2016

Homogenization of Stokes Systems with Periodic Coefficients

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Digital Object Identifier: http://dx.doi.org/10.13023/ETD.2016.302

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Homogenization of Stokes Systems with Periodic Coefficients

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Shu Gu
Lexington, Kentucky

Director: Dr. Zhongwei Shen, Professor of Mathematics
Lexington, Kentucky 2016

ABSTRACT OF DISSERTATION

Homogenization of Stokes Systems with Periodic Coefficients

In this dissertation we study the quantitative theory in homogenization of Stokes systems. We study uniform regularity estimates for a family of Stokes systems with rapidly oscillating periodic coefficients. We establish interior Lipschitz estimates for the velocity and L^∞ estimates for the pressure as well as Liouville property for solutions in $\mathbb{R}^d$. We are able to obtain the boundary $W^{1,p}$ estimates in a bounded C^1 domain for any $1 < p < \infty$. We also study the convergence rates in L^2 and H^1 of Dirichlet and Neumann problems for Stokes systems with rapidly oscillating periodic coefficients, without any regularity assumptions on the coefficients.

KEYWORDS: Homogenization, Stokes systems, convergence rates, uniform regularity, Dirichlet problem, Neumann problem

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Date: July 25, 2016

Homogenization of Stokes Systems with Periodic Coefficients

By
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Dedicated to Jing Wei

ACKNOWLEDGMENTS

My utmost thanks and deepest gratitude go to my advisor Professor Zhongwei Shen for his enormous patience, guidance, suggestions and encouragement during my Ph.D. years, without which the work would not be possible. His immense knowledge, enthusiasm and work ethic have influenced me remarkably and will continue to guide me throughout my life. Living in a different country was never easy, I am extremely grateful to Professor Shen and his wife Mrs. Liping Peng for their kindness and loads of invitations to their home.

I extend my genuine thanks and gratitude to Professor John Lewis, Professor Russell Brown, Professor Yuan Liao and Professor Dong-Sheng Yang for helpful comments and for serving on my thesis committee.

Special thanks go to Professor Russell Brown for his teaching of Real Analysis and Harmonic analysis, holding "secret" seminars, and for generous support throughout my graduate studies.

I would like to Professor Qiang Ye and Professor Qun He for all their invaluable advise and help at various stages of my mathematical education.

Most of all, I wish to thank my wife, Jing Wei. Her tremendous love and support have always been my guiding light. Finally, I thank my parents for their constant encouragement and support.

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Chapter 1 Introduction and Main Results

The theory of homogenization was introduced in part to describe the behavior of composite materials in mechanics, physics, chemistry and engineering. Composite materials are usually characterized by two scales, the microscopic one, describing the heterogeneities, and the macroscopic one, describing the global behavior of the composite. In a composite, the heterogeneities are small compared to its global dimension, while from the macroscopic points of view, the composite looks like a "homogeneous" material. The intent of homogenization theory is to replace the microscopically heterogeneous material by a *homogenized* material, whose global characteristics are a good approximation of the initial ones.

In the study of boundary value problems in media with periodic structure, if the period of the structure is small compared to the size of the region in which the system is to be studied, we will use a small parameter ε to denote the ratio of the period of the structure to a typical length in the region. In mathematics terms, a family of partial differential operators $\mathcal{L}_\varepsilon$ with rapidly oscillating coefficients, depending on the small parameter ε, is given. In a domain Ω, we have a boundary value problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F \qquad \text{in } \Omega, \\ u_\varepsilon \text{ subject to appropriate boundary conditions.} \end{cases}$$

Homogenization theory has shown that u_ε converges to u_0 as $\varepsilon \to 0$ (with suitable definition of weak type convergence), where u_0 is the solution of

$$\begin{cases} \mathcal{L}_0(u_0) = F \qquad \text{in } \Omega, \\ u_0 \text{ subject to the same kind of boundary conditions,} \end{cases}$$

where $\mathcal{L}_0$ is a partial differential operator with constant coefficients, which is called the **homogenized** operator of the family $\mathcal{L}_\varepsilon$.

Specifically in qualitative homogenization theory, for the standard elliptic system $-\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = F$ in a bounded domain Ω in $\mathbb{R}^d$, the proof of homogenization theorem was first obtained by De Diorgi and Spagnolo [16–18, 53, 54]. Here $A(y)$ is a matrix with periodic measurable coefficients satisfying ellipticity condition. Shortly thereafter, Bakhvalov [8,9] and then Lions [10,40] established the same result based on method of asymptotic expansions. Another approach to the homogenization theory, based on compensated compactness, was developed by Murat [43] and Tartar [57].

Quantitative homogenization has been studied extensively in recent years, given these qualitative results in homogenization for various types of equations with various boundary conditions. There are two main and natural tasks in quantitative homogenization theory,

1. uniform regularity estimates of solutions, which are independent of the small parameter ε;

2. sharp convergence rates, which describe the speed of convergence.

Uniform Regularity Estimates

For uniform regularity estimates, we consider a family of standard second-order elliptic operator $\mathcal{L}_\varepsilon$ in divergence form with rapidly oscillating coefficients, which are defined by

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left[a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right], \quad \varepsilon > 0. \tag{1.0.1}$$

with $1 \leq i, j, \alpha, \beta \leq d$, the *summation convention* is used throughout this thesis.

The study of uniform regularity estimates in homogenization theory was initiated by M. Avellaneda and F. Lin in 1987. In a series of paper [3–7] from 1987 to 1991, Avellaneda and Lin established interior and boundary Lipschitz estimates and also $W^{1,p}$ estimates for the standard elliptic system $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ with Dirichlet boundary condition $u_\varepsilon = g$ on $\partial\Omega$ for $C^{1,\alpha}$ domains, assuming the coefficient matrix A is elliptic, periodic and Hölder continuous. The approach called compactness method was used in [3] to prove Lipschitz estimates. We should mention that the Lipschitz estimates are sharp; in fact, even with C^∞ data, one cannot expect high-order uniform estimates for u_ε, since ∇u_ε converges to ∇u_0 only weakly.

For standard second-order elliptic system $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ with Neumann boundary condition $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$, Lipschitz estimates has been a longstanding open problem, as the boundary conditions are ε-dependent. It was only until in 2013, in [34] C. Kenig, F. Lin and Z. Shen were able to extend the boundary Lipschitz estimates to Neumann problems in $C^{1,\alpha}$ domains, with additional symmetry condition $A^* = A$. The breakthrough is based on the Rellich estimates obtained in [36,37] and nontangential maximal function estimates in [34]. Sharp $W^{1,p}$ estimates for Neumann problem were also obtained.

The symmetry condition was recently removed by S. Armstrong and Z. Shen. In [1], the uniform Lipschitz estimates and $W^{1,p}$ estimates in $C^{1,\alpha}$ were obtained for second-order elliptic system in divergence form with rapidly oscillating, almost-periodic coefficients, with either Dirichlet or Neumann data. In contrast to papers [3,34], the results were proved through constructive arguments, and thus the constants are in principle computable.

In this thesis, we study the uniform regularity estimates for a family of Stokes systems with rapidly oscillating periodic coefficients. We establish interior Lipschitz estimates for the velocity and L^∞ estimates for the pressure as well as a Liouville property for solutions in $\mathbb{R}^d$. We also obtain the boundary $W^{1,p}$ estimates in a bounded C^1 domain for any $1 < p < \infty$.

More precisely, we consider the Stokes systems in fluid dynamics,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \qquad \operatorname{div}(u_\varepsilon) = g, \end{cases} \tag{1.0.2}$$

in a bounded domain Ω in $\mathbb{R}^d$, where $\varepsilon > 0$ and $\mathcal{L}_\varepsilon$ is defined in (1.0.1). We will assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ is real, bounded measurable, satisfies the ellipticity condition:

$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \le \frac{1}{\mu}|\xi|^2, \text{ for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}, \tag{1.0.3}$$

where $\mu > 0$, and the periodicity condition:

$$A(y+z) = A(y) \text{ for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.0.4}$$

We note that the system (1.0.2), which does not fit the standard framework of second-order elliptic systems considered in [3, 34], is used in the modeling of flows in porous media.

The following is one of the main results we obtained in [30].

Theorem 1.0.1. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of the Stokes system (1.0.2) in $B(x_0,R)$ for some $x_0 \in \mathbb{R}^d$ and $R > \varepsilon$. Then, for any $\varepsilon \le r < R$,*

$$\begin{aligned} &\left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B(x_0,r)} |p_\varepsilon - \fint_{B(x_0,R)} p_\varepsilon|^2\right)^{1/2} \\ &\le C\Bigg\{\left(\fint_{B(x_0,R)} |\nabla u_\varepsilon|^2\right)^{1/2} + \|g\|_{L^\infty(B(x_0,R))} + R^\rho [g]_{C^{0,\rho}(B(x_0,R))} \\ &\qquad\qquad + CR\left(\fint_{B(x_0,R)} |F|^q\right)^{1/q}\Bigg\}, \end{aligned} \tag{1.0.5}$$

where $0 < \rho = 1 - \frac{d}{q} < 1$, and the constant C depends only on d, μ, and ρ.

The scaling-invariant estimate (1.0.5) should be regarded as a Lipschitz estimate for the velocity and L^∞ estimate for the pressure down to the microscopic scale ε, even though no smoothness assumption is made on the coefficients $A(y)$. In [30], we also obtain the following boundary $W^{1,p}$ estimates.

Theorem 1.0.2. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $1 < q < \infty$. Suppose that A satisfies ellipticity (1.0.3) and periodicity (1.0.4) conditions. Also assume that $A \in \mathrm{VMO}(\mathbb{R}^d)$. Let $f = (f_i^\alpha) \in L^q(\Omega; \mathbb{R}^{d\times d})$, $g \in L^q(\Omega)$ and $h \in B^{1-\frac{1}{q},q}(\partial\Omega; \mathbb{R}^d)$ satisfy the compatibility condition*

$$\int_\Omega g - \int_{\partial\Omega} h \cdot n = 0,$$

where n denotes the outward unit normal to $\partial\Omega$. Then the solutions $(u_\varepsilon, p_\varepsilon)$ in $W^{1,q}(\Omega; \mathbb{R}^d) \times L^q(\Omega)$ to Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = \mathrm{div}(f) & \text{in } \Omega, \\ \mathrm{div}(u_\varepsilon) = g & \text{in } \Omega, \\ u_\varepsilon = h & \text{on } \partial\Omega, \end{cases} \tag{1.0.6}$$

satisfy the estimate

$$\|\nabla u_\varepsilon\|_{L^q(\Omega)} + \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^q(\Omega)} \le C_q \left\{ \|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)} + \|h\|_{B^{1-\frac{1}{q},q}(\partial\Omega)} \right\}, \tag{1.0.7}$$

where C_q depends only on d, μ, A, and Ω.

Sharp Convergence Rates

As for the second task concerning sharp convergence rates, the primary purpose is to establish the optimal rate of convergence of solution u_ε to homogenized solution u_0 in $L^2(\Omega; \mathbb{R}^d)$ for both Dirichlet and Neumann problems.

For Dirichlet problems, consider the scalar elliptic equation $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in a Lipschitz domain Ω with Dirichlet condition $u_\varepsilon = f$ on $\partial\Omega$. By energy estimates and maximum principle, it is well known that $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \left\{ \|\nabla^2 u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^\infty(\partial\Omega)} \right\}$. More recently, using the method of periodic unfolding, Griso [26, 27] was able to establish the much sharper estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}. \tag{1.0.8}$$

In the case of elliptic systems, the estimates (1.0.8) continue to hold under the additional assumption that A is Hölder continuous. The approach was based on the uniform regularity estimates established in [3, 37] and do not apply to operators with bounded measurable coefficients. Recently, by using the Steklov smoothing operator, T.A. Suslina [55] was able to establish the $O(\varepsilon)$ estimate (1.0.8) in L^2 for a broader class of elliptic operators in $C^{1,1}$ domains without any smoothness assumptions on the coefficient matrix A.

There are relatively fewer known results for Neumann problems. Consider the Neumann problem for the scalar elliptic equation $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω with $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $\partial\Omega$, the estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon\|F\|_{H^2(\Omega)}$ was proved by Griso [27] for $C^{1,1}$ domains with bounded measurable coefficients using the "periodic unfolding" method [14, 15]. The same result was also proved by Moskow and Vogelius [42] for curvilinear convex polygons Ω in $\mathbb{R}^2$. For the system case, consider elliptic systems $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω with Neumann condition $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on $\partial\Omega$, C. Kenig, F. Lin and Z. Shen [32] have shown that the estimate (1.0.8) holds in bounded Lipschitz domain Ω, under additional assumption that A is Hölder continuous. Also recently, by using the Steklov smoothing operator, T. A. Suslina [56] was able to eliminate the smoothness condition on coefficients to establish the $O(\varepsilon)$ estimate (1.0.8) in L^2 for a broader class of elliptic operators with Neumann data.

In this thesis, we study the convergence rates in L^2 and H^1 of both Dirichlet and Neumann problems for Stokes systems with rapidly oscillating periodic coefficients in $C^{1,1}$ domains, without any smoothness assumptions on the coefficients.

More precisely, by the homogenization theory of Stokes systems (see [10, 30]), under suitable conditions on F, g and h, suppose $(u_\varepsilon, p_\varepsilon)$ is a weak solution of Stokes system (1.0.2) with either Dirichlet $u_\varepsilon = h$ or Neumann $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - p_\varepsilon \cdot n = h$ boundary conditions on $\partial\Omega$, it is known that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega; \mathbb{R}^d) \quad \text{and} \quad p_\varepsilon - \fint_\Omega p_\varepsilon \rightharpoonup p_0 - \fint_\Omega p_0 \quad \text{weakly in } L^2(\Omega),$$

where $(u_0, p_0) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ is the weak solution of the homogenized problem with constant coefficients,

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F, & \text{in } \Omega, \\ \operatorname{div}(u_0) = g, & \text{in } \Omega, \end{cases} \tag{1.0.9}$$

satisfying the same Dirichlet $u_0 = h$ or Neumann $\frac{\partial u_0}{\partial \nu_0} - p_0 \cdot n = h$ boundary condition on $\partial\Omega$. Our primary purpose is to study the rate of convergence $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$ as $\varepsilon \to 0$.

The following is the main result for Dirichlet problem we obtained in [28].

Theorem 1.0.3. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $g \in H^1(\Omega)$ and $h \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the Dirichlet compatibility condition $\int_\Omega g - \int_{\partial\Omega} h \cdot n = 0$, where n denotes the outward unit normal to $\partial\Omega$. For $F \in L^2(\Omega;\mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of Stokes systems (1.0.2), (1.0.9) respectively with Dirichlet boundary conditions $u_\varepsilon = u_0 = h$ on $\partial\Omega$. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1.0.10}$$

where the constant C depends only on d, μ, and Ω.

The next theorem is our main result for Neumann problem in [29].

Theorem 1.0.4. *Let Ω be a bounded $C^{1,1}$ domain. Suppose A satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $F \in L^2(\Omega;\mathbb{R}^d)$ and $h \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the Neumann compatibility condition $\int_\Omega F + \int_{\partial\Omega} h = 0$, for $g \in H^1(\Omega)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of Stokes systems (1.0.2), (1.0.9) respectively with Neumann boundary conditions $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - p_\varepsilon \cdot n = \frac{\partial u_0}{\partial \nu_0} - p_0 \cdot n = h$. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{1.0.11}$$

where the constant C depends only on μ, d, and Ω.

The organization of this thesis is as follows. Chapter 2 contains notations and definitions that will be used throughout the thesis. Chapter 3 is devoted to the homogenization theory of Stokes systems, including asymptotic expansions and compactness theorem. Our main results described above are presented in Chapter 4-6. In Chapter 4 and Chapter 5, we give the convergence results in L^2 and H^1 of Stokes system with Dirichlet and Neumann boundary conditions, respectively. Chapter 6 deals with uniform regularity estimates in homogenization of Stokes systems.

Chapter 2 Notations and Definitions

In this chapter we first give basic notations and definitions that will be used throughout this thesis. Then we introduce the Steklov smoothing operator and its properties, as well as a lemma that plays an vital part in the study of convergence rates.

2.1 Notations

- **1-periodic function.** We call a function f 1-periodic if it satisfies the periodicity condition we defined in (1.0.4), i.e.

$$f(y+z) = f(y) \qquad \text{for a.e. } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d.$$

- **Conormal derivative.** We define the conormal derivative of Stokes system (1.0.2) on $\partial\Omega$ by

$$\left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)^\alpha - p_\varepsilon\, n_\alpha = n_i(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial u_\varepsilon^\beta}{\partial x_j} - p_\varepsilon(x) n_\alpha(x), \tag{2.1.1}$$

  where $n = (n_1, \cdots, n_d)$ is the outward unit normal to $\partial\Omega$.

- **L^1-average.** We denote the L^1 average of f over the set E by

$$\fint_E f = \frac{1}{|E|} \int_E f.$$

- **Hölder condition.** We call $A(y)$ Hölder continuous, if it satisfies

$$|A(x) - A(y)| \leq \tau |x-y|^\lambda \qquad \text{for } x, y \in \mathbb{R}^d, \tag{2.1.2}$$

  where $\tau \geq 0$ and $\lambda \in (0,1]$.

- **Rescaling property of Stokes systems.** The technique of rescaling will be used routinely in the rest of the paper. Indeed, if $(u_\varepsilon, p_\varepsilon)$ is a weak solution of Stokes system (1.0.2) and $v(x) = u_\varepsilon(rx)$, then

$$\begin{cases} \mathcal{L}_{\frac{\varepsilon}{r}}(v) + \nabla q = \widetilde{F}, \\ \qquad \operatorname{div}(v) = \widetilde{g}, \end{cases} \tag{2.1.3}$$

  where

$$\widetilde{g}(x) = r g(rx), \qquad \widetilde{F}(x) = r^2 F(rx), \tag{2.1.4}$$

  and

$$q(x) = r p_\varepsilon(rx). \tag{2.1.5}$$

- **r-neighborhood of the boundary.** For $r > 0$, we let Ω_r and $\widetilde{\Omega}_r$ to denote the r-neighborhood of $\partial\Omega$ as

$$\begin{aligned}\Omega_r &= \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq r\},\\ \widetilde{\Omega}_r &= \{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Omega) \leq r\}.\end{aligned} \tag{2.1.6}$$

- **Hölder Space.** The Hölder space $C^{k,\lambda}(\bar{E})$ consists of all functions $u \in C^k(\bar{E})$ for which the norm

$$\|u\|_{C^{k,\lambda}(\bar{E})} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\bar{E})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\lambda}(\bar{E})} \tag{2.1.7}$$

is finite, where the λ-th semi-norm of g is denoted by

$$[g]_{C^{0,\lambda}(\bar{E})} = \sup\left\{\frac{|g(x) - g(y)|}{|x-y|^\lambda} : x, y \in \bar{E} \text{ and } x \neq y\right\}, \tag{2.1.8}$$

and $C^k(E)$ denotes the set of functions having all derivatives of order $\leq k$ continuous in E.

- **BMO Space.** A locally integrable function f will be said to belong to $\mathrm{BMO}(\mathbb{R}^d)$ if the following norm

$$\|f\|_{\mathrm{BMO}} = \sup_Q \fint_Q \left|f - \fint_Q f\right| dx \tag{2.1.9}$$

is finite.

- **VMO Space.** A function f in $\mathrm{BMO}(\mathbb{R}^d)$ is said to be $\mathrm{VMO}(\mathbb{R}^d)$, the space of functions of vanishing mean oscillation, if

$$\lim_{|Q|\to 0} \fint_Q \left|f - \fint_Q f\right| dx = 0. \tag{2.1.10}$$

2.2 Smoothing in Steklov's sense

We will use this section to introduce the Steklov smoothing operator as well as its properties, which play a crucial role in deriving convergence rates in the following chapters. More details about Steklov smoothing operator can be found in the literatures such as [45, 46, 55, 56, 59].

Let S_ε be the operator on $L^2(\mathbb{R}^d)$ given by

$$(S_\varepsilon u)(x) = \fint_Y u(x - \varepsilon z) dz \tag{2.2.1}$$

and called the *Steklov smoothing operator*. Note that

$$\|S_\varepsilon u\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)}.$$

Obviously, $D^\alpha S_\varepsilon u = S_\varepsilon D^\alpha u$ for $u \in H^s(\mathbb{R}^d)$ and any multi-index α such that $|\alpha| \leq s$. Therefore,

$$\|S_\varepsilon u\|_{H^s(\mathbb{R}^d)} \leq \|u\|_{H^s(\mathbb{R}^d)}.$$

The following are a few properties of Steklov's operator; see [55, 56].

Proposition 2.2.1. *For any $u \in H^1(\mathbb{R}^d)$ we have*

$$\|S_\varepsilon u - u\|_{L^2(\mathbb{R}^d)} \leq C\varepsilon \|\nabla u\|_{L^2(\mathbb{R}^d)},$$

where C depends only on d.

For simplicity, we will use the notation $f^\varepsilon(x) = f(x/\varepsilon)$. And we let $Y = [0,1)^d$.

Proposition 2.2.2. *Let $f(x)$ be a 1-periodic function in $\mathbb{R}^d$ such that $f \in L^2(Y)$. Then for any $u \in L^2(\mathbb{R}^d)$,*

$$\|f^\varepsilon S_\varepsilon u\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(Y)} \|u\|_{L^2(\mathbb{R}^d)}.$$

The following lemma gives us an estimate for integrals near the boundary, see [55, 56] for example.

Lemma 2.2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1 domain. Then, for any function $u \in H^1(\Omega)$ and for any $0 < r \leq \operatorname{diam}(\Omega)$,*

$$\left(\int_{\Omega_r} |u|^2 dx \right)^{1/2} \leq C\sqrt{r} \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{1/2}. \tag{2.2.2}$$

Moreover, for any 1-periodic function $f \in L^2(Y)$ and $u \in H^1(\mathbb{R}^d)$,

$$\left(\int_{\widetilde{\Omega}_{2\varepsilon}} |f^\varepsilon|^2 |S_\varepsilon u|^2 dx \right)^{1/2} \leq C\sqrt{\varepsilon} \|f\|_{L^2(Y)}^{1/2} \|u\|_{H^1(\mathbb{R}^d)}^{1/2} \|u\|_{L^2(\mathbb{R}^d)}^{1/2}, \tag{2.2.3}$$

where C depends only on Ω and S_ε is the Steklov smoothing operator defined in (2.2.1).

Chapter 3 Preliminaries

In this chapter we will first give a brief introduction to the homogenization theory of Stokes systems, including the definition of correctors and the homogenization theorem. Then we will formally derive the asymptotic expansion of Stokes system, showing the intuition behind the definition of correctors and effective matrices. Then we prove a compactness theorem for a sequence of Stokes systems with periodic coefficient matrices satisfying the ellipticity condition (1.0.3) with the same μ. At last, we will describe the homogenization of Stokes system with Neumann boundary conditions.

3.1 Homogenization Theory of Stokes Systems

In this section we will give a review of homogenization theory of Stokes systems with periodic coefficients. We refer the reader to [10, pp.76-81] for a detailed presentation.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. For $u, v \in H^1(\Omega;\mathbb{R}^d)$, we define the bilinear form as the following,

$$a_\varepsilon(u,v) = \int_\Omega a_{ij}^{\alpha\beta}\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\beta}{\partial x_j}\frac{\partial v^\alpha}{\partial x_i}dx. \tag{3.1.1}$$

For $F \in H^{-1}(\Omega;\mathbb{R}^d)$ and $g \in L^2(\Omega)$, we say that $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ is a weak solution of the Stokes system (1.0.2)

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \qquad\quad \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in Ω, if $\operatorname{div}(u_\varepsilon) = g$ in Ω and for any $\varphi \in C_0^1(\Omega;\mathbb{R}^d)$,

$$a_\varepsilon(u_\varepsilon,\varphi) - \int_\Omega p_\varepsilon \operatorname{div}(\varphi) = \langle F, \varphi\rangle.$$

The following theorem gives us the existence and uniqueness (up to constants) of weak solution of Stokes system with Dirichlet boundary condition.

Theorem 3.1.1. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Suppose A satisfies the ellipticity condition (1.0.3). Let $F \in H^{-1}(\Omega;\mathbb{R}^d)$, $g \in L^2(\Omega)$ and $h \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$ satisfy the Dirichlet compatibility condition*

$$\int_\Omega g - \int_{\partial\Omega} h \cdot n = 0, \tag{3.1.2}$$

where n is the outward unit normal to $\partial\Omega$. Then there exist a unique $u_\varepsilon \in H^1(\Omega;\mathbb{R}^d)$ and $p_\varepsilon \in L^2(\Omega)$ (unique up to constants) such that $(u_\varepsilon, p_\varepsilon)$ is a weak solution of (1.0.2) in Ω and $u_\varepsilon = h$ on $\partial\Omega$. Moreover,

$$\|u_\varepsilon\|_{H^1(\Omega)} + \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^2(\Omega)} \le C\left\{\|F\|_{H^{-1}(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)} + \|g\|_{L^2(\Omega)}\right\}, \tag{3.1.3}$$

where C depends only on d, μ, and Ω.

Proof. This theorem is well known and does not use the periodicity condition of A. First, we choose $\widetilde{h} \in H^1(\Omega;\mathbb{R}^d)$ such that $\widetilde{h} = h$ on $\partial\Omega$ and

$$\|\widetilde{h}\|_{H^1(\Omega)} \le C\|h\|_{H^{1/2}(\partial\Omega)}.$$

By considering $u_\varepsilon - \widetilde{h}$, we may assume that $h = 0$. Next, we choose a function $U(x)$ in $H^1_0(\Omega;\mathbb{R}^d)$ such that

$$\operatorname{div}(U) = g \quad \text{in } \Omega, \quad \text{and} \quad \|U\|_{H^1(\Omega)} \le C\|g\|_{L^2(\Omega)},$$

detailed proof of the existence of such functions can be found in [20]. By considering $u_\varepsilon - U$, we may further assume that $g = 0$. Finally, the case $h = 0$ and $g = 0$ may be proved by applying the Lax-Milgram Theorem to the bilinear form $a_\varepsilon(u, v)$ on the Hilbert space

$$V = \{u \in H^1_0(\Omega;\mathbb{R}^d) : \operatorname{div}(u) = 0 \text{ in } \Omega\}.$$

This completes the proof. □

Remark 3.1.2. *If Ω is $C^{1,1}$ and A is a constant matrix, the weak solution (u, p), given by Theorem 3.1.1, is in $H^2(\Omega;\mathbb{R}^d) \times H^1(\Omega)$, provided that $F \in L^2(\Omega;\mathbb{R}^d)$, $g \in H^1(\Omega)$ and $h \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$. Moreover,*

$$\|u\|_{H^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \le C\left\{\|F\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|h\|_{H^{3/2}(\partial\Omega)}\right\}, \tag{3.1.4}$$

where C depends only on d, μ, and Ω (see e.g. [24]).

Let $Y = [0, 1)^d$. We denote by $H^1_{\text{per}}(Y;\mathbb{R}^d)$ the closure in $H^1(Y;\mathbb{R}^d)$ of $C^\infty_{\text{per}}(Y;\mathbb{R}^d)$, the set of C^∞ 1-periodic and $\mathbb{R}^d$-valued functions in $\mathbb{R}^d$. Let

$$a_{\text{per}}(\psi, \phi) = \int_Y a_{ij}^{\alpha\beta}(y)\frac{\partial \psi^\beta}{\partial y_j}\frac{\partial \phi^\alpha}{\partial y_i},$$

where $\phi = (\phi^\alpha)$ and $\psi = (\psi^\alpha)$. By applying the Lax-Milgram Theorem to the bilinear form $a_{\text{per}}(\psi, \phi)$ on the Hilbert space

$$V_{\text{per}}(Y) = \{u \in H^1_{\text{per}}(Y;\mathbb{R}^d) : \operatorname{div}(u) = 0 \text{ in } Y \text{ and } \int_Y u = 0\},$$

it follows that for each $1 \leq j, \beta \leq d$, there exists a unique $\chi_j^\beta \in V_{\text{per}}(Y)$ such that

$$a_{\text{per}}(\chi_j^\beta, \phi) = -a_{\text{per}}(P_j^\beta, \phi) \qquad \text{for any } \phi \in V_{\text{per}}(Y)$$

where $P_j^\beta = P_j^\beta(y) = y_j e^\beta = y_j(0, ..., 1, ..., 0)$ with 1 in the β^{th} position. As a result, there exist 1-periodic functions $(\chi_j^\beta, \pi_j^\beta) \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) \times L^2_{\text{loc}}(\mathbb{R}^d)$, which are called the first-order correctors for the Stokes system (1.0.2), such that

$$\begin{cases} \mathcal{L}_1(\chi_j^\beta + P_j^\beta) + \nabla \pi_j^\beta = 0 & \text{in } \mathbb{R}^d \\ \operatorname{div}(\chi_j^\beta) = 0 & \text{in } \mathbb{R}^d \\ \int_Y \pi_j^\beta = 0 \text{ and } \int_Y \chi_j^\beta = 0. \end{cases} \tag{3.1.5}$$

Note that

$$\|\chi_j^\beta\|_{H^1(Y)} + \|\pi_j^\beta\|_{L^2(Y)} \leq C, \tag{3.1.6}$$

where C depends only on d and μ. Let $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$, where

$$\widehat{a}_{ij}^{\alpha\beta} = a_{\text{per}}(\chi_j^\beta + P_j^\beta, \chi_i^\alpha + P_i^\alpha) \tag{3.1.7}$$

The homogenized system for the Stokes system (1.0.2) is given by

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F, \\ \operatorname{div}(u_0) = g, \end{cases}$$

where $\mathcal{L}_0 = -\operatorname{div}(\widehat{A}\nabla)$ is a second-order elliptic operator with constant coefficients. The constant matrix $\widehat{A}$ is called the homogenized matrix or effective matrix, and satisfies the following two properties.

Remark 3.1.3. *The homogenized matrix $\widehat{A}$ satisfies the ellipticity condition*

$$\mu|\xi|^2 \leq \widehat{a}_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \leq \mu_1|\xi|^2 \tag{3.1.8}$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d\times d}$, where μ_1 depends only on d and μ. The upper bound is a consequence of the estimate $\|\nabla\chi_j^\beta\|_{L^2(Y)} \leq C(d, \mu)$, while the lower bound follows from

$$\begin{aligned} \widehat{a}_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta &= a_{per}((\chi_j^\beta + P_j^\beta)\xi_j^\beta, (\chi_i^\alpha + P_i^\alpha)\xi_i^\alpha) \\ &\geq \mu \int_Y |\nabla(\chi_i^\alpha + P_i^\alpha)\xi_i^\alpha|^2 \\ &\geq \mu|\xi|^2. \end{aligned}$$

Remark 3.1.4. *Let $\chi^* = (\chi_j^{*\beta})$ denote the matrix of correctors for the system (1.0.2) with A replaced by its adjoint A^*. Note that by definition, $\chi_j^{*\beta} \in V_{per}(Y)$ and*

$$a_{per}^*(\chi_j^{*\beta}, \phi) = -a_{per}^*(P_j^\beta, \phi)$$

where $a_{per}^*(\psi, \phi) = a_{per}(\phi, \psi)$. It follows that

$$\begin{aligned}
\widehat{a}_{ij}^{\alpha\beta} &= a_{per}(\chi_j^\beta + P_j^\beta, \chi_i^\alpha + P_i^\alpha) = a_{per}(\chi_j^\beta + P_j^\beta, P_i^\alpha) \\
&= a_{per}(\chi_j^\beta + P_j^\beta, \chi_i^{*\alpha} + P_i^\alpha) = a_{per}^*(\chi_i^{*\alpha} + P_i^\alpha, \chi_j^\beta + P_j^\beta) \\
&= a_{per}^*(\chi_i^{*\alpha} + P_i^\alpha, P_j^\beta) = a_{per}^*(\chi_i^{*\alpha} + P_i^\alpha, \chi_j^{*\beta} + P_j^\beta).
\end{aligned} \tag{3.1.9}$$

This, in particular, shows that $(\widehat{A})^* = \widehat{A^*}$.

Now we are ready to give the following homogenization theorem of Stokes systems with Dirichlet boundary conditions. It shows that the limiting solutions are actually solutions of Stokes system associated with the homogenized operator $\mathcal{L}_0$ with the same Dirichlet boundary condition.

Theorem 3.1.5. *Suppose that $A(y)$ satisfies ellipticity (1.0.3) and periodicity (1.0.4) conditions. Let Ω be a bounded Lipschitz domain. Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ be a weak solution of the following Dirichlet problem of Stokes system*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \operatorname{div}(u_\varepsilon) = g & \text{in } \Omega, \\ u_\varepsilon = h & \text{on } \partial\Omega, \end{cases} \tag{3.1.10}$$

where $F \in H^{-1}(\Omega; \mathbb{R}^d)$, $g \in L^2(\Omega)$ and $h \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ satisfying the Dirichlet compatibility condition (3.1.2). Assume that $\int_\Omega p_\varepsilon = 0$, then as $\varepsilon \to 0$,

$$\begin{cases} u_\varepsilon \to u_0 \text{ strongly in } L^2(\Omega; \mathbb{R}^d), \\ u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1(\Omega; \mathbb{R}^d), \\ p_\varepsilon \rightharpoonup p_0 \text{ weakly in } L^2(\Omega), \\ A(x/\varepsilon)\nabla u_\varepsilon \rightharpoonup \widehat{A}\nabla u_0 \text{ weakly in } L^2(\Omega; \mathbb{R}^{d\times d}). \end{cases}$$

Moreover, $\int_\Omega p_0 = 0$ and (u_0, p_0) is the weak solution of the homogenized problem

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \operatorname{div}(u_0) = g & \text{in } \Omega, \\ u_0 = h & \text{on } \partial\Omega. \end{cases} \tag{3.1.11}$$

Proof. This homogenization theorem of Stokes systems is more or less proved in [10], using Tartar's oscillating testing function method. We therefore omit the details. □

3.2 Asymptotic Expansions

In this section we will apply the multi-scale method to the study of Stokes system. As we mentioned earlier, two scales describe the model: the variable x is the "macroscopic" one, while x/ε describe the "microscopic" one. Indeed, this suggests looking for a formal asymptotic expansion of solution $(u_\varepsilon, p_\varepsilon)$ in the form:

$$\begin{aligned} u_\varepsilon(x) &= u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots, \\ p_\varepsilon(x) &= p_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon p_1\left(x, \frac{x}{\varepsilon}\right) + \cdots, \end{aligned} \tag{3.2.1}$$

where the functions $u_j(x,y)$, $p_j(x,y)$ are defined on $\Omega \times Y$ and 1-periodic in y, for any $x \in \Omega$. Note that if $\phi_\varepsilon(x) = \phi(x,y)$ with $y = x/\varepsilon$, then

$$\frac{\partial \phi_\varepsilon}{\partial x_j}(x) = \frac{1}{\varepsilon}\frac{\partial \phi}{\partial y_j}\left(x, \frac{x}{\varepsilon}\right) + \frac{\partial \phi}{\partial x_j}\left(x, \frac{x}{\varepsilon}\right).$$

Now if we consider the divergence-free Stokes system

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \operatorname{div}(u_\varepsilon) = 0, \end{cases} \tag{3.2.2}$$

which may now be rewritten as the following

$$\begin{cases} \left[(\varepsilon^{-2}L^0 + \varepsilon^{-1}L^1 + \varepsilon^0 L^2)u_\varepsilon\right]\left(x, \frac{x}{\varepsilon}\right) + \left[(\varepsilon^{-1}\nabla_y + \nabla_x)p_\varepsilon\right]\left(x, \frac{x}{\varepsilon}\right) = F, \\ \left[(\varepsilon^{-1}\operatorname{div}_y + \operatorname{div}_x)u_\varepsilon\right]\left(x, \frac{x}{\varepsilon}\right) = 0, \end{cases} \tag{3.2.3}$$

where the operators L^0, L^1, L^2 are defined by

$$\begin{cases} \left[L^0(\phi(x,y))\right]^\alpha = -\dfrac{\partial}{\partial y_i}\left(a_{ij}^{\alpha\beta}(y)\dfrac{\partial \phi^\beta}{\partial y_j}\right), \\ \left[L^1(\phi(x,y))\right]^\alpha = -\dfrac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(y)\dfrac{\partial \phi^\beta}{\partial y_j}\right) - \dfrac{\partial}{\partial y_i}\left(a_{ij}^{\alpha\beta}(y)\dfrac{\partial \phi^\beta}{\partial x_j}\right), \\ \left[L^2(\phi(x,y))\right]^\alpha = -\dfrac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(y)\dfrac{\partial \phi^\beta}{\partial x_j}\right). \end{cases}$$

We identify the coefficients of the powers ε^{-2}, ε^{-1}, ε^0. This gives the following systems to be solved. As of order $O(\varepsilon^{-2})$, we have

$$L^0(u_0) = 0, \tag{3.2.4}$$

Of order $O(\varepsilon^{-1})$, we have

$$\begin{cases} L^0(u_1) + \nabla_y p_0 = -L^1(u_0), \\ \operatorname{div}_y(u_0) = 0. \end{cases} \tag{3.2.5}$$

And of order $O(\varepsilon)$, we obtain

$$\begin{cases} L^0(u_2) + \nabla_y p_1 = F - L^1(u_1) - L^2(u_0) - \nabla_x p_0, \\ \qquad\quad \mathrm{div}_y(u_1) = -\mathrm{div}_x(u_0). \end{cases} \tag{3.2.6}$$

Using the fact that $u_0(x, y)$ is 1-periodic in y, we may derive from (3.2.4) that $u_0(x, y)$ is independent of y, i.e.,

$$u_0(x, y) = u_0(x). \tag{3.2.7}$$

Then (3.2.5) reduces to

$$L^0(u_1) + \nabla_y p_0 = -L^1(u_0).$$

The second condition in (3.2.6) implies

$$\int_Y [\mathrm{div}_y(u_1) + \mathrm{div}_x(u_0)]\, dy = 0,$$

and since the above integral equals $|Y|\mathrm{div}(u_0)$, one has

$$\mathrm{div}(u_0) = 0. \tag{3.2.8}$$

Then the second condition in (3.2.6) is equivalent to $\mathrm{div}_y(u_1) = 0$; i.e., finally for (u_1, p_0), we need to solve the following system (note that $\mathcal{L}_1 = L^0$),

$$\begin{cases} L^0(u_1) + \nabla_y p_1 = -L^0(P_j^\beta) \dfrac{\partial u_0^\beta}{\partial x_j} \\ \qquad \mathrm{div}_y(u_1) = 0 \end{cases} \tag{3.2.9}$$

Recalling that the correctors $(\chi_j^\beta, \pi_j^\beta)$ are solution to the cell problem (3.1.5), then the general solution of (3.2.9) is,

$$u_1(x, y) = \chi_j^\beta(y) \frac{\partial u_0^\beta}{\partial x_j}(x) + \widetilde{u}_1(x), \tag{3.2.10}$$

and

$$p_0(x, y) = \pi_j^\beta(y) \frac{\partial u_0^\beta}{\partial x_j}(x) + \widetilde{p}_0(x), \tag{3.2.11}$$

where $\widetilde{u}_1(x)$ and $\widetilde{p}_0(x)$ are independent of y. We now use the equations (3.2.10) and (3.2.11) in the first equation in (3.2.6) to obtain

$$\begin{aligned} \big(L^0(u_2)\big)^\alpha + \frac{\partial p_1}{\partial y_\alpha} = F^\alpha(x) + \left[a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y) \frac{\partial \chi_j^{\gamma\beta}}{\partial y_k} \right] \frac{\partial^2 u_0^\beta}{\partial x_i \partial x_j} + \frac{\partial}{\partial y_i} \left\{ a_{ij}^{\alpha\beta}(y) \frac{\partial \widetilde{u}_1^\beta}{\partial x_j} \right\} \\ - \pi_j^\beta(y) \frac{\partial^2 u_0^\beta}{\partial x_\alpha \partial x_j} - \frac{\partial \widetilde{p}_0}{\partial x_\alpha} \end{aligned}$$

The above equation can be solved in (u_2, p_1) if we integrate both sides in y over Y,

$$F^\alpha(x) = -\int_Y \left[a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y) \frac{\partial \chi_j^{\gamma\beta}}{\partial y_k} \right] dy \cdot \frac{\partial^2 u_0^\beta}{\partial x_i \partial x_j} + \frac{\partial \widetilde{p}_0}{\partial x_\alpha},$$

where we have used the fact that $\int_Y \pi_j^\beta dy = 0$. The above equation is nothing else than

$$\mathcal{L}_0(u_0) + \nabla \widetilde{p}_0 = F,$$

where $\mathcal{L}_0 = -\mathrm{div}(\widehat{A}\nabla)$ and $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$ defined the same as in (3.1.7) that

$$\widehat{a}_{ij}^{\alpha\beta} = \int_Y \left[a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y) \frac{\partial \chi_j^{\gamma\beta}}{\partial x_k} \right] dy.$$

So far by this multi-scale method, if we denote $\widetilde{p}_0(x)$ by $p_0(x)$ for simplicity, we have formally shown that the homogenization problem of Stokes system (3.2.2) is exactly

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F, \\ \mathrm{div}(u_0) = 0. \end{cases} \tag{3.2.12}$$

3.3 Compactness Theorem

We now prove a compactness theorem for a sequence of Stokes systems with coefficient matrices satisfying the same conditions and should be regarded as a compactness property of the Stokes systems with periodic coefficients. Its proof follows the Tartar's oscillating testing function method found in [10] for the proof of Theorem 3.1.5, and also uses the following observation.

Proposition 3.3.1. *Suppose that $\{\phi_k\}$ be a sequence of 1-periodic functions with $\|\phi_k\|_{L^2(Y)} \le C$ and $\varepsilon_k \to 0$, then*

$$\phi_k(x/\varepsilon_k) - \fint_Y \phi_k \rightharpoonup 0 \text{ weakly in } L^2(\Omega), \tag{3.3.1}$$

as $k \to \infty$.

Proof. Let $u_k \in H^2_{\mathrm{per}}(Y)$ such that

$$\Delta u_k = \phi_k - \fint_Y \phi_k, \qquad \text{in } Y.$$

Let $U_k = \nabla u_k$. Then $\mathrm{div}(U_k) = \phi_k - \fint_Y \phi_k$ and

$$\|U_k\|_{L^2(Y)} \le \|\phi_k - \fint_Y \phi_k\|_{L^2(Y)} \le C.$$

Then for any $\varphi \in C_0^1(\Omega)$,

$$\int_\Omega \big[\phi_k(x/\varepsilon_k) - \fint_Y \phi_k\big]\varphi(x) = -\varepsilon_k \int_\Omega U_k(x/\varepsilon_k) \cdot \nabla\varphi(x) \to 0, \tag{3.3.2}$$

as $\varepsilon_k \to 0$, since

$$\|U_k(x/\varepsilon_k)\|_{L^2(\Omega)} \le C\|U_k\|_{L^2(Y)} \le C,$$

where we have used the periodicity of U_k. We may now conclude that

$$\phi_k(x/\varepsilon_k) - \fint_Y \phi_k \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega),$$

as similarly

$$\|\phi_k(x/\varepsilon)\|_{L^2(\Omega)} \le C\|\phi_k\|_{L^2(Y)} \le C$$

The proof is now complete. $\square$

We are now ready to prove our interior compactness theorem of Stokes systems.

Theorem 3.3.2. *Let $\{A^k(y)\}$ be a sequence of 1-periodic matrices satisfies the ellipticity condition (1.0.3) (with the same μ). Let $(u_k, p_k) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(A^k(x/\varepsilon_k)\nabla u_k) + \nabla p_k = F_k, \\ \qquad\qquad\qquad\qquad \mathrm{div}(u_k) = g_k \end{cases}$$

in Ω, where $\varepsilon_k \to 0$, $F_k \in H^{-1}(\Omega;\mathbb{R}^d)$ and $g_k \in L^2(\Omega)$. We further assume that as $k \to \infty$,

$$\begin{cases} F_k \to F_0 \text{ strongly in } H^{-1}(\Omega;\mathbb{R}^d), \\ g_k \to g_0 \text{ strongly in } L^2(\Omega), \\ u_k \rightharpoonup u_0 \text{ weakly in } H^1(\Omega;\mathbb{R}^d), \\ p_k \rightharpoonup p_0 \text{ weakly in } L^2(\Omega), \\ \widehat{A^k} \to A^0, \end{cases}$$

where $\widehat{A^k}$ is the coefficient matrix of the homogenized system for the Stokes system with coefficient matrix $A^k(x/\varepsilon)$. Then, $A^k(x/\varepsilon_k)\nabla u_k \rightharpoonup A^0\nabla u_0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$, and (u_0, p_0) is a weak solution of

$$\begin{cases} -\mathrm{div}(A^0\nabla u_0) + \nabla p_0 = F_0, \\ \qquad\qquad\qquad \mathrm{div}(u_0) = g_0 \end{cases} \quad \text{in } \Omega. \tag{3.3.3}$$

Proof. Let $A^k = (a_{ij}^{k\alpha\beta})$ and

$$(\xi_k)_i^\alpha = a_{ij}^{k\alpha\beta}\left(\frac{x}{\varepsilon_k}\right)\frac{\partial u_k^\beta}{\partial x_j}$$

Note that $\|(\xi_k)_i^\alpha\|_{L^2(\Omega)} \le C$. It suffices to show that if $\{\xi_{k'}\}$ is a subsequence of $\{\xi_k\}$ and $\{\xi_{k'}\}$ converges weakly to ξ_0 in $L^2(\Omega; \mathbb{R}^{d\times d})$, then $\xi_0 = A^0\nabla u_0$. This would imply that (u_0, p_0) is a weak solution of (3.3.3) in Ω. It also implies that the whole sequence $\{\xi_k\}$ converges weakly to $A^0\nabla u_0$ in $L^2(\Omega; \mathbb{R}^{d\times d})$.

Without loss of generality we may assume that $\xi_k \rightharpoonup \xi_0$ weakly in $L^2(\Omega; \mathbb{R}^{d\times d})$. Note that

$$\langle \xi_k, \nabla\phi\rangle = \langle F_k, \phi\rangle + \langle p_k, \operatorname{div}(\phi)\rangle \tag{3.3.4}$$

for all $\phi \in H_0^1(\Omega; \mathbb{R}^d)$. Fix $1 \le j, d \le d$ and $\psi \in C_0^1(\Omega)$. Let

$$\phi_k(x) = \Big(P_j^\beta(x) + \varepsilon_k \chi_j^{k*\beta}(x/\varepsilon_k)\Big)\psi(x),$$

where $\chi_j^{k*\beta}$ (and $\pi_j^{k*\beta}$ used in the following) are the correctors for the Stokes systems with coefficient matrix $(A^k)^*(x/\varepsilon)$, introduced in Remark 3.1.4. A computation shows that

$$\begin{aligned}
\langle \xi_k, \nabla\phi_k\rangle &= \langle A^k(x/\varepsilon_k)\nabla u_k, \nabla\big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\cdot\psi\rangle \\
&\qquad + \langle A^k(x/\varepsilon_k)\nabla u_k, \big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\nabla\psi\rangle \\
&= \langle \psi\nabla u_k, (A^k)^*(x/\varepsilon_k)\nabla\big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\rangle \\
&\qquad + \langle A^k(x/\varepsilon_k)\nabla u_k, \big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\nabla\psi\rangle \\
&= \langle \nabla(\psi u_k), (A^k)^*(x/\varepsilon_k)\nabla\big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\rangle \\
&\qquad - \langle (\nabla\psi)u_k, (A^k)^*(x/\varepsilon_k)\nabla\big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\rangle \\
&\qquad + \langle \xi_k, \big(P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big)\nabla\psi\rangle.
\end{aligned} \tag{3.3.5}$$

Since

$$-\operatorname{div}\Big((A^k)^*(x/\varepsilon_k)\nabla\big[P_j^\beta(x) + \varepsilon_k\chi_j^{k*\beta}(x/\varepsilon_k)\big]\Big) = -\nabla\Big[\pi_j^{k*\beta}(x/\varepsilon_k)\Big] \qquad \text{in } \mathbb{R}^d,$$

it follows that the first term in the right hand side of (3.3.5) equals

$$\langle \pi_j^{k*\beta}(x/\varepsilon), \operatorname{div}(\psi u_k)\rangle = \langle \pi_j^{k*\beta}(x/\varepsilon) - \fint_Y \pi_j^{k*\beta}, \operatorname{div}(\psi u_k)\rangle.$$

Using the fact that

$$\operatorname{div}(\psi u_k) = \nabla\psi\cdot u_k + \psi g_k \to \nabla\psi\cdot u_0 + \psi g_0 \quad \text{strongly in } L^2(\Omega)$$

and by Proposition 3.3.1,

$$\pi_j^{k*\beta}(x/\varepsilon) - \fint_Y \pi_j^{k*\beta} \rightharpoonup 0 \text{ weakly in } L^2(\Omega),$$

we see that first term in the right hand side of (3.3.5) goes to zero. In view of the estimate

$$\|\varepsilon_k \chi_j^{k*\beta}(x/\varepsilon_k)\|_{L^2(\Omega)} \le C\varepsilon_k \|\chi_j^{k*\beta}\|_{L^2(Y)} \le C\ \varepsilon_k,$$

it is easy to see that for the third term in the right hand side of (3.3.5) goes to $\langle \xi_0, P_j^\beta \nabla\psi\rangle$.

To handle the second term in the right hand side of (3.3.5), we note that again by Proposition 3.3.1,

$$\nabla P_i^\alpha \cdot (A^k)^*(x/\varepsilon_k)\nabla\big(P_j^\beta(x) + \varepsilon_k \chi_j^{k*\beta}(x/\varepsilon_k)\big)$$

converges weakly in $L^2(\Omega)$ to

$$\lim_{k\to\infty} \int_Y \nabla P_i^\alpha \cdot (A^k)^*(y)\nabla\big(P_j^\beta + \chi_j^{k*\beta}(y)\big)dy = \lim_{k\to\infty} \widehat{a}_{ji}^{k\beta\alpha} = a_{ji}^{0\beta\alpha},$$

where $\widehat{A^k} = (\widehat{a}_{ij}^{k\alpha\beta})$, $A^0 = (a_{ij}^{0\alpha\beta})$, and we have used the definition of matrices of effective coefficients as well as the assumption that $\widehat{A^k} \to A^0$. This, together with the fact that $u_k \to u_0$ strongly in $L^2(\Omega;\mathbb{R}^d)$, shows that the second term in the right hand side of (3.3.5) goes to

$$-a_{ji}^{0\beta\alpha} \int_\Omega \frac{\partial\psi}{\partial x_i} u_0^\alpha = a_{ji}^{0\beta\alpha} \int_\Omega \psi \frac{\partial u_0^\alpha}{\partial x_i},$$

where we have used integration by parts. To summarize, we have proved that as $k \to \infty$,

$$\langle \xi_k, \nabla\phi_k\rangle \to \langle \xi_0, P_j^\beta \nabla\psi\rangle + a_{ji}^{0\beta\alpha} \int_\Omega \psi \frac{\partial u_0^\alpha}{\partial x_i}. \tag{3.3.6}$$

Finally, since $\phi_k \rightharpoonup P_j^\beta \psi$ weakly in $H_0^1(\Omega;\mathbb{R}^d)$ and $F_k \to F_0$ strongly in $H^{-1}(\Omega;\mathbb{R}^d)$, we have $\langle F_k, \phi_k\rangle \to \langle F_0, P_j^\beta\psi\rangle$. Also, since $\mathrm{div}(\chi_j^\beta) = 0$ in $\mathbb{R}^d$,

$$\langle p_k, \mathrm{div}(\phi_k)\rangle = \langle p_k, \mathrm{div}(P_j^\beta\psi)\rangle + \langle p_k, \varepsilon_k \chi_j^{k*\beta}(x/\varepsilon)\nabla\psi\rangle \to \langle p_0, \mathrm{div}(P_j^\beta\psi)\rangle.$$

Thus, the right hand side of (3.3.4) converges to

$$\langle F_0, P_j^\beta\psi\rangle + \langle p_0, \mathrm{div}(P_j^\beta\psi)\rangle = \langle \xi_0, \nabla(P_j^\beta\psi)\rangle = \langle \xi_0, P_j^\beta\nabla\psi\rangle + \langle \xi_0, \psi\nabla P_j^\beta\rangle,$$

where the first equality follows by taking the limit in (3.3.4) with $\phi = P_j^\beta\psi$. In view of (3.3.6) we obtain

$$a_{ji}^{0\beta\alpha} \int_\Omega \psi \frac{\partial u_0^\alpha}{\partial x_i} = \langle \xi_0, \psi\nabla P_j^\beta\rangle.$$

Since $\psi \in C_0^1(\Omega)$ is arbitrary, this gives $(\xi_0)_j^\beta = a_{ji}^{0\beta\alpha}\frac{\partial u_0^\alpha}{\partial x_i}$, i.e., $\xi_0 = A^0\nabla u_0$. The proof is complete. $\square$

3.4 Homogenization for Neumann problems

The homogenization theory can be extended to Neumann problem of Stokes systems, following by an analogous argument as in Section 3.1 for Dirichlet boundary value problems. Here we state the main results.

The following theorem gives us the existence and uniqueness of weak solution for Neumann problem of Stokes systems.

Theorem 3.4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose $A(y)$ satisfies the ellipticity condition (1.0.3). Let $F \in H^{-1}(\Omega;\mathbb{R}^d)$, $g \in L^2(\Omega)$ and $f \in H^{-1/2}(\partial\Omega;\mathbb{R}^d)$ satisfy the following Neumann compatibility condition*

$$\int_\Omega F + \int_{\partial\Omega} h = 0. \tag{3.4.1}$$

Then there exist a unique $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ (unique in the sense of up to constants), such that $(u_\varepsilon, p_\varepsilon)$ is a weak solution of (1.0.2) and $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - p_\varepsilon \cdot n = h$ on $\partial\Omega$, where $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}$ is defined in (2.1.1) and n is the outward unit normal. Moreover,

$$\|u_\varepsilon\|_{H^1(\Omega)} + \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^2(\Omega)} \le C\left\{\|F\|_{H^{-1}(\Omega)} + \|g\|_{L^2(\Omega)} + \|h\|_{H^{-1/2}(\partial\Omega)}\right\}, \tag{3.4.2}$$

where C depends only on d, μ, and Ω.

Proof. The existence and uniqueness of weak solutions for Neumann problem of Stokes system can be proved again by applying the Lax-Milgram Theorem. We skip the details here. □

Remark 3.4.2. *If Ω is $C^{1,1}$ and A is a constant matrix, the weak solution (u,p), given by Theorem 3.4.1, is in $H^2(\Omega;\mathbb{R}^d) \times H^1(\Omega)$, provided that $F \in L^2(\Omega;\mathbb{R}^d)$, $g \in H^1(\Omega)$ and $h \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$. Moreover,*

$$\|u\|_{H^2(\Omega)} + \|\nabla p\|_{L^2(\Omega)} \le C\Big\{\|F\|_{L^2(\Omega)} + \|g\|_{H^1(\Omega)} + \|h\|_{H^{1/2}(\partial\Omega)}\Big\}, \tag{3.4.3}$$

where C depends only on d, μ, and Ω (see e.g. [24]).

The following homogenization theorem of Stokes systems with Neumann boundary condition also shows that the limiting solutions are solutions of Stokes system associated with effective coefficients with the same Neumann boundary condition.

Theorem 3.4.3. *Suppose $A(y)$ satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let Ω be a bounded Lipschitiz domain. Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times$*

$L^2(\Omega)$ to be a weak solution of the following Neumann problem of Stokes system

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \textit{in } \Omega, \\ \operatorname{div}(u_\varepsilon) = g & \textit{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - p_\varepsilon \cdot n = h & \textit{on } \partial\Omega, \end{cases} \tag{3.4.4}$$

where $F \in H^{-1}(\Omega;\mathbb{R}^d)$, $g \in L^2(\Omega)$ and $h \in H^{-1/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the Neumann compatibility condition (3.4.1). Assume that $\int_\Omega u_\varepsilon = \int_\Omega p_\varepsilon = 0$, then as $\varepsilon \to 0$,

$$\begin{cases} u_\varepsilon \to u_0 & \text{strongly in } L^2(\Omega;\mathbb{R}^d), \\ u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(\Omega;\mathbb{R}^d), \\ p_\varepsilon \rightharpoonup p_0 & \text{weakly in } L^2(\Omega), \\ A(x/\varepsilon)\nabla u_\varepsilon \rightharpoonup \widehat{A}\nabla u_0 & \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{cases}$$

Moreover, $\int_\Omega u_0 = \int_\Omega p_0 = 0$ and (u_0, p_0) is the weak solution of the homogenized problem

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \textit{in } \Omega, \\ \operatorname{div}(u_0) = g & \textit{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} - p_0 \cdot n = h & \textit{on } \partial\Omega. \end{cases} \tag{3.4.5}$$

Proof. The proof will use the same approach as in the proof of Theorem 3.3.2. Let $\xi_\varepsilon = A(x/\varepsilon)\nabla u_\varepsilon$. Note that $\|\xi_\varepsilon\|_{L^2(\Omega)} \le C$. We say $(u_\varepsilon, p_\varepsilon)$ is a weak solution of Neumann problem (3.4.4), if

$$\langle \xi_\varepsilon, \nabla\phi\rangle = \langle F, \phi\rangle + \langle p_\varepsilon, \operatorname{div}(\phi)\rangle + \langle h, \phi\rangle_{H^{-1/2}(\partial\Omega;\mathbb{R}^d)\times H^{1/2}(\partial\Omega;\mathbb{R}^d)}, \tag{3.4.6}$$

for any $\phi \in H^1(\Omega;\mathbb{R}^d)$. By Theorem 3.4.1, we can extract a subsequence, still denoted by $\{u_\varepsilon\}$, $\{p_\varepsilon\}$ and $\{\xi_\varepsilon\}$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 && \text{weakly in } H^1(\Omega;\mathbb{R}^d); \\ p_\varepsilon &\rightharpoonup p_0 && \text{weakly in } L^2(\Omega); \\ \xi_\varepsilon &\rightharpoonup \xi_0 && \text{weakly in } L^2(\Omega;\mathbb{R}^{d\times d}). \end{aligned}$$

Similarly, as in (3.3.5) of the proof of Theorem 3.3.2, we choose

$$\phi(x) = \Big(P_j^\beta(x) + \varepsilon\chi_j^{*\beta}(x/\varepsilon)\Big)\,\psi(x),$$

where $1 \le j, d \le d$ and $\psi(x) \in C_0^1(\Omega)$ are fixed. Similarly,

$$\begin{aligned} \langle \xi_\varepsilon, \nabla\phi\rangle = {}& \langle \nabla(\psi u_\varepsilon), A^*(x/\varepsilon)\nabla\big(P_j^\beta(x) + \varepsilon\chi_j^{*\beta}(x/\varepsilon)\big)\rangle \\ & - \langle (\nabla\psi)u_\varepsilon, A^*(x/\varepsilon)\nabla\big(P_j^\beta(x) + \varepsilon\chi_j^{*\beta}(x/\varepsilon)\big)\rangle \\ & + \langle \xi_\varepsilon, \big(P_j^\beta(x) + \varepsilon\chi_j^{*\beta}(x/\varepsilon)\big)\nabla\psi\rangle. \end{aligned} \tag{3.4.7}$$

Since

$$-\text{div}\Big(A^*(x/\varepsilon)\nabla\big[P_j^\beta(x)+\varepsilon\chi_j^{*\beta}(x/\varepsilon)\big]\Big)=-\nabla\pi_j^{*\beta}(x/\varepsilon) \qquad \text{in } \mathbb{R}^d,$$

it follows that the first term in the right hand side of (3.4.7) equals

$$\langle\pi_j^{*\beta}(x/\varepsilon),\text{div}(\psi u_\varepsilon)\rangle=\langle\pi_j^{*\beta}(x/\varepsilon)-\fint_Y\pi_j^{*\beta},\text{div}(\psi u_\varepsilon)\rangle.$$

Using the fact that

$$\text{div}(\psi u_\varepsilon)=\nabla\psi\cdot u_\varepsilon+\psi g\to\nabla\psi\cdot u_0+\psi g \quad \text{strongly in } L^2(\Omega)$$

and by Proposition 3.3.1,

$$\pi_j^{*\beta}(x/\varepsilon)-\fint_Y\pi_j^{*\beta}\rightharpoonup 0 \text{ weakly in } L^2(\Omega),$$

we see that first term in the right hand side of (3.4.7) goes to zero. In view of the estimate

$$\|\varepsilon\chi_j^{*\beta}(x/\varepsilon)\|_{L^2(\Omega)}\le C\varepsilon\|\chi_j^{*\beta}\|_{L^2(Y)}\le C\,\varepsilon,$$

it is easy to see that for the third term in the right hand side of (3.4.7) goes to $\langle\xi_0,P_j^\beta\nabla\psi\rangle$.

To handle the second term in the right hand side of (3.4.7), we note that again by Proposition 3.3.1,

$$\nabla P_i^\alpha\cdot A^*(x/\varepsilon)\nabla\big(P_j^\beta(x)+\varepsilon\chi_j^{*\beta}(x/\varepsilon)\big)$$

converges weakly in $L^2(\Omega)$ to

$$\int_Y\nabla P_i^\alpha\cdot A^*(y)\nabla\big(P_j^\beta+\chi_j^{*\beta}(y)\big)dy=\widehat{a}_{ji}^{\beta\alpha},$$

where we have used the definition of matrices of effective coefficients. This, together with the fact that $u_\varepsilon\to u_0$ strongly in $L^2(\Omega;\mathbb{R}^d)$, shows that the second term in the right hand side of (3.4.7) goes to

$$-\widehat{a}_{ji}^{\beta\alpha}\int_\Omega\frac{\partial\psi}{\partial x_i}u_0^\alpha=\widehat{a}_{ji}^{\beta\alpha}\int_\Omega\psi\frac{\partial u_0^\alpha}{\partial x_i},$$

where we have used integration by parts. To summarize, we have proved that,

$$\langle\xi_\varepsilon,\nabla\phi\rangle\to\langle\xi_0,P_j^\beta\nabla\psi\rangle+\widehat{a}_{ji}^{\beta\alpha}\int_\Omega\psi\frac{\partial u_0^\alpha}{\partial x_i}. \tag{3.4.8}$$

Finally, since $\phi \rightharpoonup P_j^\beta \psi$ weakly in $H_0^1(\Omega;\mathbb{R}^d)$, we have $\langle F,\phi\rangle \to \langle F, P_j^\beta\psi\rangle$. Also, since $\operatorname{div}(\chi_j^\beta) = 0$ in $\mathbb{R}^d$,

$$\langle p_\varepsilon, \operatorname{div}(\phi)\rangle = \langle p_\varepsilon, \operatorname{div}(P_j^\beta\psi)\rangle + \langle p_\varepsilon, \varepsilon\chi_j^{*\beta}(x/\varepsilon)\nabla\psi\rangle \to \langle p_0, \operatorname{div}(P_j^\beta\psi)\rangle.$$

Thus, the right hand side of (3.4.6) converges to

$$\langle F, P_j^\beta\psi\rangle + \langle p_0, \operatorname{div}(P_j^\beta\psi)\rangle = \langle \xi_0, \nabla(P_j^\beta\psi)\rangle = \langle \xi_0, P_j^\beta\nabla\psi\rangle + \langle \xi_0, \psi\nabla P_j^\beta\rangle,$$

where the first equality follows by taking the limit in (3.4.6) with $\phi = P_j^\beta\psi$. In view of (3.4.8) we obtain

$$\widehat{a}_{ji}^{\beta\alpha}\int_\Omega \psi\frac{\partial u_0^\alpha}{\partial x_i} = \langle \xi_0, \psi\nabla P_j^\beta\rangle.$$

Since $\psi \in C_0^1(\Omega)$ is arbitrary, this gives $(\xi_0)_j^\beta = \widehat{a}_{ji}^{\beta\alpha}\frac{\partial u_0^\alpha}{\partial x_i}$, i.e. we have prove that

$$\xi_0 = \widehat{A}\nabla u_0.$$

Taking limit in (3.4.6), we see that

$$\langle \widehat{A}\nabla u_0, \nabla\phi\rangle = \langle F,\phi\rangle + \langle p_0, \operatorname{div}(\phi)\rangle + \langle h,\phi\rangle_{H^{-1/2}(\partial\Omega;\mathbb{R}^d)\times H^{1/2}(\partial\Omega;\mathbb{R}^d)},$$

this implies that (u_0, p_0) is the unique solution of Neumann problem

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \operatorname{div}(u_0) = g & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} - p_0\cdot n = h & \text{on } \partial\Omega. \end{cases}$$

satisfying $\int_\Omega u_0 = \int_\Omega p_0 = 0$. As a result we conclude that the whole sequence $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega;\mathbb{R}^d)$ and $p_\varepsilon \rightharpoonup p_0$ weakly in $L^2(\Omega)$. $\square$

Chapter 4 Convergence Rates of Dirichlet Problems in Homogenization of Stokes Systems

In this chapter we study the convergence rates in L^2 and H^1 of Dirichlet problems for Stokes systems with rapidly oscillating periodic coefficient, without any regularity assumptions on the coefficients.

4.1 Introduction

More precisely, we consider the following Dirichlet problem for Stokes systems

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \text{div } u_\varepsilon = g & \text{in } \Omega, \\ u_\varepsilon = h & \text{on } \partial\Omega, \end{cases}$$

with the Dirichlet compatibility condition

$$\int_\Omega g - \int_{\partial\Omega} h \cdot n = 0.$$

By homogenization theorem (Theorem 3.1.5), we have shown that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega; \mathbb{R}^d),$$

and

$$p_\varepsilon - \fint_\Omega p_\varepsilon \rightharpoonup p_0 - \fint_\Omega p_0 \quad \text{weakly in } L^2(\Omega),$$

where $(u_0, p_0) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ is the weak solution of the homogenized problem

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \text{div}(u_0) = g & \text{in } \Omega, \\ u_0 = h & \text{on } \partial\Omega. \end{cases}$$

The main purpose of this chapter is to investigate the rate of convergence of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$ as $\varepsilon \to 0$, which is stated in the following theorem.

Theorem 4.1.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $g \in H^1(\Omega)$ and $f \in H^{3/2}(\partial\Omega; \mathbb{R}^d)$ satisfying the Dirichlet compatibility condition (3.1.2), for*

$F \in L^2(\Omega;\mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of Dirichlet problems (3.1.10), (3.1.11), respectively. Then

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4.1.1}$$

where the constant C depends only on d, μ, and Ω.

Theorem 4.1.1 gives the optimal $O(\varepsilon)$ convergence rate for the inverses of the Stokes operators in L^2 operator norm. Indeed, let

$$T_\varepsilon : F \in L^2_\sigma(\Omega) \to u_\varepsilon,$$

where

$$L^2_\sigma(\Omega) = \left\{ F \in L^2(\Omega;\mathbb{R}^d) : \operatorname{div}(F) = 0 \text{ in } \Omega \right\},$$

and u_ε denotes the solution of (3.1.10) with $F \in L^2_\sigma(\Omega;\mathbb{R}^d)$ and $g = 0$, $f = 0$. Then it follows from (4.1.1) and the estimate $\|u_0\|_{H^2(\Omega)} \le C\|F\|_{L^2(\Omega)}$ that

$$\|T_\varepsilon - T_0\|_{L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)} \le C\varepsilon,$$

where $T_0 : F \in L^2_\sigma(\Omega) \to u_0$.

In this chapter we also obtain $O(\sqrt{\varepsilon})$ rates for a two-scale expansion of $(u_\varepsilon, p_\varepsilon)$ in $H^1 \times L^2$. Let (χ, π) denote the correctors associated with A, defined by (3.1.5), and S_ε the Steklov smoothing operator defined by (2.2.1).

Theorem 4.1.2. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies ellipticity (1.0.3) and periodicity (1.0.4) conditions. Let $(u_\varepsilon, p_\varepsilon)$ and (u_0, p_0) be the same as in Theorem 4.1.1. Then*

$$\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0)\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{4.1.2}$$

where $\chi^\varepsilon(x) = \chi(x/\varepsilon)$ and $\widetilde{u}_0$ is the extension of u_0 defined as in (4.2.1). Moreover, if $\int_\Omega p_\varepsilon = \int_\Omega p_0 = 0$, then

$$\|p_\varepsilon - p_0 - \left\{ \pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) - \fint_\Omega \pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) \right\}\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{4.1.3}$$

where $\pi^\varepsilon(x) = \pi(x/\varepsilon)$. The constants C in (4.1.2) and (4.1.3) depend only on d, μ, and Ω.

For the known results, as we mentioned earlier, consider the Dirichlet problem for the scalar elliptic equation $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in a Lipschitz domain Ω with $u_\varepsilon = f$ on $\partial\Omega$. It is well known that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \left\{ \|\nabla^2 u_0\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^\infty(\partial\Omega)} \right\}. \tag{4.1.4}$$

To see (4.1.4), one considers the difference between u_ε and its first order approximation $u_0 + \varepsilon\chi^\varepsilon \nabla u_0$ and let

$$v_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon \nabla u_0. \tag{4.1.5}$$

To correct the boundary data, one further introduces a function w_ε, where w_ε is the solution to the Dirichlet problem: $\mathcal{L}_\varepsilon(w_\varepsilon) = 0$ in Ω and $w_\varepsilon = -\varepsilon\chi^\varepsilon\nabla u_0$ on $\partial\Omega$. Using energy estimates, one may show that

$$\|v_\varepsilon - w_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon \|\nabla^2 u_0\|_{L^2(\Omega)}.$$

The estimate (4.1.4) follows from this and the estimate $\|w_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon\|\nabla u_0\|_{L^\infty(\partial\Omega)}$, which is obtained by the maximum principle (see e.g. [31]). More recently, Griso [26, 27] was able to establish the much sharper estimate (4.1.1), using the method of periodic unfolding. We mention that in the case of scalar elliptic equations with bounded measurable coefficients, one may also prove (4.1.1) by using the so-called Dirichlet corrector. In fact, it was shown in [33] that

$$\|u_\varepsilon - u_0 - \Big\{\Phi_\varepsilon - x\Big\}\nabla u_0\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4.1.6}$$

where $\Phi_\varepsilon(x)$ is the solution of $\mathcal{L}_\varepsilon(\Phi_\varepsilon) = 0$ in Ω with $\Phi_\varepsilon = x$ on $\partial\Omega$. In the case of elliptic systems, the estimates (4.1.6) and thus (4.1.1) continue to hold under the additional assumption that A is Hölder continuous. Moreover, if A is Hölder continuous and symmetric, it was proved in [32] that

$$\|v_\varepsilon\|_{H^{1/2}(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}. \tag{4.1.7}$$

The approaches used in [32, 33] rely on the uniform regularity estimates established in [3, 37] and do not apply to operators with bounded measurable coefficients. Recently, by using the Steklov smoothing operator, T. A. Suslina [55, 56] was able to establish the $O(\varepsilon)$ estimate (4.1.1) in L^2 for a boarder class of elliptic operators, which, in particular, contains the elliptic systems $\mathcal{L}_\varepsilon$ in divergence form with coefficients satisfying the ellipticity condition $a_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \ge \mu|\xi|^2$ for any $\xi = \big(\xi_i^\alpha\big) \in \mathbb{R}^{m\times d}$. Since the correctors χ may not be bounded in the case of non-smooth coefficients, the idea is to consider the two-scale expansion

$$v_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0), \tag{4.1.8}$$

where S_ε is a smoothing operator at scale ε defined in (2.2.1) and $\widetilde{u}_0$ an extension of u_0 to $\mathbb{R}^d$ (also see [45, 46, 59] and their references on the use of S_ε in homogenization). This reduces the problem to the control of the L^2 norm of w_ε, where w_ε is the solution to the Dirichlet problem: $\mathcal{L}_\varepsilon(w_\varepsilon) = 0$ in Ω and $w_\varepsilon = -\varepsilon\chi^\varepsilon S_\varepsilon \nabla(\widetilde{u}_0)$ on $\partial\Omega$. Next, one considers

$$h_\varepsilon = w_\varepsilon - \varepsilon\chi^\varepsilon \theta_\varepsilon S_\varepsilon(\nabla\widetilde{u}_0),$$

where θ_ε is a cutoff function supported in an ε neighborhood of $\partial\Omega$. Note that $h_\varepsilon = 0$ on $\partial\Omega$ and $\mathcal{L}_\varepsilon(h_\varepsilon)$ is supported in an ε neighborhood of $\partial\Omega$. This allows one to approximate h_ε in the L^2 norm by h_0, using an $O(\sqrt{\varepsilon})$ estimate in H^1 and a duality argument, where $\mathcal{L}_0(h_0) = \mathcal{L}_\varepsilon(h_\varepsilon)$ in Ω and $h_0 = 0$ on $\partial\Omega$. Finally, one estimates the L^2 norm of h_0 by another duality argument.

In this chapter we extend the approach of Suslina to the case of Stokes systems, which do not fit the standard framework of second-order elliptic systems in divergence form. As expected in the study of Stokes or Navies-Stokes systems, the main difficulty is caused by the pressure term p_ε. By carefully analyzing the systems for the correctors (χ, π) as well as their dual $(\phi_{kij}^{\alpha\beta}, q_{ij}^{\beta})$, we are able to establish the $O(\sqrt{\varepsilon})$ error estimates, given in Theorem 4.1.2, for the two-scale expansions of $(u_\varepsilon, p_\varepsilon)$ in $H^1 \times L^2$. This allows us to use the idea of boundary cutoff and duality argument in a manner similar to that in [55].

4.2 Convergence rates for u_ε in H^1

From now on we will assume that Ω is a bounded domain with boundary of class $C^{1,1}$, $F \in L^2(\Omega;\mathbb{R}^d)$, $g \in H^1(\Omega)$, and $h \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$. We fix a linear continuous extension operator

$$E_\Omega : H^2(\Omega;\mathbb{R}^d) \to H^2(\mathbb{R}^d;\mathbb{R}^d),$$

and let

$$\widetilde{u}_0 = E_\Omega u_0, \tag{4.2.1}$$

so that $\widetilde{u}_0 = u_0$ in Ω and

$$\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \le C\|u_0\|_{H^2(\Omega)}, \tag{4.2.2}$$

where C depends on Ω. We introduce a first order approximation of u_ε,

$$v_\varepsilon = u_0 + \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0).$$

Let $(w_\varepsilon, \tau_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ be a weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon) + \nabla\tau_\varepsilon = 0 & \text{in } \Omega, \\ \operatorname{div}(w_\varepsilon) = \varepsilon \operatorname{div}\big(\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0\big) & \text{in } \Omega, \\ w_\varepsilon = \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) & \text{on } \partial\Omega. \end{cases} \tag{4.2.3}$$

We will use w_ε to approximate the difference between u_ε and its first order approximation v_ε. To this end, for $1 \le i, j, \alpha, \beta \le d$, we let

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y) \frac{\partial}{\partial y_k}\big(\chi_j^{\gamma\beta}\big) - \widehat{a}_{ij}^{\alpha\beta}. \tag{4.2.4}$$

Note that $b_{ij}^{\alpha\beta}$ is 1-periodic. By the definition of χ and $\widehat{A}$, $b_{ij}^{\alpha\beta} \in L^2(Y)$ satisfies

$$\int_Y b_{ij}^{\alpha\beta}(y)\, dy = 0.$$

and, for each $1 \le \alpha, \beta, j \le d$,

$$\begin{aligned}
\frac{\partial}{\partial y_i}\big(b_{ij}^{\alpha\beta}(y)\big) &= \frac{\partial}{\partial y_i}\big(a_{ij}^{\alpha\beta}(y)\big) + \frac{\partial}{\partial y_i}\left(a_{ik}^{\alpha\gamma}(y)\frac{\partial \chi_j^{\gamma\beta}}{\partial y_k}\right)\\
&= \frac{\partial}{\partial y_i}\big(a_{ij}^{\alpha\beta}(y)\big) - \frac{\partial}{\partial y_i}\left(a_{ik}^{\alpha\gamma}(y)\frac{\partial P_j^{\gamma\beta}}{\partial y_k}\right) + \frac{\partial}{\partial y_\alpha}(\pi_j^\beta)\\
&= \frac{\partial}{\partial y_\alpha}(\pi_j^\beta).
\end{aligned} \tag{4.2.5}$$

Lemma 4.2.1. *There exist $\Phi_{kij}^{\alpha\beta} \in H^1_{\text{per}}(Y)$ and $q_{ij}^\beta \in H^1_{\text{per}}(Y)$ such that*

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}(\Phi_{kij}^{\alpha\beta}) + \frac{\partial}{\partial y_\alpha}(q_{ij}^\beta) \quad \text{ and } \quad \Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}. \tag{4.2.6}$$

Moreover,

$$\|\Phi_{kij}^{\alpha\beta}\|_{L^2(Y)} + \|q_{ij}^\beta\|_{L^2(Y)} \le C, \tag{4.2.7}$$

where C depends only on d and μ.

Proof. Fix $1 \le i, j, \beta \le d$. There exist $f_{ij}^\beta = (f_{ij}^{\alpha\beta}) \in H^2_{\text{per}}(Y;\mathbb{R}^d)$ and $q_{ij}^\beta \in H^1_{\text{per}}(Y)$ satisfying the following Stokes system,

$$\begin{cases}
\Delta f_{ij}^\beta + \nabla q_{ij}^\beta = b_{ij}^\beta & \text{in } Y,\\
\operatorname{div}(f_{ij}^\beta) = 0 & \text{in } Y,\\
\displaystyle\int_Y f_{ij}^\beta\, dy = 0,
\end{cases} \tag{4.2.8}$$

where $b_{ij}^\beta = (b_{ij}^{\alpha\beta})$. We now define

$$\Phi_{kij}^{\alpha\beta}(y) = \frac{\partial}{\partial y_k}(f_{ij}^{\alpha\beta}) - \frac{\partial}{\partial y_i}(f_{kj}^{\alpha\beta}).$$

Clearly, $\Phi_{kij}^{\alpha\beta} \in H^1_{\mathrm{per}}(Y)$ and $\Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}$. Note that, by (4.2.5) and (4.2.8), $\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i} \in H^1_{\mathrm{per}}(Y)$ satisfies

$$\begin{cases} \qquad \Delta\left(\dfrac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}\right) = -\dfrac{\partial}{\partial y_\alpha}\left(\dfrac{\partial q_{ij}^{\beta}}{\partial y_i}\right) + \dfrac{\partial b_{ij}^{\alpha\beta}}{\partial y_i} \\ \qquad\qquad\qquad = \dfrac{\partial}{\partial y_\alpha}\left(\pi_j^\beta - \dfrac{\partial q_{ij}^\beta}{\partial y_i}\right), \\ \dfrac{\partial}{\partial y_\alpha}\left(\dfrac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}\right) = 0. \end{cases} \tag{4.2.9}$$

It follows by the energy estimates that $\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}$ is constant. Hence, by (4.2.9) we obtain that

$$\begin{aligned} \frac{\partial}{\partial y_k}(\Phi_{kij}^{\alpha\beta}) &= \frac{\partial^2}{\partial y_k \partial y_k}(f_{ij}^{\alpha\beta}) - \frac{\partial}{\partial y_i}\left(\frac{\partial}{\partial y_k}(f_{kj}^{\alpha\beta})\right) \\ &= b_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_\alpha}(q_{ij}^\beta). \end{aligned}$$

Furthermore, since $\|\chi_j^\beta\|_{H^1(Y)} \le C$, then

$$\begin{aligned} \|\Phi_{kij}^{\alpha\beta}\|_{L^2(Y)} + \|q_{ij}^\beta\|_{L^2(Y)} &\le C\|b_{ij}^{\alpha\beta}\|_{L^2(Y)} \\ &\le C, \end{aligned}$$

where C depends only on d and μ. This completes the proof. $\square$

Remark 4.2.2. *Recall that π_j^β and q_{ij}^β are both 1-periodic. By (4.2.9) and the fact that $\frac{\partial f_{ij}^{\alpha\beta}}{\partial y_i}$ is constant, we see that π_j^β and $\frac{\partial q_{ij}^\beta}{\partial y_i}$ differ only by a constant. Since $\int_Y \pi_j^\beta = 0$, we obtain the following relation,*

$$\pi_j^\beta = \frac{\partial q_{ij}^\beta}{\partial y_i}. \tag{4.2.10}$$

Lemma 4.2.3. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $g \in H^1(\Omega)$ and $h \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the compatibility condition (3.1.2), for $F \in L^2(\Omega;\mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) and $(w_\varepsilon, \tau_\varepsilon)$ be weak solutions of Dirichlet problems (3.1.10), (3.1.11) and (4.2.3), respectively. Then,*

$$\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) + w_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{4.2.11}$$

where C depends only on d, μ, and Ω.

Proof. Let
$$z_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) + w_\varepsilon.$$
Through the construction of w_ε in (4.2.3), we can easily observe that
$$\operatorname{div}(z_\varepsilon) = 0 \quad \text{in } \Omega,$$
and
$$z_\varepsilon = 0 \quad \text{on } \partial\Omega.$$
Now we compute $\mathcal{L}_\varepsilon(z_\varepsilon)$, since
$$\mathcal{L}_\varepsilon(u_\varepsilon) - \mathcal{L}_\varepsilon(u_0) = \mathcal{L}_0(u_0) - \mathcal{L}_\varepsilon(u_0) + \nabla(p_0 - p_\varepsilon),$$
then by direct computation and the definition of $b_{ij}^{\alpha\beta}(y)$ in (4.2.4) we have
$$\begin{aligned}
(\mathcal{L}_\varepsilon(z_\varepsilon))^\alpha &= -\frac{\partial[p_\varepsilon - p_0 + \tau_\varepsilon]}{\partial x_\alpha} - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\frac{\partial u_0^\beta}{\partial x_j}\right) \\
&\quad + \frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\frac{\partial}{\partial x_k}\left[\varepsilon\chi_j^{\gamma\beta}(x/\varepsilon)\right]S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) + \varepsilon\frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\right) \\
&= -\frac{\partial[p_\varepsilon - p_0 + \tau_\varepsilon]}{\partial x_\alpha} - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[\frac{\partial u_0^\beta}{\partial x_j} - S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right]\right) \\
&\quad + \frac{\partial}{\partial x_i}\left(b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right) + \varepsilon\frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\right).
\end{aligned}$$
Using Lemma 4.2.1, we may write
$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right) &= \frac{\partial}{\partial x_i}\left(\left[\frac{\partial}{\partial x_k}\Big(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)\Big) + \frac{\partial}{\partial x_\alpha}\Big(\varepsilon q_{ij}^\beta(x/\varepsilon)\Big)\right]S_\varepsilon\frac{\partial\tilde{u}_0^\beta}{\partial x_j}\right) \\
&= I_1 + I_2.
\end{aligned} \tag{4.2.12}$$
Since $\Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}$, we see that
$$\begin{aligned}
I_1 &= \frac{\partial^2}{\partial x_i\partial x_k}\left(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u_0}^\beta}{\partial x_j}\right) - \varepsilon\frac{\partial}{\partial x_i}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right) \\
&= -\varepsilon\frac{\partial}{\partial x_i}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right).
\end{aligned}$$
For the second term in the RHS of (4.2.12), we have
$$\begin{aligned}
I_2 &= \frac{\partial}{\partial x_\alpha}\left(\frac{\partial}{\partial x_i}\left[\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right]\right) - \frac{\partial}{\partial x_i}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right) \\
&= I_3 - \frac{\partial}{\partial x_i}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right).
\end{aligned} \tag{4.2.13}$$

In view of (4.2.10), for the first term on the R.H.S. of (4.2.13), we obtain

$$I_3 = \frac{\partial}{\partial x_\alpha}\left(\pi_j^\beta(x/\varepsilon)S_\varepsilon \frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) + \frac{\partial}{\partial x_\alpha}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j \partial x_i}\right). \tag{4.2.14}$$

Putting altogether, we have shown that z_ε satisfies

$$\begin{aligned}
&(\mathcal{L}_\varepsilon(z_\varepsilon))^\alpha + \frac{\partial}{\partial x_\alpha}\left(p_\varepsilon - p_0 - \pi_j^\beta(x/\varepsilon)S_\varepsilon \frac{\partial \widetilde{u}_0^\beta}{\partial x_j} - \varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j \partial x_i} + \tau_\varepsilon\right)\\
&= \varepsilon \frac{\partial}{\partial x_i}\left(\left[a_{ij}^{\alpha\gamma}(x/\varepsilon)\chi_k^{\gamma\beta}(x/\varepsilon) - \Phi_{kij}^{\alpha\beta}(x/\varepsilon)\right]S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j \partial x_k}\right)\\
&\qquad - \varepsilon \frac{\partial}{\partial x_i}\left(q_{ij}^\beta(x/\varepsilon)S_\varepsilon \frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_\alpha \partial x_j}\right)\\
&\qquad - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[\frac{\partial u_0^\beta}{\partial x_j} - S_\varepsilon \frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right]\right).
\end{aligned} \tag{4.2.15}$$

Since $z_\varepsilon \in H_0^1(\Omega;\mathbb{R}^d)$ and $\mathrm{div}(z_\varepsilon) = 0$ in Ω, it follows from (4.2.15) by the energy estimate (3.1.3) that

$$\begin{aligned}
c\int_\Omega |\nabla z_\varepsilon|^2 dx \le \varepsilon^2 &\int_\Omega \Big|\big[|\chi(x/\varepsilon)| + |\Phi(x/\varepsilon)|\big]S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big|^2 dx\\
&+ \varepsilon^2 \int_\Omega \Big|q(x/\varepsilon)S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big|^2 dx\\
&+ \int_\Omega \Big|\nabla u_0 - S_\varepsilon(\nabla \widetilde{u}_0)\Big|^2 dx.
\end{aligned}$$

Now we apply Propositions 2.2.1-2.2.2 as well as (4.2.2). This gives

$$\begin{aligned}
\|\nabla z_\varepsilon\|_{L^2(\Omega)} &\le C\varepsilon\left(\|\chi\|_{L^2(Y)} + \|\Phi\|_{L^2(Y)} + \|q\|_{L^2(Y)} + 1\right)\|\nabla^2 \widetilde{u}_0\|_{L^2(\mathbb{R}^d)}\\
&\le C\varepsilon\|\nabla^2 \widetilde{u}_0\|_{L^2(\mathbb{R}^d)}\\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)},
\end{aligned}$$

where C depends only on d, μ and Ω. Hence we have proved the desired result, $\|z_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}$, and completed the proof. □

We choose two cut-off functions $\theta_\varepsilon(x)$ and $\widetilde{\theta}_\varepsilon(x)$ in $\mathbb{R}^d$ satisfying the following conditions,

$$\begin{aligned}
&\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\theta_\varepsilon) \subset \widetilde{\Omega}_\varepsilon, \quad 0 \le \theta_\varepsilon(x) \le 1,\\
&\theta_\varepsilon|_{\partial\Omega} = 1, \quad |\nabla\theta_\varepsilon| \le \kappa/\varepsilon,
\end{aligned} \tag{4.2.16}$$

and

$$\begin{aligned}
&\widetilde{\theta}_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \mathrm{supp}(\widetilde{\theta}_\varepsilon) \subset \widetilde{\Omega}_{2\varepsilon}, \quad 0 \le \widetilde{\theta}_\varepsilon(x) \le 1,\\
&\widetilde{\theta}_\varepsilon(x) = 1 \text{ for } x \in \widetilde{\Omega}_\varepsilon, \quad |\nabla\widetilde{\theta}_\varepsilon| \le \widetilde{\kappa}/\varepsilon.
\end{aligned} \tag{4.2.17}$$

Now we are ready to give the proof of the H^1 convergence (4.1.2) in Theorem 4.1.2.

Proof of estimate (4.1.2). Lemma 4.2.3 have shown that

$$\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) + w_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)},$$

therefore the problem has been reduced to estimating w_ε in H^1. Notice that by the energy estimate (3.1.3) and since $\operatorname{div}(\chi) = 0$, then

$$\begin{aligned}\|w_\varepsilon\|_{H^1(\Omega)} &\le C\varepsilon\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^{1/2}(\partial\Omega)} + C\varepsilon\|\operatorname{div}\big(\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0)\|_{L^2(\Omega)} \\ &\le C\varepsilon\|\theta_\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} + C\varepsilon\|\chi^\varepsilon\nabla S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)}.\end{aligned} \tag{4.2.18}$$

For the first term on R.H.S. of (4.2.18), using the properties of cut-off function θ_ε in (4.2.16) we obtain

$$\begin{aligned}\|\theta_\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} &\le \|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|(\nabla\theta_\varepsilon)\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} \\ &\qquad + \varepsilon^{-1}\|\theta_\varepsilon(\nabla\chi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|\chi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}, \\ &\le \|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|\chi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)} \\ &\qquad + \varepsilon^{-1}\left\{\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_\varepsilon)} + \|(\nabla\chi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_\varepsilon)}\right\}.\end{aligned} \tag{4.2.19}$$

Plug (4.2.19) back into (4.2.18),

$$\begin{aligned}\|w_\varepsilon\|_{H^1(\Omega)} &\le C\varepsilon\left\{\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|\chi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}\right\} \\ &\qquad + C\left\{\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_\varepsilon)} + \|(\nabla\chi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_\varepsilon)}\right\} \\ &\le C\varepsilon\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} + C\sqrt{\varepsilon}\|\nabla\widetilde{u}_0\|^{1/2}_{L^2(\mathbb{R}^d)}\|\nabla\widetilde{u}_0\|^{1/2}_{H^1(\mathbb{R}^d)} \\ &\le C\sqrt{\varepsilon}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)},\end{aligned} \tag{4.2.20}$$

where in the second last inequality we have used Proposition 2.2.2 for the first brace and (2.2.3) in Lemma 2.2.3 for the second. Therefore, by (4.2.2) we see

$$\begin{aligned}\|u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon\left(\nabla\widetilde{u}_0\right)\|_{H^1(\Omega)} &\le \|z_\varepsilon\|_{H^1(\Omega)} + \|w_\varepsilon\|_{H^1(\Omega)} \\ &\le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)},\end{aligned}$$

where C depends only on d, μ, and Ω. This completes the proof. □

4.3 Convergence rates for the pressure term

To prove estimate (4.1.3), we first recall that if $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega)$ is a weak solution of the Stokes system (3.1.10), then

$$\begin{aligned}\|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^2(\Omega)} &\le C\|\nabla p_\varepsilon\|_{H^{-1}(\Omega)} \\ &\le C\Big\{\|F\|_{H^{-1}(\Omega)} + \|u_\varepsilon\|_{H^1(\Omega)}\Big\},\end{aligned} \tag{4.3.1}$$

where C depends only on d, μ, and Ω (see e.g. [58]).

Proof of estimate (4.1.3). Since we are assuming

$$\int_\Omega p_\varepsilon = \int_\Omega p_0 = 0,$$

by applying (4.3.1) to system (4.2.15), we see that

$$\begin{aligned}
\|p_\varepsilon - p_0 - &\big(\pi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + \varepsilon q^\varepsilon S_\varepsilon \nabla^2 \widetilde{u}_0 - \tau_\varepsilon\big) \\
&\qquad - \fint_\Omega \big(\pi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0 + \varepsilon q^\varepsilon S_\varepsilon \nabla^2 \widetilde{u}_0 - \tau_\varepsilon\big)\|_{L^2(\Omega)} \\
&\le C\|\nabla \Big[p_\varepsilon - p_0 - \pi^\varepsilon S_\varepsilon\left(\nabla \tilde{u}_0\right) - \varepsilon q^\varepsilon S_\varepsilon\left(\nabla^2 \tilde{u}_0\right) + \tau_\varepsilon\Big]\|_{H^{-1}(\Omega)} \\
&\le C\|\nabla z_\varepsilon\|_{L^2(\Omega)} + C\|S_\varepsilon\left(\nabla \widetilde{u}_0\right) - \nabla u_0\|_{L^2(\Omega)} \\
&\qquad\qquad + C\varepsilon \left\|\big(|\chi^\varepsilon| + |\Phi^\varepsilon| + |q^\varepsilon|\big) S_\varepsilon\left(\nabla^2 \widetilde{u}_0\right)\right\|_{L^2(\Omega)} \\
&\le C\varepsilon \|u_0\|_{H^2(\Omega)},
\end{aligned} \tag{4.3.2}$$

where the last inequality follows from the proof of Lemma 4.2.3. Note that by Proposition 2.2.1 and (4.2.2),

$$\begin{aligned}
\varepsilon\|q^\varepsilon S_\varepsilon(\nabla^2 \widetilde{u}_0) - &\fint_\Omega q^\varepsilon S_\varepsilon(\nabla^2 \tilde{u}_0)\|_{L^2(\Omega)} \\
&\le C\varepsilon \|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \\
&\le C\varepsilon \|u_0\|_{H^2(\Omega)}.
\end{aligned} \tag{4.3.3}$$

Also, since $(w_\varepsilon, \tau_\varepsilon)$ is weak solution of system (4.2.3), and we applying (4.3.1) again to obtain

$$\begin{aligned}
\|\tau_\varepsilon - \fint_\Omega \tau_\varepsilon\|_{L^2(\Omega)} &\le C\|\nabla \tau_\varepsilon\|_{H^{-1}(\Omega)} \\
&\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)} \\
&\le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)},
\end{aligned} \tag{4.3.4}$$

where the last inequality follows from (4.2.20) and (4.2.2). By combining (4.3.2), (4.3.3) and (4.3.4), we have proved that

$$\|p_\varepsilon - p_0 - \Big[\pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right) - \fint_\Omega \pi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right)\Big]\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}.$$

This completes the proof. $\square$

4.4 Convergence rates for u_ε in L^2

To establish the sharp $O(\varepsilon)$ rate for u_ε in L^2, in view of (4.2.11), we have already shown that

$$\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) + w_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}.$$

By using Proposition 2.2.2 and (4.2.2) once again,

$$\begin{aligned}\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} &\leq C\|\chi\|_{L^2(Y)}\|\nabla\widetilde{u}_0\|_{L^2(\mathbb{R}^d)}\\ &\leq C\|u_0\|_{H^2(\Omega)}.\end{aligned}$$

Therefore,

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)} + \|w_\varepsilon\|_{L^2(\Omega)}, \tag{4.4.1}$$

and it remains to estimate $\|w_\varepsilon\|_{L^2(\Omega)}$.

Lemma 4.4.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose that A satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $g \in H^1(\Omega)$ and $h \in H^{3/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the compatibility condition (3.1.2), for $F \in L^2(\Omega;\mathbb{R}^d)$, let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of the Dirichlet problems (3.1.10), (3.1.11), respectively. If $\widetilde{\theta}_\varepsilon$ is the cut-off function defined as in (4.2.17), then*

$$\|u_\varepsilon - u_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^\varepsilon S_\varepsilon\left(\nabla\widetilde{u}_0\right)\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{4.4.2}$$

and

$$\|p_\varepsilon - p_0 - \Big[(1-\widetilde{\theta}_\varepsilon)\pi^\varepsilon S_\varepsilon\left(\nabla\widetilde{u}_0\right) - \fint_\Omega \pi^\varepsilon S_\varepsilon\left(\nabla\widetilde{u}_0\right)\Big]\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{4.4.3}$$

where C depends only on d, μ, and Ω.

Proof. We use the same argument as we did for (4.2.19) in the proof of (4.1.2) to treat the extra term $\varepsilon\widetilde{\theta}_\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)$ and $\widetilde{\theta}_\varepsilon\pi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)$, with θ_ε replaced by $\widetilde{\theta}_\varepsilon$. Explicitly,

$$\begin{aligned}\varepsilon\|\widetilde{\theta}_\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} &\leq C\varepsilon\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})} + C\varepsilon\|\chi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})}\\ &\quad + C\left\{\|\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})} + \|(\nabla\chi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega_{2\varepsilon})}\right\}\\ &\leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)},\end{aligned}$$

where we have used Lemma 2.2.3 and Proposition 2.2.2 for the last inequality. This, together with the H^1 convergence (4.1.2), gives us the estimate (4.4.2).

Similarly, using Lemma 2.2.3, we see that

$$\begin{aligned}\|\widetilde{\theta}_\varepsilon\pi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\|_{L^2(\Omega)} &\leq C\left(\int_{\widetilde{\Omega}_{2\varepsilon}} |\pi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)|^2 dx\right)^{1/2}\\ &\leq C\sqrt{\varepsilon}\|\pi\|^{1/2}_{L^2(Y)}\|\nabla\widetilde{u}_0\|^{1/2}_{H^1(\mathbb{R}^d)}\|\nabla\widetilde{u}_0\|^{1/2}_{L^2(\mathbb{R}^d)}\\ &\leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}.\end{aligned}$$

Combining with estimate (4.1.3), gives us the proof of (4.4.3). $\square$

Proof of Theorem 1.0.3. In view of (4.4.1), it suffices to show that

$$\|w_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}.$$

Furthermore, let

$$\phi_\varepsilon = \varepsilon \theta_\varepsilon \chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0.$$

Since $\|\phi_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}$, it is enough to show that

$$\|\eta_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{4.4.4}$$

where $\eta_\varepsilon = w_\varepsilon - \phi_\varepsilon$.

To this end, we first note that by the definition of $(w_\varepsilon, \tau_\varepsilon)$ in (4.2.3), the functions $(\eta_\varepsilon, \tau_\varepsilon) \in H_0^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ satisfy

$$\begin{cases} \mathcal{L}_\varepsilon(\eta_\varepsilon) + \nabla \tau_\varepsilon = -\mathcal{L}_\varepsilon(\phi_\varepsilon) & \text{in } \Omega, \\ \operatorname{div}(\eta_\varepsilon) = \varepsilon \operatorname{div}((1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0) & \text{in } \Omega, \\ \eta_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4.5}$$

additionally, we let $(\eta_0, \tau_0) \in H_0^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ be weak solution of the corresponding homogenized problem

$$\begin{cases} \mathcal{L}_0(\eta_0) + \nabla \tau_0 = -\mathcal{L}_\varepsilon(\phi_\varepsilon) & \text{in } \Omega, \\ \operatorname{div}(\eta_0) = \varepsilon \operatorname{div}((1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla \widetilde{u}_0) & \text{in } \Omega, \\ \eta_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4.6}$$

To estimate $\eta_\varepsilon - \eta_0$, we consider the following duality problems. For any $H \in L^2(\Omega; \mathbb{R}^d)$, let $(\rho_\varepsilon, \sigma_\varepsilon) \in H_0^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ be the weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon^*(\rho_\varepsilon) + \nabla \sigma_\varepsilon = H & \text{in } \Omega, \\ \operatorname{div}(\rho_\varepsilon) = 0 & \text{in } \Omega, \\ \rho_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4.7}$$

and $(\rho_0, \sigma_0) \in (H^2(\Omega; \mathbb{R}^d) \cap H_0^1(\Omega; \mathbb{R}^d)) \times H^1(\Omega)$ be the weak solution of

$$\begin{cases} \mathcal{L}_0^*(\rho_0) + \nabla \sigma_0 = H & \text{in } \Omega, \\ \operatorname{div}(\rho_0) = 0 & \text{in } \Omega, \\ \rho_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4.8}$$

with

$$\int_\Omega \sigma_\varepsilon = \int_\Omega \sigma_0 = 0.$$

Here we have used the notations: $\mathcal{L}_\varepsilon^* = -\text{div}\big(A^*(x/\varepsilon)\nabla\big)$ and $\mathcal{L}_0^* = -\text{div}\big(\widehat{A^*}\nabla\big)$ to denote the adjoint operator. We note that Lemma 4.4.1 continues to hold for $\mathcal{L}_\varepsilon^*$, as A^* satisfies the same conditions as A. Also, by the $W^{2,2}$ estimates (3.1.4) for Stokes systems with constant coefficients in $C^{1,1}$ domains,

$$\|\rho_0\|_{H^2(\Omega)} + \|\sigma_0\|_{H^1(\Omega)} \le C\, \|H\|_{L^2(\Omega)}.$$

As a result, we have

$$\begin{aligned} \|\rho_\varepsilon - \rho_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{H^1(\Omega)} &\le C\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)} \\ &\le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}, \end{aligned} \tag{4.4.9}$$

and

$$\|\sigma_\varepsilon - \sigma_0 - \Big[(1-\widetilde{\theta}_\varepsilon)\pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right) - \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\Big]\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}, \tag{4.4.10}$$

where (χ^*, π^*) denotes the correctors associated with the adjoint matrix A^*.

Let $\Psi = -\mathcal{L}_\varepsilon(\phi_\varepsilon)$, and

$$\Gamma = \text{div}(\varepsilon(1-\theta_\varepsilon)\chi^\varepsilon S_\varepsilon \nabla\widetilde{u}_0).$$

Note that by (4.4.5), (4.4.6), (4.4.7) and (4.4.8),

$$\begin{aligned} \int_\Omega H\cdot(\eta_\varepsilon - \eta_0) &= \langle \Psi, \rho_\varepsilon - \rho_0\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H_0^1(\Omega;\mathbb{R}^d)} - \int_\Omega \Gamma(\sigma_\varepsilon - \sigma_0) \\ &= J_1 + J_2. \end{aligned} \tag{4.4.11}$$

For the first term of the R.H.S. of (4.4.11), because $\Psi \in H^{-1}(\Omega;\mathbb{R}^d)$ is supported in $\widetilde{\Omega}_\varepsilon$, and $1-\widetilde{\theta}_\varepsilon = 0$ in $\widetilde{\Omega}_\varepsilon$, we obtain

$$J_1 = \langle \Psi, \rho_\varepsilon - \rho_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H_0^1(\Omega;\mathbb{R}^d)}.$$

Therefore,

$$\begin{aligned} |J_1| &\le \|\Psi\|_{H^{-1}(\Omega)}\|\rho_\varepsilon - \rho_0 - \varepsilon(1-\widetilde{\theta}_\varepsilon)\chi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{H^1(\Omega)} \\ &\le C\|\varepsilon\theta_\varepsilon\chi^\varepsilon S_\varepsilon\nabla\widetilde{u}_0\|_{H^1(\Omega)}\,\sqrt{\varepsilon}\|H\|_{L^2(\Omega)} \\ &\le C\varepsilon\|u_0\|_{H^2(\Omega)}\,\|H\|_{L^2(\Omega)} \end{aligned} \tag{4.4.12}$$

where the second inequality follows from (4.4.9), and the last inequality follows from the analog of (4.2.19) (with $\widetilde{\theta}_\varepsilon$ replaced by θ_ε). For the second term of the R.H.S. of (4.4.11), we recall that $\text{div}(\chi) = 0$. Hence,

$$\begin{aligned} \Gamma &= -\varepsilon\frac{\partial\theta_\varepsilon}{\partial x_\alpha}\chi_j^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j} + \varepsilon(1-\theta_\varepsilon)\chi_j^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j} \\ &= \Gamma_1 + \Gamma_2. \end{aligned}$$

Since $\int_\Omega \Gamma = 0$, for any constant E,

$$\begin{aligned} J_2 &= -\int_\Omega \Gamma(\sigma_\varepsilon - \sigma_0 + E) \\ &= -\int_\Omega [\Gamma_1 + \Gamma_2](\sigma_\varepsilon - \sigma_0 + E). \end{aligned}$$

We split J_2 as two integrals, for the first integral, again since $1 - \widetilde{\theta}_\varepsilon = 0$ in $\widetilde{\Omega}_\varepsilon$ and Γ_1 is supported in $\widetilde{\Omega}_\varepsilon$, just as we did for J_1,

$$-\int_\Omega \Gamma_1(\sigma_\varepsilon - \sigma_0 + E) = -\int_\Omega \Gamma_1\Big(\sigma_\varepsilon - \sigma_0 - (1-\widetilde{\theta}_\varepsilon)\pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right) + E\Big).$$

Now, if we choose the constant E as $E = \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\tilde{\rho}_0\right)$, then

$$\begin{aligned} &\Big|\int_\Omega \Gamma_1(\sigma_\varepsilon - \sigma_0 + E)\Big| \\ &\quad = \Big|\int_\Omega \Gamma_1\Big\{\sigma_\varepsilon - \sigma_0 - \Big[(1-\widetilde{\theta}_\varepsilon)\pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right) - \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\Big]\Big\}\Big| \\ &\quad \le C\|\Gamma_1\|_{L^2(\widetilde{\Omega}_\varepsilon)}\sqrt{\varepsilon}\|H\|_{L^2(\Omega)} \\ &\quad \le C\big(\sqrt{\varepsilon}\|\chi\|_{L^2(Y)}\|\nabla\widetilde{u}_0\|_{H^1(\mathbb{R}^d)}\big)\big(\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}\big) \\ &\quad \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}, \end{aligned} \tag{4.4.13}$$

where we have used (4.4.10), (4.2.2) and Lemma 2.2.3. For the second integral in J_2, we have

$$\begin{aligned} &\Big|\int_\Omega \Gamma_2\big(\sigma_\varepsilon - \sigma_0 + E\big)\Big| \\ &\qquad \le \|\Gamma_2\|_{L^2(\Omega)}\|\sigma_\varepsilon - \sigma_0 + \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{L^2(\Omega)} \\ &\qquad \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}, \end{aligned} \tag{4.4.14}$$

where for the last inequality we have used the fact

$$\begin{aligned} &\|\sigma_\varepsilon - \sigma_0 + \fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{L^2(\Omega)} \\ &\qquad \le \|\sigma_\varepsilon\|_{L^2(\Omega)} + \|\sigma_0\|_{L^2(\Omega)} + \|\fint_\Omega \pi^{*\varepsilon} S_\varepsilon\left(\nabla\widetilde{\rho}_0\right)\|_{L^2(\Omega)} \\ &\qquad \le C\|H\|_{L^2(\Omega)}. \end{aligned}$$

Therefore, by combining (4.4.12)-(4.4.14), we have proved

$$\Big|\int_\Omega H\cdot(\eta_\varepsilon - \eta_0)\Big| \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)} \quad \text{for any } H \in L^2(\Omega;\mathbb{R}^d). \tag{4.4.15}$$

By duality this implies that

$$\|\eta_\varepsilon - \eta_0\|_{L^2(\Omega)} \le C\varepsilon \|u_0\|_{H^2(\Omega)}. \tag{4.4.16}$$

Finally, the problem has been reduced to the estimate of $\|\eta_0\|_{L^2(\Omega)}$. This will be done by another duality argument. Let (ρ_0, σ_0) be defined by (4.4.8). Then we split the following estimates by three parts

$$\begin{aligned}
\left|\int_\Omega H \cdot \eta_0\right| &= \left|\langle \Psi, \rho_0\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H^1_0(\Omega;\mathbb{R}^d)} - \int_\Omega \Gamma\sigma_0\right| \\
&\le \left|\langle \Psi, \rho_0\rangle_{H^{-1}(\Omega;\mathbb{R}^d)\times H^1_0(\Omega;\mathbb{R}^d)}\right| + \left|\int_{\Omega_\varepsilon} \Gamma_1\sigma_0\right| + \left|\int_\Omega \Gamma_2\sigma_0\right| \\
&= K_1 + K_2 + K_3,
\end{aligned} \tag{4.4.17}$$

where Ψ, Γ, Γ_1 and Γ_2 are as denoted above. Notice that again by Lemma 2.2.3 and through the same argument of (4.2.19), we have

$$\begin{aligned}
K_1 &\le \|\Psi\|_{H^{-1}(\Omega)}\|\rho_0\|_{H^1(\Omega_\varepsilon)} \\
&\le C\|\varepsilon\theta_\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)}\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)} \\
&\le C\big(\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}\big)\big(\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)}\big) \\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},
\end{aligned} \tag{4.4.18}$$

where we have also used the first property (2.2.2) of Lemma 2.2.3 in the second inequality to obtain

$$\|\rho_0\|_{H^1(\Omega_\varepsilon)} \le C\sqrt{\varepsilon}\|\rho_0\|_{H^2(\Omega)}.$$

Similarly, by Lemma 2.2.3, we see that

$$\begin{aligned}
K_2 &\le \|\Gamma_1\|_{L^2(\widetilde{\Omega}_\varepsilon)}\|\sigma_0\|_{L^2(\Omega_\varepsilon)} \\
&\le C\big(\sqrt{\varepsilon}\|\chi\|_{L^2(Y)}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)}\big)\big(\sqrt{\varepsilon}\|\sigma_0\|_{H^1(\Omega))}\big) \\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},
\end{aligned} \tag{4.4.19}$$

and

$$K_3 \le \|\Gamma_2\|_{L^2(\Omega)}\|\sigma_0\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}. \tag{4.4.20}$$

By combining (4.4.17)-(4.4.20), we obtain

$$\left|\int_\Omega H \cdot \eta_0\right| \le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)},$$

which, by duality, leads to

$$\|\eta_0\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}. \tag{4.4.21}$$

Hence we have proved that

$$\begin{aligned}\|w_\varepsilon\|_{L^2(\Omega)} &\le \|\eta_\varepsilon - \eta_0\|_{L^2(\Omega)} + \|\eta_0\|_{L^2(\Omega)} + \|\phi_\varepsilon\|_{L^2(\Omega)} \\ &\le C\varepsilon\|u_0\|_{H^2(\Omega)}.\end{aligned} \tag{4.4.22}$$

Therefore the proof of Theorem 1.0.3 is now complete. □

Chapter 5 Convergence Rates of Neumann Problems in Homogenization of Stokes Systems

In this chapter we study the convergence rates in L^2 and H^1 of Neumann problems for Stokes systems with rapidly oscillating periodic coefficient, without any regularity assumptions on the coefficients.

5.1 Introduction

More precisely, we consider the following Neumann problem for Stokes systems

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F & \text{in } \Omega, \\ \operatorname{div}(u_\varepsilon) = g & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - p_\varepsilon \cdot n = h & \text{on } \partial\Omega, \end{cases}$$

with the Neumann compatibility condition

$$\int_\Omega F + \int_{\partial\Omega} h = 0.$$

By homogenization theorem (Theorem 3.4.3), we have shown that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega; \mathbb{R}^d),$$

and

$$p_\varepsilon - \fint_\Omega p_\varepsilon \rightharpoonup p_0 - \fint_\Omega p_0 \quad \text{weakly in } L^2(\Omega),$$

where $(u_0, p_0) \in H^1(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ is the weak solution of the homogenized problem

$$\begin{cases} \mathcal{L}_0(u_0) + \nabla p_0 = F & \text{in } \Omega, \\ \operatorname{div}(u_0) = g & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} - p_0 \cdot n = h & \text{on } \partial\Omega. \end{cases}$$

The main purpose of this chapter is to investigate the rate of convergence of $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$ as $\varepsilon \to 0$, which is stated in the following theorem.

Theorem 5.1.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose A satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $F \in L^2(\Omega; \mathbb{R}^d)$ and $h \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ satisfying the Neumann compatibility condition (3.4.1), for $g \in H^1(\Omega)$,* *let $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) be weak solutions of Neumann problems (3.4.4), (3.4.5), respectively. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}, \tag{5.1.1}$$

where the constant C depends only on μ, d, and Ω.

In this chapter we also obtain $O(\sqrt{\varepsilon})$ rates for a two-scale expansion of $(u_\varepsilon, p_\varepsilon)$ in $H^1 \times L^2$. Let (χ, π) denote the correctors associated with A, defined by (3.1.5), and S_ε the Steklov smoothing operator defined by (2.2.1).

Theorem 5.1.2. *Let Ω be a bounded $C^{1,1}$ domain. Suppose A satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let $(u_\varepsilon, p_\varepsilon)$ and (u_0, p_0) be the same as in Theorem (5.1.1). Then*

$$\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon S_\varepsilon \left(\nabla \widetilde{u}_0\right)\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{5.1.2}$$

where $\chi^\varepsilon(x) = \chi(x/\varepsilon)$ and $\widetilde{u}_0$ is the extension of u_0 defined as in (4.2.1). Moreover, if $\int_\Omega p_\varepsilon = \int_\Omega p_0 = 0$, then

$$\|p_\varepsilon - p_0 - \left[\pi^\varepsilon S_\varepsilon \left(\nabla \widetilde{u}_0\right) - \fint_\Omega \pi^\varepsilon S_\varepsilon \left(\nabla \widetilde{u}_0\right)\right]\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{5.1.3}$$

where $\pi^\varepsilon(x) = \pi(x/\varepsilon)$. The constants C in (5.1.2) and (5.1.3) depend only on μ, d, and Ω.

The problem of convergence rates has been playing an essential role in quantitative homogenization. Most recent work on the problem of convergence rates in periodic homogenization may be found in [26–28, 31, 32, 35, 44–46, 51, 52, 55, 56] and their references.

As we mentioned earlier, for the known results on L^2 convergence rates, there are relatively fewer results in the case of the Neumann boundary conditions than Dirichlet cases. Consider the Neumann problem for the scalar elliptic equation $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω with $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = 0$ on $\partial\Omega$, the estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|F\|_{H^2(\Omega)}$$

was proved by Griso [27] for $C^{1,1}$ domains with bounded measurable coefficients using the "periodic unfolding" method [14, 15], and it was also proved in [42] for curvilinear convex polygons Ω in $\mathbb{R}^2$. For the system case, consider elliptic systems $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω with Neumann condition $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g$ on $\partial\Omega$, C. Kenig, F. Lin and Z. Shen [32] have shown that the estimate (5.1.1) continue to hold in bounded Lipschitz domain

Ω, under additional assumption that A is Hölder continuous. The approaches used in [32] were based on the explicit computation of conormal derivative of v_ε and uniform regularity for the L^2 Neumann problem obtained in [36,37]. Here $v_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon \nabla u_0$ is the discrepancy between solution and its first order approximation. Moreover, if A is Hölder continuous and symmetric, it was proved in [32] that $\|v_\varepsilon\|_{H^{1/2}(\Omega)} \leq C\varepsilon \|u_0\|_{H^2(\Omega)}$.

However, the method used in [32] does not apply to operators with bounded measurable coefficients, since the correctors χ may not be bounded. Recently, by using the Steklov smoothing operator, T. A. Suslina [55,56] was able to establish the $O(\varepsilon)$ estimate (5.1.1) in L^2 for a broader class of elliptic operators, which, in particular contains the elliptic systems $\mathcal{L}_\varepsilon$ in divergence form with coefficients satisfying the ellipticity condition $a_{ij}^{\alpha\beta}\xi_i^\alpha \xi_j^\beta \geq \mu|\xi|^2$ for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{m\times d}$. Consider the following discrepancy between u_ε and its first order approximation $u_0 + \varepsilon\chi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right)$

$$v_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon\left(\nabla \widetilde{u}_0\right), \tag{5.1.4}$$

where S_ε is the Steklov smoothing operator at scale ε defined in (2.2.1) and $\widetilde{u}_0$ an extension of u_0 to $\mathbb{R}^d$ (also see [45, 46, 59] and their references on the use of S_ε in homogenization). Relying on the sharp convergence rates for the whole space $\mathbb{R}^d$ and the estimate on the boundary layer corrector term, the $O(\sqrt{\varepsilon})$ convergence (5.1.2) in H^1 can be obtained. The $O(\varepsilon)$ estimate in L^2 then can be deduced by applying the estimate (5.1.2) to an adjoint problem and a duality argument.

In this chapter we extend the approach of Suslina to the case of Stokes systems, which certainly do not fit the standard framework of second-order elliptic systems in divergence form. As expected in the study of Stokes or Navier-Stokes systems, the main difficulty is caused by the pressure term p_ε. In contrast to [55,56], we will use a more direct approach that does not require the convergence rates in the whole space $\mathbb{R}^d$ and thus we may avoid the estimates of boundary layer correctors. Like we did for Dirichlet problems, by carefully analyzing the systems for the correctors (χ, π) as well as their dual $(\phi_{kij}^{\alpha\beta}, q_{ij}^\beta)$, we are able to establish the $O(\sqrt{\varepsilon})$ error estimates for the two-scale expansions of $(u_\varepsilon, p_\varepsilon)$ in $H^1 \times L^2$, given in Theorem 5.1.2, delivered by the following key intermediate step.

$$\begin{aligned}
&\Big|\int_\Omega A^\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi\Big| \\
&\qquad\qquad \leq C\|u_0\|_{H^2(\Omega)}\left[\|\mathrm{div}(\varphi)\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla\varphi\|_{L^2(\Omega)}\right],
\end{aligned}$$

for any $\varphi \in H^1(\Omega;\mathbb{R}^d)$, and $\Omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$. This may allow us to apply this $O(\sqrt{\varepsilon})$ estimate to solutions of adjoint problems, and further obtain the $O(\varepsilon)$ estimate in L^2 by a duality argument.

We may now mention the potential applications of these results. Inspired by recent paper of Shen [51] on systems of linear elasticity, we expect to establish the boundary Lipschitz estimates in $C^{1,\alpha}$ domains for Neumann problems of Stokes systems with rapidly oscillating periodic coefficients, using convergence rates in H^1 and L^2 rather than using the compactness method. We may also use the result to investigate the C^α, $W^{1,p}$, and L^p estimates in C^1 domains with VMO or Hölder continuous coefficients.

5.2 Convergence rates for u_ε in H^1

From now on we will assume that Ω is a bounded domain with boundary of class $C^{1,1}$, $F \in L^2(\Omega;\mathbb{R}^d)$, $g \in H^1(\Omega)$, and $h \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$. For simplicity, we let

$$v_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0) \tag{5.2.1}$$

to denote the difference between u_ε and its first order approximation $u_0+\varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)$, where $\widetilde{u}_0$ is the extension of u_0 in $\mathbb{R}^d$ defined in (4.2.1), S_ε is the Steklov smoothing operator defined by (2.2.1). In order to show

$$\|v_\varepsilon\|_{H^1(\Omega)} \leq C\|u_0\|_{H^2(\Omega)},$$

for $1 \leq i, j, \alpha, \beta \leq d$, we once again let

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y)\frac{\partial}{\partial y_k}\left(\chi_j^{\gamma\beta}\right) - \widehat{a}_{ij}^{\alpha\beta},$$

be the 1-periodic function defined in (4.2.4), and let $\Phi_{kij}^{\alpha\beta}$, q_{ij}^{β} be the functions been introduced in Lemma (4.2.1). We will use the next lemma to show which system should v_ε satisfy.

Lemma 5.2.1. *Let Ω be a bounded $C^{1,1}$ domain. Suppose A satisfies ellipticity condition (1.0.3) and periodicity condition (1.0.4). Given $F \in L^2(\Omega;\mathbb{R}^d)$ and $f \in H^{1/2}(\partial\Omega;\mathbb{R}^d)$ satisfying the compatibility condition(3.4.1), for $g \in H^1(\Omega)$, $(u_\varepsilon, p_\varepsilon)$, (u_0, p_0) are weak solutions of Neumann problems (3.4.4) and (3.4.5), respectively. If*

v_ε *is defined as (5.2.1), then* v_ε *satisfies*

$$
\begin{cases}
(\mathcal{L}_\varepsilon(v_\varepsilon))^\alpha = -\dfrac{\partial[p_\varepsilon - p_0]}{\partial x_\alpha} + \dfrac{\partial}{\partial x_i}\left(\left[\widehat{a}^{\alpha\beta}_{ij} - a^{\alpha\beta}_{ij}(x/\varepsilon)\right]\left[S_\varepsilon \dfrac{\partial \widetilde{u}^\beta_0}{\partial x_j} - \dfrac{\partial u^\beta_0}{\partial x_j}\right]\right) \\
\qquad\qquad + \varepsilon \dfrac{\partial}{\partial x_i}\left(a^{\alpha\gamma}_{ik}(x/\varepsilon)\chi^{\gamma\beta}_j(x/\varepsilon) S_\varepsilon \dfrac{\partial^2 \widetilde{u}^\beta_0}{\partial x_k \partial x_j}\right) + \dfrac{\partial}{\partial x_i}\left(b^{\alpha\beta}_{ij}(x/\varepsilon) S_\varepsilon \dfrac{\partial \widetilde{u}^\beta_0}{\partial x_j}\right) \quad \text{in } \Omega, \\
\operatorname{div}(v_\varepsilon) = -\varepsilon \chi^{\alpha\beta}_j(x/\varepsilon) S_\varepsilon \dfrac{\partial^2 \widetilde{u}^\beta_0}{\partial x_\alpha \partial x_j} \quad \text{in } \Omega, \\
\left(\dfrac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\right)^\alpha = n_\alpha\left[p_\varepsilon - p_0\right] - n_i\left[\widehat{a}^{\alpha\beta}_{ij} - a^{\alpha\beta}_{ij}(x/\varepsilon)\right]\left[S_\varepsilon \dfrac{\partial \widetilde{u}^\beta_0}{\partial x_j} - \dfrac{\partial u^\beta_0}{\partial x_j}\right] \\
\qquad\qquad - \varepsilon n_i a^{\alpha\gamma}_{ik}(x/\varepsilon)\chi^{\gamma\beta}_j(x/\varepsilon) S_\varepsilon \dfrac{\partial^2 \widetilde{u}^\beta_0}{\partial x_k \partial x_j} - n_i b^{\alpha\beta}_{ij}(x/\varepsilon) S_\varepsilon \dfrac{\partial \widetilde{u}^\beta_0}{\partial x_j} \quad \text{on } \partial\Omega.
\end{cases}
\tag{5.2.2}
$$

Proof. Since $\operatorname{div}(u_\varepsilon) = \operatorname{div}(u_0)$, and recall that $\operatorname{div}(\chi) = 0$, hence

$$
\operatorname{div}(v_\varepsilon) = -\varepsilon \chi^{\alpha\beta}_j(x/\varepsilon) S_\varepsilon \frac{\partial^2 \widetilde{u}^\beta_0}{\partial x_\alpha \partial x_j}.
$$

Now we compute $\mathcal{L}_\varepsilon(v_\varepsilon)$, by using the definition of b,

$$
\begin{aligned}
(\mathcal{L}_\varepsilon(v_\varepsilon))^\alpha &= -\frac{\partial[p_\varepsilon - p_0]}{\partial x_\alpha} - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}^{\alpha\beta}_{ij} - a^{\alpha\beta}_{ij}(x/\varepsilon)\right]\frac{\partial u^\beta_0}{\partial x_j}\right) \\
&\quad + \frac{\partial}{\partial x_i}\left(a^{\alpha\gamma}_{ik}(x/\varepsilon)\frac{\partial}{\partial x_k}\left[\varepsilon\chi^{\gamma\beta}_j(x/\varepsilon)\right] S_\varepsilon \frac{\partial \widetilde{u}^\beta_0}{\partial x_j}\right) + \varepsilon \frac{\partial}{\partial x_i}\left(a^{\alpha\gamma}_{ik}(x/\varepsilon)\chi^{\gamma\beta}_j(x/\varepsilon) S_\varepsilon \frac{\partial^2 \tilde{u}^\beta_0}{\partial x_k \partial x_j}\right) \\
&= -\frac{\partial[p_\varepsilon - p_0]}{\partial x_\alpha} + \frac{\partial}{\partial x_i}\left(\left[\widehat{a}^{\alpha\beta}_{ij} - a^{\alpha\beta}_{ij}(x/\varepsilon)\right]\left[S_\varepsilon \frac{\partial \widetilde{u}^\beta_0}{\partial x_j} - \frac{\partial u^\beta_0}{\partial x_j}\right]\right) \\
&\quad + \varepsilon \frac{\partial}{\partial x_i}\left(a^{\alpha\gamma}_{ik}(x/\varepsilon)\chi^{\gamma\beta}_j(x/\varepsilon) S_\varepsilon \frac{\partial^2 \widetilde{u}^\beta_0}{\partial x_k \partial x_j}\right) + \frac{\partial}{\partial x_i}\left(b^{\alpha\beta}_{ij}(x/\varepsilon) S_\varepsilon \frac{\partial \widetilde{u}^\beta_0}{\partial x_j}\right).
\end{aligned}
$$

Then we deal with the conormal derivative of v_ε, similarly, we have

$$
\begin{aligned}
\left(\frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\right)^\alpha &= n_i a^{\alpha\beta}_{ij}(x/\varepsilon)\frac{\partial u^\beta_\varepsilon}{\partial x_j} - n_i \widehat{a}^{\alpha\beta}_{ij}\frac{\partial u^\beta_0}{\partial x_j} + n_i\left[\widehat{a}^{\alpha\beta}_{ij} - a^{\alpha\beta}_{ij}(x/\varepsilon)\right]\frac{\partial u^\beta_0}{\partial x_j} \\
&\quad - n_i a^{\alpha\gamma}_{ik}(x/\varepsilon)\frac{\partial}{\partial x_k}(\varepsilon\chi^{\gamma\beta}_j(x/\varepsilon)) S_\varepsilon \frac{\partial \widetilde{u}^\beta_0}{\partial x_j} - \varepsilon n_i a^{\alpha\gamma}_{ik}(x/\varepsilon)\chi^{\gamma\beta}_j(x/\varepsilon) S_\varepsilon \frac{\partial^2 \widetilde{u}^\beta_0}{\partial x_k \partial x_j} \\
&= n_\alpha\left[p_\varepsilon - p_0\right] - n_i\left[\widehat{a}^{\alpha\beta}_{ij} - a^{\alpha\beta}_{ij}(x/\varepsilon)\right]\left[S_\varepsilon \frac{\partial \widetilde{u}^\beta_0}{\partial x_j} - \frac{\partial u^\beta_0}{\partial x_j}\right] \\
&\quad - \varepsilon n_i a^{\alpha\gamma}_{ik}(x/\varepsilon)\chi^{\gamma\beta}_j(x/\varepsilon) S_\varepsilon \frac{\partial^2 \widetilde{u}^\beta_0}{\partial x_k \partial x_j} - n_i b^{\alpha\beta}_{ij}(x/\varepsilon) S_\varepsilon \frac{\partial \widetilde{u}^\beta_0}{\partial x_j}.
\end{aligned}
$$

Therefore we have given the system v_ε satisfied. $\square$

We choose the same cut-off function $\theta_\varepsilon(x)$ as in Section 3.1, i.e. it satisfy the following conditions,

$$\theta_\varepsilon \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}(\theta_\varepsilon) \subset \widetilde{\Omega}_\varepsilon, \quad 0 \le \theta_\varepsilon(x) \le 1,$$
$$\theta_\varepsilon|_{\partial\Omega} = 1, \quad |\nabla\theta_\varepsilon| \le \kappa/\varepsilon.$$

Lemma 2.2.3 and the following lemma will be served as key ingredients in the proof of the convergence rates in H^1.

Lemma 5.2.2. *Let v_ε be defined as in (5.2.1). Then, for any $\varphi \in H^1(\Omega;\mathbb{R}^d)$, we have the following estimate*

$$|a_\varepsilon(v_\varepsilon,\varphi)| \le C\|u_0\|_{H^2(\Omega)}\left[\|\operatorname{div}(\varphi)\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla\varphi\|_{L^2(\Omega)}\right], \tag{5.2.3}$$

where the constant C depends only on μ, d, and Ω.

Proof. For any $\varphi \in H^1(\Omega;\mathbb{R}^d)$, by Lemma 5.2.1 and integrating by parts,

$$\begin{aligned} a_\varepsilon(v_\varepsilon,\varphi) &= \int_\Omega a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial v_\varepsilon^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} = \int_\Omega \left(\mathcal{L}_\varepsilon(v_\varepsilon)\right)^\alpha \varphi^\alpha + \int_{\partial\Omega}\left(\frac{\partial v_\varepsilon}{\partial\nu_\varepsilon}\right)^\alpha \varphi^\alpha \\ &= \int_\Omega (p_\varepsilon - p_0)\operatorname{div}(\varphi) - \int_\Omega \left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j} - \frac{\partial u_0^\beta}{\partial x_j}\right]\frac{\partial\varphi^\alpha}{\partial x_i} \\ &\quad - \varepsilon\int_\Omega a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} - \int_\Omega b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i}. \end{aligned} \tag{5.2.4}$$

By Lemma 4.2.1, we may rewrite the last term in R.H.S. of (5.2.4) as

$$\begin{aligned} &-\int_\Omega b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} \\ &\quad = -\int_\Omega \frac{\partial}{\partial x_k}\left(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)\right)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} - \int_\Omega \frac{\partial}{\partial x_\alpha}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)\right)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} \\ &\quad = R_1 + R_2. \end{aligned} \tag{5.2.5}$$

For the first integral R_1, we integrate by parts,

$$\begin{aligned} R_1 &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_k}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i} + \varepsilon\int_\Omega \Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i} \\ &= R_{1a} + R_{1b}. \end{aligned} \tag{5.2.6}$$

The second term R_{1b} can be dealt in the same way as for the third term in (5.2.4), but we will need a more delicate way to treat R_{1a}, recall that $1 - \theta_\varepsilon \equiv 0$ on $\partial\Omega$, therefore

$$\begin{aligned}R_{1a} &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_k}\left(\big[\theta_\varepsilon + 1-\theta_\varepsilon\big]\Phi^{\alpha\beta}_{kij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i}\\ &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_k}\left(\theta_\varepsilon\Phi^{\alpha\beta}_{kij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i}\\ &\qquad + \varepsilon\int_\Omega \frac{\partial^2}{\partial x_k\partial x_i}\left((1-\theta_\varepsilon)\Phi^{\alpha\beta}_{kij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\varphi^\alpha\\ &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_k}\left(\theta_\varepsilon\Phi^{\alpha\beta}_{kij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i},\end{aligned}\tag{5.2.7}$$

where the second term in the second equality vanish since Φ is anti-symmetric. Similarly for R_2, we split the integral into two separate integrals after integrating by parts,

$$\begin{aligned}R_2 &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_\alpha}\left(q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i} + \varepsilon\int_\Omega q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_k\partial x_j}\frac{\partial\varphi^\alpha}{\partial x_i}\\ &= R_{2a}+R_{2b}.\end{aligned}\tag{5.2.8}$$

Again, it will need some effort on the term R_{2b}, recalling that $1-\theta_\varepsilon\equiv 0$ on $\partial\Omega$,

$$\begin{aligned}R_{2a} &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_\alpha}\left(\big[\theta_\varepsilon + 1-\theta_\varepsilon\big]q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i}\\ &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_\alpha}\left(\theta_\varepsilon q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i}\\ &\qquad + \varepsilon\int_\Omega \frac{\partial^2}{\partial x_\alpha\partial x_i}\left((1-\theta_\varepsilon)q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\varphi^\alpha\\ &= -\varepsilon\int_\Omega \frac{\partial}{\partial x_\alpha}\left(\theta_\varepsilon q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i}\\ &\qquad - \varepsilon\int_\Omega \frac{\partial}{\partial x_i}\left((1-\theta_\varepsilon)q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\mathrm{div}(\varphi).\end{aligned}\tag{5.2.9}$$

By remark 4.2.2, more precisely that,

$$\begin{aligned}R_{2a} &= \int_\Omega\Big[\big(\varepsilon\frac{\partial\theta_\varepsilon}{\partial x_i}q^\beta_{ij}(x/\varepsilon) - (1-\theta_\varepsilon)\pi^\beta_j(x/\varepsilon)\big)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\Big]\mathrm{div}(\varphi)\\ &\quad -\int_\Omega\varepsilon\Big[(1-\theta_\varepsilon)q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial^2\widetilde{u}_0^\beta}{\partial x_j\partial x_j}\Big]\mathrm{div}(\varphi) - \varepsilon\int_\Omega\frac{\partial}{\partial x_\alpha}\left(\theta_\varepsilon q^\beta_{ij}(x/\varepsilon)S_\varepsilon\frac{\partial\widetilde{u}_0^\beta}{\partial x_j}\right)\frac{\partial\varphi^\alpha}{\partial x_i}.\end{aligned}\tag{5.2.10}$$

Therefore, we have updated (5.2.4) as

$$a_\varepsilon(v_\varepsilon,\varphi) = I_1[\varphi] + I_2[\varphi] + I_3[\varphi], \qquad \text{for any } \varphi \text{ in } H^1(\Omega;\mathbb{R}^d), \tag{5.2.11}$$

where I_1, I_2, and I_3 are defined by

$$\begin{aligned}
I_1[\varphi] &= \int_\Omega \Big[p_\varepsilon - p_0 - \big[(1-\theta_\varepsilon)\pi^\varepsilon - \varepsilon\nabla\theta_\varepsilon q^\varepsilon\big] S_\varepsilon(\nabla\widetilde{u}_0) - \varepsilon(1-\theta_\varepsilon) q^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\Big]\mathrm{div}(\varphi);\\
I_2[\varphi] &= \varepsilon\int_\Omega \Big[q^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\Big]\cdot\nabla\varphi + \varepsilon\int_\Omega\Big[(\Phi^\varepsilon - A^\varepsilon\chi^\varepsilon)S_\varepsilon(\nabla^2\widetilde{u}_0)\Big]\cdot\nabla\varphi\\
&\quad - \int_\Omega\Big[(\widehat{A}-A^\varepsilon)\big(S_\varepsilon(\nabla\widetilde{u}_0) - \nabla u_0\big)\Big]\cdot\nabla\varphi;\\
I_3[\varphi] &= -\int_\Omega\Big[\nabla\big(\varepsilon\theta_\varepsilon\Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\big)\Big]\cdot\nabla\varphi - \int_\Omega\Big[\nabla\big(\varepsilon\theta_\varepsilon q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\big)\Big]\cdot\nabla\varphi.
\end{aligned}\tag{5.2.12}$$

For the estimate on I_1, by the energy estimate (3.4.2) and Proposition 2.2.2, we use Hölder's inequality to obtain

$$|I_1[\varphi]| \le C\|u_0\|_{H^2(\Omega)}\|\mathrm{div}(\varphi)\|_{L^2(\Omega)}. \tag{5.2.13}$$

Also, by Proposition 2.2.1-2.2.2, (4.2.2), and Hölder's inequality, we can obtain that

$$\begin{aligned}
|I_2[\varphi]| &\le C\varepsilon\left(\big[\|\chi\|_{L^2(Y)} + \|\Phi\|_{L^2(Y)} + \|q\|_{L^2(Y)} + 1\big]\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)}\right)\|\nabla\varphi\|_{L^2(\Omega)}\\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}.
\end{aligned}\tag{5.2.14}$$

For I_3, since supp $\theta_\varepsilon \subset \widetilde{\Omega}_\varepsilon$ and again by Hölder inequality, we have

$$\begin{aligned}
|I_3[\varphi]| &\le C\left(\|\varepsilon\theta_\varepsilon q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} + \|\varepsilon\theta_\varepsilon\Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)}\right)\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}\\
&\le C(M_1+M_2)\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)}
\end{aligned}\tag{5.2.15}$$

The term M_1 and M_2 are treated the same way as we did in (4.2.19) for Dirichlet problem, indeed for M_1,

$$\begin{aligned}
M_1 = \varepsilon\|\theta_\varepsilon q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} &\le C\varepsilon\Big\{\|q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|(\nabla\theta_\varepsilon)q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)}\\
&\qquad + \varepsilon^{-1}\|\theta_\varepsilon(\nabla q)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|q^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}\Big\}\\
&\le C\varepsilon\left\{\|q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|q^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}\right\}\\
&\qquad + C\left\{\|q^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\widetilde{\Omega}_{2\varepsilon})} + \|(\nabla q)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\widetilde{\Omega}_{2\varepsilon})}\right\}\\
&\le C\varepsilon\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} + C\sqrt{\varepsilon}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)}\\
&\le C\sqrt{\varepsilon}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)},
\end{aligned}\tag{5.2.16}$$

where we have used Proposition 2.2.2 and Lemma 2.2.3 for the second last inequality. We can treat M_2 as the same manner,

$$
\begin{aligned}
M_2 = \varepsilon\|\theta_\varepsilon \Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{H^1(\Omega)} &\leq C\varepsilon\Big\{\|\Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|(\nabla\theta_\varepsilon)\Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} \\
&\qquad + \varepsilon^{-1}\|\theta_\varepsilon(\nabla\Phi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|\Phi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}\Big\} \\
&\leq C\varepsilon\left\{\|\Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\Omega)} + \|\Phi^\varepsilon S_\varepsilon(\nabla^2\widetilde{u}_0)\|_{L^2(\Omega)}\right\} \\
&\qquad + C\left\{\|\Phi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\widetilde{\Omega}_{2\varepsilon})} + \|(\nabla\Phi)^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)\|_{L^2(\widetilde{\Omega}_{2\varepsilon})}\right\} \\
&\leq C\sqrt{\varepsilon}\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)},
\end{aligned}
\tag{5.2.17}
$$

Substituting (5.2.16)-(5.2.17 into (5.2.15), by (4.2.2), we have proved that

$$
|I_3[\varphi]| \leq C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)} \tag{5.2.18}
$$

Therefore, by gathering (5.2.13), (5.2.14) and (5.2.18) together, we have proved

$$
|a_\varepsilon(v_\varepsilon,\varphi)| \leq C\|u_0\|_{H^2(\Omega)}\left[\|\mathrm{div}(\varphi)\|_{L^2(\Omega)} + \varepsilon^{1/2}\|\nabla\varphi\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla\varphi\|_{L^2(\Omega)}\right],
$$

where C depends only on d, μ, and Ω. □

Proof of estimate (5.1.2). We will now prove (5.1.2) by energy estimates. Noticing that by Lemma 5.2.1 and (4.2.2),

$$
\|\mathrm{div}(v_\varepsilon)\|_{L^2(\Omega)} \leq C\varepsilon\|u_0\|_{H^2(\Omega)} \tag{5.2.19}
$$

If we choose $\varphi = v_\varepsilon$ in Lemma 5.2.1, therefore by ellipticity condition (1.0.3), (5.2.3) and (5.2.19), we see that

$$
\|\nabla v_\varepsilon\|^2_{L^2(\Omega)} \leq C\|u_0\|_{H^2(\Omega)}\left[\varepsilon^{1/2}\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega)}\right]
$$

This implies the desired convergence rate in H^1,

$$
\|v_\varepsilon\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}, \tag{5.2.20}
$$

where C depends only on d, μ, and Ω. □

5.3 Convergence rates of the pressure term

Before proving (5.1.3), once again we recall that if $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ is a weak solution of any Stokes system (1.0.2), then

$$
\begin{aligned}
\|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^2(\Omega)} &\leq \|\nabla p_\varepsilon\|_{H^{-1}(\Omega)} \\
&\leq C\left\{\|F\|_{H^{-1}(\Omega)} + \|u_\varepsilon\|_{H^1(\Omega)}\right\},
\end{aligned}
$$

where C depends only on d, μ, and Ω.

Proof of estimate (5.1.3). To prove the desired estimates, we need a more precise calculation of the pressure term associated with v_ε in system (5.2.2). Recalling from Lemma 5.2.1 that

$$
\begin{aligned}
(\mathcal{L}_\varepsilon(v_\varepsilon))^\alpha = & -\frac{\partial[p_\varepsilon - p_0]}{\partial x_\alpha} + \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j} - \frac{\partial u_0^\beta}{\partial x_j}\right]\right) \\
& + \varepsilon\frac{\partial}{\partial x_i}\left(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_k\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) \quad \text{in } \Omega,
\end{aligned}
\tag{5.3.1}
$$

Using Lemma 4.2.1, we may rewrite the last term on the R.H.S. of (5.3.1) as

$$
\begin{aligned}
\frac{\partial}{\partial x_i}\left(b_{ij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) &= \frac{\partial}{\partial x_i}\left(\left[\frac{\partial}{\partial x_k}\Big(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)\Big) + \frac{\partial}{\partial x_\alpha}\Big(\varepsilon q_{ij}^\beta(x/\varepsilon)\Big)\right]S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) \\
&= N_1 + N_2.
\end{aligned}
\tag{5.3.2}
$$

Since $\Phi_{kij}^{\alpha\beta} = -\Phi_{ikj}^{\alpha\beta}$, we see that

$$
\begin{aligned}
N_1 &= \frac{\partial^2}{\partial x_i\partial x_k}\left(\varepsilon\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial \widetilde{u}_0{}^\beta}{\partial x_j}\right) - \varepsilon\frac{\partial}{\partial x_i}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right) \\
&= -\varepsilon\frac{\partial}{\partial x_i}\left(\Phi_{kij}^{\alpha\beta}(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right).
\end{aligned}
$$

For the second term in the R.H.S. of (5.3.2), we have

$$
\begin{aligned}
N_2 &= \frac{\partial}{\partial x_\alpha}\left(\frac{\partial}{\partial x_i}\left[\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right]\right) - \frac{\partial}{\partial x_i}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right) \\
&= N_3 - \frac{\partial}{\partial x_i}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right).
\end{aligned}
\tag{5.3.3}
$$

In view of (4.2.10), for the first term on the R.H.S. of (5.3.3), we obtain

$$
N_3 = \frac{\partial}{\partial x_\alpha}\left(\pi_j^\beta(x/\varepsilon)S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right) + \frac{\partial}{\partial x_\alpha}\left(\varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j\partial x_i}\right).
$$

Substituting $N_1 - N_3$ into (5.3.1), we see that

$$
\begin{aligned}
&(\mathcal{L}_\varepsilon(v_\varepsilon))^\alpha + \frac{\partial}{\partial x_\alpha}\left(p_\varepsilon - p_0 - \pi_j^\beta(x/\varepsilon)S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j} - \varepsilon q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j\partial x_i}\right) \\
&= \varepsilon\frac{\partial}{\partial x_i}\left(\left[a_{ij}^{\alpha\gamma}(x/\varepsilon)\chi_k^{\gamma\beta}(x/\varepsilon) - \Phi_{kij}^{\alpha\beta}(x/\varepsilon)\right]S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_j\partial x_k}\right) \\
&\qquad - \varepsilon\frac{\partial}{\partial x_i}\left(q_{ij}^\beta(x/\varepsilon)S_\varepsilon\frac{\partial^2 \widetilde{u}_0^\beta}{\partial x_\alpha\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\left[\widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon)\right]\left[\frac{\partial u_0^\beta}{\partial x_j} - S_\varepsilon\frac{\partial \widetilde{u}_0^\beta}{\partial x_j}\right]\right).
\end{aligned}
\tag{5.3.4}
$$

By applying (4.3.1) to system (5.3.4), and recall that $\int_\Omega p_\varepsilon = \int_\Omega p_0 = 0$, we see that

$$\begin{aligned}
&\Big\| \Big[p_\varepsilon - p_0 - \pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) - \varepsilon q^\varepsilon S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big] + \fint_\Omega \Big[\pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) + \varepsilon q^\varepsilon S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big] \Big\|_{L^2(\Omega)} \\
&\le \Big\| \nabla \Big[p_\varepsilon - p_0 - \pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) - \varepsilon q^\varepsilon S_\varepsilon(\nabla^2 \widetilde{u}_0)\Big] \Big\|_{H^{-1}(\Omega)} \\
&\le C\|\nabla v_\varepsilon\|_{L^2(\Omega)} + C\varepsilon\Big[\|\chi\|_{L^2(Y)} + \|\Phi\|_{L^2(Y)} + \|q\|_{L^2(Y)} + 1\Big]\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \\
&\le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)},
\end{aligned} \tag{5.3.5}$$

where we have used Proposition 2.2.2-2.2.1 for the second last inequality, while (5.2.20) and (4.2.2) for the last, and the constant C is independent of ε. Also, by Proposition 2.2.2 and (4.2.2), we see that

$$\begin{aligned}
\varepsilon\|q^\varepsilon S_\varepsilon(\nabla^2 \widetilde{u}_0) - \fint q^\varepsilon S_\varepsilon(\nabla^2 \widetilde{u}_0)\|_{L^2(\Omega)} &\le C\varepsilon\|\widetilde{u}_0\|_{H^2(\mathbb{R}^d)} \\
&\le C\varepsilon\|u_0\|_{H^2(\Omega)}.
\end{aligned} \tag{5.3.6}$$

By combining (5.3.5) and (5.3.6), we have proved the convergence rate of pressure term

$$\|p_\varepsilon - p_0 - \Big[\pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0) - \fint_\Omega \pi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0)\Big]\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}\|u_0\|_{H^2(\Omega)}.$$

□

5.4 Convergence rates for u_ε in L^2

Now we assume that $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$. To establish the sharp $O(\varepsilon)$ rate for u_ε in L^2, noticing that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le \|v_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\chi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0)\|_{L^2(\Omega)}. \tag{5.4.1}$$

Using Proposition 2.2.2 and (4.2.2), we observe that

$$\begin{aligned}
\|\chi^\varepsilon S_\varepsilon(\nabla \widetilde{u}_0)\|_{L^2(\Omega)} &\le \|\chi\|_{L^2(Y)}\|\nabla \widetilde{u}_0\|_{L^2(\mathbb{R}^d)} \\
&\le C\|u_0\|_{H^2(\Omega)}.
\end{aligned}$$

Thus, (5.4.1) has been reduced to prove

$$\|v_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}, \tag{5.4.2}$$

for which we will use the duality argument.

Proof of Theorem (1.0.4). We consider the following duality problems, for any $H \in L^2(\Omega;\mathbb{R}^d)$, let $(\varphi_\varepsilon, \sigma_\varepsilon) \in H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$ be the weak solution of the following Neumann problem of Stokes system

$$\begin{cases} \mathcal{L}_\varepsilon^*(\varphi_\varepsilon) + \nabla\sigma_\varepsilon = H - \fint_\Omega H & \text{in } \Omega, \\ \operatorname{div}(\varphi_\varepsilon) = 0 & \text{in } \Omega, \\ \left(\dfrac{\partial\varphi_\varepsilon}{\partial\nu_\varepsilon}\right)^* - \sigma_\varepsilon \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4.3}$$

and let $(\varphi_0, \sigma_0) \in H^2(\Omega;\mathbb{R}^d) \times H^1(\Omega)$ be the weak solution of the corresponding homogenized problem

$$\begin{cases} \mathcal{L}_0^*(\varphi_0) + \nabla\sigma_0 = H - \fint_\Omega H & \text{in } \Omega, \\ \operatorname{div}(\varphi_0) = 0 & \text{in } \Omega, \\ \left(\dfrac{\partial\varphi_0}{\partial\nu_0}\right)^* - \sigma_0 \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4.4}$$

with

$$\int_\Omega \sigma_\varepsilon = \int_\Omega \sigma_0 = 0.$$

Here we have used the notations: $\mathcal{L}_\varepsilon^* = -\operatorname{div}\big(A^*(x/\varepsilon)\nabla\big)$ and $\mathcal{L}_0^* = -\operatorname{div}\big(\widehat{A^*}\nabla\big)$ to denote the adjoint operators. We note that Theorem 5.1.2 continues to hold for $\mathcal{L}_\varepsilon^*$, as A^* satisfies the same conditions as A. Also, by the $W^{2,2}$ estimates (3.4.3) for Stokes systems with constant coefficients in $C^{1,1}$ domains, we have

$$\|\varphi_0\|_{H^2(\Omega)} + \|\sigma_0\|_{H^1(\Omega)} \le C\|H\|_{L^2(\Omega)}.$$

In other words, we have obtained

$$\begin{aligned} \|\varphi_\varepsilon - \varphi_0 - \varepsilon\chi^{*\varepsilon} S_\varepsilon(\nabla\widetilde{\varphi}_0)\|_{H^1(\Omega)} &\le C\sqrt{\varepsilon}\|\varphi_0\|_{H^2(\Omega)} \\ &\le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}. \end{aligned} \tag{5.4.5}$$

where (χ^*, π^*) denotes the correctors associated with adjoint matrix A^*. Therefore through dual pairing, and integrating by parts, we get

$$\begin{aligned} \int_\Omega v_\varepsilon \cdot \Big(H - \fint_\Omega H\Big) &= \langle \mathcal{L}_\varepsilon^*(\varphi_\varepsilon), v_\varepsilon\rangle_{H_0^{-1}(\Omega;\mathbb{R}^d)\times H^1(\Omega;\mathbb{R}^d)} + \int_\Omega v_\varepsilon \cdot \nabla\sigma_\varepsilon \\ &= a_\varepsilon(v_\varepsilon, \varphi_\varepsilon) - \int_\Omega \sigma_\varepsilon \operatorname{div}(v_\varepsilon). \end{aligned} \tag{5.4.6}$$

By (3.4.2) and (5.2.19), we know that

$$\begin{aligned}\left|\int_\Omega \sigma_\varepsilon \ \mathrm{div}(v_\varepsilon)\right| &\le C\|\sigma_\varepsilon\|_{L^2(\Omega)}\|\mathrm{div}(v_\varepsilon)\|_{L^2(\Omega)}\\ &\le C\|H\|_{L^2(\Omega)}(\varepsilon\|u_0\|_{H^2(\Omega)})\\ &\le C\varepsilon\|H\|_{L^2(\Omega)}\|u_0\|_{H^2(\Omega)}.\end{aligned} \tag{5.4.7}$$

By choosing $\varphi=\varphi_\varepsilon$ in (5.2.11), we get

$$a_\varepsilon(v_\varepsilon,\varphi_\varepsilon)=I_1[\varphi_\varepsilon]+I_2[\varphi_\varepsilon]+I_3[\varphi_\varepsilon]$$

where I_1, I_2 and I_3 are defined in (5.2.12). Since $\mathrm{div}(\varphi_\varepsilon)=0$, then by Lemma 5.2.2,

$$\begin{aligned}|a_\varepsilon(v_\varepsilon,\varphi_\varepsilon)| &\le C\|u_0\|_{H^2(\Omega)}\left[\varepsilon^{1/2}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}+\varepsilon\|\nabla\varphi_\varepsilon\|_{L^2(\Omega)}\right]\\ &\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)}+C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)},\end{aligned} \tag{5.4.8}$$

where we have used (3.4.2) for the last inequality and C is independent of ε. Simply by triangle inequality, we break the latter term on the R.H.S. of (5.4.8) as

$$\begin{aligned}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le\ &\|\nabla\big(\varphi_\varepsilon-\varphi_0-\varepsilon\chi^{*\varepsilon}S_\varepsilon(\nabla\widetilde{\varphi}_0)\big)\|_{L^2(\Omega_\varepsilon)}\\ &+\|\nabla\varphi_0\|_{L^2(\Omega_\varepsilon)}+\|\nabla\big(\varepsilon\chi^{*\varepsilon}S_\varepsilon(\nabla\widetilde{\varphi}_0)\big)\|_{L^2(\Omega_\varepsilon)}.\end{aligned} \tag{5.4.9}$$

Directly deriving from (5.4.5), we know that

$$\|\nabla\big(\varphi_\varepsilon-\varphi_0-\varepsilon\chi^{*\varepsilon}S_\varepsilon(\nabla\widetilde{\varphi}_0)\big)\|_{L^2(\Omega_\varepsilon)}\le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}. \tag{5.4.10}$$

By using Lemma 2.2.3 and (4.2.2) again, we get

$$\begin{aligned}\|\nabla\varphi_0\|_{L^2(\Omega_\varepsilon)} &\le C\left(\varepsilon\|\nabla\varphi_0\|_{H^1(\Omega)}\|\nabla\varphi_0\|_{L^2(\Omega)}\right)^{1/2}\\ &\le C\sqrt{\varepsilon}\|\varphi_0\|_{H^2(\Omega)}\\ &\le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}.\end{aligned} \tag{5.4.11}$$

By the same argument as in (4.2.19), by Lemma 2.2.3, Proposition 2.2.2, 3.4.2 and (4.2.2),

$$\begin{aligned}\|\nabla\big(\varepsilon\chi^{*\varepsilon}S_\varepsilon(\nabla\widetilde{\varphi}_0)\big)\|_{L^2(\Omega_\varepsilon)} &\le C\left\{\|(\nabla\chi^*)^\varepsilon S_\varepsilon(\widetilde{\varphi}_0)\|_{L^2(\widetilde{\Omega}_\varepsilon)}+\varepsilon\|\widetilde{\varphi}_0\|_{H^2(\mathbb{R}^d)}\right\}\\ &\le C\sqrt{\varepsilon}\|\varphi_0\|_{H^2(\Omega)}\\ &\le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}\end{aligned} \tag{5.4.12}$$

Substituting (5.4.10), (5.4.11) and (5.4.12) into (5.4.9), we have proved that

$$\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}\le C\sqrt{\varepsilon}\|H\|_{L^2(\Omega)}. \tag{5.4.13}$$

Therefore,

$$
\begin{aligned}
|a_\varepsilon(v_\varepsilon,\varphi_\varepsilon)| &\le C\varepsilon\|u_0\|_{H^2(\Omega)}\|H\|_{L^2(\Omega)} + C\varepsilon^{1/2}\|u_0\|_{H^2(\Omega)}(\varepsilon^{1/2}\|H\|_{L^2(\Omega)}) \\
&\le C\varepsilon\|H\|_{L^2(\Omega)}\|u_0\|_{H^2(\Omega)}.
\end{aligned} \tag{5.4.14}
$$

Hence, by using (5.4.14) and (5.4.7), we already proved that for any $H \in L^2(\Omega;\mathbb{R}^d)$

$$
\begin{aligned}
\Big|\int_\Omega v_\varepsilon\cdot\big(H - \fint_\Omega H\big)\Big| &\le |a_\varepsilon(v_\varepsilon,\varphi_\varepsilon)| + \Big|\int_\Omega \sigma_\varepsilon\ \mathrm{div}(\varphi_\varepsilon)\Big| \\
&\le C\varepsilon\|H\|_{L^2(\Omega)}\|u_0\|_{H^2(\Omega)}.
\end{aligned} \tag{5.4.15}
$$

Since we assume $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$,

$$
\begin{aligned}
\Big|\int_\Omega v_\varepsilon\cdot H\Big| &\le \Big|\int_\Omega v_\varepsilon\cdot\big(H - \fint_\Omega H)\Big| + \Big|\int_\Omega v_\varepsilon(\fint_\Omega H)\Big| \\
&\le C\varepsilon\|H\|_{L^2(\Omega)}\|u_0\|_{H^2(\Omega)} + C\Big|\int_\Omega \varepsilon\chi^\varepsilon S_\varepsilon(\nabla\widetilde{u}_0)(\fint_\Omega H)\Big| \\
&\le C\varepsilon\|H\|_{L^2(\Omega)}\|u_0\|_{H^2(\Omega)}
\end{aligned} \tag{5.4.16}
$$

where we have used (5.4.15) for the second last inequality and Proposition 2.2.2 for the last, for any $H \in L^2(\Omega;\mathbb{R}^d)$. By duality argument, this implies

$$
\|v_\varepsilon\|_{L^2(\Omega)} \le C\varepsilon\|u_0\|_{H^2(\Omega)}.
$$

where C is independent of ε. Therefore we have complete the proof. □

Chapter 6 Uniform Regularity Estimates in Homogenization of Stokes Systems

In this chapter, we study uniform regularity estimates for a family of Stokes systems with rapidly oscillating periodic coefficients. We establish interior Lipschitz estimates for the velocity and L^∞ estimates for the pressure as well as Liouville property for solutions in $\mathbb{R}^d$. We also obtain the boundary $W^{1,p}$ estimates in a bounded C^1 domain for any $1 < p < \infty$.

6.1 Introduction

More precisely, we consider the Stokes systems in fluid dynamics,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \qquad\quad \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in a bounded domain Ω in $\mathbb{R}^d$. One of our main purpose of this chapter is to prove the following theorem.

Theorem 6.1.1. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of the Stokes system (1.0.2) in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > \varepsilon$. Then, for any $\varepsilon \le r < R$,*

$$\begin{aligned} &\left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(x_0,r)} |p_\varepsilon - \fint_{B(x_0,R)} p_\varepsilon|^2 \right)^{1/2} \\ &\le C \Bigg\{ \left(\fint_{B(x_0,R)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \|g\|_{L^\infty(B(x_0,R))} + R^\rho [g]_{C^{0,\rho}(B(x_0,R))} \\ &\qquad\qquad + CR \left(\fint_{B(x_0,R)} |F|^q \right)^{1/q} \Bigg\}, \end{aligned} \tag{6.1.1}$$

where $0 < \rho = 1 - \frac{d}{q} < 1$, and the constant C depends only on d, μ, and ρ.

The scaling-invariant estimate (6.1.1) should be regarded as a Lipschitz estimate for the velocity u_ε and L^∞ estimate for the pressure p_ε down to the microscopic scale ε, even though no smoothness assumption is made on the coefficient matrix $A(y)$. Indeed, if estimate (6.1.1) holds for any $0 < r < R$, we would be able to bound

$$|\nabla u_\varepsilon(x_0)| + |p_\varepsilon(x_0) - \fint_{B(x_0,R)} p_\varepsilon|$$

by the right hand side of (6.1.1). Here we have taken a point of view that solution should behave much better on mesoscopic scales due to homogenization and that the smoothness of coefficients only effects the solutions below the microscopic scale (see this viewpoint in the recent development on quantitative stochastic homogenization in [2, 25] and their references). In fact, under the additional assumption that $A(y)$ is Hölder continuous,

$$|A(x) - A(y)| \leq \tau |x - y|^\lambda \qquad \text{for } x, y \in \mathbb{R}^d,$$

where $\tau \geq 0$ and $\lambda \in (0, 1]$, we may deduce the full uniform Lipschitz estimate for u_ε and L^∞ estimate for p_ε from Theorem 6.1.1, by a blow-up argument.

Corollary 6.1.2. *Suppose that $A(y)$ satisfies ellipticity (1.0.3), periodicity (1.0.4) and Hölder continuity (2.1.2) conditions. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of (1.0.2) in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 0$. Then*

$$\begin{aligned}
&\|\nabla u_\varepsilon\|_{L^\infty(B(x_0,R/2))} + \|p_\varepsilon - \fint_{B(x_0,R)} p_\varepsilon\|_{L^\infty(B(x_0,R/2))} \\
&\leq C\Bigg\{\bigg(\fint_{B(x_0,R)} |\nabla u_\varepsilon|^2\bigg)^{1/2} + \|g\|_{L^\infty(B(x_0,R))} + R^\rho [g]_{C^{0,\rho}(B(x_0,R))} \\
&\qquad\qquad + CR\bigg(\fint_{B(x_0,R)} |F|^q\bigg)^{1/q}\Bigg\},
\end{aligned} \tag{6.1.2}$$

where $0 < \rho = 1 - \frac{d}{q} < 1$, and the constant C depends only on d, μ, λ, τ and ρ.

As we mentioned in the Chapter 1, for the standard second-order elliptic system $\mathcal{L}_\varepsilon(u_\varepsilon) = F$, uniform interior Lipschitz estimates as well as uniform boundary Lipschitz estimates with Dirichlet conditions in $C^{1,\alpha}$ domains, were established by M. Avellaneda and F. Lin in [3], under conditions (1.0.3), (1.0.4) and (2.1.2). Under the additional symmetry condition $A^* = A$, the boundary Lipschitz estimates with Neumann boundary conditions in $C^{1,\alpha}$ domains were obtained by C. Kenig, F. Lin and Z. Shen in [34]. This symmetry condition was recently removed by S. N. Armstrong and Z. Shen in [1], where the uniform Lipschitz estimates were studied for second-order elliptic systems in divergence form with almost-periodic coefficients.

The proof of Theorem 6.1.1, uses a compactness argument, which was introduced to the study of homogenization problems by M. Avellaneda and F. Lin [3, 6]. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of the Stokes system (1.0.2) in $B(0, 1)$. Suppose that

$$\max\left\{\bigg(\fint_{B(0,1)} |u_\varepsilon|^2\bigg)^{1/2}, \bigg(\fint_{B(0,1)} |F|^q\bigg)^{1/q}, \|g\|_{C^\rho(B(0,1))}\right\} \leq 1$$

where $\rho = 1 - \frac{d}{q}$. By the compactness argument with an iteration procedure, which is more or less the L^2 version of the compactness method used in [3], we are able to show that if $0 < \varepsilon < \theta^{\ell-1}\varepsilon_0$ for some $\ell \geq 1$, then

$$\left(\fint_{B(0,\theta^\ell)} |u_\varepsilon - (P_j^\beta(x) + \varepsilon\chi_j^\beta(x/\varepsilon))E_j^\beta(\varepsilon,\ell) - G(\varepsilon,\ell)|^2 dx \right)^{1/2} \leq \theta^{\ell(1+\sigma)} \tag{6.1.3}$$

where $0 < \sigma < \rho$, and $E_j^\beta(\varepsilon,\ell)$, $G(\varepsilon,\ell)$ are constants satisfying

$$|E_j^\beta(\varepsilon,\ell)| + |G(\varepsilon,\ell)| \leq C.$$

In (6.1.3), $P_j^\beta(y) = y_j(0,\cdots,1,\cdots)$ with 1 in the β^{th} position and $\chi = (\chi_j^\beta(y))$ is the so-called corrector associated with the Stokes system (1.0.2). We remark that estimate (6.1.3) may be regarded as a $C^{1,\sigma}$ estimate for u_ε in scales larger than ε. This estimate allow us to deduce the Lipschitz estimate for the velocity u_ε down to the scale ε. Moreover, by carefully analyzing the error terms in the asymptotic expansion of p_ε, the estimate (6.1.3) also allows us to bound

$$\left| \fint_{B(x_0,r)} p_\varepsilon - \fint_{B(x_0,R)} p_\varepsilon \right|$$

and to derive the L^∞ estimate for the pressure p_ε, one of the main novelties of this thesis. We should point out that p_ε is related to ∇u_ε by singular integrals that are not bounded on L^∞; Lipschitz estimate for u_ε in general do not imply L^∞ estimates for p_ε. Also, observe that the L^2 formulation in (6.1.3) appears to be necessary, as the correctors are not necessarily bounded without smoothness conditions on A. We further note that as a consequence of (6.1.3), we establish a Liouville property for Stokes systems with periodic coefficients.

In this chapter we also study the uniform boundary regularity estimates for Stokes system (1.0.2) in C^1 domains. The following theorem, whose proof is given in Section 6, may be regarded as a boundary Hölder estimate for u_ε down to the scale ε. We emphasize that as in the case of Theorem 6.1.1, no smoothness assumption on A is required for Theorem 6.1.3.

Theorem 6.1.3. *Suppose that $A(y)$ satisfies ellipticity (1.0.3) and periodicity (1.0.4) conditions. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$. Let $x_0 \in \partial\Omega$ and $0 < R < R_0$, where $R_0 = \operatorname{diam}(\Omega)$. Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0 & \text{in } B(x_0,R)\cap\Omega, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } B(x_0,R)\cap\Omega, \\ u_\varepsilon = 0 & \text{on } B(x_0,R)\cap\partial\Omega. \end{cases} \tag{6.1.4}$$

Suppose that $0 < \varepsilon \le R$ and $0 < R < 1$. Then

$$\left(\fint_{B(x_0,r)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2} \le C_\rho \left(\frac{r}{R}\right)^{\rho-1} \left(\fint_{B(x_0,R)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6.1.5}$$

where C_ρ depends only on d, μ, ρ, and Ω.

Theorem 6.1.3 is also proved by compactness method, though correctors are not needed here. The scaling-invariant boundary estimate (6.1.5), combined with the interior estimates in Theorem 6.1.1, allows us to establish the boundary $W^{1,p}$ estimates for Stokes systems with VMO coefficients in C^1 domains.

Let $B^{\alpha,q}(\partial\Omega;\mathbb{R}^d)$ denote the Besov space of $\mathbb{R}^d$-valued functions on $\partial\Omega$ of order $\alpha \in (0,1)$ with exponent $q \in (1,\infty)$. It is known that if $u \in W^{1,q}(\Omega;\mathbb{R}^d)$ for some $1 < q < \infty$, where Ω is a bounded Lipschitz domain, then $u|_{\partial\Omega} \in B^{1-\frac{1}{q},q}(\partial\Omega;\mathbb{R}^d)$.

Theorem 6.1.4. *Let Ω be a bounded C^1 domain in $\mathbb{R}^d$ and $1 < q < \infty$. Suppose that A satisfies ellipticity (1.0.3) and periodicity (1.0.4) conditions. Also assume that $A \in \mathrm{VMO}(\mathbb{R}^d)$. Let $f = (f_i^\alpha) \in L^q(\Omega;\mathbb{R}^{d\times d})$, $g \in L^q(\Omega)$ and $h \in B^{1-\frac{1}{q},q}(\partial\Omega;\mathbb{R}^d)$ satisfy the compatibility condition*

$$\int_\Omega g - \int_{\partial\Omega} h\cdot n = 0,$$

where n denotes the outward unit normal to $\partial\Omega$. Then the solutions $(u_\varepsilon, p_\varepsilon)$ in $W^{1,q}(\Omega;\mathbb{R}^d)\times L^q(\Omega)$ to Dirichlet problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = \operatorname{div}(f) & \text{in } \Omega, \\ \operatorname{div}(u_\varepsilon) = g & \text{in } \Omega, \\ u_\varepsilon = h & \text{on } \partial\Omega, \end{cases} \tag{6.1.6}$$

satisfy the estimate

$$\|\nabla u_\varepsilon\|_{L^q(\Omega)} + \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^q(\Omega)} \le C_q \left\{ \|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)} + \|h\|_{B^{1-\frac{1}{q},q}(\partial\Omega)} \right\}, \tag{6.1.7}$$

where C_ρ depends only on d, μ, A, and Ω.

We mention that $W^{1,p}$ estimates for elliptic and parabolic equations with continuous or VMO coefficients have been studied extensively in recent years. We refer the reader to [11–13,21,38,48] as well as their references for work on elliptic equations and systems, and to [3,7,13,22,23,34,50] for uniform $W^{1,p}$ estimates in homogenization.

6.2 Interior Lipschitz estimates for u_ε

For a ball $B = B(x_0, r) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$ in $\mathbb{R}^d$, we will use tB to denote $B(x_0, tr)$, the ball with the same center and t times the radius of B.

We start with a Cacciopoli's inequality for the Stokes system, whose proof may be found in [24].

Theorem 6.2.1. *Let $(u_\varepsilon, p_\varepsilon) \in (H^1(2B; \mathbb{R}^d) \times L^2(2B))$ be a weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F + \operatorname{div}(f), \\ \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in $2B$, where $B = B(x_0, r)$, $F \in L^2(2B, \mathbb{R}^d)$ and $f \in L^2(2B; \mathbb{R}^{d\times d})$. Then

$$\begin{aligned} &\int_B |\nabla u_\varepsilon|^2 + \int_B |p_\varepsilon - \fint_B p_\varepsilon|^2 \\ &\qquad \le C \left\{ \frac{1}{r^2} \int_{2B} |u_\varepsilon|^2 + \int_{2B} |f|^2 + \int_{2B} |g|^2 + r^2 \int_{2B} |F|^2 \right\} \end{aligned} \tag{6.2.1}$$

where C depends only on d and μ.

Lemma 6.2.2. *Let $0 < \sigma < \rho < 1$ and $\rho = 1 - \frac{d}{q}$. Then there exist $\varepsilon_0 \in (0, 1/2)$ and $\theta \in (0, 1/4)$, depending only on d, μ, σ and ρ, such that*

$$\begin{aligned} &\left(\fint_{B(0,\theta)} \left| u_\varepsilon - \fint_{B(0,\theta)} u_\varepsilon - \left(P_j^\beta(x) + \varepsilon \chi_j^\beta(x/\varepsilon) \right) \fint_{B(0,\theta)} \frac{\partial u_\varepsilon^\beta}{\partial x_j} \right|^2 dx \right)^{1/2} \\ &\le \theta^{1+\sigma} \max \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}, \end{aligned} \tag{6.2.2}$$

whenever $0 < \varepsilon < \varepsilon_0$, and $(u_\varepsilon, p_\varepsilon)$ is weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in $B(0, 1)$.

Proof. We prove the lemma by contradiction, using the same approach as in [3] for the elliptic system $\mathcal{L}_\varepsilon(u_\varepsilon) = F$. First, we note that by the interior $C^{1,\rho}$ estimates for solutions of Stokes systems with constant coefficients,

$$\begin{aligned}
&\left(\fint_{B(0,\theta)} \left|u_0 - \fint_{B(0,\theta)} u_0 - P_j^\beta \fint_{B(0,\theta)} \frac{\partial u_0^\beta}{\partial x_j}\right|^2 dx\right)^{1/2} \\
&\qquad \le C\theta^{1+\rho}\|u_0\|_{C^{1,\rho}(B(0,1/4))} \\
&\qquad \le C_0\theta^{1+\rho}\max\left\{\left(\fint_{B(0,1/2)}|u_0|^2\right)^{1/2}, \left(\fint_{B(0,1/2)}|F_0|^q\right)^{1/q}, \|g_0\|_{C^\rho(B(0,1/2))}\right\}
\end{aligned} \tag{6.2.3}$$

for any $\theta \in (0,1/4)$, where (u_0,p_0) is weak solution of

$$-\mathrm{div}(A^0\nabla u_0) + \nabla p_0 = F_0 \quad \text{and} \quad \mathrm{div}(u_0) = g_0$$

in $B(0,1/2)$ and A^0 is a constant matrix satisfying the ellipticity condition (1.0.3). We emphasize that the constant C_0 in (6.2.3) depends only on d and μ. Since $0<\sigma<\rho$, we may choose $\theta \in (0,1/4)$ such that

$$2^d C_0 \theta^\rho < \theta^\sigma. \tag{6.2.4}$$

We claim that there exists $\varepsilon_0 > 0$, depending only on d, μ, σ and ρ, such that the estimate (6.2.2) holds with this θ, whenever $0<\varepsilon<\varepsilon_0$, and $(u_\varepsilon, p_\varepsilon)$ is a weak solution of Stokes system (1.0.2) in $B(0,1)$.

Suppose this is not the case. Then there exist sequences $\{\varepsilon_k\}$, $\{A^k(y)\}$, $\{u_k\}$ and $\{p_k\}$ such that $\varepsilon_k \to 0$, $A^k(y)$ satisfies ellipticity (1.0.3) (with the same μ) and periodicity (1.0.4) conditions, such that

$$\begin{cases} -\mathrm{div}(A^k(x/\varepsilon_k)\nabla u_k) + \nabla p_k = F_k & \text{in } B(0,1), \\ \mathrm{div}(u_k) = g_k & \text{in } B(0,1), \end{cases} \tag{6.2.5}$$

$$\max\left\{\left(\fint_{B(0,1)}|u_k|^2\right)^{1/2}, \left(\fint_{B(0,1)}|F_k|^q\right)^{1/q}, \|g_k\|_{C^\rho(B(0,1))}\right\} \le 1, \tag{6.2.6}$$

and

$$\left(\fint_{B(0,\theta)} \left|u_k - \fint_{B(0,\theta)} u_k - \left(P_j^\beta(x) + \varepsilon_k\chi_j^{k\beta}(x/\varepsilon_k)\right)\fint_{B(0,\theta)} \frac{\partial u_k^\beta}{\partial x_j}\right|^2 dx\right)^{1/2} > \theta^{1+\sigma}, \tag{6.2.7}$$

where $\chi_j^{k\beta}$ denotes the correctors for the Stokes systems with coefficient matrices $A^k(x/\varepsilon)$. Note that by (6.2.6) and Cacciopoli's inequality (6.2.1), the sequence $\{u_k\}$ is bounded in $H^1(B(0,1/2);\mathbb{R}^d)$. Thus, by passing to a subsequence, we may assume that $u_k \rightharpoonup u_0$ weakly in $L^2(B(0,1);\mathbb{R}^d)$ and $u_k \rightharpoonup u_0$ weakly in $H^1(B(0,1/2);\mathbb{R}^d)$.

Similarly, in view of (6.2.6), by passing to subsequences, we may assume that $g_k \to g_0$ in $L^\infty(B(0,1))$ and $F_k \rightharpoonup F_0$ in $L^q(B(0,1);\mathbb{R}^d)$. Since $\widehat{A^k}$ satisfies the ellipticity condition (3.1.8), we may further assume that $\widehat{A^k} \to A^0$ for some constant matrix A^0 satisfying (3.1.8).

Since $\varepsilon_k \chi_j^{k\beta}(x/\varepsilon_k) \to 0$ strongly in $L^2(B(0,1);\mathbb{R}^d)$, by taking the limit in (6.2.7), we obtain

$$\left(\fint_{B(0,\theta)} \left| u_0 - \fint_{B(0,\theta)} u_0 - P_j^\beta \fint_{B(0,\theta)} \frac{\partial u_0^\beta}{\partial x_j} \right|^2 dx \right)^{1/2} \geq \theta^{1+\sigma}. \tag{6.2.8}$$

Also observe that (6.2.6) implies

$$\max \left\{ \left(\fint_{B(0,1)} |u_0|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F_0|^q \right)^{1/q}, \|g_0\|_{C^\rho(B(0,1))} \right\} \leq 1. \tag{6.2.9}$$

Finally, we note that

$$\begin{aligned} \|p_k - \fint_{B(0,1/2)} p_k\|_{L^2(B(0,1/2))} &\leq \|\nabla p_k\|_{H^{-1}(B(0,1/2))} \\ &\leq C \left\{ \|\nabla u_k\|_{L^2(B(0,1/2))} + \|F_k\|_{H^{-1}(B(0,1/2))} \right\} \\ &\leq C, \end{aligned}$$

where the first inequality holds for any $p_k \in L^2(B(0,1/2))$, and we have applied (4.3.1) to the first equation in (6.2.5) for the second inequality and Caccioppoli's inequality for the third. Clearly, we may assume $\int_{B(0,1/2)} p_k = 0$ by subtracting a constant. Thus, by passing to a subsequence, we may assume that $p_k \rightharpoonup p_0$ weakly in $L^2(B(0,1/2))$. This, together with convergence of u_k, F_k, g_k, and $\widehat{A^k}$, allow us to apply the Compactness Theorem (Theorem 3.3.2) of Stokes system to conclude that

$$\begin{cases} -\mathrm{div}(A^0 \nabla u_0) + \nabla p_0 = F_0 & \text{in } B(0,1/2), \\ \mathrm{div}(u_0) = g_0 & \text{in } B(0,1/2). \end{cases}$$

As a result, in view of (6.2.3), (6.2.8) and (6.2.9), we obtain

$$\begin{aligned} \theta^{1+\sigma} &\leq C_0 \theta^{1+\rho} \max \left\{ \left(\fint_{B(0,1/2)} |u_0|^2 \right)^{1/2}, \left(\fint_{B(0,1/2)} |F_0|^q \right)^{1/q}, \|g_0\|_{C^\rho(B(0,1/2))} \right\} \\ &\leq 2^d C_0 \theta^{1+\rho}, \end{aligned}$$

which contradicts the choice of θ in (6.2.4). This completes the proof. $\square$

Remark 6.2.3. *It is easy to see that estimate (6.2.2) continues to hold if we replace* $\fint_{B(0,\theta)} u_\varepsilon$ *by the average*

$$\fint_{B(0,\theta)} \left[u_\varepsilon - (P_j^\beta + \varepsilon \chi_j^\beta(x/\varepsilon) \fint_{B(0,\theta)} \frac{\partial u_\varepsilon^\beta}{\partial x_j} \right] dx.$$

This will be used in the next lemma.

Lemma 6.2.4. *Let* $0 < \sigma < \rho < 1$ *and* $\rho = 1 - \frac{d}{q}$. *Let* (ε_0, θ) *be given by Lemma 6.2.2. Suppose that* $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ *for some* $k \geq 1$, *and*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \qquad \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in $B(0,1)$. *Then there exist constants* $E(\varepsilon, \ell) = (E_j^\beta(\varepsilon, \ell)) \in \mathbb{R}^{d\times d}$ *for* $1 \leq \ell \leq k$, *such that*

$$\begin{aligned} &\left(\fint_{B(0,\theta^\ell)} \left| u_\varepsilon - \big(P_j^\beta + \varepsilon\chi_j^\beta(x/\varepsilon)\big) E_j^\beta(\varepsilon,\ell) - \fint_{B(0,\theta^\ell)} \left[u_\varepsilon - \big(P_j^\beta + \varepsilon\chi_j^\beta(x/\varepsilon)\big) E_j^\beta(\varepsilon,\ell) \right] \right|^2 \right)^{1/2} \\ &\quad \leq \theta^{\ell(1+\sigma)} \max\left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}. \end{aligned} \tag{6.2.10}$$

Moreover, the constants $E(\varepsilon, \ell)$ *satisfy*

$$|E(\varepsilon,\ell)| \leq C \max\left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}, \tag{6.2.11}$$

$$\begin{aligned} &|E(\varepsilon,\ell+1) - E(\varepsilon,\ell)| \\ &\qquad \leq C\theta^{\ell\sigma} \max\left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}, \end{aligned} \tag{6.2.12}$$

where C *depends only on d,* μ, σ *and* ρ, *and in particular,*

$$\sum_{j=1}^d E_j^j(\varepsilon,\ell) = \fint_{B(0,\theta^\ell)} g. \tag{6.2.13}$$

Proof. The lemma is proved by an induction argument on ℓ. The case $\ell = 1$ follows directly from Lemma 6.2.2, with

$$E_j^\beta(\varepsilon, 1) = \fint_{B(0,\theta)} \frac{\partial u_\varepsilon^\beta}{\partial x_j}$$

(see Remark 6.2.3). Suppose that the desired constants exist for all positive integers up to some ℓ, where $1 \le \ell \le k-1$. To construct $E(\varepsilon, \ell+1)$, we consider

$$
\begin{aligned}
w(x) = u_\varepsilon(\theta^\ell x) - &\Big\{ P_j^\beta(\theta^\ell x) + \varepsilon \chi_j^\beta(\theta^\ell x/\varepsilon) \Big\} E_j^\beta(\varepsilon, \ell) \\
&- \fint_{B(0,\theta^\ell)} \Big[u_\varepsilon(\theta^\ell x) - \Big\{ P_j^\beta(\theta^\ell x) + \varepsilon \chi_j^\beta(\theta^\ell x/\varepsilon) \Big\} E_j^\beta(\varepsilon, \ell) \Big].
\end{aligned}
$$

Note that by the rescaling property of the Stokes system (see in Section 2.1),

$$
\begin{cases}
\mathcal{L}_{\frac{\varepsilon}{\theta^\ell}}(w) + \nabla \Big\{ \theta^\ell p_\varepsilon(\theta^\ell x) - \theta^\ell \pi_j^\beta(\theta^\ell x/\varepsilon) E_j^\beta(\varepsilon, \ell) \Big\} = \theta^{2\ell} F(\theta^\ell x), \\
\operatorname{div}(w) = \theta^\ell g(\theta^\ell x) - \theta^\ell \displaystyle\sum_{j=1}^d E_j^j(\varepsilon, \ell),
\end{cases}
\tag{6.2.14}
$$

in $B(0,1)$, where π_j^β is defined by (3.1.5). Since $(\varepsilon/\theta^\ell) \le (\varepsilon/\theta^{k-1}) \le \varepsilon_0$, we may apply Lemma 6.2.2 to obtain

$$
\begin{aligned}
\Bigg(\fint_{B(0,\theta)} \Bigg| w - \big(P_j^\beta + \theta^{-\ell} \varepsilon \chi_j^\beta(\theta^\ell x/\varepsilon) \big) \fint_{B(0,\theta)} \frac{\partial w^\beta}{\partial x_j} \\
- \fint_{B(0,\theta)} \Big[w - \big(P_j^\beta + \theta^{-\ell} \varepsilon \chi_j^\beta(\theta^\ell x/\varepsilon) \big) \fint_{B(0,\theta)} \frac{\partial w^\beta}{\partial x_j} \Big] \Bigg|^2 dx \Bigg)^{1/2} \\
\le \theta^{1+\sigma} \max \left\{ \left(\fint_{B(0,1)} |w|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F_\ell|^q dx \right)^{1/q}, \|\operatorname{div}(w)\|_{C^\rho(B(0,1))} \right\},
\end{aligned}
\tag{6.2.15}
$$

where $F_\ell(x) = \theta^{2\ell} F(\theta^\ell x)$.

We now estimate the right hand side of (6.2.15). Observe that by the induction assumption,

$$
\begin{aligned}
&\left(\fint_{B(0,1)} |w|^2 \right)^{1/2} \\
&\le \theta^{\ell(1+\sigma)} \max \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}.
\end{aligned}
\tag{6.2.16}
$$

Also note that since $0 < \rho = 1 - \frac{d}{q}$,

$$
\left(\fint_{B(0,1)} |\theta^{2\ell} F(\theta^\ell x)|^q dx \right)^{1/q} \le \theta^{\ell(1+\rho)} \left(\fint_{B(0,1)} |F|^q \right)^{1/q}.
$$

In view of (6.2.14) and (6.2.13), we have

$$
\operatorname{div}(w) = \theta^\ell \left\{ g(\theta^\ell x) - \fint_{B(0,\theta^\ell)} g \right\},
$$

which gives

$$\|\mathrm{div}(w)\|_{C^\rho(B(0,1))} \le \theta^{\ell(1+\rho)} \|g\|_{C^\rho(B(0,1))}.$$

Thus we have proved that the right hand side of (6.2.15) is bounded by

$$\theta^{(\ell+1)(1+\sigma)} \max\left\{ \left(-\!\!\!\!\!\!\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(-\!\!\!\!\!\!\int_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}.$$

Finally, we note that the left hand side of (6.2.15) may be written as

$$\begin{aligned} \Bigg(-\!\!\!\!\!\!\int_{B(0,\theta^{\ell+1})} \Bigg| & u_\varepsilon - \big(P_j^\beta + \varepsilon \chi_j^\beta(x/\varepsilon)\big) E_j^\beta(\varepsilon, \ell+1) \\ & - -\!\!\!\!\!\!\int_{B(0,\theta^{\ell+1})} \Big[u_\varepsilon - \big(P_j^\beta + \varepsilon \chi_j^\beta(x/\varepsilon)\big) E_j^\beta(\varepsilon, \ell+1) \Big] \Bigg|^2 dx \Bigg)^{1/2} \end{aligned}$$

with

$$E_j^\beta(\varepsilon, \ell+1) = E_j^\beta(\varepsilon, \ell) + \theta^{-\ell} -\!\!\!\!\!\!\int_{B(0,\theta)} \frac{\partial w^\beta}{\partial x_j}. \tag{6.2.17}$$

Observe that by Cacciopoli's inequality (6.2.1),

$$\begin{aligned} |E(\varepsilon, \ell+1) - E(\varepsilon, \ell)| &\le \theta^{-\ell} \left(-\!\!\!\!\!\!\int_{B(0,\theta)} |\nabla w|^2 \right)^{1/2} \\ &\le C\theta^{-\ell} \max\left\{ \left(-\!\!\!\!\!\!\int_{B(0,1)} |w|^2 \right)^{1/2}, \left(-\!\!\!\!\!\!\int_{B(0,1)} |\theta^{2\ell} F(\theta^{2\ell x})|^2 \right)^{1/2}, \left(-\!\!\!\!\!\!\int_{B(0,1)} |\mathrm{div}(w)|^2 \right)^{1/2} \right\} \\ &\le C\theta^{\ell\sigma} \max\left\{ \left(-\!\!\!\!\!\!\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(-\!\!\!\!\!\!\int_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}, \end{aligned}$$

where we have used the estimates for the right hand side of (6.2.15) for the last inequality. This, together with the estimate of $E(\varepsilon, 1)$ gives (6.2.11) and (6.2.12). To see (6.2.13), we note that by (6.2.14) and (6.2.17),

$$\begin{aligned} \sum_{j=1}^d E_j^j(\varepsilon, \ell+1) = \sum_{j=1}^d E_j^j(\varepsilon, \ell) + \theta^{-\ell} -\!\!\!\!\!\!\int_{B(0,\theta)} \mathrm{div}(w) &= -\!\!\!\!\!\!\int_{B(0,\theta)} g(\theta^\ell x) dx \\ &= -\!\!\!\!\!\!\int_{B(0,\theta^{\ell+1})} g, \end{aligned}$$

This completes the proof. $\square$

The following theorem may be viewed as the Lipschitz estimate for u_ε, down to the scale ε. Recalling in (2.1.8), we use $[g]_{C^{0,\rho}(E)}$ to denote the semi-norm

$$[g]_{C^{0,\rho}(E)} = \sup\left\{ \frac{|g(x) - g(y)|}{|x-y|^\rho} : x, y \in E \text{ and } x \ne y \right\}.$$

Theorem 6.2.5. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F \\ \qquad \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > 2\varepsilon$. Then, if $\varepsilon \le r \le (R/2)$,

$$\begin{aligned} \left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \le C \Bigg\{ \frac{1}{R} \left(\fint_{B(x_0,R)} |u_\varepsilon|^2 \right)^{1/2} + R \left(\fint_{B(x_0,R)} |F|^q \right)^{1/q} \\ + \|g\|_{L^\infty(B(x_0,R))} + R^\rho [g]_{C^{0,\rho}(B(x_0,R))} \Bigg\} \end{aligned} \tag{6.2.18}$$

where $\rho \in (0,1)$, $\rho = 1 - \frac{d}{q}$, and C depends only on d, μ, and ρ.

Proof. By covering $B(x_0, r)$ with balls of radius ε, we only need to consider the case $r = \varepsilon$. By translation and dilation, we may further assume that $x_0 = 0$ and $R = 1$. Thus we would need to show that if $0 < \varepsilon \le (1/2)$,

$$\left(\fint_{B(0,\varepsilon)} |\nabla u_\varepsilon|^2 \right)^{1/2} \le C \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(0,1)} |F|^q \right)^{1/q} + \|g\|_{C^\rho(B(0,1))} \right\} \tag{6.2.19}$$

We will see that this follows readily from Lemma 6.2.4.

Indeed, let (ε_0, θ) be given by Lemma 6.2.2. The case $\theta\varepsilon_0 \le \varepsilon \le (1/2)$ follows directly from Cacciopoli's inequality. Suppose now that $0 < \varepsilon < \theta\varepsilon_0$, choose $k \ge 2$ so that $\theta^k \varepsilon_0 \le \varepsilon < \theta^{k-1}\varepsilon_0$. It follows from Lemma 6.2.4 that

$$\begin{aligned} &\left(\fint_{B(0,\theta^{k-1})} \left| u_\varepsilon - \fint_{B(0,\theta^{k-1})} u_\varepsilon \right|^2 \right)^{1/2} \\ &\quad \le C \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(0,1)} |F|^q \right)^{1/q} + \|g\|_{C^\rho(B(0,1))} \right\}. \end{aligned} \tag{6.2.20}$$

This, together with the Cacciopoli's inequality (6.2.1), implies that

$$\left(\fint_{B(0,\theta^{k-1})} |\nabla u_\varepsilon|^2 \right)^{1/2} \le C \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(0,1)} |F|^q \right)^{1/q} + \|g\|_{C^\rho(B(0,1))} \right\}.$$

from which the estimate (6.2.19) follows. $\square$

6.3 A Liouville property for Stokes systems

In this section we prove a Liouville property for global solutions of the Stokes systems with periodic coefficient. We refer the readers to [5] for the case of the elliptic systems

$\mathcal{L}_1(u) = 0$ (also see [39, 41] and their references for related work). The Liouville property for Stokes systems with constant coefficients is well known; however, we are not aware of any previous work on the Liouville property for Stokes systems with variable coefficients.

Theorem 6.3.1. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let $(u,p) \in H^1_{loc}(\mathbb{R}^d;\mathbb{R}^d) \times L^2_{loc}(\mathbb{R}^d)$ be a weak solution of*

$$\begin{cases} \mathcal{L}_1(u) + \nabla p = 0, \\ \qquad \operatorname{div}(u) = g \end{cases} \tag{6.3.1}$$

in $\mathbb{R}^d$, where g is a constant. Assume that

$$\left(\fint_{B(0,R)} |u|^2 \right)^{1/2} \le C_u R^{1+\sigma} \tag{6.3.2}$$

for some $C_u > 0$, $\sigma \in (0,1)$, and for all $R > 1$. Then

$$\begin{cases} u(x) = H + \big(P_j^\beta(x) + \chi_j^\beta(x)\big) E_j^\beta, \\ p(x) = \widetilde{H} + \pi_j^\beta(x) E_j^\beta \end{cases} \tag{6.3.3}$$

for some constants $H \in \mathbb{R}^d$, $\widetilde{H} \in \mathbb{R}$, and $E = (E_j^\beta) \in \mathbb{R}^{d\times d}$. In particular, the space of functions (u,p) that satisfy (6.3.1) and (6.3.2) is of dimension $d^2 + d + 1$.

Proof. Fix $\sigma_1 \in (\sigma, 1)$. Let (ε_0, θ) be the constants given by Lemma 6.2.2 for $0 < \sigma_1 < \rho < 1$. Suppose that (u,p) is a solution of (6.3.1) in $\mathbb{R}^d$ for some constant g. Let

$$u_\varepsilon(x) = u(x/\varepsilon) \qquad \text{and} \qquad p_\varepsilon(x) = \varepsilon^{-1} p(x/\varepsilon).$$

Then

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0, \\ \qquad \operatorname{div}(u_\varepsilon) = \varepsilon^{-1} g, \end{cases}$$

in $B(0,1)$. It follows from Lemma 6.2.4 that if $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ for some $k \ge 1$, then

$$\begin{aligned} &\inf_{\substack{E=(E_j^\beta)\in\mathbb{R}^{d\times d} \\ H\in\mathbb{R}^d}} \left(\fint_{B(0,\theta^\ell)} |u_\varepsilon - (P_j^\beta + \varepsilon \chi_j^\beta(x/\varepsilon)) E_j^\beta - H|^2 \right)^{1/2} \\ &\qquad \le \theta^{\ell(1+\sigma_1)} \max \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \varepsilon^{-1}|g| \right\}, \end{aligned}$$

where $1 \le \ell \le k$. By a change of variable this gives

$$\inf_{\substack{E=(E_j^\beta)\in\mathbb{R}^{d\times d}\\ H\in\mathbb{R}^d}} \left(\fint_{B(0,\varepsilon^{-1}\theta^\ell)} |u - (P_j^\beta + \chi_j^\beta(x))E_j^\beta - H|^2 \right)^{1/2} \le \theta^{\ell(1+\sigma_1)} \max\left\{ \left(\fint_{B(0,\varepsilon^{-1})} |u|^2 \right)^{1/2}, \varepsilon^{-1}|g| \right\}, \tag{6.3.4}$$

where $0 < \varepsilon < \theta^{k-1}\varepsilon_0$ for some $k \ge 1$ and $1 \le \ell \le k$.

Now, suppose that u satisfies the growth condition (6.3.2). For any $m \ge 1$ such that $\theta^{m+1} \le \varepsilon_0$, let $\varepsilon = \theta^{m+\ell}$ where $\ell > 1$. It follows from (6.3.4) and (6.3.2) that

$$\begin{aligned} \inf_{\substack{E=(E_j^\beta)\in\mathbb{R}^{d\times d}\\ H\in\mathbb{R}^d}} & \left(\fint_{B(0,\theta^{-m})} |u - (P_j^\beta + \chi_j^\beta(x))E_j^\beta - H|^2 \right)^{1/2} \\ & \le \theta^{\ell(1+\sigma_1)} \max\left\{ C(\varepsilon^{-1})^{1+\sigma}, \varepsilon^{-1}|g| \right\} \\ & \le \theta^{\ell(1+\sigma_1)} \max\left\{ C\theta^{-(m+\ell)(1+\sigma)}, \theta^{-(m+\ell)}|g| \right\}, \end{aligned} \tag{6.3.5}$$

for some constant C independent of m and ℓ. Since $\sigma_1 > \sigma$, we may fix m and let $\ell \to \infty$ in (6.3.5) to conclude that the left hand side of (6.3.5) is zero. Thus, for each m large, there exist constants $H^m \in \mathbb{R}^d$ and $E^m = (E_j^{m\beta}) \in \mathbb{R}^{d\times d}$ such that

$$u(x) = H^m + \left(P_j^\beta(x) + \chi_j^\beta(x)\right)E_j^{m\beta} \qquad \text{in } B(0,\theta^{-m}).$$

Finally, we observe that $\nabla u = (\nabla P_j^\beta + \nabla \chi_j^\beta)E_j^\beta$ and since $\int_Y \nabla\chi_j^\beta = 0$,

$$\int_Y \nabla u = \int_Y \nabla P_j^\beta \cdot E_j^{m\beta}.$$

This implies that $E_j^{m\beta} = E_j^{n\beta}$ for any m, n large; as a consequence, we obtain $H^m = H^n$, for any m, n large. Thus we have proved that (6.3.3) holds for some $H \in \mathbb{R}^d$ and $E = (E_j^\beta) \in \mathbb{R}^{d\times d}$, Note that if

$$H + (P_j^\beta + \chi_j^\beta)E_j^\beta = 0 \quad \text{ in } \mathbb{R}^d,$$

then $\int_Y \nabla P_j^\beta \cdot E_j^\beta = 0$. It follows that $E_j^\beta = 0$ and hence, $H = 0$. This shows that the space of functions (u,p) that satisfy (6.3.1)-(6.3.2) is of dimension $d^2 + d + 1$. $\square$

Remark 6.3.2. *Suppose that (u,p) satisfies (6.3.1) in $\mathbb{R}^d$ for some constant g and that*

$$\left(\fint_{B(0,R)} |u|^2 \right)^{1/2} \le C_\sigma R^\sigma \tag{6.3.6}$$

for some $C_u > 0$, $\sigma \in (0,1)$, and for all $R > 1$. It follows from Theorem 6.3.1 that (u,p) must be constants.

Remark 6.3.3. *One may use the results in Theorem 6.3.1 and a line of argument used in [41] to characterize all solutions of (6.3.1) in $\mathbb{R}^d$ that satisfy the growth condition*

$$\left(\fint_{B(0,R)} |u|^2 \right)^{1/2} \leq C_\sigma R^{N+\sigma} \tag{6.3.7}$$

for some $C_u > 0$, integer $N \geq 2$, $\sigma \in (0,1)$, and for all $R > 1$. In particular, by using the difference operator $\Delta_i \phi = \phi(x+e_i) - \phi(x)$ repeatedly, one may deduce from the observation in Remark 6.3.2 that

$$u^\alpha(x) = \sum_{|\nu|=N} E(\nu,\alpha) x^\nu + \sum_{0 \leq |\nu| \leq N-1} w_{v,\alpha}(x) x^\nu,$$

where $E(v,\alpha)$ is constant and $w_{\nu,\alpha}(x)$ is 1-periodic. Hence $\nu = (\nu_1, \nu_2, \cdots, \nu_d)$ is a multi-index and $x^\nu = x_1^{\nu_1} x_2 \nu_2 \cdots x_d^{\nu_d}$. We will pursue this line of research elsewhere.

6.4 L^∞ estimates for p_ε and proof of Theorem 1.0.1

In this section we prove an L^∞ estimate for p_ε, down to the scale ε. We also give the proof of Theorem 1.0.1 and Corollary 6.1.2.

Theorem 6.4.1. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and periodicity condition (1.0.4). Let $(u_\varepsilon, p_\varepsilon)$ be a weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = F, \\ \qquad \operatorname{div}(u_\varepsilon) = g \end{cases}$$

in $B(x_0,R)$ for some $x_0 \in \mathbb{R}^d$ and $R > \varepsilon$. Then, if $\varepsilon \leq r < R$,

$$\begin{aligned} \left(\fint_{B(x_0,r)} \left| p_\varepsilon - \fint_{B(x_0,R)} p_\varepsilon \right|^2 \right)^{1/2} \leq C \Bigg\{ & \left(\fint_{B(x_0,R)} |\nabla u_\varepsilon|^2 \right)^{1/2} + R \left(\fint_{B(x_0,R)} |F|^q \right)^{1/q} \\ & + \|g\|_{L^\infty(B(x_0,R))} + R^\rho [g]_{C^{0,\rho}(B(x_0,R))} \Bigg\}, \end{aligned} \tag{6.4.1}$$

where $\rho \in (0,1)$, $\rho = 1 - \frac{d}{q}$, and C depends only on d, μ and ρ.

Proof. By translation and dilation we may assume that $x_0 = 0$ and $R = 1$. Note that by applying (4.3.1) to the Stoke system (1.0.2) in $B(0,r)$, we have

$$\begin{aligned} \|p_\varepsilon - \fint_{B(0,r)} p_\varepsilon\|_{L^2(B(0,r))} &\leq C \|\nabla p_\varepsilon\|_{H^{-1}(B(0,r))} \\ &\leq C \left\{ \|\nabla u_\varepsilon\|_{L^2(B(0,r))} + \|F\|_{H^{-1}(B(0,1))} \right\}. \end{aligned} \tag{6.4.2}$$

Thus, in view of Theorem 6.2.5, it suffices to show that

$$\Big| \fint_{B(0,r)} p_\varepsilon - \fint_{B(0,1)} p_\varepsilon \Big|$$

is bounded by the right hand side of (6.4.1). This will be done by using the $C^{1,\sigma}$ estimate for u_ε down to the scale ε in Lemma 6.2.2.

Let (θ, ε_0) be the constants given by Lemma 6.2.2. By (6.4.2) we may assume that $0 < \varepsilon \le r < \varepsilon_0$. Let $\theta^k \varepsilon_0 \le \varepsilon < \theta^{k-1}\varepsilon_0$ and $\theta^t \varepsilon_0 \le r < \theta^{t-1}\varepsilon_0$ for some $1 \le t \le k$. The terms $\fint_{B(0,r)} p_\varepsilon - \fint_{B(0,\theta^t)} p_\varepsilon$ and $\fint_{B(0,1)} p_\varepsilon - \fint_{B(0,\theta)} p_\varepsilon$ can be handled by using (6.4.2). To deal with $\fint_{B(0,\theta^t)} p_\varepsilon - \fint_{B(0,\theta)} p_\varepsilon$, we write

$$\fint_{B(0,\theta^t)} p_\varepsilon - \fint_{B(0,\theta)} p_\varepsilon = \sum_{\ell=1}^{t-1} \Big\{ \fint_{B(0,\theta^{\ell+1})} p_\varepsilon - \fint_{B(0,\theta^\ell)} p_\varepsilon \Big\}. \tag{6.4.3}$$

Let

$$\begin{aligned} v_\ell &= u_\varepsilon(x) - (P_j^\beta(x) + \varepsilon \chi_j^\beta(x/\varepsilon)) E_j^\beta(\varepsilon, \ell) \\ &\quad - \fint_{B(0,\theta^\ell)} \Big\{ u_\varepsilon(x) - (P_j^\beta(x) + \varepsilon \chi_j^\beta(x/\varepsilon)) E_j^\beta(\varepsilon, \ell) \Big\}, \end{aligned}$$

where $E(\varepsilon, \ell) = (E_j^\beta(\varepsilon, \ell)) \in \mathbb{R}^{d \times d}$ are constants given by Lemma 6.2.4. Note that by Lemma 6.2.4,

$$\begin{aligned} &\left(\fint_{B(0,\theta^\ell)} |v_\ell|^2 \right)^{1/2} \\ &\le \theta^{\ell(1+\sigma)} \max \left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\}, \end{aligned} \tag{6.4.4}$$

where $0 < \sigma < \rho < 1$, and

$$\begin{cases} \mathcal{L}_\varepsilon(v_\ell) + \nabla \{ p_\varepsilon - \pi_j^\beta(x/\varepsilon) E_j^\beta(\varepsilon, \ell) \} = F, \\ \operatorname{div}(v_\ell) = g - \fint_{B(0,\theta^\ell)} g, \end{cases} \tag{6.4.5}$$

in $B(0,1)$. Observe that for any $H \in \mathbb{R}$,

$$\begin{aligned} &\left| \fint_{B(0,\theta^{\ell+1})} p_\varepsilon - \fint_{B(0,\theta^\ell)} p_\varepsilon \right| \\ &\le \left| \fint_{B(0,\theta^{\ell+1})} \big[p_\varepsilon - H - \pi_j^\beta(x/\varepsilon) E_j^\beta(\varepsilon, \ell) \big] dx \right| \\ &\qquad + \left| \fint_{B(0,\theta^\ell)} \big[p_\varepsilon - H - \pi_j^\beta(x/\varepsilon) E_j^\beta(\varepsilon, \ell) \big] dx \right| \\ &\qquad + |E_j^\beta(\varepsilon, \ell)| \left| \fint_{B(0,\theta^{\ell+1})} \pi_j^\beta(x/\varepsilon) - \fint_{B(0,\theta^\ell)} \pi_j^\beta(x/\varepsilon) \right|. \end{aligned} \tag{6.4.6}$$

Choose

$$H = \fint_{B(0,\theta^\ell)} \left[p_\varepsilon - \pi_j^\beta(x/\varepsilon) E_j^\beta(\varepsilon, \ell) \right] dx$$

so that the second term in the right hand side of (6.4.6) equals zero. Using (6.4.2), (6.4.5), Cacciopoli's inequality (6.2.1) and (6.4.4), we see that the first term in the right hand side of (6.4.6) is bounded by

$$\begin{aligned}
C &\left(\fint_{B(0,\theta^\ell)} |p_\varepsilon - H - \pi_j^\beta(x/\varepsilon) E_j^\beta(\varepsilon, \ell)|^2 dx \right)^{1/2} \\
&\qquad \le C\theta^{-d\ell/2} \left\{ \|\nabla v_\ell\|_{L^2(B(0,\theta^\ell))} + \|F\|_{H^{-1}(B(0,\theta^\ell))} \right\} \\
&\qquad \le C\theta^{\ell\sigma} \max\left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\},
\end{aligned}$$

where we have also used $q > d$, $0 < \sigma < \rho = 1 - \frac{d}{q}$, and

$$\begin{aligned}
\|F\|_{H^{-1}(B(0,\theta^\ell))} &\le C|B(0,\theta^\ell)|^{\frac{1}{2}+\frac{1}{d}} \left(\fint_{B(0,\theta^\ell)} |F|^q \right)^{1/q} \\
&\le C\theta^{\ell(\frac{d}{2}+\rho)} \left(\fint_{B(0,1)} |F|^q \right)^{1/q}.
\end{aligned}$$

Finally, we note that since π_j^β is 1-periodic,

$$\begin{aligned}
&\left| \fint_{B(0,\theta^{\ell+1})} \pi_j^\beta(x/\varepsilon) - \fint_{B(0,\theta^\ell)} \pi_j^\beta(x/\varepsilon) \right| \\
&\qquad = \left| \fint_{B(0,\varepsilon^{-1}\theta^{\ell+1})} \pi_j^\beta - \langle \pi_j^\beta \rangle \right| + \left| \fint_{B(0,\varepsilon^{-1}\theta^\ell)} \pi_j^\beta - \langle \pi_j^\beta \rangle \right| \\
&\qquad \le C\varepsilon\theta^{-\ell} \|\pi_j^\beta\|_{L^2(Y)} \\
&\qquad \le C\varepsilon\theta^{-\ell},
\end{aligned} \tag{6.4.7}$$

where $\langle \pi_j^\beta \rangle$ denotes the average of 1-periodic function π_j^β over the periodic cell, i.e.,

$$\langle \pi_j^\beta \rangle = \fint_Y \pi_j^\beta.$$

This, together with the estimate of the first two terms in the right hand side of (6.4.6), shows that the left hand side of (6.4.3) is bounded by

$$\begin{aligned}
&C \sum_{\ell=1}^{t-1} (\theta^{\ell\sigma} + \varepsilon\theta^{-\ell}) \max\left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\} \\
&\le C \max\left\{ \left(\fint_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2}, \left(\fint_{B(0,1)} |F|^q \right)^{1/q}, \|g\|_{C^\rho(B(0,1))} \right\},
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.0.1. The estimate for ∇u_ε in (1.0.5) is given by Theorem 6.2.5, while the estimate for p_ε is contained in Theorem 6.4.1. □

Proof of Corollary 6.1.2. Under the Hölder continuous condition (2.1.2), it is known that solutions of the Stokes systems are locally $C^{1,\alpha}$ for $\alpha < \lambda$ (see [24]). In particular, it follows that if (u,p) is a weak solution of

$$\begin{cases} \mathcal{L}_1(u) + \nabla p = F, \\ \quad\quad \operatorname{div}(u) = g \end{cases}$$

in $B(y,1)$ for some $y \in \mathbb{R}^d$, then

$$\begin{aligned} &\|\nabla u\|_{L^\infty(B(y,1/2))} + \|p - \fint_{B(y,1/2)} p\|_{L^\infty(B(y,1/2))} \\ &\le C\left\{ \left(\fint_{B(y,1)} |\nabla u|^2\right)^{1/2} + \left(\fint_{B(y,1)} |F|^q\right)^{1/q} + \|g\|_{C^\rho(B(y,1))}\right\}, \end{aligned} \tag{6.4.8}$$

where $0 < \rho < 1$, $\rho = 1 - \frac{d}{q}$, and the constant C depends only on d, μ, ρ, and (λ, τ) in (2.1.2).

To prove (6.1.2), by translation and dilation, we may assume that $x_0 = 0$ and $R = 1$. Now suppose $(u_\varepsilon, p_\varepsilon)$ is a weak solution of (1.0.2) in $B(0,1)$. The estimate (6.1.2) for the case $\varepsilon \ge (1/8)$ follows directly from (6.4.8), as the matrix $A(x/\varepsilon)$ satisfies Hölder continuity (2.1.2) uniformly in ε. For $0 < \varepsilon < (1/8)$, we use a blow-up argument and estimate (6.4.8) by considering

$$u(x) = \varepsilon^{-1} u_\varepsilon(\varepsilon x) \quad \text{and} \quad p(x) = p_\varepsilon(\varepsilon x).$$

This leads to

$$\begin{aligned} &\|\nabla u_\varepsilon\|_{L^\infty(B(y,\varepsilon))} + \|p_\varepsilon - \fint_{B(y,\varepsilon)} p_\varepsilon\|_{L^\infty(B(y,\varepsilon))} \\ &\le C\left\{ \left(\fint_{B(y,2\varepsilon)} |\nabla u_\varepsilon|^2\right)^{1/2} + \varepsilon\left(\fint_{B(y,2\varepsilon)} |F|^q\right)^{1/q} + \|g\|_{C^\rho(B(y,2\varepsilon))}\right\}, \end{aligned} \tag{6.4.9}$$

for any $y \in B(0,1/2)$. In view of Theorem 6.2.5 we obtain

$$\begin{aligned} &\|\nabla u_\varepsilon\|_{L^\infty(B(0,1/2))} + \|p_\varepsilon - \fint_{B(y,\varepsilon)} p_\varepsilon\|_{L^\infty(B(y,\varepsilon))} \\ &\le C\left\{ \left(\fint_{B(0,1)} |\nabla u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B(0,1)} |F|^q\right)^{1/q} + \|g\|_{C^\rho(B(0,1))}\right\}. \end{aligned} \tag{6.4.10}$$

Finally, we note that for any $y \in B(0, 1/2)$,

$$
\begin{aligned}
&\left| p_\varepsilon - \fint_{B(0,1)} p_\varepsilon \right| \\
&\le \left| p_\varepsilon - \fint_{B(y,\varepsilon)} p_\varepsilon \right| + \left| \fint_{B(y,\varepsilon)} p_\varepsilon - \fint_{B(y,1/2)} p_\varepsilon \right| + \left| \fint_{B(y,1/2)} p_\varepsilon - \fint_{B(0,1)} p_\varepsilon \right| \\
&\le \left| p_\varepsilon - \fint_{B(y,\varepsilon)} p_\varepsilon \right| + \left(\fint_{B(y,\varepsilon)} |p_\varepsilon - \fint_{B(y,1/2)} p_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(y,1/2)} |p_\varepsilon - \fint_{B(0,1)} p_\varepsilon|^2 \right)^{1/2} \\
&\le C \left\{ \left(\fint_{B(0,1)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(0,1)} |F|^q \right)^{1/q} + \|g\|_{C^\rho(B(0,1))} \right\}
\end{aligned}
$$

where we have used (6.4.9), (6.4.10), Theorem 6.4.1, and (6.4.2) for the last inequality. This completes the proof. □

6.5 Boundary Hölder estimates

In this section we establish uniform boundary Hölder estimates for the Stokes system (1.0.2) in C^1 domains and give the proof of Theorem 6.1.3.

Let $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a C^1 function and

$$
\begin{aligned}
D_r &= D(r, \psi) = \{x = (x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + 10(M+1)r\}, \\
\Delta_r &= \Delta(r, \psi) = \{x = (x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } x_d = \psi(x')\}.
\end{aligned}
\tag{6.5.1}
$$

We will always assume that $\psi(0) = 0$ and

$$
\|\nabla \psi\|_\infty \le M, \text{ and } |\nabla \psi(x') - \nabla \psi(y')| \le \omega(|x' - y'|) \qquad \text{for any } x', y' \in \mathbb{R}^{d-1} \tag{6.5.2}
$$

where $M > 0$ is a fixed constant and $\omega(r)$ is a fixed, nondecreasing continuous function on $[0, \infty)$ and $\omega(0) = 0$.

Theorem 6.5.1. *Let $0 < \rho, \eta < 1$. Let $(u_\varepsilon, p_\varepsilon) \in H^1(D_r; \mathbb{R}^d) \times L^2(D_r)$ be a weak solution of*

$$
\begin{cases}
\mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0 & \text{in } D_r, \\
\operatorname{div}(u_\varepsilon) = g & \text{in } D_r, \\
u_\varepsilon = h & \text{on } \Delta_r
\end{cases}
\tag{6.5.3}
$$

for some $0 < \varepsilon < r < r_0$, where $g \in C^\eta(D_r)$, $h \in C^{0,1}(\Delta_r)$ and $h(0) = 0$. Then for any $0 < \varepsilon \le t < r$,

$$
\begin{aligned}
&\left(\fint_{D_t} |u_\varepsilon|^2 \right)^{1/2} \\
&\le C \left(\frac{t}{r} \right)^\rho \left\{ \left(\fint_{D_r} |u_\varepsilon|^2 \right)^{1/2} + r\|g\|_{L^\infty(D_r)} + r^{1+\eta}[g]_{C^{0,\eta}(D_r)} + r[h]_{C^{0,1}(\Delta_r)} \right\},
\end{aligned}
\tag{6.5.4}
$$

where C depends only on d, μ, ρ, η, r_0, and (M, ω) in (6.5.2).

It is not hard to see that Theorem 6.1.3 follows from Theorem 6.5.1 and the following boundary Cacciopoli's inequality whose proof may be found in [24].

Theorem 6.5.2. *Suppose that A satisfies ellipticity condition (1.0.3). Let $(u, p) \in H^1(D_r; \mathbb{R}^d) \times L^2(D_r)$ be a weak solution of*

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \nabla p = F + \mathrm{div}(f) & \text{in } D_r, \\ \mathrm{div}(u) = g & \text{in } D_r, \\ u = h & \text{on } \Delta_r. \end{cases}$$

Then

$$\int_{D_{r/2}} |\nabla u_\varepsilon|^2 \leq C \left\{ \frac{1}{r^2} \int_{D_r} |u|^2 + \int_{D_r} |f|^2 + \int_{D_r} |g|^2 + r^2 \int_{D_r} |F|^2 + \|h\|^2_{H^{1/2}(\Delta_r)} \right] \tag{6.5.5}$$

where C depends only on d, μ, and M.

To prove Theorem 6.5.1 we need an analogue of Theorem 3.3.2 in the presence of boundary. The following lemma is the compactness theorem of Stokes system with Dirichlet boundary conditions.

Lemma 6.5.3. *Let $\{A^k(y)\}$ be a sequence of matrices satisfying the ellipticity condition (1.0.3) (with the same μ) and periodicity condition (1.0.4). Let $D(k) = D(r, \psi_k)$ and $\Delta(k) = \Delta(r, \psi_k)$, where $\{\psi_k\}$ is a sequence of C^1 functions satisfying $\psi_k(0) = 0$ and (6.5.2). Let $(u_k, p_k) \in H^1(D(k); \mathbb{R}^d)) \times L^2(D(k))$ be the weak solution of*

$$\begin{cases} -\mathrm{div}(A^k(x/\varepsilon_k)\nabla u_k) + \nabla p_k = 0 & \text{in } D(k), \\ \mathrm{div}(u_k) = g_k & \text{in } D(k), \\ u_k = h_k & \text{on } \Delta(k). \end{cases}$$

where $\varepsilon_k \to 0$, $f_k(0) = 0$ and

$$\|u_k\|_{H^1(D(k))} + \|p_k\|_{L^2(D(k))} + \|g_k\|_{C^\eta(D(k))} + \|h_k\|_{C^{0,1}(\Delta(k))} \leq C \tag{6.5.6}$$

Then there exist subsequences of $\{A^k\}$, $\{u_k\}$, $\{p_k\}$, $\{\psi_k\}$, $\{g_k\}$, and $\{h_k\}$, which we will still denote by the same notation, and a constant matrix A^0 satisfying (3.1.8), a function ψ_0 satisfying $\psi_0(0) = 0$ and (6.5.2), $u_0 \in H^1(D(r, \psi_0); \mathbb{R}^d)$, $p_0 \in L^2(D(r, \psi_0))$,

$g_0 \in C^{\eta}(D(r,\psi_0))$, $h_0 \in C^{0,1}(\Delta(r,\psi_0);\mathbb{R}^d)$ *such that*

$$\begin{cases} \widehat{A^k} \to A^0, \\ \psi_k(x') \to \psi_0(x') \text{ and } \nabla\psi_k(x') \to \nabla\psi_0(x') \text{ uniformly for } |x'| < r, \\ h_k(x',\psi_k(x')) \to h_0(x',\psi_0(x')) \text{ uniformly for } |x'| < r, \\ g_k(x',\psi_k(x')) \to g_0(x',\psi_0(x')) \text{ uniformly for } |x'| < r, \\ u_k(x',x_d-\psi_k(x')) \rightharpoonup u_0(x',x_d-\psi_0(x')) \text{ weakly in } H^1(Q;\mathbb{R}^d), \\ p_k(x',x_d-\psi_k(x')) \rightharpoonup p_0(x',x_d-\psi_0(x')) \text{ weakly in } L^2(Q), \end{cases} \tag{6.5.7}$$

where $Q = \{(x',x_d) : |x'| < r \text{ and } 0 < x_d < 10(M+1)r\}$. *Moreover,* (u_0,p_0) *is a weak solution of*

$$\begin{cases} -\mathrm{div}(A^0\nabla u_0) + \nabla p_0 = 0 & \text{in } D(r,\psi_0), \\ \mathrm{div}(u_0) = g_0 & \text{in } D(r,\psi_0), \\ u_0 = h_0 & \text{on } \Delta(r,\psi_0). \end{cases} \tag{6.5.8}$$

Proof. We first note that (6.5.7) follows from (6.5.2) and (6.5.6) by passing to subsequences. To prove (6.5.8), let $\Omega \subset \overline{\Omega} \subset D(r,\psi_0)$. Observe that if k is sufficiently large, $\Omega \subset D(r,\psi_k)$. We now apply Theorem 3.3.2 in Ω to conclude that $A^k(x/\varepsilon_k)\nabla u_k \rightharpoonup A^0\nabla u_0$ weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$. As a consequence, (u_0,p_0) is a weak solution of $-\mathrm{div}(A^0\nabla u_0) + \nabla p_0 = 0$ and $\mathrm{div}(u_0) = g_0$ in Ω for some domain Ω such that $\overline{\Omega} \subset D(r,\psi_0)$, and thus for $\Omega = D(r,\psi_0)$. Finally, let $v_k(x',x_d) = u_k(x',x_d+\psi(x'))$ and $v_0(x',x_d) = u_0(x',x_d+\psi_0(x'))$. That $u_0 = h_0$ on $\Delta(r,\psi_0)$ in the sense of trace follows from the fact that $v_k \rightharpoonup v_0$ weakly in $H^1(Q;\mathbb{R}^d)$, $v_k(x',0) = h_k(x',\psi_k(x'))$ and $h_k(x',\psi_k(x')) \to h_0(x',\psi_0(x'))$ uniformly on $\{|x'| < r\}$. $\square$

With the help of Lemma 6.5.3, we prove Theorem 6.5.1 by a compactness argument in the same manner as in [3].

Lemma 6.5.4. *Let* $0 < \rho,\eta < 1$. *Then there exist constants* $\varepsilon_0 \in (0,1/2)$ *and* $\theta \in (0,1/4)$, *depending only on* d, μ, ρ, η, *and* (M,ω) *in (6.5.2), such that*

$$\left(\fint_{D(\theta)} |u_\varepsilon|^2 \right)^{1/2} \le \theta^\rho \tag{6.5.9}$$

for any $0 < \varepsilon < \varepsilon_0$, *whenever* $(u_\varepsilon,p_\varepsilon) \in H^1(D_1;\mathbb{R}^d) \times L^2(D_1)$ *is a weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0 & \text{in } D_1, \\ \mathrm{div}(u_\varepsilon) = g & \text{in } D_1, \\ u_\varepsilon = h & \text{on } \Delta_1, \end{cases} \tag{6.5.10}$$

and

$$\begin{cases} h(0) = 0, \quad \|h\|_{C^{0,1}(\Delta_1)} \le 1, \\ \displaystyle\fint_{D_1} |u_\varepsilon|^2 \le 1, \quad \|g\|_{C^\eta(D_1)} \le 1. \end{cases} \tag{6.5.11}$$

Proof. We will prove the lemma by contradiction. Let $\sigma = (1+\rho)/2 > \rho$. Using the boundary Hölder Estimates for solutions of Stokes systems with constant coefficients, we obtain

$$\left(\fint_{D_r} |w|^2\right)^{1/2} \le C r^\sigma \|w\|_{C^\sigma(D_{1/4})} \le C_0 r^\sigma, \tag{6.5.12}$$

if $0 < r < \frac{1}{4}$ and (w, p_0) satisfies

$$\begin{cases} -\operatorname{div}(A^0 \nabla w) + \nabla p_0 = 0 \;\text{ in } D_{1/2}, \\ \operatorname{div}(w) = g \;\text{ in } D_{1/2}, \\ w = h \;\text{ on } \Delta_{1/2} \\ \|h\|_{C^{0,1}(\Delta_{1/2})} \le 1, \quad f(0) = 0 \\ \displaystyle\fint_{D_{1/2}} |w|^2 dx \le |D_1|, \text{ and } \|g\|_{C^\eta(D_{1/2})} \le 1, \end{cases} \tag{6.5.13}$$

where A^0 is a constant matrix satisfying the ellipticity condition (3.1.8). The constant C_0 in (6.5.12) depends only on d, μ, ρ, η, and (M, ω) in (6.5.2). We now choose $\theta \in (0, 1/4)$ so small that

$$2C_0 \theta^\sigma < \theta^\rho \tag{6.5.14}$$

We claim that the lemma holds for this θ and some $\varepsilon_0 > 0$, which depends only on d, μ, ρ, η, and (M, ω).

Suppose this is not the case, then there exist sequences $\{\varepsilon_k\}$, $\{A^k\}$, $\{p_k\}$, $\{g_k\}$, $\{h_k\}$, $\{\psi_k\}$, such that as $\varepsilon_k \to 0$, A^k satisfies ellipticity (1.0.3)(with the same μ) and periodicity (1.0.4) conditions, ψ_k satisfies (6.5.2),

$$\begin{cases} -\operatorname{div}(A^k(x/\varepsilon_k) \nabla u_k) + \nabla p_k = 0 \;\text{ in } D(k), \\ \operatorname{div}(u_k) = g_k \;\text{ in } D(k), \\ u_k = h_k \;\text{ on } \Delta(k), \\ \|h_k\|_{C^{0,1}(\Delta(k))} \le 1, \; h_k(0) = 0, \\ \displaystyle\left(\fint_{D(k)} |u_k|^2\right)^{1/2} \le 1, \text{ and } \|g_k\|_{C^\eta(D(1,\psi_k))} \le 1, \end{cases} \tag{6.5.15}$$

and

$$\left(\fint_{D(\theta,\psi_k)} |u_k|^2\right)^{1/2} > \theta^\rho \tag{6.5.16}$$

where $D(k) = D(1, \psi_k)$ and $\Delta(k) = \Delta(1, \psi_k)$. Note that by Cacciopoli's inequality (6.5.5), the sequence $\{\|u_k\|_{H^1(D(1/2,\psi_k))}\}$ is uniformly bounded. In view of Lemma 6.5.3, by passing to subsequences, we may assume that

$$\begin{cases} \widehat{A^k} \to A^0, \\ \psi_k \to \psi_0 \text{ and } \nabla\psi_k \to \nabla\psi_0 \text{ uniformly in } \{|x'| < 1\}, \\ u_k(x', x_d - \psi_k(x')) \rightharpoonup u_0(x', x_d - \psi_0(x')) \text{ weakly in } H^1(Q; \mathbb{R}^d), \\ h_k(x', \psi_k(x')) \to h_0(x', \psi_0(x')) \text{ uniformly in } \{|x'| < 1\}, \\ g_k(x', x_d - \psi_k(x')) \to g_0(x', x_d - \psi_0(x')) \text{ uniformly in } Q, \end{cases} \tag{6.5.17}$$

where $Q = \{(x', x_d) : |x'| < 1/2 \text{ and } 0 < x_d < 5(M+1)\}$. Moreover, we note that $u_0 \in H^1(D(1/2, \psi_0); \mathbb{R}^d)$ and satisfies

$$\begin{cases} -\operatorname{div}(\widehat{A}\nabla u_0) + \nabla p_0 = 0 & \text{in } D(1/2, \psi_0), \\ \operatorname{div}(u_0) = g_0 & \text{in } D(1/2, \psi_0), \\ u_0 = h_0 & \text{on } \Delta(1/2, \psi_0). \end{cases}$$

Observe that by (6.5.15) and (6.5.17),

$$h_0(0) = 0, \quad \|h_0\|_{C^{0,1}(\Delta(1/2,\psi_0))} \le 1, \quad \|g_0\|_{C^\eta(D(1/2,\psi_0))} \le 1,$$

and

$$\begin{aligned} \int_{D(1/2,\psi_0)} |u_0|^2 &= \lim_{k\to\infty} \int_{D(1/2,\psi_k)} |u_k|^2 \\ &\le \lim_{k\to\infty} |D(1, \psi_k)| \\ &= |D(1, \psi_0)|. \end{aligned}$$

It follows that $w = u_0$ satisfies (6.5.13). However, by (6.5.16),

$$\left(\fint_{D(\theta,\psi_0)} |u_0|^2 \right)^{1/2} = \lim_{k\to\infty} \left(\fint_{D(\theta,\psi_k)} |u_k|^2 \right)^{1/2} \ge \theta^\rho. \tag{6.5.18}$$

Thus, by (6.5.12), we obtain $\theta^\rho \le C_0 \theta^\sigma$, which contradicts the choice of θ. This completes the proof. $\square$

Lemma 6.5.5. *Fix $0 < \rho, \eta < 1$. Let ε_0 and θ be constants given by Lemma 6.5.4. Suppose that $(u_\varepsilon, p_\varepsilon) \in H^1(D(1,\psi); \mathbb{R}^d) \times L^2(D(1,\psi))$ is a weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0 & \text{in } D(1, \psi), \\ \operatorname{div}(u_\varepsilon) = g & \text{in } D(1, \psi), \\ u_\varepsilon = h & \text{on } \Delta(1, \psi), \end{cases}$$

where $g \in C^\eta(D(1,\psi))$, $h \in C^{0,1}(\Delta(1,\psi), \mathbb{R}^d)$ *and* $h(0) = 0$. *Then, if* $0 < \varepsilon < \varepsilon_0 \theta^{k-1}$, *for some* $k \geq 1$,

$$\left(\fint_{D(\theta^k,\psi)} |u_\varepsilon|^2 \right)^{1/2} \leq \theta^{k\rho} \max \left\{ \left(\fint_{D(1,\psi)} |u_\varepsilon|^2 \right)^{1/2}, \|g\|_{C^\eta(D(1,\psi))}, \|h\|_{C^{0,1}(\Delta(1,\psi))} \right\}. \tag{6.5.19}$$

Proof. We will prove the lemma by an induction argument on k. The case $k = 1$ follows directly from Lemma 6.5.4. Now suppose that the estimate (6.5.19) is true for some $k \geq 1$. Let $0 < \varepsilon < \varepsilon_0 \theta^k$, we apply Lemma 6.5.4 to the function

$$w(x) = u_\varepsilon(\theta^k x) \qquad \text{in } D(1, \psi_k),$$

where $\psi_k(x') = \theta^{-k}\psi(\theta^k x')$. Observe that ψ_k satisfies (6.5.2) uniformly in k, and

$$\begin{cases} \mathcal{L}_{\frac{\varepsilon}{\theta^k}}(w) + \nabla(\theta^k p_\varepsilon(\theta^k x)) = 0 & \text{in } D(1,\psi_k), \\ \operatorname{div}(w) = \theta^k g(\theta^k x) & \text{in } D(1,\psi_k), \\ w = h(\theta^k x) & \text{on } \Delta(1,\psi_k). \end{cases}$$

Since $\theta^{-k}\varepsilon < \varepsilon_0$, by the induction assumption,

$$\begin{aligned} &\left(\fint_{D(\theta^{k+1},\psi)} |u_\varepsilon|^2 \right)^{1/2} = \left(\fint_{D(\theta,\psi_k)} |w|^2 \right)^{1/2} \\ &\leq \theta^\rho \max \left\{ \left(\fint_{D(1,\psi_k)} |w|^2 \right)^{1/2}, \|\theta^k g(\theta^k x)\|_{C^\eta(D(1,\psi_k))}, \|h(\theta^k x)\|_{C^{0,1}(\Delta(1,\psi_k))} \right\} \\ &\leq \theta^\rho \max \left\{ \left(\fint_{D(\theta^k,\psi)} |u_\varepsilon|^2 \right)^{1/2}, \theta^k \|g\|_{C^\eta(D(1,\psi))}, \theta^k \|h\|_{C^{0,1}(\Delta(1,\psi))} \right\} \\ &\leq \theta^{(k+1)\rho} \max \left\{ \left(\fint_{D(1,\psi)} |u_\varepsilon|^2 \right)^{1/2}, \|g\|_{C^\eta(D(1,\psi))}, \|h\|_{C^{0,1}(\Delta(1,\psi))} \right\}. \end{aligned}$$

This completes the proof. □

We now ready to give the proof of Theorem 6.5.1 and Theorem 6.1.3.

Proof of Theorem 6.5.1. By considering the function $u_\varepsilon(rx)$ in $D(1, \psi_r)$, where $\psi_r(x') = r^{-1}\psi(rx')$, we may assume that $r = 1$. Note that

$$\|\nabla \psi_r\|_\infty = \|\nabla \psi\|_\infty \leq M$$

and

$$\begin{aligned} |\nabla \psi_r(x') - \nabla \psi_r(y')| &= |\nabla \psi(rx') - \nabla \psi(ry')| \\ &\leq \omega(|rx' - ry'|) \\ &\leq \omega(r_0|x' - y'|). \end{aligned}$$

The bounding constant C will depend on r_0, if $r_0 > 1$.

Let $\varepsilon \le t < 1$. We may assume that $t < \varepsilon_0 \theta$, for otherwise the estimate is trivial. Choose $k \ge 1$ so that $\varepsilon_0 \theta^{k+1} \le t < \varepsilon \theta^k$. Since $\varepsilon < \varepsilon_0 \theta^{k-1}$, it follows from Lemma 6.5.5 that

$$
\begin{aligned}
\left(\fint_{D(t,\psi)} |u_\varepsilon|^2 \right)^{1/2} &\le C \left(\fint_{D_{\theta^k}} |u_\varepsilon|^2 \right)^{1/2} \\
&\le C\theta^{k\rho} \left\{ \left(\fint_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^\eta(D_1)} + \|h\|_{C^{0,1}(\Delta_1)} \right\} \\
&\le Ct^\rho \left\{ \left(\fint_{D_1} |u_\varepsilon|^2 \right)^{1/2} + \|g\|_{C^\eta(D_1)} + \|h\|_{C^{0,1}(\Delta_1)} \right\}
\end{aligned}
$$

This finishes the proof. □

Proof of Theorem 6.1.3. First, we note that by Cacciopoli's inequality and Poincarè inequality, it suffices to show that

$$
\left(\fint_{B(x_0,r)\cap\Omega} |u_\varepsilon|^2 \right)^{1/2} \le \left(\frac{r}{R} \right)^\rho \left(\fint_{B(x_0,R)\cap\Omega} |u_\varepsilon|^2 \right)^{1/2} \tag{6.5.20}
$$

for $0 < r < c_0 R < R_0$. By translation we may assume that $x_0 = 0$. Next, we may assume that in a new coordinate system, obtained from the current system through a rotation by an orthogonal matrix with rational entries,

$$
\begin{aligned}
B(0,R) \cap \Omega &= B(0,R) \cap \{(x', x_d) : x_d > \psi(x')\} \\
B(0,R) \cap \partial\Omega &= B(0,R) \cap \{(x', x_d) : x_d = \psi(x')\}
\end{aligned} \tag{6.5.21}
$$

where ψ is a C^1 function satisfying $\psi(0) = 0$ and (6.5.2). Here we have used the fact that for any $d \times d$ orthogonal matrix O and $\delta > 0$, there exists a $d \times d$ orthogonal matrix T with rational entries such that $\|O - T\|_\infty < \delta$. Moreover, each entry of T has a denominator less than a constant depending only on d and δ (see [47]). Finally, we point out that if $(u_\varepsilon, p_\varepsilon)$ is a solution of the Stokes system (1.0.2) and $u^\beta(x) = T_{\gamma\beta} v^\gamma(y)$, $p(x) = q(y)$, where $T = (T_{ij})$ is an orthogonal matrix and $y = Tx$, then

$$
\begin{cases}
-\text{div}_y(B(y/\varepsilon)\nabla_y v) + \nabla_y q = G(y), \\
\qquad\qquad\qquad\qquad \text{div}_y(v) = h(y),
\end{cases} \tag{6.5.22}
$$

where $B(y) = (b_{k\ell}^{t\gamma}(y))$ with

$$
\begin{cases}
b_{k\ell}^{t\gamma}(y) = T_{t\alpha} T_{\gamma\beta} T_{\ell j} T_{ki} a_{ij}^{\alpha\beta}(x), \\
G^t(y) = T_{t\alpha} F^\alpha(x), \\
h(y) = g(x).
\end{cases}
$$

Note that the matrix $B(y)$ is periodic, if T has rational entries (a dilation may be needed to ensure that B is 1-periodic). These observations allow us to deduce estimate (6.5.20) from Theorem 6.5.1 and complete the proof. $\square$

6.6 Interior $W^{1,p}$ estimates

In this and next sections we establish uniform $W^{1,p}$ estimates for the Stokes system (1.0.2) under the additional condition that A belongs to $\mathrm{VMO}(\mathbb{R}^d)$ (see also in (2.1.10):

$$\sup_{\substack{y\in\mathbb{R}^d\\ 0<t<r}} \fint_{B(y,t)} \Big|A - \fint_{B(y,t)} A\Big| \le \omega_1(r), \tag{6.6.1}$$

where ω_1 is a (fixed) nondecreasing continuous function on $[0,\infty)$ and $\omega_1(0) = 0$.

The following two lemmas provide the local interior and boundary $W^{1,p}$ estimates of Stokes system with variable coefficients.

Lemma 6.6.1. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and smoothness condition (6.6.1). Let $(u,p) \in H^1(B(0,1);\mathbb{R}^d) \times L^2(B(0,1))$ be a weak solution to*

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \nabla p = 0, \\ \qquad\qquad \mathrm{div}(u) = 0 \end{cases} \tag{6.6.2}$$

in $B(0,1)$. Then $|\nabla u| \in L^q(B(0,1/2))$ for any $2 < q < \infty$, and

$$\left(\fint_{B(0,1/2)} |\nabla u|^q\right)^{1/q} \le C_q \left(\fint_{B(0,1)} |\nabla u|^2\right)^{1/2}. \tag{6.6.3}$$

where C_q depends only on d, μ, q, and ω_1 in (6.6.1).

Lemma 6.6.2. *Suppose that $A(y)$ satisfies the ellipticity condition (1.0.3) and smoothness condition (6.6.1). Let $(u,p) \in H^1(D_1;\mathbb{R}^d) \times L^2(D_1)$ be a weak solution to (6.6.2) in D_1 and $u = 0$ on Δ_1. Then $|\nabla u| \in L^q(D_{1/2})$ for any $2 < q < \infty$, and*

$$\left(\fint_{D_{1/2}} |\nabla u|^q\right)^{1/q} \le C_q \left(\fint_{D_1} |\nabla u|^2\right)^{1/2}. \tag{6.6.4}$$

where C_q depends only on d, μ, q, (M,ω) in (6.5.2) and ω_1 in (6.6.1).

We remark that $W^{1,p}$ estimates for elliptic equations and systems with continuous or VMO coefficients have been studied extensively in recent years. In particular, estimates in Lemma 6.6.1 and 6.6.2 are known for solutions of $-\mathrm{div}(A(x)\nabla u) = 0$ (see [11–13,38,48] and their references). To prove Lemma 6.6.1 and 6.6.2, one follows the approach in [48] and apply a real-variable argument originated in [13]. This reduces the problem to the case of Stokes systems with constant coefficients. Note that for Stokes systems with constant coefficients, the interior estimate (6.6.3) is well known, while the boundary estimate (6.6.4) in C^1 domains follows from [19]. We omit the details.

Lemma 6.6.3. *Suppose that $A(y)$ satisfies ellipticity (1.0.3), periodicity (1.0.4) and VMO continuity (6.6.1) conditions. Let $(u_\varepsilon, p_\varepsilon) \in H^1(B(x_0,r);\mathbb{R}^d) \times L^2(B(x_0,r))$ be a weak solution to*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0, \\ \qquad \operatorname{div}(u_\varepsilon) = 0 \end{cases} \tag{6.6.5}$$

in $B(x_0,r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Then for any $2 < q < \infty$,

$$\left(\fint_{B(x_0,r/2)} |\nabla u_\varepsilon|^q \right)^{1/q} \le C_q \left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6.6.6}$$

where C_q depends only on d, μ, q, and ω_1 in (6.6.1).

Proof. By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. We may also assume $\varepsilon < (1/4)$. The case $\varepsilon \ge (1/4)$ follows directly from Lemma 6.6.1, as the coefficient matrix $A(x/\varepsilon)$ satisfies (6.6.1) uniformly in ε.

Let

$$u(x) = \varepsilon^{-1} u_\varepsilon(\varepsilon x) \quad \text{and} \quad p(x) = p_\varepsilon(\varepsilon x).$$

Then (u,p) satisfies (6.6.2) in $B(0,1)$. It follows that

$$\begin{aligned} \left(\fint_{B(0,\varepsilon/2)} |\nabla u_\varepsilon|^q \right)^{1/q} &\le C \left(\fint_{B(0,\varepsilon)} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ &\le C \left(\fint_{B(0,1/2)} |\nabla u_\varepsilon|^2 \right)^{1/2}, \end{aligned}$$

where we have used Theorem 1.0.1 for the second inequality. By translation the same argument also gives

$$\left(\fint_{B(y,\varepsilon/2)} |\nabla u_\varepsilon|^q \right)^{1/q} \le C \left(\fint_{B(y,1/2)} |\nabla u_\varepsilon|^2 \right)^{1/2} \tag{6.6.7}$$

for any $y \in B(0,1/2)$. Estimate (6.6.6) now follows from (6.6.7) by covering $B(0,1/2)$ with balls $\{B(y_k,\varepsilon/2)\}$, where $y_k \in B(0,1/2)$. $\square$

The next theorem, whose proof may be found in [49], provides a real-variable argument we will need for the $W^{1,p}$ estimates.

Theorem 6.6.4. *Let B_0 be a ball in $\mathbb{R}^d$ and $F \in L^2(4B_0)$. Let $q > 2$ and $f \in L^p(4B_0)$ for some $2 < p < q$. Suppose that for each ball $B \subset 2B_0$ with $|B| \le c_1|B_0|$, there exist two measurable functions F_B and R_B on $2B$, such that $|F| \le |F_B| + |R_B|$ on $2B$,*

$$\begin{aligned}
\left(\fint_{2B} |R_B|^q \right)^{1/q} &\le C_1 \left\{ \left(\fint_{c_2 B} |F|^2 \right)^{1/2} + \sup_{4B_0 \supset B' \supset B} \left(\fint_{B'} |f|^2 \right)^{1/2} \right\}, \\
\left(\fint_{2B} |F_B|^2 \right)^{1/2} &\le C_2 \sup_{4B_0 \supset B' \supset B} \left(\fint_{B'} |f|^2 \right)^{1/2},
\end{aligned} \tag{6.6.8}$$

where $C_1, C_2 > 0$, $0 < c_1 < 1$, and $c_2 > 2$. Then $F \in L^p(B_0)$ and

$$\left(\fint_{B_0} |F|^p \right)^{1/p} \le C \left\{ \left(\fint_{4B_0} |F|^2 \right)^{1/2} + \left(\fint_{4B_0} |f|^p \right)^{1/p} \right\}, \tag{6.6.9}$$

where C depends only on C_1, C_2, c_1, c_2, p and q.

We are now ready to prove the interior $W^{1,p}$ estimates for Stokes system (1.0.2).

Theorem 6.6.5. *Suppose that $A(y)$ satisfies ellipticity (1.0.3), periodicity (1.0.4) and VMO continuity (6.6.1) conditions. Let $(u_\varepsilon, p_\varepsilon) \in H^1(B(x_0, r); \mathbb{R}^d) \times L^2(B(x_0, r))$ be a weak solution to*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = \operatorname{div}(f) \\ \operatorname{div}(u_\varepsilon) = g \end{cases} \tag{6.6.10}$$

in $B(x_0, r)$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$. Then for any $2 < q < \infty$,

$$\begin{aligned}
&\left(\fint_{B(x_0, r/2)} |\nabla u_\varepsilon|^q \right)^{1/q} + \left(\fint_{B(x_0, r/2)} \left| p_\varepsilon - \fint_{B(x_0, r/2)} p_\varepsilon \right|^q \right)^{1/q} \\
&\le C_q \left\{ \left(\fint_{B(x_0, r)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(x_0, r)} |f|^q \right)^{1/q} + \left(\fint_{B(x_0, r)} |g|^q \right)^{1/q} \right\},
\end{aligned} \tag{6.6.11}$$

where C_q depends only on d, μ, q, and ω_1 in (6.6.1).

Proof. By translation and dilation we may assume that $x_0 = 0$ and $r = 1$. Note that the estimate for p_ε in (6.6.11) follows easily from the estimate for ∇u_ε by applying (4.3.1) to the system (6.6.10). Also we may assume that $g = 0$ by considering $u_\varepsilon - \nabla w$, where w is a scalar function such that

$$\begin{cases} \Delta w = g & \text{in } B(0,1), \\ w = 0 & \text{on } \partial B(0,1). \end{cases}$$

To apply Theorem 6.6.4, for each $B = B(y, t) \subset B(0, 3/4)$ with $0 < t < (1/64)$, we write

$$u_\varepsilon = v_\varepsilon + z_\varepsilon,$$

where $v_\varepsilon \in H_0^1(4B;\mathbb{R}^d)$ and

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) + \nabla \pi_\varepsilon = \operatorname{div}(f) & \text{in } 4B, \\ \operatorname{div}(v_\varepsilon) = 0 & \text{in } 4B. \end{cases}$$

Note that

$$\fint_{4B} |\nabla v_\varepsilon|^2 \le C \fint_{4B} |f|^2. \tag{6.6.12}$$

Also, since z_ε satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(z_\varepsilon) + \nabla (p_\varepsilon - \pi_\varepsilon) = 0 & \text{in } 4B, \\ \operatorname{div}(z_\varepsilon) = 0 & \text{in } 4B, \end{cases}$$

we may apply Lemma 6.6.3 to obtain

$$\begin{aligned} \left(\fint_{2B} |\nabla z_\varepsilon|^{\bar q}\right)^{1/\bar q} &\le C \left(\fint_{4B} |\nabla z_\varepsilon|^2\right)^{1/2} \\ &\le C \left(\fint_{4B} |\nabla u_\varepsilon|^2\right)^{1/2} + C\left(\fint_{4B} |f|^2\right)^{1/2}, \end{aligned} \tag{6.6.13}$$

where $\bar q = q+1$ and we have used (6.6.12) for the last inequality.

Finally, Let $F = |\nabla u_\varepsilon|$, $F_B = |\nabla v_\varepsilon|$ and $R_B = |\nabla z_\varepsilon|$. Note that

$$|F| \le |F_B| + |R_B| \qquad \text{in } 4B,$$

and in view of (6.6.12) and (6.6.13), we have proved that

$$\begin{aligned} \left(\fint_{2B} |R_B|^{\bar q}\right)^{1/\bar q} &\le C \left(\fint_{4B} |F|^2\right)^{1/2} + C\left(\fint_{4B} |f|^2\right)^{1/2}, \\ \left(\fint_{2B} |F_B|^{\bar q}\right)^{1/\bar q} &\le C \left(\fint_{4B} |f|^2\right)^{1/2}. \end{aligned}$$

This allows us to use Theorem 6.6.4 to conclude that

$$\left(\fint_{B(x_0,1/16)} |\nabla u_\varepsilon|^q\right)^{1/q} \le C \left\{ \left(\fint_{B(0,1)} |\nabla u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B(0,1)} |f|^q\right)^{1/q} \right\}$$

for any $x_0 \in B(0,1/2)$, which gives the desired estimate for ∇u_ε by a simple covering argument. □

6.7 Uniform Boundary $W^{1,p}$ estimates and Proof of Theorem 1.0.2

In this section we establish uniform boundary $W^{1,p}$ estimates and gives the proof of Theorem 1.0.2. Throughout this section we will assume that A satisfies ellipticity

(1.0.3), periodicity (1.0.4) and VMO continuity (6.6.1) conditions and that Ω is a bounded C^1 domain.

We begin with a boundary Hölder estimate.

Lemma 6.7.1. *Let $x_0 \in \partial\Omega$ and $0 < R < R_0$, where $R_0 =$ diam(Ω). Let $(u_\varepsilon, p_\varepsilon) \in W^{1,2}(B(x_0,R)\cap\Omega;\mathbb{R}^d) \times L^2(B(x_0,R)\cap\Omega)$ be a weak solution to*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = 0 & \text{in } B(x_0,R)\cap\Omega, \\ \operatorname{div}(u_\varepsilon) = 0 & \text{in } B(x_0,R)\cap\Omega, \\ u_\varepsilon = 0 & \text{on } B(x_0,R)\cap\partial\Omega. \end{cases} \tag{6.7.1}$$

Then

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le C\left(\frac{|x-y|}{R}\right)^\rho \left(\fint_{B(x_0,R)\cap\Omega} |u_\varepsilon|^2\right)^{1/2}, \tag{6.7.2}$$

for any $x, y \in B(x_0, R/2)\cap\Omega$, where $0 < \rho < 1$ and C depends only on d, ρ, A and Ω.

Proof. By translation and dilation we may assume that $x_0 = 0$ and $R = 1$. The case $\varepsilon \ge (1/4)$ follows directly from the local boundary $W^{1,p}$ estimates in Lemma 6.6.2 by Sobolev imbedding. To treat the case $0 < \varepsilon < (1/4)$, we note that if $0 < r < \varepsilon$, we may deduce from Lemma 6.6.2 by rescaling that

$$\begin{aligned} \left(\fint_{B(0,r)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} &\le C_q \left(\frac{\varepsilon}{r}\right)^{\frac{d}{q}} \left(\fint_{B(0,\varepsilon)\cap\Omega} |\nabla u_\varepsilon|^q\right)^{1/q} \\ &\le C_q \left(\frac{\varepsilon}{r}\right)^{\frac{d}{q}} \left(\fint_{B(0,2\varepsilon)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \end{aligned} \tag{6.7.3}$$

for any $2 < q < \infty$, where we have used Hölder's inequality for the first inequality. This, together with the estimate in Theorem 6.1.3, implies that

$$\left(\fint_{B(0,r)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \le C_\rho r^{\rho-1} \left(\fint_{B(0,1)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \tag{6.7.4}$$

for any $0 < r < (1/2)$, where $0 < \rho < 1$. A similar argument gives

$$\left(\fint_{B(y,r)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \le C_\rho r^{\rho-1} \left(\fint_{B(0,1)\cap\Omega} |\nabla u_\varepsilon|^2\right)^{1/2} \tag{6.7.5}$$

for any $y \in B(0, 1/2)$ and $0 < r < (1/2)$. The estimate (6.7.2) now follows. $\square$

Lemma 6.7.2. *Let $x_0 \in \partial\Omega$ and $0 < R < R_0$, where $R_0 = \operatorname{diam}(\Omega)$. Let $(u_\varepsilon, p_\varepsilon) \in W^{1,2}(B(x_0, R) \cap \Omega; \mathbb{R}^d) \times L^2(B(x_0, R) \cap \Omega)$ be a weak solution to the system (6.7.1) given in Lemma 6.7.1. Then for any $2 < q < \infty$,*

$$\left(\fint_{B(x_0,R/2)\cap\Omega} |\nabla u_\varepsilon|^q \right)^{1/q} \leq C_q \left(\fint_{B(x_0,R)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{6.7.6}$$

where C_q depends only on d, ρ, A and Ω.

Proof. By translation and dilation we may assume that $x_0 = 0$ and $R = 1$. Let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$. It follows from the interior $W^{1,p}$ estimates in Lemma 6.6.3 that

$$\fint_{B(y,c\,\delta(y))} |\nabla u_\varepsilon(x)|^q dx \leq C \fint_{B(y,2c\,\delta(y))} \left| \frac{u_\varepsilon(x)}{\delta(x)} \right|^q dx, \tag{6.7.7}$$

for any $y \in B(0, 1/2) \cap \Omega$, where $c = c(\Omega) > 0$ is sufficiently small. Integrating both sides of (6.7.7) in y over $B(0, 1/2) \cap \Omega$ yields

$$\int_{B(0,1/2)\cap\Omega} |\nabla u_\varepsilon(x)|^q dx \leq C \int_{B(0,3/4)\cap\Omega} \left| \frac{u_\varepsilon(x)}{\delta(x)} \right|^q dx. \tag{6.7.8}$$

Finally, note that by Lemma 6.7.1,

$$|u_\varepsilon(x)| \leq C[\delta(x)]^\rho \left(\fint_{B(0,1)\cap\Omega} |u_\varepsilon|^2 \right)^{1/2} \tag{6.7.9}$$

for any $x \in B(0, 3/4) \cap \Omega$. Choosing $\rho \in (0, 1)$ so that $(1 - \rho)q < 1$, we obtain estimate (6.7.6) by substituting (6.7.9) into the right hand side of (6.7.8). $\square$

The following theorem gives the boundary $W^{1,p}$ estimates for the Stokes system (1.0.2).

Theorem 6.7.3. *Suppose that $A(y)$ satisfies ellipticity (1.0.3), periodicity (1.0.4) and VMO continuity (6.6.1) conditions. Let Ω be a bounded C^1 domain in $\mathbb{R}^d$. Let $(u_\varepsilon, p_\varepsilon) \in H^1(B(x_0, R) \cap \Omega; \mathbb{R}^d) \times L^2(B(x_0, R) \cap \Omega)$ be a weak solution to*

$$\begin{cases} -\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) + \nabla p_\varepsilon = \operatorname{div}(f), \\ \operatorname{div}(u_\varepsilon) = g \end{cases} \tag{6.7.10}$$

in $B(x_0, R) \cap \Omega$ for some $x_0 \in \partial\Omega$ and $0 < R < R_0$, where $R_0 = \operatorname{diam}(\Omega)$. Then for any $2 < q < \infty$,

$$\begin{aligned} &\left(\fint_{B(x_0,R/2)\cap\Omega} |\nabla u_\varepsilon|^q \right)^{1/q} + \left(\fint_{B(x_0,R/2)\cap\Omega} \left|p_\varepsilon - \fint_{B(x_0,R/2)\cap\Omega} p_\varepsilon\right|^q \right)^{1/q} \\ &\leq C_q \left\{ \left(\fint_{B(x_0,R)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2} + \left(\fint_{B(x_0,R)\cap\Omega} |f| \right)^{1/q} + \left(\fint_{B(x_0,R)\cap\Omega} |g|^q \right)^{1/q} \right\}, \end{aligned} \tag{6.7.11}$$

where C_q depends only on d, μ, q, ω_1 in (6.6.1) and Ω.

Proof. This theorem follows from Lemma 6.6.3 and 6.7.2 by a real-variable argument in the same manner as in the proof of Theorem 6.6.5. We omit the details and refer the reader to [48]. $\square$

Finally, we give the proof of Theorem 1.0.2.

Proof of Theorem 1.0.2. Since $h \in B^{1-\frac{1}{q},q}(\partial\Omega;\mathbb{R}^d)$ and Ω is a bounded C^1 domain, there exists $H \in W^{1,q}(\Omega;\mathbb{R}^d)$ such that

$$\|H\|_{W^{1,q}(\Omega)} \leq C\|h\|_{B^{1-\frac{1}{q},q}(\partial\Omega)}.$$

Thus, by considering $u_\varepsilon - H$, we may assume that $h = 0$. Note that if $u_\varepsilon, v_\varepsilon \in W_0^{1,2}(\Omega;\mathbb{R}^d)$ satisfy

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = \operatorname{div}(f) \\ \operatorname{div}(u_\varepsilon) = g \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}_\varepsilon^*(v_\varepsilon) + \nabla \pi_\varepsilon = \operatorname{div}(F) \\ \operatorname{div}(u_\varepsilon) = G \end{cases} \tag{6.7.12}$$

in Ω, then

$$\begin{aligned} \int_\Omega \nabla u_\varepsilon \cdot F + \int_\Omega \big(p_\varepsilon - \fint_\Omega p_\varepsilon\big) \cdot G \\ = \int_\Omega \nabla v_\varepsilon \cdot f + \int_\Omega \big(\pi_\varepsilon - \fint_\Omega \pi_\varepsilon\big) \cdot g \end{aligned} \tag{6.7.13}$$

This allows us to use a duality argument that reduces the theorem to the estimate

$$\|\nabla u_\varepsilon\|_{L^q(\Omega)} + \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^q(\Omega)} \leq C\left\{\|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}\right\} \tag{6.7.14}$$

for $2 < q < \infty$, where

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) + \nabla p_\varepsilon = \operatorname{div}(f) & \text{in } \Omega, \\ \operatorname{div}(u_\varepsilon) = g & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, by covering Ω with balls of radius $r_0 = c_0 \operatorname{diam}(\Omega)$, we may deduce from Theorem 6.6.5 and 6.7.1 that

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L^q(\Omega)} &\leq C\{\|\nabla u_\varepsilon\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}\} \\ &\leq C\{\|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}\}, \end{aligned}$$

where we have used the estimate in Theorem 3.1.1 as well as $q > 2$. Also, note that

$$\begin{aligned} \|p_\varepsilon - \fint_\Omega p_\varepsilon\|_{L^q(\Omega)} &\leq C\|\nabla p_\varepsilon\|_{W^{-1,q}(\Omega)} \\ &\leq \{\|\nabla u_\varepsilon\|_{L^q(\Omega)} + \|f\|_{L^q(\Omega)}\} \\ &\leq C\{\|f\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}\}, \end{aligned}$$

where we have used $\nabla p_\varepsilon = \mathcal{L}_\varepsilon(u_\varepsilon) - \operatorname{div}(f)$ in Ω for the second inequality. This completes the proof. $\square$

Bibliography

[1] S. N. Armstrong and Z. Shen, *Lipschitz estimates in almost-periodic homogenization*, to appear in Comm. Pure Appl. Math. (2016).

[2] S. N. Armstrong and C. K. Smart, *Quantitative stochastic homogenization of elliptic equations in nondivergence form*, Arch. Ration. Mech. Anal. **214** (2014), no. 3, 867–911. MR 3269637

[3] M. Avellaneda and F. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math. **40** (1987), no. 6, 803–847.

[4] ———, *Homogenization of elliptic problems with L^p boundary data*, Appl. Math. Optim. **15** (1987), no. 1, 93–107 (English).

[5] ———, *A Liouville theorem for elliptic equations with periodic coefficients*, C. R. Acad. Sci. Paris Sr. I Math. **309** (1989), no. 5, 245–250.

[6] ———, *Compactness methods in the theory of homogenization II: Equations in non-divergence form*, Comm. Pure Appl. Math. **42** (1989), no. 2, 139–172.

[7] ———, *L^p bounds on singular integrals in homogenization*, Comm. Pure Appl. Math. **44** (1991), no. 8-9, 897–910.

[8] N. S. Bahvalov, *Averaged characteristics of bodies with periodic structure*, Dokl. Akad. Nauk SSSR **218** (1974), 1046–1048. MR 0395435

[9] ———, *Averaging of partial differential equations with rapidly oscillating coefficients*, Dokl. Akad. Nauk SSSR **221** (1975), no. 3, 516–519. MR 0380075

[10] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 2011, Corrected reprint of the 1978 original [MR0503330]. MR 2839402

[11] S. Byun and L. Wang, *Elliptic equations with BMO coefficients in Reifenberg domains*, Comm. Pure Appl. Math. **57** (2004), no. 10, 12831310.

[12] ———, *Gradient estimates for elliptic systems in non-smooth domains.*, Math. Ann. **341** (2008), no. 3, 629650.

[13] L. A. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form*, Comm. Pure Appl. Math. **51** (1998), no. 1, 1–21.

[14] D. Cioranescu, A. Damlamian, and G. Griso, *Periodic unfolding and homogenization*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 1, 99–104.

[15] ———, *The periodic unfolding method in homogenization*, SIAM J. Math. Anal. **40** (2008), no. 4, 1585–1620.

[16] E. De Giorgi, *Sulla convergenza di alcune successioni d'integrali del tipo dell'area*, Rend. Mat. (6) **8** (1975), 277–294, Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday. MR 0375037

[17] ———, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna, 1979, pp. 131–188. MR 533166

[18] E. De Giorgi and S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. Ital. (4) **8** (1973), 391–411. MR 0348255

[19] M. Dindoš and M. Mitrea, *The stationary Navier-Stokes system in nonsmooth manifolds: the Poisson problem in Lipschitz and C^1 domains*, Arch. Ration. Mech. Anal. **174** (2004), no. 1, 1–47. MR 2092995

[20] R. G. Durán, *An elementary proof of the continuity from $L^2_0(\Omega)$ to $H^1_0(\Omega)^n$ of Bogovskii's right inverse of the divergence*, Rev. Un. Mat. Argentina **53** (2012), no. 2, 59–78.

[21] J. Geng, *$W^{1,p}$ estimates for elliptic problems with Neumann boundary conditions in Lipschitz domains*, Adv. Math. **229** (2012), no. 4, 2427–2448.

[22] J. Geng and Z. Shen, *Uniform regularity estimates in parabolic homogenization*, Indiana Univ. Math. J. **64** (2015), no. 3, 697–733.

[23] J. Geng, Z. Shen, and L. Song, *Uniform $W^{1,p}$ estimates for systems of linear elasticity in a periodic medium*, J. Funct. Anal. **262** (2012), no. 4, 1742–1758.

[24] M. Giaquinta and G. Modica, *Nonlinear systems of the type of the stationary Navier-Stokes system*, J. Reine Angew. Math. **330** (1982), 173–214.

[25] A. Gloria, S. Neukamm, and F. Otto, *A regularity theory for random elliptic operators*, arXiv:1409.2678 (2014).

[26] G. Griso, *Error estimate and unfolding for periodic homogenization*, Asymptot. Anal. **40** (2004), no. 3-4, 269–286.

[27] ———, *Interior error estimate for periodic homogenization*, Anal. Appl. (Singap.) **4** (2006), no. 1, 61–79.

[28] S. Gu, *Convergence rates in homogenization of Stokes systems*, J. Differential Equations **260** (2016), no. 7, 5796–5815.

[29] ———, *Convergence rates of Neumann problems for Stokes systems*, Submitted (2016).

[30] S. Gu and Z. Shen, *Homogenization of Stokes systems and uniform regularity estimates*, SIAM J. Math. Anal. **47** (2015), no. 5, 4025–4057.

[31] V. V. Jikov, S. M. Kozlov, and O. A. Oleynik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.

[32] C. E. Kenig, F. Lin, and Z. Shen, *Convergence rates in L^2 for elliptic homogenization problems*, Arch. Ration. Mech. Anal. **203** (2012), no. 3, 1009–1036 (English).

[33] ———, *Estimates of eigenvalues and eigenfunctions in periodic homogenization*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 5, 1901–1925.

[34] ———, *Homogenization of elliptic systems with Neumann boundary conditions*, J. Amer. Math. Soc. **26** (2013), no. 4, 901–937.

[35] ———, *Periodic homogenization of Green and Neumann functions*, Comm. Pure Appl. Math. **67** (2014), no. 8, 1219–1262.

[36] C. E. Kenig and Z. Shen, *Homogenization of elliptic boundary value problems in Lipschitz domains*, Math. Ann. **350** (2011), no. 4, 867–917 (English).

[37] ———, *Layer potential methods for elliptic homogenization problems*, Comm. Pure Appl. Math. **64** (2011), no. 1, 1–44.

[38] N. V. Krylov, *Parabolic and elliptic equations with VMO coefficient*, Comm. Partial Differential Equations **32** (2007), no. 1-3, 453–475.

[39] P. Kuchment and Y. Pinchover, *Liouville theorems and spectral edge behavior on abelian coverings of compact manifolds.*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 57775815.

[40] J.-L. Lions, *Asymptotic expansions in perforated media with a periodic structure*, Rocky Mountain J. Math. **10** (1980), no. 1, 125–140. MR 573867

[41] J. Moser and M. Struwe, *On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus.*, Bol. Soc. Brasil. Mat. (N.S.) **23** (1992), no. 1-2, 1–20.

[42] S. Moskow and M. Vogelius, *First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof.*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 6, 12631299.

[43] F. Murat, *Compacité par compensation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), no. 3, 489–507. MR 506997

[44] D. Onofrei and B. Vernescu, *Error estimates for periodic homogenization with non-smooth coefficients*, Asymptot. Anal. **54** (2007), no. 1-2, 103–123.

[45] M. A. Pakhnin and T. A. Suslina, *Operator error estimates for the homogenization of the elliptic Dirichlet problem in a bounded domain*, St. Petersburg Math. J. **24** (2013), no. 6, 949–976.

[46] S. E. Pastukhova, *On some estimates from the homogenization of problems in plasticity theory. (Russian)*, Dokl. Akad. Nauk **406** (2006), no. 5, 604–608.

[47] E. Schmutz, *Rational points on the unit sphere*, Cent. Eur. J. Math. **6** (2008), no. 3, 482–487. MR 2425007

[48] Z. Shen, *Bounds of Riesz transforms on L^p spaces for second order elliptic operators*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 173–197.

[49] ———, *The L^p boundary value problems on Lipschitz domains*, Adv. Math. **216** (2007), no. 1, 212–254.

[50] ———, *$W^{1,p}$ estimates for elliptic homogenization problems in nonsmooth domains*, Indiana Univ. Math. J. **57** (2008), no. 5, 2283–2298.

[51] ———, *Boundary estimates in elliptic homogenization*, arXiv:1505.00694 (2015).

[52] Z. Shen and J. Zhuge, *Convergence rates in periodic homogenization of systems of elasticity*, to appear in Proc. Amer. Math. Soc. (2015).

[53] S. Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore*, Ann. Scuola Norm. Sup. Pisa (3) **21** (1967), 657–699. MR 0225015

[54] ______, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche.*, Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597; errata, ibid. (3) **22** (1968), 673. MR 0240443

[55] T. A. Suslina, *Homogenization of the Dirichlet problem for elliptic systems: L_2-operator error estimates*, Mathematika **59** (2013), no. 2, 463–476.

[56] ______, *Homogenization of the Neumann problem for elliptic systems with periodic coefficients*, SIAM J. Math. Anal. **45** (2013), no. 6, 3453–3493.

[57] L. Tartar, *Homogénéisation et compacité par compensation*, Séminaire Goulaouic-Schwartz (1978/1979), École Polytech., Palaiseau, 1979, pp. Exp. No. 9, 9. MR 557520

[58] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis.*, Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[59] V. V. Zhikov and S. E. Pastukhova, *On operator estimates for some problems in homogenization theory*, Russ. J. Math. Phys. **12** (2005), no. 4, 515–524.

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