial application of the Morse theory.

Discrete Morse theory is stated in the language of CW complexes. Instead of defining a continuous function on a space as the Morse function, one can assign a single number to each cell of the complex, that is, a discrete version of the Morse function. The goal of this setting is to find a CW decomposition of a simplicial complex with fewer cells than in the original simplicial decomposition. As a consequence, one can also study the homology of the order complex of a poset, which has various applications in poset topology. We now introduce some basic concepts and results in discrete Morse theory.

We first define some topological structures that are used in the discrete Morse theory. For more details, we refer to [43, Chapter 11].

Definition 1.4.1 *Let X and Y be topological spaces, let $A \subseteq X$ be a closed subspace and let $f : A \longrightarrow Y$ be a continuous map. Then $Y \cup_f X$ denotes the quotient space $X \dot{\cup} Y/\sim$ where the equivalence relation is given by $a \sim f(a)$ for all $a \in A$. We say the space $Y \cup_f X$ is obtained from Y by attaching X along f.*

Let an *n-cell* be a topological space homeomorphic to an n-dimensional closed ball B^n. We define the CW complex constructively.

Definition 1.4.2 *A CW complex X is obtained by a construction of the skeletons in the following procedure.*

1. *Start with a discrete set X^0, whose elements are considered as 0-cells.*
2. *Construct the n-skeleton X^n from X^{n-1} inductively by attaching n-cells e^n_α to X^{n-1} via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$. Thus $X^n = X^{n-1} \cup_{\varphi_\alpha} e^n_\alpha$.*
3. *Stop the process at a finite integer n and let $X = X^n$, or set $X = \cup_{n=0}^{\infty} X^n$. In the latter case, X is equipped with weak topology. That is, $A \subseteq X$ is open if and only if $A \cap X^n$ is open in X^n for every n.*

A *finite simplicial complex* is a finite set of vertices V with a set of subsets Δ of V such that (i) $V \subseteq \Delta$, (ii) if $\alpha \in \Delta$ and $\beta \subseteq \alpha$ then $\beta \in \Delta$. The simplicial complex is denoted by Δ and the elements of Δ are called simplices.

The *face poset* of Δ, denoted $\mathcal{F}(\Delta)$, is the set of elements consisting all nonempty simplices of K ordered by inclusion.

Given a poset P be a poset, the *order complex of* P, denoted $\Delta(P)$, is the simplical complex whose vertices are all the elements of P and whose simplices are all finite chains of P, including the empty chain.

Theorem 1.4.3 *A simplicial complex Δ is homeomorphic to the order complex of its face poset, that is,*

$$\Delta \cong \Delta(\mathcal{F}(\Delta)).$$

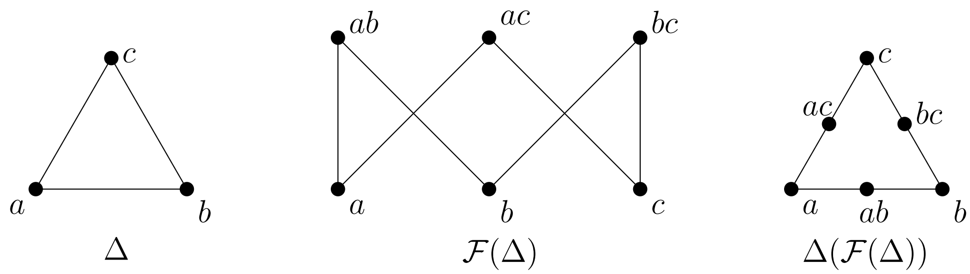


Figure 1.8: The boundary of a triangle, its face poset, and the order complex of the face poset.

See Figure 1.8 for an example.

Given a poset P, a *partial matching* is a matching $M \subseteq P \times P$ on the Hasse diagram of P satisfying (i) the ordered pair $(a, b) \in M$ implies $a \prec b$, and (ii) each element $a \in P$ belongs to at most one element in M. When $(a, b) \in M$, we write $u(a) = b$ and $d(b) = a$. A partial matching on P is *acyclic* if there does not exist a cycle

$$a_1 \prec u(a_1) \succ a_2 \prec u(a_2) \succ \cdots \succ a_n \prec u(a_n) \succ a_1$$

with $n \geq 2$, and the elements $a_1, a_2, \ldots, a_n$ are distinct.

An alternate description is to orient all the edges in the Hasse diagram of a poset downwards and then reorient all the edges occurring in the matching upwards. The acyclic condition is simply that there is no cycle on the directed Hasse diagram. For the matched edge (a, b) the notation $u(a) = b$ and $d(b) = a$ denotes the fact that in the edge oriented from a to b the element b is "upwards" from a and similarly the element a is "downwards" from b. One can use the terminology of a *gradient path* or *V-path* consisting alternatively of matched and unmatched elements from the poset [26]. A *discrete Morse matching* or *acyclic matching* is one where no gradient path forms a cycle. An element in the poset is *critical* if it is unmatched in the partial matching.

The following theorem characterizes acyclic matchings via linear extensions, see [43, Section 11.1].

Theorem 1.4.4 *A partial matching on P is acyclic if and only if there exists a linear extension L of P such that the elements a and $u(a)$ follow consecutively in L.*

With the above definitions, we are ready to state the main theorem of discrete Morse theory.

Theorem 1.4.5 (Forman, 2002) *Let M be an acyclic partial matching on $\mathcal{F}(\Delta)$ and let c_i denote the number of critical i-dimensional cells of Δ. Then Δ is homotopy equivalent to a CW complex with exactly c_i cells of dimension i for each $i \geq 0$.*

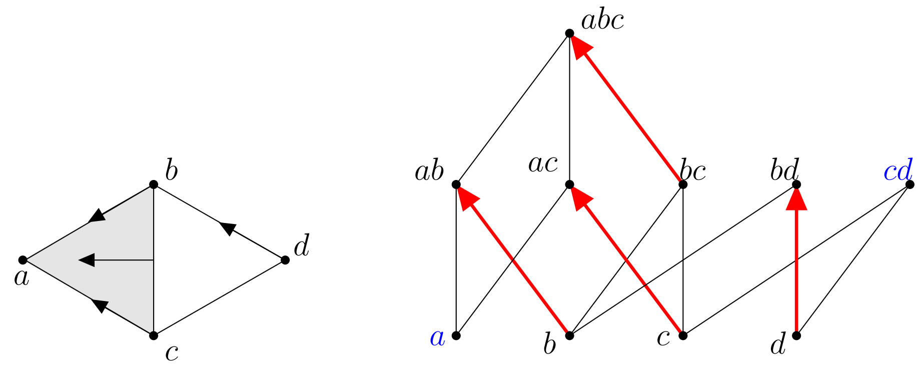


Figure 1.9: A collapse of a simplicial complex and the corresponding discrete Morse matching on the face poset with two critical cells.

See Figure 1.9 for a straightforward example. For more examples, we refer to [43, Section 11.2.3]

We have the following immediate corollary.

Corollary 1.4.6 (Forman, 2002) *If the face poset $\mathcal{F}(\Delta)$ admits an acyclic matching which has no critical elements then Δ is contractible.*

1.5 Cyclic sieving phenomenon

In [72] Stembridge studied symmetry classes of plane partitions. A *plane partition* contained in a box B with dimensions a, b, c is an array $\pi = (\pi_{ij})_{1\le i\le a, 1\le j\le b}$ such that π is weakly decreasing in rows and columns and $0 \le \pi_{ij} \le c$ for all i, j. The *size* of a plane partition π, denoted $|\pi|$, is the sum of all entries in π. Macdonald [46, Example I.5.13] defined the generating function of the plane partitions in B as

$$F_B(q) = \sum_{\pi\in B} q^{|\pi|}.$$

Stembridge found that setting $q = -1$ in $F_B(q)$ gives the number of self-complementary plane partitions, that is, a partition π that is invariant under replacing π_{ij} by $c - \pi_{ij}$ and rotating the rectangular array by 180°. As an example, Table 1.5 gives all plane partitions in the box $B = 2 \times 2 \times 2$ with the generating function

$$F_B(q) = 1 + q + 3q^2 + 3q^3 + 4q^4 + 3q^5 + 3q^6 + q^7 + q^8.$$

Setting $q = -1$ one obtains $F_B(-1) = 4$ which is the number of self-complementary plane partitions in this case. This became known as the $q = -1$ phenomenon.

In general, a finite collection of combinatorial objects X with generating function $F(q)$ exhibits the $q = -1$ *phenomenon* if one can find an involution on X such that the number of fixed points in the involution equals $F(-1)$. In [73] Stembridge provided beautiful representation-theoretic explanations for this phenomenon on plane

π	π^c	π	π^c
$\begin{pmatrix}0&0\\0&0\end{pmatrix}$	$\begin{pmatrix}2&2\\2&2\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}2&2\\2&1\end{pmatrix}$
$\begin{pmatrix}2&0\\0&0\end{pmatrix}$	$\begin{pmatrix}2&2\\2&0\end{pmatrix}$	$\begin{pmatrix}1&1\\0&0\end{pmatrix}$	$\begin{pmatrix}2&2\\1&1\end{pmatrix}$
$\begin{pmatrix}1&0\\1&0\end{pmatrix}$	$\begin{pmatrix}2&1\\2&1\end{pmatrix}$	$\begin{pmatrix}2&1\\0&0\end{pmatrix}$	$\begin{pmatrix}2&2\\1&0\end{pmatrix}$
$\begin{pmatrix}2&0\\1&0\end{pmatrix}$	$\begin{pmatrix}2&1\\2&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$\begin{pmatrix}2&1\\1&1\end{pmatrix}$
$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$\begin{pmatrix}2&1\\1&0\end{pmatrix}$	$\begin{pmatrix}2&2\\0&0\end{pmatrix}$	$\begin{pmatrix}2&2\\0&0\end{pmatrix}$
$\begin{pmatrix}2&0\\2&0\end{pmatrix}$	$\begin{pmatrix}2&0\\2&0\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\begin{pmatrix}1&1\\1&1\end{pmatrix}$

Table 1.5: Twenty plane partitions in the box $B = 2 \times 2 \times 2$ and their complementary. Four self-complementary plane partitions are listed in the last two rows.

partitions and a more general case on minuscule posets. In [74] he presented another example of the $q = -1$ phenomenon via semistandard tableaux.

The cyclic sieving phenomenon, introduced by Reiner, Stanton and White [61], is a generalization of Stembridge's $q = -1$ phenomenon. It relates the q-enumeration of combinatorial objects with its symmetry classes by evaluating the generating function at roots of unity. We now describe their theory in detail.

Let X be a finite set with an action of a cyclic group C of order n. The *stabilizer-order* of an orbit is the cardinality of the set $\{c \in C : c(x) = x\}$. Let $X(q)$ be a polynomial in q associated with the set X. $X(q)$ is usually a q-enumeration of the set X such that $X(q)$ has nonnegative integer coefficients and $X(1) = |X|$. We say the triple $(X, X(q), C)$ exhibits the *cyclic sieving phenomenon* if for every $c \in C$, $X(\omega) = |\{x \in X : c(x) = x\}|$, where ω is a complex root of unity with the same multiplicative order as c.

The following theorem is an example of the cyclic sieving phenomenon [61, Theorem 1.1(b)].

Theorem 1.5.1 (Reiner–Stanton–White, 2004) *Let X be the collections of all k-element subsets of $\{1, 2, \ldots, n\}$ and $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$. Let the generator c of C act by cycling the elements of a k-subset modulo n. Then the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

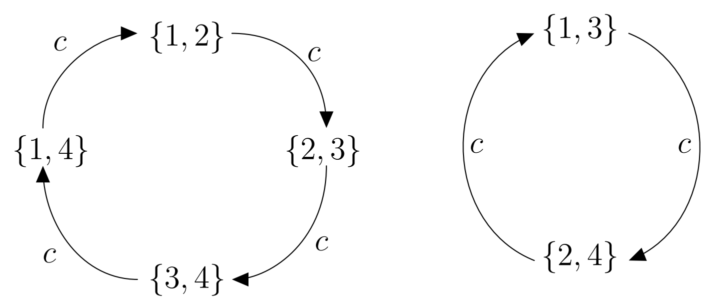


Figure 1.10: An element c of order 4 acting on the 2-subsets of $\{1, 2, 3, 4\}$.

For example, consider $X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$ and the element c of multiplicative order 4 acting on the 2-subsets of $\{1, 2, 3, 4\}$ as shown in Figure 1.10. It is straightforward to see that all six elements are fixed by c^4, giving $X(1) = 6$, while $X(-1) = 2$ counts the two subsets $\{1, 3\}$ and $\{2, 4\}$ that are fixed by c^2. There are no elements fixed by c or c^3 and we have $X(e^{\frac{2\pi i}{4}}) = 0 = X(e^{\frac{3\pi i}{4}})$.

Another example appeared in [61, Theorem 7.1] is as follows.

Theorem 1.5.2 (Reiner–Stanton–White, 2004) *Let X be the set of triangulations of a regular $(n+2)$-gon, with C a cyclic group of order $n+2$ permuting triangulations. Let*

$$X(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$

Then the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

As a remark, the polynomial $X(q)$ in this instance is the q-Catalan number considered by MacMahon [48, Page 214] as the major generating function on Dyck paths.

As a generalization of Stembridge's $q = -1$ phenomenon, some instances of the cyclic sieving phenomenon have representation-theoretic proofs. However, as the authors pointed out in [62], many known instances of the cyclic sieving phenomenon are only verified by evaluating $X(\omega^d)$ where ω is a primitive root of unity and comparing with known symmetry classes. It still remains to find a insight of these instances. We give the following example from [62].

Let X be the set of $n \times n$ alternating sign matrices, that is, the matrices with entries $0, 1$ and -1 such that the row sums and column sums are all 1, and the nonzero entries alternate in sign in rows and columns. Let C be the cyclic group of order 4 whose generator c rotates matrices by $90°$. Let

$$X(q) = \prod_{k=0}^{n-1} \frac{[3k+1]_q!}{[n+k]_q!}.$$

Then the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon. However, there is no known statistic on alternating sign matrices that gives $X(q)$ and we do not have an algebraic proof on this result.

1.6 Symmetric functions

The theory of symmetric functions plays an important role in enumerative combinatorics, as well as other aspects in mathematics such as group theory and representation theory. In [46, Chapter I] Macdonald gave formalism of symmetric functions via integer partitions. He also gave a lot examples of applications of symmetric functions, in particular, many known polynomials and formulas can be considered as specialization of symmetric functions by setting the indeterminants to different values. In [70] Stanley had many combinatorial applications and more recent development of the symmetric function theory. Here we briefly state some basic concepts in symmetric function theory.

Let $\Lambda = \mathbb{Z}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in the n variables $x_1, x_2, \ldots, x_n$ with integer coefficients. Let the symmetric group $\mathfrak{S}_n$ act on the n variables in Λ. A polynomial in Λ is *symmetric* if it is invariant under the action of symmetric group $\mathfrak{S}_n$. Let $\Lambda^n_{\mathbb{Z}}$ denote the ring of symmetric polynomials in n variables with integer coefficients. The symmetric polynomials form a subring of the polynomial ring. We now give examples of classical symmetric functions. For more details, see [46, Chapter I].

The *j-th elementary symmetric function* e_j is given by

$$e_j = \sum_{i_1 < i_2 < \cdots < i_j} x_{i_1} \cdot x_{i_2} \cdots x_{i_j},$$

with $e_0 = 1$. A *partition* of an integer $m \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of integers λ_i such that $\sum \lambda_i = m$ and $\lambda_1 \geq \lambda_2 \geq \cdots$. A partition can also be written as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ if $\lambda_i = 0$ for $i > k$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of j allowing zero parts. The *monomial symmetric function* m_λ is given by

$$m_\lambda = \sum_{\alpha} x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha^n},$$

where the sum is over all distinct permutations $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of λ and where we define $m_\emptyset = 1$. The *j-th complete homogeneous symmetric function* h_j is the sum of all distinct monomials of degree j, that is,

$$h_j = \sum_{\lambda} m_\lambda$$

where the sum is over all partitions λ of j. Define $h_0 = 1$. The elementary symmetric functions, monomial symmetric functions and complete homogeneous symmetric functions are all examples of basis of the symmetric polynomial ring.

We can also find the closed forms of the generating functions for e_j and h_j [46, Equations (I.2.2) and (I.2.5)].

$$E(t) = \sum_{j \geq 0} e_j \cdot t^j = \prod_{i=1}^{n} (1 + x_i \cdot t), \quad \text{and} \quad H(t) = \sum_{j \geq 0} h_j \cdot t^j = \prod_{i=1}^{n} (1 - x_i \cdot t)^{-1}.$$

Hence $H(t) \cdot E(-t) = 1$. Compare the coefficients of t^n on both sides gives

$$\sum_{r \geq 0} (-1)^r \cdot e_r \cdot h_{n-r} = 0 \qquad \text{for } n > 0. \tag{1.6.1}$$

The Stirling numbers of the first kind and second kind are known to be specializations of the complete homogeneous and elementary symmetric functions [46, Chapter I, Section 2, Example 11]:

$$S(n,k) = h_{n-k}(x_1, x_2, \ldots, x_k), \qquad c(n,k) = e_{n-k}(x_1, x_2, \ldots, x_{n-1}),$$

where $x_m = m$. The q-Stirling numbers are also specializations of these symmetric functions with the substitution $x_m = [m]_q$. One can rewrite Equation (1.6.1) as

$$\sum_{k=j}^{n} (-1)^{n-k} e_{n-k}(x_1, x_2, \ldots, x_{n-1}) \cdot h_{k-j}(x_1, x_2, \ldots, x_j) = \delta_{n,j}.$$

Thus the orthogonality of q-Stirling numbers follows.

Chapter 2 q-Stirling numbers: A new view

2.1 Introduction

The idea of q-analogues can be traced back to Euler in the 1700's who was studying q-series, especially specializations of theta functions. The Gaussian polynomial or q-binomial is the familiar q-analogue of the binomial coefficient given by $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, where $[n]_q = 1+q+\cdots+q^{n-1}$ and $[n]_q! = [1]_q \cdot [2]_q \cdots [n]_q$. A combinatorial interpretation due to MacMahon in 1916 [47, Page 315] is

$$\sum_{\pi \in \mathfrak{S}(0^{n-k},1^k)} q^{\mathrm{inv}(\pi)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Here $\mathfrak{S}(0^{n-k}, 1^k)$ denotes the number of 0-1 bit strings consisting of $n-k$ zeroes and k ones, and for $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}(0^{n-k}, 1^k)$ the number of inversions is $\mathrm{inv}(\pi) = |\{(i,j) : i < j \text{ and } \pi_i > \pi_j\}|$. The inversion statistic goes back to work of Cramer (1750), Bézout (1764) and Laplace (1772). See the discussion in [54, Page 92]. Netto enumerated the elements of the symmetric group by the inversion statistic in 1901 [54, Chapter 4, Sections 54 and 57], and in 1916 MacMahon [47, Page 318] gave the q-factorial expansion $\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = [n]_q!$.

The notion of the *negative q-binomial* has been recently introduced by Fu, Reiner, Stanton and Thiem [28]. It is defined by substituting $-q$ for q in the Gaussian coefficient and adjusting the sign:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q' = (-1)^{k(n-k)} \cdot \begin{bmatrix} n \\ k \end{bmatrix}_{-q} .$$

The negative q-binomial enjoys properties similar to that of the q-binomial: (i) it can be expressed as a generalized inversion number of a *subset* $\Omega(n,k)'$ of 0-1 bit strings in $\mathfrak{S}(0^{n-k}, 1^k)$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q' = \sum_{\pi \in \Omega(n,k)'} q^{a(\pi)} \cdot (q-1)^{p(\pi)}, \tag{2.1.1}$$

for statistics $a(\cdot)$ and $p(\cdot)$ [28, Theorem 1], (ii) it counts a certain subset of the k-dimensional subspaces of $\mathbb{F}_q^n$ [28, Section 6.2], (iii) it reveals a representation theory connection with unitary subspaces and a two-variable version exhibits a cyclic sieving phenomenon [28, Sections 4, 5].

An important consequence of (2.1.1) is the classical Gaussian polynomial can be expressed as sum over a *subset* of 0-1 bit strings in terms of powers of q and $1+q$ using the same statistics:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\pi \in \Omega(n,k)'} q^{a(\pi)} \cdot (1+q)^{p(\pi)}. \tag{2.1.2}$$

It is from this result that we springboard our work.

Goal 2.1.1 *Given a q-analogue*

$$f(q) = \sum_{w \in S} q^{\sigma(w)},$$

for some statistic $\sigma(\cdot)$, find a subset $T \subseteq S$ and statistics $A(\cdot)$ and $B(\cdot)$ so that the q-analogue may be expressed as

$$f(q) = \sum_{w \in T} q^{A(w)} \cdot (1+q)^{B(w)}.$$

The overall goal is not only to discover more compact encodings of classical q-analogues, but to also understand them via enumerative, poset theoretic and topological viewpoints. In this chapter we do exactly this for the q-Stirling numbers of the first and second kinds.

In Section 2.2 we recall the notion of restricted growth words or *RG*-words to encode set partitions. A weighted version yields the usual q-Stirling numbers of the second kind; see Lemma 2.2.3. In Section 2.3 we describe a subset of *RG*-words, which we call *allowable*, whose weighting gives the q-Stirling numbers of the second kind and hence a more compact presentation of the q-Stirling numbers of the second kind; see Theorem 2.3.2.

We then take a poset theoretic viewpoint in Section 2.4 where we introduce the Stirling poset of the second kind $\Pi(n, k)$. Its rank generating function is precisely the q-Stirling number $S_q[n, k]$. Using a discrete Morse theory, we show in Theorem 2.4.3 that the Stirling poset of the second kind has an acyclic matching. In Section 2.5 we give a decomposition of the Stirling poset into Boolean algebras with the minimal element of each Boolean algebra corresponding to an allowable *RG*-word; see Theorem 2.5.1. A generating function for the q-analogue of critical cells is provided.

In Section 2.6 we review the notion of an algebraic complex supported on a poset. Using Hersh, Shareshian and Stanton's homological interpretation of Stembridge's $q = -1$ phenomenon, we show in Theorem 2.6.5 that the Stirling poset $\Pi(n, k)$ supports an algebraic complex and give a basis for the integer homology, all of which occurs in even dimensions.

In Section 2.7 we review the de Médicis–Leroux rook placement interpretation of the q-Stirling numbers of the first kind. In Theorem 2.7.4 we show a subset of these boards, with the appropriate weighting, yields a compact representation of the q-Stirling number of the first kind. In Section 2.8 we introduce the Stirling poset of the first kind $\Gamma(m, n)$ whose rank generating function is precisely the q-Stirling number $c_q[n, k]$. Again, a decomposition of this graded poset is given. We show the Stirling poset of the first kind supports an algebraic complex and describe a basis for the integer homology which occurs in even dimensions. See Theorems 2.8.4 and 2.8.7. In Section 2.9 we introduce (q, t)-analogues of the Stirling numbers of the first and second kinds and show orthogonality holds combinatorially. We end with concluding remarks.

This chapter is joint work with Readdy. A preliminary version of this work appears in [10]. See also [9].

2.2 Restricted growth words

Recall a *set partition* of the n elements $\{1, 2, \ldots, n\}$ is a decomposition of this set into mutually disjoint nonempty sets called blocks. Unless otherwise indicated, throughout all set partitions will be written in standard form, that is, a partition into k blocks will be denoted by $\pi = B_1/B_2/\cdots/B_k$, where the blocks are ordered so that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. We denote the set of all partitions of $\{1, 2, \ldots, n\}$ by Π_n.

Given a partition $\pi \in \Pi_n$, we encode it using a *restricted growth word* $w(\pi) = w_1 w_2 \cdots w_n$, where $w_i = j$ if the element i occurs in the jth block B_j of π. For example, the partition $\pi = 14/236/57$ has RG-word $w = w(\pi) = 1221323$. Restricted growth words are also known as restricted growth functions. Recall a *restricted growth function* $f : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, k\}$ is a surjective map which satisfies $f(1) = 1$ and $f(i) \leq \max(f(1), f(2), \ldots, f(i-1)) + 1$ for $i = 2, 3, \ldots, n$. They have been studied by Hutchinson [37] and Milne [50, 51].

Two facts about RG-words follow immediately from using the standard form for set partitions.

Proposition 2.2.1 *The following properties are satisfied by RG-words:*

1. *Any RG-word begins with the element* 1.

2. *For an RG-word w let $\epsilon(j)$ be the smallest index such that $w_{\epsilon(j)} = j$. Then the $\epsilon(j)$ form an increasing sequence, that is,*

$$\epsilon(1) < \epsilon(2) < \cdots .$$

The *q-Stirling numbers of the second kind* are defined by

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q \cdot S_q[n-1, k], \text{ for } 1 \leq k \leq n, \tag{2.2.1}$$

with boundary conditions $S_q[n, 0] = \delta_{n,0}$ and $S_q[0, k] = \delta_{0,k}$, where $\delta_{i,j}$ is the usual Kronecker delta function. Setting $q = 1$ gives the familiar Stirling number of the second kind $S(n, k)$ which enumerates the number of partitions $\pi \in \Pi_n$ with exactly k blocks. There is a long history of studying set partition statistics [31, 45, 63] and q-Stirling numbers [11, 21, 34, 51, 79].

We begin by presenting a statistic on RG-words which generates the q-Stirling numbers of the second kind. Let $\mathcal{R}(n, k)$ denote the set of all RG-words of length n with maximum letter k, which corresponds to set partitions of $\{1, 2, \ldots, n\}$ into k blocks. For $w \in \mathcal{R}(n, k)$, let $m_i = \max(w_1, w_2, \ldots, w_i)$ and form the weight $\mathrm{wt}(w) = \prod_{i=1}^{n} \mathrm{wt}_i(w)$, where $\mathrm{wt}_1(w) = 1$ and for $2 \leq i \leq n$, let

$$\mathrm{wt}_i(w) = \begin{cases} q^{w_i - 1} & \text{if } m_{i-1} \geq w_i, \\ 1 & \text{if } m_{i-1} < w_i. \end{cases} \tag{2.2.2}$$

For example, $\mathrm{wt}(1221323) = 1 \cdot 1 \cdot q^1 \cdot q^0 \cdot 1 \cdot q^1 \cdot q^2 = q^4$. In terms of set partitions, the weight of $\pi = B_1/B_2/\cdots/B_k$ is $\mathrm{wt}(\pi) = \prod_{i=1}^{k} q^{(j-1)\cdot(|B_j|-1)}$.

Partition	RG-word w	$\mathrm{wt}(w)$
1/234	1222	$1 \cdot 1 \cdot q \cdot q = q^2$
12/34	1122	$1 \cdot 1 \cdot 1 \cdot q = q$
13/24	1212	$1 \cdot 1 \cdot 1 \cdot q = q$
14/23	1221	$1 \cdot 1 \cdot q \cdot 1 = q$
134/2	1211	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
124/3	1121	$1 \cdot 1 \cdot 1 \cdot 1 = 1$
123/4	1112	$1 \cdot 1 \cdot 1 \cdot 1 = 1$

Table 2.1: Using RG-words to compute $S_q[4,2] = q^2 + 3q + 3$.

Proposition 2.2.2 *For $w = w_1 \cdots w_n \in \mathcal{R}(n,k)$ the weight is given by*

$$\mathrm{wt}(w) = q^{\sum_{i=1}^{n} w_i - n - \binom{k}{2}}.$$

Lemma 2.2.3 *The q-Stirling number of the second kind is given by*

$$S_q[n,k] = \sum_{w \in \mathcal{R}(n,k)} \mathrm{wt}(w).$$

Proof: We show RG-words $w \in \mathcal{R}(n,k)$ satisfy the recurrence (2.2.1). Given an RG-word $w = w_1 w_2 \cdots w_n \in \mathcal{R}(n,k)$, consider the map φ defined by removing the last letter of the word, that is, $\varphi(w) = w_1 w_2 \cdots w_{n-1}$. Clearly $\varphi : \mathcal{R}(n,k) \longrightarrow \mathcal{R}(n-1,k-1) \dot{\cup} \mathcal{R}(n-1,k)$. If the only occurrence of the maximum letter k in the word w is the nth position, that is, $w_n = k$, then these words are in bijection with the set $\mathcal{R}(n-1,k-1)$. Otherwise, $\varphi(w)$ is of length $n-1$ and all the letters from $\{1,2,\ldots,k\}$ occur at least once in $\varphi(w)$. In the first case $\mathrm{wt}(\varphi(w)) = \mathrm{wt}(w)$. In the second case the letter k occurs more than once in w. Given $w' = w_1 w_2 \cdots w_{n-1} \in \mathcal{R}(n-1,k)$ there are k possibilities for the nth letter x in the inverse image $\varphi^{-1}(w') = w_1 w_2 \cdots w_{n-1} x$, namely, $x \in \{1,2,\ldots,k\}$. Each possibility respectively contributes $1, q^1, \ldots, q^{k-1}$ to the weight, giving a total weighting contribution of $[k]_q$. $\square$

See Table 2.1 for the RG-word computation of the q-Stirling number $S_q[4,2]$.

2.3 Allowable RG-words

Mirroring the negative q-binomial, in this section we define a subset of RG-words and two statistics $A(\cdot)$ and $B(\cdot)$ which generate the classical q-Stirling number of the second kind as a polynomial in q and $1+q$. We will see in Sections 2.4 through 2.6 that this has poset and topological implications.

Definition 2.3.1 *An RG-word $w \in \mathcal{R}(n,k)$ is* allowable *if every even entry appears exactly once. Denote by $\mathcal{A}(n,k)$ the set of all allowable RG-words in $\mathcal{R}(n,k)$.*

Another way to state that $w \in \mathcal{R}(n,k)$ is an allowable RG-word is that it is an initial segment of the infinite word

$$w = u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdots ,$$

where u_{2i-1} is a word on the alphabet of the odd integers $\{1, 3, \ldots, 2i-1\}$. In terms of set partitions, an RG-word is allowable if in the corresponding set partition every even indexed block is a singleton block. See Table 2.2.

For an RG-word $w = w_1 \cdots w_n$ define $\mathrm{wt}'(w) = \prod_{i=1}^{n} \mathrm{wt}'_i(w)$, where for $m_i = \max(w_1, \ldots, w_i)$

$$\mathrm{wt}'_i(w) = \begin{cases} q^{w_i - 1} \cdot (1+q) & \text{if } m_{i-1} > w_i, \\ q^{w_i - 1} & \text{if } m_{i-1} = w_i, \\ 1 & \text{if } m_{i-1} < w_i \text{ or } i = 1. \end{cases} \tag{2.3.1}$$

For completeness, we decompose the wt′ statistic into two statistics on RG-words. Let

$$A_i(w) = \begin{cases} w_i - 1 & \text{if } m_{i-1} \geq w_i, \\ 0 & \text{if } m_{i-1} < w_i \text{ or } i = 1, \end{cases} \quad \text{and} \quad B_i(w) = \begin{cases} 1 & \text{if } m_{i-1} > w_i, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.2}$$

Define

$$A(w) = \sum_{i=1}^{n} A_i(w) \quad \text{and} \quad B(w) = \sum_{i=1}^{n} B_i(w).$$

Theorem 2.3.2 *The q-Stirling numbers of the second kind can be expressed as a weighting over the set of allowable RG-words as follows:*

$$S_q[n,k] = \sum_{w \in \mathcal{A}(n,k)} \mathrm{wt}'(w) = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot (1+q)^{B(w)}. \tag{2.3.3}$$

Proof: We proceed by induction on n and k. Clearly the result holds for $S_q[n,1]$ and $S_q[n,n]$ as the corresponding allowable words are $11\cdots1$ and $12\cdots n$, each of weight 1.

For the general case it is enough to show that (2.3.3) satisfies the defining relation (2.2.1) for the q-Stirling numbers of the second kind. We first consider the case when k is even. We split the allowable words according to the value of the last letter, that is, we write $w = u \cdot w_n$. Observe that $\mathrm{wt}'(w) = \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w)$. We have

$$\begin{aligned} \sum_{w \in \mathcal{A}(n,k)} \mathrm{wt}'(w) &= \sum_{\substack{u \in \mathcal{A}(n-1,k-1) \\ w_n = k \\ m_{n-1} = k-1}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w) + \sum_{\substack{u \in \mathcal{A}(n-1,k) \\ w_n < k \\ m_{n-1} = k}} \mathrm{wt}'(u) \cdot \mathrm{wt}'_n(w) \\ &= 1 \cdot S_q[n-1,k-1] \\ &\quad + ((1+q) + q^2 \cdot (1+q) + \cdots + q^{k-2} \cdot (1+q)) \cdot S_q[n-1,k] \\ &= S_q[n-1,k-1] + [k]_q \cdot S_q[n-1,k]. \end{aligned}$$

	w	$\mathrm{wt}'(w)$		w	$\mathrm{wt}'(w)$
$\mathcal{A}(1,1)$	1	1	$\mathcal{A}(5,3)$	12311	$(1+q)^2$
$\mathcal{A}(2,1)$	11	1		12131	$(1+q)^2$
$\mathcal{A}(2,2)$	12	1		12113	$(1+q)^2$
$\mathcal{A}(3,1)$	111	1		12133	$(1+q)\cdot q^2$
$\mathcal{A}(3,2)$	121	$1+q$		12313	$(1+q)\cdot q^2$
	112	1		12331	$q^2\cdot(1+q)$
$\mathcal{A}(3,3)$	123	1		12333	$q^2\cdot q^2$
$\mathcal{A}(4,1)$	1111	1		11213	$(1+q)$
$\mathcal{A}(4,2)$	1211	$(1+q)^2$		11231	$(1+q)$
	1121	$(1+q)$		11233	q^2
	1112	1		11123	1
$\mathcal{A}(4,3)$	1213	$(1+q)$	$\mathcal{A}(5,4)$	12341	$(1+q)$
	1231	$(1+q)$		12343	$q^2(1+q)$
	1233	q^2		12134	$(1+q)$
	1123	1		12314	$(1+q)$
$\mathcal{A}(4,4)$	1234	1		12334	q^2
$\mathcal{A}(5,1)$	11111	1		11234	1
$\mathcal{A}(5,2)$	12111	$(1+q)^3$	$\mathcal{A}(5,5)$	12345	1
	11211	$(1+q)^2$			
	11121	$(1+q)$			
	11112	1			

Table 2.2: Allowable *RG*-words in $\mathcal{A}(n,k)$ and their weight for $1 \leq k \leq n \leq 5$.

where in the second sum the last letter w_n is odd. For the case when k is odd there is a similar computation, except then there are three cases:

$$\sum_{w\in\mathcal{A}(n,k)} \mathrm{wt}'(w) = \sum_{\substack{u\in\mathcal{A}(n-1,k-1)\\ w_n=k\\ m_{n-1}=k-1}} \mathrm{wt}'(u)\cdot\mathrm{wt}'_n(w) + \sum_{\substack{u\in\mathcal{A}(n-1,k-1)\\ w_n=k\\ m_{n-1}=k}} \mathrm{wt}'(u)\cdot\mathrm{wt}'_n(w)$$
$$+ \sum_{\substack{u\in\mathcal{A}(n-1,k-1)\\ w_n<k\\ m_{n-1}=k}} \mathrm{wt}'(u)\cdot\mathrm{wt}'_n(w).$$

Here in the second and third sums the last letter w_n is odd. In both parity cases for k, the result is equal to $S_q[n,k]$, as desired. □

See Table 2.2 for the allowable *RG*-words for $1 \leq n \leq 5$.

Denote by $a(n,k) = |\mathcal{A}(n,k)|$ the cardinality of allowable words, and call it the *allowable Stirling number of the second kind*. The following holds.

Proposition 2.3.3 *The allowable Stirling numbers of the second kind satisfy the recurrence*

$$a(n,k) = a(n-1,k-1) + \lceil k/2\rceil\cdot a(n-1,k) \quad \textit{for } n\geq 1 \textit{ and } 1\leq k\leq n,$$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	$a(n)$	$b(n)$
0	1											1	1
1	0	1										1	1
2	0	1	1									2	2
3	0	1	2	1								4	5
4	0	1	3	4	1							9	15
5	0	1	4	11	6	1						23	52
6	0	1	5	26	23	9	1					65	203
7	0	1	6	57	72	50	12	1				199	877
8	0	1	7	120	201	222	86	16	1			654	4140
9	0	1	8	247	522	867	480	150	20	1		2296	21147
10	0	1	9	502	1291	3123	2307	1080	230	25	1	8569	115975

Table 2.3: The allowable Stirling numbers of the second kind $a(n, k)$, the allowable Bell numbers $a(n)$ and the classical Bell numbers $b(n)$ for $0 \leq n \leq 10$.

with the boundary conditions $a(n, 0) = \delta_{n,0}$.

Proof: By definition each allowable word $w \in \mathcal{A}(n, k)$ corresponds to a set partition of $\{1, 2, \ldots, n\}$ into k nonempty subsets where each block with an even label has exactly one element in it. Let $p(w)$ be the corresponding set partition.

There are two cases. If n occurs as a singleton block in $p(w)$, then after deleting the element n we obtain a set partition of the elements $\{1, 2, \ldots, n-1\}$ into $k-1$ blocks. This corresponds to a word in $\mathcal{A}(n-1, k-1)$. Otherwise assume the element n occurs in a block with more than one element. We can first build an allowable set partition of $\{1, 2, \ldots, n-1\}$ into k blocks and then put the element n into one of the k blocks. Notice that n can only be placed into an odd numbered block, so we have $\lceil k/2 \rceil$ possible blocks to assign the element n. This gives $\lceil k/2 \rceil \cdot a(n-1, k)$ possibilities. □

We call the sum $a(n) = \sum_{k=0}^{n} a(n, k)$ the *nth allowable Bell number*. See Table 2.3. The following properties are straightforward to verify.

Proposition 2.3.4 *The allowable Stirling numbers of the second kind satisfy*

$$a(n, 2) = n - 1\,, \tag{2.3.4}$$

$$a(n, n-1) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil . \tag{2.3.5}$$

Proof: By definition any $w \in \mathcal{A}(n, 2)$ is a word of length n consisting of exactly $n-1$ 1's and one 2. Since the initial letter must be 1, there are $n-1$ choices to assign the location of 2. Thus (2.3.4) follows.

For identity (2.3.5) we wish to count allowable words of length n with maximal entry $n-1$. By definition of an allowable word, there will be exactly one odd integer that appears twice and all other integers appear exactly once in such a word. In

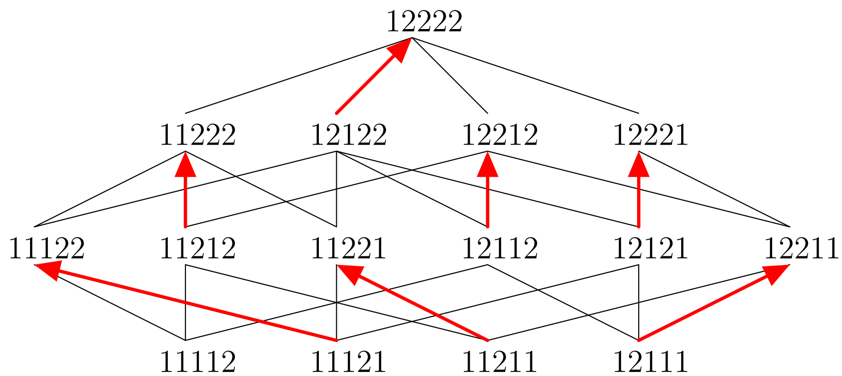


Figure 2.1: The matching of the Stirling poset $\Pi(5, 2)$.

other words, given the word $12 \cdots (n-1)$, we need to insert an odd integer less than or equal to $n-1$ so that the resulting word is still allowable. There are $\lceil (n-1)/2 \rceil = \lceil n/2 \rceil$ choices for such an odd integer. We can place this odd integer anywhere after its initial appearance in the word $12 \cdots (n-1)$. Thus we have in total $(n-1) + (n-3) + \cdots + (n - (2 \cdot \lceil (n-1)/2 \rceil - 1)) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$ ways to obtain a word in $\mathcal{A}(n, n-1)$. $\square$

Topological implications of Theorem 2.3.2 will be discussed in Section 2.6.

2.4 The Stirling poset of the second kind

In order to understand the q-Stirling numbers more deeply, we give a poset structure on $\mathcal{R}(n, k)$, which we call *the Stirling poset of the second kind*, denoted by $\Pi(n, k)$, as follows. For $v, w \in \mathcal{R}(n, k)$ let $v = v_1 v_2 \cdots v_n \prec w$ if $w = v_1 v_2 \cdots (v_i + 1) \cdots v_n$ for some index i. It is clear that if $v \prec w$ then $\mathrm{wt}(w) = q \cdot \mathrm{wt}(v)$, where the weight is as defined in (2.2.2). The Stirling poset of the second kind is graded by the degree of the weight function wt. Thus the rank of the poset $\Pi(n, k)$ is $(n-k)(k-1)$ and its rank generating function is given by $S_q[n, k]$. For basic terminology regarding posets, we refer the reader to Stanley's treatise [69, Chapter 3]. See Figures 2.1 and 2.2 for two examples of the Stirling poset of the second kind.

We next review the notion of a Morse matching [42, 43]. This will enable us to find a natural decomposition of the Stirling poset of the second kind, and to later be able to draw homological conclusions. A *partial matching* on a poset P is a matching on the underlying graph of the Hasse diagram of P, that is, a subset $M \subseteq P \times P$ satisfying (i) the ordered pair $(a, b) \in M$ implies $a \prec b$, and (ii) each element $a \in P$ belongs to at most one element in M. When $(a, b) \in M$, we write $u(a) = b$ and $d(b) = a$. A partial matching on P is *acyclic* if there does not exist a cycle

$$a_1 \prec u(a_1) \succ a_2 \prec u(a_2) \succ \cdots \succ a_n \prec u(a_n) \succ a_1$$

with $n \geq 2$, and the elements $a_1, a_2, \ldots, a_n$ are distinct.

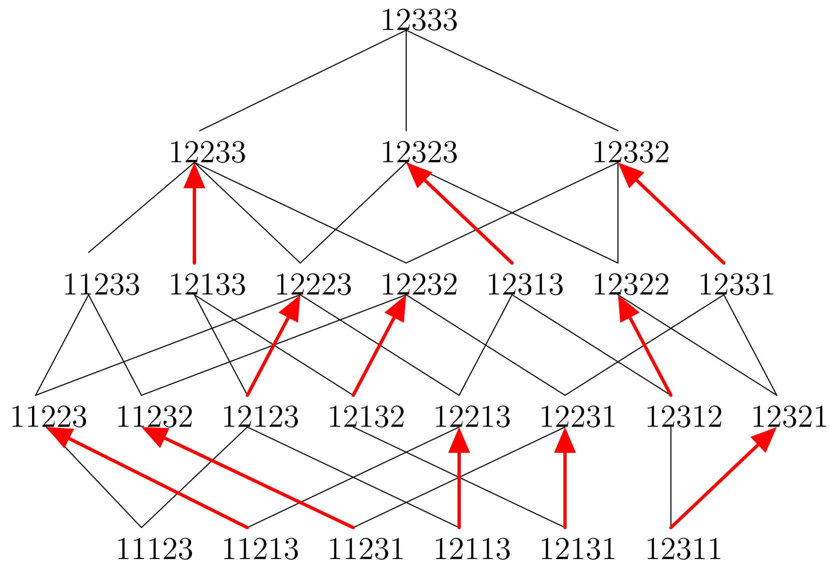


Figure 2.2: The Stirling poset $\Pi(5,3)$ and its matching. The rank generating function is the q-Stirling number $S_q[5,3] = q^4 + 3q^3 + 7q^2 + 8q + 6$. The matched elements are indicated by arrows. The unmatched elements are 11123, 11233 and 12333, and the sum of their weights is $1 + q^2 + q^4$.

An alternate manner is to orient all the edges in the Hasse diagram of a poset downwards and then reorient all the edges occurring in the matching upwards. The acyclic condition is simply that there is no cycle on the directed Hasse diagram. For the matched edge (a, b) the notation $u(a) = b$ and $d(b) = a$ denotes the fact that in the edge oriented from a to b the element b is "upwards" from a and similarly the element a is "downwards" from b. One can use the terminology of a *gradient path* or *V-path* consisting alternatively of matched and unmatched elements from the poset [26]. A *discrete Morse matching* is one where no gradient path forms a cycle.

We define a matching M on the Stirling poset $\Pi(n, k)$ in the following manner. Let w_i be the first entry in $w = w_1 w_2 \cdots w_n \in \mathcal{R}(n, k)$ such that w is weakly decreasing, that is, $w_1 \leq w_2 \leq \cdots \leq w_{i-1} \geq w_i$ and where we require the inequality $w_{i-1} \geq w_i$ to be strict unless both w_{i-1} and w_i are even. We have two subcases. If w_i is even then let $d(w) = w_1 w_2 \cdots w_{i-1}(w_i - 1)w_{i+1} \cdots w_n$. In this case we have $\mathrm{wt}(d(w)) = q^{-1} \cdot \mathrm{wt}(w)$. Otherwise, if w_i is odd then let $u(w) = w_1 w_2 \cdots w_{i-1}(w_i + 1)w_{i+1} \cdots w_n$ and we have $\mathrm{wt}(u(w)) = q \cdot \mathrm{wt}(w)$. If w is an allowable word which is weakly increasing, then w is unmatched in the poset. Again, we refer to Figures 2.1 and 2.2.

Lemma 2.4.1 *For the partial matching M described on the poset $\Pi(n, k)$ the unmatched words $U(n, k)$ are of the form*

$$w = \begin{cases} u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots u_{k-1} \cdot k & \text{for } k \text{ even,} \\ u_1 \cdot 2 \cdot u_3 \cdot 4 \cdot u_5 \cdot 6 \cdots (k-1) \cdot u_k & \text{for } k \text{ odd,} \end{cases}$$

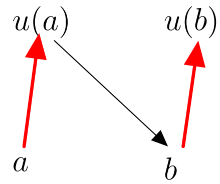


Figure 2.3: First three steps of a gradient path.

where $u_{2i-1} = (2i-1)^{j_i}$, that is, u_{2i-1} is a word consisting of $j_i \geq 1$ copies of the odd integer $2i-1$.

Proof: The result follows by observing the unmatched elements of the Stirling poset $w(n,k)$ consist of RG-words in $\mathcal{R}(n,k)$ which are always increasing and have no repeated even-valued entries. $\square$

Lemma 2.4.2 *Let a and b be two distinct elements in the Stirling poset of the second kind $\Pi(n,k)$ such that $a \prec u(a) \succ b \prec u(b)$. Then the element a is lexicographically larger than the element b.*

Proof: Suppose on the contrary that $a <_{\text{lex}} b$ with $a = a_1 \cdots a_n$. Assume that $u(a) = a_1 a_2 \cdots (a_i+1) \cdots a_n$. Then a_i is odd and the strict inequality $a_{i-1} > a_i$ holds. Since a is lexicographically smaller than b and the element b is obtained by decreasing an entry in $u(a)$ by one, the element b must be of the form $b = a_1 \cdots (a_i+1) \cdots (a_j-1) \cdots a_n$ for some index $j > i$. The first i entries in b satisfy $a_1 \leq a_2 \leq \cdots \leq a_{i-1} \geq (a_i+1)$ and a_i+1 is even, so by definition the element b is matched to an element of lower rank, contradicting the fact that $(b, u(b))$ is a matched pair in M. $\square$

Theorem 2.4.3 *The matching M described for $\Pi(n,k)$ is an acyclic matching, that is, it is a discrete Morse matching.*

Proof: By Lemma 2.4.2 one cannot find a gradient cycle of the form

$$x_1 \prec u(x_1) \succ x_2 \prec u(x_2) \succ \cdots \succ x_k \prec u(x_k) \succ x_1$$

since the elements $x_1, \ldots, x_k$ must satisfy $x_1 >_{\text{lex}} x_2 >_{\text{lex}} \cdots >_{\text{lex}} x_k >_{\text{lex}} x_1$, which is impossible. $\square$

2.5 Decomposition of the Stirling poset of the second kind

We next decompose the Stirling poset $\Pi(n,k)$ into Boolean algebras indexed by the allowable words. This gives a poset explanation for the factorization of the q-Stirling

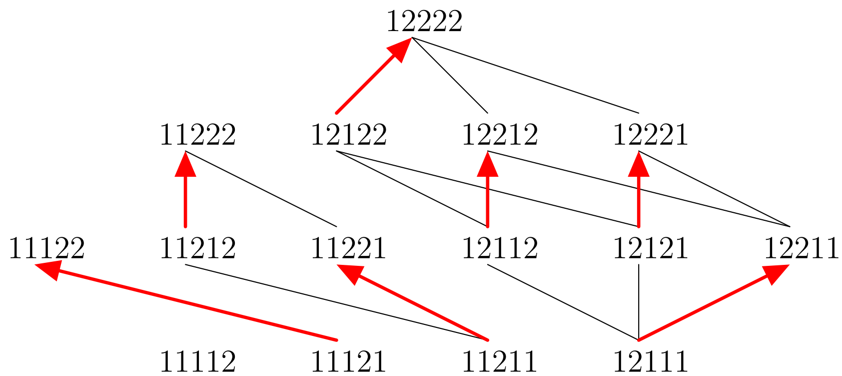


Figure 2.4: The decomposition of the Stirling poset $\Pi(5, 2)$ into Boolean algebras B_i for $i = 0, 1, 2, 3$. Based on the ranks of the minimal elements in each Boolean algebra, one obtains the weight of the poset is $1 + (1 + q) + (1 + q)^2 + (1 + q)^3$.

number $S_q[n, k]$ in terms of powers of q and $1 + q$. To state this decomposition, we need two definitions. For $w \in \mathcal{A}(n, k)$ an allowable word let $\mathrm{Inv_r}(w) = \{i : w_j > w_i \text{ for some } j < i\}$ be the set of all indices in w that contribute to the right-hand element of an inversion pair. For $i \in \mathrm{Inv_r}(w)$ such an entry w_i must be odd since in a given allowable word any entry occurring to the left of an even entry must be strictly less than it. Finally, for $w \in \mathcal{A}(n, k)$ let $\alpha(w)$ be the word formed by incrementing each of the entries indexed by the set $\mathrm{Inv_r}(w)$ by one. Additionally, for $w \in \mathcal{A}(n, k)$ and any $I \subseteq \mathrm{Inv_r}(w)$, the word formed by incrementing each of the entries indexed by the set I by one are elements of $\mathcal{R}(n, k)$ since if $i \in \mathrm{Inv_r}(w)$ then there is an index $h < i$ with $w_h = w_i$. This follows from Proposition 2.2.1 part (ii).

Theorem 2.5.1 *The Stirling poset of the second kind $\Pi(n, k)$ can be decomposed as the disjoint union of Boolean intervals*

$$\Pi(n, k) = \bigcupdot_{w \in \mathcal{A}(n,k)} [w, \alpha(w)].$$

Furthermore, if an allowable word $w \in \mathcal{A}(n, k)$ has weight $\mathrm{wt}'(w) = q^i \cdot (1 + q)^j$, then the rank of the element w is i and the interval $[w, \alpha(w)]$ is isomorphic to the Boolean algebra on j elements.

Proof: Let $w \in \mathcal{A}(n, k)$ with $\mathrm{wt}'(w) = q^i \cdot (1 + q)^j$ and $|\mathrm{Inv_r}(w)| = m$. It directly follows from the definitions that the interval $[w, \alpha(w)]$ is isomorphic to the Boolean algebra B_m. With the exception of the element w, all the other elements in the interval $[w, \alpha(w)]$ are not allowable words in $\Pi(n, k)$ since all of the newly incremented entries will have at least two equal even entries. We also claim $m = j$, since $\mathrm{wt}'(w)$ picks up a factor of $1 + q$ for each index i satisfying $w_i < m_{i-1} = \max(w_1, \ldots, w_{i-1})$. These indices are exactly the set $\mathrm{Inv_r}(w)$.

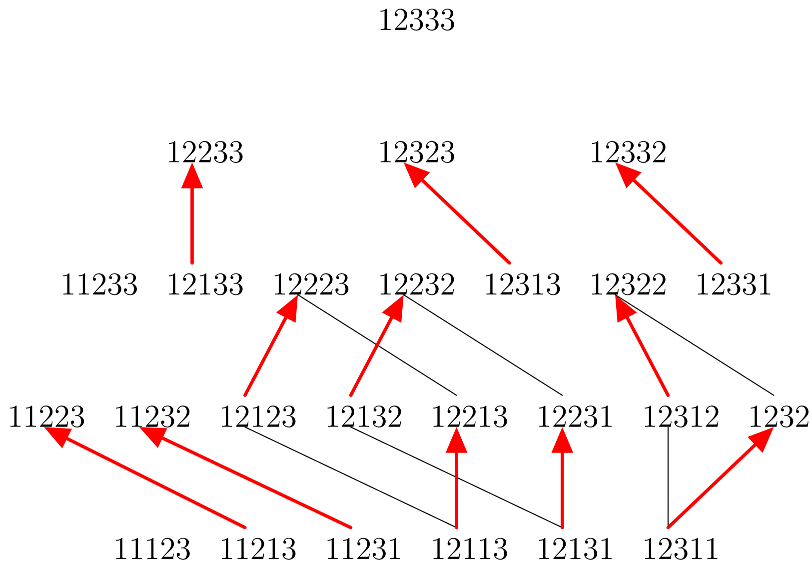


Figure 2.5: The decomposition of the Stirling poset $\Pi(5,3)$ into Boolean algebras. The weight of the poset is $1 + 2(1+q) + 3(1+q)^2 + q^2 + 3q^2(1+q) + q^4$.

We claim every element of $\Pi(n,k)$ occurs in some Boolean algebra in the decomposition. This is vacuously true if $w \in \mathcal{A}(n,k)$. Otherwise since w is not an allowable word, it has even entries which are repeated. Decrease all occurrences of these repeated entries by one except for the first occurrence of each even integer. This is the allowable RG-word associated to w. □

See Figures 2.4 and 2.5 for examples of this decomposition for the posets in Figures 2.1 and 2.2, respectively.

2.6 Homological $q = -1$ phenomenon

Stembridge's $q = -1$ phenomenon [72, 74] and the more general cyclic sieving phenomenon of Reiner, Stanton and White [61] count symmetry classes in combinatorial objects by evaluating their q-generating series at a primitive root of unity. Recently Hersh, Shareshian and Stanton [36] have given a homological interpretation of the $q = -1$ phenomenon by viewing it as an Euler characteristic computation on a chain complex supported by a poset. In the best scenario, the homology is concentrated in dimensions of the same parity and one can identify a homology basis. For further information about algebraic discrete Morse theory, see [38, 42, 68].

We will see the graded poset $\Pi(n,k)$ supports an algebraic complex $(\mathcal{C}, \partial)$. The aforementioned matching for $\Pi(n,k)$ (Theorem 2.4.3) is a discrete Morse matching for this complex. Hence using standard discrete Morse theory [27], we can give a basis for the homology.

We now review the relevant background. We follow [36] here. See also [38, 68]. Let P be a graded poset and W_i denote the rank i elements. We say the *poset P supports a chain complex* $(\mathcal{C}, \partial)$ of $\mathbb{F}$-vector spaces C_i if each C_i has basis indexed by the rank i elements W_i and $\partial_i : W_i \to W_{i-1}$ is a boundary map. Furthermore, for $x \in W_i$ and $y \in W_{i-1}$ the coefficient $\partial_{x,y}$ of y in $\partial_i(x)$ is zero unless $y <_P x$.

For $w \in \Pi(n,k)$, let

$$E(w) = \{i : w_i \text{ is even and } w_j = w_i \text{ for some } j < i\}$$

be the set of all indices of repeated even entries in the word w. Define the boundary map ∂ on the elements of $\Pi(n,k)$ by

$$\partial(w) = \sum_{j=1}^{r} (-1)^{j-1} \cdot w_1 \cdots w_{i_j-1} \cdot (w_{i_j} - 1) \cdot w_{i_j+1} \cdots w_n, \tag{2.6.1}$$

where $E(w) = \{i_1 < i_2 < \cdots < i_r\}$. For example, if $w = 122344$ then $E(122344) = \{3, 6\}$ and $\partial(122344) = 121344 - 122343$. With this definition of the boundary operator ∂, we have the following lemma.

Lemma 2.6.1 *The map ∂ is a boundary map on the algebraic complex $(\mathcal{C}, \partial)$ with the poset $\Pi(n,k)$ as support.*

Proof: By definition of ∂, we have

$$\begin{aligned} \partial^2(w) &= \sum_{i_r < i_j} (-1)^{j-1} \cdot (-1)^{r-1} \cdot w_1 w_2 \cdots w_{i_r-1} \cdots (w_{i_j} - 1) \cdots w_n \\ &\quad + \sum_{i_r > i_j} (-1)^{j-1} \cdot (-1)^{r-2} \cdot w_1 w_2 \cdots w_{i_j-1} \cdots (w_{i_r} - 1) \cdots w_n, \end{aligned}$$

where the sum is over indices i_r and i_j with $w_{i_j}, w_{i_r} \in E(w)$. These two summations cancel since after switching r and j in the second summation, the resulting expression becomes the negative of the first. Hence we have that $\partial^2(w) = 0$. $\square$

Lemma 2.6.2 *The weighted generating function of the unmatched words $U(n,k)$ in $\Pi(n,k)$ is given by the q^2-binomial coefficient*

$$\sum_{u \in U(n,k)} \mathrm{wt}(u) = \begin{bmatrix} n - 1 - \lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{bmatrix}_{q^2}.$$

Proof: Let $u = u_1 \cdots u_n \in U(n,k)$ be an unmatched word. Recall the weight is given by reading the word from left to right and gaining a multiplicative factor $q^{u_i - 1}$ for all values of i with $u_{i-1} = u_i$. Since $u_{i-1} = u_i$ can only appear when u_i is odd, the weight of an unmatched word is always q^{2m} for some non-negative integer m.

We claim that each $u \in U(n,k)$ of weight q^{2m} corresponds to an integer partition of $2m$ with at most $n-k$ parts where each part is even and where each part is at most $\rho = \lfloor (k-1)/2 \rfloor \cdot 2$. The correspondence is as follows. For each word u satisfying the condition with the odd integer j appearing m_j times, map these odd integers to $m_j - 1$ copies of $j-1$. The resulting partition of $2m$ is of the form

$$2m = \underbrace{2 + \cdots + 2}_{m_3-1} + \underbrace{4 + \cdots + 4}_{m_5-1} + \cdots + \underbrace{\rho + \cdots + \rho}_{m_\sigma-1},$$

where σ is the largest occurring odd integer in the original RG-word u and $\rho = \sigma - 1$. For example, the word 112333455 corresponds to the partition $8 = 2+2+4$. Note that the unmatched word 1 corresponds to the empty partition $\emptyset$.

An alternate way to describe these partitions is to form a partition of m into at most $n-k$ parts with each part at most $\lfloor (k-1)/2 \rfloor$. By doubling each part, we obtain the above mentioned partition. However, by [69, Proposition 1.7.3] the sum of the weight of partitions that fit into a rectangle of size $n-k$ by $\lfloor (k-1)/2 \rfloor$ is given by the Gaussian polynomial $\left[\begin{smallmatrix} \lfloor \frac{k-1}{2} \rfloor + n - k \\ \lfloor \frac{k-1}{2} \rfloor \end{smallmatrix} \right]_q$. By the substitution $q \mapsto q^2$, the result follows. $\square$

Corollary 2.6.3 *The number of unmatched words of length n that is,*

$$U(n) = \sum_{k=1}^{n} |U(n,k)|,$$

is given by the Fibonacci number F_n, where $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = F_1 = 1$.

Proof: Substituting $q^2 = 1$, that is, $q = -1$ in Lemma 2.6.2 gives the number of unmatched words $|U(n,k)|$ in the Stirling poset of the second $\Pi(n,k)$. Hence,

$$U(n) = \sum_{k=1}^{n} |U(n,k)| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} = F_n,$$

where the last equality is a well-known binomial coefficient expansion for the Fibonacci number F_n arising from compositions of n using 1s and 2s. $\square$

We have shown the graded poset $\Pi(n,k)$ supports an algebraic complex $(\mathcal{C}, \partial)$. We will need a lemma due to Hersh, Shareshian and Stanton [36, Lemma 3.2]. This is part (ii) of the original statement of the lemma.

Lemma 2.6.4 (Hersh–Shareshian–Stanton, 2014) *Let P be a graded poset supporting an algebraic complex $(\mathcal{C}, \partial)$. Assume the poset P has a Morse matching M such that for all matched pairs (y,x) with $y \prec x$ one has $\partial_{y,x} \in \mathbb{F}^*$. If all unmatched poset elements occur in ranks of the same parity, then $\dim(H_i(\mathcal{C}, \partial)) = |P_i^{\mathrm{un}\, M}|$, that is, the number of unmatched elements of rank i.*

We can now state our result.

Theorem 2.6.5 *For the algebraic complex $(\mathcal{C}, \partial)$ supported by the Stirling poset of the second kind $\Pi(n,k)$, a basis for the integer homology is given by the increasing allowable RG-words in $\mathcal{A}(n,k)$. Furthermore, we have*

$$\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i = \begin{bmatrix} n - 1 - \lfloor \frac{k}{2} \rfloor \\ \lfloor \frac{k-1}{2} \rfloor \end{bmatrix}_{q^2}.$$

Proof: By definition of the boundary map ∂, if $(x, y) \in M$ then $\partial_{y,x} = 1$ and all of the unmatched words in $\Pi(n,k)$ occur in even ranks. The conditions in Lemma 2.6.4 are satisfied. So $\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i$ is the q^2-binomial in Lemma 2.6.2. □

Remark 2.6.6 Theorem 2.6.5 can be proved without using Lemma 2.6.4. Note that the boundary map ∂ is supported on the Boolean algebras in the poset decomposition given in Theorem 2.5.1. Furthermore, the restriction to one of these Boolean algebras is the natural boundary map on that Boolean algebra. Hence the algebraic complex is a direct sum of algebraic complexes of Boolean algebras. The only summands that contribute any homology is the rank 0 Boolean algebras, that is, the unmatched elements.

2.7 q-Stirling numbers of the first kind

The (unsigned) *q-Stirling numbers of the first kind* are defined by the recurrence formula

$$c_q[n,k] = c_q[n-1, k-1] + [n-1]_q \cdot c_q[n-1, k], \tag{2.7.1}$$

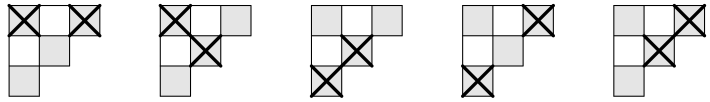
where $c_q[n, 0] = \delta_{n,0}$. When $q = 1$, the Stirling number of the first kind $c(n,k)$ enumerates permutations in the symmetric group $\mathfrak{S}_n$ having exactly k disjoint cycles. A combinatorial way to express q-Stirling numbers of the first kind is via rook placements; see de Médicis and Leroux [17]. Throughout a staircase chessboard of length m is a board with $m - i$ squares in the ith row for $i = 1, \ldots, m-1$ and each row of squares is left-justified.

Definition 2.7.1 *Let $\mathcal{P}(m,n)$ be the set of all ways to place n rooks onto a staircase chessboard of length m so that no two rooks are in the same column. For any rook placement $T \in \mathcal{P}(m,n)$, denote by $\mathrm{s}(T)$ the number of squares to the south of the rooks in T.*

Theorem 2.7.2 (de Médicis–Leroux, 1993) *The q-Stirling number of the first kind $c_q[n,k]$ is given by*

$$c_q[n,k] = \sum_{T \in \mathcal{P}(n, n-k)} q^{\mathrm{s}(T)},$$

where the sum is over all rook placements of $n-k$ rooks on a staircase board of length n.



q^2 $\quad q^2(1+q)$ $\quad (1+q)^2$ $\quad (1+q)$ $\quad (1+q)$

Figure 2.6: Computing the q-Stirling number of the first kind $c_q[4,2]$ using $\mathcal{Q}(4,2)$.

We now define a subset $\mathcal{Q}(n, n-k)$ of rook placements in $\mathcal{P}(n, n-k)$ so that the q-Stirling number of the first kind $c_q[n,k]$ can be expressed as a statistic on the subset involving q and $1+q$. The key is given any staircase chessboard, assign it a certain alternating shaded pattern.

Definition 2.7.3 *Given any staircase chessboard, assign it a chequered pattern such that every other antidiagonal strip of squares is shaded, beginning with the lowest antidiagonal. Let*

$$\mathcal{Q}(m,n) = \{T \in \mathcal{P}(m,n) : \textit{all rooks are placed in shaded squares}\}$$

For any rook placement $T \in \mathcal{Q}(m,n)$, let $\mathrm{r}(T)$ denote the number of rooks in T that are not in the first row. Define the weight to be $\mathrm{wt}(T) = q^{\mathrm{s}(T)} \cdot (1+q)^{\mathrm{r}(T)}$.

Theorem 2.7.4 *The q-Stirling number of the first kind is given by*

$$c_q[n,k] = \sum_{T \in \mathcal{Q}(n,n-k)} \mathrm{wt}(T) = \sum_{T \in \mathcal{Q}(n,n-k)} q^{\mathrm{s}(T)} \cdot (1+q)^{\mathrm{r}(T)},$$

where the sum is over all rook placements of $n-k$ rooks on an alternating shaded staircase board of length n.

Proof: We proceed by induction on n. It is straightforward to see the result holds for $n = k = 0$. Suppose the result is true for alternating shaded staircase boards of length $n-1$. Then we have

$$\begin{aligned}
\sum_{T \in \mathcal{Q}(n,n-k)} \mathrm{wt}(T) &= \sum_{\substack{T \in \mathcal{Q}(n,n-k) \\ \text{leftmost column is empty}}} \mathrm{wt}(T) + \sum_{\substack{T \in \mathcal{Q}(n,n-k) \\ \text{leftmost column is not empty}}} \mathrm{wt}(T) \\
&= \sum_{T \in \mathcal{Q}(n-1,n-k)} \mathrm{wt}(T) + \sum_{T \in \mathcal{Q}(n-1,n-k-1)} [n-1]_q \cdot \mathrm{wt}(T) \\
&= c_q[n-1,k-1] + [n-1]_q \cdot c_q[n-1,k] \\
&= c_q[n,k].
\end{aligned}$$

In the second equality, the first term follows from the fact that one can remove the leftmost column from the board, leaving a rook placement of $n-k$ rooks on a length $n-1$ shaded board. For the second term, we first consider where the rook occurs in the leftmost column. If the rook occurs in the $(2i+1)$st entry from the bottom of the leftmost column, where $0 \leq i < \lfloor (n-1)/2 \rfloor$, it contributes a weight of $q^{2i} \cdot (1+q)$ since there are $2i$ squares below it and the rook does not occur in the first row. The only way a rook in the first column can also occur in the first row of a shaded staircase board is if the leftmost column has an odd number of squares, that is, n is even. In this case the rook would contribute a weight of q^{n-2}. For n even the overall weight contribution from a rook in the first column is $1 \cdot (1+q) + q^2 \cdot (1+q) + \cdots + q^{n-4} \cdot (1+q) + q^{n-2} = [n-1]_q$ and for n odd the weight contribution is $1 \cdot (1+q) + q^2 \cdot (1+q) + \cdots + q^{n-3} \cdot (1+q) = [n-1]_q$. Hence removing the first column from the staircase board along with the rook that occurs in it leaves a shaded staircase board of length $n-1$ with $n-k-1$ rooks. The total weight lost is $[n-1]_q$. Finally, the last equality is recurrence (2.7.1). $\square$

See Figure 2.6 for the computation of $c_q[4,2]$ using allowable rook placements on length 4 shaded staircase boards.

When we substitute $q = -1$ into the q-Stirling number of the first kind, the weight $\mathrm{wt}(T)$ of a rook placement T will be 0 if there is a rook in T that is not in the first row. Hence the Stirling number of the first kind $c_q[n,k]$ evaluated at $q = -1$ counts the number of rook placements in $\mathcal{Q}(n, n-k)$ such that all of the rooks occur in shaded squares of the first row.

Corollary 2.7.5 *The q-Stirling number of the first kind $c_q[n,k]$ evaluated at $q = -1$ gives the number of rook placements in $\mathcal{Q}(n, n-k)$ where all of the rooks occur in shaded squares in the first row, that is,*

$$c_q[n,k]\Big|_{q=-1} = \binom{\lfloor n/2 \rfloor}{n-k}.$$

Let $d(n,k) = |\mathcal{Q}(n, n-k)|$. We call $d(n,k)$ the *allowable Stirling number of the first kind.* See Table 2.4 for values.

Proposition 2.7.6 *The allowable Stirling numbers of the first kind $d(n,k)$ satisfy the recurrence*

$$d(n,k) = d(n-1, k-1) + \left\lceil \frac{n-1}{2} \right\rceil \cdot d(n-1, k)$$

with boundary conditions $d(n,0) = \delta_{n,0}$, $d(n,n) = 1$ for $n \geq 0$ and $d(n,k) = 0$ when $k > n$.

Proof: For each $T \in \mathcal{Q}(n, n-k)$, there are two cases. If the leftmost column in T is empty, then after deleting this column we obtain an allowable rook placement $T' \in \mathcal{Q}(n-1, n-k)$. Otherwise assume there is a rook in the leftmost column. We can first build an allowable rook placement $T' \in \mathcal{Q}(n-1, n-k-1)$ and then add a column of $n-1$ squares with a rook in it to the left of T' to form a rook placement in $\mathcal{Q}(n, n-k)$. Notice that the rook in the leftmost column can be only put into a shaded square, so there are $\lceil (n-1)/2 \rceil$ possible squares to place the rook. Overall this case gives $\lceil (n-1)/2 \rceil \cdot d(n-1, k)$ possibilities. $\square$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	$r(n)$	$n!$
0	1											1	1
1	0	1										1	1
2	0	1	1									2	2
3	0	1	2	1								4	6
4	0	2	5	4	1							12	24
5	0	4	12	13	6	1						36	120
6	0	12	40	51	31	9	1					144	720
7	0	36	132	193	144	58	12	1				576	5040
8	0	144	564	904	769	376	106	16	1			2880	40320
9	0	576	2400	4180	3980	2273	800	170	20	1		14400	362880
10	0	2880	12576	23300	24080	15345	6273	1650	270	25	1	86400	3628800

Table 2.4: The allowable Stirling numbers of the first kind, their row sum and $n!$ for $0 \leq n \leq 10$.

Certain allowable Stirling numbers of the first kind have closed forms as follows.

Proposition 2.7.7 *The allowable Stirling numbers of the first kind satisfy*

$$d(n,1) = \begin{cases} \left(\frac{n-1}{2}\right)!^2 & \text{for } n \text{ odd,} \\ \frac{n}{2} \cdot \left(\frac{n-1}{2}\right)!^2 & \text{for } n \text{ even,} \end{cases} \tag{2.7.2}$$

$$d(n,n-1) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lceil \frac{n}{2} \right\rceil, \tag{2.7.3}$$

$$r(n) = d(n+2,1). \tag{2.7.4}$$

Proof: We first prove (2.7.4). Let $T \in \mathcal{Q}(n+2,1)$ be a rook placement on a shaded board. Since rooks are only allowed to be placed in shaded squares, the two rooks in the rightmost two columns must be in the bottommost antidiagonal. Delete the two longest anti-diagonals from T to obtain T'. Since the shaded squares are preserved, T' is still allowable with the longest column length n. The rightmost two rooks in T are deleted to form T', giving at most $n-1$ rooks in T'. Hence $d(n+2,1) \leq r(n)$.

On the other hand, for any rook placement T with at most $n-1$ rooks on a shaded staircase board of length n, we can add two anti-diagonals to T and place a rook in the bottom row for each empty column in the new chessboard to obtain T'. The board T' has $n+1$ rooks and $n+1$ columns, hence $r(n) \leq d(n+2,1)$. Hence we have the equality (2.7.4).

The expression $d(n,n-1)$ counts the number of rook placements of length n using 1 rook. This is the same as counting the number of shaded squares in a length n staircase chessboard. Counting column by column, beginning from the right, gives $1+1+2+2+\cdots+\lfloor n/2 \rfloor = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$.

Finally, the expression $d(n,1)$ counts the number of rook placements with $n-1$ columns and $n-1$ rooks. Thus each column must have a rook. For each column with k squares, there are $\lceil k/2 \rceil$ shaded squares, hence $\lceil k/2 \rceil$ choices for the rook. This gives $((n-1)/2)!^2$ ways when n is odd and $(n/2) \cdot ((n-1)/2)!^2$ ways when n is even. $\square$

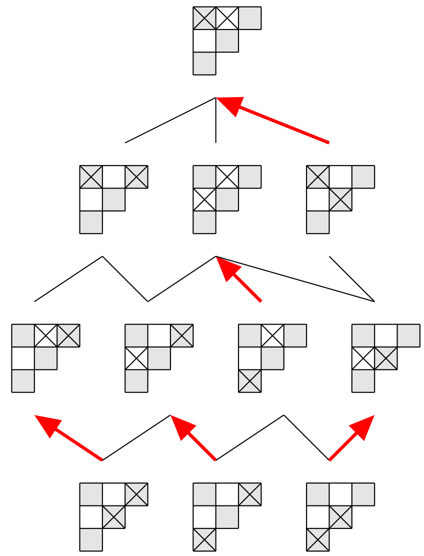


Figure 2.7: Example of $\Gamma(3, 2)$ with its matching. There is one unmatched rook placement in rank 2.

2.8 Structure and topology of the Stirling poset of the first kind

We define a poset structure on rook placements on a staircase shape board. For rook placements T and T' in $\mathcal{P}(m, n)$, let $T \prec T'$ if T' can be obtained from T by either moving a rook to the left (west) or up (north) by one square. We call this poset *the Stirling poset of the first kind* and denote it by $\Gamma(m, n)$. It is straightforward to check that the poset $\Gamma(m, n)$ is graded of rank $(m-1)+(m-2)+\cdots+(m-n) = m\cdot n-\binom{n+1}{2}$ and its rank generating function is $c_q[m+1, m+1-n]$. See Figure 2.7 for an example.

We wish to study the topological properties of the Stirling poset of the first kind. To do so, we define a matching M on the poset as follows. Given any rook placement $T \in \Gamma(m, n)$, let r be the first rook (reading from left to right) that is not in a shaded square of the first row. Match T to T' where T' is obtained from T by moving the rook r one square down if r is not in a shaded square, or one square up if r is in a shaded square but not in the first row. It is straightforward to check that the unmatched rook placements are the ones where all of the rooks occur in the shaded squares of the first row.

As an example, the matching for $\Gamma(4, 2)$ is shown in Figure 2.7, where an upward arrow indicates a matching and other edges indicate the remaining cover relations. Observe the unmatched rook placements are the ones with all the rooks occurring in the shaded squares in the first row. By the way a chessboard is shaded, the unmatched rook placements only appear in even ranks in the poset.

We have a q-analogue of Corollary 2.7.5.

Theorem 2.8.1 *For the Stirling poset of the first kind $\Gamma(m, n)$ the generating func-*

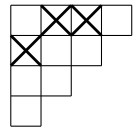


Figure 2.8: A rook placement T with rook word $w_T = 3320$.

tion for the unmatched rook placements is

$$\sum_{\substack{T\in\Gamma(m,n)\\ T\ \mathit{unmatched}}} \mathrm{wt}(T) = q^{n(n-1)} \cdot \begin{bmatrix} \lfloor \frac{m+1}{2} \rfloor \\ n \end{bmatrix}_{q^2} .$$

Proof: The number of unmatched rook placements in rank $2j$ in the poset $\Gamma(m,n)$ is the same as the number of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $2j$ into n distinct non-negative even parts, with each $\lambda_i \leq m-(2i-1)$. Alternatively, this is the number of partitions $\delta = (\delta_1, \ldots, \delta_n)$ of $2j-(0+2+\cdots+(2n-2)) = 2j-n(n-1)$ into n non-negative even parts, where each part δ_i satisfies $\delta_i = \lambda_i - (2n-(2i-2)) \leq m-2n+1$ for $i = 1, \ldots, n$. Thus we have

$$\begin{aligned}
\sum_{\substack{T\in\Gamma(m,n)\\ T\ \text{unmatched}}} \mathrm{wt}(T) &= \sum_{j\geq 0} \sum_{\substack{(\lambda_1,\ldots,\lambda_n)\vdash 2j\\ 0\leq\lambda_i\leq m-(2i-1)\\ \lambda_i\ \text{distinct even integers}}} q^{|\lambda|} \\
&= q^{n(n-1)} \cdot \sum_{2j-n(n-1)\geq 0} \sum_{\substack{\lambda\vdash 2j-n(n-1)\\ 0\leq\lambda_i\leq m-2n+1\\ i=1,\ldots,n\\ \lambda_i\ \text{even integers}}} q^{|\lambda|} \\
&= q^{n(n-1)} \cdot \sum_{j-\frac{n(n-1)}{2}\geq 0} \sum_{\substack{\lambda\vdash j-\frac{n(n-1)}{2}\\ 0\leq\lambda_i\leq\lfloor\frac{m+1}{2}\rfloor-n\\ i=1,\ldots,n}} (q^2)^{|\lambda|} .
\end{aligned}$$

The last (double) sum is over all integer partitions into at most n parts where each part is at most $\lfloor (m+1)/2 \rfloor - n$. Hence this sum is given by the Gaussian polynomial $\begin{bmatrix} \lfloor (n+1)/2 \rfloor \\ n \end{bmatrix}_{q^2}$, proving the desired identity. $\square$

Given a rook placement $T \in \mathcal{P}(m,n)$, we can associate to it a *rook word* $w_T = w_1 w_2 \ldots w_m$ where w_i is one plus the number of squares below the column i rook. If column i is empty, let $w_i = 0$. See Figure 2.8 for an example.

Lemma 2.8.2 *Let T and T' be two distinct elements in the Stirling poset of the first kind such that $T \prec u(T) \succ T' \prec u(T')$ is a gradient path. Then the rook words satisfy the inequality $w_T <_{lex} w_{T'}$.*

Proof: Let $w_T = w_1 \cdots w_n$. Since $u(T)$ is obtained from T by shifting a rook a in column i up by one square, we have $w_{u(T)} = w_1 \cdots (w_i + 1) \cdots w_n$. By definition of the matching, in the rook placement T the rook a was in a shaded square not in the first row. In the rook placement $u(T)$ the rook a is now in an unshaded square. Furthermore, all of the rooks in the leftmost $i-1$ columns of T are in shaded squares in the first row.

The rook placement T' is obtained from $u(T)$ by shifting a rook to the right or down. We first show that T' cannot be obtained by shifting a rook in $u(T)$ down by one square.

Suppose a rook b in column $j \neq i$ of $u(T)$ is shifted down to form T'. If $j < i$ since all of the rooks in columns 1 through $i-1$ occur in shaded squares of the first row, the rook b is now in an unshaded square in the rook placement T'. Hence if it is matched with another rook placement, it will be of one rank lower, contradicting the fact that we assumed T' was part of a gradient path $T \prec u(T) \succ T' \prec u(T')$. If $j > i$ then the rook a in column i of T' is in an unshaded square and hence T' should be matched to a rook placement in one lower rank. Again, this contradicts our gradient path assumption. Hence this case cannot occur.

The remaining case is when a rook in $u(T)$ occurring in the jth column for some index $j < n$ is shifted to the right to form T'. Note this implies the $(j+1)$st column of T had no rooks in it. If $j < i$, then since b in column j in $u(T)$ is in a shaded square of the first row, it is shifted to an unshaded square in T' and hence T' is matched to a rook placement in one lower rank. If $j > i$ then a in T' is the first rook that does not appear in a shaded square of the first row. Hence T' is matched to some rook placement of one rank lower, contradicting the gradient path assumption.

The only remaining possibility is when $j = i$. Then the rook a in $u(T)$ is shifted to a shaded square in T', and hence $w_T = w_1 \cdots w_{i-1} \cdot w_i \cdot 0 \cdot w_{i+2} \cdots w_n >_{\text{lex}} w_1 \cdots w_{i-1} \cdot 0 \cdot (w_i - 1) \cdot w_{i+2} \cdots w_n = w_{T'}$, as desired. $\square$

Theorem 2.8.3 *The matching M on the Stirling poset of the first kind $\Gamma(m, n)$ is an acyclic matching, that is, the Stirling poset has a discrete Morse matching.*

The proof is similar to that of Theorem 2.4.3, and thus omitted.

Next we give a decomposition of the Stirling poset of the first kind $\Gamma(m, n)$ into Boolean algebras indexed by the allowable rook placements. This will lead to a boundary map on the algebraic complex with $\Gamma(m, n)$ as the support. For any $T \in \mathcal{Q}(m, n)$, let $\alpha(T)$ be the rook placement obtained by shifting every rook that is not in the first row up by one. Then we have the following theorem.

Theorem 2.8.4 *The Stirling poset of the first kind $\Gamma(n, k)$ can be decomposed as disjoint union of Boolean intervals*

$$\Gamma(m, n) = \dot{\bigcup_{T \in \mathcal{Q}(m,n)}} [T, \alpha(T)].$$

Furthermore, if $T \in \mathcal{Q}(m,n)$ has weight $\mathrm{wt}(T) = q^i \cdot (1+q)^j$, then the rank of the element T is i and the interval $[T, \alpha(T)]$ is isomorphic to the Boolean algebra on j elements.

Proof: We first show that for any $T \in \mathcal{Q}(m,n)$ with $\mathrm{wt}(T) = q^i \cdot (1+q)^j$ that the interval $[T, \alpha(T)] \cong B_j$. Since $\mathrm{wt}(T) = q^i \cdot (1+q)^j$, the rank of T is i and there are j rooks in T that are not in the first row. The rank $i+l$ elements in the interval $[T, \alpha(T)]$ correspond to shifting l of those rooks up by one. It is straightforward to see that in the interval $[T, \alpha(T)]$ all of the elements except T are in $\mathcal{P}(m,n) - \mathcal{Q}(m,n)$ since the rook that is shifted up by one will not be in a shaded square.

We next need to show that every element $T \in \Gamma(m,n)$ occurs in some Boolean interval in this decomposition. This is vacuously true if $T \in \mathcal{Q}(m,n)$. Otherwise there are some rooks in T that are not in shaded squares. Shift all such rooks down by one to obtain an allowable rook placement associated to T. $\square$

Given a rook placement $T \in \Gamma(m,n)$, let $N(T) = \{r_1, r_2, \ldots, r_s\}$ be the set of all rooks in T that are not in shaded squares, where the rooks r_i are labeled from left to right. We define the map ∂ as follows.

Definition 2.8.5 *Let $\partial : \Gamma(m,n) \longrightarrow \mathbb{Z}[\Gamma(m,n)]$ be the map defined by*

$$\partial(T) = \sum_{r_i \in N(T)} (-1)^{i-1} \cdot T_{r_i} \,,$$

where T_{r_i} is obtained by moving the rook r_i in T down by one square.

Lemma 2.8.6 *The map ∂ in Definition 2.8.5 is a boundary map on the algebraic complex with $\Gamma(m,n)$ as the support.*

Proof: The boundary map ∂ is supported on the Boolean algebra decomposition of the Stirling poset of the first kind appearing in Theorem 2.8.4. Remark 2.6.6 applies again to show ∂ is a boundary map. $\square$

Theorem 2.8.7 *For the algebraic complex $(\mathcal{C}, \partial)$ supported by the Stirling poset of the first kind $\Gamma(m,n)$, a basis for the integer homology is given by the rook placements in $\mathcal{P}(m,n)$ having all of the rooks occur in shaded squares in the first row. Furthermore,*

$$\sum_{i \geq 0} \dim(H_i(\mathcal{C}, \partial; \mathbb{Z})) \cdot q^i = q^{n(n-1)} \cdot \begin{bmatrix} \lfloor \frac{m+1}{2} \rfloor \\ n \end{bmatrix}_{q^2} .$$

Proof: The proof follows by applying Theorems 2.8.1 and 2.8.3 and Lemmas 2.6.4 and 2.8.6. $\square$

2.9 (q,t)-Stirling numbers and orthogonality

In [78] Viennot has some beautiful results in which he gave combinatorial bijections for orthogonal polynomials and their moment generating functions. One well-known relation between the ordinary signed Stirling numbers of the first kind and Stirling numbers of the second kind is their orthogonality. A bijective proof of the orthogonality of their q-analogues via $0-1$ tableaux was given by de Médicis and Leroux [17, Proposition 3.1].

There are a number of two-variable Stirling numbers of the second kind using bistatistics on RG-words and rook placements. See [79] and the references therein. Letting $t = 1+q$ we define (q,t)-analogues of the Stirling numbers of the first and second kind. We show orthogonality holds combinatorially for the (q,t)-version of the Stirling numbers via a sign-reversing involution on ordered pairs of rook placements and RG-words.

Definition 2.9.1 *Define the (q,t)-Stirling numbers of the first and second kind by*

$$s_{q,t}[n,k] = (-1)^{n-k} \cdot \sum_{T \in \mathcal{Q}(n,n-k)} q^{\mathrm{s}(T)} \cdot t^{\mathrm{r}(T)} \tag{2.9.1}$$

and

$$S_{q,t}[n,k] = \sum_{w \in \mathcal{A}(n,k)} q^{A(w)} \cdot t^{B(w)}. \tag{2.9.2}$$

For what follows, let

$$[k]_{q,t} = \begin{cases} (q^{k-2} + q^{k-4} + \cdots + 1) \cdot t & \text{when } k \text{ is even,} \\ q^{k-1} + (q^{k-3} + q^{k-5} + \cdots + 1) \cdot t & \text{when } k \text{ is odd.} \end{cases} \tag{2.9.3}$$

Corollary 2.9.2 *The (q,t)-analogue of Stirling numbers of the first and second kind satisfy the following recurrences:*

$$s_{q,t}[n,k] = s_{q,t}[n-1,k-1] - [n-1]_{q,t} \cdot s_{q,t}[n-1,k] \quad \text{for } n \geq 1 \text{ and } 1 \leq k \leq n, \tag{2.9.4}$$

and

$$S_{q,t}[n,k] = S_{q,t}[n-1,k-1] + [k]_{q,t} \cdot S_{q,t}[n-1,k] \quad \text{for } n \geq 1 \text{ and } 1 \leq k \leq n \tag{2.9.5}$$

with initial conditions $s_{q,t}[n,0] = \delta_{n,0}$ and $S_{q,t}[n,0] = \delta_{n,0}$. For $k > n$, we set $s_{q,t}[n,k] = S_{q,t}[n,k] = 0$.

Proof: Immediate from Theorem 2.3.2 and Theorem 2.7.4. □

Recall the generating polynomials for q-Stirling numbers are

$$(x)_{n,q} = \sum_{k=0}^{n} s_q[n,k] \cdot x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_q[n,k] \cdot (x)_{k,q},$$

where $(x)_{k,q} = \prod_{m=0}^{k-1}(x - [m]_q)$. We can generalize these to (q,t)-polynomials.

Theorem 2.9.3 *The generating polynomials for the (q,t)-Stirling numbers are*

$$(x)_{n,q,t} = \sum_{k=0}^{n} s_{q,t}[n,k] \cdot x^k \,, \tag{2.9.6}$$

and

$$x^n = \sum_{k=0}^{n} S_{q,t}[n,k] \cdot (x)_{k,q,t} \,, \tag{2.9.7}$$

where $(x)_{k,q,t} = \prod_{m=0}^{k-1}(x - [m]_{q,t})$.

Proof: Both identities follow by induction on n. It is straightforward to check the case $n = 0$, so suppose the identities are true for $n-1$. Multiply the recurrence (2.9.4) for the signed (q,t)-Stirling numbers of the first kind by by x^k and sum over all $0 \le k \le n$ to give

$$\begin{aligned}
\sum_{k=0}^{n} s_{q,t}[n,k] \cdot x^k &= \sum_{k=0}^{n}(s_{q,t}[n-1,k-1] - [n-1]_{q,t} \cdot s_{q,t}[n-1,k]) \cdot x^k \\
&= x \cdot \sum_{k=0}^{n-1} s_{q,t}[n-1,k] \cdot x^k - [n-1]_{q,t} \cdot \sum_{k=0}^{n-1} s_{q,t}[n-1,k] \cdot x^k \\
&= (x)_{n-1,q,t} \cdot (x - [n-1]_{q,t}) \\
&= (x)_{n,q,t} \,,
\end{aligned}$$

which is the first identity. For the second identity, multiply the recurrence (2.9.5) for the (q,t)-Stirling number of the second kind by $(x)_{k,q,t}$ and sum over all $0 \le k \le n$ to give

$$\begin{aligned}
\sum_{k=0}^{n} S_{q,t}[n,k] \cdot (x)_{k,q,t} &= \sum_{k=0}^{n}(S_{q,t}[n-1,k-1] + [k]_{q,t} \cdot S_{q,t}[n-1,k]) \cdot (x)_{k,q,t} \\
&= \sum_{k=0}^{n} S_{q,t}[n-1,k-1] \cdot (x)_{k-1,q,t} \cdot (x - [k-1]_{q,t}) \\
&\quad + \sum_{k=0}^{n} [k]_{q,t} \cdot S_{q,t}[n-1,k] \cdot (x)_{k,q,t} \\
&= x \cdot \sum_{k=0}^{n-1} S_{q,t}[n-1,k] \cdot (x)_{k,q,t} \\
&\quad - \sum_{k=0}^{n} [k-1]_{q,t} \cdot S_{q,t}[n-1,k-1] + \sum_{k=0}^{n} [k]_{q,t} \cdot S_{q,t}[n-1,k] \,.
\end{aligned}$$

The last two summations cancel each other by shifting indices. Apply the induction hypothesis on the remaining summation yields the desired result. □

Theorem 2.9.4 *The (q,t)-Stirling numbers are orthogonal, that is, for $m \leq n$*

$$\sum_{k=m}^{n} s_{q,t}[n,k] \cdot S_{q,t}[k,m] = \delta_{m,n} \tag{2.9.8}$$

and

$$\sum_{k=m}^{n} S_{q,t}[n,k] \cdot s_{q,t}[k,m] = \delta_{m,n}. \tag{2.9.9}$$

Furthermore, this orthogonality holds bijectively.

Proof: When $m = n$ since $s_{q,t}[n,n] = S_{q,t}[n,n] = 1$, both identities are trivial. Suppose now that $n > m$. The left-hand side of (2.9.8) is the total weight of the set

$$C = \bigcup_{k=m}^{n} \mathcal{Q}(n, n-k) \times \mathcal{A}(k,m),$$

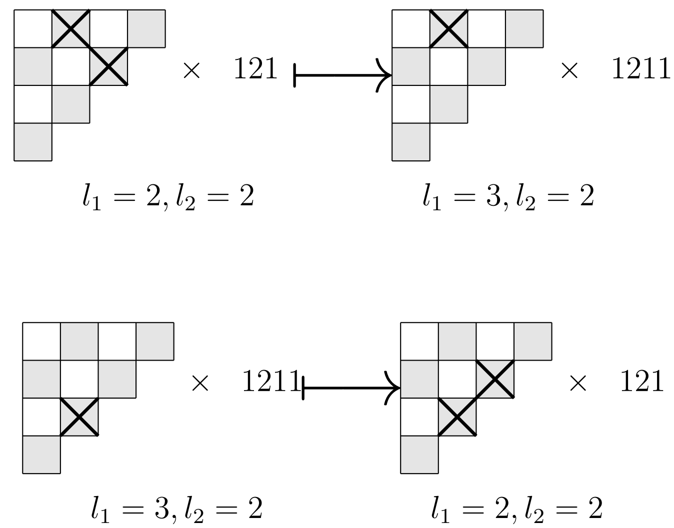
where the weight of $(T,w) \in C$ is defined by

$$\mathrm{wt}(T,w) = (-1)^{n-k} \cdot \mathrm{wt}(T) \cdot \mathrm{wt}(w).$$

Here $\mathrm{wt}(w) = q^{A(w)} \cdot t^{B(w)}$ and $\mathrm{wt}(T) = q^{\mathrm{s}(T)} \cdot t^{\mathrm{r}(T)}$ where the statistics $A(\cdot)$, $B(\cdot)$, $\mathrm{s}(\cdot)$ and $\mathrm{r}(\cdot)$ are defined in Sections 2.3 and 2.7. We wish to show that $\mathrm{wt}(C) = \sum_{(T,w)\in C} \mathrm{wt}(T,w) = 0$ by constructing a weight-preserving sign-reversing involution φ on C with no fixed points.

For any pair $(T,w) \in \mathcal{Q}(n, n-k) \times \mathcal{A}(k,m)$, define the map φ as follows. Label the columns of $T \in \mathcal{Q}(n, n-k)$ from right to left with 1 through $n-1$. Let l_1 be the label of the rightmost column in T that has a rook. If T has no rooks, let $l_1 = \infty$. Denote by $\mathrm{rb}(T)$ the number of squares below the rightmost rook in T. If $l_1 = \infty$, let $\mathrm{rb}(T) = 0$. For $w \in \mathcal{A}(k,m)$, let r be the first repeating (odd) integer reading the entries of w from left to right, and let l_2 denote the number appearing to the left of the entry r in the RG-word w. If there is no repeating integer, let $l_2 = \infty$. Note that $\mathrm{rb}(T)$ must be even.

If $l_1 \leq l_2$, remove the rightmost rook in T to form the rook placement T'. Insert the entry $\mathrm{rb}(T)+1$ to the right of the entry l_1 to obtain the word w'. Since $l_1 \leq l_2$, $\mathrm{rb}(T)+1 \leq l_1 \leq l_2$ and $\mathrm{rb}(T)+1$ is odd, so we have w' is an allowable word of length $k+1$. Hence $(t', w') \in \mathcal{Q}(n, n-k-1) \times \mathcal{A}(k+1, m)$. Also since we removed the rightmost rook in T to obtain T', we know $\mathrm{wt}(T) = q^{l_1} \cdot \mathrm{wt}(T')$ if $\mathrm{rb}(T)+1 = l_1$, that is, the rightmost rook is in the first row, or that $\mathrm{wt}(T) = q^{\mathrm{rb}(T)} \cdot t \cdot \mathrm{wt}(T')$ if $\mathrm{rb}(T)+1 < l_1$, that is, the rightmost rook is not in the first row. We also know that $\mathrm{wt}(w') = q^{l_1-1} \cdot \mathrm{wt}(w)$ if $l_1 = \mathrm{rb}(T)+1$, or $\mathrm{wt}(w') = q^{\mathrm{rb}(T)} \cdot t \cdot \mathrm{wt}(w)$ if $\mathrm{rb}(T)+1 < l_1$. Thus $\mathrm{wt}(T', w') = (-1)^{n-k-1} \cdot \mathrm{wt}(T') \cdot \mathrm{wt}(w') = -\mathrm{wt}(T,w)$.



[Example when $l_1 \le l_2$]

[Example when $l_1 > l_2$.]

Figure 2.9: Examples of the bijection proving the identity (2.9.8).

On the other hand, if $l_1 > l_2$, delete the entry r in w to obtain w'. In column l_2 of T add a rook so that there are $r-1$ empty squares below it. Similarly, one can check that $(T', w') \in \mathcal{Q}(n, n-k+1) \times \mathcal{A}(k-1, m)$ and $\mathrm{wt}(T', w') = -\mathrm{wt}(T, w)$.

Since all pairs $(T, w) \in \mathcal{Q}(n, n-k) \times \mathcal{A}(k, m)$ are mapped under φ, there are no fixed points in C, hence (2.9.8) is true.

The proof of the second identity (2.9.9) follows in a similar fashion. The left-hand side of (2.9.9) is the total weight of the set

$$D = \bigcup_{k=m}^{n} \mathcal{A}(n, k) \times \mathcal{Q}(k, k-m),$$

where $\mathrm{wt}(w, T) = (-1)^{k-m} \cdot \mathrm{wt}(w) \cdot \mathrm{wt}(T)$. We show that

$$\mathrm{wt}(D) = \sum_{(w,T)\in D} \mathrm{wt}(w, T) = 0$$

by constructing a weight-preserving sign-reversing involution ψ on D with no fixed points.

For $(w, T) \in \mathcal{A}(n, k) \times \mathcal{Q}(k, k-m)$, define the following. Let $w_i = r_1$ be the last repeated odd integer in w reading from left to right, and let l_1 be the maximum entry in w occurring before w_i. If there is no repeated entry in w, let $l_1 = 0$. Let l_2 be the label of the leftmost column in T with a rook in it and let r_2 be the number of squares above that rook. If there are no rooks in T let $l_2 = 0$. As before, we are labeling the columns right to left with 1 through $n-1$.

The bijection is built as follows. If $l_1 > l_2$, raise $w_i = r_1$ to $l_1 + 1$ and increase all of the entries to the right of w_i by 1. Denote the new word by w'. Since w_i is the last repeated odd entry, the RG-word w is of the form $w = \cdots l_1 \cdots r_1 (l_1+1)(l_1+2) \cdots k$.

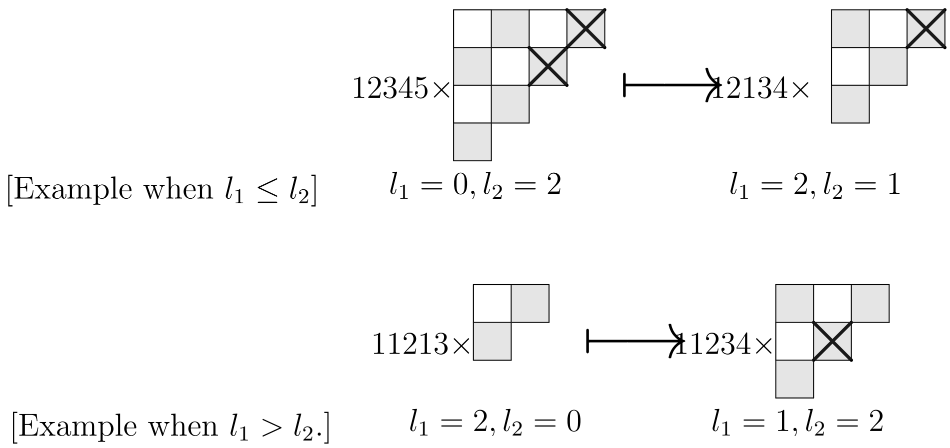


Figure 2.10: Examples of the bijection proving the identity (2.9.9).

Then by definition, the new word w' is of the form $w' = \cdots l_1 \cdots (l_1+1)(l_1+2)(l_1+3)\cdots(k+1)$. This still is an allowable word since the first $i-1$ entries in w' are the same as that in w and the remaining entries form an increasing sequence. So $w' \in \mathcal{A}(n, k+1)$. Also, in w the entries after w_i do not contribute to $\mathrm{wt}(w)$ since there are no repeated entries. When w_i is raised to l_1+1, the weight loss is q^{r_1-1} if $r_1 = l_1$ or $q^{r_1-1}\cdot t$ if $r_1 < l_1$. In the staircase board T, form a new rook placement T' by first adding a column of length k to the left, and then placing a rook in column l_1 counting from right to left such that there are $r_1 - 1$ squares below the rook. Clearly T' has k columns and $k+1-m$ rooks. Since the new rook was placed so that there are now an even number of squares below it, this rook is in a shaded square. Also since $l_1 > l_2$, there is no other rook in column l_1. Hence $T' \in \mathcal{Q}(k+1, k+1-m)$. Observe when we add a rook to obtain T', if the new rook is added in the first row, that is, $r_1 = l_1$ then the weight is increased by q^{r_1-1}. If the new rook is not in the first row, that is, $r_1 < l_1$ then the weight is increased by $q^{r_1-1}\cdot t$. Hence $\mathrm{wt}(w', T') = -\,\mathrm{wt}(w, T)$.

If $l_1 \leq l_2$, replace the entry $w_j = l_2 + 1$ in w by $l_2 - r_2$ and subtract 1 from all of the entries to the right of w_j to obtain w'. Since $w = \cdots l_1 \cdots r_1(l_1+1)\cdots k$ and $l_1 \leq l_2 \leq k-1$, we have that $w_j = l_2+1$ appears to the right of w_i and hence such an entry is unique. Also $r_2+1 \leq l_2$ gives $l_2 - r_2 \geq 1$. This difference is always odd by the fact that the rook is in a shaded square. So $w' = \cdots l_1 \cdots l_2(l_2-r_2)(l_2+1)\cdots(k-1)$ is an RG-word with even integers appearing just once, hence $w' \in \mathcal{A}(n, k-1)$. The entry $w'_{j-1} = l_2$, and $w'_j = l_2 - r_2$ contributes a weight of $q^{l_2-r_2-1}$ if $l_2 = l_2 - r_2$, that is, $r_2 = 0$ or $q^{l_2-r_2-1}\cdot t$ if $r_2 > 0$. Delete the column l_2 in T and delete one square from the bottom in all columns to the left of column l_2 to make the new staircase chessboard T'. It is straightforward to check that $T' \in \mathcal{Q}((k-1, k-1-m)$. Deleting the rook in T will decrease its weight by $q^{l_2-(r_2+1)}$ if the rook is in the first row, that is, $r_2 = 0$ or by $q^{l_2-r_2-1}\cdot t$ if the rook is not in the first row, that is, $r_2 > 0$. Hence $\mathrm{wt}(w', T') = -\,\mathrm{wt}(w, T)$. The map we described is a weight-preserving sign-reversing involution with no fixed points, so the orthogonality in (2.9.9) follows. □

See Figures 2.9 and 2.10 for examples of the bijections occurring in the proof of Theorem 2.9.4.

2.10 Concluding remarks

The Stirling numbers of the first kind and second kind are specializations of the homogeneous and elementary symmetric functions:

$$S(n,k) = h_{n-k}(x_1,\dots,x_k), \qquad c(n,k) = e_{n-k}(x_1,\dots,x_{n-1}), \tag{2.10.1}$$

where $x_m = m$. The q-Stirling numbers are also specializations of these symmetric functions with $x_m = [m]_q$. See [46, Chapter I, Section 2, Example 11]. For the (q,t)-versions take $x_m = [m]_{q,t}$ as defined in (2.9.3). A more general statement of orthogonality is

$$\sum_{k=j}^{n} (-1)^{n-k} \cdot e_{n-k}(x_1,\dots,x_{n-1}) \cdot h_{k-j}(x_1,\dots,x_j) = \delta_{n,j}. \tag{2.10.2}$$

The specializations imply orthogonality of the (q,t)-Stirling numbers, though not combinatorially as in Theorem 2.9.4. It remains to find a combinatorial proof of Theorem 2.9.3.

Stembridge's $q=-1$ phenomenon [72, 74], and the more general cyclic sieving phenomenon of Reiner, Stanton and White [61] counts symmetry classes in combinatorial objects by evaluating their q-generating series at a primitive root of unity. Are there any instances of the cyclic sieving phenomenon related to the q-Stirling numbers of the first and second kind?

Garsia and Remmel [31] have a more general notion of the q-Stirling number of the second kind as enumerating non-attacking rooks on a general Ferrers board. One may also find a q-$(1+q)$-analogue for the q-Stirling numbers of the second kind via rook placements.

It would be interesting to look deeper into the poset structure of the Stirling posets of the first and second kind, such as the interval structure and its f- and h-triangles. Park has a notion of the Stirling poset which arises from the theory of P-partitions [56]. It has no connection with the Stirling posets in this chapter.

The q-binomial has the combinatorial interpretation of counting certain subspaces over a finite field with q elements as well as the corresponding subspace lattice. Milne [50] has an interpretation of the q-Stirling number of the second kind as sequences of lines in a vector space over the finite field with q elements. Is there an analogous interpretation for the (q,t)-Stirling numbers of the second kind? Bennett, Dempsey and Sagan [3] construct families of posets which include Milne's construction. One would like a similar construction for the q-Stirling numbers of the first kind.

In [79] Wachs and White have discovered many other statistics on RG-words which generate the q-Stirling numbers. In particular, their ls and lb statistics are defined by $ls(w) = \prod_{i=1}^{n} q^{w_i - 1}$ and $lb(w) = \prod_{i=1}^{n} lb_i(w)$ where $lb_i(w) = q^{m_{i-1} - w_i}$ if $m_{i-1} \geq w_i$

and $lb_i(w) = 1$ if $m_{i-1} < w_i$. The ls statistic and the wt statistic in (2.2.2) are related by $ls(w) = q^{\binom{k}{2}} \cdot \mathrm{wt}(w)$. The authors are currently looking at these statistics, as well as White's interpolations [80] between these statistics, in view of the main Goal 2.1.1, as well as poset theoretic and homological consequences.

In terms of this research project a compact expression of q-$(1+q)$-binomial via the major index is discussed in Chapter 4.

Chapter 3 q-Stirling identities revisited

In this chapter we give combinatorial proofs of q-Stirling identities. This includes a poset theoretic proof of Carlitz's identity and a new proof of the q-Frobenius identity of Garsia and Remmel. This chapter represents joint work with Readdy, Ehrenborg collaborated on Theorems 3.6.1 and 3.6.3, and also suggested finding a new proof of Theorem 3.6.5.

3.1 Preliminaries

A set partition on n elements $[n] = \{1, 2, \ldots, n\}$ is a decomposition of this set into mutually disjoint nonempty subsets called blocks. The classical Stirling number of the second kind $S(n, k)$ is the number of set partitions of n elements into k blocks. There is a long history of studying set partitions [31, 45, 63] and Stirling numbers of the second kind [11, 21, 34, 51, 79].

Given a set partition of n elements into k blocks $\pi = B_1/B_2/\cdots/B_k$, where $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$, associate to it a restricted growth word $w = w(\pi) = w_1 w_2 \cdots w_n$, also called an *RG*-word, such that $w_i = j$ if the entry i appears in the j-th block B_j of π. The *RG*-words are also known as restricted growth functions, that is, functions of the form $f(x) : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ where $f(1) = 1$ and $f(i) \le \max(f(j) : j < i) + 1$. For further details, see work of Milne [50, 51].

Definition 3.1.1 *Given an integer sequence $w = w_1 w_2 \cdots w_n$ of length n and maximal entry k, let $\sigma(w)$ be the sum of all its entries, that is, $\sigma(w) = \sum_{i=1}^{n} w_i$. Define the weight of the word w by* $\mathrm{wt}(w) = q^{\sigma(w)-n-\binom{k}{2}}$.

Observe this weighting implies the first occurrence of the entry i is weighted by 1 and the other occurrences are weighted by q^{i-1}.

Let $\mathcal{R}(n, k)$ denote the set of all *RG*-words of length n with maximal entry k and let $w \in \mathcal{R}(n, k)$ be an *RG*-word. Following Chapter 2, define the *q-Stirling numbers of the second kind* to be

$$S_q[n, k] = \sum_{w \in \mathcal{R}(n,k)} \mathrm{wt}(w), \quad \text{for } 0 < k \le n, \tag{3.1.1}$$

with $S_q[n, 0] = \delta_{n,0}$ and $S_q[0, k] = \delta_{0,k}$. It is straightforward to verify this definition satisfies the usual defining recurrence for q-Stirling numbers of the second kind, that is,

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q \cdot S_q[n-1, k] \quad \text{for } 1 \le k \le n,$$

with the boundary conditions $S_q[n, 0] = \delta_{n,0}$ and $S_q[0, k] = \delta_{0,k}$. Here $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$ is the q-analogue of the positive integer m. For other weightings of *RG*-words which generate the q-Stirling numbers $S_q[n, k]$, see [79].

3.2 Recurrence related identities

In this section we focus on recurrence structured identities for the q-Stirling numbers of the second kind. The proofs we provide here use the combinatorics of *RG*-words.

We begin with Mercier's identity [49, Theorem 3]. Mercier's original proof of Theorem 3.2.1 was by induction. Later a combinatorial proof using 0-1 tableaux was given by de Médicis and Leroux [17]. In the same paper, de Médicis and Leroux proved Theorems 3.2.2 and 3.2.3 using 0-1 tableaux.

Theorem 3.2.1 (Mercier, 1990)

$$S_q[n+1, k+1] = \sum_{m=k}^{n} \binom{n}{m} \cdot q^{m-k} \cdot S_q[m, k]. \tag{3.2.1}$$

Proof: For any *RG*-word $w \in \mathcal{R}(n+1, k+1)$, suppose there are m entries in w that are not equal to one. Remove the $n+1-m$ entries equal to 1 in w and then subtract one from each of the remaining m entries to obtain a new word u. Observe $u \in \mathcal{R}(m, k)$ and each of the $m-k$ repeated entries loses q from its weight. We have $\mathrm{wt}(w) = q^{m-k} \cdot \mathrm{wt}(u)$. Conversely, given a word $u \in \mathcal{R}(m, k)$, one can first increase each of the m entries by one and then insert $n+1-m$ ones into the sequence to obtain an *RG*-word $w \in \mathcal{R}(n+1, k+1)$. There are $\binom{n}{n-m} = \binom{n}{m}$ ways to insert the $n+1-m$ ones since the first entry in an *RG*-word must be one. In other words, for any $u \in \mathcal{R}(m, k)$ we can obtain $\binom{n}{m}$ new *RG*-words in $\mathcal{R}(n+1, k+1)$ under the map described above, which gives the desired identity. $\Box$

Theorems 3.2.2 and 3.2.3 appear in [17, Propositions 2.3 and 2.5]. We now give straightforward proofs of each result using *RG*-words.

Theorem 3.2.2 (de Médicis–Leroux, 1993)

$$S_q[n+1, k+1] = \sum_{j=k}^{n} [k+1]_q^{n-j} \cdot S_q[j, k]. \tag{3.2.2}$$

Proof: For an *RG*-word $w \in \mathcal{R}(n+1, k+1)$ consider the factorization $w = x \cdot (k+1) \cdot y$ where $x \in \mathcal{R}(j, k)$ for some $j \geq k$. The remaining word y is of length $n-j$ and the entries can be chosen arbitrarily with integers from $\{1, 2, \ldots, k+1\}$. The sum of the weights of these words is $[k+1]_q^{n-j} \cdot S_q[j, k]$. The result follows by summing over all possible integers j. $\Box$

Theorem 3.2.3 (de Médicis–Leroux, 1993)

$$(n-k) \cdot S_q[n, k] = \sum_{j=1}^{n-k} S_q[n-j, k] \cdot ([1]_q^j + [2]_q^j + \cdots + [k]_q^j). \tag{3.2.3}$$

Proof: For an *RG*-word $w \in \mathcal{R}(n,k)$ consider a factorization $w = x \cdot y \cdot z$ with the following properties: (1) the largest letter of x is the last letter of x reading from left to right; call this letter i. (2) the word y is non-empty and all letters of y are at most i.

We claim that the number of such factorizations of w is $n-k$. Let s_i be the number of letters between the first occurrence of i and the first occurrence of $i+1$ and let s_k be the number of letters after the first occurrence of k. Thus for this particular i, we have s_i choices for the word y. But $\sum_{i=1}^{k} s_i = n-k$ since there are $n-k$ repeated letters in w. This completes the claim.

Fix integers $1 \leq j \leq n-k$ and $1 \leq i \leq k$. Given a word u in $\mathcal{R}(n-j,k)$, we can factor it uniquely as $x \cdot z$, where the last letter of x is the first occurrence of i in the word u. Pick y to be any word of length j with letters at most i. Finally, let $w = x \cdot y \cdot z$. Observe that this is a factorization satisfying the conditions from the previous paragraph. For any $w = w_1 w_2 \cdots w_n$, let $ls(w) = \prod_{i=1}^{n} q^{w_i - 1}$ be the ls-weight of w. This is a generalization of the ls-statistic of *RG*-words [79, Section 2]. Then $\mathrm{wt}(w) = \mathrm{wt}(u) \cdot ls(y)$. Summing over all words $u \in \mathcal{R}(n,k)$ and words y yields $S_q[n-j,k] \cdot [i]_q^j$. Lastly, summing over all i and j gives the desired equality. □

3.3 A poset proof of Carlitz's identity

In Section 2.4 we defined the *Stirling poset of the second kind* $\Pi(n,k)$. The elements of this graded poset are *RG*-words in $\mathcal{R}(n,k)$ with the cover relation $v = v_1 v_2 \cdots v_n \prec w$ if $w = v_1 v_2 \cdots (v_i+1) \cdots v_n$ for some index i. A Boolean algebra decomposition of this poset is given which reveals homological properties of this poset, see Theorem 2.5.1. We state here the decomposition theorem and generalize this idea to give a poset proof of Carlitz's identity. For basic poset terminology and background, we refer the reader to Stanley's treatise [69, Chapter 3].

An *RG*-word $w \in \mathcal{R}(n,k)$ is said to be *allowable* if each even entry in w appears exactly once. Denote by $\mathcal{A}(n,k)$ the set of all allowable *RG*-words of length n and maximal entry k. For an allowable *RG*-word $w = w_1 \cdots w_n$ define $\mathrm{wt}'(w) = \prod_{i=1}^{n} \mathrm{wt}'_i(w)$, where for $m_i = \max(w_1, \ldots, w_i)$

$$\mathrm{wt}'_i(w) = \begin{cases} q^{w_i - 1} \cdot (1+q) & \text{if } m_{i-1} > w_i, \\ q^{w_i - 1} & \text{if } m_{i-1} = w_i, \\ 1 & \text{if } m_{i-1} < w_i \text{ or } i = 1. \end{cases} \tag{3.3.1}$$

To state the poset decomposition, we need two definitions. For $w \in \mathcal{A}(n,k)$ an allowable word, let $\mathrm{Inv_r}(w) = \{i : w_j > w_i \text{ for some } j < i\}$ be the set of all indices in w that contribute to the right-hand element of an inversion pair. For $i \in \mathrm{Inv_r}(w)$ such an entry w_i must be odd since in a given allowable word any entry occurring to the left of an even entry must be strictly less than it. Finally, for $w \in \mathcal{A}(n,k)$ let $\alpha(w)$ be the word formed by incrementing each of the entries indexed by the set $\mathrm{Inv_r}(w)$ by one. Additionally, for $w \in \mathcal{A}(n,k)$ and any $I \subseteq \mathrm{Inv_r}(w)$, the word formed by incrementing each of the entries indexed by the set I by one are elements of $\mathcal{R}(n,k)$ since if $i \in \mathrm{Inv_r}(w)$ then there is an index $h < i$ with $w_h = w_i$.

Theorem 3.3.1 (Cai–Readdy, 2015) *The Stirling poset of the second kind* $\Pi(n,k)$ *can be decomposed as the disjoint union of Boolean intervals*

$$\Pi(n,k) = \dot{\bigcup_{w \in \mathcal{A}(n,k)}} [w, \alpha(w)].$$

Furthermore, if an allowable word $w \in \mathcal{A}(n,k)$ *has weight* $\mathrm{wt}'(w) = q^i \cdot (1+q)^j$, *then the interval* $[w, \alpha(w)]$ *is isomorphic to the Boolean algebra on* j *elements and the minimal element* w *in this interval is of rank* i *in the Stirling poset of the second kind* $\Pi(n,k)$.

Theorem 3.3.2 *The* $(n-1)$*-fold Cartesian product of the* m*-chain, where* $m \geq k$, *has the decomposition*

$$(C_m)^{n-1} = \dot{\bigcup_{1 \leq k \leq n}} \dot{\bigcup_{w \in \mathcal{A}(n,k)}} [w, \alpha(w)] \times C_{m-1} \times C_{m-2} \times \cdots \times C_{m-k+1}, \tag{3.3.2}$$

where C_j *denotes the chain on* j *elements.*

As a remark, if $\mathrm{wt}'(w) = q^i \cdot (1+q)^j$ then by Theorem 3.3.1 the interval $[w, \alpha(w)]$ in the decomposition satisfies $[w, \alpha(w)] \cong (C_2)^j$. Thus one can also view the right-hand side of (3.3.2) as a disjoint union of a Cartesian product of chains.
Proof: For each allowable word $w = w_1 w_2 \cdots w_n \in \mathcal{A}(n,k)$ with $\mathrm{wt}'(w) = q^i \cdot (1+q)^j$, we build up the lattice $[w, \alpha(w)] \times C_{m-1} \times \cdots \times C_{m-k+1}$ as follows.

Decompose w as $w = x_1 \cdot 2 \cdot y_1$ where $x_1 = 11 \cdots 1 \in \mathcal{R}(i_2 - 1, 1)$ for some index i_2. Then by the construction of the Boolean interval $[w, \alpha(w)]$, every RG-word $u \in [w, \alpha(w)]$ has a decomposition of the form $u = x_1 \cdot 2 \cdot z_1$ for some z_1. Define a chain $C_1(u)$ consisting of all the words $x_1 \cdot j \cdot z_1$ where $2 \leq j \leq m$ with the linear order on j. Moreover, if $s \in C_1(u)$ and $t \in C_1(v)$ for some $u \neq v$, then define $s \leq t$ if $u \leq v$. The union of all such chains form a poset. Moreover,

$$\dot{\bigcup_{u \in [w, \alpha(w)]}} C_1(u) \cong [w, \alpha(w)] \times C_{m-1}.$$

Denote this poset by I_1. Next, find an index i_3 such that $w = x_2 \cdot 3 \cdot y_2$ where $x_2 \in \mathcal{R}(i_3 - 1, 2)$. Again, for any $u \in I_1$, it has a decomposition of the form $x_2 \cdot 3 \cdot z_2$. Let the chain $C_2(u)$ be a chain with the linear order on all words of the form $x_2 \cdot j \cdot z_2$ for $3 \leq j \leq m$ and extend to the partial order such that if $s \in C_2(u)$ and $t \in C_2(v)$ for some $u \neq v$, then define $s \leq t$ if $u \leq v$. Similarly we have

$$\dot{\bigcup_{u \in I_1}} C_2(u) \cong I_1 \times C_{m-2} \cong [w, \alpha(w)] \times C_{m-1} \times C_{m-2}.$$

Repeat the construction until we find the index i_k, where $w = x_{k-1} \cdot k \cdot y_{k-1}$ and $x_{k-1} \in \mathcal{R}(i_k - 1, k-1)$. Then we have $I_{k-1} = [w, \alpha(w)] \times C_{m-1} \times \cdots \times C_{m-k+2}$. Define a chain $C_k(u)$ for every $u = x_{k-1} \cdot k \cdot z_{k-1} \in I_{k-1}$ consisting of the words $x_{k-1} \cdot j \cdot z_{k-1}$

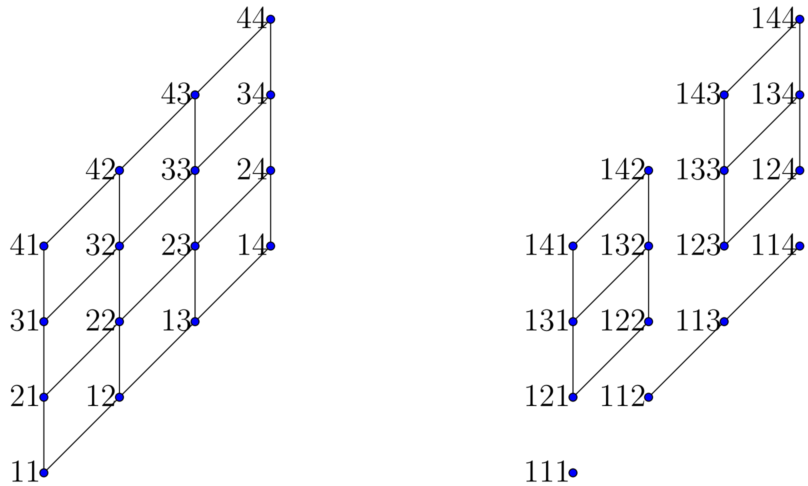


Figure 3.1: The lattice $(C_4)^2$ and its decomposition. Each subposet in the decomposition has an allowable *RG*-word as its minimal element.

where $k \leq j \leq m$ and extend the partial order similarly to the union of all chains. We obtain

$$\begin{aligned}\dot{\bigcup_{u \in I_{k-1}}} C_k(u) &\cong C_k(u) \times C_{m-k+1} \\ &\cong [w, \alpha(w)] \times C_{m-1} \times C_{m-2} \times \cdots \times C_{m-k+1}\end{aligned}$$

For $w \in \mathcal{A}(n,k)$ denote this resulting poset by $L_m(n,k;w)$

Notice that each element $u \in L_m(n,k;w)$ corresponds to an integer sequence beginning with 1. Deleting the first entry of any $u \in L_m(n,k;w)$ gives an integer sequence of length $n-1$ with maximal entry at most m. This is an element in the poset $(C_m)^{n-1}$ with the cover relation preserved. Hence each $L_m(n,k;w)$ is isomorphic to a subposet of $(C_m)^{n-1}$.

It remains to show that every element $a \in (C_m)^{n-1}$ belongs to exactly one subposet $L_m(n,k;w)$ in the decomposition for some n, k and allowable *RG*-word w. Suppose a is of the form $a_2 a_3 \cdots a_n$ with $a_i \leq m$. Let $u = 1 \cdot a$, that is, $u = u_1 u_2 \cdots u_n$ with $u_1 = 1$ and $u_i = a_i$ for $2 \leq i \leq n$. Let $i_1, i_2, \ldots, i_{m-1}$ be the indices such that $u_{i_j} \geq j+1$ and i_j is the first index satisfying the condition. Let $w = w_1 w_2 \cdots w_n$ be an integer sequence such that $w_{i_j} = j+1$ and $w_s = u_s$ for $s \neq i_j$. It is straightforward to check that w is an *RG*-word. By Theorem 3.3.1, w belongs to exactly one of the Boolean algebras in the decomposition of $\Pi(n,k)$, hence u, or a belongs to exactly one of the posets $L_m(n,k;w)$. $\square$

See Figures 3.1 and 3.2 for examples of this decomposition.

By considering the rank generating function of identity (3.3.2), we can obtain a poset theoretic proof of Carlitz's identity [12, section 3]. Other proofs of Carlitz's identity are due to Milne using finite operator techniques on restricted growth functions [51], de Médicis and Leroux via interpreting the identity as counting products

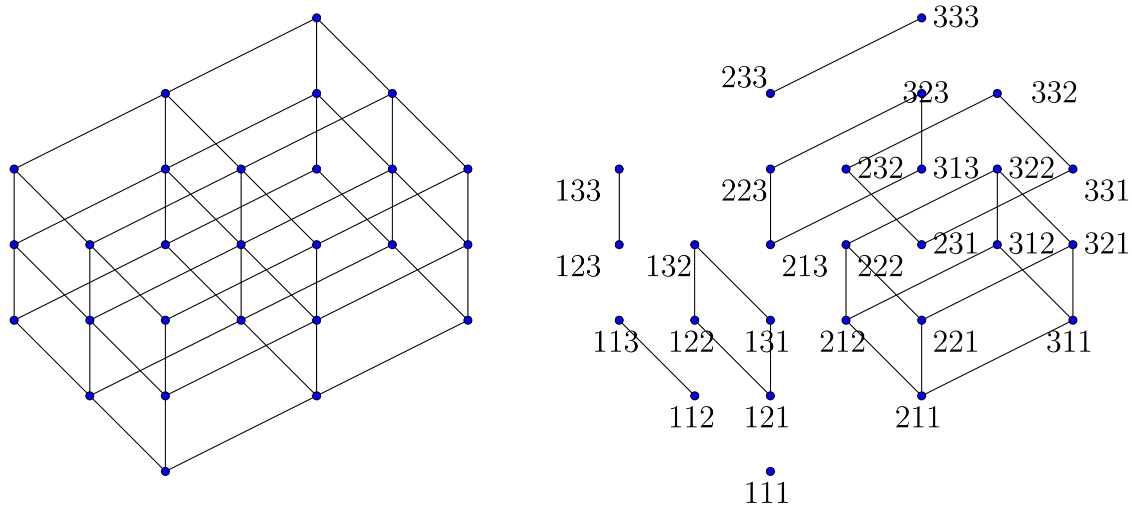


Figure 3.2: The lattice $(C_3)^3$ and its decomposition.

of matrices over $\mathbb{F}_q$ with non-zero columns [17], and Ehrenborg and Readdy using the theory of juggling [21].

Corollary 3.3.3 (Carlitz, 1948) *The following q-identity holds:*

$$[m]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} \cdot S_q[n,k] \cdot [k]_q! \cdot \begin{bmatrix} m \\ k \end{bmatrix}_q . \tag{3.3.3}$$

Proof: The cases when $n = 0$ or $m = 0$ are trivial. Now suppose $n, m \neq 0$. Equation (3.3.3) can be rewritten as

$$[m]_q^{n-1} = \sum_{k=1}^{n} q^{\binom{k}{2}} \cdot S_q[n,k] \cdot [m-1]_q \cdot [m-2]_q \cdots [m-k+1]_q . \tag{3.3.4}$$

The left-hand side of (3.3.4) is the rank generating function of the poset $(C_m)^{n-1}$. It suffices to show the right-hand side of (3.3.4) is the rank generating function of the poset decomposition $\Delta = \dot{\bigcup}_{w \in \mathcal{A}(n,k)} L_m(n,k;w)$.

For each $w \in \mathcal{A}(n,k)$, the rank generating function of $[w, \alpha(w)] \times C_{m-1} \times C_{m-2} \times \cdots \times C_{m-k+1}$ is

$$\mathrm{wt}'(w) \cdot [m-1]_q \cdot [m-2]_q \cdots [m-k+1]_q .$$

However, as an element of the poset $(C_m)^{n-1}$, the rank of w is lifted by $1 + 2 + \cdots + k - 1 = \binom{k}{2}$. Thus the subposet $L_m(n,k;w)$ of $(C_m)^{n-1}$ has generating function

$$q^{\binom{k}{2}} \cdot \mathrm{wt}'(w) \cdot [m-1]_q \cdot [m-2]_q \cdots [m-k+1]_q .$$

Summing up over $w \in \mathcal{A}(n,k)$ gives

$$\sum_{w\in\Delta} q^{\rho(w)} = \sum_{w\in\mathcal{A}(n,k)} q^{\binom{k}{2}} \cdot \mathrm{wt}'(w) \cdot [m-1]_q \cdot [m-2]_q \cdots [m-k+1]_q$$
$$= q^{\binom{k}{2}} \cdot S_q[n,k] \cdot [m-1]_q \cdot [m-2]_q \cdots [m-k+1]_q,$$

which completes the proof. □

Remark 3.3.4 Similar poset proof techniques of Theorem 3.3.2 and Corollary 3.3.3 can be applied to obtain the identities in Section 3.2. The proofs are omitted.

3.4 Generating function

Gould [34, Equation (3.4)] gave an analytic proof for the ordinary generating function of the q-Stirling numbers of the second kind. Wachs and White [79] stated a p,q-version of this generating function without proof. Here we prove Gould's q-generating function using RG-words.

Theorem 3.4.1 (Gould, 1961) *The q-Stirling numbers of the second kind $S_q[n,k]$ have the generating function*

$$\sum_{n\geq k} S_q[n,k] \cdot t^n = \frac{t^k}{\prod_{i=1}^{k}(1-[i]_q \cdot t)}. \tag{3.4.1}$$

Proof: The left-hand side is the sum of over all RG-words w of length at least k and with largest letter k, where the term is $\mathrm{wt}(w) \cdot t^{\ell(w)}$. Every such RG-word has a unique factorization as $w = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdots k \cdot u_k$ where u_j is a word in the letters $\{1, 2, \ldots, j\}$. Observe that a letter i in the word u_j contributes a factor $q^{i-1} \cdot t$ to the term, whereas a left-to-right maximum j just contributes the factor t. Since there are no restrictions of the length of jth word u_j, all the u_j words contribute the factor $1+[j]_q \cdot t+[j]_q^2 \cdot t^2+\cdots = 1/(1-[j]_q \cdot t)$. By multiplying together all the contributions from all the words u_j and the k left-to-right maximums, the identity follows. □

3.5 Convolution

Verde-Star gave a Vandermonde convolution for Stirling numbers [77, Equations (6.24), (6.25)]. Chen then gave a grammatical proof for the first identity [13, Proposition 4.1]. For q-analogues of both identitites, de Médicis and Leroux used $0,1$-tableaux for their argument [18, Equations (1.12), (1.14)]. Here we present a combinatorial proof of the de Médicis–Leroux result using RG-words.

Theorem 3.5.1 (de Médicis–Leroux, 1995) *The following q-Vandermonde convolutions hold for q-Stirling numbers of the second kind:*

$$S_q[m+n,k] = \sum_{i+j\geq k} \binom{m}{j} \cdot q^{i\cdot(i+j-k)} \cdot [i]_q^{m-j} \cdot S_q[n,i] \cdot S_q[j,k-i] \tag{3.5.1}$$

and

$$S_q[n+1,k+\ell+1] = \sum_{i=0}^{n} \sum_{j=\ell}^{i} \binom{i}{j} \cdot q^{(k+1)\cdot(j-\ell)} \cdot [k+1]_q^{i-j} \cdot S_q[j,\ell] \cdot S_q[n-i,k]. \tag{3.5.2}$$

Note that Theorem 3.2.2 is a special case of Equation (3.5.2) when one takes $\ell = 0$.

Proof: We first prove Equation (3.5.1). Given an *RG*-word $w \in \mathcal{R}(m+n,k)$ factor w as $u\cdot\bar{u}$ where u has length n and i is the largest entry in u. By assumption, $u \in \mathcal{R}(n,i)$.

The remaining segment $\bar{u} = w_{n+1} \cdot w_{n+2} \cdots w_{n+m}$ has length m and its maximal entry is at most k. In particular, if $i < k$ then the maximal entry for $\bar{u}$ is exactly k. Suppose there are j entries in $\bar{u}$ that are strictly larger than i. These j entries from $\bar{u}$ form a new integer sequence v. Denote by $v^{(-i)}$ the shift of v by subtracting i from each entry in v. It is straightforward to check that $v^{(-i)} \in \mathcal{R}(j,k-i)$.

For any *RG*-word $w \in \mathcal{R}(m+n,k)$, we can decompose it as described above. In such a decomposition, the first segment u contributes to a factor of $S_q[n,i]$. The subsequence v of the second segment $\bar{u}$ contributes to a factor of $S_q[j,k-i]\cdot q^{i\cdot(j-(k-i))}$ since the shift $v^{(-i)}$ will give a weight loss of q^i to $j-(k-i)$ repeated entries in v. Finally, the rest of the entries in $\bar{u}$ that are less than or equal to i range from 1 to i. Each will contribute to a factor of $[i]_q$. These $m-j$ entries can be assigned at any position in $\bar{u}$, which gives $\binom{m}{j}$ choices. Multiplying all these weights, we obtain the desired identity.

Identity (3.5.2) is proved in a similar fashion. For any $w \in \mathcal{R}(n+1,k+\ell+1)$, suppose w is of the form $x \cdot (k+1) \cdot y$ where $x \in \mathcal{R}(n-i,k)$ for some i. Now we consider the remaining word $y = w_{n-i+2} \cdots w_{n+1}$ of length i. The maximal entry of y is $k+\ell+1$. Suppose there are j entries in y that are at least $k+2$. These j entries form a new sequence v, and $v^{(-k-1)}$, obtained by subtracting $k+1$ from each entry in v, is an *RG*-word in $\mathcal{R}(j,\ell)$, giving a total weight of $S_q[j,\ell]$. The weight loss from the shift is $q^{(k+1)\cdot(j-\ell)}$ since there are $j-\ell$ repeated entries. The remaining $i-j$ entries in y can be any value from the set $\{1,2,\ldots,k+1\}$. Each such entry contributes to a factor of $[k+1]_q$. Finally, there are $\binom{i}{j}$ ways to place the j entries back into u. This proves identity (3.5.2). $\square$

3.6 Other identities

Decomposing *RG*-words in different ways can be used to prove more identities as well as constructing new identities. The following new identity uses this idea.

Theorem 3.6.1 *Given $n \geq 2$ and any given positive integer r, the following identity holds:*

$$\begin{aligned}
&\sum_{k=r}^{n}(-1)^{k-r} \cdot q^{(r-1)\cdot(k-r)} \cdot [k-r]_q! \cdot S_q[n,k] \\
&\quad = \sum_{c_1+c_2+\cdots c_{r-1}=n-r+1} c_{r-1} \cdot [1]_q^{c_1} \cdot [2]_q^{c_2} \cdot [3]_q^{c_3} \cdots [r-2]_q^{c_{r-2}} \cdot [r-1]_q^{c_{r-1}-1} && (3.6.1)\\
&\quad = \sum_{m=r-1}^{n-1} S[m,r-1] \cdot [r-1]^{n-m-1}. && (3.6.2)
\end{aligned}$$

Proof: For an RG-word $w \in \mathcal{R}(n,k)$, define its f-weight as

$$f(w) = (-1)^{k-r} \cdot q^{(r-1)\cdot(k-r)} \cdot [k-r]_q! \cdot \mathrm{wt}(w),$$

and $f(S) = \sum_{w \in S} f(w)$ is the f-weight of a subset S of RG-words. The left-hand side of Equation (3.6.1) is then the sum of f-weights of all RG-words in the disjoint union $C = \biguplus_{r \leq k \leq n} \mathcal{R}(n,k)$.

Consider the subset $D \subseteq \mathcal{R}(n,r)$ consisting of all RG-words $w \in \mathcal{R}(n,r)$ with the letter r occurring exactly once. Such an RG-word is of the form

$$w = 1 \cdot u_1 \cdot 2 \cdot u_2 \cdot 3 \cdots r-1 \cdot u_{r-1} \cdot r \cdot u'_{r-1},$$

where u_i and u'_i are sequences consisting of the integers $1, 2, \ldots, i$. All possible sequences u_i of length c_i give total weight of $[i]_q^{c_i}$ for $i \leq r-2$. For the sequence $u_{r-1} \cdot r \cdot u'_{r-1}$, suppose its length is c_{r-1}. Then we have c_{r-1} choices to place the letter r and the total weight of $u_{r-1} \cdot u'_{r-1}$ will be $[r-1]_q^{c_{r-1}-1}$. Hence the f-weight for the set D is

$$f(D) = \sum_{c_1+c_2+\cdots c_{r-1}=n-r+1} c_{r-1} \cdot [1]_q^{c_1} \cdot [2]_q^{c_2} \cdot [3]_q^{c_3} \cdots [r-2]_q^{c_{r-2}} \cdot [r-1]_q^{c_{r-1}-1}.$$

In particular, if $r = 1$, the set $D \subseteq \mathcal{R}(n,1)$ is empty since $n \geq 2$ and hence $f(D) = 0$.

Now we consider the set $C - D$. We will show the f-weight on this set is zero by first decomposing the set C into blocks and then constructing a matching on the blocks of $C - D$ such that the f-weights on the two matched blocks is zero.

We first discuss the case when $r \geq 2$.

For any integer k where $r \leq k \leq n$, we say two RG-words $w, v \in \mathcal{R}(n,k)$ are equivalent if there exists an index i such that $r \leq w_i, v_i \leq k$ and $w = x \cdot w_i \cdot u_{r-1}$, $v = x \cdot v_i \cdot u_{r-1}$ for some RG-word $x \in \mathcal{R}(i-1,k)$ and u_{r-1} a sequence consisting of integers from $\{1, 2, \ldots, r-1\}$. If a word $w \in \mathcal{R}(n,k)$ is of the form $w = y \cdot k \cdot u_{r-1}$ for some $y \in \mathcal{R}(i-1,k-1)$, then this word is not equivalent to any other words and hence it belongs to a singleton block. Since for any RG-word we can find a decomposition described as above, this is a partition of the set $\mathcal{R}(n,k)$ and hence the set C.

If $w = y \cdot k \cdot u_{r-1} \in \mathcal{R}(n,k)$ for some $y \in \mathcal{R}(i-1,k-1)$, we match it to the block

$$B = \{y \cdot j \cdot u_{r-1} : r \leq j \leq k-1\}.$$

Notice this implies $k \geq r+1$. It is straightforward to check that $B \subseteq \mathcal{R}(n,k-1)$. Moreover, the f-weight on the set B satisfies

$$\begin{aligned}
f(B) &= \sum_{j=r+1}^{k-1} (-1)^{k-r-1} \cdot q^{(r-1)\cdot(k-r-1)} \cdot [k-r-1]_q! \cdot \mathrm{wt}(y) \cdot q^{j-1} \cdot ls(u_{r-1}) \\
&= (-1)^{k-r-1} \cdot q^{(r-1)\cdot(k-r-1)} \cdot [k-r-1]_q! \cdot \mathrm{wt}(y) \cdot (q^{r-1} + \cdots + q^{k-2}) \cdot ls(u_{r-1}) \\
&= (-1)^{k-r-1} \cdot q^{(r-1)\cdot(k-r-1)} \cdot [k-r-1]_q! \cdot \mathrm{wt}(y) \cdot q^{r-1} \cdot [k-r]_q \cdot ls(u_{r-1}) \\
&= -(-1)^{k-r} \cdot q^{(r-1)\cdot(k-r)} \cdot [k-r]_q! \cdot \mathrm{wt}(y \cdot k \cdot u_{r-1}) \\
&= -f(w). \qquad (3.6.3)
\end{aligned}$$

Hence the weight on w and B cancel each other.

On the other hand, for the equivalence class $B = \{x \cdot j \cdot u_{r-1} : r \leq j \leq k\} \subseteq \mathcal{R}(n,k)$, where $x \in \mathcal{R}(i-1,k)$, we consider the RG-word $v = x \cdot (k+1) \cdot u_{r-1} \in \mathcal{R}(n,k+1)$. The above matching maps the block B to v and their weights cancel by a similar computation as in Equation (3.6.3).

If a singleton block $\{w\}$ is matched to some block B in the above construction, it implies that $k \geq r+1$. Thus the set D is unmatched in this matching and $f(C-D) = 0$. Hence the identity follows.

For the case $r = 1$, we define the equivalence relation based on the last letter. That is, two RG-words $w, v \in \mathcal{R}(n,k)$ are equivalent if they are of the form $w = x \cdot w_n$ and $v = x \cdot v_n$ for some $x \in \mathcal{R}(n-1,k)$. Also, an RG-word $w \in \mathcal{R}(n,k)$ belongs to a singleton block if it is of the form $w = y \cdot k$ for some $y \in \mathcal{R}(n-1,k-1)$. Notice in this case, any word in C belongs to a block. Again, each singleton block can be matched to the block

$$B = \{y \cdot j : 1 \leq j \leq k-1\}.$$

Hence the f-weight on such a block is

$$\begin{aligned}
f(B) &= \sum_{j=1}^{k-1} (-1)^{k-2} \cdot [k-2]_q! \cdot \mathrm{wt}(y) \cdot q^{j-1} \\
&= (-1)^{k-2} \cdot [k-2]_q! \cdot \mathrm{wt}(y \cdot k) \cdot (1 + q + \cdots q^{k-2}) \\
&= -(-1)^{k-1} \cdot [k-1]_q! \cdot \mathrm{wt}(w) \\
&= -f(w).
\end{aligned}$$

Therefore $f(C) = 0$ in this case. □

Setting $r = 1$ and $r = 2$ we obtain two special cases. These are originally due to Mercier [49, Theorem 2].

Corollary 3.6.2 (Mercier, 1990) *For $n \geq 2$, the following identities hold:*

$$\sum_{k=1}^{n} (-1)^k \cdot [k-1]_q! \cdot S_q[n,k] = 0. \tag{3.6.4}$$

and

$$\sum_{k=2}^{n} (-1)^k \cdot q^{k-2} \cdot [k-2]_q! \cdot S_q[n,k] = n-1, \tag{3.6.5}$$

Apply similar ideas, we also obtain the following result. This was suggested by R. Ehrenborg.

Theorem 3.6.3 *Given $n \geq 2$ and $r \geq 2$, the following identity holds:*

$$\sum_{k=r}^{n} (-1)^{k-r} \cdot q^{k-2} \cdot [k-2]_q! \cdot S_q[n,k] = q^{r-2} \cdot [r-2]_q! \cdot \sum_{i=r}^{n} S_q[i-1, r-1]. \tag{3.6.6}$$

Proof: On the set of RG-words $S = \dot{\bigcup}_{r \leq k \leq n} \mathcal{R}(n,k)$ define a weight function by

$$f(w) = (-1)^{k-r} \cdot q^{k-2} \cdot [k-2]_q! \cdot \mathrm{wt}(w).$$

Our objective is to evaluate the sum $\sum_{w \in S} f(w)$, which is the left-hand side of Equation (3.6.6). We do this in two steps. First we partition the set S into blocks and extend the weight f to a block B by $f(B) = \sum_{w \in B} f(w)$. Second on the set of blocks we define a sign-reversing involution such that if two blocks B and C are matched, their f-weights cancel, that is, $f(B) + f(C) = 0$. Hence the right-hand side is equal to the sum over all the blocks that have not been matched.

Given two words $u = u_1 u_2 \cdots u_n$ and $v = v_1 v_2 \cdots v_n$ in $\mathcal{R}(n,k)$, we say that the words u and v are equivalent if there exists an index i such that $2 \leq u_i, v_i \leq k$, $u_1 u_2 \cdots u_{i-1} = v_1 v_2 \cdots v_{i-1}$ is an RG-word in $\mathcal{R}(i-1,k)$ and $u_{i+1} u_{i+2} \cdots u_n = v_{i+1} v_{i+2} \cdots v_n = 11 \cdots 1$. Note that the letter u_i in position i in u is different from 1. Furthermore, this is not the first occurrence of this letter u_i. Most importantly, this is an equivalence relation. Hence we obtain a partition of $\mathcal{R}(n,k)$ by letting equivalent RG-words belong to the same block. Lastly, this yields a partition of the larger set S.

Observe that an RG-word u in $\mathcal{R}(n,k)$ having the form

$$u = u_1 u_2 \cdots u_{i-1} \cdot k \cdot 11 \cdots 1,$$

where $u_1 u_2 \cdots u_{i-1} \in \mathcal{R}(i-1, k-1)$ is not equivalent to any other word. That is, such a word u belongs to a singleton block. If $k \geq r+1$ we match this block $\{u\}$ with the block

$$B = \{u_1 u_2 \cdots u_{i-1} \cdot m \cdot 11 \cdots 1 \ : \ 2 \leq m \leq k-1\},$$

which is contained in $\mathcal{R}(n, k-1)$. Observe that

$$\begin{aligned}
f(B) &= \sum_{m=2}^{k-1} (-1)^{k-1-r} \cdot q^{k-3} \cdot [k-3]_q! \cdot q^{m-1} \cdot \mathrm{wt}(u_1 u_2 \cdots u_{i-1}) \\
&= (-1)^{k-1-r} \cdot q^{k-3} \cdot [k-3]_q! \cdot q \cdot [k-2]_q \cdot \mathrm{wt}(u_1 u_2 \cdots u_{i-1}) \\
&= -(-1)^{k-r} \cdot q^{k-2} \cdot [k-2]_q! \cdot \mathrm{wt}(u_1 u_2 \cdots u_{i-1} k 11 \cdots 1) \\
&= -(-1)^{k-r} \cdot q^{k-2} \cdot [k-2]_q! \cdot \mathrm{wt}(u) \\
&= -f(u).
\end{aligned} \tag{3.6.7}$$

Hence the weights of the two blocks B and $\{u\}$ cancel each other.

Consider the block $B = \{z \cdot m \cdot 11 \cdots 1 : 2 \leq m \leq k\}$, where $z \in \mathcal{R}(i-1, k)$. This yields the inequality $k \leq i-1 \leq n-1 < n$. Hence the word $u = z \cdot (k+1) \cdot 11 \cdots 1 \in \mathcal{R}(n, k+1)$ and is in the set S. By the above matching the block B is matched with the singleton block $\{u\}$ and their weights cancel by equation (3.6.7).

It remains to consider the unmatched blocks. Note that the RG-word $u = u_1 u_2 \cdots u_{i-1} \cdot r \cdot 11 \cdots 1 \in \mathcal{R}(n, r)$ where $u_1 u_2 \cdots u_{i-1} \in \mathcal{R}(i-1, r-1)$ forms a singleton block which is not matched with any other block. This is the case $k = r$. The sum of the weights of these blocks is given by

$$\begin{aligned}
\sum_{i=r}^{n} \sum_{z \in \mathcal{R}(i-1, r-1)} f(z \cdot r \cdot 111 \cdots 1) &= \sum_{i=r}^{n} \sum_{z \in \mathcal{R}(i-1, r-1)} q^{r-2} \cdot [r-2]_q! \cdot \mathrm{wt}(z \cdot r \cdot 11 \cdots 1) \\
&= q^{r-2} \cdot [r-2]_q! \cdot \sum_{i=r}^{n} \sum_{z \in \mathcal{R}(i-1, r-1)} \mathrm{wt}(z) \\
&= q^{r-2} \cdot [r-2]_q! \cdot \sum_{i=r}^{n} S_q[i-1, r-1],
\end{aligned}$$

proving the desired equality. □

We now prove a q-analogue of the Frobenius identity by Garsia and Remmel [31, Equation I.1].

Theorem 3.6.4 (Garsia–Remmel, 1986)

$$\sum_{m \geq 0} [m]_q^n \cdot x^m = \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} \cdot S_q[n, k] \cdot [k]_q! \cdot x^k}{(1-x) \cdot (1-qx) \cdots (1-q^k x)}. \tag{3.6.8}$$

Proof: It is straightforward to check the case when $n = 0$. Now suppose $n > 0$. Multiply both sides of the identity by $1 - x$ and shift the indices to give

$$\sum_{m \geq 0} ([m]_q^n - [m-1]_q^n) \cdot x^m = \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} \cdot S_q[n, k] \cdot [k]_q! \cdot x^k}{(1-qx) \cdots (1-q^k x)}. \tag{3.6.9}$$

Instead of proving the original statement of the theorem, we show Equation (3.6.9) holds. Let $\mathcal{I}(n,m)$ be the set of all integer words of length n and maximal entry m. Recall the ls-weight of a word $w = w_1w_2\cdots w_n$ is

$$ls(w) = \prod_{i=1}^{n} q^{w_i-1}.$$

Directly we have $\mathcal{R}(n,k) \subseteq \mathcal{I}(n,k)$ and $ls(w) = q^{\binom{k}{2}} \cdot \mathrm{wt}(w)$ for $w \in \mathcal{R}(n,k)$.

With this preparation, we can rewrite the left-hand side of Equation (3.6.9) as

$$\sum_{m\geq 0} \sum_{w\in\mathcal{I}(n,m)} ls(w)\cdot x^m.$$

This counts the ls-weight of all integer sequences w of length n and the power of x records the maximal entry in w.

Next we interpret the sum in another way. We begin by constructing a map

$$\varphi : \dot{\bigcup_{m\geq 0}} \mathcal{I}(n,m) \longrightarrow \dot{\bigcup_{1\leq k\leq n}} \mathcal{R}(n,k).$$

If $w \in \dot{\bigcup\limits_{m\geq 0}} \mathcal{I}(n,m)$ is an RG-word, let $\varphi(w) = w$. Otherwise let i_1 be the smallest index in w reading from left to right that makes $w_1\cdots w_{i_1}$ fail to be an RG-word. If $i_1 > 1$ then replace w_{i_1} by $\max(w_1, w_2, \ldots, w_{i_1-1}) + 1$. Otherwise $i_1 = 1$, so replace w_{i_1} by 1. Denote the newly obtained sequence by $w^{(1)}$. Let i_2 be the smallest index in $w^{(1)}$ that makes $w^{(1)}_1 \cdots w^{(1)}_{i_2}$ fail to be an RG-word. Replace w_{i_2} by $\max(w_1, w_2, \ldots, w_{i_2-1}) + 1$ to obtain $w^{(2)}$. Continue to find $i_3, i_4, \ldots$ until we obtain an RG-word $u \in \mathcal{R}(n,k)$ for some k, and let $\varphi(w) = u$. It is straightforward to check φ is a surjection. We now consider the preimage of $w \in \mathcal{R}(n,k)$ under the map φ.

For $w \in \mathcal{R}(n,k)$, decompose it as the form $w = 1\cdot u_1 \cdot 2\cdot u_2 \cdots (k-1)\cdot u_{k-1}\cdot k\cdot u_k$ where u_i is a sequence consisting of integers $\{1, 2, \ldots, i\}$. Then

$$\varphi^{-1}(w) = \{j_1\cdot u_1\cdot j_2\cdot u_2 \cdots j_{k-1}\cdot u_{k-1}\cdot j_k\cdot u_k : i \leq j_i \leq m \text{ for } i = 1, 2, \ldots, k\}.$$

To evaluate the ls-weight of the set $\varphi^{-1}(w)$, we consider the construction of any element in $\varphi^{-1}(w)$ in two steps. First from w construct an integer sequence of the form

$$v = \ell_1\cdot u_1\cdot \ell_2\cdot u_2 \cdots \ell_{k-1}\cdot u_{k-1}\cdot \ell_k\cdot u_k$$

where $i \leq \ell_i \leq k$ for all $i = 1, 2, \ldots, k-1$ and $\ell_k = k$. Denote the set of all such words v by $L(w)$. Next, we pick the last r entries in $\{\ell_1, \ell_2, \ldots, \ell_k\}$, that is, $\ell_{k-r+1}, \ell_{k-r+2}, \ldots, \ell_k$, for some integer $r \geq 0$, and increase these entries by s for some $s \geq 0$. In this step, we obtain a new word of the form

$$v' = \ell_1\cdot u_1 \cdots u_{k-r}\cdot(\ell_{k-r+1}+s)\cdot u_{k-r+1}\cdot(\ell_{k-r+2}+s)\cdot u_{k-r+2}\cdots(k+s)\cdot u_k.$$

Denote the set of all such words by $L'_{r,s}(w)$. It is straightforward to check that $\varphi^{-1}(w) = \dot{\bigcup\limits_{1\leq r\leq k}} \dot{\bigcup\limits_{s\geq 0}} L'_{r,s}(w)$.

In the first step, each letter ℓ_i is chosen from $\{i, i+1, \ldots, k\}$, giving a total weight of $[k-i+1]_q$ while the maximal letter in v remains k. Thus

$$\sum_{v \in L(w)} ls(v) \cdot x^k = ls(w) \cdot [k]_q! \cdot x^k.$$

In the second step, each increment by s contributes a factor of q^s to the ls-weight and the maximal letter is also increased by s. Thus

$$\begin{aligned}
\sum_{v' \in L'_{r,s}(w)} ls(v') \cdot x^{k+s} &= \sum_{v \in L(w)} ls(v) \cdot x^k \cdot (q^s)^r \cdot x^s \\
&= ls(w) \cdot [k]_q! \cdot x^k \cdot (q^r \cdot x)^s.
\end{aligned}$$

Summing over r and s we have

$$\begin{aligned}
\sum_{v' \in \varphi^{-1}(w)} ls(v') \cdot x^{\max(v')} &= \sum_{1 \le r \le k} \sum_{s \ge 0} \sum_{v' \in L'_{r,s}(w)} ls(v') \cdot x^{k+s} \\
&= \sum_{1 \le r \le k} \sum_{s \ge 0} ls(w) \cdot [k]_q! \cdot x^k \cdot (q^r \cdot x)^s \\
&= ls(w) \cdot [k]_q! \cdot x^k \cdot (1-qx)^{-1} \cdot (1-q^2x)^{-1} \cdots (1-q^kx)^{-1}.
\end{aligned}$$

Here $\max(v')$ denotes the maximal letter in v'. Summing over k we obtain the desired identity. $\square$

The following identity was first given by Ehrenborg [20] using juggling patterns.

Theorem 3.6.5 (Ehrenborg, 2003) *Let n and s be non-negative integers. Then the following identity holds.*

$$\det(S_q[s+i+j, s+j])_{0 \le i,j \le n} = [s]_q^0 \cdot [s+1]_q^1 \cdots [s+n]_q^n.$$

Proof: Let $\sigma \in \mathfrak{S}(0, 1, \ldots, n)$ be a permutation on $n+1$ elements and let $w(i) \in \mathcal{R}(s+i+\sigma(i), s+\sigma(i))$ be an RG-word associated to σ for any $0 \le i \le n$. Let T be the set of all $(n+1)$-tuples $(\sigma, w(0), w(1), \ldots, w(n))$ where $\sigma \in \mathfrak{S}(0, 1, \ldots, n)$ is a permutation on $n+1$ elements, and $w(i) \in \mathcal{R}(s+i+\sigma(i), s+\sigma(i))$ for all $0 \le i \le n$ is an RG-word.

The determinant can be expressed as

$$\det(S_q[s+i+j, s+j])_{0 \le i,j \le n} = \sum_{(\sigma, w(0), \ldots, w(n)) \in T} (-1)^{\mathrm{sgn}(\sigma)} \cdot q^{\mathrm{wt}(w(0)) + \cdots + \mathrm{wt}(w(n))}.$$

For any $(\sigma, w(0), \ldots, w(n)) \in T$ let a_i denote the repeated entries in the RG-word $w(i)$ after the index $s+i$, that is,

$$a_i = |\{j : j > s+i, w(i)_j = w(i)_\ell \text{ for some } \ell < j\}|.$$

Observe that there are i repeated entries in any $w(i)$ and there are $\sigma(i)$ entries after the index $s+i$, we have $0 \le a_i \le i$ and $a_i \le \sigma(i)$ for all i.

Let $T_1 \subseteq T$ consist of all tuples $(\sigma, w(0), \ldots, w(n))$ where the a_i's are distinct. This implies $a_i = i = \sigma(i)$, that is, σ is the identity permutation, and $w(i)$ is of the form $12\cdots(s+i)\cdot y_{s+i}$ where y_{s+i} is a sequence of length i with all entries from the set $\{1, 2, \ldots, s+i\}$. All such possible sequences y_{s+i} give a total weight of $[s+i]_q^i$. Thus we have

$$\sum_{(\sigma, w(0), \ldots, w(n)) \in T_1} (-1)^{\mathrm{sgn}(\sigma)} \cdot q^{\mathrm{wt}(w(0))+\cdots+\mathrm{wt}(w(n))} = \prod_{i=0}^{n} [s+i]_q^i. \tag{3.6.10}$$

Let $T_2 = T - T_1$ be the complement of T_1. We define a sign-reversing involution on T_2 as follows. For any $(\sigma, w(0), \ldots, w(n)) \in T_2$ there exist some indices i_1 and i_2 such that $a_{i_1} = a_{i_2}$. Let (j,k) be the least such pair of indices in the lexicographic order. For easier notation, denote $u = w(j) \in \mathcal{R}(s+j+\sigma(j), s+\sigma(j))$ and $v = w(k) \in \mathcal{R}(s+k+\sigma(k), s+\sigma(k))$. Define $(\sigma', w(0)', \ldots, w(n)') \in T_2$ where $\sigma'(j) = \sigma(k)$, $\sigma'(k) = \sigma(j)$ and $\sigma'(i) = \sigma(i)$ for $i \neq j, k$. Moreover, let $w(i)' = w(i)$ for $i \neq j, k$. We construct $u' = w(j)'$ and $v' = w(j)'$ such that

$$u' = u_1 u_2 \cdots u_{s+j} \cdot v_{s+k+1} v_{s+k+2} \cdots v_{s+k+\sigma(k)}$$

and

$$v' = v_1 v_2 \cdots v_{s+k} \cdot u_{s+j+1} u_{s+j+2} \cdots u_{s+j+\sigma(j)}.$$

Clearly $(-1)^{\mathrm{sgn}(\sigma')} = -(-1)^{\mathrm{sgn}(\sigma)}$. It remains to check that u', v' are RG-words and $\mathrm{wt}(u') + \mathrm{wt}(v') = \mathrm{wt}(u) + \mathrm{wt}(v)$.

Suppose there exists some $r > \max(u_1, u_2, \ldots, u_{s+j})$ such that $v_{s+k+\ell} > r$ for all $1 \le \ell \le \sigma(k)$. Then all the integers $r, r+1, \ldots, s+\sigma(j)$ will first appear in u after the index $s+j$. Hence $a_j \le \sigma(j) - (s+\sigma(j)-r+1) = r-s-1$. On the other hand the $\sigma(k)$ entries $v_{s+k+1}, v_{s+k+2}, \ldots, v_{s+k+\sigma(k)}$ take only values from $\{r+1, r+2, \ldots, s+\sigma(k)\}$. By the pigeonhole principle $a_k \ge \sigma(k) - (s+\sigma(k)-r) = r-s$. This contradicts the fact $a_j = a_k$. Hence $u' \in \mathcal{R}(s+j+\sigma(k), s+\sigma(k))$. A similar argument shows $v' \in \mathcal{R}(s+k+\sigma(j), s+\sigma(j))$.

Next we show that u', v' have the same repeats as u and v. Suppose on the contrary $s+j+\ell$ is the first index such that $u_{s+j+\ell} = b$ is a repeat in u but not a repeat in v'. Then $v_r < b$ for all $r \le s+k$ and $u_{s+j+m} < b$ for all $m < \ell$. So $a_j \ge \sigma(j) - (s+\sigma(j)-b) = b-s$ but $a_k \le \sigma(k) - (s+\sigma(k)-b+1) = b-s-1$, contradicting $a_j = a_k$. A similar argument shows that if u_{s+j+t} is a not a repeat in u then it will not be a repeat in v'. Hence u and v' have the same repeats and v and u' have the same repeats. This gives $\mathrm{wt}(u') + \mathrm{wt}(v') = \mathrm{wt}(u) + \mathrm{wt}(v)$.

The map $(\sigma, w(0), \ldots, w(n)) \mapsto (\sigma', w(0)', \ldots, w(n)')$ is a sign-reversing involution on T_2 with no fixed points. Moreover, $\mathrm{wt}(w(0))+\mathrm{wt}(w(1))+\cdots+\mathrm{wt}(w(n))$ is invariant under this map. Hence the determinant is given by Equation (3.6.10). $\square$

3.7 Concluding remarks

Is there a way to view Theorem 3.3.2 and Corollary 3.3.3 in terms of counting lattice points?

Recall the Stirling numbers of the second kind are specializations of the homogeneous symmetric function, that is, $S_q[n,k] = h_{n-k}(x_1, x_2, \ldots, x_k)$ where $x_m = [m]_q$. See [46, Chapter I, Section 2, Example 11]. Thus one can view Theorem 3.6.1 from a symmetric function perspective. To see this, denote by $[x^n]f(x)$ the coefficient of x^n in $f(x)$. We have the following lemma.

Lemma 3.7.1 *The following identity holds:*

$$\begin{aligned}
&[t^{n-r}]\frac{1}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r-1}\frac{1}{1-x_i\cdot t}\\
&\quad = h_{n-r}(x_1,\ldots,x_r)+\sum_{j=1}^{n-r}(x_{r-1}-x_r)\cdots(x_{r-1}-x_{r+j-1})\cdot h_{n-r-j}(x_1,\ldots,x_{r+j}).
\end{aligned} \tag{3.7.1}$$

Proof: We have

$$\begin{aligned}
[t^{n-r}]\frac{1}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r-1}\frac{1}{1-x_i\cdot t} &= [t^{n-r}]\frac{1-x_r\cdot t}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r}\frac{1}{1-x_i\cdot t}\\
&= [t^{n-r}]\prod_{i=1}^{r}\frac{1}{1-x_i\cdot t}+[t^{n-r}](\frac{1-x_r\cdot t}{1-x_{r-1}\cdot t}-1)\cdot\prod_{i=1}^{r}\frac{1}{1-x_i\cdot t}\\
&= h_{n-r}(x_1,\ldots,x_r)+[t^{n-r}]\frac{(x_{r-1}-x_r)\cdot t}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r}\frac{1}{1-x_i\cdot t}\\
&= h_{n-r}(x_1,\ldots,x_r)+(x_{r-1}-x_r)[t^{n-r-1}]\frac{1-x_{r+1}\cdot t}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r+1}\frac{1}{1-x_i\cdot t}
\end{aligned}$$

Apply this recursion $n-r$ times we obtain the result. $\Box$

We can now give a symmetric function proof of Theorem 3.6.1. Since

$$\frac{t}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r-1}\frac{1}{1-x_i\cdot t} = \frac{\partial}{\partial x_{r-1}}\prod_{i=1}^{r-1}\frac{1}{1-x_i\cdot t},$$

we have

$$\begin{aligned}
[t^{n-r}]\frac{1}{1-x_{r-1}\cdot t}\cdot\prod_{i=1}^{r-1}\frac{1}{1-x_i\cdot t} &= [t^{n-r+1}]\frac{\partial}{\partial x_{r-1}}\prod_{i=1}^{r-1}\frac{1}{1-x_i\cdot t}\\
&= \frac{\partial}{\partial x_{r-1}}h_{n-r+1}(x_1,\ldots,x_{r-1})\\
&= \sum_{c_1+c_2+\cdots c_{r-1}=n-r+1} c_{r-1}\cdot x_1^{c_1}\cdot x_2^{c_2}\cdots x_{r-2}^{c_{r-2}}\cdot x_{r-1}^{c_{r-1}-1}.
\end{aligned}$$

Combining this result with Lemma 3.7.1 and substituting $x_k = [k]_q$ for $k = 1, 2, \ldots, n$, we obtain Theorem 3.6.1.

Chapter 4 Major index, q-multinomials and cyclic sieving

In this chapter we describe a compact expression of the q-$(1+q)$-binomial via major index. A generalization to q-multinomial is also given as well as an instance of the cyclic sieving phenomenon involving flags of unitary spaces.

4.1 Introduction

Let $\Omega(n,k) = \mathfrak{S}(1^k, 0^{n-k})$ be the set of all 01-permutations consisting of k ones and $n-k$ zeros. For any 01-permutation $w = w_1 w_2 \cdots w_n \in \Omega(n,k)$, recall the *descent set* $D(w)$ of w is

$$D(w) = \{i : w_i > w_{i+1}\} \subseteq \{1, 2, \ldots, n-1\},$$

while the *major index* of w is defined to be the sum of all the elements of the descent set:

$$\operatorname{maj}(w) = \sum_{i \in D(w)} i.$$

MacMahon [47, Page 315] gave a closed form for the generating function of the major index

$$\sum_{w \in \Omega(n,k)} q^{\operatorname{maj}(w)} = \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{4.1.1}$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the Gaussian coefficient, that is, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!}$ where $[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$ and $[m]_q! = [m]_q \cdot [m-1]_q \cdots [1]$.

Another statistic that is closely related to major index is the inversion number of a permutation. Given a 01-permutation $w \in \Omega(n,k)$, the *inversion number* $\operatorname{inv}(w)$ of w is the cardinality of the set of *inversions* of w, that is,

$$\operatorname{inv}(w) = |\{(i,j) : i < j \text{ and } w_i > w_j\}|.$$

MacMahon [47, Page 315] showed that the two statistics $\operatorname{inv}(\cdot)$ and $\operatorname{maj}(\cdot)$ are equidistributed, that is, for any integers $n \geq k \geq 0$,

$$\sum_{w \in \Omega(n,k)} q^{\operatorname{inv}(w)} = \sum_{w \in \Omega(n,k)} q^{\operatorname{maj}(w)}.$$

Foata [24] gave the first bijective proof of this result.

The q-$(1+q)$-binomial was recently introduced by Fu–Reiner–Stanton–Thiem [28, Theorem 1] on a subset of 01-permutations $\Omega_1(n,k) \subseteq \Omega(n,k)$ with a pair of statistics $\operatorname{inv}(\cdot)$ and $r(w)$ for the powers of q and $(1+q)$. The q-binomial can be written as a more compact expression. The subset $\Omega_1(n,k)$ is defined based on a recursive pairing algorithm. Here we use the "dual" version of their algorithm in this chapter as follows.

For $w = w_1 w_2 \cdots w_n \in \Omega(n,k)$, pair the entries in w recursively in the following way.

- When $n = 1$, leave the entry w_1 unpaired.
- For $n \geq 2$ and $n \equiv k \pmod 2$, leave the last entry w_n in w unpaired, pair the remaining word $w_1 w_2 \cdots w_{n-1}$ recursively.
- For $n \geq 2$ and $n - k \equiv 1 \pmod 2$, pair the last two entries $w_{n-1}w_n$ in w and pair the remaining word $w_1 w_2 \cdots w_{n-2}$ recursively.

Let $\Omega_1(n, k) \subseteq \Omega(n, k)$ be the subset of $\Omega(n, k)$ with no paired $\underline{01}$'s in the word. For $w \in \Omega_1(n, k)$, let $r(w)$ be the number of $\underline{10}$-pairs.

Theorem 4.1.1 (Fu–Reiner–Stanton–Thiem, 2012) *The q-binomial has a q-$(1+q)$-analogue using the bistatistic $(\mathrm{inv}(\cdot), r(\cdot))$ as follows:*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega_1(n,k)} q^{\mathrm{inv}(w)-r(w)} \cdot (1+q)^{r(w)}. \tag{4.1.2}$$

The authors also studied the (q, t)-binomial evaluated at negative q values and gave an instance of the cyclic sieving phenomenon related to unitary spaces based on the (q, t)-binomial. We briefly recall the theory of cyclic sieving phenomenon here.

Let X be a finite set of cardinality n with the action of the cyclic group C of order n. Let $X(t)$ be a polynomial in t having nonnegative coefficients with the property that $X(1) = |X|$. The triple $(X, X(t), C)$ exhibits the *cyclic sieving phenomenon* if for every $c \in C$, $X(\omega) = |\{x \in X : c(x) = x\}|$, where ω is a complex root of unity with the same multiplicative order as c. For more details about the cyclic sieving phenomenon, we refer to Reiner, Stanton and White [61].

In Section 4.2 we define a map on 01-permutations which leads to a direct proof of Equation (4.1.1). In Section 4.3 we give an algorithm which is used to define a bistatistic $(\mathrm{maj}(\cdot), a(\cdot))$ which yields a compact expression for the Gaussian polynomial. In Section 4.4 we give a bijective proof between the bistatistics $(\mathrm{inv}(\cdot), r(\cdot))$ and $(\mathrm{maj}(\cdot), a(\cdot))$. In Section 4.5 we generalize the q-$(1+q)$-binomial to the q-multinomial and discuss some basic properties of the (q, t)-multinomial at negative q values. In Section 4.6 we give an instance of the cyclic sieving phenomenon on flags of unitary spaces.

4.2 The major index

In this section we construct a bijection and present a combinatorial proof of Equation (4.1.1).

Let $\Omega(n, k)$ be the set of all 01-permutations of length n with k ones. We build a bijection

$$\varphi : \Omega(n, k) \longrightarrow \{w \in \Omega(n+1, k) : w \text{ ends with a } 0\}$$

as follows. Consider a word $w \in \Omega(n, k)$ of the form $w = 0^{\ell_1} 1^{k_1} 0^{\ell_2} 1^{k_2} \cdots 0^{\ell_m} 1^{k_m} 0^{\ell_{m+1}}$ where k_1 through k_m are positive integers and ℓ_1 and ℓ_{m+1} are non-negative, that is, we can write

$$w = \underbrace{0\cdots 0}_{\ell_1}\underbrace{1\cdots 1}_{k_1}\underbrace{0\cdots 0}_{\ell_2}\underbrace{1\cdots 1}_{k_2}\cdots\underbrace{0\cdots 0}_{\ell_m}\underbrace{1\cdots 1}_{k_m}\underbrace{0\cdots 0}_{\ell_{m+1}}.$$

To build $\varphi(w)$, we first refine w into several small blocks by putting a bar in front of every 1 that is followed by a 0. This gives

$$w = \underbrace{0\cdots 0}_{\ell_1}\underbrace{1\cdots 1}_{k_1-1}|1\underbrace{0\cdots 0}_{\ell_2}\underbrace{1\cdots 1}_{k_2-1}|1\underbrace{0\cdots 0}_{\ell_3}\underbrace{1\cdots 1}_{k_3-1}|\cdots|1\underbrace{0\cdots 0}_{\ell_m}\underbrace{1\cdots 1}_{k_m-1}|1\underbrace{0\cdots 0}_{\ell_{m+1}}.$$

Next we cyclically shift the letters inside each block to the right by the number of 1's in that block. Finally add a 0 at the end of the word. Define $\varphi(w)$ to be the resulting word. That is, when $\ell_{m+1} \neq 0$, $\varphi(w)$ is of the form

$$\varphi(w) = \underbrace{1\cdots 1}_{k_1-1}\underbrace{0\cdots 0}_{\ell_1}|0\underbrace{1\cdots 1}_{k_2-1}1\underbrace{0\cdots 0}_{\ell_2-1}|0\underbrace{1\cdots 1}_{k_3-1}1\underbrace{0\cdots 0}_{\ell_3-1}|\cdots|0\underbrace{1\cdots 1}_{k_m-1}1\underbrace{0\cdots 0}_{\ell_m-1}|01\underbrace{0\cdots 0}_{\ell_{m+1}-1}|0$$

and when $\ell_{m+1} = 0$, it is

$$\varphi(w) = \underbrace{1\cdots 1}_{k_1-1}\underbrace{0\cdots 0}_{\ell_1}|0\underbrace{1\cdots 1}_{k_2-1}1\underbrace{0\cdots 0}_{\ell_2-1}|0\underbrace{1\cdots 1}_{k_3-1}1\underbrace{0\cdots 0}_{\ell_3-1}|\cdots|0\underbrace{1\cdots 1}_{k_m}1\underbrace{0\cdots 0}_{\ell_m-1}|0$$

In the above illustrations we also added a bar before the last 0 to distinguish it from the original sequence.

Example 4.2.1 *Let $w_1, w_2 \in \Omega(8,5)$ with*

$$w_1 = 011|101|10 \quad \textit{and} \quad w_2 = 1|10|10|101.$$

Then under the φ map

$$\varphi(w_1) = 110|011|01|0 \quad \textit{and} \quad \varphi(w_2) = 1|01|01|011|0.$$

We have $\mathrm{maj}(w_1) = 4+7 = 11$, $\mathrm{maj}(\varphi(w_1)) = 2+6+8 = 16 = \mathrm{maj}(w_1)+5$, *and* $\mathrm{maj}(w_2) = 2+4+6 = 12$, $\mathrm{maj}(\varphi(w_2)) = 1+3+5+8 = 17 = \mathrm{maj}(w_2)+5$.

In general we have the following lemma.

Lemma 4.2.2 *For $w \in \Omega(n,k)$ we have* $\mathrm{maj}(\varphi(w)) = \mathrm{maj}(w) + k$.

Proof: When $\ell_{m+1} \neq 0$, the descent set of w is

$$D(w) = \Big\{ \sum_{i=1}^{r} (\ell_i + k_i) : 1 \leq r \leq m \Big\},$$

while the descent set of $D(\varphi(w))$ is

$$D(\varphi(w)) = \Big\{ k_1 - 1, \sum_{i=1}^{m} (k_i + \ell_i) + 1 \Big\} \bigcup \Big\{ \sum_{i=1}^{r} (k_i + \ell_i + k_{i+1}) : 1 \leq r \leq m-1 \Big\}.$$

Hence the difference of the major indices is

$$\begin{aligned}\operatorname{maj}(\varphi(w)) - \operatorname{maj}(w) &= \sum_{t\in D(\varphi(w))} t - \sum_{s\in D(w)} s \\ &= k_1 - 1 + \sum_{i=2}^{m} k_i + 1 \\ &= k.\end{aligned}$$

For the case when $\ell_{m+1} = 0$, we have

$$D(w) = \left\{ \sum_{i=1}^{r} (\ell_i + k_i) : r = 1, 2, \ldots, m-1 \right\},$$

and

$$D(\varphi(w)) = \left\{ k_1 - 1, \sum_{i=1}^{r} (k_i + \ell_i + k_{i+1}), \sum_{i=1}^{m-1} (k_i + \ell_i) + k_m + 1 : r = 1, 2, \ldots, m-2 \right\}.$$

Again we have

$$\begin{aligned}\operatorname{maj}(\varphi(w)) - \operatorname{maj}(w) &= \sum_{t\in D(\varphi(w))} t - \sum_{s\in D(w)} s \\ &= k_1 - 1 + \sum_{i=2}^{m-1} k_i + k_m + 1 \\ &= k.\end{aligned}$$

Note in both cases, the last one occurring in each block in $\varphi(w)$ contributes to the descent set even if $\ell_1 = 0$ or $\ell_i - 1 = 0$ for $i = 2, 3, \ldots, m-1$ since the next block always begins with a zero. Hence the result follows. □

Lemma 4.2.3 *The map $\varphi : \Omega(n, k) \longrightarrow \{w \in \Omega(n+1, k) : w$ ends with a 0$\}$ is a bijection between these two sets. Denote the inversion by ψ*

Proof: For $w \in \Omega(n+1, k)$ with w ending in a 0, define $\psi(w)$ as follows. First delete the rightmost 0 of w. Next, put a bar before every 0 that is followed by a 1, then shift each block cyclically to the left by the number of 1's occurring in that block. It is straightforward to check that ψ is the inverse of φ. □

With this map we give a combinatorial proof of Equation (4.1.1).

Proposition 4.2.4 *There is a direct combinatorial proof of the identity*

$$\sum_{w\in\Omega(n,k)} q^{\operatorname{maj}(w)} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Proof: We will proceed by induction on n.

For the cases $n = 0$ and $n = 1$ one can easily verify the result. Now assume Equation (4.1.1) holds for $n-1$.

Recall the q-binomial satisfies the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{4.2.1}$$

For a 01-permutation $w \in \Omega(n,k)$, the last entry will be either 0 or 1. In the first case, $u = \psi(w) \in \Omega(n-1,k)$ and $w_n = 0$ contributes k to $\mathrm{maj}(\varphi(u)) = \mathrm{maj}(w)$, while $w_n = 1$ contributes 0 to $\mathrm{maj}(w)$, thus

$$\begin{aligned}
\sum_{w\in\Omega(n,k)} q^{\mathrm{maj}(w)} &= \sum_{\substack{w\in\Omega(n,k)\\ w \text{ ends with a } 0}} q^{\mathrm{maj}(w)} + \sum_{\substack{w\in\Omega(n,k)\\ w \text{ ends with a } 1}} q^{\mathrm{maj}(w)} \\
&= \sum_{w\in\Omega(n-1,k)} q^{\mathrm{maj}(\varphi(w))} + \sum_{w\in\Omega(n-1,k-1)} q^{\mathrm{maj}(w)} \\
&= q^k \cdot \sum_{w\in\Omega(n-1,k)} q^{\mathrm{maj}(w)} + \sum_{w\in\Omega(n-1,k-1)} q^{\mathrm{maj}(w)} \\
&= q^k \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.
\end{aligned}$$

Apply Equation (4.2.1) the result follows. □

4.3 The q-$(1+q)$-binomial using the major index

In this section we algorithmically find a subset of $\Omega(n,k)$ to generate the q-binomial as a q-$(1+q)$-analogue via major index.

Given a word $w \in \Omega(n,k)$, we define the following functions.

(i) $z(w) = n-k$ counts the number of 0's in the word w. For example, $z(1011001) = 3$.

(ii) $l(w)$ records the last entry of w. For example, $l(1011001) = 1$, and $l(1011000) = 0$.

(iii) If $l(w) = 1$, we define $\mathrm{del}(w)$ to be the 01-word obtained by deleting the last entry in w. For example, $\mathrm{del}(11011001) = 1101100$. As a remark, $\mathrm{del}(w) \in \Omega(n-1,k-1)$.

With these functions, we find a subset $\Omega_2(n,k) \subseteq \Omega(n,k)$.

Algorithm 4.3.1 *Input a word $w \in \Omega(n,k)$, determine it is in $\Omega_2(n,k)$ or not. Initialization $a = 0$.*

(a) *If* $w = \varnothing$, *return* $w \in \Omega_2(n,k)$ *and* a.
If $l(w) = 1$, *go to step (b).*
If $l(w) = 0$, *set* $w \leftarrow \psi(w)$, *go to step (d).*

(b) *If* $z(w) \equiv 0 \pmod 2$, $w \leftarrow \mathrm{del}(w)$, *go to step (a).*
If $z(w) \equiv 1 \pmod 2$, $w \leftarrow \mathrm{del}(w)$, *go to step (c).*

(c) *If* $l(w) = 0$, *return* $w \notin \Omega_2(n,k)$.
Otherwise, $w \leftarrow \mathrm{del}(w)$, *go to step (a).*

(d) *If* $z(w) \equiv 0 \pmod 2$ *and* $l(w) = 1$, *set* $a \leftarrow a+1$. *Go to step (a).*

For a word $w \in \Omega_2(n,k)$ that is obtained for the above algorithm, denote $a(w)$ the number a in the algorithm and define the weight of w to be

$$\mathrm{wt}(w) = q^{\mathrm{maj}(w)-a(w)} \cdot (1+q)^{a(w)}.$$

From the algorithm and the properties for $\varphi(w)$, one can easily derive the following properties.

Property 4.3.2 *For a word* $w \in \Omega_2(n,k)$, *its weight satisfies*

1. *If* $l(w) = 1$, *then* $\mathrm{wt}(w) = \mathrm{wt}(\mathrm{del}(w))$.
2. *If* $z(w) \equiv 0 \pmod 2$ *and* $l(w) = 1$, *then* $\mathrm{wt}(\varphi(w)) = q^{k-1} \cdot (1+q) \cdot \mathrm{wt}(w)$.
3. *If* $w \in \Omega_2(n,k)$ *does not satisfy the condition in (2), then* $\mathrm{wt}(\varphi(w)) = q^k \cdot \mathrm{wt}(w)$.

We now state our theorem.

Theorem 4.3.3 *The q-binomial satisfies the following identity.*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega_2(n,k)} \mathrm{wt}(w) = \sum_{w \in \Omega_2(n,k)} q^{\mathrm{maj}(w)-a(w)} \cdot (1+q)^{a(w)}.$$

Proof: We prove the identity by induction on n.

For the base cases $n = 0$ and $n = 1$, one can easily check that the identity holds. Now suppose the identity holds for $m \le n-1$.

We consider the set $\Omega_2(n,k)$. If $n - k \equiv 0 \pmod 2$, that is, n and k have the same parity, each $w \in \Omega_2(n,k)$ will have either $l(w) = 0$ or $l(w) = 1$. In either case $l(w)$ does not contribute to $a(w)$ and $l(w) = 0$ contributes k to $\mathrm{maj}(\varphi(u))$ where

$u = \psi(w)$. Hence

$$\begin{aligned}
\sum_{w \in \Omega_2(n,k)} \mathrm{wt}(w) &= \sum_{\substack{w \in \Omega_2(n,k) \\ l(w)=0}} \mathrm{wt}(w) + \sum_{\substack{w \in \Omega_2(n,k) \\ l(w)=1}} \mathrm{wt}(w) \\
&= \sum_{w \in \Omega_2(n-1,k)} \mathrm{wt}(\varphi(w)) + \sum_{w \in \Omega_2(n-1,k-1)} \mathrm{wt}(w) \\
&= q^k \cdot \sum_{w \in \Omega_2(n-1,k)} \mathrm{wt}(w) + \sum_{w \in \Omega_2(n-1,k-1)} \mathrm{wt}(w) \\
&= q^k \cdot \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\
&= \begin{bmatrix} n \\ k \end{bmatrix}_q.
\end{aligned}$$

On the other hand, if $n - k \equiv 1 \pmod 2$, that is, n and k have different parity, we use the following recurrence.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{2k} \cdot \begin{bmatrix} n-2 \\ k \end{bmatrix}_q + q^{k-1} \cdot (1+q) \cdot \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q. \tag{4.3.1}$$

This is obtained by applying Equation (4.2.1) twice. Again, consider the last entry in w. We have three cases.

Case 1. Both the words w and $\psi(w)$ end with 0.

Case 2. The word w ends with 0 and $\psi(w)$ ends with 1.

Case 3. The word w ends with 11.

In Case 1, each letter 0 contribute k to $\mathrm{maj}(w)$ under the φ map and zero to $a(w)$. In Case 2, $w_n = 0$ contributes k to $\mathrm{maj}(\varphi(u))$ with $u = \psi(w)$ while $l(u)$ together with w_n contribute one to $a(w)$. In Case 3, the last two entries does not contribute to $\mathrm{maj}(w)$ or $a(w)$. Hence

$$\begin{aligned}
\sum_{w \in \Omega_2(n,k)} \mathrm{wt}(w) &= \sum_{\substack{w \in \Omega_2(n,k) \\ l(w)=0}} \mathrm{wt}(w) + \sum_{\substack{w \in \Omega_2(n,k) \\ l(w)=1}} \mathrm{wt}(w) \\
&= \sum_{w \in \Omega_2(n-1,k)} \mathrm{wt}(\varphi(w)) + \sum_{w \in \Omega_2(n-2,k-2)} \mathrm{wt}(w) \\
&= \sum_{\substack{w \in \Omega_2(n-1,k) \\ l(w)=0}} \mathrm{wt}(\varphi(w)) + \sum_{\substack{w \in \Omega_2(n-1,k) \\ l(w)=1}} \mathrm{wt}(\varphi(w)) + \sum_{w \in \Omega_2(n-2,k-2)} \mathrm{wt}(w) \\
&= q^k \cdot \sum_{w \in \Omega_2(n-2,k)} \mathrm{wt}(\varphi(w)) + q^{k-1} \cdot (1+q) \cdot \sum_{w \in \Omega_2(n-2,k-1)} \mathrm{wt}(w) \\
&\quad + \sum_{w \in \Omega_2(n-2,k-2)} \mathrm{wt}(w) \\
&= q^{2k} \cdot \begin{bmatrix} n-2 \\ k \end{bmatrix}_q + q^{k-1} \cdot (1+q) \cdot \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.
\end{aligned}$$

i	π_i	$w^{(i)}$
1	1	1
2	1	11
3	0	110
4	0	1010
5	1	10101
6	0	010110

Table 4.1: The step-by-step construction of $w(\pi)$ from $\pi = 110010$.

Apply Equation (4.3.1), we obtain the theorem. □

4.4 Equidistribution of the bistatistics

In this section we show that the statistics $a(\cdot)$ and $r(\cdot)$ are equidistributed and give a bijective proof to this result.

Given any 01-permutation $\pi \in \Omega_1(n,k)$, we construct a word $w = w(\pi) \in \Omega_2(n,k)$ recursively such that $(\mathrm{inv}(\pi), r(\pi)) = (\mathrm{maj}(w), a(w))$. Denote $w^{(i)}$ the 01 sequence obtained in the construction at the i-th step, define the construction as follows: define $w_1 = \pi_1$. If $\pi_i = 1$, let $w^{(i)} = w^{(i-1)} \cdot 1$, and if $\pi_i = 0$, let $w^{(i)} = \varphi(w^{(i-1)})$.

For example, let $\pi = 110010 \in \Omega_1(6,3)$ with $q^{\mathrm{inv}(\pi)-r(\pi)} \cdot (1+q)^{r(\pi)} = q^5 \cdot (1+q)^2$, the word $w = w(\pi)$ is given in Table 4.1 with $\mathrm{wt}(w) = q^5 \cdot (1+q)^2$

Theorem 4.4.1 *For any $\pi \in \Omega_1(n,k)$ and $w(\pi) \in \Omega_2(n,k)$, we have* $(\mathrm{inv}(\pi), r(\pi)) = (\mathrm{maj}(w), a(w))$.

Proof: We prove the statement by induction on n. For the base cases $n = 0$ and $n = 1$, it is straightforward to check that $w(\pi) = \pi$ with no inversions or descents, so $(\mathrm{inv}(\pi), r(\pi)) = (\mathrm{maj}(w), a(w)) = (0,0)$.

Now suppose the theorem is true for 01-permutations of length $m \le n-1$. For any $\pi = \pi_1\pi_2\cdots\pi_n \in \Omega_1(n,k)$, denote $\sigma = \mathrm{del}(\pi)$. If $\pi_n = 1$, $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma)$ and $\mathrm{maj}(w(\pi)) = \mathrm{maj}(w(\sigma))$. If $n-k \equiv 0 \pmod 2$, by definition, π_n is unpaired and hence $r(\pi) = r(\sigma)$. From the algorithm that gives $w(\pi)$, we also have $a(w(\pi)) = a(w^{(n-1)}) = a(w(\sigma))$. If $n-k \equiv 1 \pmod 2$, it indicates that $\pi_{n-1}\pi_n = 11$, which will not contribute to $r(\pi)$. Moreover, by the construction of w, it also implies that $w^{(n)} = w^{(n-2)} \cdot 1 \cdot 1$. Hence $r(\pi) = r(\pi_1\cdots\pi_{n-2}) = a(w^{(n-2)}) = a(w^{(n)})$. This gives $(\mathrm{inv}(\pi), r(\pi)) = (\mathrm{maj}(w), a(w))$.

On the other hand, if $\pi_n = 0$, it will contribute k inversions since there are k ones in π, thus $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + k$. By Lemma 4.2.2, $\mathrm{maj}(w(\pi)) = \mathrm{maj}(\varphi(w^{(n-1)})) = \mathrm{maj}(w(\sigma)) + k$.

To show $r(\pi) = a(w)$, again we consider the cases where $n-k$ is odd or even. If $n-k \equiv 0 \pmod 2$, π_n is unpaired, and similar to the argument above, one can

i	$w^{(i-1)}$	$\pi^{(i)}$
1	010110	0
2	10101	10
3	1010	010
4	110	0010
5	11	10010
6	1	110010

Table 4.2: The step-by-step construction of $\pi(w) \in \Omega_1(6,3)$ from $w = 010110 \in \Omega_2(6,3)$.

show $r(\pi) = a(w(\pi))$. If $n-k \equiv 1 \pmod 2$, the last two digits $\pi_{n-1}\pi_n$ will either by 00 or 10. The former case gives $r(\pi) = r(\pi_1 \cdots \pi_{n-2})$ and the latter case gives $r(\pi) = r(\pi_1 \cdots \pi_{n-2}) + 1$. As for $a(w)$, by definition, $a(w(\pi)) = a(w(\pi_1 \cdots \pi_{n-2}))$ when $\pi_{n-1}\pi_n = 00$, and $a(w(\pi)) = a(w(\pi_1 \cdots \pi_{n-2})) + 1$ when $\pi_{n-1}\pi_n = 10$. So in both cases $r(\pi) = a(w)$. $\square$

To show the map is a bijection, we construct the inverse function as follows: Given a 01-permutation $w \in \Omega_2(n,k)$, let $\pi^{(i)}$ be the word obtained at the i-th step. Define $\pi^{(1)} = w_n$ and $w^{(0)} = w$. If $w^{(i-1)}_{n-i+1} = 1$, let $\pi^{(i)} = 1 \cdot \pi^{(i-1)}$ and $w^{(i)}$ be the word obtained by removing the last digit of $w^{(i-1)}$. If $w^{(i-1)}_{n-i+1} = 0$, let $\pi^{(i)} = 0 \cdot \pi^{(i-1)}$ and $w^{(i)} = \psi(w^{(i-1)})$. See Table 4.2 for an example.

4.5 The (q,t)-multinomial at negative q

In [60] Reiner and Stanton defined the (q,t)-multinomial, that is, given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ of n, define

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \begin{bmatrix} n \\ \alpha_1, \alpha_2, \ldots, \alpha_l \end{bmatrix}_{q,t} = \frac{\prod_{i=1}^{n} 1 - t^{q^n - q^{n-i}}}{\prod_{s=1}^{\ell} \prod_{i=1}^{\alpha_s} 1 - t^{q^{\sigma_s} - q^{\sigma_s - i}}}. \tag{4.5.1}$$

where $\sigma_s = \alpha_1 + \alpha_2 + \cdots + \alpha_s$ is the partial sum on the first s entries of α and $\sigma_0 = 0$.

By evaluating the limiting cases one has

$$\lim_{t \to 1} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q} = \begin{bmatrix} n \\ \alpha_1, \alpha_2, \ldots, \alpha_l \end{bmatrix}_{q} = \frac{\prod_{i=1}^{n} q^n - q^{n-i}}{\prod_{s=1}^{\ell} \prod_{i=1}^{\alpha_s} q^{\sigma_s} - q^{\sigma_s - i}}$$

and

$$\lim_{q \to 1} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q, t^{\frac{1}{q-1}}} = \begin{bmatrix} n \\ \alpha \end{bmatrix}_{t}.$$

Here $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]_q$ is the q-multinomial, that is, the q-analogue of the multinomial coefficient

$$\binom{n}{\alpha} = \frac{n!}{\alpha_1! \cdot \alpha_2! \cdots \alpha_\ell!}.$$

The (q,t)-binomial is a special case of the (q,t)-multinomial, that is,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} = \begin{bmatrix} n \\ k, n-k \end{bmatrix}_{q,t} = \prod_{i=1}^{k} \frac{1 - t^{q^n - q^{i-1}}}{1 - t^{q^k - q^{i-1}}}. \tag{4.5.2}$$

Let $M = \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, \ell^{\alpha_\ell}\}$ be a multiset of size $n = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$, and let $\mathfrak{S}_M$ be the set of all permutations on the elements of M. MacMahon [47, Page 317] gave the following combinatorial interpretation of the q-multinomial.

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \sum_{w \in \mathfrak{S}_M} q^{\mathrm{inv}(w)}.$$

In this section, we first generalize the q-$(1+q)$-binomial to q-multinomial and then discuss some properties of the (q,t)-multinomial at negative q values.

We find a subset M' of $\mathfrak{S}_M$ and give the q-$(1+q)$-statistic from a recursive pairing algorithm. For any $w \in \mathfrak{S}_M$ and $1 \le i \le \ell$, let $b_i(w)$ denote the number of elements in w that are larger than i. Let $w_{\ge i}$ denote the subsequence of w that consists of the letters $\ge i$.

Definition 4.5.1 *Given $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_M$, for every $i = 1, 2, \ldots, \ell$, pair the entries in $u := w_{\ge i} = u_1 u_2 \cdots u_m$ in the following recursive algorithm.*

- *If $m = 1$, leave the entry u_1 in u unpaired.*
- *If $m \ge 2$ and $b_i(w)$ is even, leave u_1 unpaired and pair the remaining word $u_2 u_3 \cdots u_m$ recursively.*
- *If $m \ge 2$ and $b_i(w)$ is odd, pair the first two entries u_1 and u_2 and pair the remaining word $u_3 u_4 \cdots u_m$ recursively.*

For example, let $w = 12332132, v = 31232231 \in \{1^2, 2^3, 3^3\}$, then the pairing indicated by underlining for these two words are

$$w_{\ge 1} = \underline{1}\ \underline{2}\ \underline{33}\ \underline{21}\ \underline{3}\ \underline{2}, \text{ and } v_{\ge 1} = \underline{3}\ \underline{1}\ \underline{23}\ \underline{22}\ \underline{31}.$$

$$w_{\ge 2} = \underline{23}\ \underline{3}\ \underline{23}\ \underline{2}, \text{ and } v_{\ge 2} = \underline{32}\ \underline{3}\ \underline{22}\ \underline{3}.$$

$$w_{\ge 3} = \underline{3}\ \underline{3}\ \underline{3}, \text{ and } v_{\ge 3} = \underline{3}\ \underline{3}\ \underline{3}.$$

Let $M' \subseteq \mathfrak{S}_M$ consists of all permutations w that do not contain a paired (i,j) with $j > i$ for any $w_{\ge i}$. In the above example, we have $w \notin M'$ and $v \in M'$. Moreover, for any permutation $w \in M'$, let $r(w)$ be the number of (j,i) pairs where $j > i$ over all $w_{\ge i}$.

Theorem 4.5.2 *The q-multinomial is given by*

$$\begin{bmatrix} n \\ \alpha_1, \alpha_2, \ldots, \alpha_\ell \end{bmatrix}_q = \sum_{w \in M'} q^{\mathrm{inv}(w) - r(w)} \cdot (1+q)^{r(w)}.$$

w	$w_{\geq 1}$	$w_{\geq 2}$	$r(w)$	$\mathrm{inv}(w)$
11233	$\underline{11}\ \underline{23}\ \underline{3}$	$\underline{2}\ \underline{3}\ \underline{3}$	0	0
11332	$\underline{11}\ \underline{33}\ \underline{2}$	$\underline{3}\ \underline{32}$	1	2
21133	$\underline{21}\ \underline{1}\ \underline{3}\ \underline{3}$	$\underline{2}\ \underline{3}\ \underline{3}$	1	2
21331	$\underline{21}\ \underline{3}\ \underline{31}$	$\underline{2}\ \underline{3}\ \underline{3}$	2	4
23113	$\underline{23}\ \underline{11}\ \underline{3}$	$\underline{2}\ \underline{3}\ \underline{3}$	0	4
23311	$\underline{23}\ \underline{31}\ \underline{1}$	$\underline{2}\ \underline{3}\ \underline{3}$	1	6
31132	$\underline{31}\ \underline{1}\ \underline{3}\ \underline{2}$	$\underline{3}\ \underline{32}$	2	4
31321	$\underline{31}\ \underline{3}\ \underline{21}$	$\underline{3}\ \underline{32}$	3	6
33112	$\underline{33}\ \underline{11}\ \underline{2}$	$\underline{3}\ \underline{32}$	1	6
33211	$\underline{33}\ \underline{21}\ \underline{1}$	$\underline{3}\ \underline{32}$	2	8

Table 4.3: All permutations in $M' \subseteq \mathfrak{S}_{\{1^2,2,3^2\}}$ with their pairings at each level.

Proof: The q-multinomial satisfies the recurrence

$$\begin{bmatrix} n \\ \alpha_1, \alpha_2, \ldots, \alpha_\ell \end{bmatrix}_q = \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}_q \cdot \begin{bmatrix} n-\alpha_1 \\ \alpha_2, \alpha_3, \ldots, \alpha_\ell \end{bmatrix}_q .$$

When $\ell = 2$, the algorithm gives the set $\Omega'(n,k)$ described in [28, Theorem 1]. Thus by induction on ℓ the result follows. □

For example, Table 4.3 gives all permutations in $M' \subseteq \mathfrak{S}_{\{1^2,2,3^2\}}$. Summing over all the weights gives

$$\begin{aligned} 1 + & q\cdot(1+q) + q\cdot(1+q) + q^2\cdot(1+q)^2 + q^4 + q^5\cdot(1+q) \\ & + q^2\cdot(1+q)^2 + q^3\cdot(1+q)^3 + q^5\cdot(1+q) + q^6\cdot(1+q)^2 \\ & = q^8 + 2q^7 + 4q^6 + 5q^5 + 6q^4 + 5q^3 + 4q^2 + 2q + 1 \\ & = \frac{[5]_q!}{[2]_q!\cdot[2]_q!} = \begin{bmatrix} 5 \\ 2,1,2 \end{bmatrix}_q . \end{aligned}$$

As a remark, $|\mathfrak{S}_{\{1^2,2,3^2\}}| = 30$ while the size of M' is 10 in this case, thus the q-$(1+q)$-multinomial greatly reduces the size of the permutations.

Replace t by t^{-1} in Equation (4.5.1) we obtain

$$\begin{aligned} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t^{-1}} &= \frac{\prod_{i=1}^{n}(1 - t^{-q^n + q^{n-i}})}{\prod_{s=1}^{\ell}\prod_{i=1}^{\alpha_s}(1 - t^{-q^{\sigma_s} + q^{\sigma_s - i}})} \\ &= t^{\tilde{A}} \cdot \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} , \end{aligned}$$

where the exponent $\tilde{A}$ of t is

$$\begin{aligned}\tilde{A} &= \sum_{i=1}^{n}(-q^n + q^{n-i}) - \sum_{s=1}^{\ell}\sum_{i=1}^{\alpha_s}(-q^{\sigma_s} + q^{\sigma_s - i}) \\ &= \sum_{s=1}^{\ell}(\alpha_s \cdot q^{\sigma_s} - n \cdot q^n).\end{aligned}$$

The last line holds since $\sum_{i=1}^{n} q^{n-i} = \sum_{s=1}^{\ell}\sum_{i=1}^{\alpha_s} q^{\sigma_s - i}$. Thus the (q,t)-multinomial is symmetric in t,

$$\left(\prod_{s=1}^{\ell} t^{\alpha_s \cdot (q^n - q^{\sigma_s})}\right) \cdot \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t^{-1}} = \begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t}. \tag{4.5.3}$$

In [28, Section 3] the authors studied the properties of the (q,t)-binomial at negative q values, which we now restate here.

Lemma 4.5.3 (Fu–Reiner–Stanton–Thiem, 2012) *When $q \leq -2$ is a negative integer, the (q,t)-binomial defined as a rational function in (4.5.2) is a Laurent polynomial in t whose nonzero coefficients all have sign $(-1)^{k\cdot(n-k)}$. The smallest exponent of t is*

0	*if n, k are both even,*
$k \cdot (q^n - q^k)$	*if n, k are both odd,*
$k \cdot q^n - \frac{1-q^k}{1-q}$	*if n is odd and k is even,*
$-k \cdot q^k + \frac{1-q^k}{1-q}$	*if n is even and k is odd.*

We generalize this result to the (q,t)-multinomial.

Theorem 4.5.4 *When $q \leq -2$ is a negative integer, the (q,t)-multinomial lies in $\mathbb{Z}[t,t^{-1}]$ with all nonzero coefficients having sign $\prod_{s=1}^{\ell}(-1)^{\alpha_s\cdot(n-\sigma_s)}$. Moreover, the coefficients of t are symmetric about the term $\prod_{s=1}^{\ell} t^{\alpha_i \cdot (q^n - q^{\sigma_s})}$. The smallest exponent of t is given by*

$$\sum_{\substack{0\leq i\leq \ell-2 \\ n-\sigma_i \text{ is odd}}} \alpha_{i+1}\cdot q^n - \sum_{\substack{0\leq i\leq \ell-2 \\ \alpha_{i+1} \text{ is odd}}} \alpha_{i+1}\cdot q^{\sigma_{i+1}} + \sum_{\substack{0\leq i\leq \ell-2 \\ n-\sigma_{i+1} \text{ is odd}}} \frac{q^{\sigma_i} - q^{\sigma_{i+1}}}{1-q}$$

with coefficient 1 and the largest exponent of t can be deduced thereafter using the symmetric relation as shown in Equation (4.5.3).

Proof: Since

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,t} = \begin{bmatrix} n \\ \alpha_1 \end{bmatrix}_{q,t} \cdot \begin{bmatrix} n-\alpha_1 \\ \alpha_2 \end{bmatrix}_{q,t^{q^{\alpha_1}}} \cdot \begin{bmatrix} n-\sigma_2 \\ \alpha_3 \end{bmatrix}_{q,t^{q^{\sigma_2}}} \cdots \begin{bmatrix} \alpha_{l-1}+\alpha_l \\ \alpha_{l-1} \end{bmatrix}_{q,t^{q^{\sigma_{l-2}}}},$$

apply Lemma 4.5.3 and Equation (4.5.3) recursively we obtain the theorem. □

4.6 (q,t)-multinomial and cyclic sieving phenomenon

In this section we give an instance of the cyclic sieving phenomenon on the flags of unitary spaces, this is a generalization of [28, Section 5]. We begin with some preliminaries. Throughout this section we assume n is an odd integer.

For n an odd number, define the following sums.

$$\begin{aligned} D &= \sum_{s=1}^{\ell} \frac{\alpha_s}{m} \cdot \left(\frac{n}{m} - \frac{\sigma_s}{m} \right) \\ E &= \sum_{\substack{1 \le s \le \ell \\ \sigma_s \text{ is even}}} \sum_{i=1}^{\alpha_s} 2 \cdot (q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}), \\ R &= \sum_{s=1}^{\ell} \alpha_s \cdot (n - \sigma_s). \end{aligned}$$

Let γ be a generator for the multiplicative group $\mathbb{F}_{q^{2n}}^{\times} \cong \mathbb{Z}/(q^{2n}-1)\mathbb{Z}$. The following lemma is from [28, Proposition 8].

Lemma 4.6.1 (Fu–Reiner–Stanton–Thiem, 2012) *The power γ^{q^n-1} generates a cyclic subgroup $C \cong \mathbb{Z}/(q^n+1)\mathbb{Z}$ of $\mathbb{F}_{q^{2n}}^{\times}$ which acts on $V = \mathbb{F}_{q^{2n}}$ unitarily with respect to the $\mathbb{F}_{q^{2m}}$-Hermitian form $(\cdot,\cdot)_{\mathbb{F}_{q^{2m}}}$, where n is divisible by m.*

Let X be the set of all α-flags of nondegenerate $\mathbb{F}_{q^2}$-subspace of $\mathbb{F}_{q^{2n}}$:

$$0 \subseteq V^{\sigma_1} \subseteq V^{\sigma_2} \subseteq \cdots \subseteq V^{\sigma_\ell} = \mathbb{F}_{q^{2n}}$$

where $\dim_{\mathbb{F}_{q^2}} V^{\sigma_s} = \sigma_s$. Then the cyclic group $C \cong \mathbb{Z}/(q^2+1)\mathbb{Z}$ permutes the set X.

Given $c \in C$, let $\mathbb{F}_{q^2}(c)$ denote the subfield of $\mathbb{F}_{q^{2n}}$ generated by $\mathbb{F}_{q^2}$ and c. Then there exists a unique divisor m of n such that $\mathbb{F}_{q^2}(c) = \mathbb{F}_{q^{2m}}$. Let $\beta_q(c) = |\{x \in X : c(x) = x\}|$. Notice that an α-flag described as above is fixed by c if and only if each V^{σ_s} in the flag is a nondegenerate subspace over $\mathbb{F}_{q^2}(c) = \mathbb{F}_{q^{2m}}$, that is, V^{σ_s} is a (σ_s/m)-dimensional $\mathbb{F}_{q^{2m}}$-subspace.

Lemma 4.6.2 (Fu–Reiner–Stanton–Thiem, 2012) *The number of k-dimensional $\mathbb{F}_{q^2}$-subspaces of $\mathbb{F}_{q^{2n}}$ that is fixed by c is*

$$(-q^m)^{k/m\cdot(n/m-k/m)} \cdot \begin{bmatrix} n/m \\ k/m \end{bmatrix}_{-q^m}.$$

This lemma implies that

$$\begin{aligned} \beta_q(c) &= \prod_{s=1}^{\ell} (-q^m)^{\sigma_{s-1}/m\cdot\alpha_s/m} \cdot \begin{bmatrix} \sigma_s/m \\ \sigma_{s-1}/m \end{bmatrix}_{-q^m} \\ &= (-q^m)^D \cdot \begin{bmatrix} n/m \\ \alpha_1/m, \alpha_2/m, \ldots, \alpha_\ell/m \end{bmatrix}_{-q^m}. \end{aligned} \tag{4.6.1}$$

Lemma 4.6.3 *Let $X(t)$ be a Laurent polynomial in t defined as follows.*

$$X(t) = t^E \cdot \prod_{s=1}^{\ell} \prod_{i=1}^{\alpha_s} \frac{1 - t^{q^n - (-1)^i \cdot q^{n-i}}}{1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}} \cdot \frac{\prod_{i=1}^{n} 1 - t^{q^n - (-1)^i \cdot q^{n-i}}}{\prod_{s=1}^{\ell} \prod_{i=1}^{\alpha_s} 1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}}. \tag{4.6.2}$$

Then $X(t) \in \mathbb{N}[t]$, that is, $X(t)$ is a polynomial in t with non-negative coefficients.

Proof: Denote by $v = q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}$ then we have

$$\begin{aligned} \frac{1 - t^{q^n - (-1)^i \cdot q^{n-i}}}{1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}} &= \frac{1 - t^{v \cdot q^{n - \sigma_s}}}{1 - t^v} \\ &= 1 + t^v + t^{2v} + \cdots + t^{(q^{n-\sigma_s} - 1) \cdot v}. \end{aligned}$$

So the first product in $X(t)$ is in $\mathbb{N}[t]$.

For the second product, denote it by $B(t)$. By Equation (4.5.3) we conclude

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{-q,t} = \prod_{s=1}^{\ell} t^{\alpha_s \cdot (-q^n - (-1)^{\sigma_s} \cdot q^{\sigma_s})} \cdot \begin{bmatrix} n \\ \alpha \end{bmatrix}_{-q,t^{-1}}.$$

Thus we have

$$\begin{aligned} \prod_{s=1}^{\ell} t^{\alpha_s \cdot (q^n + (-1)^{\sigma_s} \cdot q^{\sigma_s})} \begin{bmatrix} n \\ \alpha \end{bmatrix}_{-q,t} &= \frac{\prod_{i=1}^{n} (1 - t^{q^n - (-1)^i \cdot q^{n-i}})}{\prod_{s=1}^{\ell} \prod_{i=1}^{\alpha_s} (1 - t^{-(-q)^{\sigma_s} - (-1)^{\sigma_s - i} \cdot q^{\sigma_s - i}})} \\ &= \prod_{\substack{1 \le s \le \ell \\ \sigma_s \text{ is even}}} \prod_{i=1}^{\alpha_s} (-t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}) \cdot B(t). \end{aligned}$$

This gives the relation

$$t^{E/2} \cdot B(t) = \prod_{\substack{1 \le s \le \ell \\ \sigma_s \text{ is even}}} (-1)^{\alpha_s} \cdot \prod_{s=1}^{\ell} t^{\alpha_s \cdot q^n + \alpha_s \cdot (-q)^{\sigma_s}} \cdot \begin{bmatrix} n \\ \alpha \end{bmatrix}_{-q,t}. \tag{4.6.3}$$

Since n is odd, the two terms

$$\sum_{s=1}^{\ell} \alpha_s \cdot (n - \sigma_s) \qquad \text{and} \qquad \sum_{\substack{1 \le s \le \ell \\ \sigma_s \text{ is even}}} \alpha_s$$

have the same parity, thus $t^E \cdot B(t)$ have nonnegative coefficients. To show $t^E \cdot B(t) \in \mathbb{N}[t]$, we check the smallest exponent of t is nonnegative. Again, since n is odd, from Theorem 4.5.4 and Equation (4.6.3), this exponent is

$$\begin{aligned} \Delta = & \sum_{\substack{1 \le s \le \ell \\ \sigma_s \text{ is even}}} \sum_{1 \le i \le \alpha_s} (q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}) + \sum_{1 \le s \le \ell} (\alpha_s \cdot q^n + \alpha_s \cdot (-q)^{\sigma_s}) \\ & - \sum_{\substack{1 \le s \le \ell - 1 \\ \sigma_{s-1} \text{ is even}}} \alpha_s \cdot q^n - \sum_{\substack{1 \le s \le \ell - 1 \\ \alpha_s \text{ is odd}}} \alpha_s \cdot (-q)^{\sigma_s} + \sum_{\substack{1 \le s \le \ell - 1 \\ \sigma_s \text{ is even}}} \frac{(-q)^{\sigma_{s-1}} - (-q)^{\sigma_s}}{1 + q}. \end{aligned}$$

It is straightforward to check

$$\sum_{1\le s\le \ell} (\alpha_s \cdot q^n + \alpha_s \cdot (-q)^{\sigma_s}) - \sum_{\substack{1\le s\le \ell-1 \\ \sigma_{s-1} \text{ is even}}} \alpha_s \cdot q^n - \sum_{\substack{1\le s\le \ell-1 \\ \alpha_s \text{ is odd}}} \alpha_s \cdot (-q)^{\sigma_s} \ge 0.$$

Also, since we have the following expansion for the geometric sequence

$$\begin{aligned}\frac{(-q)^{\sigma_{s-1}} - (-q)^{\sigma_s}}{1+q} &= (-q)^{\sigma_{s-1}}(1 - q + q^2 - q^3 + \cdots + (-q)^{\alpha_s - 1}) \\ &= \sum_{1\le i\le \alpha_s} (-1)^i \cdot q^{\sigma_s - i},\end{aligned}$$

when σ_s is even, we have

$$\sum_{\substack{1\le s\le \ell \\ \sigma_s \text{ is even}}} \sum_{1\le i\le \alpha_s} (q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}) + \sum_{\substack{1\le s\le \ell-1 \\ \sigma_s \text{ is even}}} \frac{(-q)^{\sigma_{s-1}} - (-q)^{\sigma_s}}{1+q} \ge 0.$$

Hence we conclude $\Delta \ge 0$ and $t^E \cdot B(t) \in \mathbb{N}[t]$. Therefore $X(t)$ is in $\mathbb{N}[t]$. $\square$

Since n is odd, we also verify that

$$X(1) = (-q)^R \cdot \begin{bmatrix} n \\ \alpha \end{bmatrix}_{-q}. \tag{4.6.4}$$

Moreover, when q is odd all powers of t in $X(t)$ occur with even exponents, thus

$$X(-1) = X(1) \text{ when } q \text{ is odd}. \tag{4.6.5}$$

We have the following theorem.

Theorem 4.6.4 *The triple $(X, X(t), C)$ exhibits the cyclic sieving phenomenon.*

We first state the following proposition [28, Proposition 10], which is used as a key technique to prove the theorem.

Proposition 4.6.5 *Let c be an element in $\mathbb{F}_{q^{2n}}^\times$ with multiplicative order $r \ge 3$, and $\mathbb{F}_{q^2}(c) = \mathbb{F}_{q^{2m}}$ where m divides n. Then*

1. *The order r must divide $q^m + 1$.*
2. *The smallest positive integer d such that r divides $q^d + 1$ is m.*
3. *The order r divides $q^s + q^t$ if and only if $s - t$ is an odd multiple of m.*
4. *The order r divides $q^s - q^t$ if and only if $s - t$ is an even multiple of m.*

Apply L'Hôpital's rule we have the following limits, which will also be used in our proof.

$$\lim_{t\to\omega}\frac{1-t^a}{1-t^b}=\begin{cases} a/b & \text{if } a\equiv b\equiv 0 \pmod r,\\ 1 & \text{if } a\equiv b\not\equiv 0 \pmod r,\\ -\omega^{-b} & \text{if } a\equiv -b\not\equiv 0 \pmod r.\end{cases} \tag{4.6.6}$$

We now proof the theorem.

Proof: Assume $c\in\mathbb{F}_{q^{2n}}^{\times}$ has multiplicative order r, and $\mathbb{F}_{q^2}(c)=\mathbb{F}_{q^{2m}}$. When $r=1$ Equations (4.6.1) and (4.6.4) imply $X(1)=|X|$.

If $r=2$, this implies q is odd. By Equation (4.6.5), $X(1)=X(-1)$ hence the result holds.

Now assume $r\geq 3$. We first show that if there is some α_s such that m does not divide α, then $X(\omega)=0$. This is because $1-\omega^{q^n-(-1)^i\cdot q^{n-i}}$ is 0 if and only if i is a multiple of m, hence there are n/m zero terms on the numerator in $B(t)$. For the denominator, each term $1-\omega^{q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i}}$ is zero if i is a multiple of m, hence there are $\lfloor\alpha_s/m\rfloor$ such terms for each s. Since there is at least one s such that $m\nmid\alpha_s$ we have $n/m>\sum_{s=1}^{\ell}\lfloor\alpha_s/m\rfloor$. This implies there are more zeros in the numerator than in the denominator, hence $B(\omega)=0$ in this case.

Now assume $m|\alpha_s$ for all s. We check the two products in $X(t)$ separately. For future reference, let

$$A(t)=\prod_{s=1}^{\ell}\prod_{i=1}^{\alpha_s}\frac{1-t^{q^n-(-1)^i\cdot q^{n-i}}}{1-t^{q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i}}},$$

and rewrite

$$B(t)=\prod_{s=1}^{\ell}\prod_{i=1}^{\alpha_s}\frac{1-t^{q^n-(-1)^{i+\sigma_{s-1}}\cdot q^{n-i-\sigma_{s-1}}}}{1-t^{q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i}}}.$$

We first show that $A(\omega)=\omega^{-E/2}\cdot(-q^m)^D$.

When i is a multiple of m, each term in A takes the limit

$$\lim_{t\to\omega}\frac{1-t^{q^n-(-1)^i\cdot q^{n-i}}}{1-t^{q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i}}}=\frac{q^n-(-1)^i\cdot q^{n-i}}{q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i}}=q^{n-\sigma_s}.$$

All such terms give a total of

$$\prod_{s=1}^{\ell}\prod_{i=1}^{\alpha_s/m}q^{n-\sigma_s}=\prod_{s=1}^{\ell}q^{(n-\sigma_s)\cdot(\alpha_s/m)}=(q^m)^D.$$

If i is not a multiple of m, then when σ_s is odd, we have

$$q^n-(-1)^i\cdot q^{n-i}-(q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i})=(q^i-(-1)^i)\cdot(q^{n-i}-q^{\sigma_s-i}).$$

Since n and σ_s are both odd and divisible by m, $n-\sigma_s$ is an even multiple of m, hence $q^{n-i}-q^{\sigma_s-i}$ is divisible by r. Thus by Equation (4.6.6)

$$\lim_{t\to\omega}\frac{1-t^{q^n-(-1)^{i+\sigma_{s-1}}\cdot q^{n-i-\sigma_{s-1}}}}{1-t^{q^{\sigma_s}-(-1)^i\cdot q^{\sigma_s-i}}}=1.$$

When σ_s is even, a similar discussion gives that

$$q^n - (-1)^i \cdot q^{n-i} + (q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}) = (q^i - (-1)^i) \cdot (q^{n-i} + q^{\sigma_s - i})$$

is divisible by r. Hence the following limit holds:

$$\lim_{t\to\omega} \frac{1 - t^{q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}}}{1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}} = -\omega^{-(q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i})}.$$

Combine the above three cases and notice that when n is odd, $\sum_{\sigma_s \text{ is even}} \alpha_s$ and D have the same parity, we have

$$\lim_{t\to\omega} A(t) = \omega^{-E/2} \cdot (-q^m)^D.$$

Next we show that $B(\omega) = \omega^{-E/2} \cdot \begin{bmatrix} n/m \\ \alpha_1/m, \alpha_2/m, \ldots, \alpha_l/m \end{bmatrix}_{-q^m}$.

If i is a multiple of m, then $r | q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}$ and $r | q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}$. Hence in this case the limit is

$$\lim_{t\to\omega} \frac{1 - t^{q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}}}{1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}} = \frac{q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}}{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}.$$

All terms of this form give the product

$$\prod_{s=1}^{\ell} \prod_{i=1}^{\alpha_s/m} \frac{q^n - (-1)^{i+\sigma_{s-1}} \cdot (q^m)^{n/m - i - \sigma_{s-1}/m}}{q^{\sigma_s} - (-1)^i \cdot (q^m)^{\sigma_s/m - i}} = \\ (-1)^D \cdot \begin{bmatrix} n/m \\ \alpha_1/m, \alpha_2/m, \ldots, \alpha_l/m \end{bmatrix}_{-q^m}. \tag{4.6.7}$$

When i is not a multiple of m and σ_s is odd, we have

$$\left(q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}\right) - \left(q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}\right) = \\ \left(q^n - q^{\sigma_s}\right) - (-1)^i \cdot \left((-1)^{\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}} - q^{\sigma_s - i}\right).$$

Since $r | q^n - q^{\sigma_s}$ and $r | (-1)^{\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}} - q^{\sigma_s - i}$, the limit becomes

$$\lim_{t\to\omega} \frac{1 - t^{q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}}}{1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}} = 1. \tag{4.6.8}$$

If σ_s is even, write

$$\left(q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}\right) + \left(q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}\right) = \\ \left(q^n + q^{\sigma_s}\right) - (-1)^i \cdot \left((-1)^{\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}} + q^{\sigma_s - i}\right)$$

with $r | q^n + q^{\sigma_s}$ and $r | (-1)^{\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}} + q^{\sigma_s - i}$. Hence in this case we have

$$\lim_{t\to\omega} \frac{1 - t^{q^n - (-1)^{i+\sigma_{s-1}} \cdot q^{n-i-\sigma_{s-1}}}}{1 - t^{q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i}}} = -\omega^{-(q^{\sigma_s} - (-1)^i \cdot q^{\sigma_s - i})}. \tag{4.6.9}$$

Combining Equations (4.6.7), (4.6.8) and (4.6.9) we have

$$\lim_{t \to \omega} B(t) = \omega^{-E/2} \cdot \begin{bmatrix} n/m \\ \alpha_1/m, \alpha_2/m, \ldots, \alpha_l/m \end{bmatrix}_{-q^m}.$$

Thus

$$X(\omega) = \omega^E \cdot A(\omega) \cdot B(\omega) = (-q^m)^D \cdot \begin{bmatrix} n/m \\ \alpha_1/m, \alpha_2/m, \ldots, \alpha_l/m \end{bmatrix}_{-q^m},$$

and the theorem follows. □

Chapter 5 Future research directions

Several questions have arisen from my research.

1. Billey and Coskun [5] showed that the q-Stirling number of the second kind is the generating function of the dimension of a rank variety of the Grassmannian. Via a change of variable Theorem 2.3.2 can be expressed as a polynomial in q and $q-1$. I expect the partition corresponding to the allowable words to say something interesting about the rank variety.

2. A polynomial $F(q) = a_0 + a_1 \cdot q + \cdots + a_n \cdot q^n$ is *unimodal* if the sequence of the coefficients $(a_0, a_1, \ldots, a_n)$ is unimodal, that is, the sequence satisfies the condition

$$a_0 \le a_1 \le \cdots \le a_{i-1} \le a_i \ge a_{i-1} \ge \cdots \ge a_{n-1} \ge a_n.$$

Sylvester [75] gave the first analytic proof that the q-binomial $\left[{n \atop k}\right]_q$ is unimodal. O'Hara [57] gave the first combinatorial proof of this result using a symmetric chain decomposition. Recently, Pak and Panova [58] showed that when $m \ge 7$, the polynomial $\left[{2m \atop m}\right]_q$ is strictly unimodal, that is, if we write

$$\begin{bmatrix} 2m \\ m \end{bmatrix}_q = \sum_{i=0}^{m^2} a_i \cdot q^i,$$

then the coefficients satisfy

$$a_1 < a_2 < \cdots < a_{\lfloor m^2/2 \rfloor} = a_{\lceil m^2/2 \rceil} > \cdots > a_{m^2-2} > a_{m^2-1}.$$

It has been conjectured by Garsia and Remmel [31, Page 250] that the q-Stirling number of the second kind $S_q[n,k]$ is unimodal. Wachs and White [79, Page 45] further conjectured $S_q[n,k]$ is log-concave, a sufficient condition for unimodality. Applying O'Hara's idea to the Stirling poset of the second kind $\Pi(n,k)$, can we find a chain decomposition of this poset and therefore prove $S_q[n,k]$ is unimodal?

3. Are there other extensions of q-analogues such as q-$(1+q+q^2)$-analogues? If so, the overall goal would be to find combinatorial implications behind the expressions. Is there a cyclic sieving phenomenon for these more general q-$[n]_q$-analogues?

4. The q-$(1+q)$-analogue is defined on a smaller subset of the original combinatorial objects defining the q-analogue. However, the subsets we studied for q-Stirling numbers and q-binomials are not "minimized". In other words, we may define a bistatistic on an even smaller subset to recover the q-analogues. A natural question is then how can we measure the "minimality" of the subset. For example, one could minimize the number of elements in the subsets. Another

interesting way to look at this is via the Boolean algebra decomposition of the poset. One may want to find Boolean algebras of largest rank in such a decomposition. On the q-$(1+q)$-analogue level, this would mean finding a bistatistic in which the factor $1+q$ has the highest exponent as possible. These different measures may also give other interesting properties on different subsets for the resulting q-$(1+q)$-analogues.

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Publications

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2. "Negative q-Stirling numbers", with M. Readdy, *Proceedings of the 27th International Conference on Formal Power Series and Algebraic Combinatorics*, Daejeon, South Korea, July 2015, pp. 583–594.
3. "q-Stirling identities revisited", with M. Readdy and R. Ehrenborg, in preparation.
4. "A q-$(1+q)$ analogue of the major index", in preparation.

Sequences

I have a sequence in *The On-Line Encyclopedia of Integer Sequences*, ed. by N. J. A. Sloane. **A256161** (Triangle of allowable Stirling numbers of the second kind $a(n,k)$): 1, 1, 1, 1, 2, 1, 1, 3, 4, 1, 1, 4, 11, 6, 1, 1, 5, 26, 23, 9, 1, 1, 6, 57, 72, 50, 12, 1, 1, 7, 120, 201, 222, 86, 16, 1, 1, 8, 247, 522, 867, 480, 150, 20, 1, 1, 9, 502, 1291, 3123, 2307, 1080, 230, 25, 1, ...