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Colorings of Hamming-Distance Graphs

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Isaiah Harney
Lexington, Kentucky

Director: Dr. Heide Gluesing-Luerssen, Professor of Mathematics
Lexington, Kentucky 2017

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ABSTRACT OF DISSERTATION

Colorings of Hamming-Distance Graphs

Hamming-distance graphs arise naturally in the study of error-correcting codes and have been utilized by several authors to provide new proofs for (and in some cases improve) known bounds on the size of block codes. We study various standard graph properties of the Hamming-distance graphs with special emphasis placed on the chromatic number. A notion of robustness is defined for colorings of these graphs based on the tolerance of swapping colors along an edge without destroying the properness of the coloring, and a complete characterization of the maximally robust colorings is given for certain parameters. Additionally, explorations are made into subgraph structures whose identification may be useful in determining the chromatic number.

KEYWORDS: Hamming distance, graphs, coloring, q-ary block codes

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Date: April 28, 2017

Colorings of Hamming-Distance Graphs

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To all the nights I didn't sleep.
And all the dreams that still came true.

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Chapter 1 Introduction

1.1 Error-Correcting Codes

The study of error-correcting codes is concerned with the transmission of information via "noisy" channels in which the original message may be corrupted as it travels from sender to receiver. It is often infeasible or inefficient to simply resend the message several times. For instance, consider the cases of digital broadcasting or cell phone communication. By encoding messages before transmission, it is possible to recover the intended message even if "noise" on the channel results in an erroneous transmission. Finding the most efficient methods of transmitting data while simultaneously allowing for the correction of errors plays a central role in information theory.

A code $\mathcal{C}$ is simply a collection of known messages, called codewords. If we are wise about how we choose our code, it is possible to recover the original message if the size of the error in the transmission is not too large. A common technique is to choose a code such that all the codewords are very dissimilar from each other. The "dissimilarity" of the codewords is measured by a metric called the Hamming distance which simply counts the number of coordinates in which two codewords disagree. If all the codewords in a code have a large Hamming distance from one another, it is usually possible to determine what the intended message was as there will be a unique codeword that is closest to the received message.

A code generally has four important parameters, n, d, q, and M. An $(n, M, d)_q$-code has M codewords of length n over an alphabet of size q such that any two codewords have Hamming distance at least d. The rate of transmission increases with the size M of the code so coding theorist are concerned with finding large codes of given parameters (n, d, q). However, finding the maximal size of a code with given parameters is prohibitively difficult and energy has instead been focused of finding good bounds.

1.2 Hamming-Distance Graphs

The problem of finding large error-correcting codes can be given a graphical interpretation via the Hamming-distance graphs. $H_q(n, d)$ is defined as the graph with vertex set $\mathbb{Z}_q^n$ where two vertices are adjacent if the Hamming distance between them is at least d. Note that the cliques (complete subgraphs) of $H_q(n, d)$ form $(n, d)_q$-codes as described above. Therefore the problem of determining the largest possible size of a code of given parameters is equivalent to finding the clique number (size of largest complete subgraph) of the corresponding Hamming-distance graph.

The Hamming-distance graphs have been studied by several authors including Sloane [23] and El Rouayheb et al. [8], the latter of which applied graph theoretical techniques to $H_q(n, d)$ to reprove (and in some cases actually improve) known coding theory bounds. Although the ultimate goal was to put good bounds on the clique number of $H_q(n, d)$, their proofs involved many other properties of the graphs such as vertex transitivity and the independence number. This leads naturally to questions regarding other properties and graph invariants of $H_q(n,d)$, which we study extensively in this thesis.

The authors of [8] were also interested in determining the chromatic number of these graphs and gave the following result.

Theorem 1.2.1 ([8, Lem. 18]). *Let $q \geq n-d+2$. Then $\chi(H_q(n,d)) = q^{n-d+1}$.*

They leave as an open problem determining the chromatic number for additional parameters. We will give a partial answer using a variety of ad hoc methods to extend their results to several additional families of parameters.

There is one obvious way to color $H_q(n,d)$ and it will play a critical role in our investigations. Fixing $n-d+1$ coordinates of the vertices in $\mathbb{Z}_q^n$, assign each vertex a color corresponding with its value in those $n-d+1$ coordinates. Since two vertices must differ in at least d coordinates to be adjacent, no vertices of the same color can be adjacent and we have a proper coloring. We call this a coordinate coloring of $H_q(n,d)$ and immediately get that $\chi(H_q(n,d)) \leq q^{n-d+1}$. Comparing this with the results from Theorem 1.2.1, we see that the coordinate colorings are in fact minimal colorings if $q \geq n-d+2$.

If we restrict to the case $q \geq 3$ and $n = d$, we can strengthen this statement to the following result originally due to Greenwell/Lovász [11].

Theorem 1.2.2 ([2, Thm. 1.1, Claim 4.1]). *Let $q \geq 3$. Then $\chi(H_q(n,n)) = q$ and all q-colorings are coordinate colorings.*

We reference here the paper by Alon et al. [2] in which the authors also prove a stronger stability version of this result. In particular, they employed Fourier analytic techniques to prove results about the structure of maximal independent sets in the n-fold weak graph product of the complete graph K_q, which in our notation is simply $H_q(n,n)$. Both their result, and the original due to Greenwell/Lovász rely on this description of the graph in terms of weak graph products. Unforunately, $H_q(n,d)$ does not have such a description when $d < n$, so there is little hope of extending these results to general Hamming-distance graphs. In fact, we will give several counterexamples throughout the thesis.

While Theorem 1.2.2 is false for $d < n$, one may still ask what role coordinate colorings play among the broader set of minimal colorings (for parameters in which it is known that the coordinate colorings are indeed minimal.) Much of this thesis is dedicated to answering this question in the case $q = 2$ and $d = n-1$. In particular, we define a notion of the robustness of a coloring based on the tolerance of swapping the colors of vertices along an edge without destroying the properness of the coloring. Our central result is that, with a single exception, the maximally robust 4-colorings of $H_2(n, n-1)$ are exactly the coordinate colorings.

1.3 Outline

In Chapter 2, we derive several basic properties of the Hamming graphs which will prove useful in our later explorations of the chromatic number. In particular, we note that

$H_q(n, d)$ can be described as an undirected Cayley graph on the abelian group $\mathbb{Z}_q^n$. We calculate the girth of the graph and also give a formula for the graph distance between two vertices in terms of their Hamming distance. This in turn allows us to put bounds on the diameter of the graphs. Lastly, we show that the Hamming graphs are Hamiltonian and demonstrate interesting connections between this result and the existence of Gray codes of size q^n.

In Chapter 3, we consider the chromatic number of $H_q(n, d)$. In addition to presenting the results of El Rouyaheb et al. [8], we expand the range of known values of $\chi(H_q(n, d))$ using a combination of ad hoc methods. In particular, we apply a result due to Payan in [22] regarding the chromatic number of cubelike graphs which allows us to show that $H_2(n, n-1) = 4$. Finally, we demonstrate several relationships between the chromatic numbers of different Hamming graphs.

Chapter 4 focuses on minimal colorings of $H_2(n, n-1)$ and draws extensively from results in our paper [12]. As previously mentioned, our ultimate goal is to produce an analog to Theorem 1.2.2 for $H_2(n, n-1)$. Unlike when $n = d$ and $q \geq 3$ in which there were only the n coordinate colorings, we show that the number of distinct proper 4-colorings of $H_2(n, n-1)$ grows exponentially with n. As part of the proof, we define a notion of robustness of a coloring based on the tolerance of swapping the colors of vertices along an edge while retaining the properness of the coloring. We derive several bounds on the robustness of colorings of the Hamming graphs and show that the coordinate colorings of $H_2(n, n-1)$ are maximally robust. Due to the abundance of 4-colorings of $H_2(n, n-1)$, it is reasonable to expect there may be many other maximally robust colorings. However, the main result of Chapter 4 shows that with a single exception, the coordinate colorings are the only maximally robust 4-colorings of $H_2(n, n-1)$.

In Chapter 5, we turn instead to the fractional chromatic number of $H_q(n, d)$. After developing some standard results, we show that applying the Erdős-Ko-Rado Theorem for integer sequences already used in Chapter 3 gives the fractional chromatic number for a large range of parameters, including some cases for which the chromatic number is unknown. Moreover, we give a result which suggests that the behavior of the fractional chromatic number is truly different from that of the chromatic number in the binary case. Futhermore, we conjecture that the Hamming graphs contain an infinite family of graphs for which the difference between the chromatic and fractional chromatic numbers is arbitrarily large.

In Chapter 6 we define two graph transformations which may be useful for simplifying the calculation of the chromatic number of a graph based on knowledge of certain subgraph structures, especially if the graph is vertex-transitive. In particular, for some fixed k, the transformations preserve the set of admissible k-colorings while either adding additional edges to the graph or identifying vertices. Moreover, we show that under mild restrictions, these transformations commute. We also demonstrate that these techniques can be used to engineer critical graphs having various interesting properties.

Chapter 7 discusses unanswered questions from the thesis and outlines areas of interest for future research.

Finally, Appendix A contains a few basic results about the Mycielski construction.

Chapter 2 Basic Properties of Hamming Graphs

2.1 Notation

Given $a, b \in \mathbb{Z}_q^n$, the Hamming distance between $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ is $d(a,b) = |\{i \mid a_i \neq b_i, 1 \leq i \leq n\}| = wt(a-b) = wt(b-a)$ where $wt(a) = d(a,0)$ for all $a \in \mathbb{Z}_q^n$. An (unrestricted) $(n, M, d)_q$-code is a set of size M composed of codewords in $\mathcal{A}^n$, where $\mathcal{A}$ is an alphabet of size q, such that the Hamming distance between any two codewords is at least d. We will always set $\mathcal{A} = \mathbb{Z}_q$; however, this is merely for convenience and any alphabet of size q would suffice. The maximum M for which such a code exists is denoted $A_q(n,d)$. For any graph G we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Thus $E(G) \subseteq V(G) \times V(G)$. Unless otherwise specified, all graphs are assumed to be finite, connected, undirected, and simple (no loops or multiple edges). We also use the notation $\sim$ for adjacency, thus $x \sim y \Longleftrightarrow xy \in E(G)$. Similarly, $x \nsim y \iff xy \notin E(G)$. Let $\mathcal{N}(x) := \{y \in V(G) \mid y \sim x\}$ denote the set of neighbors of vertex $x \in V(G)$. For any $x,\ y \in V(G)$ we define $d_G(x,y)$ as the graph distance between x and y, that is, the length of a shortest path from x to y. For any graph G, we define the following invariants: $\alpha(G)$ denotes its independence number, that is, the maximal size of an independent set; $\omega(G)$ is the clique number, that is, the maximal size of a clique in G; $girth(G)$ denotes the girth, that is, the length of a shortest cycle; $diam(G)$ denotes the diameter, that is, the maximum graph distance between any two vertices in G. A proper k-coloring is a vertex coloring with k distinct colors such that adjacent vertices have different colors. We only consider proper colorings and usually omit the qualifier 'proper' unless we explicitly discuss properness. We denote by $\chi(G)$ the chromatic number, that is, the minimal k such that G has a proper k-coloring. Any $\chi(G)$-coloring is called a *minimal coloring*. A coloring is called *even* if all color classes (the sets of vertices with the same color) have the same cardinality. $Col_k(G)$ will denote the set of all proper k-colorings of G up to isomoprism. Thus $Col_k(G) = \emptyset \iff k < \chi(G)$. The vector with all entries equal to 1 is denoted by $\mathbb{1}$. We set e_i to be the standard basis vector having an entry of 1 in the i-th coordinate and a 0 in every other coordinate. Similarly, define $f_i = \mathbb{1} - e_i$ to be the vector having an entry of 0 in the i-th coordinate and a 1 in every other coordinate. For all three vectors, the length will be clear from the context. We set $[n] := \{1, \ldots, n\}$. K_n is the complete graph on n vertices. $Cay(G,S)$ denotes the undirected Cayley graph on the group G with generating set S. $\mu(G)$ will denote the Mycielskian of the graph G, which we will discuss in Appendix A.

2.2 Descriptions of Hamming-distance Graphs.

In this chapter, we explore various standard graph properties of the Hamming-distance graphs. In particular, we show that they can be given a description as an undirected Cayley graph and are therefore vertex-transitive and regular. We determine the regularity and girth, and give a formula for the graph distance between two vertices in terms of their

Hamming distance. Using this formula, we put bounds on the diameter of the graphs. We also show that the Hamming-distance graphs are Hamiltonian and note an interesting connection with the existence of Gray codes.

Definition 2.2.1. Let q, d, $n \in \mathbb{N}$. The *Hamming-distance graph*, or simply *Hamming graph*, $H_q(n, d)$ is defined as the graph with vertex set $V := \mathbb{Z}_q^n$ and edge set $E := \{(x, y) \mid x,\, y \in \mathbb{Z}_q^n,\, d(x, y) \geq d\}$.

We note that the term "Hamming graph" may refer to several different graphs in the literature (for instance, compare [15] and [8]). Throughout, we will use the term Hamming graph exclusively to refer to the graph in Definition 2.2.1.

Example 2.2.2. The graph below is $H_2(3, 2)$. Its vertices are all binary vectors of length 3 where two distinct vertices are adjacent if and only if the Hamming distance between them is at least 2.

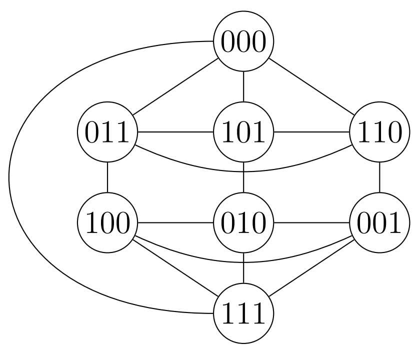

Figure 2.1: $H_2(3, 2)$

Throughout our discussion of the Hamming graphs, assume $q, d, n \in \mathbb{N}$ such that

$$q,\, n \geq 2, \quad 0 < d \leq n, \quad (q, d) \neq (2, n). \tag{2.2.1}$$

Note that the case $(q, d) = (2, n)$ is genuinely different from the general case as $H_2(n, n)$ is disconnected with 2^{n-1} components, each consisting of two vertices connected by an edge. In particular, each vertex $x \in \mathbb{Z}_2^n$ only has one neighbor, namely $x + \mathbb{1}$. We will see shortly that $H_q(n, d)$ is connected for all $(q, d) \neq (2, n)$. Thus $H_2(n, n)$ is both entirely uninteresting and truly different from the general case. Therefore we exclude it from our considerations.

Remark 2.2.3. It can be shown that the Hamming distance is actually a metric on $\mathbb{Z}_q^n$. In particular, it obeys the triangle inequality meaning $d(x, y) \leq d(x, z) + d(z, y)$ for all $z \in \mathbb{Z}_q^n$.

We begin by deriving some basic graphical properties of the Hamming graphs.

Proposition 2.2.4.

(a) $H_q(n,d)$ is simple.

(b) $H_q(n,d)$ is connected.

(c) $H_q(n,d)$ is regular of degree $\sum_{i=0}^{n-d}\binom{n}{i}(q-1)^{n-i}$.

(d) $H_q(n,d)$ is vertex-transitive.

(e) $H_q(n,d)$ is the undirected Cayley graph of the group $(\mathbb{Z}_q^n,+)$ and the subset $\mathcal{S}_d = \{v \in \mathbb{Z}_q^n \mid wt(v) \geq d\}$, which generates the group.

Proof. (e) Since $(q,d) \neq (2,n)$, the set $\mathcal{S}_d$ indeed generates the group $\mathbb{Z}_q^n$. In particular, the set of vectors $B = \{\mathbb{1}+e_1, \mathbb{1}+e_2, \dots \mathbb{1}+e_{n-1}, \mathbb{1}\} = \{b_1,\dots,b_n\}$ form a basis of the $\mathbb{Z}_q$-module $\mathbb{Z}_q^n$ and are in $\mathcal{S}_d$ for any admissible d. To see that B is indeed a basis, note that all the standard basis vectors e_i are in the span of B. Thus B generates $\mathbb{Z}_q^n$. A simple cardinality argument shows that they are linearly independent. By the definition of the Hamming graphs, x, $y \in \mathbb{Z}_q^n$ are adjacent in $H_q(n,d)$ iff $y = x+v$ for some $v \in \mathcal{S}_d$. This shows that the graph is the stated Cayley graph.
(a) This follows from (e) since $0 \notin \mathcal{S}_d$.
(b) Since $\mathcal{S}_d$ generates $\mathbb{Z}_q^n$, (b) follows immediately from (e).
(c) For any undirected Cayley graph, the regularity is simply the size of the generating set, in our case $|\mathcal{S}_d|$. Thus we simply need to count the number of vectors in $\mathbb{Z}_q^n$ with weight at least d. We sum over the number of coordinates with an entry of 0, and immediately get that $|\mathcal{S}_d| = \sum_{i=0}^{n-d}\binom{n}{i}(q-1)^{n-i}$.
(d) All Cayley graphs are vertex-transitive. □

Definition 2.2.5. A cubelike graph is a Cayley graph in which the underlying group is the elementary abelian group $\mathbb{Z}_2^n$ for some $n \in \mathbb{N}$.

Corollary 2.2.6. *$H_2(n,d)$ is a cubelike graph.*

Proof. This follows immediately from Proposition 2.2.4(e). □

2.3 Girth and Diameter

Proposition 2.3.1. *Let $G := H_q(n,d)$.*

(a) Let $q \geq 3$. Then $girth(G) = 3$.

(b) Let $q = 2$ and $d \leq \frac{2n}{3}$. Then $girth(G) = 3$.

(c) Let $q = 2$ and $\frac{2n}{3} < d < n$. Then $girth(G) = 4$. Moreover, G contains a cycle of odd length.

Proof. (a) The vertices 0, $\mathbb{1}$, $(2,\dots,2)$ form a cycle of length 3.
(b) Consider the vertices 0, x, y, where $x = (1,\dots,1,0,\dots,0)$ and $y = (0,\dots,0,1\dots,1)$ with both having exactly d entries equal to 1. If $n \geq 2d$, then $d(x,y) = 2d$ and thus 0, x, y form a cycle of length 3. Let $n < 2d$. Then x and y overlap in exactly $2d-n$ entries (which are equal to 1), and therefore $d(x,y) = 2n-2d \geq 3d-2d = d$. Again, the three vertices form a cycle of length 3.

(c) First of all, the vertices 0, $\mathbb{1}$, $(1,0,\ldots,0)$, $(0,1,\ldots,1)$ form a cycle of length 4 in the Hamming graph. By vertex-transitivity it now suffices to show that 0 is not contained in a cycle of length 3. Suppose $x, y \in \mathbb{Z}_q^n$ are adjacent to 0. Then $wt(x)$, $wt(y) \geq d$. Moreover, $x_i = 1 = y_i$ for at least $2d - n > \frac{n}{3}$ positions and therefore $d(x,y) < n - \frac{n}{3} = \frac{2n}{3} < d$. Hence x, y are not adjacent, and this shows that the graph has no cycle of length 3. A cycle of odd length is obtained as follows. For $i = 1, \ldots, n$ set $f_i := (1, \ldots, 1, 0, 1, \ldots, 1)$, where 0 is at position i. If n is odd the vertices $0, f_1, f_1 + f_2, \ldots, \sum_{i=1}^n f_i = 0$ form a cycle of length n. If n is even the cycle 0, f_1, $f_1 + f_2, \ldots, \sum_{i=1}^n f_i = \mathbb{1}$, 0 has length $n+1$. $\square$

We now proceed to prove the following relationship between the Hamming distance and graph distance of vertices in $H_q(n,d)$.

Theorem 2.3.2. *Suppose $x, y \in V(H_q(n,d))$. Then*

$$d_G(x,y) = \begin{cases} 1 & \text{if } d(x,y) \geq d \\ 2 & \text{if } q \geq 3 \text{ and } d(x,y) < d \\ \min\left\{2\left\lceil \dfrac{d(x,y)}{2(n-d)} \right\rceil, 2\left\lceil \dfrac{d-d(x,y)}{2(n-d)} \right\rceil + 1\right\} & \text{if } q = 2 \text{ and } d(x,y) < d \end{cases}$$

Before giving the proof, we will need a few definitions and lemmas.

Let δ_i denote the vector with 1's in the first i coordinates, and 0's in the other $n - i$ coordinates. That is,

$$\delta_i = \sum_{j=0}^{i} e_j = (\underbrace{1, \ldots, 1}_{i}\underbrace{0, \ldots, 0}_{n-i}).$$

Lemma 2.3.3. *Suppose $x, y \in V(H_2(n,d))$ such that $d(x,y) < d$. Then*

$$d(x,y) \leq 2(n-d) \iff d_G(x,y) = 2.$$

Proof. Since $d(x,y) < d$, we have $d_G(x,y) > 1$. Set $m := d(x,y)$. Since $H_2(n,d)$ is vertex transitive, we may assume that $x = 0$ and $y = \delta_m$.

$\Longrightarrow$ Suppose $m \leq 2(n-d)$. Consider the vertex $z = \mathbb{1} - \delta_{n-d} = (\underbrace{0, \ldots, 0}_{n-d}, \underbrace{1, \ldots, 1}_{d})$. As $d(0,z) = d$, we have that z is a neighbor of 0. If $m + d \leq n$, then $d(z, \delta_m) = d + m > d$ and we are finished since both 0 and δ_m are adjacent to z.

Therefore assume $d + m > n$ and consider the number of coordinates in which z and δ_m agree. As there are a total of $m + d$ ones but only n coordinates in which to place them, both z and δ_m have ones in $m + d - n > 0$ positions. Furthermore, it is easy to see that no coordinate contains a zero in both vectors so z and δ_m agree in $m + d - n$ coordinates. Therefore the number of coordinates in which they disagree is $d(z, \delta_m) = n - (m + d - n) = 2n - m - d \geq 2n - 2(n-d) - d = d$. This implies z and δ_m are adjacent, and since z is also adjacent to 0, we have a path of length 2 connecting 0 and δ_m.

$\Longleftarrow$ Suppose $d_G(x, y) = 2$. If $m \leq n - d \leq 2(n - d)$, we are finished, so we may also assume $n - d < m$. Since $d_G(x, y) = 2$, there exists some $w \in \mathbb{Z}_q^n$ such that $d(0, w) \geq d$ and $d(w, \delta_m) \geq d$. As w is a neighbor of 0, it must have at least d ones. Similarly, $n - d < m$ implies δ_m contains at least $n - d + 1$ ones. As there are only n coordinates and δ_m and w contain at least $m + d > n$ ones cumulatively, there must exist at least $d + m - n$ coordinates in which both δ_m and w both have a one. Thus $d(\delta_m, w) \leq n - (d + m - n) = 2n - d - m$. Then $d \leq d(\delta_m, w) \leq 2n - d - m$ and a rearrangement gives the desired result that $m \leq 2(n - d)$.

□

Lemma 2.3.4. *Suppose $x, y \in V(H_2(n, d))$ such that $d_G(x, y) = 2p$ for some $p \in \mathbb{N}$. Then $d(x, y) \leq 2p(n - d)$.*

Proof. Note that $d_G(x, y) = 2p$ implies $d(x, y) < d$. Let $P = v_0 v_1 \ldots v_{2p}$ be a path of length $2p$ such that $x = v_0$ and $y = v_{2p}$. Decompose P into p paths of length 2 of the form $v_{2i} v_{2i+1} v_{2(i+1)}$ for $i = \{0, 1, \ldots, p - 1\}$. Recall from Remark 2.2.3 that Hamming distance obeys the triangle inequality. Therefore

$$d(v_0, v_{2p}) \leq \sum_{i=0}^{p-1} d(v_{2i}, v_{2(i+1)}).$$

Since $d_G(x, y) = 2p$, P is a path of minimal length between x and y. In particular, $d(v_{2i}, v_{2(i+1)}) < d$ for all $i = \{0, 1, \ldots, p - 1\}$. Otherwise, we could omit the vertex v_{2i+1} for some i and achieve a shorter path between x and y. Then applying Lemma 2.3.3 we have $d(v_{2i}, v_{2(i+1)}) \leq 2(n - d)$ for all $i = \{0, 1, \ldots, p - 1\}$ and thus

$$d(v_0, v_{2p}) \leq \sum_{i=0}^{p-1} d(v_{2i}, v_{2(i+1)}) \leq 2p(n - d).$$

□

We are now ready to present the proof of Theorem 2.3.2.

Proof of Theorem 2.3.2.

Obviously $d(x, y) \geq d \iff d_G(x, y) = 1$. If $q \geq 3$, there exist $z_i \in \mathbb{Z}_q$ such that $x_i \neq z_i \neq y_i$ for all $i = 1, \ldots, n$. Thus $xz, zy \in E(H_q(n, d))$, and we have a path of length 2.

Now assume $d(x, y) < d$ and $q = 2$. Our goal is to show that if $d_G(x, y)$ is even then $d_G(x, y) = 2\left\lceil \dfrac{d(x, y)}{2(n - d)} \right\rceil$, and if $d_G(x, y)$ is odd then $d_G(x, y) = 2\left\lceil \dfrac{d - d(x, y)}{2(n - d)} \right\rceil + 1$. Let $j := d(x, y)$. Since the Hamming graphs are vertex-transitive, we may assume WLOG that $x = 0$ and $y = \delta_j$.

We first show that $d_G(0,\delta_j) \leq \min\left\{2\left\lceil \frac{d(x,y)}{2(n-d)} \right\rceil, 2\left\lceil \frac{d-d(x,y)}{2(n-d)} \right\rceil + 1\right\}$ by demonstrating there exist paths P_1 and P_2 of length $2\left\lceil \frac{d(x,y)}{2(n-d)} \right\rceil$ and $2\left\lceil \frac{d-d(x,y)}{2(n-d)} \right\rceil + 1$ respectively between 0 and δ_j.

We begin by constructing P_1. Set $m = \left\lceil \frac{d(0,\delta_j)}{2(n-d)} \right\rceil = \left\lceil \frac{j}{2(n-d)} \right\rceil$. If $m = 1$, then $d(0,\delta_j) \leq 2(n-d)$ and Lemma 2.3.3 gives $d_G(0,\delta_j) = 2$ as claimed. Therefore suppose $m \geq 2$. Consider the sequence of vertices $0, \delta_{2(n-d)}, \delta_{2\cdot 2(n-d)}, \ldots, \delta_{(m-1)\cdot 2(n-d)}, \delta_j$. As $(m-1)\cdot 2(n-d) + 1 \leq j \leq m \cdot 2(n-d)$, we see that each subsequent pair of vertices has Hamming distance at most $2(n-d)$. Since $m \geq 2$ we have $2(n-d) < d(0,\delta_j) < d$ so we may apply Lemma 2.3.3 to conclude that each pair of subsequent vertices has graph distance exactly two. Thus we see there exists a path P_1 of length $2m = 2\left\lceil \frac{d(0,\delta_j)}{2(n-d)} \right\rceil$ between 0 and δ_j.

Next we construct P_2 as follows. Note that $d - d(0,\delta_j) = d(\delta_d,\delta_j) < d$. Set $t = \left\lceil \frac{d-d(0,\delta_j)}{2(n-d)} \right\rceil$. If $t = 1$, then $d(\delta_d,\delta_j) \leq 2(n-d)$ so by Lemma 2.3.3 there exists a path of length exactly two between δ_d and δ_j. Furthermore, $\delta_d \in \mathcal{N}(0)$ so we immediately get that $d_G(0,\delta_j) \leq 2t + 1 = 3$. Therefore suppose $t \geq 2$. Then $2(n-d) < d(\delta_d,\delta_j) < d$. Consider the sequence of vertices $\delta_j, \delta_{j+2(n-d)}, \delta_{j+2\cdot 2(n-d)}, \ldots, \delta_{j+(t-1)\cdot 2(n-d)}, \delta_d$. Each pair of subsequent vertices has Hamming distance at most $2(n-d)$ so by Lemma 2.3.3, there exists a path of length two between them. Therefore there exists a path of length $2t = 2\left\lceil \frac{d-d(0,y)}{2(n-d)} \right\rceil$ between δ_j and δ_d. Since $\delta_d \in \mathcal{N}(0)$, we have established a path P_2 of length $2\left\lceil \frac{d-d(0,\delta_j)}{2(n-d)} \right\rceil + 1$ between 0 and δ_j.

It only remains remains to establish lower bounds on $d_G(0,\delta_j)$.

Case 1: First, suppose that $d_G(0,\delta_j) = 2p$ for some $p \in \mathbb{N}$. Then Lemma 2.3.4 gives that $d(0,\delta_j) \leq 2p(n-d)$, or equivalently $\frac{d(0,\delta_j)}{2(n-d)} \leq p$. Since $p \in \mathbb{N}$, we have $\left\lceil \frac{d(0,\delta_j)}{2(n-d)} \right\rceil \leq p$ and thus $2\left\lceil \frac{d(0,\delta_j)}{2(n-d)} \right\rceil \leq 2p = d_G(0,\delta_j)$.

Case 2: Now suppose that $d_G(0,\delta_j) = 2p+1$ for some $p \in \mathbb{N}$. Let $P = 0, v_1, \ldots, v_{2p}, \delta_j$ be a path of length $2p+1$ from 0 to δ_j. Then $d_G(0,v_{2p}) = 2p$, so applying the result from the previous case, we have $d_G(0,v_{2p}) = 2\left\lceil \frac{d(0,v_{2p})}{2(n-d)} \right\rceil$. By the triangle inequality, we have $d(0,v_{2p}) + d(0,\delta_j) \geq d(v_{2p},\delta_j)$. Since $v_{2p} \sim \delta_j$, $d(v_{2p},\delta_j) \geq d$. Thus we have $d(0,v_{2p}) \geq d - d(0,\delta_j)$ and therefore

$$d_G(0,\delta_j) = d_G(0,v_{2p}) + 1 = 2\left\lceil \frac{d(0,v_{2p})}{2(n-d)} \right\rceil + 1 \geq 2\left\lceil \frac{d-d(0,\delta_j)}{2(n-d)} \right\rceil + 1.$$

$\square$

We can now use Theorem 2.3.2 to place bounds on the diameter of $H_2(n,d)$ as follows.

Corollary 2.3.5.

$$diam(H_q(n,d)) = \begin{cases} 1 & \textit{if } d = 1 \\ 2 & \textit{if } q \geq 3 \textit{ and } d > 1 \end{cases}$$

and

$$2\left\lceil \frac{\lfloor \frac{d}{2} \rfloor}{2(n-d)} \right\rceil \leq diam(H_q(n,d)) \leq 2\left\lceil \frac{\lfloor \frac{d}{2} \rfloor}{2(n-d)} \right\rceil + 1$$

if $q = 2$ *and* $d > 1$.

Proof. The cases $d = 1$ and $q \geq 3$, $d > 1$ follow trivially from Theorem 2.3.2. Therefore let $q = 2$ and $d > 1$. Let $x, y \in \mathbb{Z}_2^n$. Then the lower bound follows immediately from considering the case $d(x,y) = \lfloor \frac{d}{2} \rfloor$ in Theorem 2.3.2.

Since $d > 1$, $H_2(n,d)$ contains pairs of vertices which are not adjacent. Thus $diam(H_2(n,d)) \geq 2$, so we need not consider x and y in $\mathbb{Z}_2^n$ such that $d(x,y) \geq d$. If $\lceil \frac{d}{2} \rceil \leq d(x,y) < d$, then $d_G(x,y) \leq 2\left\lceil \frac{d - d(x,y)}{2(n-d)} \right\rceil + 1 \leq 2\left\lceil \frac{d - \lceil \frac{d}{2} \rceil}{2(n-d)} \right\rceil + 1 = 2\left\lceil \frac{\lfloor \frac{d}{2} \rfloor}{2(n-d)} \right\rceil + 1$. Similarly, if $1 \leq d(x,y) \leq \lfloor \frac{d}{2} \rfloor$, then $d_G(x,y) \leq 2\left\lceil \frac{d(x,y)}{2(n-d)} \right\rceil \leq 2\left\lceil \frac{\lfloor \frac{d}{2} \rfloor}{2(n-d)} \right\rceil + 1$. As $d_G(x,y)$ is an integer, this covers all possible cases. □

The following example shows that both bounds on the diameter in Corollary 2.3.5 are met.

Example 2.3.6. Consider the graph $H_2(6,5)$. Suppose $x, y \in V(H_2(6,5))$ such that $d(x,y) = 4$. Then Theorem 2.3.2 gives that

$$d_G(x,y) = \min\left\{2\left\lceil \frac{4}{2(6-5)} \right\rceil, 2\left\lceil \frac{5-4}{2(6-5)} \right\rceil + 1\right\} = \min\{4, 3\} = 3.$$

Furthermore, $2\left\lceil \frac{\lfloor \frac{5}{2} \rfloor}{2(6-5)} \right\rceil = 2$, so the upper bound of Corollary 2.3.5 is met.

Next consider the graph $H_2(3,2)$. It is straightfoward to check that $diam(H_2(3,2)) = 2$ (see Figure 2.1) implying the lower bound of Corollary 2.3.5 is also met.

2.4 Hamiltonicity of $H_q(n,d)$

Definition 2.4.1. A *Hamiltonian cycle* is a simple cycle which visits each vertex in a graph exactly once. Graphs which contain a Hamiltonian cycle are called *Hamiltonian graphs.*

As noted in Proposition 2.2.4(d), $H_q(n,d)$ is vertex-transitive. The Lovász conjecture states that all finite connected vertex-transitive graphs contain a Hamiltonian cycle, with the exception of five known counterexamples. All Cayley graphs are vertex-transitive, and notably all five counterexamples mentioned above are not Cayley graphs. This has led to the following conjecture, which many references refer to as a folklore conjecture due in part to there being no clear person with whom to credit its formulation.

Conjecture 2.4.2. All finite connected Cayley graphs on at least three vertices contain a Hamiltonian cycle.

Although this conjecture is still wide open, several partial results exist. See for instance [5] for more on this topic. In particular, we have the following which applies to the Hamming graphs.

Theorem 2.4.3 ([21, Cor. 3.2])**.** *Suppose G is a finite abelian group of order at least 3. Then $Cay(G, S)$ contains a Hamiltonian cycle for all generating sets S.*

Corollary 2.4.4. *$H_q(n, d)$ is Hamiltonian.*

Proof. By Proposition 2.2.4(e), $H_q(n, d)$ is a Cayley graph on the abelian group $\mathbb{Z}_q^n$. The result then follows immediately from Theorem 2.4.3. □

In this section, we prove the following strengthened version of Corollary 2.4.4.

Theorem 2.4.5. *$H_q(n, d)$ contains a cycle of length q^i for all $i \in [n]$. In particular, $H_q(n, d)$ is Hamiltonian.*

Our proof will be constructive and is similar in spirit to known proofs of Theorem 2.4.3, although it was produced independently. It should be noted that Theorem 2.4.5 can be achieved as a corollary of Theorem 2.4.3 by noting that there exist subgraphs of $H_q(n, d)$ of size q^i which can be described as a Cayley graph on the abelian group $\mathbb{Z}_q^i$. However, analyzing the actual structure of these cycles reveals interesting fractal-like structures within the Hamming graphs, making the constructive proof appealing. Moreover, it will allow us to demonstrate an interesting connection between the Hamiltonicity of $H_q(n, d)$ and the existence of Gray codes. We first establish a few preliminary results and remarks which will be useful in simplifying the proof of Theorem 2.4.5.

Normally, paths and cycles are defined as a list of the vertices through which they pass. That is, a path of length n is written in the form

$$P = v_0 v_1 \dots v_{n-1} v_n, \tag{2.4.1}$$

where $v_i \in V(G)$. However, in constructing a Hamiltonian cycle within $H_q(n, d)$, it will prove easier to describe the cycle in terms of the labels of the edges we walk along.

Let $x, y \in V(H_q(n, d))$ such that $xy \in E(H_q(n, d))$. As $H_q(n, d)$ is a Cayley graph, there are two natural labelings for the edge xy, namely $x - y$ and $y - x$. To overcome this ambiguity, we will imagine our paths and cycles as directed. Then if the path progresses from vertex x to vertex y, the edge xy has only one natural label, namely $y - x$. To simplify notation, we will assume that all paths and cycles described begin at 0. Since $H_q(n, d)$ is vertex-transitive, this can be done without loss of generality.

In $H_q(n,d)$, the set of possible edge labels is $\mathcal{S}_d = \{v \in \mathbb{Z}_q^n \mid wt(v) \geq d\}$. Moreover, under this directional notion of edge labeling, each vertex is incident to exactly one edge of each label in $\mathcal{S}_d$ beginning at that vertex. Since $\mathcal{S}_d \subseteq V(H_q(n,d))$ as sets, we will always be careful to specify whether we are defining a path in terms of its vertices or edge labels.

Furthermore, when defining a generic path of length n in terms of its vertices, we will use the notation as in (2.4.1). When defining a generic path of length n in terms of edge labels, we will use the notation

$$P = l_1 l_2 \dots l_{n-1} l_n. \tag{2.4.2}$$

The following example shows that under the condition that all paths start at 0, it is straightforward to convert back and forth between the notations of (2.4.1) and (2.4.2).

Example 2.4.6. (a) Let $P_1 = v_0 v_1 \dots v_{n-1} v_n$ be a path of length n in $H_q(n,d)$ such that $v_0 = 0$. The edge label representation of this path is $P_1 = l_1 l_2 \dots l_{n-1} l_n$ where $l_i = v_i - v_{i-1}$ for $i \in [n]$.

(b) Let $P_2 = l_1 l_2 \dots l_{n-1} l_n$ be a path of length n in $H_q(n,d)$ beginning at 0. The vertex representation of this path is $P_2 = v_0 v_1 \dots v_{n-1} v_n$ where $v_0 = 0$ and v_i is defined iteratively as $v_i = l_i + v_{i-1}$.

The iterative definition of v_i in Example 2.4.6(b) has the following closed form.

Lemma 2.4.7. *Suppose P is a path in $H_q(n,d)$ beginning at 0 with edge labeling representation $P = l_1 \dots l_n$. Let $v_0 \dots v_n$ be the corresponding vertex representation of this path. Then $v_i = \sum_{j=1}^{i} l_j$.*

Proof. From Example 2.4.6(b), v_i has a recursive definition as $v_i = l_i + v_{i-1}$. Thus

$$v_i = l_i + v_{i-1} = l_i + l_{i-1} + v_{i-2} = \dots = \sum_{j=1}^{i} l_j + v_0 = \sum_{j=1}^{i} l_j + 0 = \sum_{j=1}^{i} l_j.$$

□

As noted in Proposition 2.2.4(e),

$$B = \{\mathbb{1} + e_1, \mathbb{1} + e_2, \dots \mathbb{1} + e_{n-1}, \mathbb{1}\} = \{b_1, \dots, b_n\} \tag{2.4.3}$$

forms a basis of the $\mathbb{Z}_q$-module $\mathbb{Z}_q^n$. Therefore given any two vertices, it is possible to find a path between them composed entirely of edges with labels in B. In fact, our construction of the Hamiltonian cycles will exclusively use edges whose labels are in B.

Restricting ourselves to vertices with edge labels in B has several advantages. From Lemma 2.4.7, we know that each vertex in a path can be described as the sum of the edge labels preceding it. As B is a basis, each vertex in $V(H_q(n,d)) = \mathbb{Z}_q^n$ can be described in terms of its basis representation. In other words, there exists a bijective map

$$\phi : \mathbb{Z}_q^n \longrightarrow \mathbb{Z}_q^n, \quad v \longmapsto (a_1, a_2, \ldots, a_n)$$

where

$$v = \sum_{i=1}^{n} a_i b_i.$$

Similarly, define

$$\phi_i : \mathbb{Z}_q^n \longrightarrow \mathbb{Z}_q, \quad v \longmapsto a_i$$

as the restriction of ϕ to the i^{th} coordinate. Thus for a path starting at 0 to end at v, we must have passed through $a_1 + m_1 q$ edges with label b_1, $a_2 + m_2 q$ edges with label b_2, and so forth where $m_i \in \{0, 1, 2, \ldots\}$. Showing that $H_q(n,d)$ is Hamiltonian amounts to finding a cycle of length q^n in $H_q(n,d)$ such that each vertex in the cycle is passed through exactly once. In other words, it will suffice to show that the basis representation (i.e. the sum of the edge labels proceeding the vertex) is unique for each vertex in the cycle.

We now proceed to construct simple paths of increasing lengths using an iterative definition based on the edge labels of the path. The motivating idea behind this construction is to create a path with a fractal-like structure which will eventually cover the whole graph. This structure may be more clear after examining the corresponding basis representation of the vertices shown in Example 2.4.8.

Each of the following paths A_i is given in terms of its edge label representation. Let

$$A_1 = \underbrace{b_1 b_1 \ldots b_1}_{q-1}$$

.

$$A_2 = \underbrace{A_1 b_2 A_1 b_2 \ldots A_1 b_2}_{q-1} A_1$$

$$= \underbrace{\underbrace{b_1 b_1 \ldots b_1}_{q-1} b_2 \underbrace{b_1 b_1 \ldots b_1}_{q-1} b_2 \ldots \underbrace{b_1 b_1 \ldots b_1}_{q-1} b_2}_{q-1} \underbrace{b_1 b_1 \ldots b_1}_{q-1}$$

and

$$A_i = \underbrace{A_{i-1} b_i A_{i-1} b_i \ldots A_{i-1} b_i}_{q-1} A_{i-1}$$

for $3 \leq i \leq n$.

We will show that each A_i is a simple path of length $q^i - 1$ and that $A_i b_i$ is a simple cycle of length q^i. It will then follow immediately that $A_n b_n$ is a Hamiltonian cycle. Before proceeding with the proof, we first give an example of the process for small parameters.

Example 2.4.8. Consider the graph $H_3(3,3)$ and construct A_3 as above. Let $v_0 v_1 \ldots v_{26}$ be the vertex representation of this path. Then we have

$$\begin{array}{lll}
\phi(v_0) = (0,0,0) & \phi(v_9) = (0,2,1) & \phi(v_{18}) = (0,1,2) \\
\phi(v_1) = (1,0,0) & \phi(v_{10}) = (1,2,1) & \phi(v_{19}) = (1,1,2) \\
\phi(v_2) = (2,0,0) & \phi(v_{11}) = (2,2,1) & \phi(v_{20}) = (2,1,2) \\
\phi(v_3) = (2,1,0) & \phi(v_{12}) = (2,0,1) & \phi(v_{21}) = (2,2,2) \\
\phi(v_4) = (0,1,0) & \phi(v_{13}) = (0,0,1) & \phi(v_{22}) = (0,2,2) \\
\phi(v_5) = (1,1,0) & \phi(v_{14}) = (1,0,1) & \phi(v_{23}) = (1,2,2) \\
\phi(v_6) = (1,2,0) & \phi(v_{15}) = (1,1,1) & \phi(v_{24}) = (1,0,2) \\
\phi(v_7) = (2,2,0) & \phi(v_{16}) = (2,1,1) & \phi(v_{25}) = (2,0,2) \\
\phi(v_8) = (0,2,0) & \phi(v_{17}) = (0,1,1) & \phi(v_{26}) = (0,0,2)
\end{array}$$

It is straightforward to check that each basis representation appears exactly once in this list, and therefore A_3 is a simple path of length $3^3 - 1 = 26$ in $H_3(3,3)$. Moreover, $v_{26} = \phi^{-1}((0,0,2)) = 222$ which is adjacent to 0 along the edge with label b_3. Thus $A_3 b_3$ is a Hamiltonian cycle in $H_3(3,3)$.

As previously mentioned, our construction of a Hamiltonian cycle in $H_q(n,d)$ is related to the existence of Gray codes. Gray codes assign to each set of contiguous integers a word of symbols such that that each two adjacent codewords differ in a single symbol. See the table below for an example of a Gray code of size 8, as well as a comparison to the more traditional binary representation.

Decimal	Binary	Gray
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

Table 2.1: (2,3)-Gray code

Gray codes have many applications including increasing machine efficiency and providing error correction. If we imagine a series of n binary switches, it is unlikely that all the switches will change position exactly simultaneously in practical applications. Therefore errors can occur when some transitional state is recorded. Gray codes have the advantage that as long as the transitional state does not differ in many coordinates from the intended state, the errors in the output are also correspondingly small. Compare this to the binary representation in which 010 and 110 differ in only a single position, but represent 2 and 6 respectively.

A Gray code using k symbols and l bits is known as an (k,l)-Gray code. We note that the basis representation of the path in Example 2.4.8 is an example of a $(3,3)$-Gray code.

Similarly, examining the basis representation of the vertices in A_i produces a (q,i)-Gray code for any $i = \{1,\ldots,n\}$.

We now prove several preliminary results which will be used in the proof of Theorem 2.4.5.

Proposition 2.4.9. *(a) If $q \geq 3$, then A_i is a simple path of length $q^i - 1$ in $H_q(n,n)$ and terminates at the vertex $v = (q-1)b_i$.*

(b) If $q = 2$, then A_i is a simple path of length $2^i - 1$ in $H_2(n,n-1)$ and terminates at the vertex $v = b_i$.

Proof. While the proof of (a) and (b) are essentially the same, the case of $q = 2$ is distinct because the basis B given in (2.4.3) contains vertices of weight $n-1$ if and only if $q = 2$. Thus the paths A_i we constructed above do not exist in $H_2(n,n)$. Of course, this is to be expected as we have already noted that $H_2(n,n)$ is either a path of length two or a disconnected graph.

We calculate the length of A_i inductively. The base case is $|A_1| = q - 1 = q^1 - 1$. Now assume $|A_j| = q^j - 1$ for all $j \leq i$. Recall $A_{i+1} = \underbrace{A_i b_{i+1} A_i b_{i+1} \ldots A_i b_{i+1}}_{q-1} A_i$

Then

$$\begin{aligned}|A_{i+1}| &= (q^i - 1 + 1)(q-1) + (q^i - 1)\\ &= q^i(q-1) + q^i - 1\\ &= q^{i+1} - 1\end{aligned}$$

Recall from Lemma 2.4.7 that each vertex can be described as the sum of the labels of the edges on the path leading to it. Thus to determine the terminal vertex, we only need to add the labels of the edges in A_i. As $A_i = \underbrace{A_{i-1} b_i A_{i-1} b_i \ldots A_{i-1} b_i}_{q-1} A_{i-1}$, each label in A_{i-1} occurs a multiple of q times, and contributes nothing to the sum. Thus A_i ends at the vertex $v = (q-1)b_i$.

It remains to show that A_i is cycle-free. This is clear for A_1. For $i \geq 2$, partition A_i into q paths,

$$P_1 = A_{i-1}, \qquad P_j = b_i A_{i-1} \;\text{ for }\; 2 \leq j \leq q$$

such that $A_i = P_1 P_2 \ldots P_q$. In this partition of the path, let each vertex be grouped in the same P_i as its preceding edge. By induction, assume that A_{i-1} is cycle-free. Since $H_q(n,d)$ is vertex-transitive (see Proposition 2.2.4(d)) and b_i is not contained in A_{i-1}, this guarantees that no cycle occurs within any P_j. Furthermore, suppose $v \in P_j$ and $v' \in P_k$ for $j \neq k$. Then $\phi_i(v) = j - 1 \neq k - 1 = \phi_i(v')$ so $v \neq v'$, and we see that the partitions of A_i are mutually disjoint. Thus A_i is a simple path. □

Corollary 2.4.10. *(a) If $q \geq 3$, then $H_q(n,n)$ contains a cycle of length q^i for all $i \in [n]$. In particular, $H_q(n,n)$ is Hamiltonian.*

(b) If $q = 2$, then $H_2(n, n-1)$ contains a cycle of length q^i for all $i \in [n]$. In particular, $H_2(n, n-1)$ is Hamiltonian.

Proof. By Proposition 2.4.9, A_i is a path of length $q^i - 1$ ending at the vertex $v = (q-1)b_i$. Note that there is an edge connecting v and 0 since $0 - (q-1)b_i = b_i$. Thus $A_i b_i$ is a cycle of length q^i. In particular, the case $i = n$ shows that the graph is Hamiltonian as $|H_q(n, d)| = q^n$ for any choice of (n, d, q). □

Theorem 2.4.11. *$H_q(n, d)$ contains a cycle of length q^i for all $i \in [n]$. In particular, $H_q(n, d)$ is Hamiltonian.*

Proof. Note that $V(H_q(n, n)) = V(H_q(n, d))$ and $E(H_q(n, n)) \subseteq E(H_q(n, d))$ for all (n, d, q). Similarly, $V(H_2(n, n-1)) = V(H_2(n, d))$ and $E(H_2(n, n-1)) \subseteq E(H_2(n, d))$ for all $d \leq n - 1$. Therefore the Hamiltonian cycle we demonstrated in Corollary 2.4.10 also exists in $H_q(n, d)$. □

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Chapter 3 On the Chromatic Number of the Hamming Graphs

In this chapter, we present various results regarding the chromatic number of the Hamming graphs. Recall from Theorem 1.2.2 that Greenwell/Lovász showed that for $q \geq 3$, $\chi(H_q(n,n)) = n$ and all minimal colorings are achieved by fixing a single coordinate and coloring each vertex according to its value in those coordinates. Part of our motivation for studying the chromatic number of the Hamming graphs was to see to what degree this result might be generalized. As noted in the discussion proceeding Theorem 1.2.1, El Rouayheb et al. give a partial result in [8] on the chromatic number of $H_q(n,d)$, but leave as an open problem determining the chromatic number for additional parameters. We present their result, and extend it slightly via a variety of ad hoc methods. Furthermore, as we proceed it will become clear that there are also a variety of interesting connections between the chromatic number of $H_q(n,d)$ and several important coding theory bounds.

3.1 Lower Bounds

For any graph G, we can bound the chromatic number below in the following ways.

$$\chi(G) \geq \omega(G) \tag{3.1.1}$$

$$\chi(G) \geq \left\lceil \frac{|G|}{\alpha(G)} \right\rceil \tag{3.1.2}$$

(3.1.1) follows from the fact that no two vertices in a clique may be assigned the same color in a proper colorings. (3.1.2) is derived from noting that each color class, that is all vertices of a given color, must be mutually independent. Thus a coloring can be thought of as a partition of the vertices into independent sets. Therefore it takes at least $\left\lceil \frac{|G|}{\alpha(G)} \right\rceil$ colors to properly color the graph.

As discussed in the introduction, $\omega(H_q(n,d)) = A_q(n,d)$; that is, the clique number of $H_q(n,d)$ is equal to the maximum size of any code of length n and minimum Hamming distance at least d over an alphabet of size q. Thus (3.1.1) gives that any lower bound on the size of such codes is a lower bound on the chromatic number of the corresponding Hamming graph. Alternatively, any upper bound on the $\chi(H_q(n,d))$ is an upper bound on $A_q(n,d)$. We will utilize this direction shortly to recover the well known Singleton Bound (see Theorem 3.2.2).

In general, upper bounds on $A_q(n,d)$ are more common that lower bounds, as lower bounds are almost always constructive. Furthermore, most lower bounds are given for linear codes, that is, codes which form a subspace of the vector space $\mathbb{F}_q^n$. Note that this requires q to be a prime power. Then $B_q(n,d)$ denotes the largest linear code of length n and distance at least d over $\mathbb{F}_q$. Since linear codes are a subset of unrestricted codes, we have $B_q(n,d) \leq A_q(n,d)$ and any lower bound on $B_q(n,d)$ is also a lower bound on

$A_q(n,d)$. The following result is known as the Gilbert Bound and is an example of a lower bound achieved through a particular "greedy" construction.

Theorem 3.1.1 ([14, Thm. 2.8.1])**.**

$$B_q(n,d) \geq \frac{q^n}{\sum\limits_{i=0}^{d-1} \binom{n}{i}(q-1)^i}.$$

Since $\chi(H_q(n,d)) \geq \omega(H_q(n,d)) = A_q(n,d) \geq B_q(n,d)$ we immediately have the following lower bound on the chromatic number of the Hamming graphs.

Corollary 3.1.2.

$$\chi(H_q(n,d)) \geq \frac{q^n}{\sum\limits_{i=0}^{d-1} \binom{n}{i}(q-1)^i}.$$

Comparing with the results we will obtain shortly in Corollary 3.1.4, we will see that the bound in Corollary 3.1.2 is quite weak. This is not so unexpected as it depended on a bound on linear codes, as opposed to the unrestricted case.

Luckily, the bound in (3.1.2) is much more useful as the value of $\alpha(H_q(n,d))$ is known for most parameters. This result is known as the Erdős-Ko-Rado Theorem for integer sequences and has been proven by Kleitman [17] for the binary case and Ahlswede/-Khachatrian [1] as well as Frankl/Tokushige [10] for the general case. In our notation it states the following.

Theorem 3.1.3.
(a) [17] or [1, Thm. K1] Let $q = 2$. Then

$$\alpha(H_2(n,d)) = \begin{cases} \sum\limits_{i=0}^{\frac{d-1}{2}} \binom{n}{i}, & \text{if } d \text{ is odd,} \\ 2\sum\limits_{i=0}^{\frac{d-2}{2}} \binom{n-1}{i}, & \text{if } d \text{ is even.} \end{cases}$$

(b) [10, Thm. 2 and pp. 57-58, Cor. 1] Let $q \geq 3$. Set $r := \lfloor \frac{n-d}{q-2} \rfloor$. Then

$$\alpha(H_q(n,d)) \leq \left\lfloor q^{d-1-2r} \sum_{i=0}^{r} \binom{n-d+1+2r}{i} (q-1)^i \right\rfloor,$$

with equality if $d \geq 2r+1$. In particular, $\alpha(H_q(n,d)) = q^{d-1}$ for $d \geq n-q+2$ (and $d > 0$).

For $d < n-q+2$, the upper bound in part (b) may not be sharp. For instance, for $d = 1$ and $n = 2q-1$ one can show that $\lfloor q^{d-1-2r} \sum_{i=0}^{r} \binom{n-d+1+2r}{i}(q-1)^i \rfloor = 2$, whereas, of course, $\alpha(H_q(n,1)) = 1$ since $H_q(n,1)$ is the complete graph on q^n vertices.

Examining the cases in which we do have equality, we achieve what we will see shortly is an important lower bound on the chromatic number.

Corollary 3.1.4.
(a) Let $q = 2$*. Then* $\chi(H_2(n,2)) \geq 2^{n-1}$ *for all* $n \in \mathbb{N}_{\geq 2}$*.*
(b) Let $q \geq 3$*. Then* $\chi(H_q(n,d)) \geq q^{n-d+1}$ *for* $d \in \{n-q+2,\ldots,n\}$ *(and* $d > 0$*).*

Proof. In all given cases, one easily verifies from Theorem 3.1.3 that $\alpha(H_q(n,d)) = q^{d-1}$. Then the results follow immediately from (3.1.2).

□

Finally, a result due to Payan [22] proved the surprising fact that no nonbipartite cube-like graph can have chromatic number less than 4. Since the binary Hamming graph $H_2(n,d)$ is the cube-like graph $Q_n[d,d+1,\ldots,n]$ we obtain the following result.

Lemma 3.1.5. $\chi(H_2(n,d) \geq 4$ *for* $d < n$*.*

Proof. All that remains is to show that $H_2(n,d)$ is nonbipartite. However, this follows immediately from Proposition 2.3.1 by the existence of a cycle of odd length. □

3.2 An Upper Bound from Coordinate Colorings

As is generally the case, upper bounds on the chromatic number are achieved via construction. There is one obvious way in which to color any Hamming graph, and it will play an important role going forward.

Definition 3.2.1. Consider the Hamming graph $H_q(n,d)$. Fix $1 \leq i_1 < i_2 < \ldots < i_{n-d+1} \leq n$. The $(i_1,\ldots,i_{n-d+1})$-*coordinate coloring* of $H_q(n,d)$ is defined as

$$K : \mathbb{Z}_q^n \longrightarrow \mathbb{Z}_q^{n-d+1},\ (x_1,\ldots,x_n) \longmapsto (x_{i_1},\ldots,x_{i_{n-d+1}}).$$

It is a proper q^{n-d+1}-coloring of $H_q(n,d)$.

The properness of K follows from the fact that vertices of the same color agree in at least $n-d+1$ coordinates, and thus differ in at most $d-1$ coordinates. By the construction of the Hamming graph, these vertices are not adjacent. Furthermore, it is easy to see that each color class has size q^{d-1}. Thus,

$$\alpha(H_q(n,d)) \geq q^{d-1} \text{ and } \chi(H_q(n,d)) \leq q^{n-d+1} \tag{3.2.1}$$

As consequences, we have the following implications:

$$\alpha(H_q(n,d)) < \left\lceil \frac{q^n}{q^{n-d+1}-1} \right\rceil \Longrightarrow \chi(H_q(n,d)) = q^{n-d+1} \tag{3.2.2}$$

and

$$\alpha(H_q(n,d)) = q^{d-1} \implies \text{ each } q^{n-d+1}\text{-coloring of } H_q(n,d) \text{ is even.} \tag{3.2.3}$$

Comparing (3.2.1) and (3.1.1), we also immediately recover the well known Singleton Bound from coding theory.

Theorem 3.2.2. *[14, Thm. 2.4.1]*

$$A_q(n,d) \le q^{n-d+1}.$$

Codes attaining this bound are called *MDS codes*, and thus they are $(n, q^{n-d+1}, d)_q$-codes. The existence of (unrestricted) MDS codes is a longstanding open problem for most parameters. We refer to the literature on details about the conjecture; see for instance [18] by Kokkala et al. and the references therein. We summarize the relation between existence of MDS codes and the chromatic number as follows.

Remark 3.2.3. If there exists an $(n, q^{n-d+1}, d)_q$-code, then $\chi(H_q(n,d)) = q^{n-d+1}$ because the MDS code in question forms a clique of size q^{n-d+1} in the Hamming graph. Again, this is only a sufficient condition for $\chi(H_q(n,d)) = q^{n-d+1}$. For instance, we will see shortly in Theorem 3.3.1 that $\chi(H_2(n,n-1)) = 4$ for all n, but there exists no $(n,4,n-1)_2$-code for $n > 3$. The latter follows from the well-known non-existence of nontrivial binary MDS codes, but can also be deduced from the girth in Proposition 2.3.1(c). Similarly, we will also see in Theorem 3.3.1 that $\chi(H_3(n,n-1)) = 9$ for all $n \ge 2$, but there exists no $(n,9,n-1)_3$-code for $n \ge 5$ thanks to the Plotkin bound [14, Thm. 2.2.1].

3.3 Known Values of $\chi(H_q(n,d))$

Greenwell/Lovász showed that for $q \ge 3$, $\chi(H_q(n,n)) = q$. Furthermore $\chi(H_q(n,1)) = q^n$ as $H_q(n,1)$ is simply the complete graph on n vertices. Combining (3.2.1) with Corollary 3.1.4, we see that just as in the case $n = d$, coordinate colorings are minimal colorings for a wide variety of parameters. Moreover, we get the following improvement to Theorem 1.2.1.

Theorem 3.3.1.

(a) Let $q = 2$. Then $\chi(H_2(n,2)) = 2^{n-1}$ and $\chi(H_2(n,n-1)) = 4$ for all $n \in \mathbb{N}_{\ge 2}$.

(b) Let $q \ge 3$. Then $\chi(H_q(n,d)) = q^{n-d+1}$ for $q \ge n-d+2$. Furthermore, each q^{n-d+1}-coloring is even.

Proof. The values of the chromatic number follow from (3.2.1), Corollary 3.1.4, and Lemma 3.1.5. As noted in the proof of Corollary 3.1.4, $\alpha(H_q(n,d)) = q^{d-1}$ for parameters considered in (b). Then the result follow from (3.2.3) □

Another case not fully covered by the above considerations can easily be dealt with in a direct fashion.

Proposition 3.3.2. *$\chi(H_q(n,2)) = q^{n-1}$ for any q and each q^{n-1}-coloring is even.*

Proof. We show that each maximal independent set is of the form $\{v + \alpha e_j \mid \alpha \in \mathbb{Z}_q\}$ for some $v \in \mathbb{Z}_q^n$ and some j, and where e_j is the j-th standard basis vector. Suppose I is a maximal independent set containing the two distinct vertices $v, w \in \mathbb{Z}_q^n$. Then $\mathrm{d}(v, w) < 2$, and thus v, w differ in exactly one position. Without loss of generality let $v_1 \neq w_1$. But then any other vertex contained in I, say z, satisfies $\mathrm{d}(z, v) = 1 = \mathrm{d}(z, w)$, and therefore z also differs from v and w only in the first coordinate. As a consequence, maximality of I yields $I = \{(\alpha, v_2, \ldots, v_n) \mid \alpha \in \mathbb{Z}_q\} = \{v + \alpha e_1 \mid \alpha \in \mathbb{Z}_q\}$, and thus $|I| = q$. Now the statements follow from (3.2.2) and (3.2.3). □

The proof shows that each maximal independent set of $H_q(n, 2)$ is the color class of a suitable coordinate coloring (based on $n-1$ coordinates). However, this does not imply that each q^{n-1}-coloring is a coordinate coloring. We will see this explicitly in Example 4.1.4 below.

The careful reader has probably noted that $\chi(H_q(n, d)) = q^{n-d+1}$ for all cases in which we have determined the chromatic number.

Question 3.3.3. Is $\chi(H_q(n, d)) = q^{n-d+1}$ for all (q, n, d)?

We currently know of no counterexamples. Computer testing has confirmed this result for several small parameters, but no general method of proof has made itself apparent. In light of Theorem 3.3.1, a counterexample would have to rely on a choice of $q = 2$, or $q \geq 3$ and $q < n - d + 2$.

3.4 Other relationships

Here we present several relationships between the chromatic numbers of different Hamming graphs. Trivially, we have the following.

$$\chi(H_q(n, d)) \leq \chi(H_q(n+1, d)) \tag{3.4.1}$$

$$\chi(H_q(n, d+1)) \leq \chi(H_q(n, d)) \tag{3.4.2}$$

$$\chi(H_q(n, d)) \leq \chi(H_{q+1}(n, d)) \tag{3.4.3}$$

These inequalities all follow immediately from noting that the Hamming graph on the left side of the inequality can be regarded as a subgraph of the one on the right side of the inequality.

Next we show that the inequalities in (3.4.1) and (3.4.3) are strict for an infinite number of values of n and q respectively. No such result is possible for (3.4.2) as d is bounded above by n.

Proposition 3.4.1.

$$\chi(H_q(n+d, d)) \geq \chi(H_q(n, d)) + 1$$

Proof. This proof will make use of the Mycielski construction (See Appendix A for additional details on this construction.) Recall that for any graph G, the Mycielskian $\mu(G)$ satisfies $\chi(\mu(G)) \geq \chi(G)+1$. Our objective will be to show that $\mu(H_q(n,d))$ is isomorphic to a subgraph of $H_q(n+d,d)$. It would then follow that $\chi(H_q(n+d,d)) \geq \chi(\mu(H_q(n,d))) = \chi(H_q(n,d))+1$.

Let $V_1 = \{(v_1,v_2,\ldots,v_{n+d}) \in \mathbb{Z}_q^{n+d} \mid v_{n+1} = 1, v_i = 0 \text{ for all } i \in \{n+2,\ldots,n+d\}\}$ and $V_2 = \{(v_1,v_2,\ldots,v_{n+d}) \in \mathbb{Z}_q^{n+d} \mid v_i = 0 \text{ for all } i \in \{n+1,n+2,\ldots,n+d\}\}$. Let H_1 and H_2 be the induced subgraph of $H_q(n+d,d)$ on V_1 and V_2 respectively. Note that H_1 and H_2 are both isomorphic to $H_q(n,d)$. Let $W = V_1 \bigcup V_2 \bigcup \{\mathbb{1}\}$ and H be the induced subgraph of $H_q(n+d,d)$ on W. We now claim $\mu(H_q(n,d))$ is a subgraph of H. To see this, note that H_1 is isomorphic to $H_q(n,d)$, $|V_1| = |V_2|$, and $\mathbb{1}$ is adjacent to every vertex in V_2 as they differ in the last d coordinates. Moreover, suppose $x_1 \in V_1$ and $x_2 \in V_2$ such that x_1 and x_2 agree in their first n coordinates (i.e. x_2 is the copy of x_1 in the Mycielski construction). We need to show that every neighbor of x_1 in V_1 is also a neighbor of x_2. However, this follows easily since two vertices in V_1 are adjacent if and only if they differ in at least d of the first n coordinates (as they agree in the last d coordinates). Since x_1 and x_2 agree perfectly in the first n coordinates, every neighbor of x_1 in V_1 is also a neighbor of x_2. Similarly, every neighbor of x_2 in V_2 is a neighbor of x_1. Therefore we see that $\mu(H_q(n,d)) \subseteq W \subset H_q(n+d,d)$ and $\chi(H_q(n+d,d)) \geq \chi(\mu(H_q(n,d)) = \chi(H_q(n,d))+1$. □

Corollary 3.4.2. *Fix q and d. Then*

$$\lim_{n\to\infty} \chi(H_q(n,d)) = \infty.$$

Proof. This follows immediately from iterative application of Proposition 3.4.1. □

Corollary 3.4.3. *Fix n and d. Then*

$$\lim_{q\to\infty} \chi(H_q(n,d)) = \infty.$$

Proof. By Theorem 1.2.2, $H_q(n,n) = q$ for $q \geq 3$. Furthermore, applying (3.4.2) we see $\chi(H_q(n,d)) \geq \chi(H_q(n,n)) = q$. □

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Chapter 4 Colorings of $H_2(n, n-1)$

4.1 Minimal Colorings of $H_2(n, n-1)$

In this chapter, we study the 4-colorings of the graphs $H_2(n, n-1)$. As noted in the outline, the majority of the results from this chapter can be found in [12]. Recall from Theorem 3.3.1 that 4-colorings of $H_2(n, n-1)$ are indeed minimal colorings. As discussed previously, the motivation of this study arose from the following result by Alon et al. [2]. It had been proven already earlier by Greenwell/Lovász in [11], but Alon et al. provide an interesting new proof based on Fourier analysis on the group $\mathbb{Z}_q^n$. We present it here again for reference.

Theorem 1.2.2.([2, Thm. 1.1, Claim 4.1]) Let $q \geq 3$. Then $\chi(H_q(n, n)) = q$ and every q-coloring of $H_q(n, n)$ is a coordinate-coloring.

In other words, the minimal colorings of $H_q(n, n)$ are exactly the coordinate colorings. It turns out that this is not the case for the Hamming graphs $H_q(n, n-1)$ so that there are indeed more q^2-colorings. We will show some small examples for $q = 2$ later in this section, and in Chapter 7 a 9-coloring of $H_3(4, 3)$ is given that is not a coordinate coloring.

The following notion will be central to our investigation.

Definition 4.1.1. Let $G = (V, E)$ be an undirected simple graph with proper k-coloring $K : V \longrightarrow [k]$. Let $(x, y) \in E$. Then the edge (x, y) is a *transition edge for* K if the map

$$K' : V \longrightarrow [k], \quad v \longmapsto \begin{cases} K(x), & \text{if } v = y, \\ K(y), & \text{if } v = x, \\ K(v), & \text{if } v \notin \{x, y\} \end{cases}$$

is a proper k-coloring of G. In other words, (x, y) is a transition edge if we may swap the colors of x and y without destroying properness of the coloring. In this case, we call K' the coloring obtained by *swapping colors along the edge* (x, y). We define

$$T(K) := \{(x, y) \mid (x, y) \text{ is a transition edge for } K\} \subseteq E$$

as the *transition space* of the coloring K. For any $x \in V$ we also define

$$T_x(K) := \{y \in V \mid (x, y) \in T(K)\}$$

as the set of all neighbors of x that are adjacent to x via a transition edge.

We have the following simple characterization of transition edges.

Remark 4.1.2. Let G and K be as in Definition 4.1.1. Let $(x, y) \in E$. Then (x, y) is a transition edge for K iff x is the only neighbor of y with color $K(x)$ and y is the only neighbor of x with color $K(y)$. In other words,

$$(x, y) \in T(K) \Longleftrightarrow K^{-1}(K(x)) \cap \mathcal{N}(y) = \{x\} \text{ and } K^{-1}(K(y)) \cap \mathcal{N}(x) = \{y\}.$$

The number of transition edges of a given coloring may be regarded as a measure of robustness of that coloring in the sense that the coloring tolerates swaps of colors along such edges. We make this precise with the following notion.

Definition 4.1.3. Let $G = (V, E)$ be a simple undirected graph.

(a) We define the *robustness* of a k-coloring $K : G \longrightarrow [k]$ as $\mathrm{rb}(K) = \frac{|T(K)|}{|E|}$.

(b) The *k-coloring robustness of G* is defined as

$$\mathrm{rb}_k(G) = \max\{\mathrm{rb}(K) \mid K \text{ is a } k\text{-coloring of } G\}.$$

Clearly, if $k \geq |V(G)|$ then $\mathrm{rb}_k(G) = 1$ because we may color each vertex differently. Now we are ready to consider some small examples.

Example 4.1.4. Consider the graph $G := H_2(3, 2)$ with vertex set $\mathbb{Z}_2^3$.

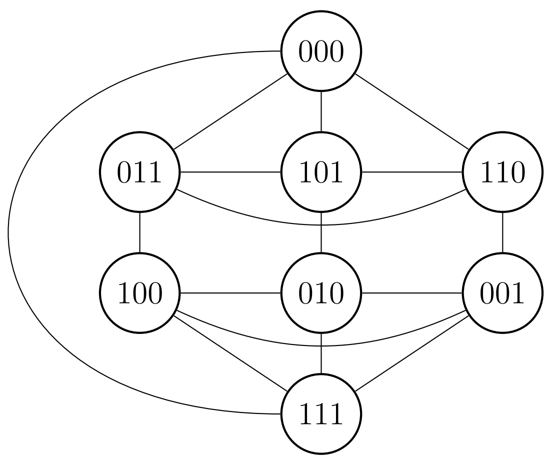


Figure 4.1: $H_2(3, 2)$

A quick investigation shows that the 9 colorings in the table below are in fact all colorings up to isomorphism. To see this explicitly, WLOG fix $K(000) = A$ and $K(111) = B$. Then exactly one vertex in $\{011, 101, 110\}$ can have color B and exactly one vertex in $\{100, 010, 001\}$ may have color A. Any combination of these choices can occur within a proper coloring, and in fact it is easy to see that such a coloring is unique for each pair of choices. Moreover, as we have fixed the color of 000 and 111, none of these 9 colorings are isomorphic.

K_1	000, 001	100, 101	010, 011	110, 111
K_2	000, 010	100, 110	001, 011	101, 111
K_3	000, 100	010, 110	001, 101	011, 111
K_4	000, 100	010, 110	001, 011	101, 111
K_5	000, 100	010, 011	001, 101	110, 111
K_6	000, 010	100, 110	001, 101	011, 111
K_7	000, 010	100, 101	001, 011	110, 111
K_8	000, 001	100, 110	010, 011	101, 111
K_9	000, 001	100, 101	010, 110	011, 111

Table 4.1: All 4-colorings of $H_2(3,2)$

The first three rows represent the $(1,2)$-, $(1,3)$-, $(2,3)$-coordinate colorings, respectively. The last 6 colorings are obtained from the coordinate colorings by swapping colors along a transition edge. For example, K_4 can be obtained from K_3 by swapping the colors of 011 and 101 or from K_2 be swapping the colors of 010 and 100. Similarly, K_5 can be obtained from K_3 by swapping the colors of 011 and 110 or from K_1 by swapping the color of 100 and 001. In the same way, each of $K_6, \dots, K_9$ can be obtained from two different coordinate colorings by swapping colors along a certain transition edge. All of this shows that each minimal coloring of G is either a coordinate coloring or just one swap away from a coordinate coloring. In particular, each minimal coloring is even.

The following two figures display the graph with two different colorings and their transition edges shown in zigzag. The left one shows the $(2,3)$-coordinate coloring (coloring K_3) and the right one shows coloring K_6. Note that for the $(2,3)$-coordinate coloring we have 8 transition edges, and they tile the graph in two 4-cycles (in the sense of Def. 4.2.1 later in this thesis). In contrast, for coloring K_6 we only have 4 transition edges. Investigating all 9 minimal colorings we obtain

$$\mathrm{rb}(K_i) = \begin{cases} \frac{1}{2}, & \text{for } i = 1,2,3, \\ \frac{1}{4}, & \text{for } i = 4,\dots,9. \end{cases}$$

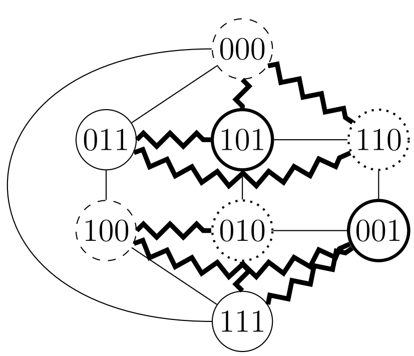


Figure 4.2: $(2,3)$-coordinate coloring with transition edges

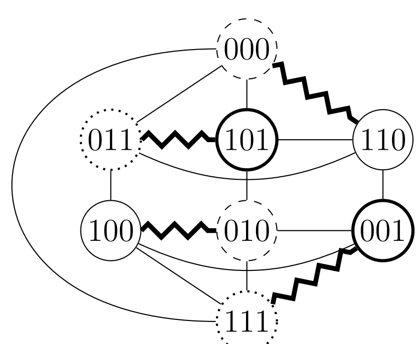


Figure 4.3: Coloring K_6 with transition edges

Example 4.1.5. Consider now the Hamming graph $G = H_2(4,3)$, shown in Figure 4.4 below. In this case one obtains an abundance of 4-colorings. In fact, we will show later in Corollary 4.1.18 that the number of proper 2^{n-d+1}-colorings of $H_2(n,d)$ grows exponential with n. We illustrate just a few particular phenomena. In addition to the 6 coordinate colorings we have for instance the colorings

K_1	0000,1000, 0100,1100	0010,1010, 0110	0001,1001, 0101,0011	1110,1101,1011, 0111,1111
K_2	0001, 0010, 0100, 1000	1110, 1101, 1011, 0111	1111, 1100 0110, 1010	0000, 0011, 0101, 1001

Table 4.2: Noncoordinate Colorings of $H_2(4,3)$

While the first one is an uneven coloring, the second one is even. Moreover, K_2 is more than one swap away from a coordinate coloring (of course, the uneven coloring K_1 cannot be obtained by any number of swaps from a coordinate coloring). Furthermore, one can check straightforwardly that coloring K_1 has 3 transition edges, and coloring K_2 has 8 transition edges. In addition, for K_2 one may move, for instance, 0000 into the first color class and/or 1111 into the second color class to obtain an uneven proper 4-coloring. Similarly, moving 0000 to the third color class of K_1 leads to yet another uneven 4-coloring.

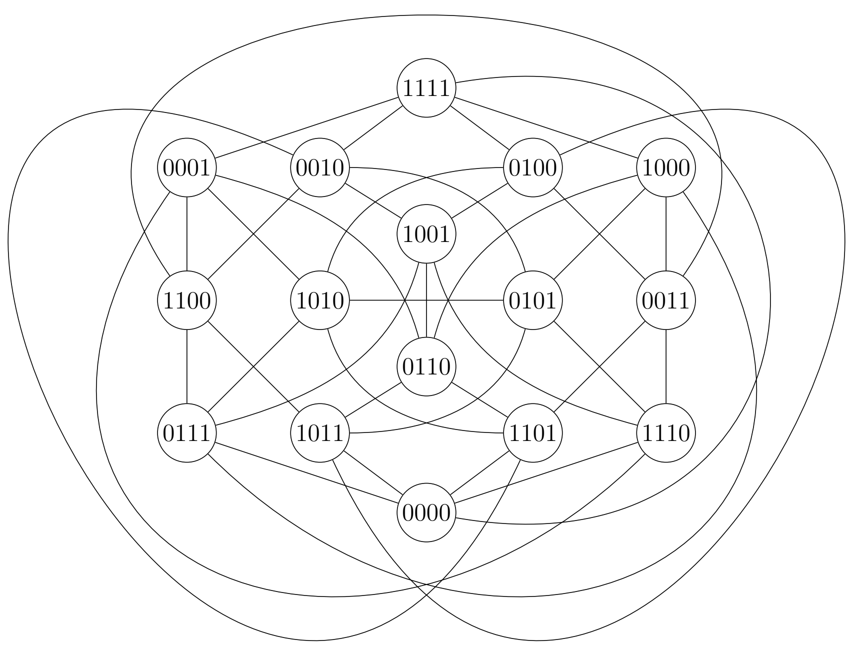


Figure 4.4: $H_2(4,3)$

In Figure 4.5 we show the $(1,2)$-coordinate coloring along with its transition edges. As was the case for coordinate colorings of $H_2(3,2)$, the transition edges tile the graph in 4-cycles (in the sense of Def. 4.2.1 later in this thesis). In particular, it has 16 transition edges and thus the highest robustness among all colorings discussed in this example.

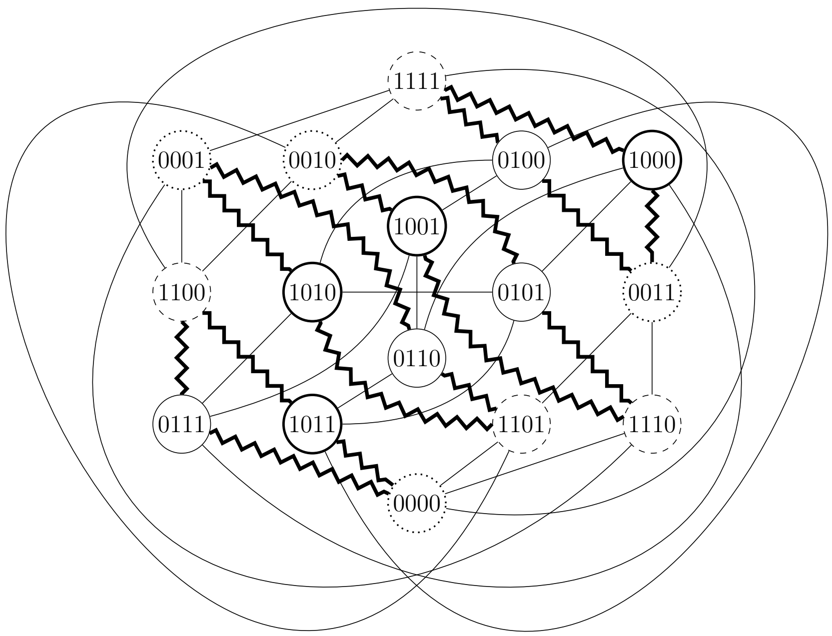


Figure 4.5: $(1, 2)$-coordinate coloring of $H_2(4, 3)$ with transition edges

We now return to the study of 4-colorings of general Hamming graphs $H_2(n, n-1)$. As the examples above suggest, the coordinate colorings appear to be more robust (that is, have more transition edges) than other 4-colorings. We will see that this is indeed true for all Hamming graphs $H_2(n, n-1)$. In the next section all maximally robust 4-colorings will be described explicitly. With one exception they are exactly the coordinate colorings.

Note that for binary Hamming graphs the edge (x, y) carries only a single natural label $x+y = x-y \in \mathbb{Z}_2^n$. For the following it is helpful to have this edge labeling in mind. Recall that the Hamming graph $G := H_2(n, d)$ is the Cayley graph of the group $\mathbb{Z}_q^n$ with generating set $\mathcal{S}_d := \{v \in \mathbb{Z}_2^n \mid wt(v) \geq d\}$. In other words, for all $x, y \in \mathbb{Z}_2^n$ we have

$$x + y \in \mathcal{S}_d \Longleftrightarrow (x, y) \in E(G) \Longleftrightarrow (x + z, y + z) \in E(G) \text{ for all } z \in \mathbb{Z}_2^n. \qquad (4.1.1)$$

Definition 4.1.6. Let K be a proper coloring of the Hamming graph $H_2(n, d)$, and let $T(K)$ be its transition space. If there exists a nonempty subset $\mathcal{B} = \{v_1, \ldots, v_t\} \subseteq \mathcal{S}_d$ such that

$$T(K) = \{(x, y) \mid x + y \in \mathcal{B}\}$$

we say that $T(K)$ *is generated by* $\mathcal{B}$ and write $T(K) = \langle \mathcal{B} \rangle = \langle v_1, \ldots, v_t \rangle$. If no set $\mathcal{B}$ exists such that $T(K) = \langle \mathcal{B} \rangle$ or if $T(K) = \emptyset$, we say that $T(K)$ *is not generated.*

Recall from Definition 4.1.1 the set $T_x(K)$ of transition neighbors of a given vertex x. With the above terminology we have

$$T(K) = \langle \mathcal{B} \rangle \Longleftrightarrow T_x(K) = \{x + b \mid b \in \mathcal{B}\} = x + \mathcal{B} \text{ for all } x \in \mathbb{Z}_2^n. \qquad (4.1.2)$$

As this identity indicates, the existence of such a set $\mathcal{B}$ is quite a strong assumption. In particular, every vertex is incident to the same number of transition edges. In most cases the transition space of a coloring is not generated by a set $\mathcal{B}$.

Example 4.1.7.

(a) Figure 3 shows that for the $(2,3)$-coordinate coloring of $H_2(3,2)$ we have $T(K) = \langle 101, 110\rangle$. For the coloring K_6 shown in Figure 4, we have $T(K) = \langle 110\rangle$.

(b) From Figure 6 we obtain that for the $(1,2)$-coordinate coloring of $H_2(4,3)$ the transition space is generated by $\{0111, 1011\}$. For the coloring K_2 in Example 4.1.5 we have $T(K_2) = \langle 1111\rangle$.

(c) An even coloring of $H_2(5,4)$ for which the transition space is nonempty and not generated can be constructed as follows. Take the $(1,2)$-coordinate coloring and swap the colors of vertices 00000 and 10111. This leads to an even 4-coloring, say K (see also Prop. 4.1.8 below). Then one checks straightforwardly that $T_{10111}(K) = \{00000\}$ and $T_{00001}(K) = \{10110, 01110\}$. This shows that $T(K)$ is not generated.

Although we are primarily interested in the transition spaces and robustness of the family $H_2(n, n-1)$, many of the results about it generalize to all $H_q(n,d)$. In what follows, we show that the transition space of any coordinate colorings of $H_2(n,d)$ is generated and give an explicit description of its generating set. We also derive several bounds on the robustness of general Hamming graphs. As a special case, we show that the coordinate colorings are maximally robust for $H_2(n, n-1)$.

Recall that for $i = 1, \ldots, n$, we defined the vectors e_i and f_i in $\mathbb{Z}_2^n$ to be

$$e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \quad\text{and}\quad f_i := e_i + \mathbb{1} = (1, \ldots, 1, 0, 1, \ldots, 1), \tag{4.1.3}$$

where 1 resp. 0 is at the i-th position.

Note that if $d = 1$, $H_q(n, 1)$ is the complete graph and thus every edge is a transition edge. For $d \geq 2$, we have the following result.

Proposition 4.1.8. *Suppose K is an $(c_1, c_2, \ldots, c_{n-d+1})$-coordinate coloring of $G = H_q(n,d)$ where $n \geq d \geq 2$.*

(a) If $q = 2$ (and thus $d < n$), then

$$T(K) = \Big\langle \Big\{ \mathbb{1} + \sum_{\substack{j=1\\ j\neq i}}^{n-d+1} e_{c_j} \;\Big|\; 1 \leq i \leq n-d+1 \Big\} \Big\rangle.$$

In particular, if K is the (i,j)-coordinate coloring of $H_2(n, n-1)$, then $T(K) = \langle f_i, f_j\rangle$.

(b) If $q > 2$, then $T(K) = \emptyset$.

Proof. Suppose $a \in V(G)$.

(a) Suppose $q = 2$. Recall from (4.1.1) that the neighbors of a are exactly the vertices $\{v \in \mathbb{Z}_2^n \mid wt(a+v) \geq d\}$. Partition the coordinates into two sets α and β where $\alpha = \{c_1, c_2, \ldots, c_{n-d+1}\}$ and $\beta = [n] \setminus \alpha$. Note that $|\alpha| = n - d + 1$ and $|\beta| = d - 1$.

In particular, $d \geq 2$ implies that both α and β are nonempty. Consider a vertex b such that $b \in \{a + \mathbb{1} + \sum_{\substack{j=1 \\ j \neq i}}^{n-d+1} e_{c_j} \mid 1 \leq i \leq n-d+1\}$. We want to show that b is the only neighbor of a of color $K(b)$. If we can do so, the symmetry of the argument immediately shows that a is the only neighbor of b with color $K(a)$. Then Remark 4.1.2 would allow us to conclude that (a, b) is a transition edge.

Assume $c \in \mathcal{N}(a)$ such that $K(c) = K(b)$. Since K is an $(c_1, c_2, \ldots, c_{n-d+1})$-coordinate coloring, the colors of b and c are determined by and therefore must agree on coordinates in α. This further implies that $a + b$ and $a + c$ agree in coordinates in α.

Note that $\mathrm{wt}(a + b) = n - ((n - d + 1) - 1) = d$. Furthermore, every coordinate of $(a + b)$ in β is nonzero but only a single coordinate of $(a + b)$ in α is nonzero. Since, $K(b) = K(c)$, there is only a single (same) coordinate of $(a + c)$ that is nonzero in α. However, the adjacency of a and c requires that $wt(a + c) \geq d$. Therefore there must be at least $d - 1$ coordinates in β that are nonzero in $(a + c)$. However, $|\beta| = d - 1$ so in fact every coordinate of $(a + c)$ in β is nonzero and we must have $b = c$.

By our argument above, we have that (a, b) must be a transition edge. Thus

$$\left\langle \left\{ \mathbb{1} + \sum_{\substack{j=1 \\ j \neq i}}^{n-d+1} e_{c_j} \,\middle|\, 1 \leq i \leq n-d+1 \right\} \right\rangle \subseteq T(K).$$

It remains to show that there exist no other transition edges. Note that $\left\{ \mathbb{1} + \sum_{\substack{j=1 \\ j \neq i}}^{n-d+1} e_{c_j} \,\middle|\, 1 \leq i \leq n-d+1 \right\}$ are exactly all vectors of weight greater than or equal to d with only a single nonzero coordinate in α. Therefore suppose $v \in \mathcal{N}(a)$ such that $a + v$ has at least two nonzero coordinates in α. By Remark 4.1.2, it suffices to demonstrate that there exists another neighbor of a with the same color as v.

Case 1. Suppose $(a + v)$ has $d - 1$ nonzero coordinates in β and let w be some coordinate in β. Then $wt(a + v + e_w) \geq 2 + (d - 2) = d$ implying $v + e_w$ is a neighbor of a. Furthermore, since K is an $(c_1, c_2, \ldots, c_{n-d+1})$-coordinate coloring, we have $K(v) = K(v + e_w)$. Therefore (a, v) is not a transition edge.

Case 2. Suppose $(a+v_1)$ has less than $d-1$ nonzero coordinates in β. Then there exists a coordinate $x \in \beta$ such that $(a+v)$ is zero in that coordinate. Then $wt(a+v+e_x) = wt(a + v) + 1 \geq d + 1$ so $v + e_x$ is a neighbor of a and $K(v) = K(v + e_x)$ since $x \in \beta$. Again, (a, v) cannot be a transition edge.

(b) Now suppose $q > 2$. If $b \in \mathcal{N}(a)$, then $\mathrm{wt}(a + b) \geq d$. Let $i \in \beta$ with β as in part (a). Then either $b + e_i$ or $b + 2e_i$ is a neighbor of a since at least one of them disagrees with a in the *i-th* coordinate and therefore has Hamming distance at least d from a. Furthermore, $K(b + e_i) = K(b + 2e_i) = K(b)$ as they agree on every coordinate in α. Therefore (a, b) is not a transition edge and since a and b were arbitrary, the transition space must be empty.

□

Of note, for $q \geq 3$ we have not yet been able to produce a q^{n-d+1}-coloring of $H_q(n,d)$ which has a nonempty transition space. Part of the difficulty is that there is no clear method (other than computer search) for producing q^{n-d+1}-colorings which are not coordinate colorings. Even for moderately sized parameters, $H_q(n,d)$ is quite large, necessitating some systemic means of coloring. A few examples of noncoordinate colorings are given in Chapter 7, but in each case it is shown that the transition space is empty.

We now prove a few preliminary results which we will use to provide an upper bound on the robustness of q^{n-d+1}-colorings of $H_q(n,d)$.

Lemma 4.1.9. *Let G be a graph and let K be a proper k-coloring of G. Let $v \in V(G)$ and $deg(v) = r$. Then*

$$|T_v(K)| \leq \begin{cases} r & \text{if } r \leq k-1 \\ k-2 & \text{if } r \geq k \end{cases}.$$

Proof. We must always have $|T_v(K)| \leq r$ regardless of the value of k as transition edges form a subset of the edge set of G. Therefore suppose $r \geq k$. Note that neighbors of v comprise at most $k-1$ colors since none of them can have the color $K(v)$. Since $r \geq k$, this implies that at least two of the neighbors of v have the same color and therefore no vertex of that color can be connected to v by a transition edge. Therefore there are at most $k-2$ transition edges incident to v (one corresponding to each remaining color). $\square$

Of note, for a fixed k the bound in Lemma 4.1.9 takes its maximum value when $r = k-1$. In this case, each of the neighbors of v may take a unique color and thus every edge incident to v is a transition edge. However, if we were to add in another vertex adjacent to v, it must take the same color as another neighbor of v, and the edges connecting each of them to v would cease to be transition edges.

Recall from Proposition 2.2.4(c) that $H_q(n,d)$ is regular of degree $\sum_{i=0}^{n-d} \binom{n}{i}(q-1)^{n-i}$. Lemma 4.1.9 suggests it would be beneficial to be able to compare this value to q^{n-d+1}.

Lemma 4.1.10. *Suppose $q \geq 2$ and $n \geq d \geq 2$ such that $(q,d) \neq (2,n)$. Then*

$$q^{n-d+1} \leq \sum_{i=0}^{n-d} \binom{n}{i}(q-1)^{n-i}.$$

Proof. If $n = d$, we simply need to show the $q \leq (q-1)^n$ which is obviously true when $q > 2$. Therefore suppose that $n > d$. Then

$$
\begin{aligned}
q^{n-d+1} &= \sum_{i=0}^{n-d+1} \binom{n-d+1}{i} (q-1)^{n-d+1-i} \\
&= 1 + (q-1)^{n-d+1} + (n-d+1)(q-1)^{n-d} + \sum_{i=2}^{n-d} \binom{n-d+1}{i} (q-1)^{n-d+1-i} \\
&\leq (q-1)^{n-d+1} + (n-d+2)(q-1)^{n-d} + \sum_{i=2}^{n-d} \binom{n-d+1}{i} (q-1)^{n-d+1-i} \\
&\leq (q-1)^{n} + n(q-1)^{n-1} + \sum_{i=2}^{n-d} \binom{n}{i} (q-1)^{n-i} \\
&= \sum_{i=0}^{n-d} \binom{n}{i} (q-1)^{n-i}
\end{aligned}
$$

□

Proposition 4.1.11. *Suppose $d \geq 2$ and $(q,d) \neq (2,n)$. Then*

$$rb_{q^{n-d+1}}(H_q(n,d)) \leq \frac{q^{n-d+1} - 2}{\sum_{i=0}^{n-d} \binom{n}{i} (q-1)^{n-i}}.$$

Proof. For all $x \in V(H_q(n,d))$, Lemma 4.1.9 and Lemma 4.1.10 combine to imply that $T_x(K) \leq q^{n-d+1} - 2$ for any proper q^{n-d+1}-coloring K. Suppose K' is a maximally robust q^{n-d+1}-coloring of $H_q(n,d)$. Then $|T(K')| \leq \frac{q^n}{2}(q^{n-d+1} - 2)$ since each edge is incident to two vertices. By Proposition 2.2.4(c), each vertex is incident to exactly $\sum_{i=0}^{n-d} \binom{n}{i}(q-1)^{n-i}$ edges so $|E(H_q(n,d)| = \frac{q^n}{2} \sum_{i=0}^{n-d} \binom{n}{i}(q-1)^{n-i}$. The result then follows immediately from Definition 4.1.3. □

In the binary case, Proposition 4.1.8 immediately provides a lower bound on the robustness of 2^{n-d+1}-colorings as it implies that in coordinate colorings each vertex is adjacent to $n-d+1$ transition edges. Then a similar argument to the proof of Proposition 4.1.11 gives the following lemma.

Lemma 4.1.12. *Suppose $n \geq d \geq 2$. Then*

$$rb_{2^{n-d+1}}(H_2(n,d)) \geq \frac{n-d+1}{\sum_{i=0}^{n-d} \binom{n}{i}}.$$

Corollary 4.1.13. *For $n \geq 3$,*

$$rb_4(H_2(n, n-1)) = \frac{2}{n+1}.$$

In particular, the coordinate colorings of $H_2(n, n-1)$ are maximally robust.

Proof. This follows immediately from Proposition 4.1.11 and Lemma 4.1.12 as both bounds reduce to $\frac{2}{n+1}$ when $(n, d, q) = (n, n-1, 2)$, and the lower bound came directly from the transition space of a coordinate coloring (see Proposition 4.1.8 and the proof of Proposition 4.1.11). □

Remark 4.1.14. It is clear from the proof of Proposition 4.1.11 that a q^{n-d+1}-coloring of $H_q(n, d)$ meets the upper bound in Proposition 4.1.11 if and only if $|T_x(K)| = q^{n-d+1} - 2$ for all $x \in V(H_q(n, d))$. In light of Corollary 4.1.13, we see that a 4-colorings K of $H_2(n, n-1)$ is maximally robust if and only if $rb(K) = \frac{2}{n+1}$ if and only if $|T_x(K)| = 2$ for all $x \in V(H_2(n, n-1))$.

It is not known whether coordinate colorings of $H_2(n, d)$ are maximally robust when $d < n-1$. While it is obvious that the difference between the bounds in Proposition 4.1.11 and Lemma 4.1.12 grows exponentially with the size of $n-d$, there is currently no evidence that the upper bound can be attained when $d < n-1$. Thus the divergence of the bounds cannot be taken as evidence for or against the claim.

Lastly, we demonstrate that the number of nonisomorphic 2^{n-d+1}-colorings of $H_2(n, d)$, and in particular the number of 4-colorings of $H_2(n, n-1)$, grows exponentially with n. The following definitions will be useful in our investigations.

Definition 4.1.15. Let G be a graph and $L = K_0, K_1, \ldots, K_m$ be a sequence of proper k-colorings of G such that for $i = \{1, \ldots, m-1\}$, K_{i+1} can be produced from K_i by swapping the colors of two vertices in K_i along a transition edge. Then we say L is a *transition sequence* of length m between K_0 and K_m. We denote this length as $|L| = m$.

Recall that we consider any two isomorphic colorings to be the same. Let K_0 and K_m be as in Definition 4.1.15 and K'_m be any coloring isomorphic to K_m. Then any transition sequence between K_0 and K_m is considered to also be a transition sequence between K_0 and K'_m, and vice versa.

Definition 4.1.16. Let G be a graph and let K and K' be two proper nonisomorphic k-colorings of G. Then we define the transition sequence distance between K and K' as

$$d_S(K, K') = \min\{\ |L| \mid L \text{ is a transition sequence between } K \text{ and } K'\ \}$$

If K and K' are isomorphic colorings, then we set $d_S(K, K') = 0$. If there exists no transition sequence between K and K', we set $d_S(K, K') = \infty$.

Proposition 4.1.17. *Let C_1 and C_2 be two distinct coordinate colorings of $H_2(n, d)$. Then*

$$d_S(C_1, C_2) \geq 2^{n-2}$$

Proof. If there is no transition sequence between C_1 and C_2, then $d_S(C_1, C_2) = \infty \geq 2^{n-2}$ and we are done. Therefore suppose there exists some transition sequence between C_1 and C_2.

Each coordinate coloring has exactly 2^{n-d+1} color classes, each with size 2^{d-1}. Let $\kappa_1, \ldots, \kappa_{2^{n-d+1}}$ and $\kappa'_1, \ldots, \kappa'_{2^{n-d+1}}$ be the color classes of C_1 and C_2 respectively. WLOG we may assume that C_1 is the $(1, 2, \ldots, n-d+1)$-coordinate coloring and C_2 is the $(c_1, \ldots, c_{n-d+1})$-coordinate coloring such that $c_j = n-d+2$ for some $j \in \{1, \ldots, n-d+1\}$.

We claim that $|\kappa_a \cap \kappa'_b| \leq 2^{d-2}$ for all (a, b). To see this, note that the 2^{d-1} vertices of κ_a are exactly the vectors in $\mathbb{Z}_2^n$ with some fixed value in the first $n-d+1$ coordinates. Moreover, exactly half of these vertices have a 0 in the $(n-d+2)^{th}$ coordinate while the other half have a 1 in this coordinate. Since C_2 is a coordinate coloring that depends on the $(n-d+2)^{th}$ coordinate, at most half of the vertices in κ_a can be in any given color class of C_2, proving the claim.

Note that this claim implies that at least half the vertices in each color class must change to a new color class at some point in a transition sequence between C_1 and C_2. Thus $2^{d-2} \cdot 2^{n-d+1} = 2^{n-1}$ must change color classes. As each swap of colors along a transition edge changes the color class of two vertices, we get that $d_S(C_1, C_2) \geq \dfrac{2^{n-1}}{2} = 2^{n-2}$. $\square$

Recall that for a graph G, we defined $Col_k(G)$ to be the set of proper k-colorings of G. The following result shows that unlike the case $q \geq 3$ and $d = n$ from Theorem 1.2.2 in which the n coordinate colorings were the only minimal colorings, the number of 2^{n-d+1}-colorings of $H_2(n, d)$ grows exponentially with n when $d < n$.

Corollary 4.1.18. *Let $n \geq 4$ and $d \geq 3$. Then*

$$\left| Col_{2^{n-d+1}}\left(H_2(n, d)\right) \right| \geq d \cdot \binom{n}{d} \cdot 2^{n-1} + \binom{n}{d-1}$$

Proof. First, note that there are $\binom{n}{n-d+1}$ coordinate colorings of $H_2(n, d)$. For a given coordinate coloring, Proposition 4.1.11 gives that each vertex is incident to $n-d+1$ transition edges. Thus a chosen coordinate coloring has $(n-d+1)2^{n-1}$ transition edges, as each edge is incident to two vertices. Moreover since $n \geq 4$, Proposition 4.1.17 gives that any two distinct coordinates colors C_1 and C_2 have $d_S(C_1, C_2) \geq 2^{4-2} = 4$. This implies that no two colorings achieved via swapping colors along transition edges in C_1 and C_2 respectively can be be the same, or else we would have $d_S(C_1, C_2) \leq 3$.

Moreover, given two distinct transition edges in the same coordinate coloring, the colorings achieved by swapping colors along those edges cannot be the same. To see this, note that the color classes have size $2^{d-1} \geq 2^{3-1} = 4$. As the edges are distinct, the resulting colorings cannot be identical, and thus we need only check that they are not isomorphic. However, for this to be true, we would have had to exchange all vertices from one color class with all vertices from a second color class. As the color classes have size at least 4, this would have required at least 4 swaps to have occurred. As only 2 were used, the colorings cannot be isomorphic and are thus distinct.

Since there were $\binom{n}{n-d+1} = \binom{n}{d-1}$ choices of coordinate colorings, each of which has $(n-d+1)2^{n-1}$ colorings associated to it by swapping colors along a transition edge, we have

$$\left|Col_{2^{n-d+1}}(H_2(n,d))\right| \geq \binom{n}{n-d+1}(n-d+1)2^{n-1} + \binom{n}{d-1}$$

However,

$$\binom{n}{n-d+1}(n-d+1) = \frac{n!(n-d+1)}{(n-d+1)!(d-1)!} = \frac{n!}{(n-d)!d!} \cdot d = \binom{n}{d} \cdot d.$$

□

4.2 Maximally Robust 4-colorings of $H_2(n, n-1)$

In the last section, we showed that the coordinate colorings of $H_2(n, n-1)$ are maximally robust minimal colorings. Unlike the case of $d = n$ studied by Greenwell/Lovász [11] and Alon et. al. [2] in which there where exactly n minimal colorings (namely the coordinate colorings), the number of proper 4-coloring of $H_2(n, n-1)$ grows exponentially with n (see Corollary 4.1.18). This naturally leads to the question of whether there might exist other maximally robust 4-colorings of $H_2(n, n-1)$.

In this section, a characterization of the maximally robust 4-colorings of $H_2(n, n-1)$ is derived in terms of their transition spaces and all these colorings are presented explicitly (up to isomorphism). In order to formulate these results, we need the following definition.

Definition 4.2.1. Let $G = (V, E)$ be a finite graph and $M \subseteq E$ be a set of edges in G. We say that M *tiles the graph in 4-cycles* if M is the union of pairwise vertex-disjoint and edge-disjoint 4-cycles and every vertex appears on (exactly) one cycle.

For example, in Figures 4.2 and 4.5 the transition edges of the given coloring tile the graph in 4-cycles.

Now we are ready to present our main results. The first theorem provides a characterization of maximally robust colorings in terms of the transition spaces. The second theorem presents for each such transition space the unique coloring associated with it or shows that no such coloring exists. The proofs will be presented afterwards.

Note that $H_2(2, 1)$ is simply K_4 and has only one 4-coloring up to isomorphism. In particular, every edge is a transition edge.

Theorem 4.2.2. *Let $n \geq 3$ and K be a 4-coloring of $H_2(n, n-1)$. The following are equivalent.*

i) There exist $v, w \in \mathcal{N}(0)$ such that $T(K) = \langle v, w\rangle$.

ii) $rb(K) = \frac{2}{n+1}$.

iii) The transition edges tile the graph in 4-cycles.

If one, hence any of the above is true, then the coloring of a single 4-cycle uniquely determines the coloring of the entire graph.

Recall that if the transition space of a coloring is generated by a set $\mathcal{B}$, then $\mathcal{B}$ is contained in $\mathcal{N}(0) = \{\mathbb{1}, f_1, \ldots, f_n\}$. Therefore the requirement in i) that the vertices v, w be in $\mathcal{N}(0)$ is not a restriction.

Theorem 4.2.3. *Let $n \geq 3$ and K be a 4-coloring of $H_2(n, n-1)$.*
1) If $T(K) = \langle f_i, f_j \rangle$ for some $1 \leq i < j \leq n$, then K is the (i,j)-coordinate coloring.
2) If $T(K) = \langle \mathbb{1}, f_j \rangle$ for some j, then n is even and K is the coloring with color sets

$$\mathcal{A}_\mu^\nu := \{v \in \mathbb{Z}_2^n \mid wt(v) \equiv \nu \bmod 2,\ v_j = \mu\} \text{ for } \nu,\ \mu \in \{0, 1\}.$$

As a consequence, any 4-coloring of $H_2(n, n-1)$ with maximal robustness is an even coloring.

Note that Part 2) tells us that if n is odd, then sets of the form $\{\mathbb{1}, f_j\}$ do not generate the transition space of a 4-coloring.

We prove the results in several steps and first need some preliminary results. Before doing so, let us point out that for any 4-coloring the graph $H_2(n, n-1)$ does not have cycles consisting of transition edges of length 3. For $n \geq 4$ this is obvious by Proposition 2.3.1(c), while for $n = 3$ this can be verified via the Figures 4.2 and 4.3, which cover all 4-colorings of $H_2(3, 2)$ up to graph isomorphism. The following lemmas will lead to information about the existence, colors, and interrelation of 4-cycles consisting of transition edges.

Lemma 4.2.4. *Consider the graph $G := H_2(n, n-1)$, where $n \geq 3$. Let $x_1, x_2, x_3, x_4 \in V(G)$ be distinct vertices such that*

$$(x_1, x_2),\ (x_2, x_3),\ (x_3, x_4),\ (x_4, x_1) \in E(G), \tag{4.2.1}$$

that is, $x_1, \ldots, x_4$ are on a 4-cycle. Then $x_1 + x_2 = x_3 + x_4$ and thus $x_1 + x_4 = x_2 + x_3$.

Proof. As all vectors are in $\mathbb{Z}_2^n$ it suffices to show $x_1 + x_2 = x_3 + x_4$. Since $x_1, \ldots, x_4$ are distinct, we have

$$x_1 + x_2 \neq x_2 + x_3 \neq x_3 + x_4.$$

By (4.2.1) and (4.1.1) all these vectors are in $\mathcal{N}(0) = \{\mathbb{1}, f_1, \ldots, f_n\}$. Using that $x_1 + x_4 = (x_1+x_2)+(x_2+x_3)+(x_3+x_4)$, we have $\mathrm{wt}\big((x_1+x_2)+(x_2+x_3)+(x_3+x_4)\big) = wt(x_1+x_4) \geq n-1$. But since the sum of any three distinct vectors from $\mathcal{N}(0)$ has weight at most $n-2$, we conclude $x_1 + x_2 = x_3 + x_4$. $\square$

Lemma 4.2.5. *Let $n \geq 3$ and let K be a 4-coloring of $H_2(n, n-1)$.*
- *(a) Suppose there exists a path of length 3 consisting of transition edges and passing through the vertices x_1, x_2, x_3, x_4. Then the colors of x_1, x_2, x_3, x_4 are distinct.*
- *(b) There exists no simple path in $H_2(n, n-1)$ of length greater than 3 consisting of transition edges.*

Proof. Both claims are easily verified if $n = 3$ as there are only 9 colorings to check; see Figure 2. Indeed, Figures 3 and 4 show (up to graph isomorphism) the only possible

scenarios of transition edges. Thus we may assume $n \geq 4$.
(a) Since $(x_1, x_2), (x_2, x_3), (x_3, x_4)$ are transition edges, the only vertices that may share a color are x_1 and x_4. Pick a vector $y \in \mathcal{N}(0) \setminus \{x_1 + x_2,\, x_2 + x_3,\, x_3 + x_4\}$. This is possible because $n \geq 3$. By (4.1.1) the vertices $x_2 + y$ and $x_3 + y$ are adjacent. Since the girth is 4 thanks to Proposition 2.3.1(c), the choice of y implies that the vertices $x_1, x_2, x_3, x_4, x_2 + y, x_3 + y$ are distinct. Thus we have the subgraph

$$\begin{tikzcd}
x_1 \arrow[r, dash] & x_2 \arrow[r, dash] \arrow[d, dash] & x_3 \arrow[r, dash] \arrow[d, dash] & x_4 \\
 & x_2 + y \arrow[r, dash] & x_3 + y &
\end{tikzcd}$$

Since (x_1, x_2) and (x_2, x_3) are transition edges, we conclude $K(x_2 + y) = K(x_4)$. In the same way $K(x_3 + y) = K(x_1)$. Now the adjacency of $x_2 + y$ and $x_3 + y$ implies $K(x_1) \neq K(x_4)$. All of this shows that x_1, x_2, x_3, x_4 assume distinct colors.
(b) Suppose there exists a path consisting of transition edges passing through the vertices $x_1, \ldots, x_5$, and where $x_i \neq x_j$ for $i \neq j$. From Part (a) we know that x_1, x_2, x_3, x_4 have distinct colors and the same is true for x_2, x_3, x_4, x_5. Thus we conclude $K(x_1) = K(x_5)$. Consider the vertex $x_2 + x_3 + x_4$, which is adjacent to x_2 and x_4 by (4.1.1). Thus we have

$$\begin{tikzcd}
x_1 \arrow[r, dash] & x_2 \arrow[r, dash] \arrow[dr, dash] & x_3 \arrow[r, dash] & x_4 \arrow[r, dash] \arrow[dl, dash] & x_5 \\
 & & x_2 + x_3 + x_4 & &
\end{tikzcd}$$

The fact that the upper row consists of transition edges implies that $x_2 + x_3 + x_4$ must equal x_1 or x_5 for otherwise there is no color left for $x_2 + x_3 + x_4$. Assume without loss of generality that $x_2 + x_3 + x_4 = x_1$. But then $K(x_5) = K(x_1) = K(x_2 + x_3 + x_4)$, contradicting the fact that (x_4, x_5) is a transition edge. Thus we have shown that no such path exists. $\square$

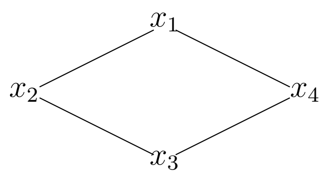
By the previous lemma there are no cycles of length greater than 4 in $H_2(n, n-1)$ consisting of transition edges. The next lemma provides information about cycles of length 4 consisting of transition edges.

Lemma 4.2.6. *Let $n \geq 3$ and let K be a 4-coloring of $G := H_2(n, n-1)$. Let $x_1, x_2, x_3, x_4 \in V(G)$ be distinct vertices such that*

$$(x_1, x_2),\ (x_2, x_3),\ (x_3, x_4),\ (x_4, x_1) \in T(K),$$

that is, $x_1, \ldots, x_4$ are on a 4-cycle whose edges are transition edges of K. Then
(a) The colors $K(x_1), \ldots, K(x_4)$ are distinct.
(b) Let $y \in V(G) \setminus \{x_1, \ldots, x_4\}$ be any other vertex of G. Then y is adjacent to at most one of the vertices $x_1, \ldots, x_4$. Furthermore, if y is adjacent to x_i, then $K(y) = K(\tilde{x}_i)$, where $\tilde{x}_i$ denotes the vertex opposite to x_i in the 4-cycle.

Proof. Consider the given 4-cycle with its opposite vertices



$$\begin{aligned}\tilde{x_1} &= x_3\\ \tilde{x_3} &= x_1\\ \tilde{x_2} &= x_4\\ \tilde{x_4} &= x_2\end{aligned} \tag{4.2.2}$$

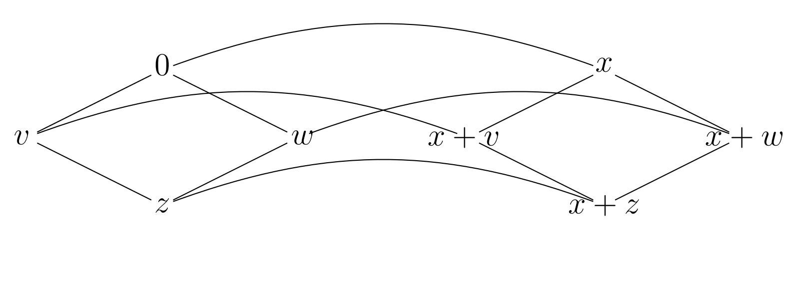
(a) follows from Lemma 4.2.5(a).
(b) Let $y \in V(G)\backslash\{x_1, \ldots, x_4\}$. Assume $x_1 \sim y \sim x_2$. Then $K(x_1) \neq K(y) \neq K(x_2)$. But since (x_1, x_4) and (x_2, x_3) are transition edges, we also have $K(x_4) \neq K(y) \neq K(x_3)$. This contradicts that K is a 4-coloring. In the same way one shows that y cannot be adjacent to two opposite vertices of the 4-cycle. Assume now $y \sim x_1$. Then, again, because (x_1, x_2) and (x_1, x_4) are transition edges, y must have color $K(x_3) = K(\tilde{x_1})$. □

Now we are ready to prove the main theorems.

Proof of Theorem 4.2.2. i) $\Rightarrow$ ii) Since $T(K) = \langle v, w\rangle$, we have $|T_x(v)| = 2$ for all $v \in \mathbb{Z}_2^n$. Then the result follows immediately from Remark 4.1.14.
ii) $\Rightarrow$ iii) Example 4.1.4 tells us that the only colorings of $H_2(3, 2)$ with maximal robustness are the coordinate colorings. Therefore the implication is easily verified for $n = 3$; see Figure 4.2. Thus let $n \geq 4$ and hence $\text{girth}(H_2(n, n-1)) = 4$ due to Proposition 2.3.1(c). By Remark 4.1.14, we have $|T_x(K)| = 2$ for all $x \in \mathbb{Z}_2^n$, that is, every vertex is incident to exactly two transition edges. The finiteness of the graph then implies that every vertex is on a unique cycle consisting of transition edges, and due to Lemma 4.2.5(b) all these cycles have length 4.
iii) $\Rightarrow$ i) Let $(0, v), (v, z), (z, w), (w, 0)$ be the 4-cycle consisting of transition edges containing the vertex 0. From Lemma 4.2.4 we obtain $v = w + z$, thus $z = v + w$. This shows that $T_x(K) = x + \{v, w\}$ for $x \in \{0, v, w, z\}$. Note also that $\text{wt}(v)$, $wt(w) \geq n - 1$. Let now $x \in V(H_2(n, n-1))\backslash\{0, v, w, z\}$ be any other vertex. We have to show that $T_x(K) = x + \{v, w\}$. We consider two cases.
<u>Case 1:</u> x is adjacent to one of the vertices $\{0, v, w, z\}$. In this case it is adjacent to exactly one of these vertices thanks to Lemma 4.2.6(b). Without loss of generality we may assume $x \sim 0$, which then implies $\text{wt}(x) \geq n - 1$. Then by (4.1.1)

$$(x, x+v),\ (x+v, x+z),\ (x+z, x+w),\ (x+w, x) \tag{4.2.3}$$

is a 4-cycle in $H_2(n, n-1)$. It can be thought of as a shift by x of the original cycle. Again by (4.1.1) we obtain that $x + y \sim y$ for $y \in \{0, v, w, z\}$. Thus we have the subgraph



Now Lemma 4.2.6(b) tells us that

$$K(x) = K(z),\ K(x+v) = K(w),\ K(x+z) = K(0),\ K(x+w) = K(v). \tag{4.2.4}$$

Using that $x \sim 0$ we see that x has neighbors of the three distinct colors $K(0),\ K(w),\ K(v)$. Now we can identify the transition edges incident to x. The fact that $T_0(K) = \{v, w\}$ tells us that $(x, 0)$ is not a transition edge. But then x cannot be incident to any transition edge (x, y) such that $K(y) = K(0)$. Since $K(v)$ and $K(w)$ are neighboring colors of x and by assumption x is on a 4-cycle consisting of transition edges, we conclude that $(x, x+v)$ and $(x, x+w)$ are the transition edges incident to x. Thus $T_x(K) = x + \{v, w\}$. By generality of x all edges in the cycle (4.2.3) are transition edges. Note also that the coloring of the cycle (4.2.3) is uniquely determined by (4.2.4).
Case 2: x is not adjacent to any vertex of the original 4-cycle. As we have shown in Proposition 2.2.4(e) the graph $H_2(n, n-1)$ is connected and therefore there exists a path from 0 to x. Assume the vertices of this path are $0, y_1, \ldots, y_t = x$. With the aid of this path, we can now translate the original cycle across the graph and obtain new 4-cycles consisting of transition edges and whose colors are uniquely determined by the coloring of the original 4-cycle. Case 1 guarantees that in each step we obtain $T_{y_i}(K) = y_i + \{v, w\}$. Thus we conclude $T_x(K) = x + \{v, w\}$, as desired. This part also showed that the coloring K is uniquely determined by a single 4-cycle, and thus the proof is complete. □

Proof of Theorem 4.2.3. 1) From Theorem 4.2.2 we know that there exists (up to isomorphism) at most one 4-coloring K such that $T(K) = \langle f_i, f_j \rangle$. Now the result follows from Proposition 4.1.8.

2) In what follows the parity of the Hamming weight will play a central role, and so we introduce $\mathrm{wt}_2(x) := wt(x) \bmod 2 \in \mathbb{Z}_2$ for any $x \in \mathbb{Z}_2^n$ and thus compute with these weights modulo 2. Recall that in $\mathbb{Z}_2^n$ we have $\mathrm{wt}(x+y) = wt(x) + wt(y) - 2\big|\{i \mid x_i = 1 = y_i\}\big|$, and therefore

$$wt_2(x+y) = wt_2(x) + wt_2(y) \text{ for all } x,\ y \in \mathbb{Z}_2^n. \tag{4.2.5}$$

We will also compute modulo 2 with the indices of the set $\mathcal{A}_\mu^\nu$ introduced in the theorem.
Case 1: Assume n is even. By Theorem 4.2.2 and Lemma 4.2.6(a) there is at most one coloring K such that $T(K) = \langle \mathbb{1}, f_j \rangle$. Thus it suffices to verify that the given K satisfies the statements. First of all, it is clear that the sets $\mathcal{A}_\mu^\nu$ are pairwise disjoint and partition the vertex set $\mathbb{Z}_2^n$. Next, since n is even, we have $wt_2(\mathbb{1}) = 0$ and $wt_2(f_i) = 1$ for all $i = 1, \ldots, n$. This allows us to describe the behavior of the sets $\mathcal{A}_\mu^\nu$ along edges of the graph. Indeed, (4.2.5) leads to

$$x + \mathcal{A}_\mu^\nu = \begin{cases} \mathcal{A}_{\mu+1}^\nu, & \text{if } x = \mathbb{1}, \\ \mathcal{A}_\mu^{\nu+1}, & \text{if } x = f_j, \\ \mathcal{A}_{\mu+1}^{\nu+1}, & \text{if } x = f_i \text{ for } i \neq j \end{cases} \tag{4.2.6}$$

for all $\nu,\ \mu \in \mathbb{Z}_2$. This shows that the sets $\mathcal{A}_\mu^\nu$ are independent and thus form the color classes of a proper 4-coloring, say K. It remains to show that $T_x(K) = x + \{\mathbb{1}, f_j\}$ for all $x \in \mathbb{Z}_2^n$. Due to (4.2.6) the vectors $\mathbb{1}$ and f_j induce 4-cycles in $H_2(n, n-1)$ of the form

$$
\begin{tikzcd}
& \mathcal{A}_1^1 \arrow[dr, dash, "f_j"] & \\
\mathcal{A}_0^1 \arrow[ur, dash, "\mathbb{1}"] \arrow[dr, dash, "f_j"'] & & \mathcal{A}_1^0 \\
& \mathcal{A}_0^0 \arrow[ur, dash, "\mathbb{1}"'] &
\end{tikzcd}
$$

Furthermore, the last case in (4.2.6) tells us that the only other neighbors of a vertex in $\mathcal{A}_\mu^\nu$ are in $\mathcal{A}_{\mu+1}^{\nu+1}$ and thus all have the same color. Since this color is different from the neighboring colors in the cycle, the edges in these 4-cycles are indeed all the transition edges of K, that is, $T(K) = \langle \mathbb{1}, f_j \rangle$. Identity (4.2.6) shows that the sets $\mathcal{A}_\mu^\nu$ all have the same cardinality and hence K is an even coloring.

<u>Case 2:</u> Assume n is odd. We have to show that there is no proper 4-coloring with transition space $\langle \mathbb{1}, f_j \rangle$. Without loss of generality let $j = n$. It will be beneficial to work again with the vertex partition $\mathbb{Z}_2^n = \mathcal{A}_0^0 \cup \mathcal{A}_1^0 \cup \mathcal{A}_0^1 \cup \mathcal{A}_1^1$ (which are not color classes at this point). As n is odd we have for all ν, $\mu \in \mathbb{Z}_2$

$$
x + \mathcal{A}_\mu^\nu = \begin{cases} \mathcal{A}_{\mu+1}^{\nu+1}, & \text{if } x = \mathbb{1}, \\ \mathcal{A}_\mu^\nu, & \text{if } x = f_n, \\ \mathcal{A}_{\mu+1}^\nu, & \text{if } x = f_i \text{ for } i < n \end{cases} \tag{4.2.7}
$$

Thus $\langle \mathbb{1}, f_j \rangle$ induces 4-cycles with vertices in the sets $\mathcal{A}_\mu^\nu$ as follows

$$
\begin{tikzcd}
& \mathcal{A}_0^0 \arrow[dr, dash, "f_n"] & & & & \mathcal{A}_1^0 \arrow[dr, dash, "f_n"] & \\
\mathcal{A}_1^1 \arrow[ur, dash, "\mathbb{1}"] \arrow[dr, dash, "f_n"'] & & \mathcal{A}_0^0 & & \mathcal{A}_0^1 \arrow[ur, dash, "\mathbb{1}"] \arrow[dr, dash, "f_n"'] & & \mathcal{A}_1^0 \\
& \mathcal{A}_1^1 \arrow[ur, dash, "\mathbb{1}"'] & & & & \mathcal{A}_0^1 \arrow[ur, dash, "\mathbb{1}"'] &
\end{tikzcd} \tag{4.2.8}
$$

Assume now that K is a proper 4-coloring such that $T(K) = \langle \mathbb{1}, f_n \rangle$. Then the above 4-cycles consist of transition edges. Lemma 4.2.6 yields that the vertices of any such cycle attain distinct colors. Denote the 4 colors by A, B, C, D. Using $v = 0 \in \mathcal{A}_0^0$ as an instance of the left cycle, we have without loss of generality

$$
K(0) = A, \; K(\mathbb{1}) = B, \; K(e_n) = C, \; K(f_n) = D. \tag{4.2.9}
$$

Theorem 4.2.2 tells us that this determines uniquely the coloring of each other transition 4-cycle (and thus of every vertex), and Lemma 4.2.6 specifies how this is done. In order to describe this precisely, the following sets will be helpful. For ν, $\mu \in \mathbb{Z}_2$ and $i = 0, \ldots, n$ define

$$
B_\mu^\nu(i) := \{ v \in \mathcal{A}_\mu^\nu \mid wt(v) = i \}.
$$

Clearly, $B_\mu^\nu(i) = \emptyset$ if $i \not\equiv \nu \bmod 2$. One easily verifies

$$
\mathbb{1} + B_\mu^\nu(i) = B_{\mu+1}^{\nu+1}(n-i), \quad f_n + B_0^\nu(i) = B_0^\nu(n-1-i), \quad f_n + B_1^\nu(i) = B_1^\nu(n+1-i). \tag{4.2.10}
$$

This shows that the vertices in the 4-cycles in (4.2.8) specify to

$$
\begin{tikzcd}[column sep=small]
 & B_0^0(i) \arrow[dr, dash, "f_n"] & & & B_1^0(n-i) \arrow[dr, dash, "f_n"] & \\
B_1^1(n-i) \arrow[ur, dash, "\mathbb{1}"] \arrow[dr, dash, "f_n"'] & & B_0^0(n-1-i) & B_0^1(i) \arrow[ur, dash, "\mathbb{1}"] \arrow[dr, dash, "f_n"'] & & B_1^0(i+1) \\
 & B_1^1(i+1) \arrow[ur, dash, "\mathbb{1}"'] & & & B_0^1(n-1-i) \arrow[ur, dash, "\mathbb{1}"'] &
\end{tikzcd}
\tag{4.2.11}
$$

The cycles on the left (resp. right) hand side exist only if i is even (resp. odd). Let $n = 2t+1$. We claim that the sets $B_\mu^\nu(i)$ attain the following colors:

$$
\left.
\begin{array}{ll}
K(B_0^0(i)) = A, & K(B_0^1(n-1-i)) = A, \\
K(B_1^1(n-i)) = B, & K(B_1^0(i+1)) = B, \\
K(B_1^1(i+1)) = C, & K(B_1^0(n-i)) = C, \\
K(B_0^0(n-1-i)) = D, & K(B_0^1(i)) = D,
\end{array}
\right\} \text{ for all } 0 \le i \le t. \tag{4.2.12}
$$

If we can prove this, we arrive at a contradiction because $B_0^0(t) = B_0^0(n-1-t)$.

In order to prove (4.2.12) we induct on i. For $i = 0$ we have

$$
B_0^0(0) = \{0\},\ B_1^1(n) = \{\mathbb{1}\},\ B_1^1(1) = \{e_n\},\ B_0^0(n-1) = \{f_n\},
$$

and (4.2.9) establishes the left hand column of (4.2.12). The sets in the right hand column are all empty for $i = 0$.

Let us now assume (4.2.12) for all $i < i'$. Suppose i' is even and let $v \in B_0^0(i')$. Let $\alpha \in \{1, \ldots, n-1\}$ be such that $v_\alpha = 1$. Then $w := v + f_\alpha \in B_1^0(n+1-i') = B_1^0(n-(i'-1))$. This means that w is the top vertex of a cycle as on the right hand side in (4.2.11), whereas v is the top vertex of a cycle as on the left hand side. These two cycles are obtained by a shift with f_α. By assumption we know that the vertices of the right hand cycle assumes the colors C, D, A, B counterclockwise with $K(w) = C$. Now Lemma 4.2.6 establishes the colors of the left hand cycle as in the first column of (4.2.12). The same reasoning applies to the case where j is odd, in which one starts with an arbitrary element in $B_0^1(i')$. This concludes the proof. □

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Chapter 5 Fractional Chromatic Number of the Hamming Graphs

The fractional chromatic number is a member of a relatively new branch of study in discrete mathematics known as fractional graph theory and has applications to activity scheduling. Fractional coloring problems can be thought of as the linear programming relaxation of classical coloring problems. See [24] for a thorough treatment of fractional graph theory.

In this section, we derive the fractional chromatic number of $H_q(n,d)$ for various parameters. In particular, we calculate the fractional chromatic number of $H_2(n,n-a)$ for a fixed a as $n \to \infty$ and give evidence which suggest there are members of this family of graphs which contain an arbitrarily large gap between their chromatic and fractional chromatic numbers. Finally, we demonstrate an interesting relationship between the maximum size of error-correcting codes and the fractional chromatic number of the graph complement of $H_q(n,d)$.

5.1 Definitions and Known Values

Definition 5.1.1. Let G be a simple graph and $b,k \in \mathbb{N}$. A proper b-fold coloring of G with k colors is an assignment of b colors to each vertex such that no adjacent vertices share any colors.

Definition 5.1.2. Let $\chi_b(G)$ be the smallest natural number such that there exists a proper b-fold coloring of G. The fractional chromatic number is defined as

$$\chi_f(G) = \inf_b \frac{\chi_b(G)}{b}.$$

It is not immediately clear that the infimum in Definition 5.1.2 exists. To show this, one employs the following result, commonly known as Fekete's Subadditive Lemma.

Lemma 5.1.3 ([9]). *For every subadditive sequence $\{a_n\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists and $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\to\infty} \frac{a_n}{n}$.*

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **subadditive** if $a_{s+t} \le a_s + a_t$. Note that $\chi_b(G)$ is subadditive since we may simply consider two disjoint color sets used for the s-fold and t-fold colorings respectively. Superimposing the two colorings produces an $(s+t)$-fold coloring using at most $\chi_s(G) + \chi_t(G)$ colors. Thus we have the following result.

Corollary 5.1.4. $\chi_f(G) = \inf_b \frac{\chi_b(G)}{b} = \lim_b \frac{\chi_b(G)}{b}$

It is clear that when $b = 1$, we have recovered the classical chromatic number. Furthermore, any b-fold coloring of a complete graph on n vertices requires $n \cdot b$ colors. Thus

$$\omega(G) \leq \chi_f(G) \leq \chi(G). \tag{5.1.1}$$

We present the following standard result which will be useful to our investigations going forward.

Proposition 5.1.5 ([24, Prop. 3.1.1])**.** *Let G be a graph. Then*

$$\chi_f(G) \geq \frac{|G|}{\alpha(G)}.$$

If G is a vertex-transitive graph then equality holds.

Recall from Proposition 2.2.4(d) that $H_q(n,d)$ is vertex-transitive. Therefore Proposition 5.1.5 implies

$$\chi_f(H_q(n,d)) = \frac{q^n}{\alpha(H_q(n,d))}. \tag{5.1.2}$$

In Section 3.1, we presented the Erdős-Ko-Rado Theorem for integer sequences which gives the value of $\alpha(H_q(n,d))$ for many parameters. In particular, inputting the values of $\alpha(H_q(n,d))$ given in Theorem 3.1.3 to (5.1.2), and comparing with Theorem 3.3.1 gives the following results.

Theorem 5.1.6. *Let $q \geq 3$ and $q \geq n-d+2$. Then*

$$\chi_f(H_q(n,d)) = q^{n-d+1} = \chi(H_q(n,d)).$$

In the binary case, we get the following complete solution.

Theorem 5.1.7.

$$\chi_f(H_2(n,d)) = \begin{cases} \dfrac{2^n}{\sum\limits_{i=0}^{\frac{d-1}{2}} \binom{n}{i}} & \text{if } d \text{ is odd} \\[2em] \dfrac{2^{n-1}}{\sum\limits_{i=0}^{\frac{d-2}{2}} \binom{n-1}{i}} & \text{if } d \text{ is even} \end{cases}$$

When studying the classical chromatic number, the Erdős-Ko-Rado Theorem for integer sequences was very profitable when $q \geq 3$, but far less useful in the binary case. In looking at the fractional chromatic number, the exact same cases are settled for $q \geq 3$, and in fact show that the chromatic number and fractional chromatic number match for a large variety of parameters. However, we see that applying the Erdős-Ko-Rado Theorem completely solves the binary case for the fractional chromatic number, while the chromatic number is still unknown for most parameters.

5.2 Gap between $\chi_f(H_2(n,d))$ and $\chi(H_2(n,d))$

Theorem 5.1.6 showed that if $q \geq 3$ and $q \geq n-d+2$, then $\chi_f(H_q(n,d)) = \chi(H_q(n,d))$. In this section, we will prove the following theorem which demonstrates that the fractional chromatic number behaves quite differently from the chromatic number in the binary case. Moreover, we present evidence which suggests that there exist binary Hamming graphs with an arbitrarily large gap between their chromatic and fractional chromatic number.

Theorem 5.2.1. *Fix $a \in \mathbb{N}$. Then* $\lim_{n\to\infty} \chi_f(H_2(n, n-a)) = 2$.

Recall from Section 3.3 that for every case in which the chromatic number has been determined, $\chi(H_q(n,d)) = q^{n-d+1}$. While we have not shown this to be true for most binary cases, it is reasonable to expect that the chromatic number would be of a similar magnitude. At the very least, Theorem 3.3.1(a) and Equation (3.4.2) give that $\chi(H_2(n, n-a)) \geq 4$ for all $1 \leq a \leq n-1$, showing there exists a gap between the chromatic number and the fractional chromatic number.

Before proving this theorem, we will need a few definitions and lemmas. Let Γ be the usual gamma function which extends the factorial function such that $\Gamma(n+1) = n!$. Given two functions $f(x)$ and $g(x)$, we say that $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. We will also make use of the following result commonly known as Stirling's Formula (see [19]).

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n \tag{5.2.1}$$

Stirling's formula can be extended to the gamma function to give the following analogous result. Let r be a positive real number. Then

$$\Gamma(r+1) \sim \sqrt{2\pi r}(\frac{r}{e})^r \tag{5.2.2}$$

Lemma 5.2.2.

$$\frac{\Gamma(n+1)}{\left(\Gamma(\frac{n}{2}+1)\right)^2} \sim 2^n\sqrt{\frac{2}{\pi n}}.$$

Proof. Applying Stirling's Formula, we have

$$\frac{\Gamma(n+1)}{\left(\Gamma(\frac{n}{2}+1)\right)^2} \sim \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\left(\sqrt{2\pi(\frac{n}{2})}(\frac{n}{2e})^{\frac{n}{2}}\right)^2} = \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\pi n(\frac{n}{2e})^n} = \sqrt{\frac{2\pi n}{(\pi n)^2}}2^n = 2^n\sqrt{\frac{2}{\pi n}}$$

□

Lemma 5.2.3. *Fix $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$,*

$$\binom{n}{i} < 2^n\sqrt{\frac{2}{\pi n}}(1+\epsilon)$$

for all $i = 0, 1, 2, \ldots, n$.

Proof. Our goal will be to show that

$$\binom{n}{i} < \frac{\Gamma(n+1)}{(\Gamma(\frac{n}{2}+1))^2}(1+\epsilon)$$

after which the result will follow immediately from Lemma 5.2.2.

If n is even, then $\binom{n}{i}$ is maximized at $i = \frac{n}{2}$ and $\binom{n}{\frac{n}{2}} = \dfrac{\Gamma(n+1)}{\Gamma(\frac{n}{2}+1))^2}$ so we are finished. Therefore suppose n is odd. Then $\binom{n}{i}$ has a maximimum at $i = \frac{n-1}{2}$. Thus we have

$$\binom{n}{i} \le \binom{n}{\frac{n-1}{2}} = \frac{\Gamma(n+1)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+3}{2})}$$

for all $i = 0, 1, 2, \ldots, n$.

Thus

$$\binom{n}{i}\frac{(\Gamma(\frac{n}{2}+1))^2}{\Gamma(n+1)} \le \frac{\Gamma(n+1)}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+3}{2})}\frac{(\Gamma(\frac{n}{2}+1))^2}{\Gamma(n+1)} = \frac{(\Gamma(\frac{n}{2}+1))^2}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+3}{2})}$$

Applying Stirling's formula,

$$\frac{(\Gamma(\frac{n}{2}+1))^2}{\Gamma(\frac{n+1}{2})\Gamma(\frac{n+3}{2})} \sim \frac{\left(\sqrt{2\pi\frac{n}{2}}\left(\frac{n}{2e}\right)^{\frac{n}{2}}\right)^2}{\sqrt{2\pi\frac{n-1}{2}}\left(\frac{n-1}{2e}\right)^{\frac{n-1}{2}}\sqrt{2\pi\frac{n+1}{2}}\left(\frac{n+1}{2e}\right)^{\frac{n+1}{2}}}$$

$$= \frac{\pi n\left(\frac{n}{2e}\right)^n}{\pi\sqrt{n^2-1}\left(\frac{n^2-1}{4e^2}\right)^{\frac{n-1}{2}}\left(\frac{n+1}{2e}\right)} = \frac{n\left(\frac{n}{2e}\right)^{n-1}\left(\frac{n}{2e}\right)}{\sqrt{n^2-1}\left(\frac{\sqrt{n^2-1}}{2e}\right)^{n-1}\left(\frac{n+1}{2e}\right)}$$

$$= \frac{n}{\sqrt{n^2-1}}\left(\frac{n}{\sqrt{n^2-1}}\right)^{n-1}\frac{n}{n+1} = \left(\frac{n}{\sqrt{n^2-1}}\right)^n\frac{n}{n+1}.$$

It is straightforward to show that $\lim_{n\to\infty}\left(\frac{n}{\sqrt{n^2-1}}\right)^n = 1$. Thus for sufficiently large n

$$\binom{n}{i}\frac{(\Gamma(\frac{n}{2}+1))^2}{\Gamma(n+1)} < (1+\epsilon)$$

and rearranging the inequality, we get the desired result. □

Lemma 5.2.4. *Fix $a \in \mathbb{N}$. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \ge N$,*

$$\alpha(H_2(n, n-a)) \ge 2^{n-1}(1-\epsilon).$$

Proof. By Lemma 5.2.3, we can choose N_1 sufficiently large such that if $n \ge N_1$, then $\binom{n}{i} < 2^n\sqrt{\frac{2}{\pi n}}(1+\frac{1}{2})$ for all $i = 0, 1, 2, \ldots, n$. Choose N_2 sufficiently large that $\frac{3}{2}a\sqrt{\frac{2}{\pi n}} < \epsilon$ for $n \ge N_2$. Set $N = max\{N_1, N_2\}$.

First, suppose $n-a$ is odd. Then Theorem 3.1.3 gives that $\sum_{i=0}^{\frac{n-a-1}{2}} \binom{n}{i} = \alpha(H_2(n, n-a))$. Furthermore for $n \geq N$,

$$\begin{aligned} 2^n &= \sum_{i=0}^{\frac{n-a-1}{2}} \binom{n}{i} + \sum_{i=\frac{n-a+1}{2}}^{\frac{n+a-1}{2}} \binom{n}{i} + \sum_{i=\frac{n+a+1}{2}}^{n} \binom{n}{i} \\ &= 2 \sum_{i=0}^{\frac{n-a-1}{2}} \binom{n}{i} + \sum_{i=\frac{n-a+1}{2}}^{\frac{n+a-1}{2}} \binom{n}{i} \\ &\leq a \left(2^n \sqrt{\frac{2}{\pi n}} (\frac{3}{2}) \right) + 2 \sum_{i=0}^{\frac{n-a-1}{2}} \binom{n}{i}. \end{aligned}$$

Then

$$2^n \left(1 - \frac{3}{2} a \sqrt{\frac{2}{\pi n}} \right) \leq 2 \sum_{i=0}^{\frac{n-a-1}{2}} \binom{n}{i}$$

and

$$2^{n-1}(1-\epsilon) \leq \sum_{i=0}^{\frac{n-a-1}{2}} \binom{n}{i} = \alpha(H_2(n, n-a)).$$

The case when $n-a$ is even requires only a slight modification of a few indices, but the proof is essentially the same. $\square$

We are now finally ready to give the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1. First we note that $\chi_f(H_2(n, n-a)) \geq 2$ for all (n, a) by (5.1.1) since $\chi_f(H_2(n, n-a)) \geq \omega(H_2(n, n-a)) \geq 2$. Together, (5.1.2) and Lemma 5.2.4 imply that $\chi_f(H_2(n, n-a)) \leq \frac{2}{1-\epsilon}$ for any $\epsilon > 0$ given sufficiently large n. Thus $\lim_{n\to\infty} \chi_f(H_2(n, n-a)) = 2$. $\square$

We mentioned previously that Theorem 5.2.1 shows that there is a gap between $\chi(H_2(n, n-a))$ and $\chi_f(H_2(n, n-a))$, and we speculated that this gap was in fact arbitrarily large for sufficiently large n and a. While several families of graphs such as the Kneser graphs are know to have arbitrarily large gaps between the chromatic and fractional chromatic number, the number of such examples does not appear to be expansive (see [24, Section 3.3] for more on this topic.) Therefore it would be quite interesting if we were able to prove the following conjecture.

Conjecture 5.2.5. Given $b \in \mathbb{N}$, there exists (n, a) such that

$$\chi(H_2(n, n-a)) - \chi_f(H_2(n, n-a)) \geq b.$$

While answering Question 3.3.3 in the affirmative and showing $\chi(H_2(n,d)) = 2^{n-d+1}$ would accomplish this objective, it currently seems unlikely that such a result can be easily obtained. In what follows, we provide two much weaker conditions, either of which if shown to be true, would prove Conjecture 5.2.5.

(1.) $\chi(H_2(n,d-1)) > \chi(H_2(n,d))$ for all (n,d).

(2.) $\chi(H_2(n+1,d)) > \chi(H_2(n,d))$ for all (n,d).

Proposition 5.2.6. *If either (1.) or (2.) above holds, then Conjecture 5.2.5 is true.*

Proof. Suppose (1.) holds. Fix $b \in \mathbb{N}$. Choose (n,a) sufficiently large such that $\chi_f(H_2(n,n-a)) < 3$ and $n > a > b+1$. Recall that $\chi(H_2(n,n)) = 2$. Then applying (1.) inductively we have

$$2 = \chi(H_2(n,n)) < \chi(H_2(n,n-1)) < \ldots < \chi(H_2(n,n-a))$$

As each inequality is strict, we have $\chi(H_2(n,n-a)) \geq a+2 > b+3$. Then

$$\chi(H_2(n,n-a)) - \chi_f(H_2(n,n-a)) \geq b+3-3 = b.$$

Now assume (2.) holds. Again fix $b \in \mathbb{N}$. Choose (n,a) sufficiently large such that $\chi_f(H_2(n,n-a)) < 3$ and $n > a > b+1$. Then applying (2.) inductively, we have

$$2 = \chi(H_2(n-a,n-a)) < \chi(H_2(n-a+1,n-a)) < \ldots < \chi(H_2(n,n-a)).$$

Thus $\chi(H_2(n,n-a)) \geq a+2 > b+3$ so

$$\chi(H_2(n,n-a)) - \chi_f(H_2(n,n-a)) \geq b+3-3 = b.$$

□

5.3 Expressing $A_q(n,d)$ in terms of fractional colorings.

Finally, we make note of an interesting description of the maximal size of error-correcting codes in terms of the fractional chromatic number of the graph complement of $H_q(n,d)$.

Definition 5.3.1. Let G be simple a graph. We denote by $\overline{G}$ the graph complement of G, constructed as follows.

Let $V(\overline{G}) = V(G)$ and $x,y \in V(\overline{G})$. Then $x \sim_{\overline{G}} y \iff x \neq y$ and $x \nsim_G y$.

In other words, G and $\overline{G}$ are both simple graphs on the same vertex set such that for distinct $x,y \in V(G) = V(\overline{G})$, the edge xy is in either G or $\overline{G}$, but not both.

Lemma 5.3.2. *Let G be a simple graph. Then*
(a) $\alpha(G) = \omega(\overline{G})$ *and* $\omega(G) = \alpha(\overline{G})$.

(b) G is vertex-transitive $\iff$ $\overline{G}$ is vertex-transitive.

Proof. (a) As noted above, two vertices are adjacent in G if and only they are not adjacent in $\overline{G}$ and vice versa. Therefore cliques in G correspond to independent sets in $\overline{G}$ and vice versa, so the maximum size of each must be the same.

(b) Let $x, y \in V(G)$. If G is vertex-transitive, there exists an automorphism ϕ of G such that $\phi(x) = y$. Then it is easy to check that ϕ is also an automorphism of $\overline{G}$ sending x to y.

□

Recall from the introduction that $A_q(n, d)$ is the maximum size of a q-ary code of length n and minimum Hamming distance at least d.

Proposition 5.3.3. $A_q(n, d) = \dfrac{q^n}{\chi_f(\overline{H_q(n, d)})}$.

Proof. We know from Proposition 5.1.5 that for a vertex-transitive graph G,

$$\chi_f(G) = \frac{|G|}{\alpha(G)} \quad \text{or equivalently} \quad \alpha(G) = \frac{|G|}{\chi_f(G)}.$$

Recall that much of the original interest in studying $H_q(n, d)$ centered around the fact that $\omega(H_q(n, d)) = A_q(n, d)$. Proposition 2.2.4(d) gives us that $H_q(n, d)$ is vertex-transitive, so by Lemma 5.3.2 $\overline{H_q(n, d)}$ is vertex-transitive and

$$A_q(n, d) = \omega(H_q(n, d)) = \alpha(\overline{H_q(n, d)}) = \frac{|\overline{H_q(n, d)}|}{\chi_f(\overline{H_q(n, d)})} = \frac{q^n}{\chi_f(\overline{H_q(n, d)})}.$$

□

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Chapter 6 Implicit Edges and Entanglements

Before presenting the definitions and properties of implicit edges and entanglements, we take a second to explain the process through which we came to study them. Prior to becoming aware of the result by Payan [22] regarding the chromatic number of cubelike graphs discussed in Section 3.1, effort was put into proving that $\chi(H_2(n, n-1)) = 4$ for all $n \geq 2$. A naive initial attempt involved trying to show the following untrue statement.

False Claim 1: *$H_2(n, n-1) \subseteq H_2(n+1, n)$ for all $n \geq 4$.*

If this were true, then the fact that $\chi(H_2(4, 3)) = 4$ would prove that $\chi(H_2(n, n-1)) \geq 4$ for all n, and the existence of coordinate colorings would give us the desired result. Unfortunately, one can straighforwardly show that the odd girth (length of shortest odd cycle) of $H_2(n, n-1)$ is n if n is odd and $n+1$ if n is even. In particular, this implies $H_2(n, n-1)$ is not a subgraph of $H_2(n+2, n+1)$ for any n.

Rather than admit an obvious defeat, the idea of implicit edges amounts to the following wishful thinking.

"Ok, my graph doesn't have all the edges I need to prove what I want to show. Maybe those edge are there and I just can't see them."

The key idea is that for some fixed k, an edge may be "implicitly" contained within a graph in the sense that the set of k-colorings remains unchanged after adding in the edge. Therefore, while we may not have true subgraph containment between $H_2(n, n-1)$ and $H_2(n+1, n)$, it's possible that if we added in all the implicit edges of $H_2(n+1, n)$, achieving a graph we will later denote $Exp_k(H_2(n+1, n))$, then $H_2(n, n-1)$ would in fact be a subgraph of $Exp_k(H_2(n+1, n))$. Moreover, since we designed $Exp_k(H_2(n+1, n))$ to have the same set of proper k-colorings as $H_2(n+1, n)$, $Exp_k(H_2(n+1, n))$ is k-colorable if and only $H_2(n+1, n)$ is k-colorable. In particular, we would consider the case $k = 3$ with the goal being to show that that $H_2(n, n-1)$ is not 3-colorable for any $n \in \mathbb{N}$. Then (3.2.1) would give that $H_2(n, n-1) = 4$ for all $n \geq 2$.

In the following sections, we formalize this idea of implicit edges and define a dual notion of entanglements. Although the result of Payan made our attempts to apply these techniques to the family of Hamming graphs $H_2(n, n-1)$ unnecessary, implicit edges and entanglements are of interest on their own merit. In particular, these techniques are especially attractive when our graph is vertex-transitive as the discovery of a single implicit edge or entanglement within the graph implies the existence of many more. While we were motivated by our study of the Hamming graphs, the definitions and results apply to all finite, simple, connected graphs. Throughout this chapter, assume all graphs are of this type unless otherwise specified.

6.1 Implicit Edges

Definitions and Basic Properties

Definition 6.1.1. Let G be a graph and $x, y \in V(G)$. Suppose G satisfies the following:
(a) $x \nsim y$
(b) $\chi(G) = k$
(c) For all $K \in Col_k(G)$, $K(x) \neq K(y)$.
Then we say that G is an implicit edge of order k between vertices x and y and denote this by $xy \in I_k(G)$.

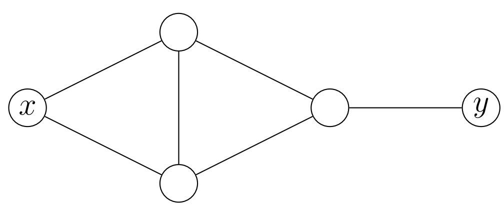
Note that there may be many pairs of vertices for which G is an implicit edge of a given order, giving rise to the notation $xy \in I_k(G)$ as opposed to $xy = I_k(G)$. Therefore

$$I_k(G) = \{xy \in V(G) \times V(G) \mid G \text{ is an implicit edge of order } k \text{ between } x \text{ and } y\}.$$

Intuitively, $I_k(G)$ can be thought of as the set of edges which can be added to G without changing the set of admissible k-colorings.

Remark 6.1.2. One could equivalently formulate (b) in Definition 6.1.1 as $\chi(G) \leq k$. However, it is easy to show that if $\chi(G) < k$, then $I_k(G) = \emptyset$. Furthermore, although we are assuming all graphs are connected, (c) of Definition 6.1.1 already implies that x and y must be in the same connected component of G.

Example 6.1.3. The graph below has an implicit edge of order 3 between vertices x and y.



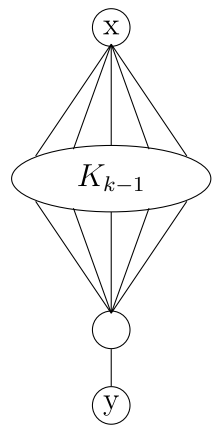
In other word, the graph is 3-colorable but despite x and y not being adjacent, there exists no proper 3-coloring in which x and y are assigned the same color. Thus, in regards to the set of 3-colorings of this graph, it is as if there is an edge connecting x and y.

We can generalize to the following graph that has an implicit edge of order $k \geq 2$ between vertices x and y in the following way.

Definition 6.1.4. The *canonical implicit edge of order* k, denoted $Can^i(k)$, is constructed as follows.

i) Begin with a copy of K_{k-1}.
ii) Add two additional vertices, each of which are connected to every vertex in the original K_{k-1}.

iii) Add one additional vertex and connect it to one of the vertices added in step (ii).



Again one can straightforwardly check that the graph is k-colorable, but that no proper k-coloring can assign the same colors to x and y. Thus the graph is an implicit edge of order k between the vertices x and y.

Often we will want want to discuss implicit edges of order k occurring inside graphs that may not themselves be k-colorable. For this reason, we adopt the following convention.

Definition 6.1.5. Let G be a graph and $x, y \in V(G)$ such that $x \nsim y$. We say that G contains an implicit edge of order k between x and y if there exists a subgraph $H \subseteq G$ such that $xy \in I_k(H)$.

Example 6.1.6.

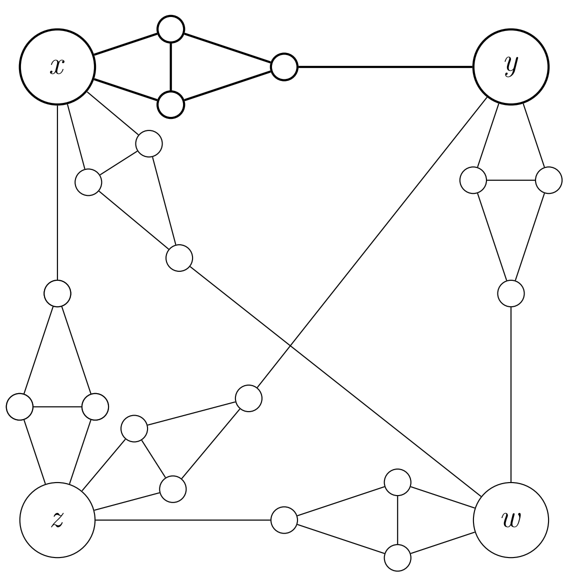


Figure 6.1: Implicit edge K_4

Consider the graph in Figure 6.1. Given any pair of vertices in $\{x, y, w, z\}$, there exists a canonical implicit edge of order 3 between them. (For instance, the one between x and y is highlighted in bold). By way of contradiction, assume that G is 3-colorable. Given any proper 3-coloring of the whole graph, there is an induced coloring on every subgraph. Since there is an implicit edge of order 3 between each pair of vertices in $\{x, y, w, z\}$, each of these vertices must be different colors. As this is impossible, there can be no proper 3 colorings of the G. Thus its chromatic number is at least 4.

Example 6.1.6 illustrates one motivation for studying implicit edges. When attempting to determine if a given graph is k-colorable, knowledge of implicit edges of order k in the graph may often simplify or even trivialize the problem. The following definition and results will make this notion more precise.

Definition 6.1.7. Let G be a graph. We define the graph transformation Exp_k as follows:

i. $V(Exp_k(G)) = V(G)$

ii. Two vertices $x, y \in Exp_k(G)$ are adjacent if either

  a) $xy \in E(G)$, or

  b) $\exists H \subseteq G$ such that $xy \in I_k(H)$.

We will refer to $Exp_k(G)$ as the k-explicitization of G.

Remark 6.1.8. We see immediately from the definitions that if $\chi(G) < k$, then $Exp_k(G) = G$.

Example 6.1.9. Below is the canonical implicit edge of order 3 and its image under Exp_3.

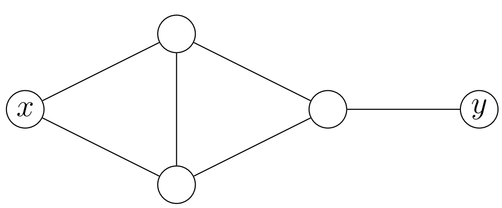


Figure 6.2: $Can^i(K)$

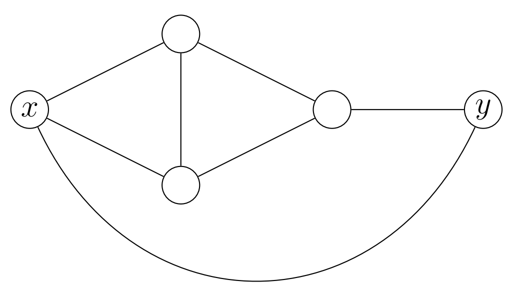


Figure 6.3: $Exp_k(Can^i(K))$

In the case where $\chi(G) = k$, Exp_k is a graph operation which inserts all edges into a graph which can be added without affecting the set of possible k-colorings (as we will show in Proposition 6.1.14).

The same cannot be said if $\chi(G) > k$. While it is still true that every edge added by Exp_k does not change the set of k-colorings, this statement is trivial as that set is already empty. Moreover, Exp_k does not necessarily add every possible edge to the graph, even though any edge could be added without altering the set of k-colorings.

Recall that for a graph G, we denote by $Col_k(G)$ the set of proper k-colorings of G.

Definition 6.1.10. Given two graphs G and H on n vertices labeled $\{1, 2, \ldots, n\}$, we say $Col_k(G) = Col_k(H)$ if there exists some permutation $\rho : [n] \to [n]$ and some bijection $\phi : Col_k(G) \to Col_k(H)$ such that $K(t) = \phi(K)(\rho(t))$ for all $t \in [n]$ and $K \in Col_k(G)$.

We now turn to proving our assertion that including implicit edges in a graph does not change the set of possible k-colorings. While all of our results can be derived directly from the definitions, some of the proofs can be simplified by considering the chromatic polynomials of the graphs involved. See [3] for more on the role of chromatic polynomials in graph colorings.

Definition 6.1.11. For a graph G, let $P(G, t)$ denote the chromatic polynomial of G evaluated at t. For an edge $e \in E(G)$, let G/e and $G - e$ denote the graphs which result from contracting and deleting the edge e respectively. Similarly, for vertices $x, y \in V(G)$ such that $x \nsim y$, let $G + xy$ denote the graph which results from adding the edge xy into the graph G and G/xy denote the graph resulting from merging the vertices x and y.

The deletion-contraction recursion for the chromatic polynomial states that for any $e \in E(G)$,

$$P(G, t) = P(G - e, t) - P(G/e, t) \tag{6.1.1}$$

If two vertices $x, y \in V(G)$ are not adjacent, we can restate the deletion-contraction formula as

$$P(G, t) = P(G + xy, t) + P(G/xy, t) \tag{6.1.2}$$

These relationships can be used to recursively calculate the chromatic polynomial. It should be noted that the chromatic polynomial only outputs the number of colorings of a graph that can be achieved using a specified number of colors. It is possible for two non-isomorphic graphs to have the same chromatic polynomial. Thus the chromatic polynomials can not tell us if two graphs have the same set of k-colorings, but it can show that these sets of k-colorings have the same cardinality. It is this property that we will utilize in the following proof, and several other proofs throughout this chapter.

Although the following lemma is perhaps fairly obvious from the definitions, we provide a short proof using chromatic polynomials as it will demonstrate a proof technique we will use again shortly in more intricate settings.

Lemma 6.1.12. *Let G be a graph. If $xy \in I_k(G)$, then $Col_k(G) = Col_k(G + xy)$.*

Proof. $P(G + xy, k) = P(G, k) - P(G/xy, k)$ by deletion-contraction. However, since $xy \in I_k(G)$, every k-coloring K of G, and therefore of $(G + xy)$, satisfies $K(x) \neq K(y)$. Thus $P(G/xy, k) = 0$ implying $P(G + xy, k) = P(G, k)$. Since $G \subseteq (G + xy)$, we have $Col_k(G+xy) \subseteq Col_k(G)$. However, $|Col_k(G)| = P(G, k) = P(G+xy, k) = |Col_k(G+xy)|$ so $Col_k(G) = Col_k(G + xy)$. □

Lemma 6.1.13. *Let G and H be graphs on n vertices such that $\chi(G), \chi(H) \leq k$. Suppose $Col_k(G) = Col_k(H)$ and let ρ be the permutation defined in Definition 6.1.10. Then $I_k(G) \cup E(G) = I_k(H) \cup E(H)$, meaning for $a, b \in V(G)$, $ab \in I_k(G) \cup E(G) \iff \rho(a)\rho(b) \in I_k(H) \cup E(H)$.*

Proof. Assume $Col_k(G) = Col_k(H)$. WLOG of generality, we may assume ρ is the identity map. Suppose $xy \in I_k(G) \cup E(G)$. Then for all $K \in Col_k(G)$, $K(x) \neq K(y)$. Since $Col_k(G) = Col_k(H)$, we also have $K'(x) \neq K'(y)$ for all $K' \in Col_k(H)$. Thus either $xy \in E(H)$ or $xy \in I_k(H)$. Therefore $I_k(G) \cup E(G) \subseteq I_k(H) \cup E(H)$. A symmetric argument gives $I_k(G) \cup E(G) \supseteq I_k(H) \cup E(H)$ so $I_k(G) \cup E(G) = I_k(H) \cup E(H)$. □

Proposition 6.1.14. *Let G be a graph. Then $Col_k(G) = Col_k(Exp_k(G))$. In particular, a graph G is k-colorable if and only if $Exp_k(G)$ is k-colorable.*

Proof. If $\chi(G) > k$, then $Col_k(G) = \emptyset = Col_k(Exp_k(G))$. If $\chi(G) < k$, then Remark 6.1.8 gives that $G = Exp_k(G)$ and we are finished. Therefore suppose $\chi(G) = k$. Since G is finite, $I_k(G)$ is a finite set, say $\{e_1, \ldots, e_n\}$. By Lemma 6.1.12, $Col_k(G) = Col_k(G + e_1)$. Furthermore, Lemma 6.1.13 imples that $I_k(G + e_1) = \{e_2, \ldots, e_n\}$. Similarly $Col_k(G) = Col_k(G + e_1) = Col_k(G + \{e_1, e_2\})$ and inductively we get that $Col_k(G) = Col_k(G + \{e_1, \ldots, e_n\}) = Col_k(Exp_k(G))$.

Therefore, $\chi(G) \leq k \iff |Col_k(G)| > 0 \iff |Col_k(Exp_k(G))| > 0 \iff \chi(Exp_k(G)) \leq k$. □

Thus questions of k-colorability of G are equivalent to k-colorability of $Exp_k(G)$. However, $Exp_k(G)$ will often have many more edges than G, making it easier to show $Exp_k(G)$ is not k-colorable. Implicit edges serve less purpose in showing that a graph is k-colorable, except in possibly reducing the time necessary to do an exhaustive search by removing choices which will eventually lead to failure.

It is worth noting that finding $Exp_k(G)$ may often be more computationally expensive than actually determining if G is k-colorable. Luckily, combining Lemma 6.1.12 and Lemma 6.1.13 allows us to work with partial results in the sense that identification and addition of any implicit edges to a graph preserves the set of k-colorings. It is therefore not necessary to check that we have in fact found every implicit edge, i.e. determined $Exp_k(G)$, before making use of knowledge which a few may provide us. For instance, consider the argument presented in Example 6.1.6. The identification of only $\binom{4}{2} = 6$ implicit edges of order 3 was enough to conclude the graph was not 3-colorable. However, many more implicit edges exist in that graph.

Graphs of Large Girth and k-critical Graphs

The only examples of implicit edges we have considered so far are the canonical implicit edges, but these are by no means the only types. In what follows, we give several results which show that implicit edges are present in a large range of graphs. In particular, we show that every k-critical graph except K_k contains an implicit edge of order $k-1$, and give an example of a family of graphs in which there is an implicit edge of order k between every pair of nonadjacent vertices. Additionally, we give a nonconstructive proof for the existence of implicit edges of order k in graphs with arbitrarily high girth.

Example 6.1.15. The graph below is triangle-free and is an implicit edge of degree 3 between vertices x and y.

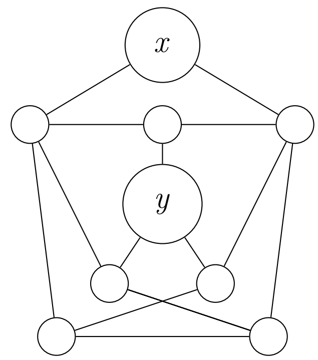


Figure 6.4: K_3-free implicit edge of order 3

To verify this is an implicit edge of order 3 between x and y, we need to show that the graph is 3-colorable and that there exists no proper 3-coloring such that x and y have the same color. We represent the three colors by dotted, dashed, and dash-dotted vertices.

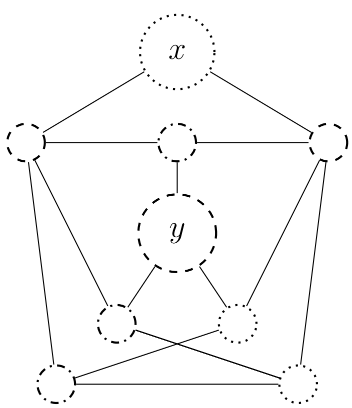


Figure 6.5: Proper 3-coloring of Figure 6.4

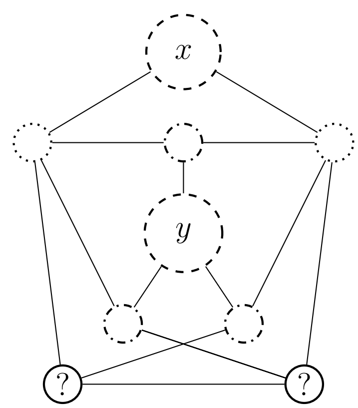


Figure 6.6: Failed 3-coloring of Figure 6.4

Figure 6.5 establishes that the graph is indeed 3-colorable. Figure 6.6 demonstrates that if x and y are assigned the same color, there is only a single choice for the color of each of their neighbors. However, it is then impossible to color both the vertices in the bottom row. Therefore no proper 3-coloring can assign the same colors to x and y.

Definition 6.1.16. A graph G is said to be k-critical if $\chi(G) = k$ and the removal of any edge or vertex decreases the chromatic number.

A well-known result due to Dirac [7] states that every k-critical graph is $k-1$ connected, meaning a minimum of $k-1$ edges must be removed to cause the graph to become disconnected. In particular, the following result will utilize the fact that if $k \geq 3$, the removal of a single edge from a k-critical graph cannot cause the graph to become disconnected.

Proposition 6.1.17. *Let G be a k-critical graph with $k \geq 3$ and $x, y \in V(G)$ such that $d_G(x, y) = 2$. Then G contains an implicit edge of order $k-1$ between x and y.*
As a consequence, every k-critical graph except K_k contains an implicit edge of order $k-1$.

Proof. Since $d_G(x, y) = 2$, there exists $z \in V(G)$ such that $xz, yz \in E(G)$. Consider the subgraph of G obtained by removing the the edge xz, that is $H := G - \{xz\}$. As G is k-critical, we have $\chi(H) = k - 1$. Suppose K is a $(k-1)$-coloring of H. If $K(x) \neq K(z)$ then K would also be a proper coloring of G which contradicts that $\chi(G) = k$. Therefore $K(x) = K(z)$. Since $yz \in E(H)$, it follows that $K(y) \neq K(z) = K(x)$. Thus H is an implicit edge of order $k-1$ between x and y.

Finally for any connected graph G on at least three vertices, there exists $x, y \in V(G)$ such that $d_G(x, y) = 2$ if and only if $G \neq K_k$. □

Lemma 6.1.18. *Every finite k-chromatic graph G contains a k-critical subgraph H.*

Proof. If G is k-critical, we are finished. Otherwise, there exist some $e_1 \in E(G)$ such that if we define $G_1 = G - \{e_1\}$, we have $\chi(G_1) = k$. If G_1 is k-critical, we are finished. If not, there exist $e_2 \in V(G_1)$ such that $G_2 = G_1 - \{e_2\}$ satisfies $\chi(G_2) = k$. Continuing this process iteratively and removing any vertices which become isolated in the process, G being finite implies that we eventually arrive at a k-critical graph. □

Finally, we present the following famous result of Erdős.

Theorem 6.1.19. *[6, Thm. 5.2.5] Given any $k, l \in \mathbb{N}$, there is a graph G such that $\chi(G) > k$ and girth$(G) > l$.*

Erdős' proof was nonconstructive and in particular relied on a probabilistic argument on random graphs. As a result, our proof of implicit edges of high order and girth will also be nonconstructive. However, we do give a constructive proof for triangle-free implicit edges of arbitrarily high order in Proposition 6.1.21.

Corollary 6.1.20. *Given any $k, l \in \mathbb{N}$, there exists a graph G with girth$(G) > l$ such that G contains an implicit edge of order t for some $t \geq k$.*

Proof. Noting that deletion of edges cannot decrease the girth of a graph, applying Lemma 6.1.18 allows us to assume that the graph described in Theorem 6.1.19 is $t+1$-critical for some $t + 1 > k$. We may assume that $l, k \geq 3$. Then there exist two vertices x and y such that $d_G(x, y) = 2$. Then Proposition 6.1.17 gives us the existence of an implicit edge of order t. □

Finally, we consider the famous Mycielski graphs as an example of a family of implicit edge dense graphs. See Appendix A for more on Mycielski graphs and their basic properties. In particular, we have the following result.

Proposition 6.1.21. *Suppose $k \geq 3$. Then the Mycielski graph of order k is k-critical, triangle-free, and has diameter 2.*

Proof. This is easily verified when $k = 3$. Then Proposition A.2 from Appendix A inductively gives the result. $\square$

Proposition 6.1.22. *Let G be the Mycielski graph of order $k \geq 3$ and let $x, y \in V(G)$ be two nonadjacent vertices. Then there exist a triangle-free implicit edge of order $k - 1$ between x and y.*

Proof. From Proposition 6.1.21, we have that G is triangle free, k-critical, and has diameter 2. Thus for any two nonadjacent vertices $x, y \in V(G)$, we have $d_G(x, y) = 2$. Then Proposition 6.1.17 implies the existence of an implicit edge of order $k - 1$ between x and y. Since this implicit edge must be a subgraph of G, it is also triangle-free. $\square$

Iterating Exp_k and the Implicit Closure

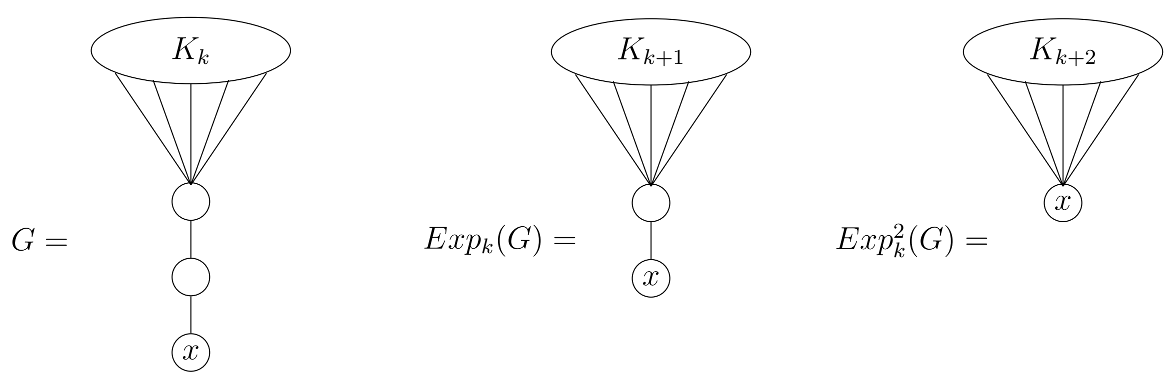
In Definition 6.1.7, we defined a graph transformation Exp_k which can be thought of as adding in all the implicit edges of order k. A natural question arises as to whether we always have $Exp_k(Exp_k(G)) = Exp_k(G)$. The following example and propositions will show that the answer depends on the chromatic number of the graph in question. To avoid burdensome notation, we introduce the following.

Definition 6.1.23. Define $Exp_k^0(G) := G$ and $Exp_k^t(G) := Exp_k(Exp_k^{t-1}(G))$ for all $t \in \mathbb{N}$. As G is assumed to be finite, there must exist some $p \in \mathbb{N}$ such that $Exp_k^{p+1}(G) = Exp_k^p(G)$. In particular, if new edges continue to be added, we will eventually arrive at the complete graph and have $Exp_k(K_{|G|}) = K_{|G|}$. Then $Exp_k^\infty(G) := Exp_k^p(G)$ for p as described. We will call $Exp_k^\infty(G)$ the k-implicit closure of G.

Note that applying Proposition 6.1.14 iteratively, we immediately get the following result.

Corollary 6.1.24. *Let G be a graph. Then $Col_k(G) = Col_k(Exp_k^\infty(G))$. In particular, a graph G is k-colorable if and only if $Exp_k^\infty(G)$ is k-colorable.*

Example 6.1.25. Below is a graph G such that $Exp_k(G) \neq Exp_k^2(G)$.



To see this, note that multiple copies of the canonical implicit edge $Can^i(k)$ exist as subgraphs of G and $Exp_k(G)$.

We can generalize Example 6.1.25 to a graph G such that $Exp_k^t(G) \neq Exp_k^{t-1}(G)$ for any $t \in \mathbb{N}$ by attaching a path of length $t-2$ to the vertex x in G above.

Note that these examples contain a copy of K_{k+1} and therefore are not k-colorable. It is reasonable to wonder if such an example can be produced in which the initial graph is k-colorable. The followings result answers this question in the negative.

Proposition 6.1.26. *Let G be a graph. If $\chi(G) \leq k$, then $Exp_k^\infty(G) = Exp_k(G)$.*

Proof. If $\chi(G) < k$, then $Exp_k(G) = G$ by Remark 6.1.8 and we are done. Therefore suppose $\chi(G) = k$.

By way of contradiction, suppose there exists an edge $e \in E(Exp_k^2(G))$ such that $e \notin E(Exp_k(G))$. If x and y are the vertices incident to e, then there exists a subgraph $H \subseteq Exp_k(G)$ such that $e \in I_k(H)$. Moreover, since $e \notin E(Exp_k(G))$, H cannot be a subgraph of G. Let $e_1, \ldots e_n$ be all edges in $E(H)$ but not in $E(G)$, and let x_i, y_i be the vertices adjacent to e_i. Then for each $i \in [n]$, there must exists $J_i \subseteq G$ such that $e_i \in I_k(J_i)$. Consider

$$L = (H \cap G) \cup \left(\bigcup_{i=1}^{n} J_i \right).$$

Since $\chi(G) = k$ and $L \subseteq G$, it follows that $\chi(L) \leq k$. Therefore suppose K is some proper k-coloring of L. Since $J_i \subseteq L$, we have $K|_{J_i}$ is a proper K-coloring of J_i. Since $e_i \in I_k(J_i)$, it follows that $K(x_i) \neq K(y_i)$ for all $i \in [n]$. Finally consider $K|_{H\cap G}$. Note that $H = (H \cap G) \cup \{e_1, \ldots, e_n\}$. However, since $K|_{H\cap G}$ satisifes $K|_{H\cap G}(x_i) \neq K|_{H\cap G}(y_i)$ for all i, we see that $K|_{H\cap G}$ extends in a trivial way to a proper k-coloring $K|_H$ of H (as $V(H) = V(H \cap G)$). Since $e \in I_k(H)$, we must have $K(x) \neq K(y)$. Since K was arbitrary, we see that $e \in I_k(L)$. However, since $L \subseteq G$, we then have that $e \in Exp_k(G)$, contradicting our assumption. $\square$

6.2 Entanglements

In this section, we develop the notion of entanglements. Many of the following definitions and results will have a flavor very similar to that of implicit edges, and in fact entanglements can be thought of as a dual notion to implicit edges.

Definition 6.2.1. Let G be a graph and $x, y \in V(G)$. Suppose G satisfies the following:
(a) $x \neq y$
(b) $\chi(G) = k$
(c) For all $K \in Col_k(G)$, $K(x) = K(y)$.
Then we say that G is an entanglement of order k between vertices x and y and denote this by $(x, y) \in \mathcal{E}_k(G)$.

As was the case with implicit edges, the same graph G may be an entanglement between many different pairs of vertices. Therefore

$$\mathcal{E}_k(G) = \{(x, y) \in V(G) \times V(G) \mid G \text{ is an entanglement of order } k \text{ between } x \text{ and } y\}.$$

Intuitively, $\mathcal{E}_k(G)$ can be thought of as the set of nonadjacent pairs of vertices which could be identified without affecting any k-colorings.

As with implicit edges, we would like to be able to discuss entanglements of order k occurring inside graphs that may not be k-colorable.

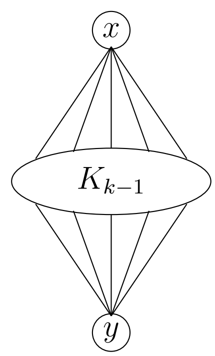
Definition 6.2.2. Let G be a graph and $x, y \in V(G)$. We say that G contains an entanglement of order k between x and y if there exists a subgraph $H \subseteq G$ such that $(x, y) \in \mathcal{E}_k(H)$.

Remark 6.2.3. As with implicit edges, condition (b) of Definition 6.2.1 could be reformulated as $\chi(G) \leq k$. However, just as before, it is not hard to show that if $\chi(G) < k$, then $\mathcal{E}_k(G) = \emptyset$.

Definition 6.2.4. The *canonical entanglement of order k*, denoted $Can^e(k)$, is defined as follows.

i) Begin with a copy of K_{k-1}.

ii) Add two additional vertices, each connected to every vertex in the original K_{k-1}.



It is easy to see that this graph is k-colorable and that every proper k-coloring must assign the same color to x and y, namely the single color not used in coloring K_{k-1}.

Just as with implicit edges and Exp_k, we can associate to entanglements a graph transformation that preserves the set of proper k-colorings of a graph. We first present the definition and a few examples of this transformation before proceeding to show it has the desired property of preserving the set of proper k-colorings.

Definition 6.2.5. Let G be a simple graph. We define the graph $DE_k(G)$ as follows:

1. Begin with the graph G. Given two vertices $x, y \in V(G)$, we say $x \sim_{DE_k} y$ if there exists a sequence $x = x_1, x_2, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in \mathcal{E}_k(G)$ for all $1 \leq i \leq n-1$. Then $\sim_{DE_k}$ is an equivalence relation on $V(G)$. Combine into a

single vertex all vertices in an equivalence group. All edges between vertices are preserved. Note that loops may have been introduced in this process.

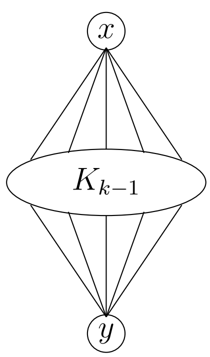
2. If there exist multiple edges between a pair of vertices or multiple loops on the same vertex, delete edges until only a single edge or loop remains.

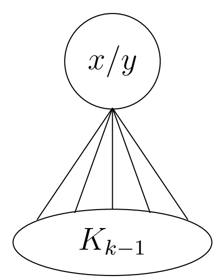
We call $DE_k(G)$ the k-disentanglement of G. Note that $DE_k(G)$ may not be a simple graph since loops may be introduced.

Definition 6.2.6. Let $q : V(G) \to V(DE_k(G))$ be the map defined by the identifications of vertices in step 1 of Definition 6.2.5. As identification in the process constitutes an equivalence relation, q is a quotient map and we will refer to it as the *defining quotient map* of $DE_k(G)$.

Remark 6.2.7. Although the identification of vertices in the construction of $DE_k(G)$ constitutes an equivalence relation, one should note that the property of having a k-entanglement is not necessarily transitive if $\chi(G) > k$. Suppose $(x, y) \in \mathcal{E}_k(H_1)$ and $(y, z) \in \mathcal{E}_k(H_2)$ where H_1 and H_2 are subgraphs of G. If $\chi(G) \leq k$, then $\chi(H_1 \cup H_2) \leq k$ and $(x, z) \in \mathcal{E}_k(H_1 \cup H_2)$. However, if $\chi(G) > k$, it is possible $\chi(H_1 \cup H_2) > k$ in which case $H_1 \cup H_2$ cannot serve as the k-entanglement between x and z. In fact, it is not clear such an entanglement even exists as a subgraph of G.

Example 6.2.8. The canonical entanglement of order k and its image under DE_k are shown below.





Remark 6.2.3 give that $\mathcal{E}_k(G) = \emptyset$ if $\chi(G) < k$. The construction of DE_k then immediately gives us the following result.

Corollary 6.2.9. *If $\chi(G) < k$, then $DE_k(G) = G$.*

We now proceed to show that DE_k has the desired property of preserving the set of proper k-colorings of a graph. Of course, G and $DE_k(G)$ may not have the same number of vertices as many vertices from G may have been identified into a single vertex in $DE_k(G)$. Therefore, we introduce the following expanded definition for the equivalence of sets of k-colorings.

Definition 6.2.10. Suppose G and H are two graphs on n and m vertices respectively where $n \geq m$. We say $Col_k(G) \gtrsim Col_k(H)$ if there exist some surjection $\tau : [n] \hookrightarrow [m]$ and some bijection $\phi : Col_k(G) \to Col_k(H)$ such that $K(t) = \phi(K)(\tau(t))$ for all $t \in [n]$ and $K \in Col_k(G)$.

Remark 6.2.11. Note that $Col_k(G) \gtrsim Col_k(H)$ implies $|Col_k(G)| = |Col_k(H)|$. However, the inverse implication does not hold.

Lemma 6.2.12. *Suppose $(x, y) \in \mathcal{E}_k(G)$. Then $Col_k(G) \gtrsim Col_k(G/xy)$.*

Proof. Let $\tau : V(G) \to V(G/xy)$ be the natural quotient map identifying x and y. Let $t \in V(G/xy)$ and $K \in Col_k(G)$. Define $\phi(K)(t) = K(\tau^{-1}(t))$. This map is well-defined as K is constant on $\tau^{-1}(t)$ for all $t \in V(G/xy)$ since the only preimage of size greater than one contains exactly the two k-entangled vertices x and y. Furthermore $\phi(K) \in Col_k(G/xy)$. Thus ϕ is an injection from $Col_k(G)$ to $Col_k(G/xy)$ satisfying $K(v) = K(\tau^{-1}(\tau(v))) = \phi(K)(\tau(v))$ for all $v \in V(G)$.

By deletion-contraction, $P(G, k) = P(G + xy, k) + P(G/xy, k)$, where $P(G, k)$ is chromatic polynomial evaluated at k (See Definition 6.1.11). Since $xy \in \mathcal{E}_k(G)$, every k-coloring K of G and therefore $(G + xy)$ satisfies $K(x) = K(y)$. Thus $P(G + xy, k) = 0$ implies $P(G, k) = P(G/xy, k)$. Therefore $|Col_k(G)| = P(G, k) = P(G/xy, k) = |Col_k(G/xy)|$. Thus ϕ must be a bijection and therefore $Col_k(G) \gtrsim Col_k(G/xy)$. □

Lemma 6.2.13. *Let G and H be k-colorable graphs on n and m vertices respectively, where $n \geq m$. Suppose that $Col_k(G) \gtrsim Col_k(H)$ and let τ be the surjection defined in Definition 6.2.10. Then*

$$\frac{V(G)}{\mathcal{E}_k(G)} \gtrsim \frac{V(H)}{\mathcal{E}_k(H)},$$

meaning for $x, y \in V(G)$, we have $(x, y) \in \mathcal{E}_k(G) \iff [\tau(x) = \tau(y)$ or $(\tau(x), \tau(y)) \in \mathcal{E}_k(H)]$.

Proof. $\Longrightarrow$ Suppose $(x, y) \in \mathcal{E}_k(G)$. Then for all $K \in Col_k(G)$, $K(x) = K(y)$. Since $Col_k(G) \gtrsim Col_k(H)$, we have $K'(\tau(x)) = K'(\tau(y))$ for all $K' \in Col_k(H)$. Thus either $\tau(x) = \tau(y)$ or $(\tau(x), \tau(y)) \in \mathcal{E}_k(H)$.

$\Longleftarrow$ Suppose $\tau(x) = \tau(y)$ or $(\tau(x), \tau(y)) \in \mathcal{E}_k(H)$. Then $K(\tau(x)) = K(\tau(y))$ for every $K \in Col_k(H)$. Suppose $K' \in Col_k(G)$. Since $Col_k(G) \gtrsim Col_k(H)$, this implies $K'(x) = \phi(K')(\tau(x)) = \phi(K')(\tau(x)) = K'(y)$. Hence $(x, y) \in \mathcal{E}_k(G)$. □

Proposition 6.2.14. *Let G be a graph. Then $Col_k(G) \gtrsim Col_k(DE_k(G))$. As a consequence, G is k-colorable $\iff$ $DE_k(G)$ is k-colorable.*

Proof. If $\chi(G) < k$, then Corollary 6.2.9 gives $G = DE_k(G)$ and we are finished.

If $\chi(G) > k$, then $Col_k(G) = \emptyset$. Suppose by way of contradiction that $K \in Col_k(DE_k(G))$. Then define $K' : V(G) \to [k]$ by $K'(v) = K(q(v))$ where q is the defining quotient map. Suppose $x, y \in V(G)$ such that $x \sim y$. Then $K'(x) = K(q(x)) \neq K(q(y)) = K(y)$ since $x \sim y \implies q(x) \sim q(y)$ and K is a proper k-coloring. Since

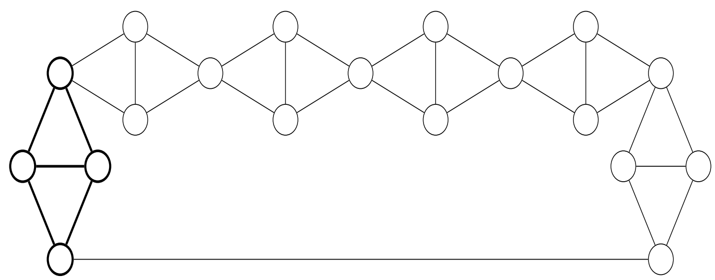
x, y were arbitrary, K' is a proper k-coloring of G, contradicting $\chi(G) > k$. Thus $Col_k(DE_k(G)) = \emptyset = Col_k(G)$.

If $\chi(G) = k$. Then Lemma 6.2.12 and Lemma 6.2.13 combine to inductively show $Col_k(G) \gtrsim Col_k(DE_k(G))$.

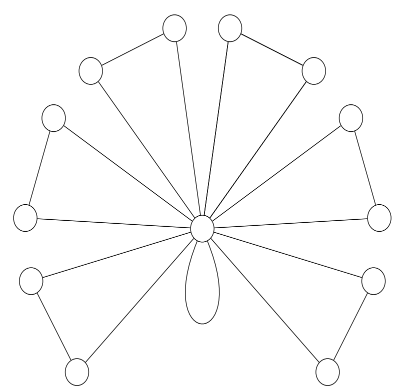
Finally, $\chi(G) \leq k \iff |Col_k(G)| > 0 \iff |Col_k(DE_k(G))| > 0 \iff \chi(DE_k(G)) \leq k$. □

Example 6.2.15. Consider the graph G below. It contains many copies of the canonical entanglement of order 3. For instance, one copy is presented in bold. In its image under DE_3, each copy becomes a triangular wedge, all joined at a central vertex. Since two vertices that had an entanglement of order three were adjacent, a loop is created. It is then obvious that $DE_3(G)$ is not 3-colorable. By Proposition 6.2.14, we see that G is not 3-colorable.

G =



$DE_3(G)$ =



We did not explicitly show that the graph in the example above contained no other entanglements than those noted, although the symmetry of the graph makes an exhaustive search fairly simple. Just as with Exp_k, determining DE_k can sometimes be computationally difficult and it is easier to work with partial results. That is, applying Lemma 6.2.12 and Lemma 6.2.13 it is also possible to draw conclusions about the original graph by simply identifying some vertices which have entanglements between them, without being concerned with ensuring we have found all entanglements in the graph. For instance, in the example above, even if one is not convinced that the second graph in Example 6.2.15 is in fact $DE_3(G)$, it is still clear that $\chi(G) > 3$ as the identification of vertices with an entanglement of order 3 between them results in a graph containing a loop, which is obviously not 3 colorable.

Additional Results about Entanglements

Many of the results concerning implicit edges have an analog for entanglements, and their proofs follow along extremely similar lines. We present here a list of these results but omit the proofs.

Proposition 6.2.16. *Let G be a k-critical graph with $k \geq 3$ and $e \in E(G)$. Then $G - e$ is an entanglement of order $k-1$ between the two vertices incident to e.*

Proposition 6.2.17. *Given any $k, l \in \mathbb{N}$, there exists a graph G with girth$(G) > l$ such that G contains an entanglement of order t for some $t \geq k$.*

Definition 6.2.18. Define $DE_k^0(G) := G$ and $DE_k^t(G) := DE_k(DE_k^{t-1}(G))$ for all $t \in \mathbb{N}$. As G is assumed to be finite, there must exist some $p \in \mathbb{N}$ such that $DE_k^{p+1}(G) = DE_k^p(G)$. In particular, if vertices continue to be identified, we will eventually arrive at a single vertex graph. Then $DE_k^\infty(G) := DE_k^p(G)$ for p as described. We will call $DE_k^\infty(G)$ the k-disentanglement closure of G.

Proposition 6.2.19. *For any $t \in \mathbb{N}$, there exists a graph G such that $DE_k^t(G) \neq DE_k^{t-1}(G)$.*

Proposition 6.2.20. *Let G be a graph. Then $Col_k(G) \gtrsim Col_k(DE_k^\infty(G))$. In particular G is k-colorable $\iff$ $DE_k(G)$ is k-colorable.*

Proposition 6.2.21. *Let G be a graph. If $\chi(G) \leq k$, then $DE_k^\infty(G) = DE_k(G)$.*

6.3 Results and Applications

In this section, we show that under the condition that $\chi(G) \leq k$, the two graph transformations $Exp_k(G)$ and $DE_K(G)$ actually commute. In other words, $Exp_k(DE_k(G)) = DE_k(Exp_k(G))$. We also show that replacing edges in k-critical graphs by implicit edges of a certain type produces a larger k-critical graph. In particular, this allows one to produce k-critical graphs with arbitrarily high diameter.

Proposition 6.3.1. *Suppose $\chi(G) \leq k$. Then*

(i) $I_k(DE_k(G)) = I_k(G)$

(ii) $\mathcal{E}_k(Exp_k(G)) = \mathcal{E}_k(G)$

Proof. (i) "$\supseteq$" Suppose $xy \in I_k(G)$. Note that this implies that x and y do not get identified by DE_k. By Proposition 6.2.14, $Col_k(G) \gtrsim Col_k(DE_k(G))$. Let ϕ and τ be as in Definition 6.2.10. Recall from the proof of Proposition 6.2.14 that τ is the defining quotient map of DE_k. In particular, we may assume WLOG that $\tau(x) = x$ and $\tau(y) = y$. Thus since any k-coloring K of G satisfies $K(x) \neq K(y)$ and ϕ is a bijection, we get $K'(x) = \phi(K)(\tau(x)) = K(x) \neq K(y) = \phi(K)(\tau(y)) = K'(y)$ for any $K' \in Col_k(DE_k(G))$. Thus $xy \in I_k(DE_k(G))$.

"$\subseteq$" Suppose $xy \in I_k(DE_k(G))$. Note this implies $x \neq y$ in $DE_k(G)$. As before, we may assume WLOG that $\tau(x) = x$ and $\tau(y) = y$. Suppose $K \in Col_k(G)$ and $K' \in Col_k(DE_k(G))$ such that $\phi(K) = K'$. Then $K(x) = \phi(K)(\tau(x)) = K'(x) \neq K'(y) = \phi(K)(\tau(y)) = K(y)$. Thus $xy \in I_k(G)$.

(ii) By Proposition 6.1.14, $Col_k(G) = Col_k(Exp_k(G))$. Then the proof of (ii) follows similarly to (i).

$\square$

Theorem 6.3.2. *If $\chi(G) \leq k$, then $Exp_k(DE_k(G)) = DE_k(Exp_k(G))$.*

Proof. We first show that $V(DE_k(Exp_k(G))) = V(Exp_k(DE_k(G)))$. As $V(Exp_k(G)) = V(G)$ and Proposition 6.3.1(ii) gives $\mathcal{E}_k(Exp_k(G)) = \mathcal{E}_k(G)$, we get $V(DE_k(Exp_k(G))) = V(DE_k(G))$. Finally, $V(DE_k(G)) = V(Exp_k(DE_k(G)))$ so $V(DE_k(Exp_k(G))) = V(Exp_k(DE_k(G)))$.

Thus we have shown $V(Exp_k(DE_k(G))) = V(DE_k(Exp_k(G)))$. Note that this result can be restated in the following way:

- For $x, y \in V(G)$, $q_1(x) = q_1(y) \iff q_2(x) = q_2(y)$, where q_1 is the defining quotient map when applying DE_k to G in the left side of the equation, while q_2 is the defining quotient map when applying DE_k to $Exp_k(G)$ in the right side of the equation. As a consequence, we may assume WLOG that $q_1(x) = q_2(x)$ for $x \in V(G)$.

Having established this notation, we now proceed to show that $E(Exp_k(DE_k(G))) = E(DE_k(Exp_k(G)))$, or rather the equivalent statement:

- For all $x, y \in V(G)$,

$$q_1(x)q_1(y) \in E(Exp_k(DE_k(G))) \iff q_2(x)q_2(y) \in E(DE_k(Exp_k(G))).$$

Suppose $q_1(x)q_1(y) \in E(Exp_k(DE_k(G)))$. Then either $q_1(x)q_1(y) \in E(DE_k(G))$ or $q_1(x)q_1(y) \in I_k(DE_k(G))$. If $q_1(x)q_1(y) \in E(DE_k(G))$ then there exist $w \in q_1^{-1}(q_1(x))$ and $z \in q_1^{-1}(q_1(y))$ such that $wz \in V(G)$. Then $q_1(w)q_1(z) = q_1(x)q_1(x) = q_2(x)q_2(y) \in E(DE_k(Exp_k(G)))$. If $q_1(x)q_1(y) \in I_k(DE_k(G))$, then Proposition 6.3.1(i) gives that $xy \in I_k(G)$. Thus $xy \in E(Exp_k(G))$ and therefore $q_2(x)q_2(y) \in E(DE_k(Exp_k(G)))$.

Now suppose $q_2(x)q_2(y) \in E(DE_k(Exp_k(G)))$. Then there exist $w \in q_2^{-1}(q_2(x))$ and $z \in q_2^{-1}(q_2(y))$ such that either $wz \in E(G)$ or $wz \in I_k(G)$. If $wz \in E(G)$, then $q_2(w)q_2(z) = q_2(x)q_2(y) = q_1(x)q_1(y) \in E(DE_k(G))$ and therefore $q_1(x)q_1(y) \in E(Exp_k(DE_k(G)))$. If $wz \in I_k(G)$, Proposition 6.3.1(i) gives that $q_1(w)q_1(z) = q_1(x)q_1(y) \in I_k(DE_k(G))$ implying $q_1(x)q_1(y) \in E(Exp_k(DE_k(G)))$. $\square$

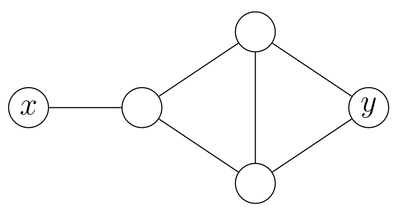
We now define one last graph transformation which replaces each edge in a graph with a copy of another graph. In particular, we will be interested in replacing the edges of k-critical graphs with a certain type of implicit edge of order $k - 1$.

Definition 6.3.3. Let G be a directed graph, meaning each edge is directed toward one of its incident vertices, and let H be a doubly-rooted graph. We define the graph $T_H(G)$ in the following way.

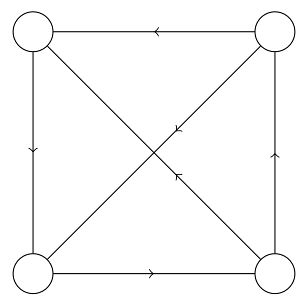
i) Beginning with G, remove all edges from the graph such that only the vertices of G remain.
ii) Add n copies of H labeled $H_1, \ldots, H_n$, where $n = |E(G)|$. Denote the two roots of H_i by x_i and y_i. For an edge $e_i \in E(G)$, let v_i and w_i be the vertices incident to e_i such that e_i is directed toward w_i.
iii) Identify x_i with v_i and y_i with w_i for all $i \in [n]$.

Example 6.3.4. Let $G = K_4$ with edges directed as pictured and $H = Can^i(3)$ with roots x and y.

$H =$

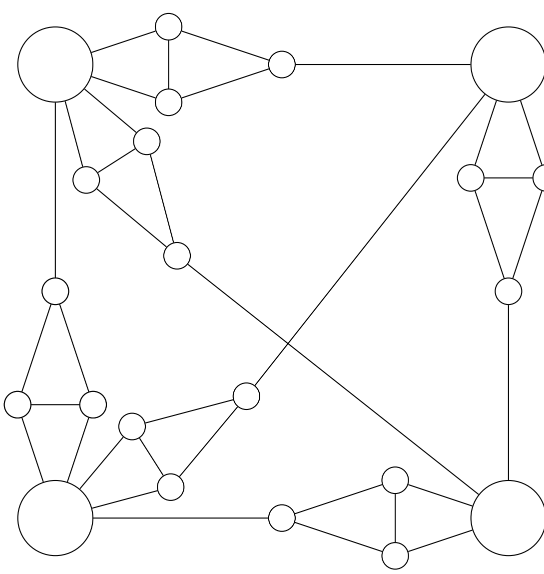


$G =$



Then

$T_H(G) =$



While we assumed that G was a directed graph so that the construction would be well-defined, in the applications for which the construction will be utilized, it will usually be unimportant what the direction of the edges are.

Definition 6.3.5. Suppose G is a graph and $x, y \in V(G)$ such that $xy \in I_k(G)$. We say G is critical with respect to x and y if the the deletion of any edge from G would cause it to no longer be an implicit edge of order k between x and y.

Lemma 6.3.6. *Let G be a k-critical graph and H be a critical implicit edge of order $k-1$ between its two roots. Then $T_H(G)$ is k-critical.*

Proof. We first show that $\chi(T_H(G)) \leq k$. Let x and y be the two roots of H. Since H is an implicit edge of order $k-1$, there exist a $(k-1)$-coloring K of H with $K(x) \neq K(y)$. Labeling the colors $\{1, 2, \ldots, k\}$, we can also create a k-coloring K' of H defined by

$$K'(v) = \begin{cases} K(v) & \text{if } v \notin \{x, y\} \\ k & \text{if } v \in \{x, y\} \end{cases}.$$

We can then produce a k-coloring of $T_H(G)$ by first coloring all vertices from G (which are exactly the collection of roots from each of the H_i) in any fashion we like, then coloring the various H_i using either K or K' (up to permutation of the color set) depending on whether or not their roots were given the same color. Therefore $\chi(T_H(G)) \leq k$.

Next we show that $\chi(T_H(G)) \geq k$. By construction, $E(G) \subseteq I_{k-1}(T_H(G))$ so $\chi(Exp_{k-1}(T_H(G))) \geq \chi(G) = k$. By Proposition 6.1.14 $T_H(G)$ is $(k-1)$-colorable $\iff$ $Exp_{k-1}(T_H(G))$ is $(k-1)$-colorable so we must also have that $\chi(T_H(G)) \geq k$.

Having established that $\chi(T_H(G)) = k$, we must now show that $T_H(G)$ is k-critical. Suppose we remove an edge a from $T_H(G)$. WLOG, assume $a \in E(H_i)$. Let $e_i \in E(G)$ be the edge that was replaced by H_i in the construction of $T_H(G)$. Since G was k-critical, every proper $(k-1)$-coloring L of $G - e_i$ must have $L(v_i) = L(w_i)$ where v_i and w_i are the vertices incident to e_i in G. We can then define a proper $k-1$ coloring on $T_H(G)$ in the following way. Begin by coloring all roots according to their induced coloring from L. Note that for all $j \neq i$, H_j is an implicit edge of order $k-1$. Therefore there exists a $(k-1)$-coloring L_1 of H_j such that $L_1(x_j) \neq L_1(y_j)$. Since H_i is critical, $H_i - \{a\}$ is not an implicit edge between x_i and y_i meaning there exists some $(k-1)$-coloring L_2 of $H_i - \{a\}$ such that $L_2(x_i) = L_2(y_i)$. Therefore we see that up to an appropriate permutation of the coloring set, for $j \neq i$ we can color each H_j using L_1 so that it agrees with the induced coloring from L on the roots. Similarly, coloring $H_i - \{e\}$ using L_2 (again up to proper permutation of the color set) gives a proper $(k-1)$-coloring of $T_H(G)$.

Therefore $\chi(T_H(G) - a) = k - 1$ and since a was arbitrary, $T_H(G)$ is k-critical. $\square$

As an easy corollary of this result, we get a constructive proof for the existence of k-critical graphs with arbitrarily high diameter.

Corollary 6.3.7. *There exists a k-critical graph G with arbitrarily high diameter.*

Proof. Let $G = K_k$ and $H = Can^i(k-1)$. Then define $G_0 := G$ and $G_t := T_H(G_{t-1})$ for $t \geq 1$. One can easily check that H is a critical implicit edge of order $k-1$ between x and y. Since G is a k-critical graph, Lemma 6.3.6 inductively gives us that G_t is k-critical graph for all $t \geq 1$. Moreover, since $d_G(x, y) = 3$, we see that $diam(G_t) = 3 \cdot diam(G_{t-1})$. Since $diam(G_0) = 1$, we have $diam(G_t) = 3^t$. $\square$

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Chapter 7 Open Problems

7.1 Concerning Hamming Graphs

Question 7.1.1. First of all, it remains to answer Question 3.3.3 for the cases not covered in Section 3.3.

Question 7.1.2. Can one describe the maximally robust 2^{n-d+1}-colorings of $H_2(n,d)$? Are their transition spaces generated? Proposition 4.1.8 provides a baseline for this investigation, but is only a first step.

Question 7.1.3. From Theorem 3.3.1 we know that $\chi(H_q(n,n-1)) = q^2$ for all $q \geq 3$ and all q^2-colorings are even. Can one derive further information about these minimal colorings?

Not surprisingly, just as in the binary case not all minimal colorings are coordinate colorings. Here is a 9-coloring of $H_3(4,3)$ that is not a coordinate coloring. Consider the independent set $\mathcal{I} := \{x \in \mathbb{Z}_3^4 \mid wt(x) \leq 1\}$ and the linear MDS code (see Remark 3.2.3 and the paragraph preceeding it)

$$\mathcal{C} = \text{rowspace}\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} := \{a(0111) + b(1012) \mid a, b \in \mathbb{Z}_3\} \subseteq \mathbb{Z}_3^4. \tag{7.1.1}$$

Then each of the shifts $v + \mathcal{I}$, $v \in \mathcal{C}$, is clearly an independent set, and these shifts are pairwise disjoint. Thus they form the color classes of a 9-coloring that is not a coordinate coloring. Note that $\mathcal{I}$ is the set of coset leaders of the code $\mathcal{C}$ (that is, each $v \in \mathcal{I}$ is the unique vector of smallest weight in the coset $v+\mathcal{C}$, and these cosets partition $\mathbb{Z}_3^4$). In fact, $\mathcal{C}$ is a perfect code, see [14, Sec. 1.12]. This example thus generalizes whenever we have a linear perfect MDS code over a finite field. The latter are exactly the Hamming codes with parameter $r = 2$, hence the q-ary Hamming codes of length $n = q + 1$, dimension $q-1$, and distance $d = 3$ and where q is a prime power and the alphabet the finite field $\mathbb{F}_q$ of order q; see for instance [14, Thm. 1.12.3]. As a consequence, we obtain a q^{q-1}-coloring of $H_q(q+1,3)$ that is not a coordinate coloring. As before, the color classes are the shifts of the independent set $\{x \in \mathbb{F}_q^{q+1} \mid wt(x) \leq 1\}$.

Incidentally, the example above can be utilized to create an uneven 27-coloring of $H_3(5,3)$ (recall however, that it is not clear whether this graph has chromatic number 27). Using $\mathcal{C}$ as in (7.1.1) we define the sets

$$\begin{aligned} \hat{\mathcal{C}} &:= \{(v_1, \ldots, v_4, 0) \mid (v_1, \ldots, v_4) \in \mathcal{C}\} \subseteq \mathbb{Z}_3^5, \\ \mathcal{J} &:= \{v \in \mathbb{Z}_3^5 \mid wt(v) \leq 1\}, \\ \mathcal{B}_\alpha &:= \{(v_1, \ldots, v_4, \alpha) \mid wt(v_1, \ldots, v_4) = 1\} \text{ for } \alpha = 1, 2. \end{aligned}$$

Then $\mathcal{J}$ is an independent set of cardinality 11 (showing that the upper bound in Theorem 3.1.3(b) is attained for $H_3(5,3)$), and $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent sets of size 8. It is straightforward to see that the 27 sets $v+\mathcal{J}$, $v+\mathcal{B}_1$, $v+\mathcal{B}_2$, where $v \in \hat{\mathcal{C}}$, form a partition of $\mathbb{Z}_3^5$ and thus serve as the color classes of an uneven 27-coloring.

Question 7.1.4. Do the q^{n-d+1}-colorings of $H_q(n,d)$ for $q \geq 3$ have transition edges?

For $d = 1$ the graph is the complete graph on q^n vertices and thus $T(K) = E(H_q(n,1))$ for every q^n-coloring. Furthermore, for $q \geq 3$, $d \geq 2$ we showed in Proposition 4.1.8(b) that the q^{n-d+1}-coordinate colorings of $H_q(n,d)$. have no transition edges. Furthermore, the even colorings obtained from MDS Hamming codes as described above do not have transition edges either. This can be shown by a similar argument as that used in the proof of Proposition 4.1.8(b). Finally, it is straightforward to verify that the uneven 27-coloring of $H_3(5,3)$ given above has no transition edges.

Since for the binary Hamming graphs $H_2(n, n-1)$ the coordinate colorings have the maximum possible number of transition edges (that is, maximum robustness), we conjecture that no q^{n-d+1}-coloring of $H_q(n,d)$, $q \geq 3$, has any transition edges.

Question 7.1.5. Suppose K is a q^{n-d+1}-coloring of $H_q(n,d)$ such that its transition space $T(K)$ is generated. Does this imply that K is even?

7.2 Concerning Implicit Edges and Entanglements

Question 7.2.1. If $\chi(G) > k$, is $Exp_k^\infty(G)$ the complete graph on $|G|$ vertices?

This has been the case for the few simple examples we have examined. For large graphs, $Exp_k(G)$ is difficult to calculate. However, we suspect there may be a relatively simple argument which would give this result.

Question 7.2.2. Is the following claim true?

$$\chi(G) \geq k \iff Exp_{k-1}(G) \text{ contains } K_k \text{ as a subgraph.}$$

Question 7.2.2 is reminiscent of the well-known Hadwiger conjecture which states that $\chi(G) \geq k \iff K_k$ is a graph minor of G. Of note, it is not hard to show that if the claim in Question 7.2.2 is true, then Question 7.2.1 is also answered in the affirmative.

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Chapter A The Mycielski Construction

Definition A.1. The Mycielskian of a graph G, denoted $\mu(G)$, is a graph formed using the following construction (pictured in Figure A.1).

1. Begin with a copy of G and label its vertices $v_1, \ldots, v_n$.
2. Add n vertices labeled $u_1, \ldots, u_n$.
3. For each edge $v_iv_j \in E(G)$, add edges u_iv_j and u_jv_i.
4. Add a vertex w and place an edge between w and each u_i.

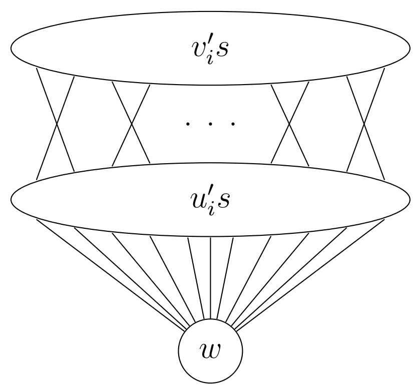

Figure A.1: Mycielski Construction

While the Mycielskian can be applied to any graph, there exists a family of graphs known as the Mycielskian graphs and constructed in the following way. The Mycielski graph of order 2 is two vertices connected by an edge. The Mycielski graph of order k for $k > 2$ is defined inductively as the Mycielskian of the Mycielski graph of order $k-1$. The Mycielski graphs of order 2, 3, and 4 are pictured below.

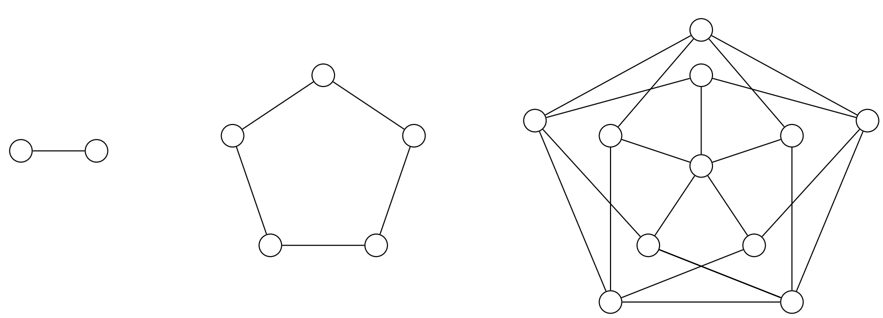

Figure A.2: Mycielski graph of orders 2, 3, and 4

Mycielski is famous for using these graphs in [20] to give a simple and constructive proof for the existence of triangle-free graphs of arbitrarily high chromatic number. Recall that a triangle-free graph is a graph which does does not contain K_3. Additionally, a graph is called k-critical if $\chi(G) = k$ and the removal of any edge or vertex would decrease the chromatic number by one.

Proposition A.2. Let $\mu(G)$ denote the Mycielskian of G. Then
(a) If $\chi(G) = k$, then $\chi(\mu(G)) = k + 1$.
(b) If G is triangle-free, then $\mu(G)$ is triangle-free.
(c) If G is k-critical, then $\mu(G)$ is $(k + 1)$-critical.
(d) If the $diam(G) = 2$, then $diam(\mu(G)) = 2$.

Proof. Parts (a) and (b) are the classical results from [20], and (c) can be found in [4, Lem. 1] . For (d), partition the vertices of $\mu(G)$ into 3 groups based on the label v, u, or w they received in the construction. Firstly, w is connected to every u_i and each v_j is connected to at least one u_i, so w has graph distance at most two from every other vertex. Furthermore, given two vertices u_i and u_k, they are both adjacent to w and therefore have a graph distance of two from each other. By assumption two vertices v_j and v_l have graph distance at most two.

Therefore it remains to show that there exists a path of length at most two between any given u_i and v_j. If there exist an edge v_iv_j, then by construction u_iv_j is an edge. If v_i is not adjacent to v_j, there exist v_k such that $v_iv_k, v_kv_j \in E(G)$ since $diam(G) = 2$. Then $v_ku_i \in E(\mu(G))$ by construction, so there is a path of distance two between u_i and v_j, namely $u_iv_kv_j$. □

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