2014

HOMOLOGICAL ALGEBRA WITH FILTERED MODULES

Raymond Edward Kremer
University of Kentucky, ray.kremer13@gmail.com

STUDENT AGREEMENT:

REVIEW, APPROVAL AND ACCEPTANCE

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's dissertation including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Raymond Edward Kremer, Student

Dr. Edgar Enochs, Major Professor

Dr. Peter Perry, Director of Graduate Studies

HOMOLOGICAL ALGEBRA WITH FILTERED MODULES

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Raymond Edward Kremer
Lexington, Kentucky

Director: Dr. Edgar Enochs, Professor of Mathematics
Lexington, Kentucky 2014

ABSTRACT OF DISSERTATION

HOMOLOGICAL ALGEBRA WITH FILTERED MODULES

Classical homological algebra is done in a category of modules beginning with the study of projective and injective modules. This dissertation investigates analogous notions of projectivity and injectivity in a category of filtered modules. This category is similar to one studied by Sjödin, Năstăsescu, and Van Oystaeyen. In particular, projective and injective objects with respect to the restricted class of strict morphisms are defined and characterized. Additionally, an analogue to the injective envelope is discussed with examples showing how this differs from the usual notion of an injective envelope.

KEYWORDS: filtered module, projective object, injective object, injective envelope, strict morphism

Author's signature: Raymond Edward Kremer

Date: May 3, 2014

HOMOLOGICAL ALGEBRA WITH FILTERED MODULES

By
Raymond Edward Kremer

Director of Dissertation: Edgar Enochs

Director of Graduate Studies: Peter Perry

Date: May 3, 2014

In memory of my father,
Robert A. Kremer Sr.

ACKNOWLEDGMENTS

I would first like to acknowledge my advisor, Dr. Edgar Enochs. Thank you for your willingness to help at any time, your patience in answering questions, and your guidance throughout my time in graduate school. I am honored to have had the opportunity to work with you.

I would also like to acknowledge the rest of my committee: Dr. Alberto Corso, Dr. Paul Eakin, Dr. Cidambi Srinivasan, and Dr. Tim Gorringe. Thank you for your time spent throughout the process of defending this dissertation.

Finally, I want to acknowledge my family and friends. To my mother, Paula, thank you for your constant support in all aspects of my life and for showing me the true meaning of perseverance and strength. To my siblings, Rob and Regina, thank you for believing in me even when I did not. To all my friends, there are too many to list, but I appreciate each one of you. Thank you for all your support as I pursued this project.

TABLE OF CONTENTS

Chapter 1 Introduction

In 1956 Cartan and Eilenberg published the first book on the subject of homological algebra [2]. Their book introduced the notion of a projective module and uses projective and injective resolutions to develop derived functors. The importance of projective and injective modules lead to much work being done to study these objects. For example, Kaplansky showed in [10] that every projective module over a local ring is free. There was also the famous Serre conjecture of whether or not finitely generated projective modules over a polynomial ring were free. This was eventually settled in the affirmative by Quillen and Suslin.

On the injective side there was also much work being done. Reinhold Baer discovered a test to determine whether a module is injective. This is now known simply as "Baer's Criterion." Baer also characterized all injectives modules over a principal ideal domain as the divisible modules and showed that every module is a submodule of an injective module. Eckmann and Schopf introduced the notion of an injective envelope and proved that every module has one. Trying to answer similar type questions about projective and injective objects in a category of filtered modules is a main goal of this work.

The category of filtered modules considered is similar to one studied by Sjödin, Năstăsescu, and Van Oystaeyen. Sjödin studied the relationship between a filtered module and its associated graded module in relation to weak dimension, injectivity and projectivity in [17]. Năstăsescu and Van Oystaeyen continued this study in [13]. While the relationship between a filtered module and its associated graded module is not a main focus, a relationship between exact sequences of filtered modules and exact sequences of their associated graded modules is used in discussing the analogue to a strict injective envelope in Chapter 5.

Chapters 2-4 are background material. Chapter 2 consists of definitions and examples of different types of modules that will be encountered throughout the rest of the work as well as definitions of important objects used in homological algebra . Chapter 3 focuses on infinite direct sums and products. These objects are integral to two of the main results in Chapter 5: the characterizations of strict projective and strict injective modules. Chapter 3 also includes a famous result of Ernst Specker from [18] and some discussion of reflexive and slender modules. These topics give a feel for some of the objects in the category of filtered modules studied in Chapter 5 as well as showing the interplay between the algebraic and topological structure on filtered modules. The notion of completeness is of particular importance. Chapter 4 discusses some background material on Category Theory. This language is used in subsequent chapters and guides the initial discussion of the category of filtered modules in Chapter 5.

Chapter 5 contains the bulk of this dissertation. In this chapter a category of filtered modules, denoted **R-filt**, equipped with descending filtrations such that the topology induced on the module by the filtration is both Hausdorff and complete is considered. The general category theory objects discussed in Chapter 4 are now discussed in the context of this specific category of filtered modules. The main results come when we get to the discussion of projective and injective objects in **R-filt**. The typical categorical definitions of these objects lead to similar or uninteresting results when compared to the usual category of modules. So a modified approach is taken. Rather than looking at projectivity and injectivity with respect to all epimorphisms and monomorphisms, we consider the projective and injective objects with respect to the restricted class of strict morphisms. In doing so, very concrete characterizations of these strict projective and strict injective modules are established.

Chapter 5 concludes with a discussion of an analogue to an injective envelope. This strict injective envelope is analogous to a characterization of injective envelopes given by Enochs in [7]. Two examples are given to show how the strict injective envelope differs from the usual injective envelope.

The final chapter of this works discusses further generalizations for the work done in Chapter 5. Here we consider indexing the filtration of a filtered module by powers of the natural numbers. These multifiltrations allow for similar results to those in Chapter 5 with an additional condition on the filtrations that must be considered.

Chapter 2 Background Material

In this chapter we introduce some ideas from module theory that will be necessary in later chapters. The basic notion of a module is defined and is followed by a brief discussion of several more specific types of modules that will appear later. These include free, injective, and projective modules. This leads naturally to defining projective and injective resolutions. For this the language of commutative diagrams is necessary. Finally, filtered modules are of particular importance to us so we spend the remainder of this chapter on them. We will discuss certain types of filtrations, infinite products of filtered modules, and one way to topologize filtered modules.

2.1 Modules

The following definitions can be found in almost any abstract algebra book that includes a chapter on modules. See [4] for example.

The primary mathematical object of concern to us is the module. Unless otherwise stated, we will assume that every ring R is commutative with identity.

Definition 2.1.1. An *R-module* is an additive abelian group M together with a map, called scalar multiplication, $R \times M \to M$ denoted $(r, m) \mapsto rm$ for all $r \in R$ and $m \in M$ satisfying:

(i) $(r + s)m = rm + sm$ for all $r, s \in R$, $m \in M$,

(ii) $(rs)m = r(sm)$ for all $r, s \in R$, $m \in M$,

(iii) $r(m + n) = rm + rn$ for all $r \in R$, $m, n \in M$, and

(iv) $1m = m$ for all $m \in M$.

Definition 2.1.2. A subgroup N of an R-module M is an *R-submodule*, or simply *submodule*, if R is clear from context, if $rn \in N$ for all $r \in R$ and $n \in N$.

Definition 2.1.3. Let R be a ring and M, N be R-modules. A map $\varphi : M \to N$ is an *R-module homomorphism* if it respects the R-module structures of M and N, that is;

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M$ and

(ii) $\varphi(rx) = r\varphi(x)$ for all $r \in R$, $x \in M$.

Two important submodules related to a given R-module homomorphism are the following.

Definition 2.1.4. Let $\varphi : M \to N$ be an R-module homomorphism. The *kernel of* φ, denoted $\ker(\varphi)$, is the set of elements of M that map to zero. That is,

$$\ker(\varphi) = \{x \in M | \varphi(x) = 0\}.$$

Definition 2.1.5. Let $\varphi : M \to N$ be an R-module homomorphism. The *image of* φ, denoted $\text{Im}(\varphi)$, is the set of elements in N that can be mapped onto by an element of M via φ. That is,

$$\text{Im}(\varphi) = \{n \in N | n = \varphi(m) \text{ for some } m \in M\}.$$

Certain types of R-module homomorphisms occur frequently and are integral in defining more advanced terms in later chapters. These types of homomorphisms include:

Definition 2.1.6. Let $\varphi : M \to N$ be an R-module homomorphism. The map φ is *injective* if $\varphi(x) = \varphi(y)$ implies $x = y$ for all $x, y \in M$.

A common characterization of an injective homomorphism is the following lemma given without proof.

Lemma 2.1.7. *An R-module homomorphism φ is an injection if and only if* $\ker(\varphi) = 0$.

Definition 2.1.8. Let $\varphi : M \to N$ be an R-module homomorphism. The map φ is *surjective* if for any $n \in N$ there exists an $m \in M$ such that $\varphi(m) = n$.

Definition 2.1.9. An R-module homomorphism $\varphi : M \to N$ is said to be *bijective* if it is both injective and surjective.

Definition 2.1.10. An R-module homomorphism is an *isomorphism* if it is bijective. Two modules M and N are said to be *isomorphic*, denoted $M \cong N$, if there is some R-module isomorphism $\varphi : M \to N$.

In order to be able to define the last term in this section, we first need the idea of a quotient module. Quotient modules are very similar to quotients of other algebraic objects (groups, rings, fields, or vector spaces for example) and are defined similarly.

Definition 2.1.11. Let N be a submodule of the R-module M. Then the *quotient module* M/N is the additive, abelian group M/N made into an R-module by defining the scalar multiplication as

$$r(x+N) = (rx) + N \text{ for all } r \in R, x+N \in M/N.$$

Quotient modules are needed at this point to define the dual notion of the kernel of a map φ.

Definition 2.1.12. Let $\varphi : M \to N$ be an R-module homomorphism. The *cokernel of* φ, denoted $\mathrm{Coker}(\varphi)$, is the quotient module $N/\mathrm{Im}(\varphi)$.

2.2 Commutative Diagrams

Diagrams are a way to pictorially represent information about mathematical objects and maps between them. They provide a condensed way to give a lot of information about a situation. Below are the basic conventions that I will follow along with some illustrative examples.

The simplest form of a diagram that we will see represents a single R-module homomorphism between two R-modules. Let $\varphi : M \to N$ be an R-module homomorphism. As a diagram this will be represented as

$$M \xrightarrow{\varphi} N.$$

Diagrams of this type may be of any length including possibly infinite sequences of modules and maps.

Example 2.2.1. The diagram

$$\ldots \xrightarrow{\varphi_{-2}} M_{-1} \xrightarrow{\varphi_{-1}} M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \ldots$$

represents an infinite sequence of R-module homomorphisms $\varphi_i : M_i \to M_{i+1}$.

Here are definitions of several types of such sequences.

Definition 2.2.2. Consider the diagram (where the length may be finite or infinite)

$$\ldots \xrightarrow{\varphi_{-2}} M_{-1} \xrightarrow{\varphi_{-1}} M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \ldots$$

(i) This sequence is said to be a *complex* if $\mathrm{Im}(\varphi_i) \subset \ker(\varphi_{i+1})$ for all i.

(ii) The sequence is said to be *exact at* M_i if $\mathrm{Im}(\varphi_{i-1}) = \ker(\varphi_i)$.

(iii) The sequence is said to be an *exact sequence* if $\mathrm{Im}(\varphi_i) = \ker(\varphi_{i+1})$ for all i.

The following proposition will be needed later.

Proposition 2.2.3. *Consider the short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of R-modules and R-homomorphisms. The following are equivalent:

1. *There exists a homomorphism $\alpha : B \to A$ such that $\alpha f = \mathrm{id}_A$*
2. *There exists a homomorphism $\beta : C \to B$ such that $g\beta = \mathrm{id}_C$*
3. $\mathrm{Im}(f)$ *is a direct summand of B.*

Definition 2.2.4. An exact sequence which satifies any of the equivalent conditions of Proposition 2.2.3 is called a *split exact sequence* and one simply says the exact sequence splits.

If the exact sequence from 2.2.3 splits then $B = f(A) \oplus \beta(C) = f(A) \oplus \ker(g) \cong A \oplus C$. That is, the module in the middle of the short exact sequence is isomorphic to the direct sum of the other two modules in the exact sequence. This means one can essentially *split* B into a direct sum of A and C, thus explaining the terminology. Split exact sequences are particularly important when dealing with projective and injective modules due to the relationships between them and direct sums. This will be discussed in the next couple of sections.

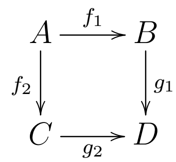
Diagrams need not only be sequences of modules and homomorphisms, but they may take any shape or size. In general, a diagram consists of objects (usually labeled with uppercase letters) and maps between those objects (usually labeled with lowercase or Greek letters). This generality allows one to talk about diagrams in any category, a fact that will be necessary for the main results of this work. For now, all diagrams will use R-modules and R-modules homomorphisms for a fixed ring R.

Some diagrams give more information than simply showing objects and maps. One such type of diagram is a commutative diagram.

Definition 2.2.5. A *commutative diagram* is a diagram such that any two sequences of maps between the same two objects are equal.

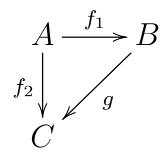
Here are two examples that illustrate the above definition.

Example 2.2.6. The diagram



is commutative if and only if $g_1 f_1 = g_2 f_2$.

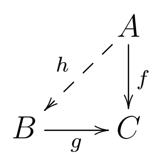
Example 2.2.7. The diagram



is commutative if and only if $f_2 = g f_1$.

It is often the case that in the middle of an argument one needs to construct a portion of a commutative diagram. In this case I will use dashed arrows to indicate where such a map is desired, but that it is not yet clear what that map is.

Example 2.2.8. To say, "the diagram



can be completed to a commutative diagram" means that there exists a map h such that $g \circ h = f$.

2.3 Free, Projective, and Injective Modules

Free modules are modules that have a basis and behave most like vector spaces. Projective modules are defined through a particular commutative diagram and are a generalization of free modules in the sense that every free module satisfies the defining property of projective modules, but the converse is not true. Injective modules are defined dually to projective modules. Projective and injective modules are necessary to the development of homological algebra.

Definition 2.3.1. An R-module F is said to be *free* on the subset A of F if for every nonzero element x of F, there exist unique nonzero elements $r_1, r_2, \ldots, r_n$ of R and unique $a_1, a_2, \ldots, a_n$ in A such that $x = r_1 a_1 + r_2 a_2 + \ldots + r_n a_n$, for some positive integer n. The set A is called a *basis of free generators* for F.

Proposition 2.3.2 (Universal Property of Free Modules)**.** *For any set A there is a free R-module $F(A)$ on the set A and $F(A)$ satisfies the following* universal property*: if M is any R-module and $\varphi : A \to M$ is any map of sets, then there is a unique R-module homomorphism $\Phi : F(A) \to M$ such that $\Phi(a) = \varphi(a)$, for all $a \in A$.*

Definition 2.3.3. A module P is said to be a *projective module* if given a diagram

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle g} & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

of R-modules and R-module homomorphisms with exact row, there exists an R-module homomorphism h such that the diagram commutes, that is, $g = fh$.

Proposition 2.3.4. *Every free module is projective.*

Proposition 2.3.5. *An R-module M is a projective module if and only if M is a direct summand of a free module.*

Proposition 2.3.6. *For every R-module M there exists a surjection $\varphi : P \to M$ such that P is projective. This is commonly referred to by saying* there are enough projective modules*.*

Definition 2.3.7. A module E is said to be an *injective module* if given a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N \\ & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ & & E & & \end{array}$$

of R-modules and R-homomorphisms with exact row, there exists an R-module homomorphism h such that the diagram commutes, that is, $g = hf$

Proposition 2.3.8. *For every R-module M there exists an injection $\varphi : M \to E$ such that E is injective. This is commonly referred to by saying* there are enough injective modules*.*

2.4 Projective and Injective Resolutions

Projective and injective resolutions are complexes in which most of the modules in the complex are projective (resp. injective). These resolutions are used in homological algebra to define derived functors. Construction of projective and injective resolutions relies on the fact that there are enough projective modules and enough injective modules.

Definition 2.4.1. Let M be an R-module. An exact sequence of the form

$$\ldots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \ldots \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

in which every P_n is projective is called a *projective resolution* of M.

Theorem 2.4.2. *Every R-module M has a projective resolution.*

Dually,

Definition 2.4.3. Let M be an R-module. An exact sequence of the form

$$0 \to M \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \to \ldots$$

in which every E^n is injective is called an *injective resolution* of M.

Theorem 2.4.4. *Every R-module M has an injective resolution.*

2.5 Injective Envelopes

Knowing that every module can be embedded in an injective module leads to the natural question of if this can be done in a minimal way. The injective envelope of a module M is simultaneously the smallest injective module larger than M and the largest essential extension over M.

Definition 2.5.1. A module I is the *injective envelope* of the module M if I satisfies one of the three equivalent conditions in the next theorem.

Theorem 2.5.2. *For modules $M \subset I$, the following are equivalent:*

1. *I is maximal essential over M.*
2. *I is injective, and is essential over M.*
3. *I is minimal injective over M.*

Here are a couple of examples:

Example 2.5.3. The injective envelope of $\mathbf{Z}/(p)$ is $\mathbf{Z}(p^\infty)$.

Proof. The group

$$\mathbf{Z}(p^\infty) = \left\{ \frac{a}{p^n} + \mathbf{Z} \middle| n \geq 0, a \in \mathbf{Z} \right\} \subset \mathbf{Q}/\mathbf{Z}.$$

I will show that $\mathbf{Z}(p^\infty)$ is injective and essential over $\mathbf{Z}/(p)$. Recall that an abelian group A is divisible if and only if A is an injective $\mathbf{Z}$-module. First note that $\mathbf{Z}(p^\infty)$ is p-divisible for if $\frac{a}{p^n} + \mathbf{Z} \in \mathbf{Z}(p^\infty)$ then $p^k(\frac{a}{p^{n+k}} + \mathbf{Z}) = \frac{a}{p^n} + \mathbf{Z}$ for all $k \geq 1$. Let $\frac{a}{p^n} + \mathbf{Z} \in \mathbf{Z}(p^\infty)$ and $0 \neq r \in \mathbf{Z}$. If $p \nmid r$ then $p^n \nmid r$ and hence there exist integers s and t such that $sp^n + tr = 1$. So $sp^n a + tra = a$. Thus,

$$\begin{aligned} \frac{a}{p^n} + \mathbf{Z} &= \frac{sp^n a + tra}{p^n} + \mathbf{Z} \\ &= \frac{tra}{p^n} + \mathbf{Z} \\ &= r\left(\frac{ta}{p^n} + \mathbf{Z}\right). \end{aligned}$$

If $p|r$, then let $r' = \frac{r}{p^k}$ where k is the largest positive integer such that $p^k|r$. Similar to the above one can show that

$$\frac{a}{p^n} + \mathbf{Z} = r'\left(\frac{t'a}{p^n} + \mathbf{Z}\right)$$

and thus

$$\frac{a}{p^n} + \mathbf{Z} = r\left(\frac{t'a}{p^{n+k}} + \mathbf{Z}\right)$$

showing that $\mathbf{Z}(p^\infty)$ is divisible.

All that remains is to show that $\mathbf{Z}(p^\infty)$ is essential over $\mathbf{Z}/(p)$. First note that we identify $\mathbf{Z}/(p)$ with the subgroup of $\mathbf{Z}(p^\infty)$ generated by $\frac{1}{p} + \mathbf{Z}$ because they are isomorphic. Let $\frac{a}{p^n} + \mathbf{Z}$ be a non-zero element of $\mathbf{Z}(p^\infty)$. Then $p^{n-1}\left(\frac{a}{p^n} + \mathbf{Z}\right)$ is an element in subgroup generated by $\frac{1}{p} + \mathbf{Z}$ and hence $\mathbf{Z}(p^\infty)$ is essential over $\mathbf{Z}/(p)$. □

Example 2.5.4. The injective envelope of $\widehat{\mathbf{Z}_p}$ is $\widehat{\mathbf{Q}_p}$, the field of p-adic numbers.

Proof. This is a special case of the fact that if R is a commutative domain with quotient field K then K is the injective envelope of R. The quotient field K is essential over R because if $0 \neq \frac{a}{b} \in K$, then $b(\frac{a}{b}) = a \neq 0 \in R$. Also, K is easily seen to be divisible and torsion-free and thus K is an injective R-module. □

2.6 Filtered Modules

Work done later in this dissertation takes some of the standard objects from homological algebra discussed previously in this chapter and looks at them in terms of filtered modules. This section defines filtered modules and some of the language necessary to use them later.

There are different types of filtrations of algebraic structures. I will solely be using descending filtrations on modules indexed by the natural numbers.

Definition 2.6.1. Let M be an R-module. Then a *descending filtration* on M is a family of submodules $\{M_i\}_{i \in I}$ of M for some well-ordered index set I such that $M_\alpha \subset M_\beta$ for all α, $\beta \in I$ where $\alpha > \beta$.

Definition 2.6.2. A descending filtration is called *exhaustive* if $\cup_{i \in I} M_i = M$.

Definition 2.6.3. A descending filtration is is called *separated* if $\cap_{i \in I} M_i = 0$.

A filtered module has a natural topology on it. There is a whole field of study called topological algebra (see [1] for more information) which begins by taking algebraic structures and putting a topology on them which respects the algebraic structure.

Definition 2.6.4. In a topological space X, a *fundamental system of neighborhoods of a point x* (or *a countable basis at x*) is any set $\mathcal{S}$ of neighborhoods of x such that for each neighborhood V of x there is a neighborhood $W \in \mathcal{S}$ such that $W \subset V$. A space X that has a countable basis at each of its points is said to satisfy the *first countability axiom* (or the space X is simply called *first countable*).

Given a module M with filtration $\{M_i\}_{i \in I}$ for some index set I, consider the topology on M defined by taking a basis of the topology to be all translates of the submodules in the filtration. That is,

$$\mathcal{B} = \{x + M_i | x \in M, i \in I\}$$

is a basis for the topology.

With this topology, it is clear that the set $\{M_i\}$ is a fundamental system of neighborhoods of zero for the filtered module M. Sometimes the phrase "topology defined by taking $\{M_i\}$ as a fundamental system of neighborhoods of zero" is used to mean the same topology as above.

Definition 2.6.5. If the filtration $\{M_i\}$ is indexed by a countable set, then the topology defined by taking $\{M_i\}$ as a fundamental system of neighborhoods of zero is called a *linear toplogy.*

The term *separated* in Definition 2.6.3 is not chosen arbitrarily. It corresponds to a topological property of the same name.

Definition 2.6.6. A topological space X is said to be *separated* (or *Hausdorff*) provided that if $x \neq y$ are two distinct points in X then there exist two open sets U_x and U_y such that $U_x \cap U_y = \emptyset$.

Lemma 2.6.7. *Let M be an R-module with filtration $\{M_i\}$. Then $\cap_{i\in I} M_i = 0$ if and only if M is a Hausdorff topological space when given the topology associated with $\{M_i\}$.*

Proof. If M is not Hausdorff, then there exist $x, y \in M$ with $x \neq y$ such that if any open sets U_x, U_y containing x, y respectively then $U_x \cap U_y \neq \emptyset$. In particular, this is true for $U_x = x + M_i$ and $U_y = y + M_i$. Therefore, $x - y \in M_i$ for all $i \in I$ and thus $\cap_{i\in I} M_i \neq 0$. Conversely, if $x \in \cap_{i\in I} M_i$ is non-zero, then the points 0 and x can not be separated by open sets. □

The basic ideas of any topological space can be transferred over to filtered modules once one puts a topology on the module. The notion of completeness is of particular importance to the work in these pages. Completions will be defined analogously to the standard topological definition, but one first needs to define what it means for a sequence to converge and what a Cauchy sequence is in a topological module.

Definition 2.6.8. Let M be an R-module with a topology defined by taking the separated descending filtration $\{M_i\}_{i\in I}$ for some index set I as a fundamental system of neighborhoods of zero. A sequence $\{m_j\}_{j=1}^{\infty}$ of module elements is said to *converge to m* if for any open set U containing m there exists a positive integer N such that if $j > N$ then $m_j \in U$.

Definition 2.6.9. Let M be an R-module with a topology defined by taking the separated descending filtration $\{M_i\}_{i\in I}$ for some index set I as a fundamental system of neighborhoods of zero. A sequence $\{m_j\}_{j=1}^{\infty}$ of module elements is said to be *Cauchy* if for any open set U containing 0 there exists a positive integer N such that if $j, j' > N$ then $m_j - m_{j'} \in U$.

Note that it is enough to use the set $\{M_i\}$ instead of arbitrary open sets U in the above definitions because every open set U completely contains M_i for some $i \in I$.

Definition 2.6.10. An R-module M is *complete* if there is a Hausdorff topology defined by a filtration on M such that every Cauchy sequence of elements of M converges to an element of M.

Remark 2.6.11. Some authors use *complete* to mean complete in a non-discrete topology. I will not exclude the discrete case, but rather explicitly state *complete in a non-discrete topology* when necessary.

In topology there is the following well known lemma and theorem.

Lemma 2.6.12 (The sequence lemma)**.** *Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is metrizable.*

Theorem 2.6.13. *Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $\{x_n\} \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable.*

In fact, the assumption of metrizability for the converses can be weakened to requiring that the the space be first countable instead.

Remark 2.6.14. A filtered module M with filtration $\{M_i\}$ indexed by a countable set is first countable as a topological space when given the topology defined by taking $\{M_i\}$ as a fundamental system of neighborhoods of zero.

Lemma 2.6.15 (Alternative sequence lemma)**.** *Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$. The converse holds if X is first countable.*

Proof. Suppose that $\{x_n\}$ converges to x, where $x_n \in A$ for all n. Then every neighborhood U of x contains a point of A, so $x \in \bar{A}$. Conversely, suppose that X is first countable, $x \in \bar{A}$, and $\{U_n\}$ is a countable basis at the point x. For each n I can choose an element $x_n \in U_1 \cap \ldots \cap U_n$. Then any open set containing x contains one of the U_n, and therefore contains all of the $x_j s$ for $j > n$ by construction. Thus, the sequence $\{x_n\}$ converges to x. □

Theorem 2.6.16. *Let $f : X \to Y$. If the function f is continuous, then for every convergent sequence $\{x_n\} \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first countable.*

Proof. Suppose f is continuous and that $\{x_n\}$ is a sequence converging to x. Let V be an open set containing $f(x)$. Then $f^{-1}(V)$ is an open set containing x and thus, there exists a natural number N such that $x_n \in f^{-1}(V)$ for all $n > N$. This implies that $f(x_n) \in V$ for all $n > N$. Hence, $\{f(x_n)\}$ converges to $f(x)$.

Conversely, I will show that f is continuous by showing that $f(\bar{A}) \subset \overline{f(A)}$ for all $A \subset X$. If $a \in \bar{A}$, then there is a sequence $\{a_n\}$ in A that converges to a by Lemma 2.6.15. Then the sequence $\{f(a_n)\}$ converges to $f(a)$ by assumption. Therefore, $f(a) \in \overline{f(A)}$ again using Lemma 2.6.15. □

Chapter 3 Infinite Direct Products and Sums

Throughout this work infinite direct products and sums of modules play an integral role. This chapter begins with defining these objects and giving examples of both. Describing homomorphisms into and out of these objects is also discussed. This includes Specker's famous result (Proposition 3.3.3). Finally, the last two sections of the chapter discuss reflexive and slender modules. These types of modules provide further examples of infinite direct sums and products as well as provide a context for combining infinite direct sums and products with filtrations, topology, and completeness. All of these are notions that are basic to the remaining parts of this work.

3.1 Definitions and Examples

Let R be a commutative ring with 1. We begin with the finite case.

Definition 3.1.1. Let $N_1, N_2, \ldots, N_k$ be R-modules. The *direct product* of the modules $N_1, N_2, \ldots, N_k$, denoted $N_1 \times N_2 \times \ldots \times N_k$, is the module with elements as ordered k-tuples $(n_1, n_2, \ldots, n_k)$ with $n_i \in N_i$, addition defined componentwise, and scalar multiplication given by $r(n_1, n_2, \ldots, n_k) = (rn_1, rn_2, \ldots, rn_k)$.

Definition 3.1.2. If an R-module M is the direct product of the modules N_1, N_2, $\ldots$, N_k then the *restricted direct product of* N_1, N_2, $\ldots$, N_k (or *(external) direct sum*) is the submodule of M consisting of the elements which are non-zero in only finitely many components.

In this finite case it is easy to see that the direct sum and direct product of modules are the same. However, extending these definition to the case of an arbitrary indexing set, rather than only using a finite number of modules, leads to some differences between the two objects.

Definition 3.1.3. Let $\{M_i\}_{i \in I}$ be a family of R-modules for some indexing set I. Then the *direct product* of the family, denoted $\prod_{i \in I} M_i$, is the module whose elements are families $(m_i)_{i \in I}$ such that $m_i \in M_i$ for every $i \in I$ with addition done componentwise and scalar multiplication by elements in R is given by $r(m_i)_{i \in I} = (rm_i)_{i \in I}$.

Definition 3.1.4. Let $\{M_i\}_{i \in I}$ be a family of R-modules for some indexing set I. Then the *direct sum* of the family, denoted $\oplus_{i \in I} M_i$, is the submodule of the direct product, $\prod_{i\in I} M_i$, consisting of families which have only finitely many non-zero elements.

Example 3.1.5. Let R be a ring, considered as an R-module over itself. Then the product $\prod_{i\in\mathbf{N}} R$ consists of all the countable sequences $(r_i)_{i\in\mathbf{N}}$ where $r_i \in R$ for all i. This module will be denoted R^ω. The direct sum, $\oplus_{i\in\mathbf{N}} R$, consists of all the countable sequences $(r_i)_{i\in\mathbf{N}}$ such that $r_i \neq 0$ for only finitely many i. This direct sum will be denoted $R^{(\omega)}$. In both cases, ω is chosen because it is the symbol typically used to represent the first infinite cardinal number, that is, the cardinality of $\mathbf{N}$.

The module $R^{(\omega)}$ is one special submodule of R^ω that will show up often. The next example describes another set of submodules of R^ω that will be important throughout the remainder of this work.

Example 3.1.6. Consider the R-module R^ω and define

$$W_n := 0 \times 0 \times \ldots \times 0 \times R \times R \times \ldots$$

to be the submodules of the given form with n zeros where n is finite.The family $\{W_n\}_n \in \mathbf{N}$ are a descending filtration of R^ω turning R^ω into a filtered module and thus they form a fundamental system of neighborhoods for a topology on R^ω. In fact, the topology in this case is the standard product topology for a product of topological spaces (here each copy of R has the discrete topology). This will be the topology that R^ω is assumed to have unless otherwise stated. More generally, if $\prod_{i\in I} M^i$ is the direct product of any family $\{M^i\}$ of R-modules, then we will assume this direct product has the topology defined by a fundamental system of neighborhoods of zero given by $M_n := \prod_{i\geq n} M^n$; that is, the product topology.

Lemma 3.1.7. *R^ω with the topology induced by $\{W_n\}_n \in \boldsymbol{N}$ is Hausdorff and complete.*

Proof. Clearly $\cap_{n\in\mathbf{N}} W_n = 0$ so that the topology is Hausdorff. To show completeness it is enough to consider series of the form $\sum_{i=0}^{\infty} r_{n_i}$ where $\{n_i\}$ is a strictly increasing sequence of natural numbers and $r_{n_i} \in W_{n_i}$ for all n_i. Each r_{n_i} is zero in the first n_i components so this series converges to the element r with the jth component of r given by the finite sum $(r)_j = \sum (r_{n_i})_j$. □

Now that we have defined R^ω and $R^{(\omega)}$, we can talk about R-module homomorphisms using R^ω or $R^{(\omega)}$ as the domain or codomain.

Example 3.1.8. There are maps $\pi_i : R^\omega \to R$ given by $\pi_i(x_0, x_1, \ldots) = x_i$. Such a map is called a *canonical projection*, or more specifically, the projection onto the ith component.

Example 3.1.9. There are maps $\iota_i : R \to R^\omega$ given by $\iota_i(r) = (0, \ldots, 0, r, 0, \ldots)$ where the r is in the ith position. Such a map is called a *canonical injection*, or more specifically, the injection into the ith component.

Example 3.1.10. Let $\varphi : R^{(\omega)} \to R$ be an R-module homomorphism. Let $e_i \in R^{(\omega)}$ be the elements $e_i = (0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the ith position and there are zeros elsewhere. The family $\{e_i\}$ forms a base for the free module $R^{(\omega)}$ and therefore the map φ is completely determined by $\varphi(e_i)$ for all i. Suppose $\varphi(e_i) = r_i$ for all i. Then the map φ is defined by $(x_0, x_1, x_2, \ldots) = \sum_{i=0}^{\infty} x_i e_i \mapsto \sum_{i=0}^{\infty} x_i r_i$ which makes sense because only finitely many x_i are non-zero. That is, $\sum_{i=0}^{\infty} x_i r_i = \sum_{i=0}^{N} x_i r_i$ for some positive integer N.

The previous example shows that every map $\varphi : R^{(\omega)} \to R$ is given by an inner product. The final step relies on the fact that only finitely many x_i are non-zero because $(x_0, x_1, \ldots)$ is an element from $R^{(\omega)}$. Clearly this need not still be the case when we transition to R^ω. There are certain cases when a map $\varphi : R^\omega \to R$ will look like a finite inner product, if R is slender for example, but this is not always true. The next two examples show that either case is possible.

Example 3.1.11. Consider the map $f : \mathbf{Z}^\omega \to \mathbf{Z}$ defined by $(x_0, x_1, \ldots) \mapsto \sum_{i=0}^{\infty} a_i x_i$ where $(a_0, a_1, \ldots)$ is any element in $\mathbf{Z}^{(\omega)}$. These maps are clearly given by inner products because only finitely many a_i are non-zero.

Example 3.1.12. Let $R = F[[x]]$ be the ring of formal power series with coefficient in a field F. In this setting there is the notion of order with $\operatorname{ord}(s(x)) = n$ if

$$s(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n + \ldots$$

where $\alpha_0 = \ldots = \alpha_{n-1} = 0$ and $\alpha_n \neq 0$. Also, $\operatorname{ord}(0) = +\infty$.

Now if T_0, T_1, T_2, … is a sequence in $F[[x]]$ such that

$$\lim_{n\to\infty} \operatorname{ord}(T_n) = +\infty$$

then the infinite sum

$$T_0 + T_1 + T_2 + \ldots$$

makes sense and, in fact, if $S_0,\ S_1,\ S_2,\ \ldots$ is any sequence in $F[[x]]$ you can make sense out of

$$S_0T_0 + S_1T_1 + S_2T_2 + \ldots$$

since $\operatorname{ord}(S_nT_n) = \operatorname{ord}(S_n) + \operatorname{ord}(T_n) \geq \operatorname{ord}(T_n)$.

So with $R = F[[x]]$ and a sequence $\{T_n\}$ as above we have the R-linear map, $f : R \times R \times \ldots \to R$ defined by

$$(S_0, S_1, S_2, \ldots) \mapsto S_0T_0 + S_1T_1 + S_2T_2 + \ldots = \sum_{i=0}^{\infty} S_iT_i$$

which is clearly well-defined even when $T_n \neq 0$ for all n. Taking $T_n = x^n$ is such an example. Note the resemblance to an inner product.

The two final examples in this section come from Dimitric's Paper [3]. They are both endomorphisms of R^ω and they emphasize the fact that a sum in R^ω converges as long as each component of the sum has only finitely many non-zero terms being added together.

Example 3.1.13. Let M be an R-module, $f : R^\omega \to M$, (y_n) a sequence with $y_n \in W_n$, and $z \in M$ a non-zero element such that $f(y_n) = z$, for every n. Then the correspondence $g : R^\omega \to R^\omega$ defined by

$$g((x_n)_{n\in\mathbf{N}}) = \left(\sum_{i=1}^{n} x_i(y_{i+1} - y_i)_n\right)_{n\in\mathbf{N}}$$

defines an endomorphism of R^ω.

Example 3.1.14. For an arbitrary $a = (a_0, a_1, \ldots, a_n, \ldots) \in R^\omega$ one can define an endomorphism of R^ω by

$$g((x_n)_{n\in\mathbf{N}}) = \left(\sum_{i=1}^{n} x_i a_n\right)_{n\in\mathbf{N}}$$

Definition 3.1.15. Let $f : R^\omega \to M$ be any homomorphism where R^ω has the product topology (see Example 3.1.6). The topology on M defined by taking $f(W_n)$ to be a fundamental system of neighborhoods of zero is called the *topology induced by f* and will be denoted T_f.

Lemma 3.1.16. *If $f : R^\omega \to M$ is such that T_f is Hausdorff, then M is complete in T_f.*

Proof. Let $\{m_i\}_{i=0}^{\infty}$ be a Cauchy sequence in T_f. The convergence of this sequence in equivalent to the convergence of the sequence of partial sums for the series $m_0 + \sum_{i=0}^{\infty} m_{i+1} - m_i$. In fact, every such series may be thought of as a series of the form $\sum_{i=0}^{\infty} m_{n_i}$ where $m_{n_i} \in f(W_{n_i})$ for all i and $\{n_i\}$ is a strictly increasing sequence of natural numbers by grouping terms together.

If $m_{n_i} \in f(W_{n_i})$ then there exists a $y_{n_i} \in W_{n_i}$ such that $f(y_{n_i}) = m_{n_i}$. But then the series $\sum_{i=0}^{\infty} y_{n_i}$ converges, say to the element y in R^ω because each y_{n_i} is zero in the first n_i components. Thus, the series $\sum_{i=0}^{\infty} m_{n_i}$ converges to $f(y)$ because f is linear and $\sum m_{n_i} = \sum f(y_{n_i}) = f(\sum y_{n_i}) = f(y)$. □

3.2 Reflexive Modules

Reflexive modules are modules that are self-bidual. This notion is made precise in Definition 3.2.2 and in what follows. The idea is that there is a natural way to relate elements in a module to homomorphisms from the dual of that module into the underlying ring.

Again we assume that every ring R is commutative with 1.

Definition 3.2.1. Let R be a ring and let N be a fixed R-module. Then for any R-module M we consider $\mathrm{Hom}_R(M, N)$, the group of all R-module homomorphisms from M to N, as an R-module to be the *dual of M with respect to N*. The dual of M with respect to R will be denoted M^*.

Definition 3.2.2. An R-module M is *reflexive with respect to N* if M is naturally isomorphic to $\mathrm{Hom}_R(\mathrm{Hom}_R(M, N), N)$, the bidual (or double dual) of the module M with respect to N.

These two modules being *naturally* isomorphic means that there is a particular map which is an isomorphism between the two modules. The map takes an element x of M to the map given by evaluation at x. More formally, let x be an element in M. The map $f : M \to M^{**}$ is given by $f(x) = f_x$ where $f_x : M^* \to R$ is given by $f_x(\varphi) = \varphi(x)$. The next two examples show that this natural map need not be injective nor surjective. That is, the next two examples are examples of modules that are not reflexive.

Example 3.2.3. Let $R = \mathbf{Z}$ be the ring of integers and M be the R-module $\mathbf{Z}/(n)$ for some $n > 1$. Then $M^* = 0$ and thus $M^{**} = 0$. Hence, the map $M \to M^{**}$ is not injective because $M \neq 0$.

The next example requires the use of a different natural homomorphism. This natural homomorphism goes from the direct sum of dual modules to the dual of the direct product of modules. That is, let $(M_i)_{i\in I}$ be a family of left R-modules and N be another R-module. Then there is a natural homomorphism

$$g : \bigoplus_{i\in I} \mathrm{Hom}(M_i, N) \to \mathrm{Hom}\left(\prod_{i\in I} M_i, N\right)$$

such that $g((f_i)_{i\in I}) = f$ where $f : \prod M_i \to N$ satisfies $f((m_i)_{i\in I}) = \sum_{i\in I} f_i(m_i)$.

Lemma 3.2.4. *The natural homomorphism g is injective.*

Proof. Suppose $(f_i)_{i\in I} \in \ker(g)$. Then $f = g((f_i)_{i\in I}) = 0$ and thus $f((m_i)_{i\in I}) = 0$ for all $(m_i)_{i\in I} \in \prod M_i$. Also, $f_i = 0$ for all but finitely many i because $(f_i)_{i\in I}$ is an element in a direct sum. For those finitely many i, notice that $f_i(m_i) = f(m_i e_i) = 0$ for all $m_i \in M_i$. That is, $f_i = 0$ for all i, including the indicies that were excluded in the previous case. □

Example 3.2.5. Consider the situation in Example 3.1.12. There we had $R = F[[x]]$ and the R-linear map, $f : R^\omega \to R$ defined by

$$(S_0, S_1, S_2, \ldots) \mapsto S_0T_0 + S_1T_1 + S_2T_2 + \ldots = \sum_{i=0}^{\infty} S_iT_i.$$

Now this map, f, is in the codomain of the natural map g when considering the situation with the R-module $M = R^{(\omega)}$ so each $M_i = R$, $N = R$, and the sequence $\{T_n\}$ of elements of R where $T_n = x^n$ for all n. However, this map is not in the image of g because $f_i \neq 0$ for all i.

Finally, note that $M^* \cong R^\omega$ and $\mathrm{Hom}(R, R) \cong R$. Therefore the natural map $M \to M^{**}$ is g composed with isomorphisms on both sides. So this example shows that the natural map $M \to M^{**}$ need not be a surjection because g is not.

Now that we have examples of modules that are not reflexive, let's look at some examples of modules that are.

Proposition 3.2.6. *A free module of finite rank is reflexive.*

Proof. Let M be a free R-module of finite rank with base $\{e_1, \ldots, e_n\}$. Then we have the dual base $\{e_1^*, \ldots, e_n^*\}$ of M^* where each e_i^* is the function

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We need to show that the natural map $f : M \to M^{**}$ is both injective and surjective.

Suppose $m = a_1e_1 + \ldots + a_ne_n$ is in the kernel of f. Then $\varphi(m) = 0$ for all $\varphi : M \to R$. In particular, this is true for the projection maps $\pi_i : M \to R$ defined by $\pi_i(m) = a_i$. Therefore, if $f(m) = 0$, then $m = 0$.

Let $g \in M^{**}$ be any element of the double dual of M. I want g to be given by evaluation at an element of m. That is, I want to find an m such that $g(\varphi) = \varphi(m)$ for all $\varphi \in M^*$. Each φ can be written as an R-linear combination of the dual base $\{e_i^*\}$, so one can write $\varphi = a_1e_i^* + \ldots + a_ne_n^*$. Let $r_i = g(e_i^*)$ for $i = 1, \ldots, n$. Then $g(\varphi) = \sum_{i=1}^n a_ir_i = \varphi(\sum_{i=1}^n r_ie_1)$. Thus, $\sum_{i=1}^n r_ie_1$ is the desired m. □

Corollary 3.2.7. *As special cases of Proposition* 3.2.6 *we have the following:*

(a) Every ring R is reflexive as an R-module.

(b) Every finite-dimensional vector space is reflexive.

In [6] Enochs proved the following proposition which gives more examples of reflexive modules.

Proposition 3.2.8. *Let A be a discrete valuation ring with unique prime π and let E be a free A-module with a countable base. Then E is reflexive if and only if A is not complete.*

3.3 Slender Modules

Slender modules provide a context for combining infinite direct products and infinite direct sums with filtrations. In this section, Specker's famous result showing that **Z** is slender is proved followed by several other results and characterizations of slender modules. The goal of this section is to get a feel for some of the interplay between algebra and topology as well as study infinite direct sums and products which will appear again later.

Definition 3.3.1. An R-module M is *slender* if every R-homomorphism $f : R^\omega \to M$ is such that $f(W_n) = 0$ for some $n \geq 0$. Equivalently, M is slender if there exists a natural number n_0 such that $f(e_n) = 0$ for all $n \geq n_0$.

Example 3.3.2. Let $E \neq 0$ be an injective R-module. Consider $f : R^{(\omega)} \to E$ defined by $f(e_i) = e$ for all i where e is a fixed non-zero element of E. Then f can be extended to a map $F : R^\omega \to E$ which is non-zero on every e_n. That is, injective modules are not slender.

In his article [18], Ernst Specker proved that the additive group of integers, here denoted $\mathbf{Z}$, is slender. The next proposition provides a version of this argument. In fact, this argument shows that every homomorphism from $\mathbf{Z}^\omega$ into $\mathbf{Z}$ is given by a finite inner product.

Proposition 3.3.3. *The additive group of integers, $\mathbf{Z}$, is slender. Furthermore, every $h : \mathbf{Z}^\omega \to \mathbf{Z}$ is of the form $h(A) = \sum_{i=0}^{n} a_i x_i$ for some $X = (x_0, x_1, \ldots)$ and some positive integer n where $A = (a_0, a_1, \ldots)$ is any element of $\mathbf{Z}^\omega$.*

Proof. Let $e_k = (0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the kth position be the standard base for $\mathbf{Z}^\omega$. Let $h : \mathbf{Z}^\omega \to \mathbf{Z}$ be a homomorphism and define $x_k := h(e_k)$. We will show that there exists $n \geq 0$ such that $x_k = 0$ for all $k > n$ and then that h is of the desired form.

Choose an element $C = (c_0, c_1, \ldots)$ in $\mathbf{Z}^\omega$ satisfying the following three conditions:

(i) $0 < c_0 < c_1 < c_2 < \ldots$,

(ii) $c_k | c_{k+1}$ for all $k \geq 0$, and

(iii) $|c_k| + |c_0 x_0| + |c_1 x_1| + \ldots + |c_k x_k| < c_{k+1}$ for all $k \geq 0$.

Let $w = h(C)$. Then for some n we have $|w| < c_n$. Note that

$$\begin{aligned} |h(C) - h(c_0, \ldots, c_n, 0, \ldots)| &= \left| w - \sum_{k=0}^{n} c_k x_k \right| \\ &< c_n + \sum_{k=0}^{n} |c_k x_k| \\ &< c_{n+1} \end{aligned}$$

and also that

$$h(C) - h(c_0, \ldots, c_n, 0, \ldots) = h(0, \ldots, 0, c_{n+1}, c_{n+2}, \ldots)$$
$$= c_{n+1} h\left(0, \ldots, 0, 1, \frac{c_{n+2}}{c_{n+1}}, \frac{c_{n+3}}{c_{n+1}}, \ldots\right).$$

So if $h(0, \ldots, 0, 1, \frac{c_{n+2}}{c_{n+1}}, \ldots) \neq 0$ then $|h(0, \ldots, 0, c_{n+1}, c_{n+2}, \ldots)| \geq c_{n+1}$. If $|w| < c_n$ then the above implies that $h(0, \ldots, 0, c_{n+1}, c_{n+2}, \ldots) = 0$. But $c_n < c_{n+1}$ and thus, $h(0, \ldots, 0, c_{n+2}, c_{n+3}, \ldots) = 0$ as well. Subtracting these two lines gives $h(c_{n+1}e_{n+1}) = 0$ which implies that $h(e_{n+1}) = x_{n+1} = 0$. In the same way, one can show that $h(e_k) = 0$ for all $k > n$. That is, any n such that $|w| < c_n$ may be taken to be the desired n.

It remains to show that every $h : \mathbf{Z}^\omega \to \mathbf{Z}$ is of the form $h(A) = \sum_{i=0}^{n} a_i x_i$ for some $X = (x_0, x_1, \ldots)$ and some positive integer n where $A = (a_0, a_1, \ldots)$ is any element of $\mathbf{Z}^\omega$. First note that $\mathbf{Z}^\omega$ is a ring with coordinate multiplication. I will use the notation $A \cdot B$ to mean $(a_0b_0, a_1b_1, \ldots)$ when $A = (a_0, a_1, \ldots)$ and $B = (b_0, b_1, \ldots)$ are any two elements of $\mathbf{Z}^\omega$.

Let $h' : \mathbf{Z}^\omega \to \mathbf{Z}$ be the homomorphism defined by

$$A = (a_0, a_1, \ldots) \mapsto h(A) - \sum_{k=0}^{n} a_k x_k$$

where $x_k = h(e_k)$ for all k and n is chosen such that $x_k = 0$ for all $k > n$. It now suffices to show that $h' = 0$. Certainly $h'(e_k) = 0$ for all k. Let $B = (b_0, b_1, \ldots)$ be such that

(i) $0 < b_0 < b_1 < \ldots$ and

(ii) $b_k | b_{k+1}$ for all $k \geq 0$.

But since $h'(e_0) = 0$ we get that $h'(b_0, 0, 0, \ldots) = 0$. Therefore, $h'(b_0, b_1, b_2, \ldots) = h'(0, b_1, b_2, \ldots)$ and continuing in this fashion we see that

$$h'(b_0, b_1, \ldots) = h'(0, \ldots, 0, b_m, b_{m+1}, \ldots)$$
$$= b_m \left(0, \ldots, 0, 1, \frac{b_{m+1}}{b_m}, \frac{b_{m+2}}{b_m}, \ldots\right)$$

for each $m \geq 0$. So $b_m | h'(B)$ for every m, but by our choice of b_m we get that $h'(B) = 0$. Then for any $S = (s_0, s_1, \ldots)$ in $\mathbf{Z}^\omega$ a similar argument shows that $h'(S \cdot B) = 0$.

Finally, let B be as above and let C be another element of $\mathbf{Z}^\omega$ satisfying analogous conditions. Additionally, let B and C be such that $\gcd(b_m, c_m) = 1$ for all m. Then given any A in $\mathbf{Z}^\omega$ one can find $S, T \in \mathbf{Z}^\omega$ so that $A = S \cdot B + T \cdot C$. But then $h'(A) = h'(S \cdot B) + h'(T \cdot C) = 0$ and hence $h(A) = \sum_{k=0}^{n} a_k x_k$ as desired. □

Slenderness is integrally tied to topological algebra. The next few lemmas show some relationships between slenderness and topology. In what follows I will consider R as a module over itself and R will have the discrete topology unless otherwise stated.

Lemma 3.3.4. *An R-module M is slender if and only if every $\varphi : R^\omega \to M$ is continuous for the product topology on R^ω and the discrete topology on M.*

Proof. Suppose M is slender, let $\mathsf{r} = (r_0, r_1, \ldots)$ be an element in R^ω, let $m = \varphi(\mathsf{r})$, and let N be such that $\varphi(e_n) = 0$ for all $n > N$. Then

$$U = \{r_0\} \times \{r_1\} \times \ldots \times \{r_N\} \times R \times R \times \ldots$$

is an open set in R^ω containing r such that $\varphi(U) = \{m\}$. Thus, φ is continuous.

Conversely, if every $\varphi : R^\omega \to M$ is continuous, then in particular the zero map is continuous. Therefore, there exists an open set U in R^ω containing 0 such that $\varphi(U) = \{0\}$. Any such open set U contains a set of the form

$$\{0\} \times \{0\} \times \ldots \times \{0\} \times R \times R \times \ldots$$

for some finite number of zeros, say N. Then $\varphi(e_n) = 0$ for all $n > N$. □

Using this lemma, one can easily see that submodules of slender modules are slender. The next proposition shows that taking direct sums also preserves slenderness. This was shown for abelian groups by Fuchs [8] and for modules by Lady in [11]. The next proposition and its corollaries come from [11].

Proposition 3.3.5. *A direct sum of slender modules is slender.*

Proof. Let $\varphi : R^\omega \to \oplus_{i \in I} M_i$ where each M_i is slender and I is any indexing set. Let $\varphi_k : R^\omega \to M_k$ be the composition of φ and the projection onto the kth component of the direct sum. Each φ_k is continuous for the product and discrete topologies respectively because each M_k is slender.

For a fixed n, $\varphi_k(a_n)) \neq 0$ for finitely many k. Therefore, there are at most countably many k for which $\varphi_k(a_n) \neq 0$ for some n. Therefore we may assume that we are dealing with $\oplus_{i=0}^{\infty} M_i$ instead of using an arbitrary indexing set I.

For any n, $\varphi^{-1}(\oplus_{i=1}^{\infty} M_i) = \cap_{k>n}\varphi_k^{-1}(0)$ which is closed by the continuity of each φ_k. Also, $R^\omega = \cup_{n=1}^{\infty}\varphi^{-1}(\oplus_{i=1}^{n} M_i)$. Thus, we may apply the Baire Category Theorem (R^ω is a complete metric space) to get $\varphi(W_m) \subset \oplus_{i=1}^{n} M_i$ for some m and n. A finite direct sum $\oplus_{k=1}^{n} M_k$ is slender and therefore $\varphi(e_i) \in \oplus_{k=1}^{n} M_k$ for all $i \geq m$ implies $\oplus_{i\in I} M_i$ is slender. □

Corollary 3.3.6. *If R is a slender ring, then so is the polynomial ring $R[x]$, the $n\times n$ matrices $M_n(R)$ over R, and the group ring $R[G]$, for any group G. In addition, if R is an algebra over a field K and F is an extension field of K, then $F \otimes_K R$ is slender.*

Proof. In each case, the larger ring is a free R-module and therefore isomorphic to a direct sum of copies of R. □

Corollary 3.3.7. *A ring R is slender if and only if projective left R-modules are slender.*

Proof. Suppose R is slender. Then every free R-module is slender (being a direct sum of copies of R). Hence every projective R-module is slender because every projective module R-module is a direct summand (and therefore a submodule) of a free module. If every projective R-module is slender, then every free R-module is slender. Therefore, R itself is slender as it is a free R-module. □

The first part of Corollary 3.3.6 was generalized further by O'Neill in [15] who showed the following:

Proposition 3.3.8. *For any ring R, the polynomial ring $R[x]$ is slender.*

Proof. Suppose $R[x]$ is not slender. Then there is an $R[x]$-homomorphism $f : R[x]^\omega \to R[x]$ such that $f(e_n) \neq 0$ for each n. As an R-module $R[x] = \oplus_\omega Rx^n$ and we let $g_n : R[x] \to Rx^n$ be the nth projection. Next I wish to construct a sequence of integers $\{n_k\}_{k=0}^{\infty}$ so that for each k, $g_n f(x^{n_k} e_k) = 0$ for $n \geq n_{k+1}$. Let $n_0 = 0$. Then for $k = 0$ note that $f(x^{n_0} e_0)$ is in $R[x]$ so there are only finitely many non-zero coefficients when considered as an element of $R[x]^\omega$ and hence I can choose n_1 to be large enough such that $g_n f(x^{n_0} e_0) = 0$ for $n \geq n_1$. One may continue inductively to choose the remainder of the sequence $\{n_k\}$. Let $v_k = x^{n_k} e_k$ and $v = \sum_\omega v_k$.

Similarly, there exists a value of k such that $g_n f(v) = 0$ for all $n \geq n_k$. Let $u = v - v_0 - v_1 - \ldots - v_k$ and observe that $u = x^{n_{k+1}} z$ for some z in $R[x]^\omega$. Also recall that $f(e_n) \neq 0$ for all n because $R[x]$ is not slender. Therefore, $g_m f(v_k) \neq 0$ for some m with $n_k \leq m < n_{k+1}$. So $g_m f(u) = g_m f(v) - g_m f(v_0) - \ldots - g_m f(v_k) =$

$-g_m f(v_k) \neq 0$. On the other hand, $f(u) = x^{n_{k+1}} f(z)$ and thus $g_m f(u) = 0$. This is a contradiction and thus, $R[x]$ is slender. □

Also note the following,

Proposition 3.3.9. *For any ring R, the ring of formal power series $R[[x]]$ is not slender.*

Proof. In Example 3.1.12 one sees that the ring of formal power series with coefficients in a field is not slender. If the coefficients come from any ring (not necessarily commutative, but still with 1) then choosing $T_n = x^n$ still makes $\operatorname{ord}(S_n T_n) \geq \operatorname{ord}(T_n)$ and hence the R-linear map $f : R[[x]]^\omega \to R[[x]]$ defined by

$$(S_0, S_1, \ldots) \mapsto S_0 T_0 + S_1 T_1 + \ldots = \sum_{i=0}^{\infty} S_i T_i$$

is still non-zero on all e_i. That is, $R[[x]]$ is not slender. □

It is interesting to compare the previous two propositions. Proposition 3.3.8 shows that polynomial rings, which are isomorphic to infinite direct sums, over any ring R are slender. On the other hand, Proposition 3.3.9 shows that formal powers series rings, which are isomorphic to infinite direct products, over any non-zero ring R are not slender. So in this sense, slenderness is preserved by taking infinite direct sums, but not when taking infinite direct products.

Slender groups, rings, and modules have been studied extensively and I will conclude this chapter by mentioning several other known results about slender modules. The next four results all characterize slender groups or modules in some way.

Theorem 3.3.10 ([14])**.** *A torsion-free group is slender if and only if it is reduced, contains no copy of the p-adic integers for any prime p, and contains no copy of $P = \boldsymbol{Z}^\omega$.*

Theorem 3.3.11 ([11])**.** *Let R be a slender Dedekind domain such that the set $\mathcal{M}$ of non-zero ideals in R is countable. Then an R-module B is slender if and only if B is reduced, torsion-free, and contains no subgroup which is isomorphic to $P = R^\omega$ or $\mathcal{M}$-adically complete.*

Theorem 3.3.12 ([3])**.** *A torsion-free R-module is slender if and only if*

(i) $\operatorname{Hom}_R(R^\omega / R^{(\omega)}, M) = 0$, *and*

(ii) M is not metrizable and complete in any non-discrete linear topology.

Theorem 3.3.13 ([5]). *A finitely generated module M is slender if and only if $Soc(M) = 0$ and M is not complete in any non-discrete topology defined by a filtration.*

Chapter 4 Category Theory

Category theory is about abstraction. It provides a language to discuss ideas that simultaneously show up in many areas of mathematics. This chapter provides definitions and examples of the abstract notions necessary to discuss the transition from the usual category of R-modules to a category of filtered modules in the next chapter.

4.1 Basic Definitions and Examples

Definition 4.1.1. A *category* $\mathcal{C}$ consists of the following data:

1. A collection of objects, denoted $ob(\mathcal{C})$ or more simply $\mathcal{C}$.
2. A collection of morphisms (or arrows, or maps) between the objects, denoted $Mor(\mathcal{C})$.
3. A weak partition of $Mor(\mathcal{C})$ indexed by $ob(\mathcal{C}) \times ob(\mathcal{C})$ where the set indexed by (X, Y) is denoted $Hom_{\mathcal{C}}(X, Y)$ (or $Hom(X, Y)$ when the category is clear).
4. A composition law, that is, for every three objects X, Y, and Z in $\mathcal{C}$ there is a function $Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$ where the image of (g, f) is denoted $g \circ f$ (or simply gf).

satisfying the following axioms:

1. Associativity of morphisms: $h \circ (g \circ f) = (h \circ g) \circ f$ whenever both sides are defined.
2. Identity morphism: For each $X \in ob(\mathcal{C})$ there is a morphism $e \in Hom(X, X)$ such that $f \circ e = f$ and $e \circ g = g$ whenever the compositions are defined.

Remark 4.1.2. Some notation and terminology for a category $\mathcal{C}$.

1. To indicate $f \in Hom(X, Y)$ one writes $f : X \to Y$.
2. If $f : X \to Y$, then X is the *domain* of f and Y is the *codomain* of f.
3. If $f : X \to Y$ and $g : Y' \to Z$, then one says gf is defined if $Y = Y'$.
4. If $f \in Hom(X, Y)$ and $f' \in Hom(X', Y')$ then to say $f = f'$ requires $X = X'$ and $Y = Y'$.

5. The identity morphisms in $\mathcal{C}$ are unique and the identity morphism for an object X in $\mathcal{C}$ is denoted id_X.

Example 4.1.3. Here are some examples of categories:

1. **Sets** - The objects of this category are sets and the morphisms are the usual set functions.

2. **Groups** - The objects of this category are groups and the morphisms are the group homomorphisms.

3. **Rings** - The objects of this category are rings and the morphisms are the rings homomorphisms.

4. **R-mod** - The objects of this category are left R-modules and the morphisms are the R-homomorphisms.

5. **Top** - The objects of this category are topological spaces and the morphisms are continuous functions.

The last two examples that I will mention are not as common:

6. **R-filt** - The objects of this category are modules M with a descending filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

such that M is Hausdorff and complete in the topology defined by taking the submodules in the filtration as a fundamental system of neighborhoods of zero. If N is another object in **R-filt** with filtration

$$N = N_0 \supset N_1 \supset N_2 \supset \dots$$

then a morphism in **R-filt** is a map $f : M \to N$ which is linear and such that $f(M_j) \subset N_j$ for all $j \geq 0$.

7. **gr(R-mod)** - The objects in this category are left R-modules, S, with a direct sum decomposition

$$S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$$

and morphisms

$$g : S \to T = T_0 \oplus T_1 \oplus T_2 \oplus \dots$$

which are linear and such that $g(S_n) \subset T_n$ for all $n \geq 0$.

The category **R-filt** will be the focus of Chapter 5. The category **gr(R-mod)** is connected to **R-filt** through a grading functor that is discussed at the end of this chapter.

Remark 4.1.4. One type of category that is used in this chapter is a *concrete category*. To give a full definition of a concrete category requires defining a particular type of functor (see Definition 4.5.4). In lieu of giving the full definition at this time, I will just comment that in order for a category to be concrete, it is enough to have an underlying set structure to the category. For example, the category **R-mod** is concrete because all of the objects are sets (with additional structure) and all of the morphisms are set functions (again with additional structure).

Next I will define some specific types of objects and morphisms in a category that play important roles moving forward.

Definition 4.1.5. An object X of a category $\mathcal{C}$ is called an *initial object* of $\mathcal{C}$ if for any object Y of $\mathcal{C}$, $Hom(X, Y)$ has exactly one element.

Definition 4.1.6. An object X of a category $\mathcal{C}$ is called a *terminal object* of $\mathcal{C}$ if for any object Y of $\mathcal{C}$, $Hom(Y, X)$ has exactly one element.

Definition 4.1.7. An object X of a category $\mathcal{C}$ is called a *zero object* of $\mathcal{C}$ if X is both initial and terminal.

Example 4.1.8. Examples of zero objects:

1. The category **Groups** has the trivial group consisting of only the identity element as its zero object.

2. The category **R-mod** has the 0 module as its zero object.

3. The category **R-filt** has the 0 module (with the trivial filtration) as its zero object.

Definition 4.1.9. A category $\mathcal{C}$ is called a *category with zero morphisms* if for every two objects A and B in $\mathcal{C}$ there is a fixed morphism $0_{AB} : A \to B$ such that for all objects X, Y, Z in $\mathcal{C}$ and all morphisms $f : X \to Y$, $g : Y \to Z$ the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{0_{XY}} & Y \\ {\scriptstyle f}\downarrow & \searrow^{0_{XZ}} & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[0_{YZ}]{} & Z \end{array}$$

Remark 4.1.10. Any category with a zero object has zero morphisms. The zero morphism 0_{XY} in such a category is the composition $X \to 0 \to Y$ of the only maps $X \to 0$ and $0 \to Y$. Therefore, **Groups**, **R-mod**, and **R-filt** are all categories with zero morphisms.

Definition 4.1.11. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be a *monomorphism* if for any object $W \in ob(\mathcal{C})$ and any $g_1, g_2 : W \to X$ then $fg_1 = fg_2$ implies $g_1 = g_2$.

Definition 4.1.12. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be an *epimorphism* if for any object $Z \in ob(\mathcal{C})$ and any $g_1, g_2 : Y \to Z$ then $g_1 f = g_2 f$ implies $g_1 = g_2$.

Definition 4.1.13. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be a *bimorphism* if f is both a monomorphism and an epimorphism.

Definition 4.1.14. A morphism $f : X \to Y$ in $\mathcal{C}$ is said to be an *isomorphism* if there exists a morphism $g : Y \to X$ in $\mathcal{C}$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Remark 4.1.15. Clearly an isomorphism is a bimorphism, but the converse need not be true.

Monomorphisms and epimorphisms are generalizations of the notions of injective and surjective maps in the category **Sets**. These notions need not coincide in every category. For example:

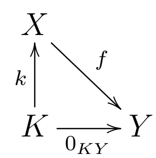
Example 4.1.16. Consider the ring homomorphism $\mathbf{Z} \hookrightarrow \mathbf{Q}$, the inclusion homomorphism from the ring of integers to the ring of rational numbers. This is clearly not a surjection. However, suppose $g_1, g_2 : \mathbf{Q} \to R$ are any two ring homomorphisms such that $g_1|_{\mathbf{Z}} = g_2|_{\mathbf{Z}}$. Then if $q = \frac{x}{y}$ is any rational number, $g_1(q) = g_1(x)g_1(y)^{-1} = g_2(x)g_2(y)^{-1} = g_2(q)$. Therefore, the inclusion map is an epimorphism, but not a surjection.

Example 4.1.17. An example showing that a morphism can be a monomorphism without being injective is harder to come by. It can be shown in the category of divisible abelian groups with group homomorphisms that the quotient map $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ is a monomorphism which is not injective.

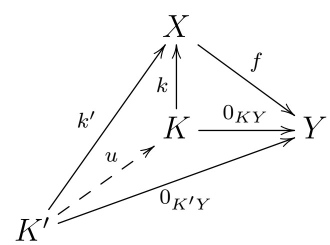
4.2 Kernels and Cokernels

Throughout this section I will be assuming that the category $\mathcal{C}$ is a category with zero morphisms. Kernels in category theory are generalizations of the kernel of a group homomorphism. Recall that the kernel of a group homomorphism $\varphi : G \to H$ is the set of all elements of G that map to the identity of H. To be more explicit,

Definition 4.2.1. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. A *kernel of* f is an object K along with a morphism $k : K \to X$ such that $fk = 0_{KY}$, i.e. the diagram



is commutative, and satisfying the following universal property: given any object K' and morphism $k' : K' \to X$ such that $fk' = 0_{K'Y}$, there is a unique morphism $u : K' \to K$ such that $ku = k'$, i.e. the diagram



is commutative.

Remark 4.2.2. A few observations about kernels

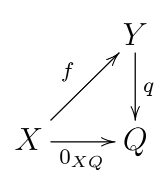
1. In a concrete category the object K is typically referred to as the kernel of f rather than the pair (K, k).

2. A kernel is always a monomorphism.

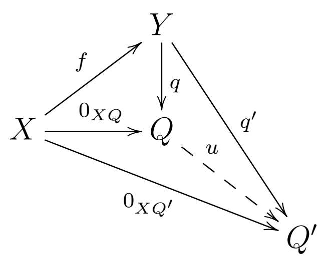
3. In an arbitrary category kernels do not always have to exist, but they are unique up to isomorphism when they do exist.

The dual notion to a kernel is that of a cokernel. Thus, a cokernel is defined as follows:

Definition 4.2.3. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. A *cokernel of* f is an object Q together with a morphism $q : Y \to Q$ such that $qf = 0_{XQ}$, that is the diagram



is commutative, and satisfying the following universal property: given any object Q' and morphism $q' : Y \to Q'$ such that $q'f = 0_{XQ'}$, there is a unique morphism $u : Q \to Q'$ such that $uq = q'$, i.e. the diagram



is commutative.

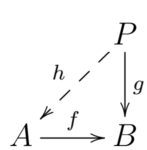
Remark 4.2.4. A few observations about cokernels

1. In a concrete category the object Q is typically referred to as the cokernel of f rather than the pair (Q, q).
2. A cokernel is always an epimorphism.
3. In an arbitrary category cokernels do not always have to exist, but they are unique up to isomorphism when they do exist.

4.3 Projective and Injective Objects

Two of the objects necessary to begin the development of homological algebra are the projective and injective objects. In a categorical sense, these are abstractions of the notions of the projective and injective modules in **R-mod**. These objects are defined diagramatically in relation to epimorphisms and monomorphisms.

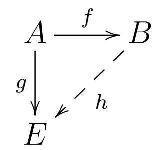
Definition 4.3.1. Let $\mathcal{C}$ be a category. An object P in $\mathcal{C}$ is said to be a *projective object* if for any epimorphism $f : A \to B$ and any morphism $g : P \to B$ then there exists a morphism $h : P \to A$ such that $fh = g$. In diagram form, one must be able to complete the diagram



to a commutative diagram by a morphism in $\mathcal{C}$.

Dual to this definition is the following definition of an injective object.

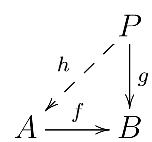
Definition 4.3.2. Let $\mathcal{C}$ be a category. An object E in $\mathcal{C}$ is said to be an *injective object* if for any monomorphism $f : A \to B$ and any morphism $g : A \to E$ then there exists a morphism $h : B \to E$ such that $hf = g$. In diagram form, one must be able to complete the diagram



to a commutative diagram by a morphism in $\mathcal{C}$.

On occasion one may only want to look at projectivity and injectivity with respect to a different class of morphisms. This was done in [16] with regards to injectivity. I will be using the following definitions for this situation.

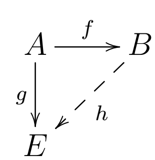
Definition 4.3.3. Let $\mathcal{C}$ be a category and let $\mathcal{H}$ be a class of morphisms in $\mathcal{C}$. An object P in $\mathcal{C}$ is said to be $\mathcal{H}$-projective if for any morphism $f : A \to B$ in $\mathcal{H}$ and any morphism $g : P \to B$ (in $\mathcal{C}$) then there exists a morphism $h : P \to A$ (in $\mathcal{C}$) such that $fh = g$. In diagram form, one must be able to complete the diagram



to a commutative diagram by a morphism in $\mathcal{C}$.

Remark 4.3.4. Clearly, if $\mathcal{H}$ is all epimorphisms, then the $\mathcal{H}$-projective objects are the usual projective objects.

Definition 4.3.5. Let $\mathcal{C}$ be a category and let $\mathcal{H}$ be a class of morphisms in $\mathcal{C}$. An object E in $\mathcal{C}$ is said to be $\mathcal{H}$-injective if for any morphism $f : A \to B$ in $\mathcal{H}$ and any morphism $g : A \to E$ (in $\mathcal{C}$) then there exists a morphism $h : B \to E$ (in $\mathcal{C}$) such that $hf = g$. In diagram form, one must be able to complete the diagram



to a commutative diagram by a morphism in $\mathcal{C}$.

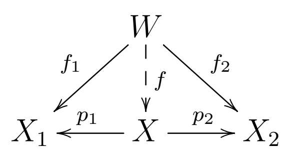
Remark 4.3.6. Clearly, if $\mathcal{H}$ is all monomorphisms, then the $\mathcal{H}$-injective objects are the usual injective objects.

4.4 Products and Coproducts

In many contexts one would like to combine objects in a category to get more objects in the category. Depending on what the objects in a given category are, this may be done in several different ways. The following definitions give two categorical ways to combine objects. Following that, it is shown that these objects do not necessarily exist in every category, but that they satisfy several nice properties with respect to projectivity and injectivity if they do exist.

Definition 4.4.1. If X_1 and X_2 are objects in $\mathcal{C}$, then by a *product* (in $\mathcal{C}$) of the pair X_1 and X_2 we mean an object X and two morphisms $p_1 : X \to X_1$ and $p_2 : X \to X_2$ such that for any W and any $f_1 : W \to X_1$ and $f_2 : W \to X_2$ there is a unique morphism $f : W \to X$ such that $p_1 f = f_1$ and $p_2 f = f_2$. Such an X is often denoted $X_1 \times X_2$.

Diagrammatically, a product X, with morphisms p_1 and p_2, is such that for any W, f_1, and f_2 as above, the diagram



can be completed to a commutative diagram by a unique morphism f in $\mathcal{C}$.

Example 4.4.2. Here are just a few examples where categorical products agree with a well-known notion of a product.

1. In **Sets**, the product of two sets is the Cartesian product.

2. In **Groups**, the product of two groups is the usual direct product of groups.

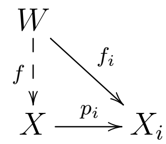
3. In **Top**, the product of two topological spaces is the Cartesian product of the two underlying sets given the product topology.

4. In **R-mod**, the product of two R-modules is the usual direct product of modules.

More generally, one may define the product of an arbitrary collection of objects in a category as follows.

Definition 4.4.3. If $(X_i)_{i \in I}$ is a family of objects in a category $\mathcal{C}$, then a *product* of the family is an object X and morphisms $p_i : X \to X_i$ such that if W is an object in $\mathcal{C}$ and $f_i : W \to X_i$ are morphisms, then there is a unique morphism $f : W \to X$ such that $p_i f = f_i$ for all $i \in I$. Such a product is usually denoted $\prod_{i \in I} X_i$.

Diagrammatically, a product X, with morphisms $p_i : X \to X_i$ is such that for any W, and morphisms f_i, the unique morphism f from above completes the diagrams



to commutative diagrams for all $i \in I$.

Remark 4.4.4. Note that products need not exist in a category. Consider an empty product in a category (i.e. a product where $I = \emptyset$) which is the same as a terminal object. Then the category of infinite abelian groups does not have all products because it does not have a terminal object. This is because there are infinitely many morphism $\mathbf{Z} \to G$ for G an infinite abelian group. Thus G can not be terminal.

Proposition 4.4.5. *Suppose that X, with morphisms $p_i : X \to X_i$, and X', with morphisms $p'_i : X' \to X_i$, are two products of the same family $\{X_i\}_{i \in I}$ in a category $\mathcal{C}$. Then X and X' are isomorphic in $\mathcal{C}$.*

Proof. Since X, with morphisms p_i, is a product in $\mathcal{C}$ and the p'_i are morphisms, there exists a unique morphism $f : X' \to X$ such that $p_i f = p'_i$ for all $i \in I$. Similarly, since X', with morphisms p'_i, is a product in $\mathcal{C}$ and the p_i are morphisms, there exists a unique morphism $g : X \to X'$ such that $p'_i g = p_i$ for all $i \in I$. Thus, $p'_i g f = p'_i$ and $p_i f g = p_i$ for all $i \in I$. However, id_X and $\mathrm{id}_{X'}$ also satisfy the equations $p'_i \mathrm{id}_{X'} = p'_i$ and $p_i \mathrm{id}_X = p_i$ for all $i \in I$. Therefore, by the uniqueness of the morphisms from the definitions of a product we get that $gf = \mathrm{id}_{X'}$ and $fg = \mathrm{id}_X$. That is, X is isomorphic to X' because f is an isomorphism with g as its inverse. □

Proposition 4.4.6 (Taking products preserves injectivity.)**.** *Let $\{E_i\}_{i \in I}$ be a family of injective objects in a category $\mathcal{C}$. Then the product (if it exists) of this family, E, with morphisms $p_i : E \to E_i$, is injective.*

Proof. Let $\alpha : A \to B$ be a monomorphisms and $g : A \to E$ be any morphism. Then for every $i \in I$ the morphism $p_i g : A \to E_i$ may be extended to a morphism $f_i : B \to E_i$ because E_i is injective. This means $f_i \alpha = p_i g$ for all $i \in I$. Now since E, with morphisms p_i, is a product of $\{E_i\}$, there exists a unique morphism $f : B \to E$ such that $p_i f = f_i$ for all $i \in I$.

Now consider the morphisms $f_i \alpha : A \to E_i$. There exists a unique morphism $h : A \to E$ such that $f_i \alpha = p_i h$; again because E, with morphisms p_i, is a product of $\{E_i\}$. Note that $f_i = p_i f$ for all $i \in I$ implies $f_i\alpha = p_i(f\alpha)$ for all $i \in I$ and also that $f_i\alpha = p_i g$ from the previous paragraph. Therefore, $f\alpha = g$ by the uniqueness of h. □

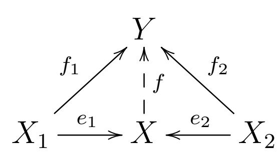
Proposition 4.4.7 (Taking products preserves $\mathcal{H}$-injectivity.)**.** *Let $\mathcal{H}$ be a class of morphisms in a category $\mathcal{C}$ and let $\{E_i\}_{i\in I}$ be a family of $\mathcal{H}$-injective objects in $\mathcal{C}$. Then the product (if it exists) of this family, E, with morphisms $p_i : E \to E_i$, is $\mathcal{H}$-injective.*

Proof. The proof goes exactly like the proof to Proposition 4.4.6 except changing "monomorphism" to "morphism in $\mathcal{H}$" and "injective" to "$\mathcal{H}$-injective." □

Dual to the notion of a product in a category is a coproduct, or categorical sum.

Definition 4.4.8. If X_1 and X_2 are objects in $\mathcal{C}$, then by a *coproduct* (in $\mathcal{C}$) (also called a *categorical sum*) of the pair X_1 and X_2 we mean an object X of $\mathcal{C}$ along with morphisms $e_1 : X_1 \to X$ and $e_2 : X_2 \to X$ such that if Y is an object of $\mathcal{C}$ and $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are morphisms in $\mathcal{C}$ then there is a unique $f : X \to Y$ such that $fe_1 = f_1$ and $fe_2 = f_2$.

Diagrammatically, a coproduct X, with morphisms e_1 and e_2, is such that for any Y, f_1, and f_2 as above, the diagram



can be completed to a commutative diagram by a unique morphism $f : X \to Y$.

Example 4.4.9. Coproducts are often more complicated objects than products.

1. In **Sets**, the coproduct of two sets is the disjoint union with the e_i being the inclusion maps.

2. In **Groups**, the coproduct of two groups is the their free product.

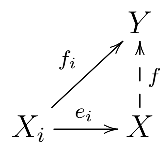
3. In **Top**, the coproduct of two topological spaces is the disjoint union of the two underlying sets with the disjoint union topology.

4. In **R-mod**, the coproduct of two R-modules is the direct sum.

More generally, one may define the coproduct of an arbitrary collection of objects in a category as follows.

Definition 4.4.10. If $(X_i)_{i \in I}$ is a family of objects in $\mathcal{C}$, then a *coproduct* of the family is an object X and morphisms $e_i : X_i \to X$ for each $i \in I$ such that if Y is an object in $\mathcal{C}$ and $f_i : X_i \to Y$ are morphisms, then there is a unique $f : X \to Y$ such that $fe_i = f_i$ for each $i \in I$. Such an X is sometimes denoted $\coprod_{i \in I} X_i$.

Diagrammatically, a coproduct X, with morphisms $e_i : X_i \to X$ is such that for any Y, and morphisms $f_i : X_i \to Y$, the unique morphism f from above completes the diagrams



to commutative diagrams for all $i \in I$.

Remark 4.4.11. Note that coproducts also need not exist in a category. Consider an empty coproduct in a category (i.e. a coproduct where $I = \emptyset$) which is the same as an initial object. Then the category of non-empty sets does not have all coproducts because it does not have an initial object. Suppose a is an element of a set A in this category. Then for any set B with $|B| > 1$ there multiple morphisms (i.e. set functions) $A \to B$ can be defined by sending a to different elements of B.

Proposition 4.4.12. *Suppose that X, with morphisms $e_i : X_i \to X$, and X', with morphisms $e'_i : X'_i \to X$, are two coproducts of the same family $\{X_i\}_{i \in I}$ in a category $\mathcal{C}$. Then X and X' are isomorphic in $\mathcal{C}$.*

Proof. Since X, with morphisms e_i, is a coproduct in $\mathcal{C}$ and the e'_i are morphisms, there exists a unique morphism $f : X \to X'$ such that $fe_i = e'_i$ for each $i \in I$. Similarly, since X', with morphisms e'_i, is a coproduct in $\mathcal{C}$ and the e_i are morphisms, there exists a unique morphism $g : X' \to X$ such that $ge'_i = e_i$ for all $i \in I$. Thus, $gfe_i = e_i$ and $fge'_i = e'_i$ for all $i \in I$. However, id_X and $\mathrm{id}_{X'}$ also satisfy the equations $\mathrm{id}_X e_i = e_i$ and $\mathrm{id}_{X'} e'_i = e'_i$ for all $i \in I$. Therefore, by the uniqueness of the morphisms from the definitions of a product we get that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_{X'}$. That is, X is isomorphic to X' because f is an isomorphism with g as its inverse. □

Proposition 4.4.13 (Taking coproducts preserves projectivity.)**.** *Let $\{P_i\}_{i \in I}$ be a family of projective objects in a category $\mathcal{C}$. Then the coproduct (if it exists) of this family, P, with morphisms $e_i : P_i \to P$, is projective.*

Proof. Let $\alpha : A \to B$ be an epimorphism and $g : P \to B$ be any morphism. Then for every $i \in I$ the morphism $ge_i : P_i \to B$ factors through α because P_i is projective.

That is, there exists morphisms $f_i : P_i \to A$ such that $\alpha f_i = ge_i$ for all $i \in I$. Now since P, with morphisms e_i, is a coproduct in $\mathcal{C}$ there exists a unique morphism $f : P \to A$ such that $fe_i = f_i$ for all $i \in I$.

Now consider the morphisms $\alpha f_i : P_i \to B$. There exists a unique morphism $h : P \to B$ such that $he_i = \alpha f_i$ for all $i \in I$; again because P, with morphisms e_i, is a coproduct of $\{P_i\}$. Clearly, g is such a morphism. Note also that $fe_i = f_i$ for all $i \in I$ implies $(\alpha f)e_i = \alpha f_i$ for all $i \in I$. But then $\alpha f_i = he_i$ for all $i \in I$ implies $(\alpha f)e_i = he_i$ for all $i \in I$. Therefore, $\alpha f = g$ by the uniqueness of h and it follows that P is projective. □

Proposition 4.4.14 (Taking coproducts preserves $\mathcal{H}$-projectivity.)**.** *Let $\mathcal{H}$ be a class of morphisms in a category $\mathcal{C}$ and let $\{P_i\}_{i\in I}$ be a family of $\mathcal{H}$-projective objects in $\mathcal{C}$. Then the coproduct (if it exists) of this family, P, with morphisms $e_i : P_i \to P$ is $\mathcal{H}$-projective.*

Proof. The proof goes exactly the same as that for Proposition 4.4.13 except changing "epimorphism" to "morphism in $\mathcal{H}$" and "projective" to "$\mathcal{H}$-projective." □

4.5 Functors

Functors are an abstraction of functions. Functions take elements in one mathematical object to elements in another mathematical object within some particular setting. Functors do this on a categorical level. A functor is a function which takes objects (resp. morphisms) in one category to objects (resp. morphisms) in another category. This section defines functors and gives two examples of functors which will be used next chapter.

Definition 4.5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* F from $\mathcal{C}$ to $\mathcal{D}$, denoted $F : \mathcal{C} \to \mathcal{D}$, is a mapping that

1. associates an object $F(X)$ in $\mathcal{D}$ to each object X in $\mathcal{C}$,
2. associates a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$ to each morphism $f : X \to Y$ in $\mathcal{C}$ such that
   a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$ and
   b) $F(gf) = F(g)F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

Definition 4.5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *contravariant functor* F from $\mathcal{C}$ to $\mathcal{D}$, denoted $F : \mathcal{C} \to \mathcal{D}$, is a mapping that

1. associates an object $F(X)$ in $\mathcal{D}$ to each object X in $\mathcal{C}$,
2. associates a morphism $F(f) : F(Y) \to F(X)$ in $\mathcal{D}$ to each morphism $f : X \to Y$ in $\mathcal{C}$ such that
    a) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all objects X in $\mathcal{C}$ and
    b) $F(gf) = F(f)F(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$.

Definition 4.5.3. A functor (either covariant or contravariant) $F : \mathcal{C} \to \mathcal{D}$ is said to be *additive* if $F(f + g) = F(f) + F(g)$ for all pairs of morphisms f, g in $\mathcal{C}$.

Definition 4.5.4. A concrete category is a pair $(\mathcal{C}, U)$ such that

1. $\mathcal{C}$ is a category and
2. $U : \mathcal{C} \to$ **Sets** is a faithful functor from $\mathcal{C}$ into the category of sets and functions.

The functor U is usually referred to as a *forgetful* functor which assigns to each object in $\mathcal{C}$ the underlying set and each morphism the underlying set function.

In concrete categories there is a relationship between monomorphisms (resp. epimorphisms) and injections (resp. surjections). The two notions still need not coincide, but one direction of their equivalence is always true.

Lemma 4.5.5. *In a concrete category, every morphism whose underlying function is injective (resp. surjective) is a monomorphism (resp. epimorphism).*

Proof. Let $\mathcal{C}$ be a concrete category. Then all of the morphisms in $\mathcal{C}$ are set functions, along with any additional structure imposed by the category. Therefore, if $f : X \to Y$ is a morphism in $\mathcal{C}$ whose underlying function is injective and $g_1, g_2 : W \to X$ are any morphisms in $\mathcal{C}$ such that $fg_1 = fg_2$ then $f(g_1(w)) = f(g_2(w))$ implies $g_1(w) = g_2(w)$. Similarly, if $h : X \to Y$ is a morphism in $\mathcal{C}$ whose underlying function is surjective and $k_1, k_2 : Y \to Z$ are any morphisms in $\mathcal{C}$ such that $k_1h = k_2h$ then $k_1 = k_2$ because the image of h is all of Y. □

I will end this section by discussing two functors: a shift functor which will be denoted

$$T(n) : \textbf{R-filt} \to \textbf{R-filt}$$

and a grading functor which will be denoted

$$gr : \textbf{R-filt} \to \textbf{gr(R-mod)}$$

that will be important later.

Example 4.5.6 (The Shift Functor)**.** Suppose M is an object in **R-filt** with filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

There is a covariant functor, called the *shift functor*, denoted

$$T(n) : \textbf{R-filt} \to \textbf{R-filt}$$

that changes the filtration on M in the following ways:

1. $(T(n)(M))_0 = M_0 = M$ for all n.
2. If $n = 0$, then $T(n)$ does not change the filtration on M.
3. If $n = 1$, then $(T(1)(M))_p = M_{p+1}$ for all $p \neq 0$.
4. If $n = -1$, then $(T(-1)(M))_p = M_{p-1}$ for all $p \neq 0$.

Then the functor $T(n)$ is defined as the composition of $T(1)$ n times if $n > 0$ and as the composition of $T(-1)$ $|n|$ times if $n < 0$.

Also, for any morphism $f : M \to N$ in **R-filt**, $T(n)(f) : T(n)(M) \to T(n)(N)$ is essentially the same map as f because both M and N have their filtrations shifted so $T(n)(f)$ still satisfies the filtration agreement condition imposed on morphisms in **R-filt**. which is clearly still in **R-filt** and satisfies the two requirements from the definition of a functor.

Note that the map $\mathrm{id}_M : T(-1)(M) \to M$ is not always a morphism in **R-filt** because it is not always true that $M_0 \subset M_1$. However, $\mathrm{id}_M : M \to T(-1)(M)$ is always a monomorphism in **R-filt**.

Example 4.5.7. The functor

$$gr : \textbf{R-filt} \to \textbf{gr(R-mod)}$$

maps the object $M = M_0 \supset M_1 \supset M_2 \supset \dots$ to

$$gr(M) = \frac{M_0}{M_1} \oplus \frac{M_1}{M_2} \oplus \frac{M_2}{M_3} \oplus \dots$$

and maps the morphism $f : M \to N$ in **R-filt** to

$$gr(f) : gr(M) = \frac{M_0}{M_1} \oplus \frac{M_1}{M_2} \oplus \ldots \to gr(N) = \frac{N_0}{N_1} \oplus \frac{N_1}{N_2} \oplus \ldots$$

which is defined by $gr(f)((m_i + M_{i+1})_{i=0}^{\infty}) = (f(m_i) + N_{i+1})_{i=0}^{\infty}$.

Remark 4.5.8. Note that if $(\overline{m_i}) = (\overline{m'_i})$ in $gr(M)$, then $m_i - m'_i \in M_{i+1}$ for all $i \geq 0$ and hence

$$f(m_i - m'_i) = f(m_i) - f(m'_i) \in N_{i+1}.$$

That is, the functor gr is well-defined.

Remark 4.5.9. The functor gr is additive because

$$\begin{aligned} gr(f+g)(m_i + M_{i+1}) &= ((f+g)(m_i) + N_{i+1}) \\ &= ((f(m_i) + g(m_i)) + N_{i+1}) \\ &= ((f(m_i) + N_{i+1}) + (g(m_i)) + N_{i+1}) \\ &= gr(f)(m_i + M_{i+1}) + gr(g)(m_i + M_{i+1}). \end{aligned}$$

Lemma 4.5.10. *If $gr(f)$ is injective (resp. surjective) then f is injective (resp. surjective).*

Proof. Suppose that $gr(f)$ is injective and also that $f(m) = 0$. Then

$$gr(f)(\overline{m}, 0, 0, \ldots) = (\overline{f(m)}, 0, 0, \ldots) = 0$$

which means that $(\overline{m}, 0, 0, \ldots) \in \ker(gr(f)) = 0$ and hence $m \in M_1$. By a similar argument (changing the element we start with to $(0, \overline{m}, 0, \ldots)$) we get that $m \in M_2$. Continuing this process we see that $m \in \cap_{i=0}^{\infty} M_i = 0$. Therefore, f is injective.

Now suppose that $gr(f)$ is surjective and let $n \in N$. Then there exists an element $(\overline{m_0}, \overline{m_1^{(0)}}, \overline{m_2^{(0)}}, \ldots)$ in $gr(M)$ such that

$$gr(f)((\overline{m_0}, \overline{m_1^{(0)}}, \overline{m_2^{(0)}}, \ldots)) = (\overline{n}, 0, 0, \ldots).$$

Applying the definition of $gr(f)$ and subtracting gives that

$$(\overline{n} - \overline{f(m_0)}, -\overline{f(m_1^{(0)})}, \ldots) = 0.$$

Thus, $n - f(m_0) \in N_1$. Next choose an element $(\overline{m_0^{(1)}}, \overline{m_1}, \overline{m_2^{(1)}}, \ldots)$ such that

$$gr(f)((\overline{m_0^{(1)}}, \overline{m_1}, \overline{m_2^{(1)}}, \ldots)) = (0, \overline{n - f(m_0)}, 0, \ldots)$$

and we similarly get that

$$n - f(m_0) - f(m_1) \in N_2.$$

Continuing this process we get that

$$n - \sum_{j=0}^{\infty} f(m_j) \in \cap_{i=0}^{\infty} N_i = 0.$$

Therefore, $n = f\left(\sum_{j=0}^{\infty} m_j\right)$ which is well-defined because M is complete. That is, f is surjective. □

Remark 4.5.11. The converse of the previous lemma is not true in general as the next two examples show. That is, the functor gr does not preserve injections and surjections unless we strengthen the filtration agreement condition imposed upon morphisms in **R-filt**.

Example 4.5.12. Let

$$M = \mathbf{Z} \supset 4\mathbf{Z} \supset 0 \supset \dots$$

and

$$N = \mathbf{Z} \supset 2\mathbf{Z} \supset 0 \supset \dots$$

Let $f : M \to N$ be the map in **R-filt** defined by $f(z) = z$. This map is injective because it is just the identity on $\mathbf{Z}$. However, the map $gr(f) : gr(M) \to gr(N)$ is not injective because the element $(2, 0, \dots)$ maps to zero in $gr(N)$. □

Example 4.5.13. Let

$$M = \mathbf{Z} \supset 2\mathbf{Z} \supset 0 \supset \dots$$

and

$$N = \mathbf{Z}/(4) \supset \mathbf{Z}/(4) \supset 0 \supset \dots$$

Let $f : M \to N$ be the canonical surjection of $\mathbf{Z}$ onto $\mathbf{Z}/(4)$. This map is surjective, but $gr(f)$ is not surjective because the element $(0, \bar{3}, 0, \dots)$ in $gr(N)$ is not in the image of $gr(f)$. □

Definition 4.5.14. A morphism $f : M \to N$ in **R-filt** is said to be *strict* if

$$f(M_j) = f(M) \cap N_j \text{ for all } n \geq 0.$$

Lemma 4.5.15. *If $f : M \to N$ in **R-filt** is strict, then*

1. *f is an injection if and only if $gr(f)$ is an injection, and*

2. *f is a surjection if and only if $gr(f)$ is a surjection.*

Proof. One direction of both statements was shown in Lemma 4.5.10.

Suppose that f is injective, f is strict and that $gr(f)(\overline{m_0}, \overline{m_1}, \ldots) = 0$. Then by the definition of $gr(f)$ this means $(\overline{f(m_0)}, \overline{f(m_1)}, \ldots) = 0$ and thus $f(m_i) \in N_{i+1}$ for all i. So we have $f(m_i) \in f(M) \cap N_{i+1} = f(M_{i+1})$. Therefore, $f(m_i) = f(m)$ for some $m \in M_{i+1}$. But $f(m_i) = f(m)$ implies $f(m_i - m) = 0$ and thus $m_i = m$ because f is injective. That is, $m_i \in M_{i+1}$ for all i. Hence, $gr(f)$ is injective.

Now suppose that f is surjective, f is strict, and let $(\overline{n_0}, \overline{n_1}, \ldots)$ be an element of $gr(N)$. For each i there exists an $m_i \in M$ such that $f(m_i) = n_1$ because f is surjective. Now, $f(m_i) \in f(M) \cap N_i = f(M_i)$ so there is an element $m'_i \in M_i$ such that $f(m'_i) = f(m_i)$. Therefore,

$$\begin{aligned} gr(f)(\overline{m'_0}, \overline{m'_1}, \ldots) &= (\overline{f(m'_0)}, \overline{f(m'_1)}, \ldots) \\ &= (\overline{n_0}, \overline{n_1}, \ldots) \end{aligned}$$

and hence $gr(f)$ is surjective. □

Chapter 5 A Category of Filtered Modules

In this chapter we transition fully to a category of filtered modules. Many of the categorical objects from last chapter are discussed in the specific context of **R-filt**. The chapter culminates with several new results. These are concrete characterizations of strict projective and strict injective modules, as well as defining and discussing strict injective envelopes. Two examples are given to show differences between injective envelopes in **R-mod** and strict injective envelopes in **R-filt** as well as an example minimal strict injective resolution of the p-adic integers in **R-filt**.

5.1 The Category R-filt

Let R be a commutative ring with 1 and let "module" mean "left R-module". The category that I will be considering, denoted **R-filt**, is the category of modules with a descending filtration of submodules indexed by the natural numbers for which the topology induced by this filtration is both Hausdorff and complete. That is, an object in the category **R-filt** is a module M with a filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

where $\cap_{j=0}^{\infty} M_j = 0$ and M is complete in the topology defined by taking the submodules in the filtration as a fundamental system of neighborhoods of zero. Recall that the condition $\cap_{j=0}^{\infty} M_j = 0$ is equivalent to saying the this topology is Hausdorff (see Lemma 2.6.7). Let N be another object in **R-filt** with filtration

$$N = N_0 \supset N_1 \supset N_2 \supset \dots .$$

Then a morphism $f : M \to N$ in **R-filt** is a linear map such that $f(M_j) \subset N_j$ for all $j \geq 0$.

Example 5.1.1. The following types of objects in **R-filt** will be used throughout this chapter.

1. Let M be an R-module and filter M trivially. That is,

   $$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

   where $M_i = 0$ for $i \neq 0$. This is clearly Hausdorff and complete in the given topology.

2. Slightly more generally, let M be an R-module and consider the filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots$$

where $M_i = 0$ for all $i \geq K$ where K is a positive integer. These modules are Hausdorff and complete in the given topology. In fact, the given topology makes M a discrete topological space.

3. Let $\{M^i\}_{i=0}^{\infty}$ be a sequence of R-modules. Then the product $M = M^0 \times M^1 \times M^2 \times \ldots$ filtered by the submodules

$$M_n = 0 \times \ldots \times 0 \times M^n \times M^{n+1} \times \ldots$$

is Hausdorff and complete in the given topology. Formal power series are an example of this type of object in **R-filt**.

4. The p-adic integers, $\widehat{\mathbf{Z}_p}$, with the filtration

$$\widehat{\mathbf{Z}_p} = (p^0) \supset (p^1) \supset (p^2) \supset \ldots$$

forms a Hausdorff and complete topological space.

5.2 Submodules and Quotients in R-filt

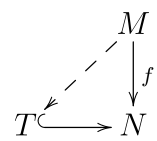
Lemma 5.2.1. *Let N be an object in **R-filt** and let T be a submodule (in **R-mod**) of N with the induced filtration $T_j = T \cap N_j$. Then T is an object in **R-filt** (i.e. T is complete) if and only if T is closed (as a subspace of N as a topological space).*

Proof. Suppose T is closed and let $\{t_j\}$ be a Cauchy sequence of elements in T. This sequence must converge in N because N is complete and since T is closed, it must contain all of its limit points. Therefore, the limit in N must be in N. Hence, T is complete.

Conversely, suppose that T is not closed. This means $N \setminus T$ is not open. So there exists an $x \in N \setminus T$ such that for any $j \in \mathbf{N}$ there exists a $y_j \in T$ such that $x - y_j \in N_j$. Consider the sequence $\{y_j\}$. This sequence is Cauchy because for any $m \in \mathbf{N}$, $y_m - y_n = (y_m - x) + (x - y_n) \in N_m$ for all $n > m$ and this sequence converges to x by construction. Thus, T is not complete. □

Thus we have shown that the induced filtration given to a submodule of an object in **R-filt** results in another object in **R-filt** if and only if the submodule is closed. Unless otherwise stated, any time I refer to a submodule of an object in **R-filt** I will be assuming that the submodule is closed so it is an object in **R-filt**. This induced filtration is unique in the following sense.

Lemma 5.2.2. *The filtration $T_j = T \cap N_j$ is the unique filtration on T such that $T \hookrightarrow N$ is a morphism in **R-filt** and such that if $f : M \to N$ is a morphism in the category such that $f(M) \subset T$ then we can complete the diagram*



*to a commutative diagram by a morphism in **R-filt**.*

Proof. Given that $f(M) \subset T$ it is clear that such an $h : M \to T$ exists in **R-mod** by simply defining $h(m) := f(m)$ for all $m \in M$. In **R-filt**, let

$$M = M_0 \supset M_1 \supset \dots$$

be the filtration of M and consider the filtration

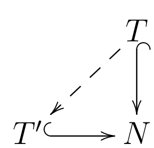
$$T = T_0 \supset T_1 \supset \dots$$

of T where $T_j = T \cap N_j$ for all j. We can again complete the diagram defining $h : M \to T$ by $h(m) = f(m)$, but now we have a morphism in **R-filt**.

To show the uniqueness, suppose

$$T' = T'_0 \supset T'_1 \supset \dots$$

is a different filtration of the same base module T satisfying the above conditions. Since $T' \hookrightarrow N$ is a morphism in **R-filt**, then we know that $T'_j \subset N_j$. Therefore $T'_j \subset T_j$. Now using the second condition, we can complete the diagram



with a morphism in **R-filt**. Since the two given maps are inclusions, the map that completes the diagram must be id_T. Thus, $T_j \subset T'_j$. That is, the two filtrations are in fact the same. ∎

Due to this uniqueness, I will assume that a submodule of an object in **R-filt** has this induced filtration unless otherwise noted.

Lemma 5.2.3. *Let M with filtration $\{M_j\}$ be an object in **R-filt** and N be a submodule of M in **R-mod**. Then the quotient module M/N with the induced filtration*

$$\left(\frac{M}{N}\right)_j = \frac{N + M_j}{N}$$

*is an object in **R-filt** if and only if N is an object in **R-filt** when given the induced filtration of Lemma 5.2.1.*

Proof. Suppose N is an object in **R-filt** when given the induced filtration as a submodule of M. The induced filtration on M/N is certainly descending because $\{M_j\}$ is descending. To show M/N is Hausdorff, it suffices to show that

$$\bigcap_{j=0}^{\infty} \frac{N + M_j}{N} = 0.$$

Suppose $\bar{x} = x + N$ is in this intersection. Then for each $j \in \mathbf{N}$ there is an element $m_j \in M_j$ such that $\bar{x} = \bar{m_j}$. This means $x - m_j \in N$ for each m_j chosen. Also, $m_i - m_{i+1} \in N \cap M_i$ for all i. Note that N has the induced filtration as a submodule of M, thus $m_i - m_{i+1} \in N_i$. Therefore,

$$x - m_0 + \sum_{i=0}^{\infty} m_i - m_{i+1}$$

converges to $x \in N$ because N is complete. Thus, $\bar{x} = \bar{0}$ which means M/N is Hausdorff.

To show that M/N is complete it is enough to show that all series of the form $\sum_{l=0}^{\infty} (m_l + N)$ where $m_l \in M_l$ for all l converge. Let $m := \sum_{l=0}^{\infty} m_l$ (which exists because M is complete). Note that each M_j is closed in addition to being open because $M \setminus M_j$ is the union of the translates of M_j that are not M_j itself. Then the above series converges to $m + N \in M/N$ because for any k, $(m + N) - (\sum_{l=0}^{L-1} (m_l + N)) = (\sum_{l=L}^{\infty} m_l) + N \in (M/N)_k$ for all $L \geq k$.

Conversely, suppose that M/N is an object in **R-filt**. It is enough to show that N is closed in M by Lemma 5.2.1. But N is the inverse image of the closed set $\{\bar{0}\}$ in M/N under the continuous map $M \to M/N$ and thus, N is closed. □

Quotients of objects in **R-filt** will be given the induced filtration unless otherwise noted. Quotients are used in the discussion of epimorphisms and they play an important role in the characterization of strict injective modules later in this chapter.

5.3 More on Morphisms in R-filt

Example 5.3.1. The following are the canonical injections and surjections in **R-filt**.

1. (Canonical Injection) Suppose T is a submodule of the object M in **R-filt**. Then the inclusion map $i : T \to M$ is an injective morphism in **R-filt**.

2. (Canonical Surjection) Suppose N is a submodule of M in **R-filt**. Then the map $M \to M/N$ given by $m \mapsto m + N$ is a surjective morphism in **R-filt**.

Here are two important facts about morphisms in **R-filt**.

Lemma 5.3.2. *Every morphism $f : M \to N$ in **R-filt** is a continuous map between M and N thought of as topological spaces. Moreover, f is uniformly continuous.*

Proof. Let $m \in M$. Then $V = f(m) + N_j$ is a basis open set of N containing $f(m)$. Note that $f(m + M_j) \subset f(m) + N_j$ because f is linear and therefore f is continuous. Moreover, if $m_1 - m_2 \in M_j$ then $f(m_1) - f(m_2) = f(m_1 - m_2) \in N_j$ because f is linear. Therefore f is also uniformly continuous. □

One particular type of morphism is used throughout the rest of the work. These are the so-called *strict* morphisms. Recall the following definition (the same as Definition 4.5.14)

Definition 5.3.3. Let $f : M \to N$ be a morphism in **R-filt**. If f satisfies

$$f(M_j) = f(M) \cap N_j$$

(a stronger condition than $f(M_j) \subset N_j$) then we say f is a *strict* morphism in **R-filt**.

Example 5.3.4. The canonical injection and canonical surjection from above are both strict for all M, N, and T in **R-filt**.

Lemma 5.3.5 (Factorization Through a Strict Surjective Morphism in **R-filt**)**.** *Let $f : A \to B$ be any morphism in **R-filt** and $g : A \to C$ be a strict surjective morphism. Then there exists a morphism $h : C \to B$ in **R-filt** such that $f = hg$ if and only if $\ker(g) = \ker(f)$.*

Proof. The forward direction is obvious. For the reverse direction, let $c \in C$. Then there exists an $a \in A$ such that $g(a) = c$ because g is surjective. Then use this a to define $h : C \to B$ by $h(c) = f(a)$. Suppose $c \in C$ and there exists a_1 and

a_2 in A such that $c = g(a_1) = g(a_2)$. Then $a_1 - a_2 \in \ker(g) \subset \ker(f)$. Thus, $f(a_1 - a_2) = f(a_1) - f(a_2) = 0$ showing that h is a function.

Now let $c_1, c_2 \in C$. Then there exists elements $a_1, a_2 \in A$ such that $g(a_1) = c_1$, $g(a_2) = c_2$, and hence $g(a_1 + a_2) = c_1 + c_2$. Thus, by the definition of h and the linearity of f, $h(c_1 + c_2) = f(a_1 + a_2) = f(a_1) + f(a_2) = h(c_1) + h(c_2)$. Similarly, if $c \in C$ and $a \in A$ such that $g(a) = c$ then $g(ra) = rg(a) = rc$. Therefore, $h(rc) = f(ra) = rf(a) = rh(c)$ using the definition of h and the fact that f is an R-homomorphism. All this shows that g is an R-homomorphism.

Finally, note that if $c \in C_n$, then there is an element $a_n \in A_n$ such that $g(a_n) = c$ because g is strict. Therefore, $h(c) = f(a_n) \in B_n$ because f is a morphism in **R-filt**. This shows that h is also a morphism in **R-filt**. Such an h is clearly unique. □

The category **R-filt** with the forgetful functor is a concrete category because each module has an underlying set structure. Therefore, from Chapter 4 every injective function is a monomorphism and every surjective function is an epimorphism. The converse of this statement for monomorphisms is true.

Lemma 5.3.6. *In **R-filt**, injective functions coincide with monomorphisms.*

Proof. Note that the above discussion gives one direction.

Let $f : M \to N$ be a monomorphism in **R-filt** and K be the kernel of f filtered as a submodule of M. Note that K is closed because it is the inverse image of the closed set $\{0\}$ under the continuous map f. Then consider the two morphisms $g_1, g_2 : K \to M$ defined by $g_1(k) = 0$ and $g_2(k) = k$. Clearly $fg_1 = fg_2$ and thus $g_1 = g_2$ because f is a monomorphism. Therefore, the kernel of f must be zero which implies that f is an injection.

□

On the other hand, the dual statement regarding epimorphisms does not work out the same way. This is because the dual argument would require the use of a cokernel instead of a kernel. The next remark describes why this could be a problem.

Remark 5.3.7. Typically, the cokernel of an R-homomorphism $f : M \to N$ is defined to be the quotient module $N/\mathrm{Im}f$. Defining a cokernel in **R-filt** takes a little more work. Suppose $f : M \to N$ is a morphism in **R-filt**. For an arbitrary morphism, it is not guaranteed that the image of f will be a closed submodule of N. Therefore, it is reasonable to expect that the cokernel of $f : M \to N$ in **R-filt** is the quotient object $N/\overline{\mathrm{Im}f}$. This will guarantee that the quotient object exists and indeed it satisfies the requirements to be a cokernel as the next proposition will show.

Proposition 5.3.8. *If $f : M \to N$ is a morphism in **R-filt**, then $N/\overline{f(M)}$ (where $\overline{f(M)}$ is the closure of the image of f) with the induced quotient filtration is the cokernel of f.*

Proof. Recall that the categorical definition of a cokernel includes a morphism with the object. In this case the morphism is taken to be the canonical surjection $\pi : N \to N/\overline{f(M)}$ defined by $\pi(n) = n + \overline{f(M)}$. Clearly, $\pi f = 0$. Suppose that $q : N \to Q$ is another morphism in **R-filt** such that $qf = 0$. Then, because π is a strict surjection, there exists a unique morphism $u : N/\overline{f(M)} \to Q$ such that $u\pi = q$ by Lemma 5.3.5 and the proof is complete. □

Proposition 5.3.9. *A morphism $f : M \to N$ in **R-filt** is an epimorphism if and only if $f(M)$ is dense in N.*

Proof. Suppose $f : M \to N$ is an epimorphism. Consider the canonical surjection $\pi : N \to N/\overline{f(M)}$ and the zero morphism with the same domain and codomain. Then clearly $\pi f = 0$ and $0f = 0$. Therefore, $\pi = 0$ as morphisms in **R-filt**. This means that $N = \overline{f(M)}$ and thus $f(M)$ is dense in N.

Conversely, suppose that $f : M \to N$ and that $g, h : N \to L$ are morphisms in **R-filt** such that $gf = hf$. Also suppose that $f(M)$ is dense in N. This means that $\overline{f(M)} = N$. That is, every element of N is the limit of a sequence of elements in $f(M)$ by Lemma 2.6.15. Let $n \in N$ be arbitrary and let $\{f(m_j)\}$ be a sequence converging to n. Then

$$g(n) = g(\lim f(m_j)) = \lim g(f(m_j)) = \lim h(f(m_j)) = h(\lim f(m_j)) = h(n)$$

by Theorem 2.6.16 and because $gf = hf$ as above. □

In the category **R-mod** every R-module homomorphism that is a bijection is an isomorphism. The same is not true for the category **R-filt** because of the filtration agreement condition imposed upon morphisms in this category.

Example 5.3.10. Let M be a non-zero R-module with filtration

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

where $M_j \neq 0$ for all j in addition to the other requirements for an object in **R-filt**. This module M can also be considered as an object of **R-filt** with the trivial filtration. That is,

$$M = M \supset 0 \supset 0 \supset \dots$$

has $M_0 = M$ and $M_j = 0$ for all $j \neq 0$. Let A be the object in **R-filt** with base module M and trivial filtration and B be the object in **R-filt** with base module M and the non-trivial filtration described above. The R-module homomorphism id_M can be thought of as a morphism in $\mathrm{Hom}(A, B)$ that is bijective. If id_M has an inverse in **R-filt**, then that inverse must also be the identity map on M (as an R-module). However, if $g : B \to A$ is such a map, then $g(B_j) = 0$ for all $j \neq 0$ because of the filtration agreement requirement. Therefore, $\mathrm{id}_M : A \to B$ is not an isomorphism in **R-filt** even though it is a bijection. □

If one considers a strict bijective morphism in **R-filt**, then it is true that this map is an isomorphism. This provides a situation similar to those in the category **Set** and **R-mod**.

Lemma 5.3.11. *In the category **Sets**, bijective morphisms and isomorphisms are the same.*

Lemma 5.3.12. *An R-homomorphism (i.e. morphism in **R-mod**) $\varphi : M \to N$ is bijective if and only if φ is an isomorphism in **R-mod**.*

Lemma 5.3.13. *A bijective strict morphism in **R-filt** is an isomorphism in **R-filt**.*

Proof. Let $f : M \to N$ be a bijective morphism in **R-filt**. Then from above we know that f has an inverse in **R-mod**, say $g : N \to M$, and I claim that this inverse is actually in **R-filt** when f is strict. Recall that f strict means $f(M_j) = f(M) \cap N_j$ and when f is also surjective this implies $f(M_j) = N_j$.

In order for g to be in **R-filt**, I must show that $g(N_j) \subset M_j$ for all j. Since $N_j = f(M_j)$ we have $g(N_j) = g(f(M_j))$. But g is the inverse of f in **R-mod** (and also in **Sets**) so $g(f(M_j)) = M_j$. Thus, $g(N_j) = M_j$ and therefore g is a morphism in **R-filt**. In fact, g is a strict morphism. □

Definition 5.3.14. By an *exact sequence* in **R-filt** I mean a sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where A, B, and C are objects in **R-filt**; f is an injective morphism in **R-filt**; and g is a surjective morphism in **R-filt**.

Definition 5.3.15. By a *strict exact sequence* in **R-filt** I mean a sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where A, B, and C are objects in **R-filt**; f is a strict injective morphism in **R-filt**; and g is a strict surjective morphism in **R-filt**.

Lemma 5.3.16. *If*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

*is a strict exact sequence in **R-filt** with $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ as the respective filtrations of A, B, and C, then there exists an induced exact sequence*

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

*in **R-mod** for each n.*

Proof. For each n we can restrict the domains of f and g to A_n and B_n respectively. Since f and g are morphisms in **R-filt** we know that $f(A_n) \subset B_n$ and $g(B_n) \subset C_n$ for all n. So we can then corestrict the codomains of the already restricted maps to B_n and C_n respectively for each n. Therefore, for each n define $f_n : A_n \to B_n$ as f with domain restricted to A_n and codomain corestricted to B_n. In particular, f_n agrees with f on A_n. Define $g_n : B_n \to C_n$ similarly.

Now, if $f_n(a_1) = f_n(a_2)$, then $f(a_1) = f(a_2)$ because f_n agrees with f, and hence $a_1 = a_2$ because f is injective. Thus, f_n is injective. Also, $g_n(B_n) = g(B_n) = g(B) \cap C_n = C \cap C_n = C_n$ so we see g_n is surjective. Since f_n and g_n agreed with f and g respectively, it is easy to see that $g_n(f_n(a)) = 0$ for all $a \in A_n$. Finally, if $x \in \ker(g_n) \subset B_n$ then $g_n(x) = g(x) = 0$. Thus, x is also in $\ker(g) = f(A)$. Therefore, $x \in f(A) \cap B_n = f(A_n)$ because f is strict. Hence, the sequence is exact. □

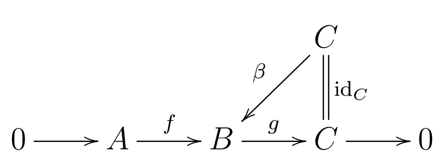
Strict exact sequences in **R-filt** lead to results similar to the standard splitting lemmas in **R-mod**. Note that the next two lemmas use the notion of a direct sum in **R-filt**. The direct sum of two objects M and N in **R-filt** is the usual module direct sum $M \oplus N$ with filtration $(M \oplus N)_j = M_j \oplus N_j$. Direct sums are discussed in depth in Section 5.4.

Lemma 5.3.17. *Consider the strict exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

*in **R-filt**. If there exists a map $\beta : C \to B$ in **R-filt** such that $g \circ \beta = \mathrm{id}_C$, then $B = A \oplus C$ in **R-filt**.*

Proof. Consider the diagram



with strict exact row. Furthermore, this diagram implies that the row is a split exact sequence in **R-mod**. So, $B = A \oplus C$. However, we want B to be a direct sum in **R-filt**, not just in **R-mod**. To see this, we use Lemma 5.3.16 to construct a diagram of the form

$$\begin{array}{ccccccccc} & & & & & & C_n & & \\ & & & & & \overset{\beta_n}{\swarrow} & \Big\| \mathrm{id}_{C_n} & & \\ 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \end{array}$$

in which the row is exact. The maps f_n, g_n, and β_n are defined by appropriately restricting the domains and corestricting the codomains of f, g, and β respectively. Therefore, this sequence also splits in **R-mod** which implies that $B_n = A_n \oplus C_n$. That is, $B = A \oplus C$ in **R-filt**. □

Lemma 5.3.18. *Consider the strict exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

*in **R-filt**. If there exists a map $\alpha : B \to A$ in **R-filt** such that $\alpha f = \mathrm{id}_A$, then $B = A \oplus C$ in **R-filt**.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \mathrm{id}_A \Big\| & \underset{\alpha}{\swarrow} & & & & & \\ & & A & & & & & & \end{array}$$

with strict exact row in **R-filt**. This diagram implies that the row is a split exact sequence in **R-mod**. So, $B = A \oplus C$. However, we want B to be a direct sum in **R-filt**, not just in **R-mod**. To see this, we use Lemma 5.3.16 to construct a diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\ & & \mathrm{id}_{A_n} \Big\| & \underset{\alpha_n}{\swarrow} & & & & & \\ & & A_n & & & & & & \end{array}$$

in which the row is exact. The maps f_n, g_n, and α_n are defined by appropriately restricting the domains and corestricting the codomains of f, g, and α respectively. Therefore, this sequence also splits in **R-mod** which implies that $B_n = A_n \oplus C_n$. That is, $B = A \oplus C$ in **R-filt**. □

Some morphisms in **R-filt** can be described using the following matrix notation. Let

$$M = M^0 \times M^1 \times M^2 \times \dots \text{ and}$$
$$N = N^0 \times N^1 \times N^2 \times \dots$$

with the usual filtration and $f : M \to N$ be a morphism in **R-filt**. One can give a matrix representation of f as follows. Let x be an element of M and write x as the vector

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

where $x_i \in M^i$ for $i \geq 0$. Then let $f_{ji} : M^i \to N^j$ be f restricted to M^i and corestricted to N^j. The morphism f can then be represented as the $\mathbf{N} \times \mathbf{N}$ lower triangular matrix

$$\begin{bmatrix} f_{00} & 0 & 0 \\ f_{10} & f_{11} & 0 \\ \vdots & \vdots & \ddots \end{bmatrix}$$

where evaluating $f(x)$ works just like multiplying a matrix times a vector and composing two functions works like multiplying two matrices except we are using functional composition instead of multiplication.

Lemma 5.3.19. *A morphism $f : M \to N$ written in this matrix form is an isomorphism if and only if f_{ii} is an isomorphism for all i.*

Proof. Suppose each $f_{ii} : M^i \to N^i$ is an isomorphism. Let $g_{ii} : N^i \to M^i$ be the inverse of f_{ii} for each $i \geq 0$. Then we can compute the inverse of f similar to how one might find the inverse of any non-singular matrix. Suppose $g : N \to M$ is a morphism of this kind represented by the $\mathbf{N} \times \mathbf{N}$ matrix

$$\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ g_{20} & g_{21} & g_{22} & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

If g is going to be the inverse of f then it must be true that $f_{10}g_{00} + f_{11}g_{10} = 0$. Thus, $g_{10} = f_{11}^{-1}(-f_{10}g_{00})$. For all other off-diagonal entries, a similar procedure gives g_{ji}

for $j > i$. Then it is not hard to check that $g \circ f$ is represented by the matrix

$$\begin{bmatrix} g_{00} \circ f_{00} & 0 & 0 \\ 0 & g_{11} \circ f_{11} & 0 \\ 0 & 0 & \ddots \end{bmatrix} = \mathrm{id}_M$$

and $f \circ g$ is represented by the matrix

$$\begin{bmatrix} f_{00} \circ g_{00} & 0 & 0 \\ 0 & f_{11} \circ g_{11} & 0 \\ 0 & 0 & \ddots \end{bmatrix} = \mathrm{id}_N$$

Therefore, f and g are inverses. Notice that, f and g are actually inverses in **R-filt** because both are represented by lower triangular matrices meaning that they are morphisms in **R-filt** satisfying the filtration requirements $f(M_j) \subset N_j$ and $g(N_j) \subset M_j$.

Conversely, suppose $f : M \to N$ is an isomorphism in **R-filt**. Then there exists another morphism, $g : N \to M$, in **R-filt** such that $f \circ g = \mathrm{id}_N$ and $g \circ f = \mathrm{id}_M$. But the diagonal entries of these compositions are $f_{ii} \circ g_{ii}$ and $g_{ii} \circ f_{ii}$ respectively. Also, the diagonal entries of id_N and id_M are $\mathrm{id}_{N_{ii}}$ and $\mathrm{id}_{M_{ii}}$ respectively. Therefore, f_{ii} and g_{ii} are inverses of each other as well. Thus, we have shown that a morphism $f : M \to N$ in **R-filt** is an isomorphism if and only if each $f_{ii} : M^i \to N^i$ is an isomorphism. □

5.4 Products and Coproducts in R-filt

Proposition 5.4.1. *Suppose that M and N are objects in **R-filt**. Then the direct product $M \times N$ in **R-mod** given the filtration $(M \times N)_j = M_j \times N_j$ is a product in **R-filt** in the categorical sense. That is, there exists morphisms $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ such that for any W in **R-filt** and any $f_1 : W \to M$ and $f_2 : W \to N$ there is a unique morphism $f : W \to M \times N$ such that $p_1 f = f_1$ and $p_2 f = f_2$.*

Proof. First note that $M \times N$ is both Hausdorff and complete. The maps p_1 and p_2 are the usual projection maps defined by $p_1(m, n) = m$ and $p_2(m, n) = n$ which are clearly maps in **R-filt** with the given filtrations. Then define a map $f : W \to M \times N$ by $f(w) = (f_1(w), f_2(w))$. This is a morphism in **R-filt** because both f_1 and f_2 are morphisms in **R-filt**. Clearly $p_1 f = f_1$ and $p_2 f = f_2$. Finally note that f is unique because if g were another such map then $p_2 f = f_2 = p_2 g$ implies $g = f$ since p_2 is a surjection (and hence an epimorphism). □

Proposition 5.4.2. *If $\{M^i\}$ is a family of objects in **R-filt** then the direct product $\prod M^i$ in **R-mod** with filtration $(\prod M^i)_j = \prod M^i_j$ is the categorical product in **R-filt**.*

Proof. Suppose $\{M^i\}$ is a family of objects in **R-filt**. Consider the direct product $\prod M^i$ in **R-mod** with filtration $(\prod M^i)_j = \prod M^i_j$. The argument above showing that $M \times N$ is a categorical product generalizes to the arbitrary case with no problem. The only question is whether or not $\prod M^i$ is Hausdorff and complete. If $(m^0, m^1, m^2, \ldots)$ is an element of $\cap(\prod M^i)_j$ then each m^i is in $\cap M^i_j$ which equals 0 because each M^i is Hausdorff. Unlike the coproduct case, there is no problem with completeness when generalizing the index set.

Suppose $((m^i_j)_{i\in I})_{j=0}^{\infty}$ is a cauchy sequence in $\prod M^i$. Then each component sequence $(m^i_j)_{j=0}^{\infty}$ is cauchy in M^i. Thus, the original sequence converges to $(n^i)_{i\in I}$ where n^i is the limit of the sequence $(m^i_j)_{j=0}^{\infty}$. Hence, $\prod M^i$ is complete. □

Coproducts (direct sums) in **R-filt** play an important role in the characterization of strict projective modules later this chapter. Here I will discuss direct sums in **R-filt** in both the finite and arbitrary cases.

Proposition 5.4.3. *Suppose that M and N are objects in **R-filt**. Then the direct sum $M \oplus N$ in **R-mod** given the filtration $(M \oplus N)_j = M_j \oplus N_j$ is a direct sum in **R-filt** in the categorical sense. That is, there exists morphisms*

$$e_1 : M \to M \oplus N \text{ and } e_2 : N \to M \oplus N$$

*in **R-filt** that satisfy the following universal property: If $f_1 : M \to L$ and $f_2 : N \to L$ are two morphisms in **R-filt**, then there exists a unique morphism $f : M \oplus N \to L$ in **R-filt** such that $f \circ e_i = f_i$ for $i = 1, 2$.*

Proof. The maps e_1 and e_2 are the obvious maps in **R-filt** defined by $e_1(m) = (m, 0)$ and $e_2(n) = (0, n)$. Suppose that f_1 and f_2 are maps as described in the statement of the proposition. Then define the map $f : M \oplus N \to L$ by

$$f(m, n) := f_1(m) + f_2(n).$$

If $(m, n) \in (M \oplus N)_j = M_j \oplus N_j$, then $m \in M_j$ and $n \in N_j$. Therefore $f_1(m) \in L_j$ and $f_2(n) \in L_j$ because f_1 and f_2 are morphisms in **R-filt**. Thus, $f(m, n) = f_1(m) + f_2(n) \in L_j$ and hence $f((M \oplus N)_j) \subset L_j$ so that f is also a morphism in **R-filt**. Clearly, $f \circ e_i = f_i$ for $i = 1, 2$.

For uniqueness, suppose there exists a map g in **R-filt** such that $g \circ e_i = f_i$ for $i = 1, 2$. Then

$$\begin{aligned} f(m,n) &= f_1(m) + f_2(n) \\ &= g(e_1(m)) + g(e_2(n)) \\ &= g(m,0) + g(0,n) \\ &= g(m,n). \end{aligned}$$

Thus, $f = g$ and the map defined above is unique. □

It is important to note that the above argument requires that the module $M \oplus N$ be both Hausdorff and complete when using the given filtration. If one uses this type of filtration for any direct sum then the Hausdorff condition is satisfied. On the other hand, the completeness requirement only holds in certain situations as the next proposition shows.

Proposition 5.4.4. *Let I be any index set and M^i be an object in **R-filt** for each $i \in I$. The sum $\bigoplus_{i \in I} M^i$ with the filtration*

$$\left(\bigoplus_{i \in I} M^i \right)_n = \bigoplus_{i \in I} M^i_n$$

is complete if and only if $M^i = 0$ except for a finite number of $i \in I$.

Proof. Suppose that $M^i = 0$ except for a finite number of $i \in I$. Then we can enumerate the M^i such that

$$\bigoplus_{i \in I} M^i = M^{i_0} \oplus M^{i_1} \oplus \ldots \oplus M^{i_k}$$

for a finite positive integer k. Let $\{\bar{m}_j\}_{j=0}^{\infty} = \{(m_j^{i_0}, m_j^{i_1}, \ldots, m_j^{i_k})\}_{j=0}^{\infty}$ be a Cauchy sequence in $\bigoplus_{i \in I} M^i$. Then with the filtration we chose for $\bigoplus_{i \in I} M^i$ it is not hard to see that the component sequences

$$\{m_j^{i_0}\}, \{m_j^{i_1}\}, \ldots$$

are Cauchy in $M^{i_0}, M^{i_1}, \ldots$ respectively. Each M^i is in **R-filt** and therefore complete. Thus, each component sequence converges to an element, say m^{i_l}, in the respective M^{i_l}. Putting these all together gives the element $(m^{i_0}, m^{i_1}, \ldots, m^{i_k})$ in $\bigoplus_{i \in I} M^i$ to which the original Cauchy sequence converges.

Conversely suppose that infinitely many of the M^i we started with are non-zero. We wish to construct a Cauchy sequence in $\bigoplus_{i\in I} M^i$ that converges to an element outside of $\bigoplus_{i\in I} M^i$. Since there are infinite number of $M^i \neq 0$, we can choose a countably infinite subset of these M^i, enumerated as $M^{i_0}, M^{i_1}, \ldots$, to work with. We may also reorder the indicies of the direct sum so that these M^{i_k} come first.

Then we choose a Cauchy sequence $\{m_j^{i_l}\}$ in each of the chosen $M^{i_l} \neq 0$ that converges to a non-zero element. Next, choose a subsequence $\{m_{j_k}^{i_l}\}$ of each $\{m_j^{i_l}\}$ which is still Cauchy and such that $m_{j_k}^{i_l}$ is an element for which $m_{j_a}^{i_l} - m_{j_b}^{i_l} \in M_k^{i_l}$ for all $a, b \geq k$. Finally, we define a sequence $\{\overline{m_k}\}$ in $\bigoplus_{i\in I} M^i$ where $\overline{m_k}$ is the element in $\bigoplus_{i\in I} M^i$ that has $m_{j_k}^{i_l}$ in the first k places corresponding to $M^{i_l} \neq 0$ and zeros elsewhere. That is,

$$\overline{m_k} = (m_{j_k}^{i_1}, m_{j_k}^{i_2}, \ldots, m_{j_k}^{i_k}, 0, \ldots).$$

This sequence is Cauchy by construction, but it converges to an element that has an infinite number of non-zero entries. Thus, this sequence does not converge in $\bigoplus_{i\in I} M^i$ and hence $\bigoplus_{i\in I} M^i$ is not complete. □

Since the above filtration does not always make the usual direct sum a complete module then one may ask how you could get a categorical direct sum of an arbitrary number of modules in this setting. The following module will be a candidate for such a direct sum.

Definition 5.4.5. Suppose $\{M^i\}_{i\in I}$ is a collection of objects in **R-filt** where I is an arbitrary index set. Let $\overline{M} \subset \prod_{i\in I} M^i$ consist of all $(x_i)_{i\in I}$ such that at most countably many $x_i \neq 0$ and if there are countably many $x_i \neq 0$, then if we arrange them in a sequence $x_{i_0}, x_{i_1}, \ldots$ (so i_0, i_1, … are all distinct) then given any $n \geq 0$ we have $x_{i_j} \in (M^{i_j})_n$ for all but a finite number of the indices i_j. Define the filtration on $\overline{M}$ by letting $(\overline{M})_n$ be the set of all $(x_i)_{i\in I} \in \overline{M}$ as above but with the added restriction that $x_{i_j} \in (M^{i_j})_n$ for all indices i_j.

If this module, $\overline{M}$, is going to be the categorical direct sum of $\{M^i\}$ then $\overline{M}$ must be complete, $\overline{M}$ must be a categorical sum (i.e. it must satisfy an analogous universal property as in Proposition 5.4.3), and $\overline{M}$ should be the completion of the usual direct sum (with the filtration given in Proposition 5.4.4) so that this sum agrees with the sum in the finite case.

Proposition 5.4.6. $\overline{M}$ *is an object in* ***R-filt***. *That is,* $\overline{M}$ *is Hausdorff and complete in the given filtration.*

Proof. $\overline{M}$ being Hausdorff follows easily by noting that every component of an element in $\cap_{n=0}^{\infty}(\overline{M})_n$ is in $(M^{i_j})_n$ for all n. Therefore, every component is 0 because each M^{i_j} is Hausdorff individually.

Let $\{(x_i)_k\}$ be a Cauchy sequence in $\overline{M}$. Each (x_i) in the sequence need not have the same indices for which $x_i \neq 0$, but since there are a countable number of elements in the sequence and each one has at most countably many non-zero entries then there are still at most a countable number of elements in I for which $x_i \neq 0$ for some (x_i) in the given sequence. Enumerate these elements of I as $i_0, i_1, i_2, \ldots$.

Now, $\{(x_i)_k\}$ being Cauchy means for every $n \geq 0$ there exists an integer N_n such that $(x_i)_a - (x_i)_b \in (\overline{M})_n$ for all $a, b \geq N_n$. Therefore, the component sequences corresponding to $i_0, i_1, i_2, \ldots$ are Cauchy in $M^{i_0}, M^{i_1}, M^{i_2}, \ldots$ respectively. Each M^{i_j} is complete so each component sequence converges. Let $\{(\bar{x}_i)\}$ be the element in $\overline{M}$ consisting of the limit of each component sequence at the appropriate index. Then $\{(x_i)_k\} \to (\bar{x}) \in \overline{M}$. □

Proposition 5.4.7. $\overline{M}$ *is a categorical sum (in* ***R-filt****).*

Proof. All modules and maps in this argument are assumed to be in **R-filt**.

To prove $\overline{M}$ is a categorical sum we must show that the obvious maps $e_i : M^i \to \overline{M}$ satisfy the universal property that given maps $f_i : M^i \to N$ for all $i \in I$ there exists a unique map $f : \overline{M} \to N$ such that $f \circ e_i = f_i$ for all i.

Let $(x_i) \in \overline{M}$ and enumerate the indices for which $x_i \neq 0$ as $i_0, i_1, i_2, \ldots$. Then for every $n \geq 0$, $f_{i_j}(x_{i_j}) \in N_n$ for all but finitely many indices i_j because each f_i is a morphism in **R-filt**. Thus the map $f : \overline{M} \to N$ defined by $f((x_i)) := \sum f_{i_j}(x_{i_j})$ is well-defined because N is complete.

Suppose there is another map $g : \overline{M} \to N$ satisfying $g \circ e_i = f_i$ for all i. Then

$$f((x_i)) = \sum f_{i_j}(x_{i_j}) = \sum g(e_{i_j}(x_{i_j})) = g\left(\sum e_{i_j}(x_{i_j})\right) = g((x_i))$$

shows that $f = g$. □

Proposition 5.4.8. $\overline{M}$ *is the completion of the usual direct sum* $M = \bigoplus_{i \in I} M^i$ *with filtration* $M_n = \bigoplus_{i \in I} M^i_n$.

Proof. Let $\{(x_i)\}$ be a Cauchy sequence in M. Each (x_i) has finitely many non-zero terms and there are countably many terms in the sequence, so in total there are a countable number of indices in I that correspond to a non-zero entry in one of the elements of the sequence. Enumerate these as $i_0, i_1, i_2, \ldots$.

Consider the sequences consisting of the entries in each of these positions in their respective M^{i_j}. Each of these sequences is Cauchy because the original sequence is Cauchy. Therefore each sequence converges in its respective M^{i_j}. Putting these together gives an element, $\bar{x}$, that we wish to be the limit of $\{(x_i)\}$. The issue being that $\bar{x}$ is not necessarily in M. However, $\bar{x}$ is the limit of the given sequence in $\prod M^i$ given the filtration $(\prod M^i)_n = \prod M^i_n$ and having at most countably many non-zero components, $\bar{x}$ may be in $\overline{M}$.

In fact, if we let $n \geq 0$, then $\bar{x}$ being the limit of $\{(x_i)\}$ in $\prod M^i$ implies that there exists an integer N_n such that $\bar{x} - (x_i)_j \in \prod M^i_n$ for all $j \geq N_n$. Choose any $j > N_n$. Then $\bar{x} - (x_i)_j \in \prod M^i_n$. But $(x_i)_j$ has only finitely many non-zero entries. Therefore, $\bar{x} - (x_i)_j$ is the same as $\bar{x}$ in every other position. That is, $\bar{x}_{i_j} \in (M^{i_j})_n$ for all but the finitely many indices for which $(x_i)_j$ is non-zero. Thus, $\bar{x} \in \overline{M}$. □

Remark 5.4.9. Note that while these arbitrary direct sums are different than one might expect, they still preserve projectivity and $\mathcal{H}$-projectivity (see Propositions 4.4.13 and 4.4.14).

The next proposition I will give is one of the results that I will need later to prove a characterization of strict projective modules. This particular result shows that direct summands of a certain type (the type in Example 3.1.6) of object in **R-filt** are of that same type. I will break down the proof of the proposition into a series of lemmas.

Suppose that $M = M^0 \times M^1 \times M^2 \times \ldots$ is filtered by the submodules $M_n = 0 \times \ldots \times 0 \times M^n \times M^{n+1} \times \ldots$ and that M has a direct sum decomposition $M = S \oplus T$ in **R-filt**. Recall that this means $M = S + T$, $S \cap T = 0$ and $M_n = S_n \oplus T_n$ for each $n \geq 0$.

Lemma 5.4.10. *The submodule M_{n+1} is a direct summand (in **R-mod**) of the submodule M_n.*

Proof. Consider the map $e : M_n \to M_n$ given by

$$e(0, \ldots, 0, m^n, m^{n+1}, \ldots) = (0, \ldots, 0, m^{n+1}, \ldots).$$

Then $\mathrm{Im}(e) = M_{n+1}$ and e is idempotent. Thus, M_{n+1} is a direct summand of M_n. □

Lemma 5.4.11. *S_{n+1} is a direct summand (in **R-mod**) of S_n and T_{n+1} is a direct summand (in **R-mod**) of T_n for each $n \geq 0$.*

Proof. Since M_{n+1} is a direct summand of M_n we can write $M_n = M_{n+1} \oplus K$ for some module K. But also $M_{n+1} = S_{n+1} \oplus T_{n+1}$. So $M_n = S_{n+1} \oplus T_{n+1} \oplus K$. Let $e : S_n \to S_n$ be the map that takes an element $x \in S_n$, thought of as an element of M_n, to the projection of x onto S_{n+1} using the direct sum decomposition $M_n = S_{n+1} \oplus T_{n+1} \oplus K$. Then, similar to the above, $\mathrm{Im}(e) = S_{n+1}$ and e is idempotent so that S_{n+1} is a direct summand of S_n. The proof to show T_{n+1} is a direct summand of T_n is similar. □

Now there are modules $S^0, S^1, \ldots$ and $T^0, T^1, \ldots$ such that $S_n = S^n \oplus S_{n+1}$ and $T_n = T^n \oplus T_{n+1}$ for all $n \geq 0$.

Define the modules

$$\bar{S} := S^0 \times S^1 \times S^2 \times \ldots$$

and

$$\bar{T} := T^0 \times T^1 \times T^2 \times \ldots .$$

These will be filtered as usual when we consider them as objects in **R-filt**.

There are natural maps

$$\sigma_S : \bar{S} \to S \text{ defined by } (s_0, s_1, \ldots) \mapsto \sum s_i$$

and

$$\sigma_T : \bar{T} \to T \text{ defined by } (t_0, t_1, \ldots) \mapsto \sum t_i$$

where both sums are well-defined because S^n can be thought of as a submodule of S_n, T^n can be thought of as a submodule of T_n, and S and T are complete. In fact,

Lemma 5.4.12. *σ_S and σ_T are isomorphisms when we consider S, T, $\bar{S}$, and $\bar{T}$ as filtered modules.*

Proof. Let $s \in S$ and decompose s using the direct sum $S_0 = S^0 \oplus S_1$. Inductively continue this process to get elements, s_n, in each S^n for $n \geq 0$ that sum to s. Then the element $(s_0, s_1, s_2 \ldots)$ maps to s via σ_S.

In fact, the direct sum decompositions are unique, so each element of S comes from a unique element of $\bar{S}$. Thus, σ_S is also injective. The proof for σ_T is similar. □

Then we get an isomorphism

$$\bar{S} \times \bar{T} \to S \oplus T = M$$

i.e. essentially

$$(S^0 \times T^0) \times (S^1 \times T^1) \times \ldots \cong M^0 \times M^1 \times \ldots$$

This gives that direct summands of an $M = M^0 \times M^1 \times \ldots$ are of the same form. That is,

Proposition 5.4.13. *Direct summands of modules of the form*

$$M^0 \times M^1 \times M^2 \times \ldots$$

with filtration $M_n = 0 \times \ldots \times 0 \times M^n \times M^{n+1} \times \ldots$ *are of the form*

$$S^0 \times S^1 \times S^2 \times \ldots$$

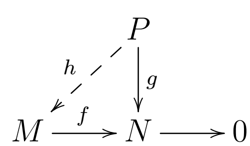
with filtration $S_n = 0 \times \ldots \times 0 \times S^n \times S^{n+1} \times \ldots$.

5.5 Projective and Injective Objects in R-filt

To begin talking about homological algebra in a category, one must first discuss the projective and injective objects in that category. This section will briefly discuss the projective and injective objects of **R-filt**.

The usual definition of a projective object translated to **R-filt** is the following:

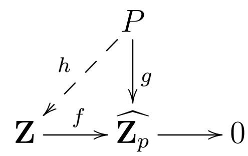
Definition 5.5.1. An object P in **R-filt** is said to be a *projective object* if for every morphism $g : P \to N$ and every epimorphism $f : M \to N$ the diagram



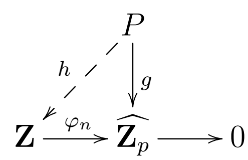
can be completed to a commutative diagram in **R-filt**.

However, these objects are not as easy to deal with as the projective modules in **R-mod** because the epimorphisms in **R-filt** are the maps with dense image, not the surjective maps. In order to be able to define a lifting I would then need to require that $g(P) \subset f(M)$. As examples of how things could potentially work out consider the following:

Example 5.5.2. Consider the case where $R = \mathbf{Z}$ and $N = \widehat{\mathbf{Z}_p}$, the ring of p-adic integers. The ring of integers $\mathbf{Z}$ is dense in $\widehat{\mathbf{Z}_p}$ so if P is projective in **R-filt** then being able to complete the diagram

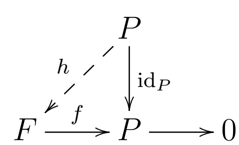


means that $g(P) \subset \mathbf{Z}$. Note that the element $u_n = 1 + p^n$ is a unit of $\widehat{\mathbf{Z}_p}$ for all $n \geq 0$ and hence the maps $\varphi_n : \widehat{\mathbf{Z}_p} \to \widehat{\mathbf{Z}_p}$ defined by $\varphi_n(x) = u_n x$ are group isomorphisms. Also, these φ_ns are isometries using the p-adic metric. Therefore, $u_n\mathbf{Z} = (1 + p^n)\mathbf{Z} \subset u_n\widehat{\mathbf{Z}_p} = \widehat{\mathbf{Z}_p}$ is dense because $\mathbf{Z} \subset \widehat{\mathbf{Z}_p}$ is dense. Then if P is projective it is possible to complete the diagrams



for all $n \geq 0$ and thus $g(P) \subset (1 + p^n)\mathbf{Z}$ for all $n \geq 0$. But if $m \neq 0$ is in $(1 + p^n)\mathbf{Z}$ then $|m| > 1 + p^n$. Thus $\cap_{n=0}^{\infty}(1 + p^n)\mathbf{Z} = 0$. Therefore, if P is projective in this setting then $g(P) = 0$ and the lifting h must be the zero map. □

Example 5.5.3. Suppose P is projective in **R-filt** with $R = \mathbf{Z}$. Consider

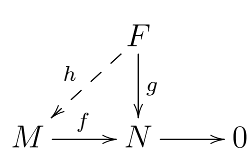


where F is the free module on the set P with the trivial filtration and where f is the usual surjection. Then the lifting h will be an injection and hence P is free as a $\mathbf{Z}$-module. Also note that P will have the trivial filtration because F does. □

In order to conclude that P is free above one really only needs that the ring R is a PID. That is, the previous example actually says the following about projectives in **R-filt**:

Lemma 5.5.4. *Suppose R is a PID. If P is projective in **R-filt**, then P is a free module and has the trivial filtration.*

Conversely, if an object F in **R-filt** is free with the trivial filtration, then in order to be projective we must be able to complete a diagram of the form



whenever $f(M)$ is dense in N. This would require again that $g(F) \subset f(M)$ for all f with dense image, g any morphism, and M any object of **R-filt**. While being rather restrictive, this does give the following:

Lemma 5.5.5. *Let R be a PID. An object P in **R-filt** is projective if and only if both*

1. *$g(P) \subset f(M)$ for any object M in **R-filt**, any epimorphism $f : M \to N$, and any morphism $g : P \to N$; and*

2. *P is free and has the trivial filtration.*

Since the projective objects are not easy to work with, one could instead discuss projective objects in relation to *surjections* rather than epimorphisms. Let $\mathcal{S}$ be the set of all surjections in **R-filt**. Then the $\mathcal{S}$-projective objects in **R-filt** can be characterized in the following way.

Lemma 5.5.6. *The $\mathcal{S}$-projective objects in **R-filt** are exactly the projective modules in **R-mod** with the trivial filtration.*

Proof. Let P be an object in **R-filt** such that the underlying module is projective in **R-mod** and the filtration on P is the trivial one. Then the diagram

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle g} & & \\ M & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

can be completed in **R-mod** when $f \in \mathcal{S}$. Note that $h(P) \subset M$ and $h(P_n) \subset M_n$ for all $n > 0$ because P has the trivial filtration. Thus the $h : P \to M$ which completes the diagram in **R-mod** actually completes the diagram in **R-filt** too since h satisfies the filtration agreement condition imposed on all morphisms in **R-filt**.

Now suppose that N is an object in **R-filt** with a non-trivial filtration. Let M be the object in **R-filt** with the same base module as N, but with the trivial filtration. Then consider the diagram

$$\begin{array}{ccccc} & & N & & \\ & \swarrow & \Vert & & \\ M & \xrightarrow{id} & N & \longrightarrow & 0 \end{array}$$

Clearly the identity map $\mathrm{id}_{MN} : M \to N$ is surjective, and the only way to complete this diagram is to use the $\mathrm{id}_{NM} : N \to M$ as the dashed map. However, the identity map id_{NM} is not a morphism in **R-filt** because $\mathrm{id}(N_n) \not\subset M_n$ for at least one $n > 0$ using the given filtrations. $\square$

This lemma shows that the projective modules in **R-mod** and the $\mathcal{S}$-projective objects in **R-filt** are essentially the same. Therefore, using these objects to build up the usual tools of homological algebra will not result in anything different than the usual case. In the next section I will consider projective objects with respect to an even more restrictive class of morphisms to investigate how this changes the usual results.

Definition 5.5.7. An object E in **R-filt** is said to be an *injective object* if for every morphism $g : M \to E$ and every monomorphism (i.e. injection) $f : M \to N$ the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{f} & N \\ & & {\scriptstyle g}\downarrow & \swarrow & \\ & & E & & \end{array}$$

can be completed to a commutative diagram in **R-filt**.

Lemma 5.5.8. *Let E be an injective object in **R-filt**. Then $E = 0$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \xrightarrow{\mathrm{id}_E} & T(-1)(E) \\ & & {\scriptstyle \mathrm{id}_E}\downarrow & \swarrow & \\ & & E & & \end{array}$$

which can be completed by assumption and where T is the shift functor described in Example 4.5.6. Then looking at the index $n = 1$ of the filtrations one can see that $E_1 \subset E_0 = E$ from the top row and $E = E_0 \subset E_1$ from the dashed map. A similar argument using the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E & \xrightarrow{\mathrm{id}_E} & T(-n)(E) \\ & & {\scriptstyle \mathrm{id}_E}\downarrow & \swarrow & \\ & & E & & \end{array}$$

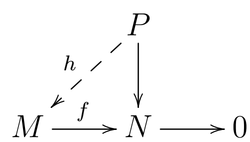
shows that $E = E_1 = E_2 = \ldots = E_n$ for all $n > 0$. Now since all modules in **R-filt** are Hausdorff, we must have that $\cap E_n = 0$. In this case $\cap E_n = E$ and thus, $E = 0$. $\square$

This lemma shows that the only injective object in **R-filt** is the trivial 0 module. Therefore, the usual notions of injective resolutions and injective envelopes can not be defined in a way that makes sense. In section 5.8, I will consider injective objects with respect to a more restrictive class of morphisms and show that this leads to analogues to the usual injective objects.

5.6 Strict Projective Objects

In this section I will discuss projectivity with respect to strict morphisms instead of all morphisms in the category **R-filt**. One reason for doing this is to look for something more interesting than the categorical projective objects in **R-filt** which were discussed last section. The main result of this section is a characterization of strict projective modules.

Definition 5.6.1. A module P in the category **R-filt** is called a **strict projective** module if we can complete a diagram of the form



to a commutative diagram whenever f is a strict surjection. These are the so-called $\mathcal{H}$-projectives where $\mathcal{H}$ is the set of all strict surjective morphisms in **R-filt**.

Remark 5.6.2. Any projective object in **R-filt** is a strict projective module because every surjection is an epimorphism. Similarly, every projective module in **R-mod** with the trivial filtration is a strict projective module.

Example 5.6.3. Consider the module $\mathbf{Z}^\omega = \mathbf{Z} \times \mathbf{Z} \times \ldots$ with filtration

$$\mathbf{Z}^\omega_n = 0 \times \ldots \times 0 \times \mathbf{Z} \times \mathbf{Z} \times \ldots$$

where there are n zeros. This module is not projective in **R-mod** (i.e. not $\mathcal{S}$-projective in **R-filt**) with $R = \mathbf{Z}$ because projective and free are equivalent when R is a PID and $\mathbf{Z}^\omega$ is not a free $\mathbf{Z}$-module. To see that $\mathbf{Z}^\omega$ is not a free $\mathbf{Z}$-module first suppose $\mathbf{Z}^\omega$ is free. Note that $\mathbf{Z}^\omega$ is uncountable. This can be shown using a diagonal argument similar to that in Cantor's proof that the real numbers are uncountable. Therefore, if $\mathbf{Z}^\omega$ is free, then the basis for $\mathbf{Z}^\omega$ must be uncountable. But then the dual group $(\mathbf{Z}^\omega)^*$ would be uncountable. However, Specker's theorem (Proposition 3.3.3) implies that $(\mathbf{Z}^\omega)^* \cong \mathbf{Z}^{(\omega)}$ which is countable. Thus, $\mathbf{Z}^\omega$ can not be free over $\mathbf{Z}$.

The module $\mathbf{Z}^\omega = \mathbf{Z} \times \mathbf{Z} \times \ldots$ is however a strict projective module. Therefore, restricting to the class of strict surjections rather than either the set of epimorphisms or the set of surjective morphisms results in an increase in the number of projective objects.This is actually a special case of the next proposition which shows that any direct product of projective modules in **R-mod**, with a certain filtration, is strictly projective.

Proposition 5.6.4. *Let P^0, P^1, P^2, ... be any sequence of projective modules in **R-mod**. Let*

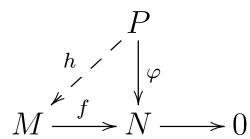
$$P = P^0 \times P^1 \times P^2 \times \dots$$

and filter P by the submodules

$$P_n = 0 \times \dots \times 0 \times P^n \times P^{n+1} \times \dots$$

Then P is strictly projective.

Proof. Consider the diagram

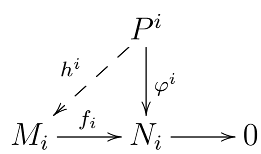


where f is a strict surjection. We wish to define h so that the diagram is commutative.

Since φ and f are morphisms in **R-filt**, we know that $\varphi(P_i) \subset N_i$ and $f(M_i) \subset N_i$. In fact, $f(M_i) = N_i$ because f is a strict surjection. Therefore, we can define two new maps,

1. $\varphi^i : P^i \to N_i$ by $\varphi^i(p_i) := \varphi(0, \dots, 0, p_i, 0, \dots)$
2. $f_i : M_i \to N_i$ by $f_i(m_i) := f(m_i)$

Note that f_i is surjective because f is a strict surjection. Therefore, we can complete the diagram



in **R-mod** for each $i \geq 0$ because P^i is projective and f_i is surjective.

Therefore,

$$\begin{aligned}
\varphi(p) = \varphi(p_0, p_1, \dots) &= \sum \varphi(0, \dots, 0, p_i, 0, \dots) \\
&= \sum \varphi^i(p_i) \\
&= \sum f_i(h^i(p_i)) \\
&= \sum f(h^i(p_i)) \\
&= f\left(\sum h^i(p_i)\right).
\end{aligned}$$

So define $h : P \to M$ by $h(p) := \sum h^i(p_i)$ which clearly gives a commutative diagram. Furthermore, each $h^i(p_i) \in M_i$ so that if $p \in P_n$, then $h(p) = \sum_{i \geq n} h^i(p_i) \in M_n$. That is, h is in fact a morphism in **R-filt**, not just in **R-mod**. □

The converse to Proposition 5.6.4 is also true. That is, any strict projective module in **R-filt** can be written as a direct product of projective modules with this type of filtration. In order to prove this result, and therefore characterize strict projective modules, I will first need the results that follow.

Proposition 5.6.5. *There are enough strict projectives in **R-filt**. That is, given an object M in **R-filt** there exists a strict surjection $\Phi : P \to M$ such that P is a strict projective.*

Proof. Let M be any module in **R-filt** with the filtration $M = M_0 \supset M_1 \supset \ldots$. For each M_i one can define a free module on its elements by Proposition 2.3.2 which we will call $F(M_i)$ as follows. The elements of $F(M_i)$ are functions

$$\sigma : M_i \to R$$

such that $\sigma(m_i) = 0$ for all but finitely many $m_i \in M_i$. The functions $\sigma_{m_i} : M_i \to R$ defined by $\sigma_{m_i}(m) = 1$ for $m = m_i$ and $\sigma_{m_i}(m) = 0$ elsewhere form a basis for $F(M_i)$. So we can write any element σ in $F(M_i)$ as a linear combination of these σ_{m_i}. We can then define a map $\Phi_i : F(M_i) \to M_i$ by

$$\Phi_i : \sum_{i=1}^{n} r_i \sigma_{m_i} \mapsto \sum_{i=1}^{n} r_i m_i.$$

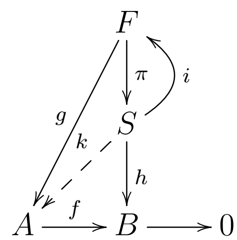
Note that each of the free modules $F(M_i)$ is projective. Let $P = F(M_0) \times F(M_1) \times F(M_2) \times \ldots$ with the usual filtration for a countable direct product. Then by Proposition 5.6.4, we know that P is strictly projective. For $\sigma_j = \sum_{i=1}^{n_j} r_i \sigma_{m_i} \in F(M_j)$ define a map as follows:

$$\Phi : P \to M$$
$$(\sigma_j)_{j=0}^{\infty} \mapsto \sum_{j=0}^{\infty} \Phi_j(\sigma_j).$$

This is well-defined because each $\Phi_i(\sigma_i) \in M_i$ and M is complete. This is also a map in **R-filt** because $\Phi(P_i) \subset M_i$ since each $\Phi_i(\sigma_i) \in M_i$ and each M_i is closed. The map is surjective because the elements $(\sigma_m, 0, 0, \ldots) \mapsto m$. Finally, Φ is strict because if $\bar{p}$ is in both $\Phi(P)$ and M_i, then $\bar{p} = \Phi(0, \ldots, 0, \sigma_{\bar{p}}, 0, \ldots)$ where $\sigma_{\bar{p}}$ is in the i-th position is in $\Phi(P_i)$. Thus, there are enough strict projective modules in **R-filt**. $\square$

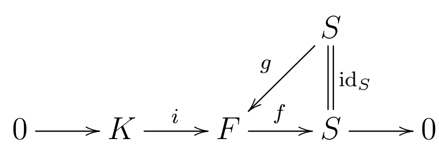
Proposition 5.6.6. *A module $S \in$ **R-filt** is a strict projective module if and only if S is a direct summand (in **R-filt**) of a module of the form $F^0 \times F^1 \times F^2 \times \ldots$ where each F^i is a free R-module. This is analogous to Proposition* 2.3.5 *for **R-mod**.*

Proof. Suppose that S is a direct summand of a module, F, of the given form. Then consider the diagram



where the row is exact, f is strict, π is the projection of F onto the summand S, i is the canonical injection of S into F, and g exists because F is strictly projective. Define the map $k := gi$. Then $fk = fgi = h\pi i = h$ so the diagram is commutative and thus S is strictly projective.

Next suppose that S is a strict projective module. Consider the diagram



where the map f comes from there being enough strict projectives, K is the kernel of f filtered by the submodules $K_n = K \cap F_n$, and g comes from S being strictly projective. By construction, the row in the above diagram is a strict exact sequence. In fact, this diagram also verifies the existence of a section of $f : F \to S$. Hence, by Lemma 5.3.17, S is a direct summand of F in **R-filt**. $\square$

I now have enough to completely characterize the strict projective modules in the following way.

Theorem 5.6.7. *An object $S \in$ **R-filt** is a strict projective module if and only if*

$$S = S^0 \times S^1 \times S^2 \times \dots$$

filtered by the submodules $S_n = 0 \times \dots \times 0 \times S^n \times S^{n+1} \times \dots$ where S^n is projective for all $n \geq 0$.

Proof. I have already shown the reverse direction of this theorem in Proposition 5.6.4. That is, a module of the given form with this particular filtration is a strict projective whenever each S^n is projective.

Conversely, suppose that S is a strict projective. Then S is a direct summand (in **R-filt**) of a module of the form $F^0 \times F^1 \times F^2 \times \dots$ where each F^i is free by Proposition 5.6.6. Even further, S must be of the form

$$S = S^0 \times S^1 \times S^2 \times \dots$$

with filtration

$$S_n = 0 \times \dots \times 0 \times S^n \times S^{n+1} \dots$$

by Proposition 5.4.13 and all that remains to show is that each S^n is projective.

Suppose we have exact sequences

$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

of modules in **R-mod** for each $n \geq 0$. Define A, an object in **R-filt**, as

$$A := \prod_{n=0}^{\infty} A^n$$

filtered by $A_n = 0 \times \dots \times 0 \times A^n \times A^{n+1} \times \dots$ as usual. Define two more objects B and C in **R-filt** similarly as

$$B := \prod_{n=0}^{\infty} B^n$$

filtered by $B_n = 0 \times \dots \times 0 \times B^n \times B^{n+1} \times \dots$ and

$$C := \prod_{n=0}^{\infty} C^n$$

filtered by $C_n = 0 \times \dots \times 0 \times C^n \times C^{n+1} \times \dots$. Next, define the map $f : A \to B$ in **R-filt** by $f((a^n)_{n=0}^{\infty}) := (f^n(a^n))_{n=0}^{\infty}$ and also the map $g : B \to C$ by $g((b^n)_{n=0}^{\infty}) := (g^n(b^n))_{n=0}^{\infty}$. These objects and morphisms are induced by those from the above exact sequences and thus

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is a strict exact sequence.

Let $h^n : S^n \to C^n$ be any linear map for each $n \geq 0$ and define $h : S \to C$ by $h((s^n)_{n=0}^{\infty}) := (h^n(s^n))_{n=0}^{\infty}$. Then consider the diagram

$$\begin{array}{ccccccccc} & & & & & & S & & \\ & & & & & \swarrow^{k} & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

where k exists because S is a strict projective. Now restricting g, h, and k induces the commutative diagrams

$$\begin{array}{ccc} & & S_n \\ & \swarrow^{k_n} & \downarrow{\scriptstyle h_n} \\ B_n & \xrightarrow{g_n} & C_n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & & S_{n+1} \\ & \swarrow^{k_{n+1}} & \downarrow{\scriptstyle h_{n+1}} \\ B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \end{array}$$

These two diagrams then induce the following maps between quotient modules.

1. $$\bar{g}_n : \frac{B_n}{B_{n+1}} \to \frac{C_n}{C_{n+1}}$$
$$b_n + B_{n+1} \mapsto g_n(b_n) + C_{n+1}$$

2. $$\bar{h}_n : \frac{S_n}{S_{n+1}} \to \frac{C_n}{C_{n+1}}$$
$$s_n + S_{n+1} \mapsto h_n(s_n) + C_{n+1}$$

3. $$\bar{k}_n : \frac{S_n}{S_{n+1}} \to \frac{B_n}{B_{n+1}}$$
$$s_n + S_{n+1} \mapsto k_n(s_n) + B_{n+1}$$

Additionally, there are obvious isomorphisms

$$\iota_S^n : S^n \xrightarrow{\cong} \frac{S_n}{S_{n+1}}, \qquad \iota_B^n : B^n \xrightarrow{\cong} \frac{B_n}{B_{n+1}}, \qquad \iota_C^n : C^n \xrightarrow{\cong} \frac{C_n}{C_{n+1}}$$

between the modules in the direct product and quotients of subsequent modules in the filtration. Note that $g^n = (\iota_C^n)^{-1} \circ \bar{g}_n \circ \iota_B^n$ and $h^n = (\iota_C^n)^{-1} \circ \bar{h}_n \circ \iota_S^n$. Therefore, I similarly define a map $k^n : S^n \to B^n$ by $k^n := (\iota_B^n)^{-1} \circ \bar{k}_n \circ \iota_S^n$. With this newly defined map, it follows that $g^n k^n = h^n$ because

$$\begin{aligned} g^n k^n &= ((\iota_C^n)^{-1} \circ \bar{g}_n \circ \iota_B^n)((\iota_B^n)^{-1} \circ \bar{k}_n \circ \iota_S^n) \\ &= (\iota_C^n)^{-1} \circ \bar{g}_n \circ \bar{k}_n \circ \iota_S^n \end{aligned}$$

and

$$\begin{aligned} \bar{g}_n(\bar{k}_n(s_n + S_{n+1})) &= \bar{g}_n(k_n(s_n) + B_{n+1}) \\ &= g_n(k_n(s_n)) + C_{n+1} \\ &= h_n(s_n) + C_{n+1} \\ &= \bar{h}_n(s_n + S_{n+1}). \end{aligned}$$

Thus, each S^n is projective because the exact sequences

$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

and the linear maps

$$h^n : S^n \to C^n$$

were arbitrary. □

5.7 Strict Projective Resolutions

The fact that there are enough projectives in **R-mod** is used to construct projective resolutions. Similarly, we can use the fact that there are enough strict projective modules in **R-filt** to construct strict projective resolutions.

Definition 5.7.1. Let M be a module in **R-filt**. A strict exact sequence of the form

$$\mathbf{P} : \cdots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \to \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

in which every P_n is a strict projective module in **R-filt** is called a **strict projective resolution** of the module M.

Theorem 5.7.2. *Every module M in **R-filt** has a strict projective resolution.*

Proof. The proof is exactly analogous to the proof in **R-mod**. Since there are enough strict projective modules, there is a strict exact sequence

$$P_0 \xrightarrow{\epsilon} M \to 0$$

where P_0 is a strict projective module. For the inductive step, we let K_n be the kernel of the map $P_n \to P_{n-1}$ filtered by the submodules $K_n^{(i)} = K_n \cap P_n^{(i)}$. Then, again using the fact that there are enough strict projectives, we construct the diagram

$$\begin{array}{ccccccccccc} P_{n+1} & \dashrightarrow & & \dashrightarrow & P_n & \xrightarrow{d_n} & \cdots \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M & \longrightarrow 0 \\ & \searrow^{\pi_n} & & \nearrow^{i_n} & & & & & & & \\ & & K_n & & & & & & & & \end{array}$$

where P_{n+1} is a strict projective module, π_n is a strict surjection, and let $d_{n+1} = i_n \circ \pi_n$. In fact, d_{n+1} is strict because

$$\begin{aligned} d_{n+1}(P_{n+1}^{(i)}) &= \pi_n(P_{n+1}^{(i)}) = \pi_n(P_{n+1}) \cap K_n^{(i)} \\ &= K_n \cap P_n^{(i)} = d_{n+1}(P_{n+1}) \cap P_n^{(i)}. \end{aligned}$$

The addition of P_{n+1} and d_{n+1} in this fashion clearly keep the sequence exact. Therefore, every module M in **R-filt** has a strict projective resolution. □

In **R-mod** we have the following comparison theorem for projective resolutions.

Theorem 5.7.3. *Given a diagram*

$$\begin{array}{lccccccccc} \boldsymbol{P}: & \cdots \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \longrightarrow \cdots \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow 0 \\ & & & & & & & & \downarrow{\scriptstyle f} & \\ \boldsymbol{B}: & \cdots \longrightarrow & B_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \longrightarrow \cdots \xrightarrow{\partial_1} & B_0 & \xrightarrow{\eta} & B & \longrightarrow 0 \end{array}$$

of R-modules and homomorphisms in which the lower row is exact and every P_n in the upper row is a projective R-module, then there exists a chain map $f = \{f_n\} : \boldsymbol{P}_A \to \boldsymbol{B}_B$ over f and two such chain maps are homotopic.

We next check that this result carries over into the category **R-filt**. First we need the following lemma.

Lemma 5.7.4. *Let S be a strict projective R-module. If in the diagram*

$$\begin{array}{ccccc} & & S & & \\ & & \downarrow{\scriptstyle f} & & \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \end{array}$$

*of modules and homomorphisms in **R-filt** the row is strict exact and $\beta f = 0$, then there exists a homomorphism $g : S \to A$ such that $\alpha g = f$.*

Proof. Let $\bar{B} = \alpha(A) = \ker \beta$ filtered by the submodules $\bar{B}_n = B_n \cap \alpha(A)$. Let $\bar{\alpha}$ be the map defined by α with the codomain restricted to $\alpha(A) = \bar{B}$. That is, $\bar{\alpha} : A \to \bar{B}$ and $\bar{\alpha}(a) = \alpha(a)$ for all $a \in A$. By doing so, $\bar{\alpha}$ is surjective. Also note that

$$\bar{\alpha}(A_n) = \alpha(A_n) = B_n \cap \alpha(A) = \bar{B}_n \cap \bar{\alpha}(A)$$

and thus $\bar{\alpha}$ is a strict morphism. Since $\beta f = 0$, we have that $f(S) \subset \ker(\beta) = \alpha(A) = \bar{B}$ so we can also restrict the codomain of f to get a function $\bar{f} : S \to \bar{B}$ defined by $\bar{f}(s) = f(s)$. Thus we have a diagram

$$\begin{array}{ccccc} & & S & & \\ & & \downarrow{\scriptstyle \bar{f}} & & \\ A & \xrightarrow{\bar{\alpha}} & \bar{B} & \longrightarrow & 0 \end{array}$$

with S being a strict projective module and $\bar{\alpha}$ a strict surjection. Therefore, there exists a map $g : S \to A$ such that $\bar{\alpha} g = \bar{f}$. But if $i : \bar{B} \to B$ is the inclusion map, then

$$\alpha g = (i\bar{\alpha})g = i(\bar{\alpha}g) = i\bar{f} = f$$

as desired. □

The analogous comparison theorem in **R-filt** would say

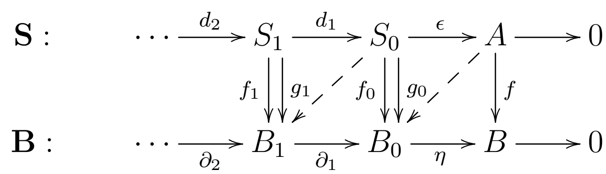
Theorem 5.7.5. *(Comparison Theorem in **R-filt**) Given a diagram*

$$\begin{array}{lccccccccccccc} \boldsymbol{S}: & \cdots & \longrightarrow & S_{n+1} & \xrightarrow{d_{n+1}} & S_n & \longrightarrow & \cdots & \xrightarrow{d_1} & S_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\ & & & & & & & & & & & \downarrow{\scriptstyle f} & & \\ \boldsymbol{B}: & \cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial_{n+1}} & B_n & \longrightarrow & \cdots & \xrightarrow{\partial_1} & B_0 & \xrightarrow{\eta} & B & \longrightarrow & 0 \end{array}$$

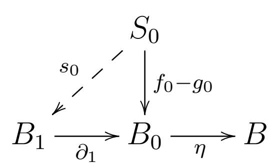
*of R-modules and homomorphisms in **R-filt** in which the lower row is strict exact and every S_n in the upper row is a strict projective R-module, then there exists a chain map $f = \{f_n\} : \boldsymbol{S}_A \to \boldsymbol{B}_B$ over f and two such chain maps are homotopic.*

Proof. Since S_0 is a strict projective module, $f\epsilon$ is a homomorphism, and η is a strict surjection there is a homomorphism $f_0 : S_0 \to B_0$ such that $f\epsilon = \eta f_0$. Now suppose that we have constructed functions $f_0, \ldots, f_n$ such that the given diagram commutes. Then the map $\partial_n(f_n d_{n+1}) = (\partial_n f_n)d_{n+1} = (f_{n-1}d_n)d_{n+1} = f_{n-1}(d_n d_{n+1}) = 0$. Now using Lemma 5.7.4 there exists a function $f_{n+1} : S_{n+1} \to B_{n+1}$ making the diagram commute. Therefore a chain map $f = \{f_n\} : \mathbf{S}_A \to \mathbf{B}_B$ over f exists by induction.

Suppose that $g = \{g_n\}$ is another chain map over f. Let us look at the beginning of the above diagram:



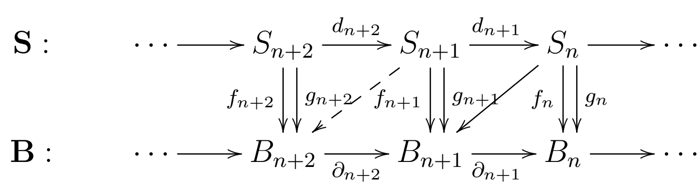
and define the map $s_{-1} : A \to B_0$ by $s_{-1}(a) = 0$ and $s_0 : S_0 \to B_1$ using the fact that S_0 is strictly projective to complete



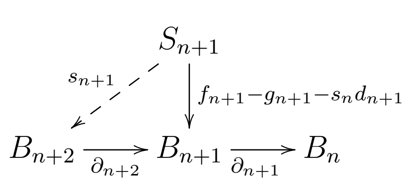
Then certainly, $f_0 - g_0 = \partial_1 s_0 + s_{-1}\epsilon$ because s_{-1} is the zero map. Now for the inductive step we want to focus on a portion of the diagram further down the line. So assume that we have already constructed $s_i : S_i \to B_{i+1}$ such that

$$\begin{cases} f_0 - g_0 = \partial_1 s_0 + s_{-1}\epsilon \\ f_i - g_i = \partial_{i+1} s_i + s_{i-1} d_i \quad 1 \le i \le n; \end{cases}$$

and consider the following diagram:



Then we construct the dotted map by completing



which can be done using Lemma 5.7.4 because S_{n+1} is strictly projective, the row is strict exact, and $\partial_{n+1}(f_{n+1} - g_{n+1} - s_n d_{n+1}) = 0$. Thus we can construct $s = \{s_n\}$ by induction and s is a chain homotopy between f and g because

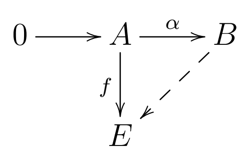
$$f_n - g_n = \partial_{n+1} s_n + s_{n-1} d_n$$

for all $n \geq 0$ by construction. □

5.8 Strict Injective Objects

In this section I will discuss injectivity with respect to strict morphisms instead of all morphisms in the category **R-filt**. One reason for doing this is to look for something more interesting than the categorical injective objects in **R-filt** which were discussed last section. The main result of this section is a characterization of strict injective modules.

Definition 5.8.1. A module E in the category **R-filt** is called a **strict injective** module if we can complete a diagram of the form



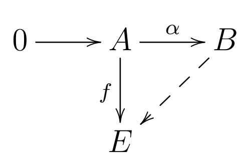
to a commutative diagram whenever α is a strict injection. These are the so-called $\mathcal{H}$-injectives where $\mathcal{H}$ is the set of all strict injective maps (i.e. strict monomorphisms).

Proposition 5.8.2. *Let $E^0, E^1, E^2, \ldots$ be any sequence of injective (in **R-mod**) modules. Let*

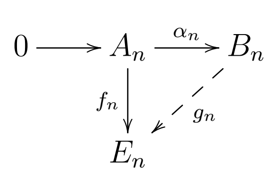
$$E = E^0 \times E^1 \times \ldots$$

filtered by $E_n = 0 \times \ldots \times 0 \times E^n \times E^{n+1} \times \ldots$. Then E is a strict injective module.

Proof. Consider the diagram



in **R-filt** where α is a strict injection. Restricting the maps f and α gives the diagram



in **R-mod** where $f_n =_{E_n} |f|_{A_n}$ and $\alpha_n =_{B_n} |\alpha|_{A_n}$ is injective. Also, E_n is injective so the map g_n exists. Consider the induced map

$$\bar{\alpha}_n : \frac{A_n}{A_{n+1}} \to \frac{B_n}{B_{n+1}}$$
$$a_n + A_{n+1} \mapsto \alpha_n(a_n) + B_{n+1}.$$

This map is injective because α is strictly injective. Similarly, consider the induced map $\bar{f}_n : \dfrac{A_n}{A_{n+1}} \to \dfrac{E_n}{E_{n+1}}$ and let $\iota_E^n : E^n \overset{\cong}{\to} \dfrac{E_n}{E_{n+1}}$ be the obvious isomorphism. Then the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \dfrac{A_n}{A_{n+1}} & \xrightarrow{\bar{\alpha}_n} & \dfrac{B_n}{B_{n+1}} \\ & & {\scriptstyle \iota_E^{n\,-1}\circ \bar{f}_n}\downarrow & \swarrow {\scriptstyle \bar{g}_n} & \\ & & E^n & & \end{array}$$

can be completed in **R-mod**. Let $j_n : B_n \to \dfrac{B_n}{B_{n+1}}$ be the canonical surjection. Then define $h_n : B_n \to E^n$ by $h_n := \bar{g}_n \circ j_n$. It is important to note that $h_n(B_{n+1}) = 0$ and therefore $h_n(B_l) = 0$ for all $l > n$.

Now for each h_n we want to extend the domain to B. Such maps exist because the diagrams

$$\begin{array}{ccccc} 0 & \longrightarrow & B_n & \hookrightarrow & B \\ & & {\scriptstyle h_n}\downarrow & \swarrow {\scriptstyle k_n} & \\ & & E^n & & \end{array}$$

can be completed (E^n is injective) in **R-mod**. Next, each $k_n : B \to E^n$ can be thought of as a map from $B \to E$, by composing with the canonical injection $E^n \to E$, such that $k_n(B) \subset 0 \times \ldots \times E^n \times 0 \times \ldots$. Finally, define $g : B \to E$ to be the sum of the k_n. Then $g(b) = \sum k_n(b)$ which converges because $k_n(b) \in 0 \times \ldots \times E^n \times 0 \times \ldots \subset E_n$ for all n.

Finally note that the map $g : B \to E$ is actually a map in **R-filt** because if $b_m \in B_m$ then

$$g(b_m) = \sum_{n=0}^{\infty} k_n(b_m) = \sum_{n=m}^{\infty} k_n(b_m) \in E_m.$$

□

Remark 5.8.3. Recall that there are enough injective modules in **R-mod**. Any R-module A is an abelian group and can therefore be embedded into a divisible abelian group D. Then it can be shown that A can be embedded into the injective

R-module $Hom(R, D)$ (see [19], for example). I use this idea as the building block to constructing a strict injective object E in **R-filt** for each object M so that there exists a strict injection $\theta : M \to E$.

Proposition 5.8.4. *There are enough strict injective modules in* ***R-filt****. That is, given an object $M \in$* ***R-filt*** *there exists a strict injection $\varepsilon : M \to E$ such that E is a strict injective module.*

Proof. Let M be an object in **R-filt** with filtration $\{M_i\}$ and consider the modules $K_i = M/M_{i+1}$. Each K_i is an abelian group and may therefore be embedded in a divisible abelian group D_i by an injection $\alpha_i : K_i \to D_i$. Also, K_i can be embedded in the injective module $Hom(R, D_i)$ by the injection $\theta_i : K_i \to Hom(R, D_i)$ defined by $\theta_i(k_i) = \alpha_i f_{k_i}$ where $f_{k_i} : R \to K_i$ is the map defined by $f_{k_i}(r) = rk_i$

The object

$$E := \mathrm{Hom}(R, D_0) \times \mathrm{Hom}(R, D_1) \times \mathrm{Hom}(R, D_2) \times \ldots$$

in **R-filt** with D_i from above with the filtration

$$E_n = 0 \times 0 \times \ldots \times 0 \times Hom(R, D_n) \times Hom(R, D_{n+1}) \times \ldots$$

is strictly injective by Proposition 5.8.2. Define a map

$$\begin{array}{rrl} \theta : & M \to & \mathrm{Hom}(R, D_0) \times \mathrm{Hom}(R, D_1) \times \mathrm{Hom}(R, D_2) \times \ldots \\ & m \mapsto & (\alpha_0 f_{k_0}, \alpha_1 f_{k_1}, \alpha_2 f_{k_2}, \ldots) \\ & \text{where} & k_i = m + M_{i+1}. \end{array}$$

First note that if $m \in M_n$, then $k_i = 0$ for $i = 0, 1, \ldots, n-1$. Thus,

$$\begin{aligned} \theta(m) &= (\alpha_0 f_{k_0}, \alpha_1 f_{k_1}, \alpha_2 f_{k_2}, \ldots) \\ &= (0, \ldots, 0, \alpha_n f_{k_n} \ldots) \in E_n. \end{aligned}$$

That is, θ is a morphism in **R-filt**.

Next, if $\theta(m_1) = \theta(m_2)$, then $r\alpha_i(m_1 + M_{i+1}) = r\alpha_i(m_2 + M_{i+1})$ for all i and for all $r \in R$. In particular this is true for $r = 1$. Therefore $m_1 + M_i = m_2 + M_i$ for all i and hence $m_1 = m_2$ (i.e. θ is injective).

Finally, suppose $e \in \theta(M) \cap E_n$. Then $e = (\alpha_i f_{k_i})_{i=0}^{\infty}$ for some $m \in M$ (again where $k_i = m + M_{i+1}$). In fact, $\alpha_i f_{k_i} = 0$ for $i = 0, 1, \ldots, n-1$ because $e \in E_n$.

That is, $\alpha_i(rk_i) = 0$ for all $r \in R$ and $i = 0, 1, \ldots, n-1$. In particular, $\alpha_i(k_i) = 0$ for $i = 0, 1, \ldots, n-1$. But α_i is injective so $k_i = 0$ for $i = 0, 1, \ldots, n-1$. Thus, $m \in M_n$ because $k_{n-1} = m + M_n$. Hence, $e \in \theta(M_n)$ and θ is strict. □

Corollary 5.8.5. *Every strict injective module in **R-filt** is a direct summand of a strict injective module of the form*

$$E = E^0 \times E^1 \times \ldots$$

filtered by $E_n = 0 \times 0 \times \ldots \times 0 \times E^n \times E^{n+1} \times \ldots$ *where each* E^i *is injective in **R-mod**.*

Proof. Let Q be a strict injective object in **R-filt**. Then we can embed Q into a strict injective module E of the form in Proposition 5.8.4 by using a strict injection. Note that doing so means that there is an isomorphic copy of Q in E which has the filtration induced by being a submodule of E. Thus the quotient E/Q with the induced filtration is actually an object in **R-filt**. Therefore we can use the quotient module E/Q to construct a strict exact sequence

$$0 \to Q \xrightarrow{i} E \xrightarrow{p} E/Q \to 0$$

in **R-filt**. Since Q is strict injective, there is a map, $g : E \to Q$, in **R-filt** such that $gi = \mathrm{id}_Q$. This implies that $E = Q \oplus E/Q$ in **R-filt** by Lemma 5.3.18. □

This shows that every strict injective module is a direct summand of another strict injective module. It is also true that direct summands of strict injective modules are strict injective. Since I don't use this again, I will just state it without proof:

Lemma 5.8.6. *If* $E = M \oplus N$ *in **R-filt** with* E *strictly injective, then* M *and* N *are strictly injective.*

Next we get the actual characterization of the strict injective modules.

Theorem 5.8.7. *A module* Q *in **R-filt** is a strict injective module if and only if*

$$Q = Q^0 \times Q^1 \times Q^2 \times \ldots$$

filtered by the submodules $Q_n = 0 \times \ldots \times 0 \times Q^n \times Q^{n+1} \times \ldots$ *where* Q^n *is injective (in **R-mod**) for all* $n \geq 0$.

Proof. We now have that every strict injective module is a direct summand of another strict injective module of the form

$$E = E^0 \times E^1 \times E^2 \times \ldots$$

filtered by $E_n = 0 \times \ldots \times 0 \times E^n \times E^{n+1} \ldots$ from Corollary 5.8.5. Then using Proposition 5.4.13 we see that Q must have the form $Q = Q^0 \times Q^1 \times Q^2 \times \ldots$ filtered as usual.

Suppose we have exact sequences

$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

of modules for each $n = 0, 1, 2, \ldots$. Define A, an object in **R-filt**, as

$$A := \prod_{n=0}^{\infty} A^n$$

filtered by $A_n = 0 \times \ldots \times 0 \times A^n \times A^{n+1} \times \ldots$ as usual. Define two more objects B and C in **R-filt** similarly as

$$B := \prod_{n=0}^{\infty} B^n$$

filtered by $B_n = 0 \times \ldots \times 0 \times B^n \times B^{n+1} \times \ldots$ and

$$C := \prod_{n=0}^{\infty} C^n$$

filtered by $C_n = 0 \times \ldots \times 0 \times C^n \times C^{n+1} \times \ldots$. We can construct the strict exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where $f((a^n)) := (f^n(a^n))$ and $g((b^n)) := (g^n(b^n))$. Let $h^n : A^n \to Q^n$ be any linear map for each $n \geq 0$ and $h : A \to Q$ be the linear map such that $h((a^n)) := (h^n(a^n))$. Then we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & \swarrow{\scriptstyle k} & & & & & \\ & & Q & & & & & & \end{array}$$

where k exists because Q is strictly injective. Now restricting f, h, and k there are also commutative diagrams

$$\begin{array}{ccc} A_n & \xrightarrow{f_n} & B_n \\ {\scriptstyle h_n}\downarrow & \swarrow{\scriptstyle k_n} & \\ Q_n & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} \\ {\scriptstyle h_{n+1}}\downarrow & \swarrow{\scriptstyle k_{n+1}} & \\ Q_{n+1} & & \end{array}$$

From these we define the maps

1. $$\begin{aligned} \bar{f}_n : \frac{A_n}{A_{n+1}} &\to \frac{B_n}{B_{n+1}} \\ a_n + A_{n+1} &\mapsto f_n(a_n) + B_{n+1} \end{aligned}$$

2. $$\begin{aligned} \bar{h}_n : \frac{A_n}{A_{n+1}} &\to \frac{Q_n}{Q_{n+1}} \\ a_n + A_{n+1} &\mapsto h_n(a_n) + Q_{n+1} \end{aligned}$$

3. $$\begin{aligned} \bar{k}_n : \frac{B_n}{B_{n+1}} &\to \frac{Q_n}{Q_{n+1}} \\ b_n + B_{n+1} &\mapsto k_n(b_n) + Q_{n+1} \end{aligned}$$

Additionally, there are obvious isomorphisms

$$\iota_Q^n : Q^n \xrightarrow{\cong} \frac{Q_n}{Q_{n+1}}, \qquad \iota_A^n : A^n \xrightarrow{\cong} \frac{A_n}{A_{n+1}}, \qquad \iota_B^n : B^n \xrightarrow{\cong} \frac{B_n}{B_{n+1}}$$

which we will use to create a map $k^n : B^n \to Q^n$. One can check that

$$f^n = (\iota_B^n)^{-1} \circ \bar{f}_n \circ \iota_A^n \qquad \text{and} \qquad h^n = (\iota_Q^n)^{-1} \circ \bar{h}_n \circ \iota_A^n.$$

Therefore, we similarly define a map $k^n : B^n \to Q^n$ by

$$k^n := (\iota_Q^n)^{-1} \circ \bar{k}_n \circ \iota_B^n.$$

All that remains to be shown is that $k^n f^n = h^n$. This follows because

$$\begin{aligned} k^n f^n &= ((\iota_Q^n)^{-1} \circ \bar{k}_n \circ \iota_B^n)((\iota_B^n)^{-1} \circ \bar{f}_n \circ \iota_A^n) \\ &= (\iota_Q^n)^{-1} \circ \bar{k}_n \circ \bar{f}_n \circ \iota_A^n \end{aligned}$$

and

$$\begin{aligned} \bar{k}_n(\bar{f}_n(a_n + A_{n+1})) &= \bar{k}_n(f_n(a_n) + B_{n+1} \\ &= k_n(f_n(a_n)) + Q_{n+1} \\ &= h_n(a_n) + Q_{n+1} \\ &= \bar{h}_n(a_n + A_{n+1}). \end{aligned}$$

Because the exact sequences

$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

and the linear maps

$$h^n : A^n \to Q^n$$

were arbitrary, we now have that each Q^n is injective.

Proposition 5.8.2 gives the reverse direction. □

5.9 Strict Injective Envelopes

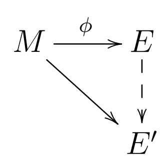
This section considers the idea of an injective envelope in **R-filt**.

Remark 5.9.1. The typical definition of an injective envelope (Definition 2.5.1) does not translate well into **R-filt**. One problem is that in showing the equivalence of the three conditions for an injective envelope a quotient is taken by a union of modules is taken. Taking such a union with filtered modules does give a filtered submodule, but the desired quotient will not be Hausdorff unless the union is closed.

In [7] Enochs gave the following characterization of an injective envelope.

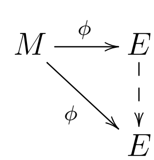
Proposition 5.9.2. *An injective envelope of a left R-module M can be characterized as a linear map $\phi : M \to E$ into an injective R-module E with two properties:*

(a) Any diagram



where E' is an injective left R-module can be completed (or equivalently, ϕ is an injection).

(b) The diagram



can be completed only by automorphisms of E (equivalently, E is an essential extension of $\phi(M)$).

Remark 5.9.3. This characterization leads to a strict analogue of an injective envelope in **R-filt** as follows:

Let M be an object in **R-filt**. Then for each $n \geq 0$ there is an injective envelope (in **R-mod**) of M_n/M_{n+1}, denoted $E(M_n/M_{n+1})$. The composition $M_n \to M_n/M_{n+1} \to E(M_n/M_{n+1})$ where the first map is the natural projection and the second is the natural injection can be extended to a map $g_n : M \to E(M_n/M_{n+1})$. Using these $g_n s$, we construct the map

$$\begin{aligned} g : &M \to \prod_{n=0}^{\infty} E(M_n/M_{n+1}) \\ &x \mapsto (g_0(x), g_1(x), \ldots) \end{aligned}$$

and define an object I in **R-filt** as

$$I = \prod_{n=0}^{\infty} E(M_n/M_{n+1})$$

with filtration

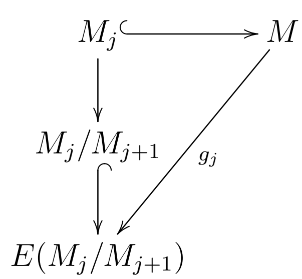
$$I_n = 0 \times \ldots \times 0 \times E(M_n/M_{n+1}) \times E(M_{n+1}/M_{n+2}) \times \ldots$$

as usual. This object I is strict injective from Theorem 5.8.7 and it turns out that g satisfies analogous conditions to those in [7].

Lemma 5.9.4. *This map $g : M \to I$ is a morphism in **R-filt**, injective, and strict.*

Proof. Let $g : M \to I$ be as above.

(a) If $m_n \in M_n$ (and hence $m_n \in M_{j+1}$ for all $j < n$), then $g_j(m_n) = 0$ for all $j < n$ because the diagrams



commute. Therefore, $g(M_n) \subset I_n$, i.e. g is a morphism in **R-filt**.

(b) Note that if $x \in M_j$ then $g_j(x) = 0$ implies $x \in M_{j+1}$. Suppose $g(x) = 0$. Then $g_j(x) = 0$ for all $j \geq 0$. Therefore, $x \in \cap_{n=0}^{\infty} M_n$ by induction and thus $x = 0$, i.e. g is an injection.

(c) Now suppose $i \in I_n \cap g(M)$. Then i can be written in the form

$$i = (0, \ldots, 0, i_n, i_{n+1} \ldots)$$

where $i_n \in E(M_n/M_{n+1})$ or $i = g(x)$ for some $x \in M$. Putting these together shows that $g_j(x) = 0$ for $j < n$ and $g_j(x) = i_n$ otherwise. This means $x \in M_n$ by a similar argument as the previous part except that the induction may stop at the nth position.

□

Proposition 5.9.5. *If $f : I \to I$ is a morphism in* ***R-filt*** *such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{g} & I \\ & \searrow^{g} & \downarrow f \\ & & I \end{array}$$

is commutative, then f is an automorphism (in ***R-filt****) of I.*

Proof. The given diagram induces a commutative diagram (in **R-mod**)

$$\begin{array}{ccc} M_n & \xrightarrow{g_n} & I_n \\ & \searrow^{g_n} & \downarrow f_n \\ & & I_n \end{array}$$

for each $n \geq 0$. Furthermore, each of these diagram then induces another commutative diagram (in **R-mod**)

$$\begin{array}{ccc} M_n/M_{n+1} & \xrightarrow{\bar{g_n}} & I_n/I_{n+1} \\ & \searrow^{\bar{g_n}} & \downarrow \bar{f_n} \\ & & I_n/I_{n+1} \end{array}$$

for each $n \geq 0$. But there is an obvious isomorphism

$$\sigma_n : I_n/I_{n+1} \to E(M_n/M_{n+1})$$

so the previous diagrams can be extended to the larger commutative diagrams

$$\begin{array}{ccccc} M_n/M_{n+1} & \xrightarrow{\bar{g_n}} & I_n/I_{n+1} & \xrightarrow{\sigma_n} & E(M_n/M_{n+1}) \\ & \searrow^{\bar{g_n}} & \downarrow \bar{f_n} & & \downarrow \sigma_n \circ \bar{f_n} \circ \sigma_n^{-1} \\ & & I_n/I_{n+1} & \xrightarrow{\sigma_n} & E(M_n/M_{n+1}) \end{array}$$

for each $n \geq 0$. Now since $E(M_n/M_{n+1})$ is the injective envelope of M_n/M_{n+1} and $\sigma_n \circ \bar{g}_n$ is injective we have that $\sigma_n \circ \bar{f}_n \circ \sigma_n^{-1}$ is an isomorphism, and thus $\bar{f}_n$ is an isomorphism (in **R-mod**) as well.

The map $gr(f) : gr(I) \to gr(I)$ is therefore easily seen to be a bijection using the relation between $gr(f)$ and the $\bar{f}_n$s. Back in Lemma 4.5.10 we showed that $gr(f)$ being a bijection implies that the associated map $f : I \to I$ in **R-filt** is a bijection. Then the only remaining thing to check is that the map f is actually an isomorphism and not just a bijection.

If f is strict, then the bijection f is actually an isomorphism in **R-filt** by Lemma 5.3.13. To show f is strict it is enough to show that $I_n \subset f(I_n)$. Furthermore, it is enough to show that $f(i) \notin I_{n+1}$ for all $i \in I_n \backslash I_{n+1}$. That is, if $(0, \ldots, 0, z_n, z_{n+1}, \ldots) \in I$ with $z_n \neq 0$ then $f(0, \ldots, 0, z_n, z_{n+1}, \ldots) = (0, \ldots, 0, w_n, w_{n+1})$ implies $w_n \neq 0$. Note that $f(0, \ldots, 0, z_{n+1}, \ldots)$ is an element in I_{n+1} so w_n is completely determined by $f(0, \ldots, 0, z_n, 0, \ldots)$. But since $\sigma_n \circ \bar{f}_n \circ \sigma_n^{-1}$ is an isomorphism and induced by f we have $(\sigma_n \circ \bar{f}_n \circ \sigma_n^{-1})(z_n) = w_n$ and therefore $w_n \neq 0$ when $z_n \neq 0$. □

Here are two ways to see how this strict injective envelope differs from the standard injective envelope.

Example 5.9.6. Consider the p-adic integers, denoted $\widehat{\mathbf{Z}_p}$, with the filtration

$$\widehat{\mathbf{Z}_p} = (p^0) \supset (p) \supset (p^2) \supset \ldots .$$

Note that $\frac{(p^n)}{(p^{n+1})} \cong \mathbf{Z}/(p)$ and the injective envelope of $\mathbf{Z}/(p)$ (as a $\mathbf{Z}$-module) is isomorphic to $\mathbf{Z}(p^\infty)$ (see Example 2.5.3). Therefore, the strict injective envelope of $\widehat{\mathbf{Z}_p}$ with this filtration is

$$\mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots ,$$

but the injective envelope of $\widehat{\mathbf{Z}_p}$ is $\widehat{\mathbf{Q}_p}$, the field of p-adic numbers (see Example 2.5.4).

Another way to see that these objects differ is to compare the Galois group of the strict injective envelope

$$\widehat{\mathbf{Z}_p} \subset \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots$$

and the Galois group for the injective envelope $\widehat{\mathbf{Z}_p} \subset \widehat{\mathbf{Q}_p}$. Recall the following definitions.

Definition 5.9.7. If $k \subset K$ are fields, then the Galois group $\mathrm{Gal}(K/k)$ is the group of automorphisms of the field K that fix all $\alpha \in k$.

Definition 5.9.8. The Galois group of the injective envelope $M \subset E(M)$ is the group of automorphisms of $E(M)$ that fix all $m \in M$.

Example 5.9.9. For the injective envelope $\widehat{\mathbf{Z}_p} \subset \widehat{\mathbf{Q}_p}$ the Galois group is trivial.

Proof. Let f be an element of the Galois group of the injective envelope $\widehat{\mathbf{Z}_p} \subset \widehat{\mathbf{Q}_p}$. That is, $f : \widehat{\mathbf{Q}_p} \to \widehat{\mathbf{Q}_p}$ is an automorphism and $f(x) = x$ for all $x \in \widehat{\mathbf{Z}_p}$. In particular, $f(p) = p$. All elements in $\widehat{\mathbf{Q}_p}$ can be written in the form

$$x = \sum_{i \geq n} a_i p^i$$

where n is an integer such that $a_n \neq 0$ and $0 \leq a_i \leq p - 1$. Then,

$$\begin{aligned} f\left(\sum_{i \geq n} a_i p^i\right) &= f\left(\sum_{i=n}^{-1} a_i p^i\right) + f\left(\sum_{i=0}^{\infty} a_i p^i\right) \\ &= \sum_{i=n}^{-1} f(a_i p^i) + f\left(\sum_{i=0}^{\infty} a_i p^i\right) \\ &= \sum_{i=n}^{-1} a_i f(p)^i + f\left(\sum_{i=0}^{\infty} a_i p^i\right) \\ &= \sum_{i=n}^{-1} a_i p^i + \sum_{i=0}^{\infty} a_i p^i \\ &= \sum_{i \geq n} a_i p^i. \end{aligned}$$

That is, $f = \mathrm{id}_{\widehat{\mathbf{Q}_p}}$. □

I additionally need a couple of remarks about the mappings involved here before I can get the result that I want.

Definition 5.9.10. Given a p-adic integer $x = a_0 + a_1 p + a_2 p^2 + \ldots \in \widehat{\mathbf{Z}_p}$ and $n \geq 0$ define

$$\frac{x}{p^n} + \mathbf{Z} := \frac{a_0 + a_1 p + \ldots + a_{n-1} p^{n-1}}{p^n} + \mathbf{Z}$$

to be the cosets in $\mathbf{Z}(p^\infty)$. This is a natural extension of $\mathbf{Z}(p^\infty)$ because if a p-adic integer has finitely many $a_i \neq 0$ then this agrees with the usual coset for the given integer.

Definition 5.9.11. Given a p-adic integer $x = a_0 + a_1 p + a_2 p^2 + \ldots \in \widehat{\mathbf{Z}_p}$ and $n \geq 0$ define a map (written multiplicatively) from $\widehat{\mathbf{Z}_p} \times \mathbf{Z}(p^\infty) \to \mathbf{Z}(p^\infty)$ by

$$x\left(\frac{a}{p^n} + \mathbf{Z}\right) := \frac{ax}{p^n} + \mathbf{Z}.$$

Lemma 5.9.12. *Any map (i.e. homomorphism) $\phi : \mathbf{Z}(p^\infty) \to \mathbf{Z}(p^\infty)$ is given by multiplication by a p-adic integer in the above sense. Moreover, the map ϕ is an isomorphism if and only if x is a unit in $\widehat{\mathbf{Z}_p}$.*

Proof. Consider any map $\phi : \mathbf{Z}(p^\infty) \to \mathbf{Z}(p^\infty)$. We see that the restricted map $\phi_1 : \left\langle \frac{1}{p} + \mathbf{Z} \right\rangle \to \mathbf{Z}(p^\infty)$ is completely determined by where $\frac{1}{p} + \mathbf{Z}$ is mapped. Suppose $\phi_1\left(\frac{1}{p} + \mathbf{Z}\right) = \frac{s_0}{p^n} + \mathbf{Z}$. Then

$$\begin{aligned}
0 &= \phi_1\left(\left(\frac{1}{p} + \mathbf{Z}\right) + \left(\frac{p-1}{p} + \mathbf{Z}\right)\right) \\
&= \phi_1\left(\frac{1}{p} + \mathbf{Z}\right) + \phi_1\left(\frac{p-1}{p} + \mathbf{Z}\right) \\
&= \left(\frac{s_0}{p^n} + \mathbf{Z}\right) + (p-1)\left(\frac{s_0}{p^n} + \mathbf{Z}\right) \\
&= \frac{ps_0}{p^n} + \mathbf{Z} \\
&= \frac{s_0}{p^{n-1}} + \mathbf{Z}.
\end{aligned}$$

So in fact, $p^{n-1} | s_0$ and thus we can write $\phi_1\left(\frac{1}{p} + \mathbf{Z}\right) = \frac{a_0}{p} + \mathbf{Z}$ where $0 \leq a_0 < p$.

Similarly, one can show that the restriction of ϕ to

$$\phi_2 : \left\langle \frac{1}{p^2} + \mathbf{Z} \right\rangle \to \mathbf{Z}(p^\infty)$$

is completely determined by knowing

$$\phi_2\left(\frac{1}{p^2} + \mathbf{Z}\right) = \frac{s_1}{p^n} + \mathbf{Z} = \frac{b_1}{p^2} + \mathbf{Z}$$

where $0 \leq b_1 < p^2$. Then we can write the p-adic expansion of b_1, that is, write $b_1 = a_0' + a_1 p$ where $0 \leq a_0', a_1 < p$. Then note that

$$\phi_2\left(\frac{p}{p^2} + \mathbf{Z}\right) = \phi_2\left(\frac{1}{p} + \mathbf{Z}\right) = \frac{a_0}{p} + \mathbf{Z}$$

and

$$\begin{aligned}\phi_2\left(\frac{p}{p^2}+\mathbf{Z}\right) &= \frac{pb_1}{p^2}+\mathbf{Z}\\ &= \frac{b_1}{p}+\mathbf{Z}\\ &= \frac{a_0'+a_1p}{p}+\mathbf{Z}\\ &= \frac{a_0'}{p}+\mathbf{Z}.\end{aligned}$$

Therefore, $p|a_0-a_0'$, but this implies $a_0=a_0'$ because $0\le a_0, a_0' < p$.

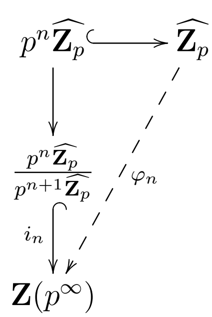
Using a similar procedure for each $n\ge 0$ gives a way to associate the given map ϕ to the p-adic integer $x=a_0+a_1p+a_2p^2+\ldots$. In fact, the map ϕ is essentially given by multiplication by x because

$$\begin{aligned}\phi\left(\frac{a}{p^n}+\mathbf{Z}\right) &= a\phi\left(\frac{1}{p^n}+\mathbf{Z}\right)\\ &= a\left(\frac{a_0+a_1p+a_2p^2+\ldots+a_{n-1}p^{n-1}}{p^n}+\mathbf{Z}\right)\\ &= a\left(\frac{x}{p^n}+\mathbf{Z}\right)\\ &= x\left(\frac{a}{p^n}+\mathbf{Z}\right).\end{aligned}$$

Therefore, x is a unit in $\widehat{\mathbf{Z}_p}$ (i.e. $a_0\neq 0$) if and only if ϕ is an isomorphism. □

Finally, before putting all this together I need to say what it means for a morphism $\phi:\mathbf{Z}(p^\infty)\times\mathbf{Z}(p^\infty)\times\ldots\to\mathbf{Z}(p^\infty)\times\mathbf{Z}(p^\infty)\times\ldots$ to fix $\widehat{\mathbf{Z}_p}$.

Remark 5.9.13. Suppose $x=a_0+a_1p+\ldots$ is a p-adic integer. As in Lemma 5.9.4, consider the diagram



where $\varphi_n : \widehat{\mathbf{Z}_p} \to \mathbf{Z}(p^\infty)$ is the map defined by

$$a \mapsto \frac{a}{p^{n+1}} + \mathbf{Z}$$

and

$$i_n : \frac{p^n \widehat{\mathbf{Z}_p}}{p^{n+1}\widehat{\mathbf{Z}_p}} \to \mathbf{Z}(p^\infty)$$

is defined by

$$p^n x + p^{n+1}\widehat{\mathbf{Z}_p} \mapsto \frac{p^n x}{p^{n+1}} + \mathbf{Z} = \frac{a_0}{p} + \mathbf{Z}.$$

Therefore, one choice for a strict, injective morphism in **R-filt** from $\widehat{\mathbf{Z}_p} \to \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots$ is the map $\varphi : \widehat{\mathbf{Z}_p} \to \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots$ where

$$x \mapsto \left(\frac{x}{p} + \mathbf{Z}, \frac{x}{p^2} + \mathbf{Z}, \dots \right).$$

Then to say that a map $\psi : \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots \to \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots$ fixes $\widehat{\mathbf{Z}_p}$ means that ψ fixes elements of the form $\left(\frac{x}{p} + \mathbf{Z}, \frac{x}{p^2} + \mathbf{Z}, \dots \right)$ where x is a p-adic integer. □

The previous discussion has shown the following result:

Proposition 5.9.14. *The group of automorphisms of $\mathbf{Z}(p^\infty)^\omega$ is the group of $\mathbf{N} \times \mathbf{N}$ lower triangular matrices with p-adic units on the diagonal.*

Proof. Any map $\phi : \mathbf{Z}(p^\infty)^\omega \to \mathbf{Z}(p^\infty)^\omega$ has a matrix representation as discussed in Section **??**. Moreover, ϕ is an isomorphism if and only if each ϕ_{ii} in the matrix representation of ϕ is an isomorphism by Lemma 5.3.19. Finally, each ϕ_{ii} is an isomorphism if and only if each ϕ_{ii} is given by multiplication by a p-adic unit by Lemma 5.9.12. □

Example 5.9.15. The Galois group for the strict injective envelope

$$\widehat{\mathbf{Z}_p} \subset \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots$$

is larger than the Galois group for the injective envelope $\widehat{\mathbf{Z}_p} \subset \widehat{\mathbf{Q}_p}$.

Proof. Consider the morphism $\varphi : \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots \to \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots$ with matrix representation

$$\begin{bmatrix} \varphi_{00} & 0 & 0 & \dots \\ \varphi_{10} & \varphi_{11} & 0 & \dots \\ \varphi_{20} & \varphi_{21} & \varphi_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where φ_{00} is given by multiplication by $1+p$, φ_{ii} is given by multiplication by 1 for $i > 0$, and φ_{ij} given by multiplication by zero for $i \neq j$. This map is not the identity map, but it does fix $\widehat{\mathbf{Z}_p}$ so it is in the Galois group for the strict injective envelope of $\widehat{\mathbf{Z}_p}$.

Additionally, morphisms of the form

$$\begin{bmatrix} \varphi_{00} & 0 & 0 & \dots \\ \varphi_{10} & \varphi_{11} & 0 & \dots \\ \varphi_{20} & \varphi_{21} & \varphi_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with φ_{ii} given by multiplication by u_i, a p-adic unit with first component 1, for all i, $\varphi_{i,i-1}$ given by multiplication by $-(u_i - 1)/p$ for all $i > 0$, and φ_{ij} given by multiplication by zero for all $j < i-1$ when $i > 1$ are in the Galois group of the strict injective envelope.

However, the Galois group for the strict injective envelope does not contain all automorphisms of $\mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \dots$ with diagonal entries given by multiplication by p-adic units with first component 1. Consider the automorphism φ with matrix representation

$$\begin{bmatrix} \varphi_{00} & 0 & 0 & \dots \\ \varphi_{10} & \varphi_{11} & 0 & \dots \\ \varphi_{20} & \varphi_{21} & \varphi_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with φ_{ii} given by multiplication by 1 for all i, φ_{10} given by multiplication by 1, and φ_{ij} given by multiplication by zero for all other pairings of indices. This is an automorphism with diagonal entries all having first component 1, but this will not fix any element of $\widehat{\mathbf{Z}_p}$ that is non-zero in the first component.

□

5.10 Strict Injective Resolutions

The fact that there are enough injectives in **R-mod** is used to construct injective resolutions. Similarly, we can use the fact that there are enough strict injective modules in **R-filt** to construct strict injective resolutions.

Definition 5.10.1. Let M be a module in **R-filt**. A strict exact sequence of the form

$$\mathbf{E} : 0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \to \cdots$$

in which every E^n is a strict injective module in **R-filt** is called a **strict injective resolution** of the module M.

Remark 5.10.2. The usual procedure to construct an injective resolution is to embed a module into an injective module, take the cokernel of the embedding, and embed the cokernel into another injective module inductively. Cokernels in **R-filt** do not behave exactly like cokernels in **R-mod**, but in this particular case they do. In the proof of Corollary 5.8.5 I mentioned that embedding an object in **R-filt** into a strict injective module by a strict injection meant that we could form the quotient module, that is, the cokernel. This is in fact true for any strict injection. Therefore, the usual procedure of constructing an injective resolution will carry through in **R-filt**.

Theorem 5.10.3. *Every module M in **R-filt** has a strict injective resolution.*

Proof. Since there are enough strict injectives there is a strict exact sequence

$$0 \to M \xrightarrow{\eta} E^0$$

where E^0 is a strict injective module. For the inductive step, let C_n be the cokernel of the map $E^{n-1} \xrightarrow{d^{n-1}} E^n$ filtered by $C_n^{(i)} = (d^{n-1}(E^{n-1}) + E_i^n)/d^{n-1}(E^{n-1})$. Then again using the fact that there are enough strict injectives construct the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & M & \xrightarrow{\eta} & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & \cdots & \xrightarrow{d^{n-1}} & E^n & \dashrightarrow & E^{n+1} \\ & & & & & & & & & & & \searrow^{\pi_n} \quad C_n \quad \nearrow^{i_n} & \end{array}$$

where E^{n+1} is a strict injective module, i_n is a strict injection, and $d^n = i_n \circ \pi_n$. In fact, d_n is strict because

$$\begin{aligned} d^n(E_i^n) &= i_n(\pi_n(E_i^n)) = i_n(C_n) \cap E_i^{n+1} \\ &= d^n(E^n) \cap E_i^{n+1}. \end{aligned}$$

The addition of E^{n+1} and d^n in this fashion clearly keep the sequence exact. Therefore, every module M in **R-filt** has a strict injective resolution. $\square$

Theorem 5.10.4 (Comparison Theorem for Injective Resolutions)**.** *Given a diagram*

$$\begin{array}{ccccccccccccc}
\boldsymbol{X} : 0 & \longrightarrow & A & \xrightarrow{\eta} & X^0 & \xrightarrow{d^0} & \cdots & \longrightarrow & X^n & \xrightarrow{d^n} & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow f & & & & & & & & & & \\
\boldsymbol{E} : 0 & \longrightarrow & B & \xrightarrow[\zeta]{} & E^0 & \xrightarrow{\delta^0} & \cdots & \longrightarrow & E^n & \xrightarrow{\delta^n} & E^{n+1} & \longrightarrow & \cdots
\end{array}$$

*of R-modules and homomorphisms in **R-filt** in which the upper row is strict exact and every E^n in the lower row is a strict injective R-module, then there exists a chain map $f = \{f_n\} : \boldsymbol{X}_A \to \boldsymbol{E}_B$ over f and two such chain maps are homotopic.*

Proof. The proof runs similar to that for the strict projective resolution comparison test. $\square$

The following discussion will exhibit a minimal strict injective resolution of the p-adic integers $\widehat{\mathbf{Z}_p}$ in **R-filt**. We begin by taking a strict injective map $\varphi : \widehat{\mathbf{Z}_p} \to \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots$. Such a map was constructed in Remark 5.9.13 and was defined by

$$x \mapsto \left(\frac{x}{p} + \mathbf{Z}, \frac{x}{p^2} + \mathbf{Z}, \ldots \right).$$

This gives the beginning of a strict injective resolution of $\widehat{\mathbf{Z}_p}$ as

$$0 \to \widehat{\mathbf{Z}_p} \xrightarrow{\varphi} \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots \to \ldots$$

Note that a zero can not be the next object in this resolution because φ is not a surjection. Instead, consider extending the resolution to

$$0 \to \widehat{\mathbf{Z}_p} \xrightarrow{\varphi} \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots \xrightarrow{\psi} \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots \to 0$$

where ψ is defined by the matrix representation

$$\begin{pmatrix} p & 0 & 0 & 0 & 0 & \ldots \\ -1 & p & 0 & 0 & 0 & \ldots \\ 0 & -1 & p & 0 & 0 & \ldots \\ 0 & 0 & -1 & p & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, ψ is defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (px_0, px_1 - x_0, px_2 - x_1, \ldots).$$

I will show that this map is surjective, strict, and that $\ker(\psi) = \mathrm{Im}(\varphi)$, thus completing the strict injective resolution (add zeros after this point).

If
$$x = \left(\frac{a_0}{p^{n_0}} + \mathbf{Z}, \frac{a_1}{p^{n_1}} + \mathbf{Z}, \frac{a_2}{p^{n_2}} + \mathbf{Z}, \ldots\right) \in \mathbf{Z}(p^\infty) \times \mathbf{Z}(p^\infty) \times \ldots$$
then the element
$$\left(\frac{a_0}{p^{n_0+1}} + \mathbf{Z}, \frac{a_0}{p^{n_0+2}} + \frac{a_1}{p^{n_1+1}} + \mathbf{Z}, \frac{a_0}{p^{n_0+3}} + \frac{a_1}{p^{n_1+2}} + \frac{a_2}{p^{n_2+1}} + \mathbf{Z}, \ldots\right)$$
maps to x under ψ. Therefore ψ is surjective.

In fact, if
$$x = \left(0, \ldots, 0, \frac{a_j}{p^{n_j}} + \mathbf{Z}, \frac{a_{j+1}}{p^{n_j+1}} + \mathbf{Z}, \frac{a_{j+2}}{p^{n_{j+2}}} + \mathbf{Z}, \ldots\right)$$
then similarly
$$\left(0, \ldots, 0, \frac{a_j}{p^{n_j+1}} + \mathbf{Z}, \frac{a_j}{p^{n_j+2}} + \frac{a_{j+1}}{p^{n_{j+1}+1}} + \mathbf{Z}, \ldots\right)$$
maps to x under ψ. Thus, ψ is strict.

Next consider the composition of φ followed by ψ. Suppose $a \in \widehat{\mathbf{Z}_p}$. Then
$$\varphi(a) = \left(\frac{a}{p} + \mathbf{Z}, \frac{a}{p^2} + \mathbf{Z}, \ldots\right).$$
Also,
$$\begin{aligned}\psi(\varphi(a)) &= \left(\frac{pa}{p} + \mathbf{Z}, \frac{pa}{p^2} - \frac{a}{p} + \mathbf{Z}, \ldots, \frac{pa}{p^{j+1}} - \frac{a}{p^j} + \mathbf{Z}, \ldots\right)\\ &= (0, 0, \ldots).\end{aligned}$$

Therefore, $\operatorname{Im}(\varphi) \subset \ker(\psi)$ and all that remains to be shown is that $\ker(\psi) \subset \operatorname{Im}(\varphi)$.

Let $(x_0, x_1, x_2, \ldots) \in \ker(\psi)$ where
$$x_i = \frac{b_i}{p^{n_i}} + \mathbf{Z}$$
for some integers $0 \leq b_i < p^{n_i}$ and $n_i \geq 0$ such that $\gcd(b_i, p) = 1$ if $b_i \neq 0$. Note that if $px_0 = 0$, then $\frac{b_0}{p^{n_0-1}} \in \mathbf{Z}$. That is, $p^{n_0-1} | b_0$. Thus, either $b_0 = 0$ or $n_0 = 1$. In either case I can write x_0 in the form
$$x_0 = \frac{b_0}{p} + \mathbf{Z}$$
described above. That is, I can take $n_0 = 1$. Now suppose that $i \geq 1$ and $n_{i-1} = i$. Then starting with $(x_0, x_1, x_2, \ldots)$ in $\ker(\psi)$ means $px_i - x_{i-1} = 0$. That is,
$$\frac{b_i}{p^{n_i-1}} - \frac{b_{i-1}}{p^i} \in \mathbf{Z}.$$

If $n_i - 1 \neq i$ then both $b_i = 0$ and $b_{i-1} = 0$. Otherwise, $n_i = i + 1$. Then, again in either case, I can write x_i in the form

$$x_i = \frac{b_i}{p^{i+1}} + \mathbf{Z}.$$

Therefore, by induction, $n_i = i + 1$ for all $n \geq 0$.

To end the argument I want to construct a p-adic integer, call it a, such that $\varphi(a) = (x_0, x_1, x_2, \ldots)$. I will do so by inductively finding each a_i so that $a = \sum_{i \geq 0} a_i p^i$ is as desired. Let $a_0 = b_0 (\text{mod } p)$. Then the first component of $\varphi(a)$ is

$$\frac{a}{p} + \mathbf{Z} = \frac{a_0}{p} + \mathbf{Z} = x_0.$$

Next I want a_1 such that

$$\frac{a_0 + a_1 p}{p^2} + \mathbf{Z} = \frac{b_1}{p^2} + \mathbf{Z}.$$

That is, I want a_1 such that $p^2 | b_1 - (a_0 + a_1 p)$. But $px_1 - x_0 = 0$ so $\frac{b_1}{p} - \frac{a_0}{p} \in \mathbf{Z}$. Thus, $p | b_1 - a_0$. Define $q_1 = \frac{b_1 - a_0}{p}$. Then $p^2 | b_1 - (a_0 + a_1 p)$ is equivalent to $p | q_1 - a_1$. So I let $a_1 = q_1 (\text{mod } p)$.

Now suppose I have found elements $a_0, \ldots, a_{i-1}$ by taking $a_j = q_j (\text{mod } p)$ for all $0 \leq j < i$ where q_j is the quotient after dividing $b_j - (a_0 + a_1 p + \ldots + a_{j-1} p^{j-1})$ by p^j (I know the quotient is an integer because x is in $\ker(\psi)$). Then I want to find a_i such that

$$x_i = \frac{a_0 + a_1 p + \ldots + a_{i-1} p^{i-1} + a_i p^i}{p^{i+1}} + \mathbf{Z}$$

which is equivalent to

$$p^{i+1} | b_i - (a_0 + a_1 p + \ldots a_i p^i).$$

But

$$p^i | b_i - (a_0 + a_1 p + \ldots + a_{i-1} p^{i-1})$$

because x is in $\ker(\psi)$. So if q_i is as above, then I want a_i such that

$$p^{i+1} | p^i (q_i - a_i).$$

That is, I want $p | q_i - a_i$. Clearly, taking $a_i = q_i (\text{mod } p)$ meets this criteria.

Therefore, by induction, the p-adic integer $a = a_0 + a_1 p + a_2 p^2 + \ldots$ where $a_i = q_i (\text{mod } p)$ for all i is the desired element.

Chapter 6 Strict Injective Envelopes in Other Filtered Categories

The goal of this chapter is to discuss further generalizations for strict injective envelopes in other filtered categories.

6.1 A Well-Ordering on N^p

One way to get a well-ordering on $\mathbf{N}^p$ is to use the lexicographical order. That is, $(a_1, a_2 \ldots, a_p) < (b_1, b_2, \ldots, b_p)$ provided that $a_1 < b_1$ or $a_i = b_i$ for $i = 1, \ldots j-1$ and $a_j < b_j$ for some $2 \leq j \leq p$. The well-ordering could then, in part, be listed out as

$$\begin{aligned}(0,\ldots,0) &< (0,\ldots,0,0,0,1) < (0,\ldots,0,0,0,2) < \ldots \\ &< (0,\ldots,0,0,1,0) < (0,\ldots,0,0,1,1) < \ldots \\ &< (0,\ldots,0,0,2,0) < \ldots \\ &< (0,\ldots,0,0,3,0) < \ldots \\ &< (0,\ldots,0,1,0,0) < \ldots\end{aligned}$$

So we will look at a filtration on a module M indexed by this set. Such a filtration will be called a multifiltration (as in Gomez-Torrecillas [9]) and is defined below.

6.2 The Category R-multifilt

Next, I want to extend the ideas from our previous work in **R-filt** to a category of multifiltered modules. Much like the category **R-filt**, we will be considering the category whose objects are R-modules, say M, with a descending filtration of submodules

$$M = M_{(0,\ldots,0,0)} \supset M_{(0,\ldots,0,1)} \supset \ldots$$

indexed by the lexicographical well-ordering on $\mathbf{N}^p$ such that M is complete and Hausdorff when given the topology induced by taking the filtration to be a fundamental system of neighborhoods of zero.

In addition, we require that if γ is a limit ordinal such that $\gamma < \beta$ for some fixed ordinal β then M/M_γ is complete where the filtration on M/M_γ is given by

$$\left(\frac{M}{M_\gamma}\right)_\alpha = \frac{M_\alpha}{M_\gamma} \text{ for } \alpha < \gamma.$$

Note that $M_\gamma \subset M_\alpha$ for all $\alpha < \gamma$ and hence M_γ is contained in the intersection of all such M_α. Conversely, suppose $\bar{m}$ is in M_α/M_γ for all $\alpha < \gamma$. Then $\bar{m}$ is in $(M/M_\gamma)_\alpha$ for all $\alpha < \gamma$. But requiring (M/M_γ) to be complete means that (M/M_γ) is Hausdorff (because completeness requires a metric and every metric space is Hausdorff). Therefore $\bar{m} = \bar{0}$ and thus $m \in M_\gamma$. Hence,

$$M_\gamma = \bigcap_{\alpha<\gamma} M_\alpha$$

is equivalent to the condition that M/M_γ is complete in the given topology.

Definition 6.2.1. A multifiltered module M is called *continuous* if it satisfies the two equivalent conditions:

1. M/M_γ is complete where the filtration on M/M_γ is given by
$$\left(\frac{M}{M_\gamma}\right)_\alpha = \frac{M_\alpha}{M_\gamma} \text{ for } \alpha < \gamma$$

2.
$$M_\gamma = \bigcap_{\alpha<\gamma} M_\alpha$$

If M and N are multifiltered R-modules (i.e. objects in **R-multifilt**) then a morphism in **R-multifilt** is a morphism $\phi : M \to N$ as R-modules such that $\phi(M_\alpha) \subset N_\alpha$ for all $\alpha \in \mathbf{N}^p$. Such morphisms are called *multifiltered*. Furthermore, such a morphism is called *strict* if $\phi(M_\alpha) = \phi(M) \cap N_\alpha$ for all $\alpha \in \mathbf{N}^p$.

Remark 6.2.2. I will be using the following notation. Let $\alpha = (a_1, a_2, \ldots, a_p)$ be an element of $\mathbf{N}^p$. Then by $\alpha + 1$, I mean the element

$$(a_0, a_1, \ldots, a_{p-1}, a_p + 1).$$

Most of our work in **R-filt** carries over in a straightforward fashion. The following result is of particular interest because we use it to show the existence of strict injective envelopes in **R-multifilt**. The proof is completely analogous to that of Proposition 5.8.2 in Chapter 5.

Proposition 6.2.3. *If $\{E^\alpha\}_{\alpha\in\mathbf{N}^p}$ is a family of injective modules, then the module*

$$E = \prod_{\alpha\in\mathbf{N}^p} E^\alpha$$

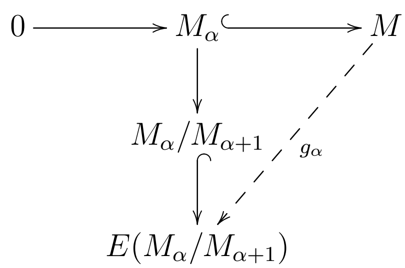
with multifiltration given by

$$E_\alpha = \prod_{\beta<\alpha} 0 \times \prod_{\beta\geq\alpha} E^\beta$$

*is a strict injective module in **R-multifilt**.*

6.3 Strict Injective Envelopes of N^p Multifiltered Modules

Let M be a multifiltered module. For each $\alpha \in \mathbf{N}^p$, consider the module $M_\alpha/M_{\alpha+1}$ and its injective envelope in **R-mod** denoted $E(M_\alpha/M_{\alpha+1})$. Then we may construct a map $g_\alpha : M \to E(M_\alpha/M_{\alpha+1})$ by completing the diagram:



Let $I = \prod_{\alpha \in \mathbf{N}^p} E(M_\alpha/M_{\alpha+1})$ have the multifiltration given by

$$I_\alpha = \prod_{\beta<\alpha} 0 \times \prod_{\beta \geq \alpha} E(M_\beta/M_{\beta+1}).$$

This module I will be the strict injective envelope of M in **R-multifilt** in the sense of the following proposition. But first a lemma,

Lemma 6.3.1. *Let $g : M \to I$ be the map defined by $m \mapsto (g_\alpha(m))_{\alpha \in \mathbf{N}^p}$. This map g is multifiltered, injective, and strict.*

Proof. If $m \in M_\alpha$, then $g_\beta(m) = 0$ for all $\beta < \alpha$ and hence $g(m) \in I_\alpha$. Therefore, g is multifiltered.

Suppose $g(m) = 0$. Then $g_\alpha(m) = 0$ for all $\alpha \in \mathbf{N}^p$. In particular, $g_{(0,\ldots,0)}(m) = 0$. Then from the above diagram we see that $m \in M_{(0,\ldots,0,1)}$. More generally, if $m \in M_\alpha$ then we see that $m \in M_{\alpha+1}$. Furthermore, if α is a limit element, then $m \in M_\alpha$ because $m \in M_\beta$ for all $\beta < \alpha$ and the multifiltration on M is continuous. Thus by transfinite induction, g is injective.

Let $i_\alpha \in g(M) \cap I_\alpha$. Then $i_\alpha = g(m)$ for some $m \in M$ and the β-components of i_α with $\beta < \alpha$ are zero. It follows that $g_\beta(m) = 0$ for all $\beta < \alpha$. Then using a similar argument to the above, one sees that $m \in M_\beta$ for all $\beta \leq \alpha$, i.e. $m \in M_\alpha$. Thus g is strict. $\square$

Proposition 6.3.2. *If $f : I \to I$ is a morphism in **R-multifilt** such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{g} & I \\ & \searrow^{g} & \downarrow f \\ & & I \end{array}$$

*is commutative, then f is an automorphism (in **R-multifilt**) of I.*

Proof. The proof of this proposition is very similar to the proof in the **R-filt** case. The only change in the argument of the first part is to change all the "n"s to "α"s. The map $gr(f) : gr(I) \to gr(I)$ defined by $(i_\alpha + I_{\alpha+1})_{\alpha \in \mathbf{N}^p} \mapsto (f(i_\alpha) + I_{\alpha+1})_{\alpha \in \mathbf{N}^p}$ is still an isomorphism because $\bar{f}_\alpha$ is an isomorphism for every $\alpha \in \mathbf{N}^p$. In order to show that $gr(f)$ being an isomorphism implies f is a bijection in **R-multifilt**, one must use transfinite induction instead of standard induction. The idea of the proof in both cases is the same and the continuity assumption is needed for the limit ordinal case of the transfinite induction.

If f is strict, then the inverse of f in **R-mod** is actually the inverse of f in **R-multifilt**. To show f is strict it is enough to show that $I_\alpha \subset f(I_\alpha)$. Furthermore, it is enough to show that $f(i) \notin I_{\alpha+1}$ for all $i \in I_\alpha \setminus I_{\alpha+1}$. That is, if $0 \neq z_\alpha \in E(M_\alpha/M_{\alpha+1})$ then $f(0, \ldots, 0, z_\alpha, z_{\alpha+1}, \ldots) = (0, \ldots, 0, w_\alpha, w_{\alpha+1})$ implies $w_\alpha \neq 0$. Note that $f(0, \ldots, 0, z_{\alpha+1}, \ldots) \in I_{\alpha+1}$ so w_α is completely determined by $f(0, \ldots, 0, z_\alpha, 0, \ldots)$. But since $\sigma_\alpha \circ \bar{f}_\alpha \circ \sigma_\alpha^{-1}$ is an isomorphism and induced by f we have $(\sigma_\alpha \circ \bar{f}_\alpha \circ \sigma_\alpha^{-1})(z_\alpha) = w_\alpha$ and therefore $w_\alpha \neq 0$ when $z_\alpha \neq 0$. □

6.4 Further Generalization

Thus far I have shown the existence of strict injective envelopes for two different categories of filtered modules; one with filtrations indexed by $\mathbf{N}$ (with ordinal number ω) and the other with filtrations indexed by $\mathbf{N}^p$ (with ordinal number $\omega + \omega + \ldots + \omega$ where there are n copies of ω). These results generalize even further to any well-ordered set. Note that there is an ordinal number $\omega + \omega + \ldots$ which is an infinite sum. Also note that any well-ordered set is isomorphic to the set of ordinal numbers $\{\alpha | \alpha < \beta\}$ where β is a fixed ordinal number.

So to generalize further, consider a category of complete, Hausdorff, and filtered modules where the filtration is indexed by any well-ordered set. Then all of the above arguments follow through by simply translating to indices from a set of ordinals $\{\alpha | \alpha < \beta\}$ which is isomorphic to the well-ordered set.

Appendix A: A Note on Strictness

The notion of strictness plays an important role throughout the preceding pages. One may wonder why the category **R-filt** is not assumed to have all of its morphisms be strict. The point of this note is to show several examples which indicate reasons why **R-filt** was not defined in this way.

If one were to require morphisms in **R-filt** to be strict and check the axioms of a category, then the first problem one encounters is the following:

Example A.0.1 (Composition of two strict morphisms need not be strict.)**.** Let $M \neq 0$ be a left R-module. Let $\bar{M} = M \oplus M$ with the filtration

$$\bar{M} \supset 0 \oplus M \supset 0 \supset 0 \supset \dots$$

i.e. $\bar{M}_1 = 0 \oplus M$ and $\bar{M}_n = 0$ for $n \geq 2$. Let $S = M \oplus 0 \subset \bar{M}$ have the filtration induced by the filtration on $\bar{M}$. That is,

$$S = S \cap \bar{M} \supset S \cap (0 \oplus M) = 0 \supset 0 \supset 0 \supset \dots$$

So $S_n = 0$ for $n \geq 1$. Then the inclusion map $S \hookrightarrow \bar{M}$ is strict.

Let $\Delta = \{(x,x) | x \in M\}$ which is closed in $\bar{M}$ because $\bar{M}$ is Hausdorff. Then let $\bar{M}/\Delta$ have the filtration

$$\frac{\bar{M}}{\Delta} \supset \frac{\bar{M}_1 + \Delta}{\Delta} \supset \frac{\bar{M}_2 + \Delta}{\Delta} \supset \dots$$

Then the canonical surjection $\bar{M} \to \bar{M}/\Delta$ is strict.

Finally, consider the composition $S \to \bar{M} \to \bar{M}/\Delta$. Let $x \in M$ such that $x \neq 0$. Then $(x,0) \in S$ and the composition maps $(x,0)$ to $(x,0)+\Delta$ which is also non-zero. But, $(x,0)+\Delta = ((x,0)+(-x,-x))+\Delta = (0,-x)+\Delta \in (\bar{M}/\Delta)_1$. However, there is no element in $S_1 = 0$ that maps to $(x,0)+\Delta$.

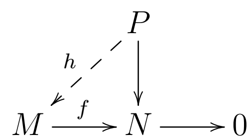
Therefore, the law of compositions required in the definition of a category could not be defined for all pairs of morphisms $f : X \to Y$ and $g : Y \to Z$. In the same setting, the following example shows that the sum of two strict morphisms need not be strict either.

Example A.0.2 (Sum of two strict morphisms need not be strict.)**.** In the same setting as above, consider the canonical injection $f : S \to \bar{M}$ and a map $g : S \to \bar{M}$

defined by $g(x,0) = (-x,-x)$. The map g is strict because $g(S) = \Delta$ and hence $g(S) \cap (0 \oplus M) = 0$. Suppose $x \neq 0$ is an element of M. Then $(f+g)(x,0) = (0,-x)$ is in $\bar{M}_1$. However, $S_1 = 0$ so $(f+g)(S) = 0$ and therefore, $f+g$ is not strict.

Another place where requiring all morphisms in **R-filt** to be strict differs from the results shown before is in regards to the notion of *strict projectivity.* Recall that a strict projective module was defined as follows:

Definition A.0.1. A module P in the category **R-filt** is called a **strict projective** module if we can complete a diagram of the form



to a commutative diagram whenever f is a strict surjection. These are the so-called $\mathcal{H}$-projectives where $\mathcal{H}$ is the set of all strict surjective morphisms in **R-filt**.

Were I to require that all morphisms in **R-filt** be strict, then it would make sense that this definition should additionally require that the lifting map h be strict as well. However, a direct analogue of Proposition 5.6.4 does not exist in this case. That is, the object

$$P = P^0 \times P^1 \times P^2 \times \dots$$

with each P^i projective and filtered by the submodules

$$P_n = 0 \times \dots \times 0 \times P^n \times P^{n+1} \times \dots$$

is not necessarily strictly projective if one requires h to be strict. For example,

Example A.0.3. Let $R = \mathbf{Z}$ be the ring of integers,

$$P = \mathbf{Z} \times 0 \times 0 \times \dots$$

be filtered as usual,

$$M = \mathbf{Z}/(4) \supset 2\mathbf{Z}/(4) \supset 0 \supset \dots,$$

$$S = 2\mathbf{Z}/(4) \supset 2\mathbf{Z}/(4) \supset 0 \supset 0 \supset \dots$$

(i.e. the induced filtration as a submodule of M), and

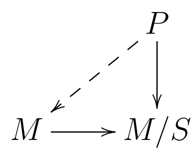
$$M/S = \frac{\mathbf{Z}/(4)}{2\mathbf{Z}/(4)} \cong \mathbf{Z}/(2)$$

be filtered by

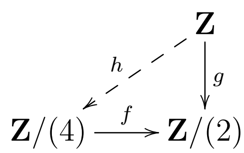
$$M/S \supset (M/S)_1 = \frac{M_1 + S}{S} = 0 \supset 0 \supset 0 \supset \dots .$$

Note that the canonical projection $\pi : \mathbf{Z} \to \mathbf{Z}/(2)$ induces a strict morphism in **R-filt** $\pi : P \to M/S$ given by $\pi(z, 0, 0, \dots) = z + 2\mathbf{Z}$. Also note that the canonical projection $M \to M/S$ given by $m \mapsto m + S$ is a strict surjection in this setting.

So consider the diagram



with the given maps. Any lifting is essentially a lifting in the diagram



where $f(x + 4\mathbf{Z}) = x + 2\mathbf{Z}$ and $g(x) = x + 2\mathbf{Z}$. Now g is surjective which means that fh will also be surjective. Therefore, $h(x) = 1 + 4\mathbf{Z}$ or $h(x) = 3 + 4\mathbf{Z}$ for at least one $x \in \mathbf{Z}$ in order for $1 + 2\mathbf{Z}$ to be in the range of fh. Suppose $h(x) = 1 + 4\mathbf{Z}$ for some $x \in \mathbf{Z}$. Then $h(2x) = 2 + 4\mathbf{Z}$, $h(3x) = 3 + 4\mathbf{Z}$, and $h(4x) = 0 + 4\mathbf{Z}$. On the other hand, if $h(x) = 3 + 4\mathbf{Z}$ for some $x \in \mathbf{Z}$ then $h(2x) = 2 + 4\mathbf{Z}$, $h(3x) = 1 + 4\mathbf{Z}$, and $h(4x) = 0 + 4\mathbf{Z}$. Thus, in either case, the map h is surjective. Therefore, $M_1 \neq 0$ will be in the image of any lifting. However, $P_1 = 0$ so the lifting can not be strict. $\square$

Thus, the characterization of projective modules given (Theorem 5.6.7) would not hold if we require that all morphisms be strict.

Bibliography

[1] Arnautov, V., Glavatsky, S., & Mikhalev, A. A. (1995). Introduction to the theory of topological rings and modules (Vol. 197). CRC Press.

[2] Cartan, H., & Eilenberg, S. (1956). Homological Algebra. Princeton Univ.

[3] Dimitrić, R. (1983). Slender modules over domains. Communications in Algebra, 11(15), 1685-1700.

[4] Dummit, D. S., & Foote, R. M. (2004). Abstract algebra.

[5] El Bashir, R., & Kepka, T. (1997). On when commutative noetherian rings are slender. Communications in Algebra, 25(8), 2585-2591.

[6] Enochs, E. (1964). A note on reflexive modules. Pacific J. Math, 14(3), 879-881.

[7] Enochs, E. E. (1981). Injective and flat covers, envelopes and resolvents. Israel journal of mathematics, 39(3), 189-209.

[8] Fuchs, L. Infinite abelian groups, vol. I (1970), vol. II (1973). Pure and Applied Math., Series, 36.

[9] Gómez-Torrecillas, J., & Lobillo, F. J. (2000). Global homological dimension of multifiltered rings and quantized enveloping algebras. Journal of Algebra, 225(2), 522-533.

[10] Kaplansky, I. (1958). Projective modules. Annals of Mathematics, 372-377.

[11] Lady, E. (1973). Slender rings and modules. Pacific J. of Math, 49, 397-406.

[12] Munkres, J. Topology. 2000. Prenctice Hall, US.

[13] Năstăcescu, C., & Van Oystaeyen, F. (1974). Graded and Filtered Rings and Modules. Lecture Notes in Math, 758.

[14] Nunke, R. J. (1961). Slender groups. Bulletin of the American Mathematical Society, 67(3), 274-275.

[15] O'Neill, J. D. (1991). Slender modules over various rings. Indian J. pure appl. Math, 22, 287-293.

[16] Rosický, J. (2002). Injectivity and accessible categories. Retrieved December 1, 2013, from http://www.math.muni.cz/~rosicky/ .

[17] Sjödin, G. (1973). On Filtered Modules and their Associated Graded Modules. Mathematica Scandinavica, 33, 229-249.

[18] Specker, E. (1950). Additive Gruppen von Folgen ganzer Zahlen. Portugaliae Mathematica, 9(3), 131-140.

[19] Vermani, L. R. (2004). An elementary approach to homological algebra. CRC Press.

Vita

Raymond Edward Kremer
Born in Fort Thomas, Kentucky

Education

- **University of Kentucky**, Lexington, Kentucky
  - M.A., Mathematics, December 2010
  - B.S., Mathematics, May 2008

Teaching Experience

- **Teaching Assistant:**
  - University of Kentucky, August 2008 - May 2014

Awards and Fellowships

- A&S Certificate for Outstanding Teaching, Spring 2011
- Daniel R. Reedy Quality Achievement Award, Fall 2008 - Spring 2011