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Rees Products of Posets and Inequalities

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ABSTRACT OF DISSERTATION

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Rees Products of Posets and Inequalities

ABSTRACT OF DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
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Lexington, Kentucky

Director: Dr. Margaret A. Readdy, Professor of Mathematics
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ABSTRACT OF DISSERTATION

Rees Products of Posets and Inequalities

In this dissertation we will look at properties of two different posets from different perspectives. The first poset is the Rees product of the face lattice of the n-cube with the chain. Specifically we study the Möbius function of this poset. Our proof techniques include straightforward enumeration and a bijection between a set of labeled augmented skew diagrams and barred signed permutations which label the maximal chains of this poset. Because the Rees product of this poset is Cohen-Macaulay, we find a basis for the top homology group and a representation of the top homology group over the symmetric group both indexed by the set of labeled augmented skew diagrams. We also show that the Möbius function of the Rees product of a graded poset with the t-ary tree and the Rees product of its dual with the t-ary tree coincide. We discuss labelings for Rees and Segre products in general, particularly the Rees product of the face lattice of a polytope with the chain. We also look at cases where the Möbius function of a poset is equal to the permanent of a matrix and we consider local h-vectors for the barycentric subdivision of the n-cube. In each section we state open conjectures. The second poset in this dissertation is the Dowling lattice. In particular we look at the $k = 1$ case, that is, the partition lattice. We study inequalities on the flag vector of the partition lattice via a weighted boustrophedon transform and determine a more generalized version for the Dowling lattice. We generalize a determinantal formula of Niven and conclude with conjectures and avenues of study.

KEYWORDS: algebraic combinatorics, commutative algebra, Möbius function, poset topology, representation theory

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Dedicated to my family
and
in memory of my father
John Dement Muldoon, III

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Chapter 1 Introduction

In combinatorics we value using different tools and techniques to study a single object. In particular, interaction between disciplines in mathematics is a valuable aim. In this dissertation we will look at properties of two different posets from many angles. The first will be studied using combinatorics, topology, and representation theory and the second through a transformation of sequences, determinants, and analysis.

In Chapter 1 we begin with an overview of the themes and results in this dissertation. We then introduce some tools and concepts from algebra, topology, and combinatorics which we will use in later chapters. It is meant to be a quick orientation to these topics and references will be given for further research.

Chapter 2 is a study of the Rees product of the cubical lattice and the chain. This poset has an interesting sequence of Möbius values given by a multiple of the permanent of a certain matrix. The permanent can be viewed as a signed analogue of the derangement numbers. Furthermore, this poset is Cohen-Macaulay, and hence has only nonzero homology in the top dimension. We can describe the top homology of the order complex of the Rees product of the cubical lattice with the chain using a set of labeled skew diagrams. These diagrams lead us to a representation of the top homology as a direct sum of skew Specht modules.

Chapter 3 contains various results that came up during the course of studying Rees products. R-, EL-, and CL-labelings are given for Segre and Rees products of any two posets each having one of these labelings. A description is given for the top homology group of the Rees product of any poset homotopic to a sphere and the chain using the skew diagrams discussed in the second chapter. We will also give a partial solution to a conjecture from Chapter 1 about permanents of a particular class of matrices, and we will state a conjecture about the coefficients of the local h-vector of the barycentric subdivision of the cube.

In Chapter 4 we leave the Rees product behind to study the flag f-vector of the Dowling lattice in general and the partition lattice in particular. This data, encoded using **ab**-words, can be counted using different methods. We will use a weighted boustrophedon transformation and a determinantal formula to determine this descent set statistic. The latter technique leads to an interesting relationship between matrices of Whitney numbers of the first and second kinds for classes of posets which satisfy regularity conditions in their lower or upper ideals. We conclude with some partial results on maximizing descent words in the partition lattice and some conjectures.

1.1 Dissertation Overview: Themes and Results

The Möbius function μ of a partially ordered set (poset) is an important invariant in combinatorics. It is a generalization of the well-known number theoretic Möbius function. (Recall for an integer n that $\mu(n) = 0$ if $p^2|n$ for some prime p and $\mu(n) = (-1)^k$ if n is a product of k distinct primes.) More formally, for an interval $[x, y]$ in a poset the Möbius function is given recursively by

$$\mu(x,y) = -\sum_{x \le z < y} \mu(x,z),$$

with $\mu(x,x) = 1$. This poset generalization was popularized by Rota and can be found in his seminal paper on the Möbius function [40] and is originally due to Hall [27]. To recapture the number-theoretic Möbius function $\mu(n)$, one simply computes the Möbius function of $\mu(\hat{0}, \hat{1})$ in the divisor lattice.

The Möbius function links combinatorics to the field of algebraic topology. Philip Hall's Theorem gives an interpretation of the Möbius function as the alternating sum

$$\mu(x,y) = c_o - c_1 + c_2 - c_3 + \cdots,$$

where c_i is the number of chains $x = x_0 < x_1 < \cdots < x_i = y$ of length i in the interval $[x,y]$. This is precisely the Euler characteristic of the order complex of the interval $[x,y]$.

An important class of posets are the Eulerian posets. A graded poset P with rank function ρ is *Eulerian* if $\mu(x,y) = (-1)^{\rho(x,y)}$ for every interval $[x,y]$ in P where $\rho(x,y) = \rho(y) - \rho(x)$, that is, the length of the interval $[x,y]$. Examples of Eulerian posets include the face lattice of a convex polytope and the strong Bruhat order [51]. The Eulerian condition says that every interval of the poset satisfies the Euler-Poincaré relation.

In this dissertation we consider a poset operation called the Rees product. This is a poset product defined by Björner and Welker [13] to model a particular case of the commutative algebra version of the Rees algebra. Björner and Welker conjectured that the Rees product of the Boolean algebra with the chain is given by the nth derangement number. Recall the nth derangement number is the number of fixed point free permutations in the symmetric group on n elements. Jonsson proved this conjecture using poset matching techniques [32].

Since the cubical lattice is a signed analogue of the Boolean algebra, it is natural to next consider the Rees product of the face lattice of the n-dimensional cube with the chain. Using the recursive definition of the Möbius function we obtained the following expression for this Möbius function. (See Proposition 2.3.2.)

Proposition 1.1.1 *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathscr{C}_n, C_{n+1})) = -1 + \sum_{i=0}^{n} (-1)^{n-i} \cdot 2^{n-i} \binom{n}{i} (i+1)(n-i)!.$$

Notice these Möbius values are given by an alternating sum. We would prefer to have an expression where all the individual terms are positive. Finding an R-labeling for this poset will enable us to find such an expression.

Given a poset P one can associate to it a Hasse diagram. Loosely speaking this is a directed graph whose vertices are the elements of P and (x,y) is a directed edge in the graph if x is covered by y in the poset. See Figure 1.2 for an example of a Hasse diagram. An edge labeling of the Hasse diagram of a poset P is a map from the set of edges of P to another poset. An edge labeling is called an *R-labeling* if in every interval there exists a unique labeled maximal chain whose labels are increasing. A falling chain is one whose labels are nonincreasing, that is, adjacent labels are either decreasing or incomparable.

The following theorem allows one to compute the Möbius function of a poset having an R-labeling by counting falling chains. It is due to Stanley in the case of admissible lattices [46], Björner for R-labelings and edge lexicographic labelings [9], and Björner–Wachs for non-pure posets with a CR-labeling [12].

Theorem 1.1.2 (Stanley, Björner, Björner–Wachs) *Let P be a graded poset of rank n with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. Suppose P has an R-labeling. Then with respect to this R-labeling,*

$$\mu(\hat{0}, \hat{1}) = (-1)^n \cdot \text{number of falling maximal chains in } P.$$

Computing Möbius functions using this theorem is a standard technique in algebraic combinatorics. There are many classes of posets which have an R-labeling, including supersolvable and semimodular lattices [48, Chapter 3].

In Section 2.4 we find an R-labeling for the Rees product of the cubical lattice with the chain. This gives a second expression for the Möbius function. (See Theorem 2.5.3.)

Theorem 1.1.3 *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathscr{C}_n, C_{n+1})) = (-1)^n \cdot \sum_{c} 2^{n-c_1} \binom{n}{c_1, \ldots, c_k} \cdot c_1 \cdot \prod_{i=2}^{k} (c_i - 1),$$

where the sum is over all compositions $c = c_1 + \cdots + c_k$ of n and $1 \leq k \leq n$.

Although Theorem 1.1.3 expresses the Möbius function of the Rees product of the cubical lattice with the chain using positive terms, one still cannot recognize this as some signed derangement number. To do this we give a bijective proof of the Möbius function, that is, we find a class of objects which are in a one-to-one correspondence with the falling chains occurring in the R-labeling. More precisely we associate to each falling chain a labeled augmented skew diagram. We find a bijection between the set of unsigned and unaugmented diagrams and fixed point free permutations. This bijection extends to a bijection between the falling chains of the Rees product of the cubical lattice with the chain and augmented permutations where the 1-cycles are signed one way and all other cycles are signed in one of two ways. (See Theorem 2.6.1.)

Theorem 1.1.4 *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathscr{C}_n, C_{n+1})) \quad = \quad (-1)^n \cdot n \cdot \text{per} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 1 & 2 & \cdots & 2 \\ 2 & 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 1 \end{bmatrix}.$$

that is, n times the permanent of a square $(n-1) \times (n-1)$ matrix having 1's on the diagonal and 2's everywhere else.

Observe that the usual derangement numbers are given by the permanent of the matrix with zeros on the diagonal and ones on the off diagonals. Thus the permanent appearing in this theorem can be viewed as a signed analogue of the derangement numbers. As a corollary our bijection specializes to give a short and elegant proof of Jonsson's theorem.

We have discovered a relationship between the Rees product of a graded poset P with a t-ary tree and the Rees product of the dual poset P^* with a t-ary tree. A t-ary tree $T_{t,n+1}$ is a graded poset of rank n having t^k elements of rank k for $0 \leq k \leq n$ with every non-leaf element covered by precisely t children. We show that the Möbius function of $\text{Rees}(P, T_{t,n+1})$ is equal to the Möbius function of $\text{Rees}(P^*, T_{t,n+1})$. As a special case this says $\mu(\text{Rees}(P, C_n)) = \mu(\text{Rees}(P^*, C_n))$ for the chain C_n on n elements.

Independently Shareshian and Wachs study the Rees product of a number of posets with the t-ary tree including the Boolean algebra, its q-analogue, and a truncated face lattice of the n-dimensional cross-polytope [43]. The goal of their work was to study joint distributions of various permutation statistics and their q-analogues. Unlike our proofs, all of their Möbius function results are enumerative but not bijective. Although the face lattice of the cross-polytope and the cubical lattice are dual posets, the truncated version is missing the factor of n we have in Theorem 1.1.4.

The Möbius function of a poset P also has an interpretation in terms of the reduced Euler characteristic of a simplicial complex called the order complex of P. Recall that for Δ an abstract simplicial complex the *reduced Euler characteristic* of Δ is

$$\widetilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i,$$

where f_i is the number of i-dimensional faces in Δ and $f_{-1} = 1$ provided $\Delta \neq \emptyset$. For any graded poset P with a maximal element $\hat{1}$ and minimal element $\hat{0}$, we can form an abstract simplicial complex $\Delta(P)$ called the *order complex* of P where the k-faces in the order complex correspond to the chains of length k in $\widehat{P} = P - \{\hat{0}, \hat{1}\}$. Restating Philip Hall's Theorem, we have Proposition 3.8.6 in [48] as follows.

Proposition 1.1.5 (Hall) *The Möbius function of a poset P is given by*

$$\mu_P(\hat{0}, \hat{1}) = \widetilde{\chi}(\Delta(P)).$$

Given an abstract simplicial complex Δ, the reduced Euler characteristic χ can be defined for X, that is, the geometric realization of the abstract simplicial complex Δ. We have

$$\widetilde{\chi}(X) = \sum_i (-1)^i \text{rank } \widetilde{H}_i(X, \mathbb{Z}),$$

where $\widetilde{H}_i(X, \mathbb{Z})$ is the ith reduced homology group of X over the integers $\mathbb{Z}$. This result is particularly useful when a poset P is Cohen-Macaulay. Recall if a poset P is *Cohen-Macaulay* then the reduced homology groups of its order complex vanish in all but the top dimension. In this case the rank of the top homology group is given up to a sign by the Möbius function of the poset.

Björner and Welker [13] proved that the order complex of the Rees product of two posets whose order complexes are each Cohen-Macaulay is also Cohen-Macaulay. They also showed that if two posets have order complexes which are homotopically Cohen-Macaulay then so is their Rees product. Recall an abstract simplicial complex is *homotopically Cohen-Macaulay* if every interval is homotopic to a wedge of spheres of dimension the length of the interval minus two. It is well-known that the order complex of the cubical lattice and the order complex of the chain are each Cohen-Macaulay. Hence as a corollary the order complex $\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1}))$ has the homotopy type of a wedge of n times a signed derangement number of spheres of dimension n.

Next it is natural to find a basis for the top homology group, that is, a set of fundamental cycles which index the spheres describing the homotopy type of the order complex of the poset. More interestingly, one would like to find a natural combinatorial object to describe a basis. In 1982 Björner constructed an integer basis for the reduced homology of the order complex of the partition lattice, and more generally, of any geometric lattice [10]. The fundamental cycles are formed from taking joins of atoms in the lattice. Other results in this vein are due to Wachs, who gave a basis for the top homology group of the order complex of the d-divisible partition lattice using a set of labeled skew diagrams [58], Sundaram–Wachs, who found a basis for the top homology group of the order complex of the k-equal partition lattice [55], and Gottlieb, who gave a basis for the h, k-Equal Dowling lattice using a set of labeled trees [26].

In this dissertation, we find a basis for the top homology group of the order complex of the Rees product of the cubical lattice with the chain. It is indexed by the set of falling chains determined by the R-labeling of the poset. For each falling chain we describe a subposet whose order complex is homotopic to the suspension of the barycentric subdivision of the boundary of the n-dimensional cube, that is, homotopic to an n-dimensional sphere. For more details, see Theorem 2.7.5.

In [55] and [58], respectively, the authors find a representation of the symmetric group on the order complex of the d-divisible, respectively, k-equal partition lattices. For a brief description of representation theory, see Section 1.4. In this dissertation we give a representation for the top homology group of the order complex of the Rees product of the cubical lattice with the chain. (See Theorem 2.8.1.)

Theorem 1.1.6 *There exists an $\mathfrak{S}_n$-module isomorphism between*

$$H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$$

and $\bigoplus 2^{n-|\lambda_1|} S^\lambda$ *where the direct sum is over all partitions* λ *with each* λ_i *shaped into hooks as described in Section 2.6 taken with multiplicity* $2^{n-|\lambda_1|}$.

The second focus of this dissertation is to study flag vector inequalities for posets. In order to do this we need a generalization of the Principle of Inclusion and Exclusion known as the Möbius Inversion Theorem.

Given a field k, the *incidence algebra* of a poset P is the k-algebra of all functions from the set of intervals $[x, y]$ of the poset to the field k. The identity in this k-algebra is given by the Kroenecker delta function, that is,

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

See [5] and [18] for more examples of functions on the incidence algebra. Two very important functions in the incidence algebra of a poset are the zeta function ζ and the Möbius function μ. For $x, y \in P$ the *zeta function* is given by $\zeta(x, y) = 1$ if $x \leq y$ and zero otherwise. The zeta function can be used to count the number of chains (or multichains) of a given length in a poset. Because $\zeta(x, x) \neq 0$ for all $x \in P$, the zeta function has a unique inverse function [48, Proposition 3.6.2]. The Möbius function is precisely the inverse of the zeta function, that is, $\mu\zeta = \delta$.

The following result is known as the Möbius Inversion Theorem. It is originally due to Weisner [60] and independently by Hall [27]. It can be found in Chapter 3 Proposition 2 in Rota's treatise about the Möbius function [40] and, in a slightly different form, appears in Proposition 3.7.1 in [48].

Theorem 1.1.7 (Möbius Inversion Theorem) *Let* P *be a finite poset and let* f *and* g *be functions defined on elements of* P*, that is,* $f, g : P \longrightarrow \mathbb{C}$*. Then*

$$g(x) = \sum_{y \leq x} f(y) \text{ holds for all } x \in P$$

if and only if

$$f(x) = \sum_{y \leq x} g(y)\mu(y, x) \text{ holds for all } x \in P.$$

This combinatorial Möbius Inversion Theorem is analogous to the classical Möbius Inversion Theorem of number theory.

We now apply the Möbius Inversion Theorem to derive the flag h-vector from the flag f-vector of a poset. For a graded poset P of rank n a *flag* is a chain $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ of elements in P. For $S = \{s_1 < s_2 < \cdots < s_{k-1}\} \subseteq [n-1] = \{1, 2, \ldots, n-1\}$ the *flag f-vector* consists of the 2^n values $(f_S)_{S \subseteq [n-1]}$ where the entry f_S is given by $f_S = \#\{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ with the rank $\rho(x_i) = s_i$ for $i = 1, \ldots, k-1$. The *flag h-vector* is then defined as

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T.$$

Equivalently, by the Möbius Inversion Theorem

$$f_S = \sum_{T \subseteq S} h_T.$$

In the case of the face lattice of a convex polytope, we have that $h_S = h_{\overline{S}}$ where $\overline{S}$ is the complement of S. This result has been proven in special cases by Björner–Garsia–Stanley [11] and Stanley [47]. Danilov and Kouchnirenko independently prove this relation using the Stanley-Reisner ring [16, 33]. Bayer–Biller show that the only linear relations which hold among the flag h-vector entries for polytopes, and more generally, completely balanced homology spheres are $h_\emptyset = 1$ and $h_S = h_{\overline{S}}$ [3]. This means that the flag h-vector is a more compact way to encode the face incidence data of a polytope. See Chapter 4 for a more detailed background on flag f- and flag h-vectors of polytopes.

The flag h-vector can be used to describe the descent set statistic of a graded poset P of rank n having an R-labeling λ which labels the edges in the Hasse diagram. Let $m : \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ be a maximal labeled chain in the poset P. To this chain associate a word $w(m) = w_1 w_2 \cdots w_{n-1}$ in the noncommutative letters **a** and **b** where $w_i = \mathbf{a}$ if there is an ascent at the ith position in the maximal chain, that is, $\lambda(x_{i-1}, x_i) < \lambda(x_i, x_{i+1})$ and $w_i = \mathbf{b}$ otherwise. For an **ab**-word w the *descent set statistic* $[w]_P$ is the number of maximal labeled chains in P having descent word w. The descent set statistic is another way to describe the flag h-vector. We have

$$[w]_P = h_S \text{ where } S = \{i : w_i = \mathbf{b}\}.$$

One natural question to ask is which words maximize the descent statistic. The earliest result in this area is due to Niven. We state it in more modern language.

Theorem 1.1.8 (Niven) *For the Boolean algebra B_n on n elements, equivalently the symmetric group $\mathfrak{S}_n$, the following inequality holds. For **ab**-words u and v*

$$[u\boldsymbol{aa}v]_{B_n} < [u\boldsymbol{ab}\overline{v}]_{B_n},$$

*where $\overline{v}$ denotes the **ab**-word v with the **a**'s and **b**'s uniformly exchanged.*

For example, Niven's inequality shows that in the symmetric group $\mathfrak{S}_6$ the number of permutations with descents at positions 1 and 5 is less than the number of permutations with descents at positions 1, 3, and 4, that is,

$$[\mathbf{baaab}]_{B_6} < [\mathbf{babba}]_{B_6}.$$

It is well-known that an R-labeling λ of the Boolean algebra is given by the unique element in $T - S$, where T covers S in the Boolean algebra. Thus the maximal chains in the Boolean algebra B_n are in a one-to-one correspondence with the $n!$ permutations in the symmetric group $\mathfrak{S}_n$.

The face lattice of the n-dimensional simplex is isomorphic to the Boolean algebra B_n. Niven's inequality implies the following maximization result for the flag h-vector of the Boolean algebra.

Theorem 1.1.9 (Niven, de Bruijn) *For the Boolean algebra B_n, equivalently the symmetric group $\mathfrak{S}_n$, the alternating words $\boldsymbol{bab}\cdots$ and $\boldsymbol{aba}\cdots$ of length $n-1$ maximize the descent set statistic.*

This theorem has been proven in many different ways. It is a direct consequence of Niven's inequality. Niven also proved this theorem using a determinantal formula for permutations having descent set S. See Section 4.5 and Niven's paper [37]. Independently de Bruijn described a simple algorithm which we will discuss in detail below [17]. Viennot [56] gave a recursive algorithm to construct the set of alternating permutations. Stanley [51, Corollary 2.9] and Readdy [39, Theorem 3.0.4] showed it follows from the nonnegativity of the **cd**-index. The **cd**-index is another poset invariant which encodes the flag f-vector data of the face lattice of a polytope, and more generally, of an Eulerian poset by removing all the linear relations among the flag f-vector entries. For a proof of the existence of the **cd**-index, see [4] for the case of the face lattice of a polytope and see [52] for Eulerian posets. For the first proof of the nonnegativity of the **cd**-index, see [52]. Bayer–Billera determined all linear relations which hold among the flag f-vector entries, known as the generalized Dehn-Sommerville relations [3]. Sagan–Yeh–Ziegler consider families of subposets of the Boolean algebra and prove Niven's inequality injectively [42]. Each of these techniques is helpful for calculating individual statistics as well as calculating the maximum descent word.

There is one more way to calculate the descent statistic of a given **ab**-word w. The *boustrophedon transformation* is an operation that takes a sequence $\{a_i\}_{i\geq 0}$ to another sequence $\{b_i\}_{i\geq 0}$ via a triangular array. It has been given in various forms by de Bruijn [17], Arnol'd [1], Millar–Sloane–Young [35], and Ehrenborg–Mahajan [22].

Ehrenborg and Mahajan use the following boustrophedon transformation to calculate the descent set statistics in the Boolean algebra. Given an **ab**-word $w = w_1 w_2 \cdots w_{n-1}$, define a triangular array with elements $s_{m,i}$, $1 \leq i \leq m$ by

$$s_{1,1} = 1$$

and

$$s_{m,i} = \begin{cases} \displaystyle\sum_{j=1}^{i-1} s_{m-1,j} & \text{if } w_{m-1} = \mathbf{b}, \\ \displaystyle\sum_{j=i}^{m-1} s_{m-1,j} & \text{if } w_{m-1} = \mathbf{a}, \end{cases}$$

where $m = 2, \ldots, n$ and $i = 1, \ldots, m$. In other words, if w ends with a **b**, add all the elements from the $(n-1)$st row which are located above and to the left of $s_{n,i}$ and if w ends with an **a**, add all the elements from the $(n-1)$st row which are above and to the right of $s_{n,i}$. This boustrophedon transform can be thought of as a discrete form of integration.

Lemma 1.1.10 (Ehrenborg–Mahajan) *If w is an $\boldsymbol{ab}$-word of length $n-1$ then*

$$[w]_{B_n} = \sum_{j=1}^{n} s_{n,j}.$$

Furthermore, the entry $s_{n,j}$ equals the number of permutations in the symmetric group having descent word w beginning with the element j. In [22] the authors also prove a conjecture of Gessel concerning runs in a permutation. Loosely speaking, we say a permutation $\pi \in \mathfrak{S}_n$ has k runs if its descent word is of the form $a^{i_1}b^{i_2}a^{i_3}\cdots b^{i_k}$ (or $b^{i_1}a^{i_2}b^{i_3}\cdots a^{i_k}$) if k is even and $a^{i_1}b^{i_2}a^{i_3}\cdots a^{i_k}$ (or $b^{i_1}a^{i_2}b^{i_3}\cdots b^{i_k}$) if k is odd. The descent set statistic over all subsets with k runs is maximized when each run is about n/k long. Ehrenborg and Majahan give an analogous result for the subspace lattice, that is, the q-analogue of the Boolean algebra.

In this dissertation we study flag vector inequalities for the partition lattice. Although the partition lattice is not the face lattice of a polytope, it is geometrically motivated. It is the intersection lattice of the hyperplane arrangement consisting of the n coordinate hyperplanes $x_i = 0$ for $1 \leq i \leq n$ and the $\binom{n}{2}$ hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$. The partition lattice is a nontrivial example of a Cohen-Macaulay poset which is not the face lattice of a polytope. Furthermore the partition lattice is not Eulerian, so its flag vector data, encoded using the **ab**-index, cannot be reduced to the more compact **cd**-index. However, the partition lattice does enjoy many favorable properties, including being supersolvable. This latter result is due to Stanley who also showed how to obtain EL-labelings for supersolvable lattices [45]. In particular, this EL-labeling gives an R-labeling for the partition lattice.

Using the R-labeling of the partition lattice, we show the flag h-vector of the partition lattice is given by a weighted boustrophedon transform. We now briefly describe this. Given an **ab**-word $w_1 w_2 \cdots w_{n-1}$ define a new triangular array $\{t_{m,i}\}$ to count the **ab**-words in the partition lattice by

$$t_{1,1} = 1$$

and

$$t_{m,i} = \begin{cases} i \displaystyle\sum_{j=1}^{i-1} t_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{b}, \\ i \displaystyle\sum_{j=i}^{m-1} t_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{a}, \end{cases}$$

where $m = 2, \ldots, n$ and $i = 1, \ldots, m$. We have

Corollary 1.1.11 *The descent statistic for an **ab**-word w in Π_n is given by*

$$[w]_{\Pi} = \sum_{i=1}^{n} t_{n,i}.$$

See Corollary 4.3.4.

In [22] further generalizations are made for transformations to enumerate statistics for the subspace lattice, that is, the lattice of subspaces of an n-dimensional vector space over a finite field with q elements.

We can also define a weighted boustrophedon transform for a class of lattices which generalize the partition lattice. These are known as the Dowling lattices. Let

ζ be a kth root of unity. Let H_n be the hyperplane arrangement in $\mathbb{C}^n$ consisting of the hyperplanes

$$\begin{array}{ll} z_i = \zeta^h z_j & \text{for } 1 \leq i < j \leq n \text{ and } 0 \leq h \leq k-1, \\ z_i = 0 & \text{for } 1 \leq i \leq n. \end{array}$$

Definition 1.1.12 *The* Dowling lattice L_n *is the intersection lattice of the hyperplane arrangement* H_n *ordered by reverse inclusion.*

The Dowling lattice is a generalization of the partition lattice since when $k = 1$ we have $L_n \cong \Pi_{n+1}$. We now define a generalized weighted boustrophedon transformation to compute the descent set statistics of the Dowling lattice. Given an **ab** word $w = w_1 w_2 \cdots w_{n-1}$ of length $n-1$, define a new triangular array with elements $t^k_{m,i}$ recursively by

$$t^k_{1,1} = 1$$

and

$$t^k_{m,i} = \begin{cases} (k(i-1)+1) \displaystyle\sum_{j=1}^{i-1} t^k_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{b}, \\ (k(i-1)+1) \displaystyle\sum_{j=i}^{m-1} t^k_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{a}, \end{cases}$$

for $m = 2, \ldots, n$ and $i = 1, \ldots m$. In a similar manner as the partition lattice, we show the descent set statistic may be computed using this weighted transform. See Corollary 4.4.6.

Corollary 1.1.13 *The descent statistic for an **ab**-word w in the kth Dowling lattice is given by the row sum*

$$[w]_L = \sum_{i=1}^{n} t^k_{n,i}.$$

Ehrenborg and Readdy have conjectured the extremal configuration for maximizing the descent set statistic for the partition lattice [23].

Conjecture 1.1.14 (Ehrenborg–Readdy) *For the partition lattice Π_n, the descent statistic is maximized when*

$$w = \begin{cases} (\boldsymbol{ba})^k \cdot \boldsymbol{b} & \text{when } n = 2k+3, \\ (\boldsymbol{ba})^k \cdot \boldsymbol{bb} & \text{when } n = 2k+4. \end{cases}$$

Notice that the extremal configuration is an alternating **ab**-word or almost alternating **ab**-word, depending on the parity of n.

Using flag vector techniques we obtain the following inequalities. See Theorem 4.6.3.

Theorem 1.1.15 *The flag h-vector of the partition lattice satisfies the following inequalities:*

$$\begin{aligned} [\boldsymbol{aa} \cdot w]_\Pi &< [\boldsymbol{ab} \cdot w]_\Pi, \\ [\boldsymbol{bbb} \cdot w]_\Pi &< [\boldsymbol{bba} \cdot w]_\Pi, \\ [\boldsymbol{ba} \cdot \boldsymbol{a}^i]_\Pi &< [\boldsymbol{bb} \cdot \boldsymbol{a}^i]_\Pi \quad (i \geq 1). \end{aligned}$$

We also obtain termwise inequalities using the weighted transform. See Theorem 4.6.8.

Theorem 1.1.16 *The flag h-vector of the partition lattice satisfies the following inequalities for any* $\boldsymbol{ab}$*-word* w*.*

1. $[w \cdot \boldsymbol{aa}]_\Pi < [w \cdot \boldsymbol{ab}]_\Pi$,
2. $[w \cdot \boldsymbol{aab}]_\Pi < [w \cdot \boldsymbol{abb}]_\Pi$,
3. $[w \cdot \boldsymbol{aba}]_\Pi < [w \cdot \boldsymbol{abb}]_\Pi$,
4. $[w \cdot \boldsymbol{bba}]_\Pi < [w \cdot \boldsymbol{bab}]_\Pi$,
5. $[w \cdot \boldsymbol{bbb}]_\Pi < [w \cdot \boldsymbol{bab}]_\Pi$.

Unfortunately these inequalities do not imply Conjecture 1.1.14. However after computing small examples, we make a similar conjecture for the Dowling lattice. See Conjecture 4.6.15.

Conjecture 1.1.17 *For the Dowling lattice* L_n*, the descent statistic is maximized when*

$$w = \begin{cases} (\boldsymbol{ba})^k \cdot \boldsymbol{b} & \text{when } n = 2k+2, \\ (\boldsymbol{ba})^k \cdot \boldsymbol{bb} & \text{when } n = 2k+3. \end{cases}$$

1.2 Poset preliminaries

A partially ordered set or *poset* P is a finite set of elements with an order relation denoted $<_P$ or $<$ for short satisfying the axioms of reflexivity, antisymmetry, and transitivity. Let $\hat{0}$ denote the unique minimal element and let $\hat{1}$ denote the unique maximal element in the poset P. A *chain* is any string $c : x_0 < x_1 < \cdots < x_k$ of comparable elements in P. We say y *covers* x and write $x \prec y$ when there does not exist an element $z \in P$ such that $x < z < y$. A chain in P is *maximal* or *saturated* if we have $\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_k = \hat{1}$. Let $\leq_P$ denote the order relation less than or equal to in the poset P. For $x, y \in P$ an interval $[x, y]$ is the set of all elements $z \in P$ such that $x \leq z \leq y$. A poset is *graded* if all its maximal chains have the same length in any interval of the poset. The *rank function* of a graded poset P is the map $\rho : P \longrightarrow \mathbb{N} \cup \{0\}$ from the elements of P to the set of natural numbers including zero where $\rho(x)$ equals the length of any maximal chain from the unique minimal element $\hat{0}$ to x. All of the posets under consideration in this dissertation will be graded and bounded, that is, each has a unique minimal element $\hat{0}$, a unique

Figure 1.1: The chain of length three.

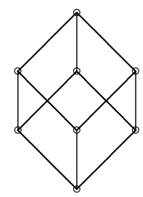

Figure 1.2: The Boolean algebra on three elements.

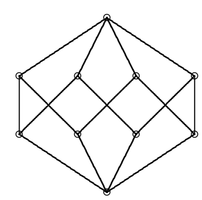

Figure 1.3: The face lattice of the square.

maximal element $\hat{1}$ and a rank function ρ. The *Hasse diagram* of a poset is a graph whose vertices are the elements of the poset and whose edges are the cover relations. Figure 1.1, Figure 1.2, and Figure 1.3 are examples of the Hasse diagrams of graded and bounded posets. The maximal chains are any path from the bottom element to the top element.

Sometimes it is of interest to consider particular subposets of P. Let $\widehat{P}$ denote the poset $P - \{\hat{0}, \hat{1}\}$. For any poset P of rank n and a set $S \subseteq [n-1]$, we can define the *rank-selected subposet* P_S by $P_S = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$ with the partial order inherited by P. Finally, the *Möbius function* is defined inductively by $\mu(x, x) = 1$ for all x in P and $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$ for all $x < y$ in P. There is a slight abuse of notation here where we write $\mu(x, y)$ to denote $\mu([x, y])$. Please see Rota [40] and Stanley [48, Chapter 3] for more details.

In our notation, C_n will denote the chain on n elements and $\mathscr{C}_n$ will be the face lattice of the n-cube, that is, the lattice whose elements are faces of the n-dimensional cube ordered by inclusion.

Definition 1.2.1 *An* abstract simplicial complex Δ *is set of vertices* V *where* $V = \{v_1, v_2, \ldots, v_k\}$ *and a family of subsets* $\Delta \subseteq 2^V$ *satisfying*

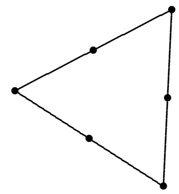

Figure 1.4: The order complex of the Boolean algebra on three elements.

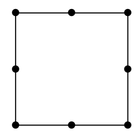

Figure 1.5: The order complex of the face lattice of the square.

1. *$v_i \in \Delta$ for all $v_i \in V$,*

2. *if $F, G \subseteq V$ are such that $F \subseteq G$ and $G \in \Delta$ then $F \in \Delta$.*

Given a bounded poset P of rank n there is an associated topological complex, called its order complex. The *order complex* of P, denoted $\Delta(P)$, is the abstract simplicial complex whose k-faces correspond exactly to the chains of length k in $\widehat{P} = P - \{\hat{0}, \hat{1}\}$. In particular, elements of the poset $\widehat{P}$ correspond to vertices of $\Delta(P)$ and maximal chains in $\widehat{P}$ correspond to facets in $\Delta(P)$. It is straightforward to see that $\Delta(P)$ is a simplicial complex, that is, a complex whose faces are all simplices. The order complex of the Boolean algebra on three elements is given in Figure 1.4 and the order complex of the square is given in Figure 1.5. The former is the barycentric subdivision of the boundary of the simplex on three elements and the latter is the barycentric subdivision of the boundary of the square.

1.3 Labelings

We begin with some facts about R-labelings.

Given a poset P an *edge labeling* is a map $\lambda : E(P) \to \Lambda$, where $E(P)$ denotes the edges in the Hasse diagram of P and the labels form a poset Λ with order relation $<_\Lambda$. An edge labeling λ is said to be an *R-labeling* if in every interval $[x, y]$ of P there is a unique saturated chain $c : x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ whose labels are rising, that is, which satisfies $\lambda(x_0, x_1) <_\Lambda \lambda(x_1, x_2) <_\Lambda \cdots <_\Lambda \lambda(x_{k-1}, x_k)$. Given a maximal chain $m : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$ in P of rank n, the *descent set* of m is the set $D(m) = \{i : \lambda(x_{i-1}, x_i) \not<_\Lambda \lambda(x_i, x_{i+1})\} \subseteq \{1, 2, \ldots, n-1\}$. Alternatively, when we view the labels of the maximal chain as the word $\lambda(m) = \lambda_1 \cdots \lambda_n$, where $\lambda_i = \lambda(x_{i-1}, x_i)$, there is a descent in the ith position of $\lambda(m)$ if the labels λ_i and λ_{i+1}

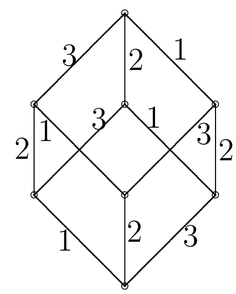

Figure 1.6: An EL-labeling on the Boolean algebra on three elements.

are either incomparable in the label poset Λ or satisfy $\lambda_i >_\Lambda \lambda_{i+1}$. In particular, a maximal chain m is said to be *rising* if its descent set satisfies $D(m) = \emptyset$ and *falling* if $D(m) = \{1, \ldots, n\}$.

The existence of an R-labeling for a poset is useful in that it gives an alternate way to compute the Möbius function μ of a poset. The following theorem is due in various cases to Stanley [46], Björner [9], and Björner–Wachs [12].

Theorem 1.3.1 (Stanley, Björner, Björner–Wachs) *Let P be a graded poset of rank n with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. Suppose P has an R-labeling. Then with respect to this R-labeling,*

$$\mu(\hat{0}, \hat{1}) = (-1)^n \cdot \textit{number of falling maximal chains in } P$$

There are other types of edge labelings. An edge labeling λ is said to be an *edge-lexicographical labeling* or EL-labeling of P if for any interval $[x, y]$ in P there exists an increasing maximal chain which lexicographically precedes all other maximal chains in the interval.

For example, the Boolean algebra on n elements, B_n, is the set of subsets of $[n]$ ordered by inclusion. For $A, B \in B_n$, an edge labeling on $\lambda : \mathscr{E}(B_n) \longrightarrow [n]$ is given by

$$\lambda(A, B) = i \text{ where } i \text{ is the unique element in } B - A.$$

Figure 1.6 shows an EL-labeling of the Boolean algebra on three elements.

EL-labelings are useful as they give a shelling order for the order complex of P, that is, a way to linearly order the facets of the complex so that one can build the complex one facet at a time and preserve the property of being spherical at all shelling steps except possibly the last. Please see the section which follows for more details.

Another poset labeling is a *chain-edge labeling*. For a poset P, let $\mathscr{E}^*(P)$ be the set of pairs $(c, x \prec y)$ consisting of a maximal chain c and an edge $x \prec y$ of that chain. A *chain-edge labeling* is a map $\delta : \mathscr{E}^*(P) \longrightarrow \Lambda$ where Λ is some poset which satisfies the property that if two chains agree along their first k edges then their labels also coincide along these edges. Further, there are CR- and CL-labelings, which are specific cases of chain-edge labelings defined similarly to the R- and EL-labelings. CL-labeling also imply shellability of the poset.

Variations of Theorem 1.3.1 are due to Stanley in the case of admissible lattices, Björner for R-labelings and edge lexicographic labelings, and Björner–Wachs for non-pure posets with a CR-labeling. See [12] for historical details. For a more comprehensive coverage of edge labelings and chain-edge labelings see [36].

1.4 Homology and shellability

Topological combinatorics is a branch of mathematics which provides a link between algebraic combinatorics and many other disciplines including group theory, commutative algebra, representation theory, and topology. We refer the reader to Wachs' paper [59] for a more comprehensive overview.

Given a n-dimensional simplicial complex Δ, we can define its homology groups. Let A_k be the linear span of the set of k-faces of the simplicial complex Δ. Let ∂_k be the boundary map from A_k to A_{k-1} which takes a face to its boundary, that is,

$$\partial_k(\{x_1, x_2, \ldots, x_{k+1}\}) = \sum_{i=1}^{k+1} (-1)^i \{x_1, \ldots, \widehat{x}_i, \ldots, x_{k+1}\}.$$

The sets of faces and boundary maps form a chain complex:

$$0 \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0.$$

Since $\partial^2 = \partial_k \circ \partial_{k-1} = 0$, we have $\mathrm{im}(\partial_{k-1}) \subseteq \ker(\partial_k)$. The *kth homology group* is given by the quotient

$$H_k(\Delta) = \frac{\ker(\partial_k)}{\mathrm{im}(\partial_{k+1})}.$$

We can dualize the chain complex above replacing each A_k with its dual group $A_k^* = \mathrm{Hom}(A_k; \mathbb{Z})$ and replacing the boundary map ∂_k with it dual coboundary map $\delta_k = \partial_k^* : A_{k-1}^* \longrightarrow A_k^*$. We have the cochain complex

$$0 \xleftarrow{\delta_{n+1}} A_n \xleftarrow{\delta_n} \cdots \xleftarrow{\delta_2} A_1 \xleftarrow{\delta_1} A_0 \xleftarrow{\delta_0} 0,$$

with the *kth cohomology group* given by

$$H^k(\Delta) = \frac{\ker(\delta_{k+1})}{\mathrm{im}(\delta_k)}.$$

We are concerned with the homology and cohomology groups because homotopically Cohen-Macaulay complexes with isomorphic homology groups have the same homotopy type. In particular we will be considering such complexes, that is, complexes whose homology groups vanish in all but the top dimension and simplicial complexes with a property called shellability.

Definition 1.4.1 *An abstract simplicial complex* Δ *is* pure *if all of its maximal faces have the same dimension. It is* shellable *if its facets can be arranged in a linear order* $F_1, F_2, \ldots, F_k$ *so that the subcomplex* $\{G : G \in F_i \cap (\overline{F_1 \cup \cdots \cup F_i}), 1 \leq i \leq k-1\}$ *is*

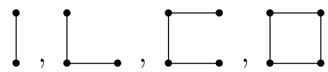

Figure 1.7: A shelling of the boundary of the square.

a pure complex and is (dim $F_k - 1$)-dimensional. In the case Δ is of dimension zero, that is, consists of a finite number of vertices, any ordering of the vertices is defined to be a shelling order.

See Figure 1.7 for an example of a shelling.

Shellable simplicial complexes are well-behaved because they are always homotopic to a wedge of spheres in varying dimensions. Two topological spaces X and Y are homotopic to each other if there exists two continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the composition $g \circ f$ is homotopic to the identity map on X and the composition $f \circ g$ is homotopic to the identity map on Y. In this case f and g are called *homotopy equivalences.* Loosely speaking two continuous functions are homotopic if there exists a deformation between them. In the case of the $\mathscr{C}_2$, the order complex $\Delta(\mathscr{C}_2)$ is homotopic to wedge of one 1-dimensional sphere. More details can be found in [36].

Björner and Wachs [12] developed the theory of lexicographic shellability which uses decreasing labeled chains in a poset to describe the topology of its order complex. An important theorem is the following.

Theorem 1.4.2 (Björner and Wachs) *Suppose P is a graded poset with an EL-labeling. Then the order complex $\Delta(P)$ has the homotopy type of a wedge of spheres, where the number of i-dimensional spheres is the number of decreasing maximal chains of length $i + 2$ in P. Furthermore, these chains form a basis for the cohomology $\widetilde{H}^i(\Delta(P); \mathbb{Z})$.*

Recall the number of decreasing maximal chains in a poset P is, up to a sign, equal to the Möbius function of P.

In this thesis, we consider posets which are not only shellable, but also Cohen-Macaulay over a field k. A poset P is said to be *Cohen-Macaulay* if for all $x < y$ in $\widehat{P}$ the reduced simplicial homology $\widetilde{H}_i(\Delta([x, y]); k)$ vanishes for $i \neq \rho(y) - \rho(x) - 2$. In particular, a poset P is *homotopically Cohen-Macaulay* when the order complex of the interval $[x, y]$ is homotopic to a wedge of sphere of dimension $\rho(y) - \rho(x) - 2$. In this case, because the homology groups are finitely generated and torsion free, the decreasing maximal chains of the poset P will form a basis for the top homology and cohomology groups, that is, the Möbius function determines the dimension of the top homology and cohomology groups. Figures 1.1, 1.2, and 1.3 are all examples of posets which are shellable and Cohen-Macaulay.

9			
5	10		
2	6		
8	3	4	1

Figure 1.8: A Young tableau of shape $\lambda = (1, 2, 2, 4)$.

1.5 Representations and Specht modules

We will also need some tools from combinatorial representation theory. For an introduction, we refer the reader to Sagan's book [41].

A *representation* Φ of a group G on a vector space V over a field k is a group homomorphism

$$\Phi : G \longrightarrow GL(V)$$

where $GL(V)$ is the general linear group of the vector space. When G has such a representation, we call V a *G-module*. Representations enable one to describe group elements as matrices and the group operation as matrix multiplication. The general linear group of V is often much easier to understand and study than the group itself. A representation is *irreducible* if the G-module V does not contain a nontrivial submodule. In particular, irreducible representations of the symmetric group have been well-studied. See [30] or [41] for further information.

In Chapter 2 we will briefly review the construction of the irreducible representations of the symmetric group $\mathfrak{S}_n$ on n elements over the complex numbers. It can be shown that the number of irreducible representations of a group is equal to the number of conjugacy classes. For the symmetric group on n elements, these conjugacy classes can be indexed by partitions of n.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n, that is, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ where $\lambda_i \in \mathbb{Z}^+ = \{1, 2, \ldots\}$ and $\sum_{1 \leq i \leq l} \lambda_i = n$. (Note, these partitions are constructed in "French" notation rather than the "English" notation where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$.) The *Ferrers diagram* associated to λ is an array of n boxes having l left-justified rows with row i containing λ_i boxes where we read rows from top to bottom. A *Young tableau of shape* λ is a Ferrers diagram of shape λ labeled with positive integers. Figure 1.8 is a Young diagram of shape $\lambda = (1, 2, 2, 4)$. In Chapter 2 we will use these partitions to index the irreducible representations.

Sometimes we wish to consider skew Ferrers diagrams. If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_p)$ are partitions such that $\mu \subseteq \lambda$, i.e., $\mu_{i-\ell+p} \leq \lambda_i$ for $\ell - p + 1 \leq i \leq \ell$, then the *skew diagram*, λ/μ is the set of boxes

$$\lambda/\mu = \{c : c \in \lambda \text{ and } c \notin \mu\}.$$

For example, if μ is the Ferrers diagram of shape $\mu = (1, 1, 2)$ in Figure 1.9, then Figure 1.10 gives the skew diagram λ/μ.

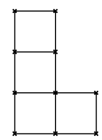

Figure 1.9: The Ferrers diagram $\mu = (1, 1, 2)$.

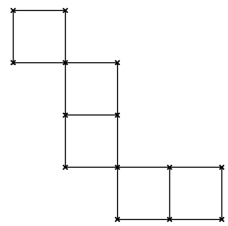

Figure 1.10: The skew diagram $\lambda/\mu = (1, 2, 2, 4)/(1, 1, 2)$.

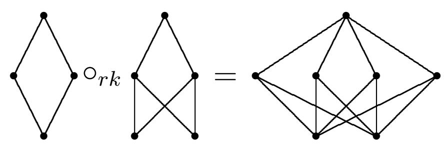

Figure 1.11: The Segre product of two posets.

Skew Specht modules can be similarly defined as the submodules spanned by the polytabloids e_t, where t has skew shape λ/μ. These submodules are not always irreducible, but the Littlewood-Richardson Rule allows one to decompose these modules into a series of irreducible Specht modules. See James' work [31] for further details.

1.6 Segre and Rees products

Following Björner–Welker [13], we have the following definitions. For two graded and bounded posets P and Q of rank n, the *Segre product* or *rank product* $P \circ_{rk} Q$ consists of the elements $\{(p, q) : p \in P, q \in Q, \rho(p) = \rho(q)\}$ partially ordered by $(p, q) \leq (p', q')$ if $p \leq_P q'$ and $q \leq_Q q'$. The rank of $P \circ_{rk} Q$ is n. See Figure 1.11 for an example.

For two graded posets P and Q with rank function ρ, the Rees product $P \star Q$, is the set of ordered pairs (p, q) in the Cartesian product $P \times Q$ with $\rho(p) \geq \rho(q)$. These pairs are partially ordered by $(p, q) \leq (p', q')$ if $p \leq_P p'$, $q \leq_Q q'$, and $\rho(p') - \rho(p) \geq \rho(q') - \rho(q)$. The Rees product is actually a special case of the Segre product. The

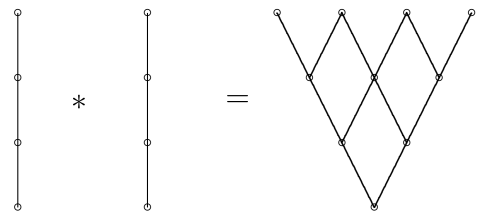

Figure 1.12: The Rees product of two posets.

Rees product between two posets P and Q is equivalent to the Segre product $P \circ_{rk} \widetilde{Q}$ where $\widetilde{Q}$ is the rank-selected subposet of the Cartesian product $Q \times C_n$ selecting elements of rank at most n. That is

$$P \star Q \simeq P \circ_{rk} \widetilde{Q} \simeq P \circ_{rk} (Q \times C_n)_{[0,n]}$$

See Figure 1.12 for an example.

The definitions of these two posets are based upon the commutative algebraic notions of the Segre products of rings and the Rees algebra. Following [57], let R be a Noetherian ring and let I be an ideal defined by a set of generators $I = (f_1, \ldots, f_k)$. For an indeterminate variable t, the *Rees algebra* is the subring $R[f_1t, \ldots, f_kt] \subset R[t]$. A fundamental question in commutative algebra is: when is the Rees algebra Cohen-Macaulay. See [57] for more information on Rees algebras.

Björner and Welker consider the special case when $R = \oplus_{i\geq 0} R_i$ is a standard graded k-algebra and $I = \oplus_{i\geq 1} R_i$ is the maximal ideal $\mathfrak{m}_R$. If A is another standard graded k-algebra then, as in [13], we define the Rees product of R and A as

$$R * A = \bigoplus_{i\geq 0} \mathfrak{m}_R^i \otimes_k A_i.$$

When $A = k[t]$, the Rees product $R * A$ is the Rees algebra $R[t]$ for $I = \mathfrak{m}$.

In order to describe some applications of the poset-theoretic Rees product to the commutative algebraic Rees algebra, we need some more terminology. A *semigroup* is a set with an operation on the set which satisfies associativity. An *affine semigroup* is a finitely generated semigroup containing a zero element. The Rees product of two posets has applications to commutative algebra when the posets considered are intervals of affine semigroup posets Λ whose order is given by $\alpha \leq \beta$ when there exists an element $\gamma \in \Lambda$ such that $\alpha + \gamma = \beta$. In this situation Björner–Welker [13] consider when this product is Cohen-Macaulay. A result of Peeva–Reiner–Sturmfels states in the case where the poset is an interval in the affine semigroup, this interval is Cohen-Macaulay if and only if the ring $k[\Lambda]$ over a field k if Koszul, that is, if Λ has a linear resolution over the ring [38]. Björner and Welker apply this result to show the weighted Segre product of two affine semigroup rings is Koszul.

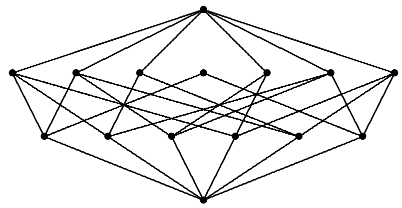

Figure 1.13: The partition lattice on four elements.

1.7 Partition lattice

We have previously described the partition lattice as the intersection lattice of a hyperplane arrangement. See Section 1.2 for details. See Figure 1.13 for the partition lattice on four elements. Notice that the upper intervals of the form $[x, \hat{1}]$ are isomorphic to smaller partition lattices. This behavior is true in general and will be exploited in Chapter 4.

The partition lattice is the classic example of a supersolvable lattice. A lattice L is said to be *supersolvable* if there is a maximal chain m such that for every chain c in L the sublattice generated by m and c is a distributive lattice. This notion is due to Stanley [45]. In the case G is a supersolvable finite group and L is its lattice of subgroups, the maximal chain m is given by the maximal chain of normal subgroups corresponding to a chief series of G.

The canonical way to obtain an R-labeling λ for a supersolvable lattice with M-chain $m : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$ is

$$\lambda(x, y) = \min\{i : x \vee x_i = y \vee x_i\}.$$

Using the maximal chain $m : 1/2/3/\cdots/m \prec 12/3/\cdots/m \prec \cdots \prec 123\cdots n$ in the partition lattice Π_n we obtain an R-labeling as follows. Suppose the two blocks B_1 and B_2 of a partition π are merged to form a partition σ. We have $\pi \prec \sigma$ and

$$\lambda(\pi \prec \sigma) = \max\{\min\{B_1, B_2\}\} - 1.$$

Notice that the maximal chains in the partition lattice are labeled with permutations from the symmetric group $\mathfrak{S}_{n-1}$. Furthermore there are maximal chains which are labeled by the same permutation. This observation will be critical in determining the weighted version of the boustrophedon transform that we develop to compute the flag h-vector of the partition lattice explicitly.

Chapter 2 Rees Product of the Cubical Lattice and the Chain

2.1 Introduction

Björner and Welker [13] initiated a study to generalize concepts from commutative algebra to the area of poset topology. Motivated by the ring-theoretic Rees algebra, one of the new poset operations they define is the Rees product.

Definition 2.1.1 *For two graded posets P and Q with rank function ρ the* Rees product*, denoted $P \star Q$, is the set of ordered pairs (p, q) in the Cartesian product $P \times Q$ with $\rho(p) \geq \rho(q)$. These pairs are partially ordered by $(p, q) \leq (p', q')$ if $p \leq_P p'$, $q \leq_Q q'$, and $\rho(p') - \rho(p) \geq \rho(q') - \rho(q)$.*

The rank of the resulting poset is $\rho(P \star Q) = \rho(P)$. For more details concerning the Rees product and other poset products, see [13].

From the perspective of topological combinatorics, one of the most important results that Björner and Welker show in their paper is that the poset theoretic Rees product preserves the Cohen-Macaulay property; see [13].

Theorem 2.1.2 (Björner–Welker) *If P and Q are two Cohen-Macaulay posets then so is the Rees product $P * Q$.*

Very little is known about the Rees product of specific examples of Cohen-Macaulay posets. However, what has been studied has yielded rich combinatorial results. The first example in this vein is due to Jonsson [32], who settled an open question of Björner and Welker concerning the Rees product of the Boolean algebra with the chain. For brevity, throughout we will use the notation Rees(P, Q) to denote the Rees product

$$\text{Rees}(P, Q) = ((P - \{\hat{0}\}) * Q) \cup \{\hat{0}, \hat{1}\}.$$

As usual, we will assume that P and Q are graded posets with P having unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$.

Theorem 2.1.3 (Jonsson) *The Möbius function of the Rees product of the Boolean algebra B_n on n elements with the n element chain C_n is given by the nth derangement number, that is,*

$$\mu(\text{Rees}(B_n, C_n)) = (-1)^n \cdot D_n.$$

Recall the nth derangement number D_n is the number of permutations in the symmetric group on n elements having no fixed points ($D_0 = 1$ and $D_n = n \cdot D_{n-1} + (-1)^n$ for $n \geq 1$). Jonsson's original proof uses an non-acyclic element matching to show the Euler characteristic vanishes appropriately. Figures 2.1 and 2.2 are two examples.

In this chapter we study the signed version of Jonsson's results, that is, the Rees product of the rank $n+1$ cubical lattice $\mathscr{C}_n$, (i.e., the face lattice of the n-dimensional

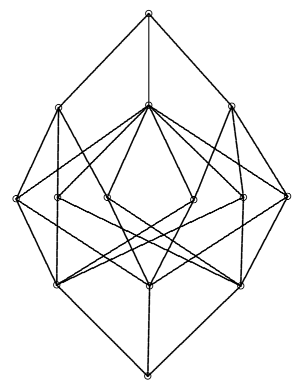

Figure 2.1: Rees(B_3, C_3).

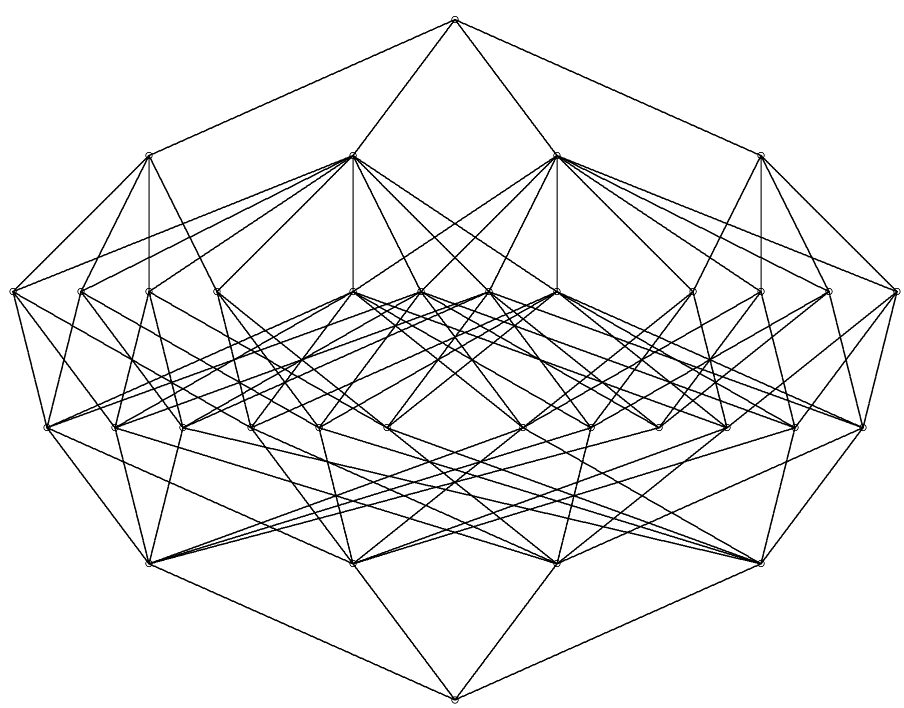

Figure 2.2: Rees(B_4, C_4).

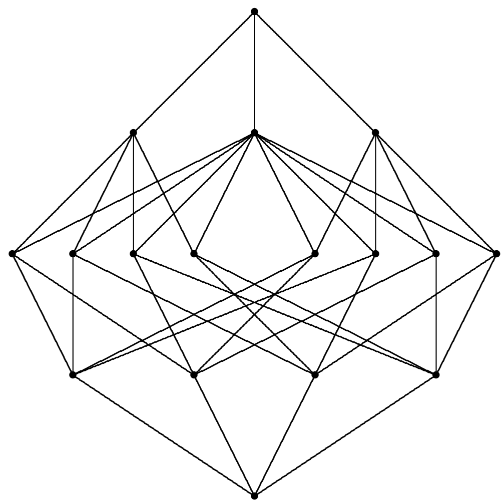

Figure 2.3: Rees($\mathscr{C}_2, C_3$).

cube) with the chain. See Figure 2.3 and Figure 2.4 for examples. In each of these examples, notice there are 2^n copies of $\mathscr{C}_n$ in Rees($\mathscr{C}_n, C_{n+1}$).

In the next section we begin by expressing the flag f-vector of the Rees product of any graded poset with a t-ary tree in terms of the flag f-vector of the original poset. We obtain the surprising conclusion that the Möbius function of the poset with the tree coincides with the Möbius function of its dual with the tree.

Besides using various poset techniques to give explicit formulas for the Möbius function of Rees($\mathscr{C}_n, C_{n+1}$), we use poset homology techniques to find an explicit basis for the reduced homology and determine a representation of the homology of the order complex of Rees($\mathscr{C}_n, C_{n+1}$) over the symmetric group. As a corollary to our enumerative results, we give a bijective proof of Jonsson's results. We end this chapter with further questions.

2.2 Rees product of graded posets with a tree

In this section we determine the flag vector of the Rees product of any graded poset P with a t-ary tree. As a consequence we show the Möbius function of the Rees products Rees($P, T_{t,n+1}$) and Rees($P^*, T_{t,n+1}$) coincide, although the posets are not isomorphic in general.

For nonnegative integers n and t, let $T_{t,n+1}$ be the poset corresponding to an t-ary tree of rank n, that is, the poset consisting of t^k elements of rank k for $0 \leq k \leq n$ with each nonleaf element covered by exactly t children. Observe that the 1-ary tree $T_{1,n+1}$ is precisely the $(n+1)$-chain C_{n+1}. Recall for a graded poset P of rank $n+1$ and $S = \{s_1, \ldots, s_k\} \subseteq \{1, \ldots, n\}$ with $s_1 < \cdots < s_k$, the *flag f-vector* $f_S = f_S(P)$ is the number of chains $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ with $\rho(x_i) = s_i$.

We now define two weight functions. Here we use the notation $[k]$ to denote the t-analogue of the nonnegative integer k, i.e., $[k] = 1 + t + \cdots + t^{k-1}$.

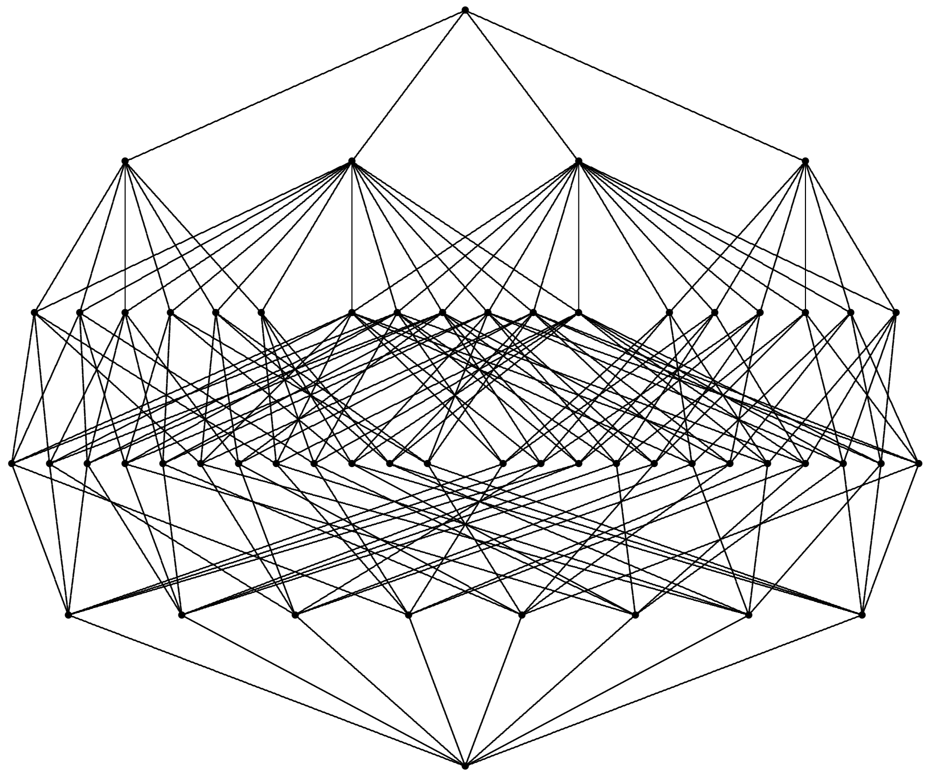

Figure 2.4: $\mathrm{Rees}(\mathscr{C}_3, C_4)$.

Definition 2.2.1 *For a nonempty subset $S = \{s_1 < \cdots < s_k\} \subseteq \mathbb{P}$ define*

$$\mathrm{w}(S) = [s_1] \cdot [s_2 - s_1 + 1] \cdots [s_k - s_{k-1} + 1]$$

with $\mathrm{w}(\emptyset) = 1$. *For a nonempty subset $S = \{s_1 < \cdots < s_k\} \subseteq \{1, \ldots, n\}$ define*

$$\mathrm{v}(S) = \mathrm{w}(S \cup \{n+1\}) - \mathrm{w}(S) = t \cdot \mathrm{w}(S) \cdot [(n+1) - s_k]$$

with $\mathrm{v}(\emptyset) = t \cdot [n]$.

Lemma 2.2.2 *For a graded poset P of rank $n+1$, let $R = \mathrm{Rees}(P, T_{t,n+1})$. Then*

$$f_S(R) = \mathrm{w}(S) \cdot f_S(P), \tag{2.1}$$
$$f_{S \cup \{n+1\}}(R) = \mathrm{w}(S \cup \{n+1\}) \cdot f_S(P), \tag{2.2}$$

for $S \subseteq \{1, \ldots, n\}$.

Proof: Consider first $S = \{s_1 < \cdots < s_k\} \subseteq \{1, \ldots, n\}$. Given an element x_1 of rank $\rho(x_1) = s_1$ from the poset P, there are $[s_1]$ copies of it in the Rees poset R. Each of these copies has $[s_2 - s_1 + 1]$ elements in R of rank s_2 which are greater than it with respect to the partial order of the Rees poset R. In general, each rank s_i

element in R has $[s_{i+1} - s_i + 1]$ elements greater than it in the Rees poset R. Hence relation (2.1) holds.

To show (2.2), note the maximal element $\hat{1}$ of P gets mapped to the $[n+1]$ coatoms of the Rees poset R. In particular $[(n+1) - s_k + 1]$ of these elements will cover a given element of rank s_k in R. Hence the result follows. $\square$

Lemma 2.2.3 *For a graded poset P of rank $n+1$, let $R = \text{Rees}(P, T_{t,n+1})$. Then*

$$\mu(R) = \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} \cdot \text{v}(S) \cdot f_S(P).$$

Proof: By Philip Hall's theorem, we have

$$\begin{aligned} \mu(R) &= \sum_{S \subseteq \{1,\ldots,n+1\}} (-1)^{|S|-1} f_S(R) \\ &= \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} f_S(R) + \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} f_{S \cup \{n+1\}}(R) \\ &= \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} \text{w}(S) \cdot f_S(P) + \sum_{S \subseteq \{1,\ldots,n\}} (-1)^{|S|} \text{w}(S \cup \{n+1\}) \cdot f_S(P), \end{aligned}$$

where we have rewritten the sum using Lemma 2.2.2. Combining the two sums proves the desired identity. $\square$

Theorem 2.2.4 *For a graded poset P of rank $n+1$ we have*

$$\mu(\text{Rees}(P, T_{t,n+1})) = \mu(\text{Rees}(P^*, T_{t,n+1})),$$

where P^* is the dual of P. In particular, for the chain on $n+1$ elements we have

$$\mu(\text{Rees}(P, C_{n+1})) = \mu(\text{Rees}(P^*, C_{n+1})).$$

Proof: Let $S = \{s_1 < \cdots < s_k\} \subseteq \{1, \cdots, n\}$. The result follows by noting that $\text{v}(S) = \text{v}(S^{\text{rev}})$, where the reverse of S is $S^{\text{rev}} = \{n+1-s_k < n+1-s_{k-1} < \cdots < n+1-s_1\}$ and applying Lemma 2.2.3. $\square$

It is clear from the definition of the weight $\text{v}(S)$ that the Möbius function $\mu(\text{Rees}(P, T_{t,n+1}))$ is divisible by t. When the poset has odd rank we can say more.

Corollary 2.2.5 *For a graded poset P of odd rank $n+1$, the Möbius function $\mu(\text{Rees}(P, T_{t,n+1}))$ is divisible by $[2] = 1+t$. In particular, for a graded poset P of odd rank $n+1$, the Möbius function $\mu(\text{Rees}(P, C_{n+1}))$ is even.*

Proof: Observe that $1+t$ divides $[k]$ if and only if k is even. Hence $1+t$ does not divide $\text{v}(S)$ for a set $S = \{s_1 < \cdots < s_k\}$ implies that s_1 is odd, s_i has the same parity as s_{i+1} and $n+1-s_k$ is odd. This implies that n is odd. Hence that n is even implies that the weight $\mathrm{v}(S)$ is divisible by $1+t$ for all subsets S, including the empty set. Thus by Lemma 2.2.3 the Möbius function of $\mathrm{Rees}(P, T_{t,n+1})$ is divisible by $1+t$. $\square$

2.3 Rees product of the cubical lattice with the chain

In this section we give an explicit formula for the Möbius function of the poset $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$. After finding an R-labeling in Section 2.4, we relate the Möbius function with a class of permutations, that is, the double augmented barred signed permutations. These are in a one-to-one correspondence with certain skew diagrams. We will return to these when we consider homological questions for $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$. In Section 2.6 we give a bijective proof of the Möbius function result expressed as a permanent of a certain matrix.

We will represent an element $(x, i) \in \mathrm{Rees}(\mathscr{C}_n, C_{n+1}) \backslash \{\hat{0}, \hat{1}\}$ as an ordered pair where $x = (x_1, x_2, \ldots, x_n) \in \{0, 1, *\}^n$ and $i \in \{1, \ldots, n\}$. Observe that such an element (x, i) has rank k if there are exactly $k-1$ stars appearing in its first coordinate, $1 \leq i \leq k$.

For a graded poset P with minimal element $\hat{0}$ and maximal element $\hat{1}$, throughout we will use the shorthand $\mu(P)$ to denote the Möbius function $\mu_P([\hat{0}, \hat{1}])$.

Proposition 2.3.2 gives an explicit formula for the Möbius function of the poset $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$. The proof will require the following lemma.

Lemma 2.3.1 *The following identity holds:*

$$1 + \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} k! (n-k+1) = 0.$$

Proof: Define sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ by $a_n = (-1)^{n+1} n!$ and $b_n = n+1$. These sequences have exponential generating functions

$$A(x) = \sum_{n\geq 0} (-1)^{n+1} x^n = -\frac{1}{1+x}$$

and

$$B(x) = \sum_{n\geq 0} (1+n) \frac{x^n}{n!} = (1+x)e^x.$$

Thus, $D(x) = A(x)B(x) = -e^x$. But

$$\begin{aligned} D(x) &= \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!} \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} k! (n-k+1) \frac{x^n}{n!}, \end{aligned}$$

which proves the claim. $\square$

Proposition 2.3.2 *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathscr{C}_n, C_{n+1})) = -1 + \sum_{i=0}^{n} (-1)^{n-i} \cdot 2^{n-i} \binom{n}{i} (i+1)(n-i)!.$$

Proof: Let x be an element of corank k from $\text{Rees}(\mathscr{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$. First note that the number of elements of corank i in the half-open interval $[x, \hat{1})$ is $\binom{k-1}{i-1} \cdot (k-i+1)$. This follows from the fact that the element $x = (b, p)$ has $k-1$ non-stars appearing in b, so a corank i element $y = (c, q) \in [x, \hat{1})$ has $i-1$ more stars appearing in c and the second coordinate q satisfying $p \leq q \leq p + k - i + 1$. Hence there are $\binom{k-1}{i-1} \cdot (k-i+1)$ such elements y. Secondly, we claim that for a corank k element $x \in \text{Rees}(\mathscr{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$, we have

$$\mu([x, \hat{1}]) = (-1)^k \cdot (k-1)!. \tag{2.3}$$

We induct on the corank k. The case $k = 0$ is clear, as then x is a coatom. For the general case, we have

$$\begin{aligned}
\mu([x, \hat{1}]) &= -\sum_{x < y \leq \hat{1}} \mu([y, \hat{1}]) \\
&= -\left(1 + \sum_{\substack{x < y \leq \hat{1}, \\ 1 \leq \text{corank}(y) \leq k-1}} \mu([y, \hat{1}])\right) \\
&= -\left(1 + \sum_{i=1}^{k-1} (-1)^i \cdot (i-1)! \cdot \text{number of elements of corank } i \text{ in } [x, \hat{1})\right),
\end{aligned}$$

where the third equality is applying the induction hypothesis. The number of corank i elements in the half-open interval $[x, \hat{1})$ is $\binom{k-1}{i-1} \cdot (k-i+1)$, giving

$$\mu([x, \hat{1}]) = -\left(1 + \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i-1} \cdot (i-1)! \cdot (k-i+1)\right) = (-1)^k \cdot (k-1)!$$

by Lemma 2.3.1.

To finish the argument, there are $2^{n-k} \cdot \binom{n}{k} \cdot (k+1)$ elements of rank $k+1$, each having Möbius value $\mu(x, \hat{1}) = (-1)^{n-k+1} \cdot (n-k)!$. Hence the lemma follows the fact that for a poset P with $\hat{0}$ and $\hat{1}$, the identity $\mu_P(\hat{0}, \hat{1}) = -\sum_{\hat{0} < x \leq \hat{1}} \mu_P(x, \hat{1})$ holds. $\square$

n	$D_n = (-1)^n\mu(\text{Rees}(B_n, C_n)$	$(-1)^n\mu(\text{Rees}(\mathscr{C}_n, C_{n+1}))$		Factorization
0	1	0	=	0
1	0	1	=	$1 \cdot 1$
2	1	2	=	$2 \cdot 1$
3	2	15	=	$3 \cdot 5$
4	9	116	=	$4 \cdot 29$
5	44	1165	=	$5 \cdot 233$
6	265	13974	=	$6 \cdot 2329$
7	1854	195643	=	$7 \cdot 27949$
8	14833	3130280	=	$8 \cdot 391285$
9	133496	56345049	=	$9 \cdot 6260561$
10	1334961	1126900970	=	$10 \cdot 112690097$

Table 2.1: Table of Möbius values for the Rees product of the Boolean algebra with the chain and the Rees product of the cubical lattice with the chain.

2.4 Edge labeling

We begin by recalling some facts about R-labelings. For a complete overview, we refer the reader to Section 5 of Björner and Wachs' paper [12].

Given a poset P an *edge labeling* is a map $\lambda : E(P) \to \Lambda$, where $E(P)$ denotes the edges in the Hasse diagram of P and the labels are elements from a poset Λ. An edge labeling λ is said to be an *R-labeling* if in every interval $[x, y]$ of P there is a unique saturated chain $c : x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ whose labels are rising, that is, which satisfies $\lambda(x_0, x_1) <_\Lambda \lambda(x_1, x_2) <_\Lambda \cdots <_\Lambda \lambda(x_{k-1}, x_k)$. Given a maximal chain $m : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}$ in P, the *descent set* of m is the set $D(m) = \{i : \lambda(x_{i-1}, x_i) \not<_\Lambda \lambda(x_i, x_{i+1})\}$. Alternatively, when we view the labels of the maximal chain as the word $\lambda(m) = \lambda_1 \cdots \lambda_n$, where $\lambda_i = \lambda(x_{i-1}, x_i)$ and the rank of P is n, there is a descent in the ith position of $\lambda(m)$ if the labels λ_i and λ_{i+1} are either incomparable in the label poset Λ or satisfy $\lambda_i >_\Lambda \lambda_{i+1}$. In particular, a maximal chain m is said to be *rising* if its descent set satisfies $D(m) = \emptyset$ and *falling* if $D(m) = \{1, \ldots, n\}$.

The usefulness of an R-labeling is that it gives an alternate way to compute the Möbius function μ of a poset. Variations of this result are due to Stanley in the case of admissible lattices, Björner for R-labelings and edge lexicographic labelings, and Björner–Wachs for non-pure posets with a CR-labeling. See [12] for historical details.

Theorem 2.4.1 *Let P be a poset of rank n with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$. Suppose P has an R-labeling. Then with respect to this R-labeling,*

$$\mu(\hat{0}, \hat{1}) = (-1)^n \cdot \textit{number of falling maximal chains in } P$$

Let $\lambda : E(\text{Rees}(\mathscr{C}_n, C_{n+1})) \to \{0, \pm 1, \pm 2, \ldots, \pm n, n + 1\} \times \{0, 1\}$ be a labeling of the edges of the Hasse diagram of $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$ defined by

Edge			Condition	$\lambda(E)$	Notation
(x,i)	$\prec$	(y,i)	$x_a = 1, y_a = *$	$(a,0)$	a
(x,i)	$\prec$	(y,i)	$x_a = 0, y_a = *$	$(-a,0)$	$-a$
(x,i)	$\prec$	$(y,i+1)$	$x_a = 1, y_a = *$	$(a,1)$	$\overline{a}$
(x,i)	$\prec$	$(y,i+1)$	$x_a = 0, y_a = *$	$(-a,1)$	$\overline{-a}$
$\hat{0}$	$\prec$	$(x,1)$		$(0,0)$	0
(x,i)	$\prec$	$\hat{1}$		$(n+1,0)$	$n+1$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Elements $\{0, \pm 1, \ldots, \pm n, n+1\} \times \{0,1\}$ are partially ordered with the product order, that is $(x,i) \le (y,j)$ if $x \le y$ and $i \le j$.

Proposition 2.4.2 *The labeling λ is an R-labeling of* $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$.

Proof: Let $I = [(x,i),(y,j)]$ be an interval in $\mathrm{Rees}(\mathscr{C}_n, C_{n+1}) - \{\hat{0}, \hat{1}\}$ of length m with $x = (x_1, \ldots, x_n)$ and $y = (y_1 \ldots, y_n)$. We wish to find a saturated chain $c : (x,i) = (z_0, p_0) \prec (z_1, p_1) \prec \cdots \prec (z_m, p_m) = (y,j)$ in the interval I with increasing edge labels.

Let $S_0 = \{k : x_k = 0 \text{ and } y_k = *\}$ and $S_1 = \{k : x_k = 1 \text{ and } y_k = *\}$. Let $s = j - i$ and $t = |S_0|$. Without loss of generality, we may assume $S_0 = \{i_1, \ldots, i_t\}$ and $S_1 = \{i_{t+1}, \ldots, i_m\}$ with $i_1 > \cdots > i_t$ and $i_{t+1} < \cdots < i_m$. Set $(z_0, p_0) = (x,i)$. For $1 \le k \le m$, let $(z_k, p_k) = ((z_{1,k}, \ldots, z_{n,k}), p_k)$ where

$$z_{i,k} = \begin{cases} * & \text{if } i = i_k, \\ z_{i,k-1} & \text{otherwise,} \end{cases}$$

and

$$p_k = \begin{cases} p_{k-1} & \text{if } 1 \le k \le m-s, \\ p_{k-1}+1 & \text{otherwise.} \end{cases}$$

The first coordinate of the edge labels of the chain c form the strictly increasing sequence $-i_1 < \cdots < -i_t < i_{t+1} < \cdots < i_m$ as the i_j's are all positive, while the second coordinate of the edge labels form the weakly increasing sequence $0 \le \cdots \le 0 \le 1 \le \cdots \le 1$. Hence the chain c constructed is increasing.

We also claim that the chain c is the unique such chain that is increasing in the interval I. For any maximal chain in this interval, each $i \in S_0$ appears as the first coordinate in an edge label with a negative sign and every $i \in S_1$ must appear with a positive sign. Hence there is exactly one way to linearly order these m values. The second coordinate of the labels of any maximal chain in I is a permutation of the multiset $\{0^{m-s}, 1^s\}$. Again, there is exactly one way to order these m values in a weakly increasing fashion. Hence the increasing chain c is unique.

For the case when the interval is $[\hat{0}, (y,j)] \in \mathrm{Rees}(\mathscr{C}_n, C_{n+1})$ with $(y,j) \neq \hat{1}$, the first edge label in any saturated chain is always $(0,0)$. Hence the first coordinate of the labels in any increasing chain in this interval must all be non-negative, implying an increasing chain must pass through the atom $(a,1) = ((1,\ldots,1),1)$. The remainder of the increasing chain is given by the unique increasing maximal chain in the interval $[(a,1),(y,j)]$.

For an interval of the form $[(x,i),\hat{1}]$, since the last edge label of any saturated chain has label $(n+1,0)$, this forces all the elements of such a chain to be of the form (y,i) with $x \leq_{\mathscr{C}_n} y$. In particular, the rank n element of such a chain is precisely the element $(b,i) = ((*,\ldots,*),i)$. Hence the increasing maximal chain in $[(x,i),\hat{1}]$ is given by the increasing maximal chain guaranteed in $[(x,i),(b,i)]$ concatenated with the element $\hat{1}$. □

2.5 Falling chains

Define the set of (*double augmented*) *barred signed permutations* $\overline{\mathfrak{S}_n^{\pm}}$ to be those permutations $\pi = \pi_0\pi_1\cdots\pi_{n+1}$ satisfying (i) $\pi_0 = 0$ and $\pi_{n+1} = n+1$, (ii) for $1 \leq i \leq n$, π_i is equal to one of a_i, $-a_i$, $\overline{a_i}$ or $\overline{-a_i}$ for some $a_i \in \{1,\ldots,n\}$, and (iii) $a_1\cdots a_n$ is a permutation in the symmetric group $\mathfrak{S}_n$ on n elements. Given a double augmented barred signed permutation $\pi = \pi_0\pi_1\cdots\pi_{n+1}$, a *descent* at position i occurs when $|\pi_i| > |\pi_{i+1}|$, where $|\pi_j|$ denotes the element π_j with its (possible) bar removed.

Proposition 2.5.1 *With respect to the R-labeling λ of the poset* $\text{Rees}(\mathscr{C}_n, C_{n+1})$*, the falling chains are described as the set of double augmented barred signed permutations $\pi = \pi_0\pi_1\cdots\pi_{n+1} \in \overline{\mathfrak{S}_n}^{\pm}$ satisfying*

1. *if π_i is unbarred then there must be a descent at the ith position.*
2. *if π_i is barred, then either (i) π_{i+1} is unbarred or (ii) π_{i+1} is barred and there is a descent at the ith position.*

Example 2.5.2 *The permutation* $(0,-3,\overline{-4},2,\overline{-1},5) \in \overline{\mathfrak{S}_4^{\pm}}$ *corresponds to the falling chain* $\hat{0} \prec (0100,1) \prec (01*0,1) \prec (01**,2) \prec (0***,2) \prec (****,3) \prec \hat{1}$ *in the poset* $\text{Rees}(\mathscr{C}_4, C_5)$.

Proof: Given a barred signed permutation satisfying the conditions of the proposition, we wish to find a falling chain $c : \hat{0} \prec (x_1,i_1) \prec \cdots \prec (x_n,i_n) \prec \hat{1}$ in $\text{Rees}(\mathscr{C}_n, C_{n+1})$. For $1 \leq k \leq n$, if $\pi_k < 0$ then set $x_{1,k} = 1$; otherwise set $x_{1,k} = 0$. To find (x_k,i_k) recursively, set $i_1 = 0$, let $x_{w_k,k} = *$, and set

$$i_k = \begin{cases} i_{k-1}+1 & \text{if } \pi_k \text{ is barred,} \\ i_{k-1} & \text{if } \pi_k \text{ is not barred.} \end{cases}$$

Observe that c is a falling chain. The labels on the barred signed permutation correspond to the labels on the falling chain. Note that if the unbarred signed permutation does not have a descent at some position k, then π_k is barred and π_{k+1} is not, implying the second coordinate in the labeling $\lambda((x_k,i_k),(x_{k+1},i_{k+1}))$ is 1, while the second coordinate in the labeling $\lambda((x_{k+1},i_{k+1}),(x_{k+2},i_{k+2}))$ is 0. Hence, the chain is not rising in the kth position. Otherwise, the unbarred permutation has a descent and hence the first coordinate in the labeling $\lambda((x_k, i_k), (x_{k+1}, i_{k+1}))$ is greater than the first coordinate in the labeling $\lambda((x_{k+1}, i_{k+1}), (x_{k+2}, i_{k+2}))$ and hence the chain is not rising. $\square$

Throughout we will use $\mathcal{F}_n$ to denote the set of all the falling double augmented barred signed permutations in $\overline{\mathfrak{S}_n^{\pm}}$.

Theorem 2.5.3 *The Möbius function of the Rees product* $\text{Rees}(\mathscr{C}_n, C_{n+1})$ *is given by*

$$\mu(\text{Rees}(\mathscr{C}_n, C_{n+1})) = (-1)^n \cdot \sum_c 2^{n-c_1} \binom{n}{c_1, \ldots, c_k} \cdot c_1 \cdot \prod_{i=2}^{k} (c_i - 1),$$

where the sum is over all compositions $c = c_1 + \cdots + c_k$ *of* n *and* $1 \le k \le n$.

Proof: By Theorem 2.4.1, to determine the Möbius function of the Rees product $\text{Rees}(\mathscr{C}_n, C_{n+1})$ it is enough to count the number of falling chains in $\text{Rees}(\mathscr{C}_n, C_{n+1})$. Proposition 2.5.1 allows one to separate the double augmented barred signed permutations corresponding to falling chains into substrings which consist of a sequence of unbarred elements followed by a sequence of barred elements.

By Proposition 2.5.1, the element 0 will alway be part of the first substring and the last substring will consist only of the element $n+1$. Determining the size of each substring is equivalent to taking a composition $c = (c_1, c_2, \ldots, c_k)$ of n. Note that the first substring will be of size $c_1 + 1$ to account for the element 0 and the $(k+1)$st substring will consist only of the element $n+1$.

In each substring there is a sequence of elements without bars followed by a sequence of elements with bars. Given the size of each substring we determine at what place the barred elements begin. In the first substring we can begin the bars at any place, so there are c_1 ways. For all the other substrings the first element cannot be barred, for otherwise it would belong to the previous substring. Thus, we can begin the sequence of barred elements in $c_i - 1$ ways for $i = 2, \ldots, k$. The total number of ways to place bars over elements is $c_1 \cdot \Pi_{i=2}^{k}(c_i - 1)$.

Next, we choose the elements that will be in each substring. This is done in $\binom{n}{c_1, c_2, \ldots, c_k}$ ways. Now we must sign these elements. Note that the elements in each substring must be arranged in decreasing order. Once we have chosen the signs, this can be done in exactly one way. Furthermore, all of the elements in the first block must be negative because the falling double augmented signed permutation begins with the element 0. This leaves 2^{n-c_1} ways to sign the remaining elements. $\square$

2.6 Skew diagrams and a bijective proof

In this section we give a bijective proof of the following theorem.

Theorem 2.6.1 *The Möbius function of the Rees product of the cubical lattice with the chain is given by*

$$\mu(\text{Rees}(\mathscr{C}_n, C_{n+1})) \quad = \quad (-1)^n \cdot n \cdot \text{per} \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 1 & 2 & \cdots & 2 \\ 2 & 2 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 1 \end{bmatrix},$$

that is, n times the permanent of a square $(n-1)\times(n-1)$ matrix having 1's on the diagonal and 2's everywhere else.

Recall that the derangement number D_n can be expressed as the permanent of an $(n-1)\times(n-1)$ matrix having 0's on the diagonal and 1's everywhere else. As a corollary to Theorem 2.6.1, we can slightly modify our proofs to give a bijective proof of Jonsson's result (Theorem 2.1.3).

Corollary 2.6.2 *There is an explicit bijection implying that*

$$\mu(\text{Rees}(B_n, C_n)) = (-1)^n \cdot D_n.$$

In order to prove Theorem 2.6.1, we will work with skew diagrams associated to falling double augmented barred and signed permutations. In Section 2.7 we will use these skew diagrams to describe $\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1}))$, the order complex of the Rees product of the cubical lattice with the chain, in the spirit of Wachs' work with the d-divisible partition lattice [58]. We will also use these diagrams to construct an explicit basis for the homology of $\text{Rees}(\mathscr{C}_n, C_{n+1})$.

Besides the interest in the bijection itself to prove Theorem 2.6.1, these diagrams allow us to find explicit bases for the integer homology $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})), \mathbb{Z})$ indexed by the falling augmented signed barred permutations.

We begin by recalling some objects from combinatorial representation theory. For background material in this area, we refer to Sagan's book [41]. Let $(\lambda_1, \ldots, \lambda_k) \vdash n$ be a partition of the integer n with $\lambda_1 \leq \cdots \leq \lambda_k$. Recall the Ferrers diagram of the partitionλ consists of n boxes where row i has λ_i boxes for $i = 1, \ldots, k$ and all the rows are left-justified. Given two Ferrers diagrams $\mu \subseteq \lambda$, the *skew diagram* λ/μ is the set of all boxes $\lambda/\mu = \{b \ : \ b \in \lambda \text{ and } b \notin \mu\}$.

For us, a *hook* is a skew diagram of the form λ/μ where $\lambda = ((h+1)^v)$ and $\mu = (h^{(v-1)})$. We will be interested in skew diagrams consisting of a disjoint union of hooks. More precisely, let $c = (c_1, \ldots, c_k)$ be a composition of n with $c_i = u_i + b_i$, for $i = 1, \ldots, k$ where $u_1 \geq 0$, $u_i > 0$ for $i = 2, \ldots, k$, and $b_i > 0$ for $i = 1, \ldots, k$. Form the partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ where $\lambda_i = (u_1 + \cdots + u_i + i)^{b_i}$ for $1 \leq i \leq k$, $\mu_i = ((u_1 + \cdots + u_i + i - 1)^{b_i - 1}, u_1 + \cdots + u_i + i)$ for $1 \leq i \leq k-1$, and $\mu_k = (u_1 + \cdots + u_k + k - 1)^{b_k - 1})$. The skew diagram λ/μ is then a union of k hooks where the southeast corner of the last box of the ith hook touches the northwest corner of the first box of the $(i+1)$st hook. We call such a diagram an *unsigned barred permutation skew diagram*. We call a filling of the n boxes with the

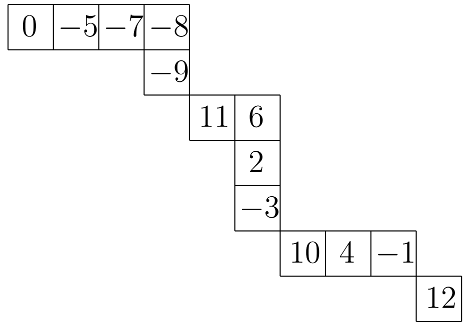

Figure 2.5: The skew diagram corresponding to the falling double augmented barred signed permutation $\pi = 0 \;\; -5 \;\; -7 \; \overline{-8} \; \overline{-9} \; 11 \; \overline{6} \; \overline{2} \; \overline{-3} \; 10 \; 4 \; \overline{-1} \; 12$ in $\overline{\mathfrak{S}}^{\pm}_{11}$.

elements $\{1, \ldots, n\}$ *standard* if the rows are decreasing when read from left to right and the columns are decreasing when read from top to bottom. If we insert a box labeled 0 in front of the first horizontal row and add a box labeled $n+1$ as the new last hook, then we call such a filled diagram a *standard double augmented unsigned barred skew diagram.* Given a double augmented unsigned barred permutation that is falling, recall that it consists of strings of unbarred and barred elements concatenated together. Given such a falling permutation, one forms the standard skew diagram by representing the first string of unbarred elements as the first horizontal string of boxes in the first hook concatenated with the same number of vertical boxes as the number of barred elements in the first string of the permutation. Note that the ith hook has $u_i + 1$ horizontal boxes, where u_i is the number of unbarred elements in the first string of the permutation. See Figure 2.5.

Theorem 2.6.3 *There exists a bijection between the set of all fixed point free permutations in the symmetric group on n elements and the set of all standard skew diagrams λ/μ having n boxes and hooks of size greater than* 1*.*

Proof: We describe an algorithm to move between these two sets. The idea is to first break a cycle at the end of each of its descent runs to form blocks. Each of these blocks will become a hook in the resulting skew diagram. The next step is to use the first element of each block (for the first block, use the second element) to determine which elements will be barred in a given block. The third step is to reverse the order of the blocks. The fact that the original first block contained the smallest element in the given cycle will enable us to recover the complete cycle decomposition of a permutation from its skew diagram in the general case when a permutation has more than one cycle.

We first consider the case where $\pi = (\pi_1, \ldots, \pi_n) \in \mathfrak{S}_n$ consists of a single cycle of length n with $\pi_1 = 1$, that is, the smallest element of the set $\{\pi_1, \ldots, \pi_n\}$.

1. Identify the descents within the cycle. For each run of consecutive descents, say $[i, j] = i, i+1, \ldots, j$, break the permutation in front of the last descent in the

run, that is, the $(j-1)$st position provided this does not create a first block having size one.

2. Suppose reading from left to right the first element in the ith block is m_{i_j}, where the elements in this block have the linear order $m_{i_1} < m_{i_2} < \cdots$. (For the case of the first block, let m_{1_j} be the second element in this block when reading from left to write and where the block elements have linear order $m_{1,1} < m_{1,2} < \cdots$.) Rewrite the elements in the block in decreasing order and place bars over each of the last $j-1$ elements.

3. Reverse the order of the blocks, that is, if $B_1|B_2|\cdots|B_k$ is the original block decomposition, reverse this to $B_k|B_{k-1}|\cdots|B_1$. Finally, remove the vertical block separators.

   This yields the union of unsigned hooks, where a hook consists of the run of unbarred elements followed by the run of barred elements.

Example 2.6.4 *As an example, let* $\pi = (135764928) \in \mathfrak{S}_9$. *We have*

$$\begin{aligned} \pi &\rightarrow 1357|64|928 \\ &\rightarrow 1357|46|298 \\ &\rightarrow 753\bar{1}|6\bar{4}|9\bar{8}\bar{2} \\ &\rightarrow 9\bar{8}\bar{2}6\bar{4}753\bar{1} \end{aligned}$$

If a permutation consists of more than one cycle, without loss of generality we may assume the permutation is written in standard cycle notation, that is, each cycle is written so that it begins with the smallest element in its cycle and the cycles are then ordered in increasing order by the smallest element in each cycle. Given such a permutation, apply the algorithm to each individual cycle. Concatenate the resulting barred words using the original order of the cycles.

We can reverse this process beginning with a standard unsigned skew diagram.

1. Given a standard unsigned skew diagram, we will separate it into cycles based on the minimal element. Break the diagram after the hook containing the element 1. Next, break the diagram after the hook containing the smallest element occurring to the right of the first break. Then break after the hook containing the smallest element to the right of the second break. Continue this process until there is a break at the end of the diagram. These breaks now correspond to individual cycles in the final permutation.

2. Within each of these breaks, put parentheses around the elements of each hook and reverse the order of the hooks, that is, if break i has hooks $h_{i,1}h_{i,2}\cdots h_{i,j}$ then reverse these to $h_{i,j}h_{i,j-1}\cdots h_{i,1}$.

3. In each parenthetical piece, remove the bars and reorder the elements by the following rule. The now unbarred elements in each parenthesis can be linearly ordered, say $m_{i_1} < m_{i_2} < \cdots < m_{i_k}$. If there were bars over j numbers in this piece, reorder the elements as $m_{i_1}, m_{i_{j+1}}, m_{i_2}, \ldots, m_{i_k}$ if $j \neq k$ and $m_{i_1}, m_{i_k}, m_{i_2}, \ldots, m_{i_{k-1}}$ if $j = k$.

4. Within each cycle, leave the vertical bars fixed for the moment and switch the first two numbers of all the parenthetical pieces except the first piece which begins the cycle. Remove the inner parentheses and concatenate the pieces within each vertically barred piece into one cycle.

These processes we have described are the inverse of each other. Thus we have a bijection. □

Example 2.6.5 *Let* $8\bar{7}\bar{2}6\bar{1}9\bar{5}4\bar{3}$ *be a falling barred permutation. The algorithm gives:*

$$\begin{aligned} 8\bar{7}\bar{2}6\bar{1}9\bar{5}4\bar{3} &\rightarrow 8\bar{7}\bar{2}6\bar{1}|9\bar{5}4\bar{3}| \\ &\rightarrow (6\bar{1})(8\bar{7}\bar{2})|(4\bar{3})(9\bar{5}) \\ &\rightarrow (16)(287)|(34)(59) \\ &\rightarrow (16827)(3495). \end{aligned}$$

Let $F \subseteq [n-1]$ be the set of fixed points for a permutation $\pi \in \mathfrak{S}_{n-1}$. We will build n ordered pairs, (F_i, τ) where $i = 1, \ldots, n$ and τ is a partial permutation on $n - |F| - 1$ elements from the set $[n]$. Set

$$F_i = \begin{cases} F \cup \{i\} & \text{if } i \notin F, \\ F \cup \{n\} & \text{if } i \in F, \end{cases}$$

where $i = 1, \ldots, n$. To define τ, consider the partial permutation $\widehat{\pi}$ consisting of the cycles of π with sizes greater than 1. The elements in these cycles can be linearly ordered as $m_{i_1} < m_{i_2} < \cdots < m_{i_{n-|F|-1}}$. The elements of $[n] - F_i$ also can be linearly ordered as $l_{i_1} < \cdots < l_{i_{n-|F|-1}}$. Define a map Ψ which sends $m_{i_j} \mapsto l_{i_j}$. Set $\tau = \Psi(\widehat{\pi})$. Let $F_\pi = \{(F_i, \tau) : i = 1, \ldots, n\}$ so that $|F_\pi| = n$.

Proposition 2.6.6 *There exists a bijection between* $\{F_\pi : \pi \in \mathfrak{S}_n\}$ *and the set of standard unsigned skew diagrams where each hook except the first has size greater than one.*

Proof: Given a permutation π with fixed point set F and one ordered pair (F_i, τ), we will define a map which sends F_i to the first hook of the diagram and which sends τ to the rest of the diagram. To create the first part of the map, write the elements of F_i in decreasing order. To place the bars, consider two cases.

1. If $i \notin F$, place bars over the element i and every element less than i.

2. If $i \in F$ we use the linear total order on F, say $f_1 < \cdots < f_{|F|}$. We have $i = f_j$ for some $j = 1, \ldots, |F|$. Place bars over the smallest j elements.

This map can be reversed given the first piece of some unsigned skew diagram.

To determine the rest of the diagram, we use τ, a partial permutation on an $n - |F| - 1$ element subset of $[n]$. There is a bijection between all such partial permutations and the set of fixed point free permutations in $\mathfrak{S}_{n-|F|-1}$. Use the linear order on the elements of τ, that is, these elements can be written $m_{i_1} < \cdots < m_{i_k}$. Let Φ be a map between these two sets where $\Phi(m_{i_j}) = j$. Note that because the partial permutation τ can be written as a product of cycles with no one-cycles, then $\Phi(\tau)$ is also a fixed point free product of cycles. Composing Φ with the algorithm above, we can go from a partial permutation τ to the rest of the diagram having hook sizes greater than 1. $\Box$

To prove Theorem 2.6.1, we sign the first hook (which consists of the horizontal piece 0 concatenated with the vertical piece) in one way, that is, with all negative signs, and then reorder the elements in decreasing order. For the remaining hooks, we can sign these remaining elements in $2^{n-|F|-1}$ ways and within each hook reorder them in a decreasing manner in one way.

As a corollary, we can slightly modify our proofs to give a bijective proof of Jonsson's result (Theorem 2.1.3) for the Möbius function of the Rees product of the Boolean algebra with the chain.

Proof of Theorem 2.1.3: It is enough to observe that $\text{Rees}(B_n, C_n)$ is isomorphic to the upper order ideal generated by any atom of $\text{Rees}(\mathscr{C}_n, C_{n+1})$. Hence $\text{Rees}(B_n, C_n)$ inherits the R-labeling of $\text{Rees}(\mathscr{C}_n, C_{n+1})$. The maximal chains in $\text{Rees}(B_n, C_n)$ are described by augmented barred permutations, that is, permutations of the form $\pi = \pi_1 \cdots \pi_n \pi_{n+1}$ with $\pi_{n+1} = n + 1$, $|\pi| = |\pi_1| \cdots |\pi_n| \in \mathfrak{S}_n$ (again, $|\pi_j|$ denotes removing any bar occurring in π_j), π_1 not barred and each of the elements $\pi_2, \ldots, \pi_n$ may be barred. The falling chains correspond to unsigned labeled skew diagrams having hooks of size greater than or equal to 2 which are augmented at the end with a block containing the element $n + 1$. Theorem 2.6.3 now applies to prove the result. $\Box$

Shareshian and Wachs [44] have proved a dual version of Proposition 2.3.2 where they instead work with a truncated face lattice of the cross-polytope $\mathscr{O}_n$. To state their results we use $\text{Rees}^{-}(P, Q)$ to indicate the maximal and minimal elements are removed from P before taking the Rees product of two graded posets P and Q, that is, $\text{Rees}^{-}(P, Q) = ((P - \{\hat{0}, \hat{1}\}) * Q) \cup \{\hat{0}, \hat{1}\}$.

Corollary 2.6.7 (Shareshian–Wachs) *For all n,*

$$\dim \widetilde{H}_{n-1}(\Delta(\text{Rees}^{-}(\mathscr{O}_n, C_n))) = d_n^{BC},$$

where d_n^{BC} denotes the number of derangements occurring from reflections acting on the dual of the n-dimensional cross polytope.

Shareshian and Wachs' original proof follows from the Björner–Welker Theorem 2.1.2 and from the fact that the reduced homology of a Cohen-Macaulay poset vanishes everywhere except the top dimension, where the dimension is given by the Möbius function of the poset.

Shareshian and Wachs also have q-analogues of Theorems 2.1.3 and 2.6.7, where they work with the subspace lattice $B_n(q)$ and the isotropic subspace lattice $\mathscr{O}_n(q)$. See [44, Theorem 2.1.6 and Theorem 2.4.5].

Theorem 2.6.8 (Shareshian–Wachs) *For all n, let $d_n(q) = \sum_{\sigma\in\mathcal{D}_n} q^{comaj}(\sigma) + exc(\sigma)$, where $\mathcal{D}_n$ denotes the set of derangements in $\mathfrak{S}_n$. Then*

$$\dim \widetilde{H}_{n-1}(\Delta(\mathrm{Rees}(B_n(q), C_{n+1}))) = d_n(q).$$

and

$$\dim \widetilde{H}_{n-1}(\Delta(\mathrm{Rees}^-(\mathscr{O}_n(q), C_{n+1}))) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} \prod_{i=k+1}^{n} (1+q^i)\cdot d_{n-k}(q).$$

Hence $\dim \widetilde{H}_{n-1}(\Delta(\mathrm{Rees}(B_n(q), C_{n+1})))$ *and* $\dim \widetilde{H}_{n-1}(\Delta(\mathrm{Rees}^-(\mathscr{O}_n(q), C_{n+1})))$ *are polynomials in q with non-negative integer coefficients.*

2.7 A Basis for the homology

In this section we consider homological questions for the poset $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$. A similar analysis for the the d-divisible partition lattice was done by Wachs [58],

Proposition 2.7.1 *The order complex $\Delta(\mathrm{Rees}(\mathscr{C}_n, C_{n+1}))$ is a Cohen-Macaulay complex and has vanishing homology groups in every dimension except in top dimension.*

This follows by a result of Björner and Welker [13] that the Rees product of any two Cohen-Macaulay posets is also Cohen-Macaulay. Furthermore, the absolute value of the Möbius function of the poset $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$ gives the dimension of the top homology group.

We next give an explicit basis for the homology $H(\Delta(\mathrm{Rees}(\mathscr{C}_n, C_{n+1})), \mathbb{Z})$ indexed by the falling augmented signed barred permutations. Recall that $\mathcal{F}_n$ denotes the set of falling augmented signed barred permutations from $\overline{\mathfrak{S}_n^{\pm}}$. For each $\sigma \in \mathcal{F}_n$ we define a subposet $\mathscr{C}_\sigma$ of $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$ as follows. Let $m_\sigma = m_{\sigma,0} \prec m_{\sigma,1} \prec \cdots \prec m_{\sigma,n}$ be the chain in $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})\backslash\{\hat{0}, \hat{1}\}$ labeled by $\sigma \in \mathcal{F}_n$. For example, for the double augmented barred signed permutation $\sigma = \sigma_0 \cdots \sigma_6 = 0\ \ -1\ \overline{-3}\ 5\ \overline{2}\ \overline{-4}\ 6$, we have

$$m_\sigma = (01001, 1) \prec (*1001, 1) \prec (*1*01, 2) \prec (*1*0*, 2) \prec (***0*, 3) \prec (*****, 4)$$

We define the elements of $\mathscr{C}_\sigma$ recursively. The rank 0 elements of $\mathscr{C}_\sigma$ are of the form $(x, 1)$, where x is a 0-dimensional face of the n-cube. For $1 \le i \le n-1$, the rank i elements of $\mathscr{C}_\sigma$ are of the form (x, j), where x is an i-dimensional face of the n-cube and the second coordinate j is determined according to the following rules:

i. If σ_{i-1} is not barred, σ_i is not barred, and σ_{i+1} is either barred or unbarred, then $j = k$ where (y, k) is any rank $i-1$ element of $\mathscr{C}_\sigma$.

ii. If σ_{i-1} is either barred or unbarred, and both σ_i and σ_{i+1} are barred, then $j = k+1$ where (y, k) is any rank $i-1$ element of $\mathscr{C}_\sigma$.

iii. If σ_{i-1} is either barred or unbarred, σ_i is barred and σ_{i+1} is not barred, then $j = k$ where (y, k) is a rank $i-1$ element of $\mathscr{C}_\sigma$. The exception to this rule is for the i-dimensional element x occurring in the chain m_σ, that is, $m_{\sigma,i} = (x, r)$. In this case, $m_{\sigma,i}$ becomes the element $(x, k+1)$ in $\mathscr{C}_\sigma$.

iv. If σ_{i-1} is barred, σ_i is not barred,and σ_{i+1} is either barred or unbarred, then $j = k+1$ where (y,k) is any rank $i-1$ element of $\mathscr{C}_\sigma$ different from $m_{\sigma,i-1}$. Notice that both $m_{\sigma,i-1}$ and $m_{\sigma,i}$ have the same second coordinate, namely $k+1$.

Finally, there are two rank n elements $(*\cdots*,k)$ and $(*\cdots*,k+1)$, where k is the second coordinate of any rank $n-1$ element of $\mathscr{C}_\sigma$.

Define $\widetilde{\mathscr{C}_n}$ to be the poset $\mathscr{C}_n \backslash \{\hat{0}\} \cup \{\hat{1}'\}$, that is, the face lattice of the n-dimensional cube with its minimal element removed and adjoined with a second maximal element $\hat{1}'$ which also covers all the coatoms in $\mathscr{C}_n \backslash \{\hat{0}\}$.

Theorem 2.7.2 *For $\sigma \in \mathcal{F}_n$, the order complex $\Delta(\mathscr{C}_\sigma)$ is isomorphic to the suspension of the barycentric subdivision of the boundary of the n-cube.*

Proof: It is enough to show the posets $\mathscr{C}_\sigma$ and $\widetilde{\mathscr{C}_n}$ are isomorphic. Define the "forgetful" map $f : \mathscr{C}_\sigma \to \widetilde{\mathscr{C}_n}$ which sends an element $(x,k) \in \mathscr{C}_\sigma$ to the element x for elements of ranks 1 through $n-1$ in $\mathscr{C}_{n-1}$. For the two rank n elements, let $f(*\cdots*,j_n) = \hat{1}$ and $f(*\cdots*,j_n+1) = \hat{1}'$. Clearly the map f is a bijection from the elements of $\mathscr{C}_\sigma$ to those of $\widetilde{\mathscr{C}_n}$. Additionally, f is order-preserving since for $(y,k) \prec (x,j)$ in $\mathscr{C}_\sigma$, one has $y \prec x$ in the cubical lattice $\mathscr{C}_n$

To define the inverse map f^{-1}, one follows the described scheme to determine the second coordinate as above. Note that for elements x and y with $y \prec x$ in $\widetilde{\mathscr{C}_n}$ and $\rho(y) < n$, the inverse map satisfies $f^{-1}(y) = (y,k) \prec f^{-1}(x) = (x,j)$ since $k \leq j$ by construction. The two maximal elements of $\widetilde{\mathscr{C}_n}$ are easily seen to be mapped to the two maximal elements of $\mathscr{C}_\sigma$, so the bijection is order-preserving as desired. □

Corollary 2.7.3 *For $\sigma \in \mathcal{F}_n$, $\Delta(\mathscr{C}_\sigma)$ is homotopy equivalent to the suspension of the $(n-1)$-dimensional sphere, S^{n-1}.*

Proof: The order complex of $\mathscr{C}_n - \{\hat{0},\hat{1}\}$ is the barycentric subdivision of the boundary of the n-cube. The boundary of the n-cube is homotopic to S^{n-1}. The poset $\widetilde{\mathscr{C}_n}$ differs from $\mathscr{C}_n - \{\hat{0},\hat{1}\}$ by the addition of two maximal elements. Therefore, the order complex of $\widetilde{\mathscr{C}_n}$ is found from $\Delta(\mathscr{C}_n - \{\hat{0},\hat{1}\})$ by adding two $(k+1)$-dimensional faces for each k-face in $\Delta(\mathscr{C}_n - \{\hat{0},\hat{1}\})$ composed of the k-dimensional face with each maximal element. This is a suspension over the barycentric subdivision of the boundary of the n-cube which is homotopic to the suspension of S^{n-1}. Thus, by Theorem 2.7.2 we then have $\Delta(\mathscr{C}_\sigma) \simeq \Delta(\widetilde{\mathscr{C}_n})$ and we have proven the corollary. □

The suspension of S^{n-1} is homotopic to S^n, and as a result $\Delta(\mathscr{C}_\sigma)$ is a triangulation of the n-sphere. Let ρ_σ denote a fundamental cycle of the spherical complex $\Delta(\mathscr{C}_\sigma)$. To show that the set set $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ forms a basis for $H(\text{Rees}(\mathscr{C}_n, C_{n+1}))$, we first place a total order on $\mathcal{F}_n$. Let $\sigma = 0\sigma_1 \cdots \sigma_n\, n+1$ and $\tau = 0\tau_1 \cdots \tau_n\, n+1$ be two permutations from $\mathcal{F}_n$. If the entries $\sigma_1, \ldots, \sigma_{i-1}$ and $\tau_1, \ldots, \tau_{i-1}$ are unbarred, σ_i is barred and τ_i is unbarred, then we say $\sigma > \tau$. Otherwise, if σ and τ are barred and unbarred at exactly the same places and the permutation σ without the bars is lexicographically greater than the permutation τ without its bars, then we say $\sigma > \tau$. We then have

Lemma 2.7.4 *If m_τ is a chain in $\mathscr{C}_\sigma$, then $\tau \leq \sigma$.*

Proof: Let σ and τ be permutations in $\mathcal{F}_n$ with $\tau > \sigma$. We want to show m_τ is not a chain in $\mathscr{C}_\sigma$. There are two cases to consider.

First suppose σ and τ are barred at precisely the same locations and that the unbarred permutation τ is lexicographically greater than the unbarred permutation σ. Let i be the least index where σ_i is barred and σ_{i+1} is not barred. If such an i does not exist, then each permutation corresponds to a diagram consisting of one hook and as such has a labeling $-1 \cdots - n$, implying $\sigma = \tau$, a contradiction. So we may assume such an i satisfying $1 \leq i < n$ exists. We see the first i elements in the chain m_τ are elements in $\mathscr{C}_\sigma$. However, the ith element $m_{\tau,i} = (x, r_{\sigma,i} + 1)$ where x is the unique rank i element in $\mathscr{C}_n \backslash \{\hat{0}\}$ given by the unbarred permutation τ and $r_{\sigma,i}$ is the number of bars over elements $\sigma_1, \sigma_2, \ldots, \sigma_i$, will not be an element in $\mathscr{C}_\sigma$. (Note, the second coordinate $r_{\sigma,i} + 1$ in $m_{\sigma,i}$ is the same as the ith second coordinate $r_{\tau,i} + 1$ in $m_{\tau,i}$ for all i.) We can see this by observing that the only element in $\mathscr{C}_\sigma$ with first coordinate a rank i element in $\mathscr{C}_n \backslash \{\hat{0}\}$ and with second coordinate $r_{\sigma,i} + 1 = r_{\tau,i} + 1$ corresponds to the unique element given by the unbarred σ. Thus, since the unbarred σ is not equal to the unbarred τ, m_τ is not a chain in $\mathscr{C}_\sigma$.

For the second case, suppose that σ_j is barred if and only if τ_j is barred for $j = 1, \ldots, i-1$ and τ_i is barred while σ_i is not barred. We claim the ith element $m_{\tau,i}$ in m_τ is not an element of $\mathscr{C}_\sigma$. Note the second coordinate $r_{\tau,j} + 1$ in $m_{\tau,j}$ is the same as the second coordinate $r_{\sigma,j} + 1$ in $m_{\sigma,j}$ where $j = 1, \ldots, i-1$ because the pattern of bars coincide for the first $i-1$ terms in the permutations. However, in $\mathscr{C}_\sigma$ all rank i elements $\mathscr{C}_n \backslash \{\hat{0}\}$ have second coordinate r_{i-1}. As there is no bar over σ_i, the second coordinate does not increase. Since the element τ_i is barred, the element $(x, r_{\sigma,i-1} + 1)$ is an element in the chain m_τ but not in the poset $\mathscr{C}_\sigma$. $\square$

Theorem 2.7.5 *The set $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ forms a basis for $H(\Delta(\mathrm{Rees}(\mathscr{C}_n, C_{n+1})))$ over $\mathbb{Z}$.*

Proof: To show that $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ are linearly independent, let $\sum_{\sigma \in \mathcal{F}_n} a_\sigma \rho_\sigma = 0$ where $a_\sigma \in \mathbf{k}$. With respect to the total order we have described above, suppose τ is the greatest element of $\mathcal{F}_n$ for which $a_\tau \neq 0$. We apply Lemma 2.7.4 to derive a contradiction. We have

$$\begin{aligned} 0 &= \sum_{\sigma \in \mathcal{F}_n} a_\sigma \rho_\sigma|_{m_\tau} \\ &= \sum_{\sigma \in \mathcal{F}_n, \sigma \leq \tau} a_\sigma \rho_\sigma|_{m_\tau} \\ &= a_\tau \rho_\tau|_{m_\tau} \\ &= \pm a_\tau, \end{aligned}$$

since the fundamental cycle evaluated at a facet has coefficient ± 1. However, this gives a contradiction. Since the rank of $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ is equal to the cardinality of $\mathcal{F}_n$, we have proven the basis result when $\mathbf{k}$ is a field.

When $\mathbf{k} = \mathbb{Z}$, linear independence of $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ implies this set is also linearly independent over the rationals $\mathbb{Q}$ and hence that it spans $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ over $\mathbb{Q}$. Let $\rho \in H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$. Then $\rho = \sum_{\sigma \in \mathcal{F}_n} c_\sigma \rho_\sigma$ for $c_\sigma \in \mathbb{Q}$. We will show $c_\sigma \in \mathbb{Z}$ for all $\sigma \in \mathcal{F}_n$. Suppose α is the greatest element of $\mathcal{F}_n$ for which $c_\alpha \neq 0$. Then by Lemma 2.7.4

$$\begin{aligned} \rho|_{m_\alpha} &= \sum_{\sigma \in \mathcal{F}_n, \sigma \leq \alpha} c_\sigma \rho_\sigma|_{m_\alpha} \\ &= c_\alpha \rho_\alpha|_{m_\alpha} \\ &= \pm c_\alpha. \end{aligned}$$

Since $\rho \in H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$, we have $\rho|_{m_\alpha} \in \mathbb{Z}$. Thus $c_\alpha \in \mathbb{Z}$ and $\rho - c_\alpha\rho_\alpha \in H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$. Repeat this argument for $\rho - c_\alpha\rho_\alpha$ to conclude $c_\beta \in \mathbb{Z}$ and $\rho - c_\alpha\rho_\alpha - c_\beta\rho_\beta \in H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ for β the next to the last element in the total order on $\mathcal{F}_n$ for which $c_\beta \neq 0$. Since there are finitely-many elements in $\mathcal{F}_n$, we may conclude that $c_\sigma \in \mathbb{Z}$ for all $\sigma \in \mathcal{F}_n$. Hence $\{\rho_\sigma : \sigma \in \mathcal{F}_n\}$ spans $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ over $\mathbb{Z}$ and thus is a basis for $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ over $\mathbb{Z}$. □

2.8 Representation over $\mathfrak{S}_n$

In this section we develop a representation of $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ over the symmetric group. This can be done using a set of skew Specht modules.

The homology of the order complex of $\text{Rees}(\mathscr{C}_n, C_{n+1})$ is an $\mathfrak{S}_n$-module in the following manner. A signed permutation $\pi \in \overline{\mathfrak{S}_n^\pm}$ corresponds to a labeled maximal chain of the poset $\text{Rees}(\mathscr{C}_n, C_{n+1})$. A permutation $\tau \in \mathfrak{S}_n$ acts on the chains of $\text{Rees}(\mathscr{C}_n, C_{n+1})$ by sending the maximal chain labeled with π to the maximal chain whose labels are $\tau\pi$. Note that under the action of τ the placement of the bars is fixed and the signs remain attached to the same numbers. This action induces an action on the faces of $\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1}))$ and even further on the homology group itself whose basis is indexed by a subset of chains in $\text{Rees}(\mathscr{C}_n, C_{n+1})$. We have $\tau\rho_\pi = \rho_{\tau\pi}$ for any basis element $\rho_\pi \in H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$.

Theorem 2.8.1 *There exists an $\mathfrak{S}_n$-module isomorphism between*

$$H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$$

and $\bigoplus 2^{n-|\lambda_1|} S^\lambda$ where the direct sum is over all partitions λ with each λ_i shaped into hooks as described in Section 2.6 taken with multiplicity $2^{n-|\lambda_1|}$.

To prove this result, we will need some tools from combinatorial representation theory. For more details and background information, see [41].

Recall that two tableaux t_1 and t_2 of shape λ are *row equivalent*, written $t_1 \sim t_2$, if the entries in each row of t_1 are a permutation of the entries in the corresponding row of t_2. A *tabloid of shape* λ (λ-*tabloid* or *tabloid*, for short) is then an equivalence class $\{t\}$. For a fixed partition λ we denote by M^λ the k-vector space having λ-tabloids as a basis. In the usual way a permutation $\sigma \in \mathfrak{S}_n$ acts on a λ-tableau by replacing each entry by its image under σ. Thus σ acts on a λ-tabloid $\{t\}$ by $\sigma\{t\} = \{\sigma t\}$. For a tableau t of shape λ, the *polytabloid* corresponding to t is

$$e_t = \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\{\sigma t\},$$

where the sum is over all permutations belonging to the column stabilizer C_t of t.

The *Specht module* S^λ is the submodule of M^λ spanned by the polytabloids e_t, where t has shape λ. The Specht module S^λ is an $\mathfrak{S}_n$-module in the following manner. A permutation $\tau \in \mathfrak{S}_n$ acts linearly on the elements of S^λ by permuting the entries of t, that is, $\tau e_t = e_{\tau t}$.

Specht modules were developed to construct all irreducible representations of the symmetric group over $\mathbb{C}$. We will use these modules to give a representation of $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ over $\mathfrak{S}_n$.

Recall that a tableau t is said to be *standard* if the entries are increasing in each row and column of t. The following theorem is originally due to Young, though not in this form. It is also due to Specht.

Theorem 2.8.2 *The set*

$$\{e_t : t \textit{ is a standard } \lambda\textit{-tableau}\}$$

is a basis for S^λ.

With these definitions in mind, we can begin the proof of Theorem 2.8.1. Let $\nu = \lambda - \mu$ be a skew diagram consisting of the union of k hooks where the ith hook has size $|\nu_i|$. We consider the case where $\nu = \nu_1 \cdots \nu_k$ is fixed.

Define a set $\mathcal{F}_n^- = \{-\sigma : \sigma = \sigma_1 \cdots \sigma_n \in \mathcal{F}_n\}$ where $-\sigma = -\sigma_1 - \sigma_2 \cdots - \sigma_n$. It is easily noted there exists a bijection between $\mathcal{F}_n^-$ and $\mathcal{F}_n$. We use this bijection to move between basis elements of $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$ which correspond to decreasing labeled skew shapes and standard tableaux which have increasing labels.

Consider the usual unsigned Specht module S^λ in the case λ is composed of hooks of size at least two and is augmented at the end with a block containing the element $n+1$. It is generated by polytabloids which are indexed by standard labelings of λ. Define an $\mathfrak{S}_n$-module homomorphism

$$\theta : S^\lambda \to H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$$

where $e_t \mapsto \rho_{-\sigma}$ for t a standard λ-tableau and $\sigma \in \mathcal{F}_n^-$ is found by writing the labels on t from left to right and by placing bars over all elements which occur in the rightmost columns of a hook.

We wish to extend this map over "signings" of S^λ. Given a standard polytabloid $e_t \in S^\lambda$ where t is a standard tableau, we sign the elements occurring in the last $k-1$ hooks of t, that is, sign the labels on $\lambda_2, \ldots, \lambda_k$. This can be written as a subset $A \subset [n] - \{\lambda_1\}$ where A corresponds to the elements in t labeled with a negative sign. For each e_t there are $\sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j}$ such signings, or equivalently, such subsets A. We let e_t^A denote the signing by A of the polytabloid e_t. Using the binomial theorem, there is an isomorphism

$$\bigoplus_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda \simeq 2^{n-|\lambda_1|} S^\lambda.$$

This is an $\mathbb{C}\mathfrak{S}_n$-module with action $\pi e_t^A = e_{\pi t}^A$ where $\pi \in \mathfrak{S}_n$ permutes the labels of the tableau t.

To show each Specht module S^λ occurs with multiplicity $2^{n-|\lambda_1|}$ in the top homology group, we extend the map θ to $\theta : \sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda \to H(\Delta(\mathrm{Rees}(\mathscr{C}_n, C_{n+1})))$ where basis elements are mapped by $e_t^A \mapsto \rho_{-\sigma}$. The permutation σ is found by attaching negative signs to the labels in t which are also in A. Then the labels in each hook written in increasing order. As before, we form the permutation σ by writing down the labels reading from left to right with bars placed over the rightmost element in every row. Note that when A is empty, we are back in the usual unsigned case.

Set $E^\lambda = \{e_t^A\}$ where t ranges over all standard Young tableaux of shape λ and A ranges over all subsets of $[n] - \{\lambda_1\}$.

Proposition 2.8.3 *The map*

$$\theta : E^\lambda \longrightarrow \{\rho_\sigma | \sigma \in \mathcal{F}_n \textit{ and } sh(\sigma) = \lambda\}$$

is a set of bijections.

Proof: Let θ' be a map from $\{\rho_\sigma : \sigma \in \mathcal{F}_n \text{ and } sh(\sigma) = \lambda\}$ to E^λ. Given $\sigma \in \mathcal{F}_n$ with shape λ, we will define $\theta'(\rho_\sigma) = e_t^A$ such that $\theta(e_t^A) = \rho_\sigma$.

Set $\theta'(\rho_\sigma) = e_t^A$ by labeling λ from left to right with the elements of $-\sigma$. Then in each hook, rearrange the labels so the absolute value of these labels is increasing. Call this labeling t'. The subset A is determined by the negatively-labeled elements in t' and t is given by the absolute value of t'.

One can check $\theta(\theta'(\rho_\sigma)) = \rho_\sigma$ and $\theta'(\theta(e_t^A)) = e_t^A$. □

Proposition 2.8.3 can be extended by linearity to a vector space isomorphism between the two spaces.

We sum over all possible partitions and signings of λ to get a bijection between basis elements of $\sum_{j=0}^{n-|\lambda_1|} \binom{n-|\lambda_1|}{j} S^\lambda$ and the basis elements of $H(\Delta(\mathrm{Rees}(\mathscr{C}_n, C_{n+1})))$ to make the following corollary.

Corollary 2.8.4 *The map*

$$\theta : \{E^\lambda\}_\lambda \longrightarrow \{\rho_\sigma | \sigma \in \mathcal{F}_n\}$$

is a set bijection where λ ranges over all skew diagrams which are finite unions of hooks of size at least two augmented at the end by a block containing the element $n+1$.

Again we extend by linearity to a vector space isomorphism between these two spaces. It is left to prove the module isomorphism properties in order to prove Theorem 2.8.1. First, we look at which elements of $\mathfrak{S}_n$ fix basis elements of $\sum_{j=0}^{n-|\lambda_1|}\binom{n-|\lambda_1|}{j}S^\lambda$ and $H(\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1})))$.

For a given tableau t, define $S_t = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ where the λ_i are subsets of $[n]$ corresponding to the labels of the ith hook t.

Claim 2.8.5 *The polytabloid e_t^A satisfies $e_t^A = e_{\pi t}^A$ for all $\pi \in S_t$.*

Proof: Given such a permutation $\pi \in S_t$, it acts on t by permuting labels only within individual hooks of t. If the labels within a row are permuted, the polytabloid is fixed because each tabloid is a row equivalence class. If a labels within a column are permuted, π acts as an element of the column stabilizer C_t. For such an element π we have $e_t = e_{\pi t}$. Lastly, if an element in a column is moved out of its column but within its row because of the equivalence class, we can rewrite the tabloid with that element occurring at the end of the row, leaving π to act as an element of C_t. $\square$

Claim 2.8.6 *The fundamental cycle ρ_σ satisfies $\rho_\sigma = \rho_{\pi\sigma}$ for all $\pi \in S_t$.*

Proof: It is enough to show the equality of the posets $\mathscr{C}_\sigma$ and $\mathscr{C}_{\pi\sigma}$ to prove the equality of the fundamental cycles of their order complexes ρ_σ and $\rho_{\pi\sigma}$. The posets $\mathscr{C}_\sigma$ and $\mathscr{C}_{\pi\sigma}$ have bars in the same places and negative signs with the same numbers, so it is left to consider the ranks in the poset where one element of rank i for some i has a different second coordinate from all other elements of that rank. If the set of ranks with this property is the same in $\mathscr{C}_\sigma$ and $\mathscr{C}_{\pi\sigma}$, the two posets are equal. In σ or $\pi\sigma$ the rank i must correspond to a label at the end of a piece j for some j. In $\mathscr{C}_\sigma$, this element will have stars in positions corresponding to labels in the first j places in the permutation. This is the same in $\mathscr{C}_{\pi\sigma}$ because π only permutes elements within individual pieces. The fixed negative signs assure the non-starred elements are the same in both. Thus, we have $\mathscr{C}_\sigma = \mathscr{C}_{\pi\sigma}$. $\square$

We now prove Theorem 2.8.1. For $\pi \in S_t$, we have $\pi\theta(e_t^A) = \theta(\pi e_t^A)$. That is,

$$\pi\theta(e_t^A) = \pi\rho_\sigma = \rho_{\pi\sigma} = \rho_\sigma = \theta(e_t^A) = \theta(e_{\pi t}^A) = \theta(\pi e_t^A)$$

It is left to show this relationship holds for $\tau \in S_n - S_t$. In fact, it is enough to show $\theta(\tau e_t^A) = \pi\tau\rho_\sigma$ for some $\pi \in S_t$.

Consider $\theta(\tau e_t^A)$ and $\tau\rho_{-\sigma}$ for some $\tau \in S_n - S_t$ and some e_t^A such that $\theta(e_t^A) = \rho_\sigma$. The permutation τ acts on t by permuting the labels. The polytabloid $e_{\tau t}^A$ is a sum of tabloids under action by the column stabilizer C_t. Hence, we are only concerned with cycles of τ which move labels from one hook of t to another hook of t. Let θ take $e_{\tau t}^A$ onto $\rho_{-\widehat{\sigma}}$. We know $\widehat{\sigma}$ is found by attaching the signs from A to t and reordering so each piece is decreasing, and τ acts on σ also by permuting the labels. There is no guarantee that $\tau\sigma$ will have hooks each of which having labels in decreasing order. However, we can find a permutation $\pi \in S_t$ such that $\pi\tau\sigma$ will have hooks whose labels are in decreasing order. Since there is only one way to write a set of integers in decreasing order, it is left to show the labels on each hook of $\widehat{\sigma}$ are the same as the labels on the corresponding hook of $\tau\sigma$. (Hooks of $\sigma \in \overline{\mathfrak{S}_n^{\pm}}$ correspond to the hooks in the λ associated with σ.) If label l is in a different hook in $\widehat{\sigma}$ than in $\tau\sigma$, then τt mapped l to a different hook than $\tau\sigma$. This is a contradiction because labels in t are in the same corresponding hooks as labels in σ. Hence, $\widehat{\sigma} = \pi\tau\sigma = \tau\sigma$ and $\theta(\tau e_t^A) = \rho_{-\tau\sigma}$.

The isomorphism $\oplus_{j=0}^{n-|\lambda_1|}\binom{n-|\lambda_1|}{j} S^\lambda \simeq 2^{n-|\lambda_1|}S^\lambda$ induces a module isomorphism $2^{n-|\lambda|}S^\lambda \simeq H(\Delta(\mathrm{Rees}(\mathscr{C}_n, C_{n+1})))$. Thus, we have proved Theorem 2.8.1.

2.9 Concluding remarks

For future study we list a few open questions. Several of these will be addressed in Chapter 3.

1. What poset P would have its Möbius function related to the permanent

$$\mathrm{per}\begin{bmatrix} s & r & r & \cdots & r \\ r & s & r & \cdots & r \\ r & r & s & \cdots & r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & s \end{bmatrix}?$$

2. The derangement numbers occur as the local h-vector of the barycentric subdivision of the n-simplex. Is there a relation between the local h-vector and the Rees product?

3. Shareshian has pointed out though it is not the case the Möbius numbers in Theorem 2.6.1 count derangements in the action on the n-cube, they do in fact count a class of type BC permutations, namely, those that fix (setwise) exactly two sides of the given cube. Equivalently, those that fix exactly two points of the dual cross-polytope. (Those two points are necessarily antipodal.) To obtain the $n \times \mathrm{per}(A)$ formula, where $\mathrm{per}(A)$ is the permanent of the $(n-1)\times(n-1)$ matrix with ones on the diagonal and twos off the diagonal, there are n ways to pick a pair of antipodal points. It is not hard to show that $\mathrm{per}(A)$ counts derangements in the action of the stabilizer of these two points on the vertices of the $(n-1)$-cross-polytope spanned by the remaining $2n-2$ points using the fact that for any given permutation w in S_{n-1}, there are $2^{n-1-\mathrm{fix}(w)}$ barred permutations associated with w that are derangements in the given action.

4. The next most natural Cohen-Macaulay poset to study is the Rees product of the partition lattice with the chain. Table 2.2 contains the first few Möbius

n	$\mu(\text{Rees}(\Pi_n, C_{n-1}))$		Prime factorization
1	1	=	1
2	0	=	0
3	2	=	2
4	12	=	$2^2 \cdot 3$
5	142	=	$2 \cdot 71$
6	2400	=	$2^5 \cdot 3 \cdot 5^2$
7	55834	=	$2 \cdot 27917$
8	1708420	=	$2^2 \cdot 5 \cdot 7 \cdot 12203$
9	66514550	=	$2 \cdot 5^2 \cdot 43 \cdot 30937$
10	3210526440	=	$2^3 \cdot 3^2 \cdot 5 \cdot 11 \cdot 107 \cdot 7577$

Table 2.2: Table of Möbius values for Rees product of the partition lattice with the chain.

values of the Rees product of the partition lattice with the chain. Do these values have any combinatorial interpretation?

Chapter 3 Other Facts About Poset Products

In personal communication with Shareshian and Wachs, they claim that if two graded posets P and Q are shellable then all the intervals in the Rees product $P * Q$ are also shellable. Here "shellable" loosely means that both posets are EL-shellable or CL-shellable. In this chapter we will describe R-labelings and EL-labelings on Segre products and Rees products inherited by the labelings on the factors of the products. We will give an explicit labeling for the Rees product of two CL-shellable posets and we will generalize $\text{Rees}(\mathscr{C}_n, C_{n+1})$ to the Rees product of the face poset of a polytope with the chain. Further we will discuss some partial results for the question about permanents of matrices from Chapter 2. After defining local h-vectors, we state a few conjectures about the local h-vector of the barycentric subdivision of the cube. We conclude with a bijective proof of Theorem 2.3.1.

3.1 R-labelings and EL-labelings of Segre and Rees products

Proposition 3.1.1 *Let P and Q be graded and bounded posets of rank n. If both P and Q have an R-labeling, respectively EL-labeling, then the Segre product $P \circ_{rk} Q$ has an R-labeling, respectively EL-labeling.*

Proof: Let $\lambda_P : E(P) \to \Lambda_P$ be an R-labeling on the poset P and let $\lambda_Q : E(Q) \to \Lambda_Q$ be an R-labeling on the poset Q. Let $\Lambda_P \times \Lambda_Q$ be ordered by $(a_1, b_1) < (a_2, b_2)$ if $a_1 <_{\Lambda_P} a_2$ and $b_1 <_{\Lambda_Q} b_2$. Construct a labeling $\lambda : E(P \circ_{rk} Q) \to \Lambda_P \times \Lambda_Q$ as follows. If $x = (a_1, a_2)$ and $y = (b_1, b_2)$ are two elements in the Segre product $P \circ_{rk} Q$ where $x \prec y$ then

$$\lambda(x, y) = (\lambda_Q(a_1 \prec a_2), \lambda_Q(b_2 \prec b_2)).$$

In the poset $P \circ_{rk} Q$, the edge label (x, y) is less than (x', y') if $x <_{\Lambda_P} x'$ and $y <_{\Lambda_Q} y'$. We check that this is an R-labeling. For any interval $[x, y] \in P \circ_{rk} Q$ one finds the unique increasing maximal chain $m : x = x_0 < x_1 < \cdots < x_k = y$ given by $x_i = (a_i, b_i)$ where $a_0 \prec a_1 \prec \cdots \prec a_k$ is the unique increasing maximal chain in the interval $[a_0, a_k] \in P$ and $b_0 \prec b_1 \prec \cdots \prec b_k$ is the unique increasing maximal chain in the interval $[b_0, b_k] \in Q$. In fact, m is the only increasing maximal chain in $[x, y]$. All other chains are the Segre product of pairs of chains with the first coordinates from the chain in P and the second coordinates from the chain in Q. By the uniqueness of the increasing chains in P and Q respectively, there cannot exist another increasing chain.

Now suppose further that λ_P and λ_Q are EL-labelings. The lexicographically least increasing chain in the interval $[x, y]$ consists of taking the Segre product of the lexicographically least increasing chain in the interval $[a_0, a_k]$ in P with the lexicographically least increasing chain in the interval $[b_0, b_k]$ in Q. Again, it is straightforward to check this chain in $[x, y]$ is the unique such lexicographic least chain, for otherwise this would contradict the uniqueness of the lexicographic least chain in either P or Q. □

This R-labeling scheme also gives an R-labeling for the poset $Q \circ_{rk} P$ simply by switching the coordinates of the labeling and that the posets $P \circ_{rk} Q$ and $Q \circ_{rk} P$ have the same set of falling chains. The Segre product operation is commutative up to isomorphism. Furthermore extend this labeling to a Segre product of a finite number of posets. To label an edge in $P_1 \circ_{rk} P_2 \circ_{rk} \cdots \circ_{rk} P_n$ we use an n-tuple of the labels on respective edges in each individual poset.

The Rees product is more complicated. Unlike the Segre product, the labeling does not commute.

Proposition 3.1.2 *Let P and Q be graded and bounded posets where the rank of P is n and the rank of Q is $n-1$. If both P and Q have an R-labeling, respectively EL-labeling, then the Rees product* $\mathrm{Rees}(P, Q)$ *has an R-labeling, respectively EL-labeling.*

Proof: Let $\lambda_P : \mathscr{E}(P) \longrightarrow \Lambda_P$ be an R-labeling on P and let $\lambda_Q : \mathscr{E}(Q) \longrightarrow \Lambda_Q$ be an R-labeling on Q. Define an R-labeling $\lambda : \mathrm{Rees}(P, Q) \to (\Lambda_P \cup \{M\}) \times (\Lambda_Q \cup \{0\})$ with the convention

$$a < M \text{ for all } a \in \Lambda_P \text{ and } 0 < b \text{ for all } b \in \Lambda_Q.$$

Order the labels in the Cartesian product $(\Lambda_P \cup \{M\}) \times (\Lambda_Q \cup \{0\})$ so that

$$(a_1, b_1) < (a_2, b_2) \text{ if } a_1 <_{\Lambda_P \cup \{M\}} a_2 \text{ and } b_1 <_{\Lambda_Q \cup \{0\}} b_2.$$

For the R-labeling λ, label the edge $\hat{0} \prec (\alpha, \hat{0}_Q)$ with $(\lambda_P(\hat{0}_P \prec \alpha), 0)$ for any atom $\alpha \in P$. Label the edges $(\hat{1}_P, b) \prec \hat{1}$ with $(M, 0)$ for all $b \in Q$. Label all other edges $x \prec y$ where $x = (a_1, b_1)$ and $y = (a_2, b_2)$ by

$$\lambda(x, y) = \begin{cases} (\lambda_P(a_1 \prec a_2), \lambda_Q(b_1 \prec b_2)) & \text{if } b_1 \neq b_2, \\ (\lambda_P(a_1 \prec a_2), 0) & \text{if } b_1 = b_2. \end{cases}$$

We claim this is an R-labeling. Consider any interval of the form $[\hat{0}, y] \in \mathrm{Rees}(P, Q)$ where $y = (a, b) \neq \hat{1}$. Look at the first coordinates of all the labeled chains in this interval. These are exactly the set of labeled chains in $[\hat{0}_P, a] \in P$. Hence, there is exactly one increasing labeled chain of first coordinates. Say this chain is $c : \hat{0}_P \prec a_1 \prec a_2 \prec \cdots \prec a_k$. Similarly, there is exactly one increasing labeled chain on the interval $[\hat{0}_Q, b] \in Q$ of length $j \leq k$. Say this chain is $c' : \hat{0}_Q \prec b_1 \prec \cdots \prec b_j$. We see that the rank of the interval in Q may be smaller than the rank of the interval in P. This chain in Q will supply the second coordinate for the last $j + 1$ elements of the maximal chain in $[\hat{0}, y]$. These elements of Q must remain at the end of the chain in $[\hat{0}, y]$ because the labels on the edges between them are all greater than the zeros which are the second coordinate label for all other edges. For $[\hat{0}, y]$, we have the increasing chain

$$\hat{0} \prec (a_1, \hat{0}_Q) < \cdots < (a_{k-j}, \hat{0}_Q) < (a_{k-j+1}, b_1) < \cdots < (a_k, b_j) = (a, b).$$

This is the unique rising chain. The first coordinates are fixed by the unique chain in P. The second coordinate must be $\hat{0}_Q$ for the first $k-j$ elements after $\hat{0}$, otherwise the chain would not be increasing at some further index. The last j elements are fixed by the unique chain in Q.

Increasing maximal chains on intervals of the form $[x, y] = [(a_1, b_1), (a_2, b_2)]$ where $x \neq \hat{0}$ and $y \neq \hat{1}$ can be found in the same way. For the first coordinates choose the elements from the increasing labeled chain in $[a_1, a_2] \in P$. Let the second coordinate be b_1 for the first $k-j$ elements and then choose the elements from the increasing labeled chain in $[b_1, b_2] \in Q$. As above, one can check that this is the unique increasing chain.

Now consider any interval of the form $[x, \hat{1}] \in \text{Rees}(P, Q)$ where $x = (a, b_1) \neq \hat{0}$. We find the first coordinates using the increasing chain found in $[a, \hat{1}_P]$ and the label M for the last edge $(\hat{1}_P, b_2) \prec \hat{1}$. Because the edges from all coatoms to $\hat{1}$ have 0 as a second coordinate, the second element in the maximal chain stays fixed. This can only happen one way. Thus the maximal chain is

$$x = (a, b) \prec (a_1, b) \prec \cdots \prec (\hat{1}_P, b) \prec \hat{1}.$$

It is left to find the unique increasing chain in the entire poset. This is done as above producing the chain

$$\hat{0} \prec (a_1, \hat{0}_Q) \prec \cdots \prec (a_n, \hat{0}_Q) \prec \hat{1}.$$

Suppose further that λ_P and λ_Q are EL-labelings of P and Q, respectively. In the Rees product $P*Q$ the increasing maximal chain in a given interval is composed of the lexicographically smallest labeled chain from an interval in P on the first coordinates and a string of zeros followed by the lexicographically smallest labeled chain from the appropriate interval in Q on the second coordinates. Any other chain in the interval of $P * Q$ will have the same number of zeros in the chain of labels on the second coordinate but will either have a nonzero element earlier in this chain or will change the order of one of the lexicographic smallest chains in P or Q. In either of these cases this chain is not lexicographically less than the unique increasing chain. □

Sometimes we do not want to remove the minimal element of P when we take the Rees product.

Proposition 3.1.3 *Let P and Q be graded bounded posets where the rank of P and Q is n. If both P and Q have an R-labeling, respectively EL-labeling, then their Rees product $(P \star Q) \cup \{\hat{1}\}$ has an R-labeling, respectively EL-labeling.*

Proof: As before, define a labeling $\lambda' : (P \star Q) \cup \{\hat{1}\} \to (\Lambda_P \cup \{M\}) \times (\Lambda_Q \cup \{0\})$ with the same order relations. The edges $(\hat{1}_P, b) \prec \hat{1}$ are labeled with $(M, 0)$ for all $b \in Q$. All other edges $x \prec y$ where $x = (a_1, b_1)$ and $y = (a_2, b_2)$ are labeled

$$\lambda'(x, y) = \begin{cases} (\lambda_P(a_1 \prec a_2), \lambda_Q(b_1 \prec b_2)) & \text{if } b_1 \neq b_2, \\ (\lambda_P(a_1 \prec a_2), 0) & \text{if } b_1 = b_2. \end{cases}$$

Similarly, we prove this in an R-labeling. Consider an interval $[x, y] = [(a_1, b_1), (a_2, b_2)]$ in $(P \star Q) \cup \{\hat{1}\}$ where $y \neq \hat{1}$. There is exactly one increasing chain forming the first coordinates of the labeling λ' and that is the increasing labeled chain in $[a_1, a_2] \in P$. There is exactly one increasing chain forming the second coordinates. That is, a string of the appropriate number of zeros followed by the unique increasing labeled chain in $[b_1, b_2] \in P$. If the interval is of the form $[x, \hat{1}] = [(a, b), \hat{1}]$, the λ' gives the labels on the increasing chain on $[a, \hat{1}_P]$ augmented with M at the end for the first coordinates and a string of zeros as the second coordinates. Again, if the labelings on P and Q are EL-labelings, then these increasing labeled chains forming the first and second coordinates of $(P \star Q) \cup \{\hat{1}\}$ are lexicographically least. The string of zeros must appear before any nonzero elements labeling chains in Q. □

Corollary 3.1.4 *Let P be a rank n graded and bounded poset with an R-labeling and let C_n be the chain of length n. Then $\mu((C_n \star P) \cup \{\hat{1}\}) = -\mu(P)$.*

Proof: We use the R-labeling λ_P for P and label each of the edges of the Hasse diagram of C_n with 1. Form the labeling $\lambda_P \times \{0\}$ on the Rees product. The first coordinates of the labeling of any chain are never decreasing. Thus, the non-increasing chains in $C_n * P$ are determined by non-increasing chains of P. Given a maximal chain in $C_n * P$, if at any point in the chain $(i, p) \prec (i + 1, p)$ then there will be a zero in the second coordinate of the label on this edge. Therefore this labeled maximal chain does not correspond to a non-increasing chain in the Rees product. Thus, $|\mu((C_n \star P) \cup \{\hat{1}\})| = |\mu(P)|$. The rank of $(C_n \star P) \cup \{\hat{1}\}$ is one greater than the rank of P, so the resulting Möbius function is shifted by a sign. □

3.2 CL-labelings of Rees products

Recall a chain-edge labeling of a poset is a map from the set of pairs $(c, x \prec y)$ consisting of a maximal chain c and an edge $x \prec y$ of the chain to a poset. See Section 1.2. Suppose we want to restrict a chain-edge labeling to an interval $[x, y]$ of the poset. There may be several ways to label this interval depending on which maximal chains we consider. For any maximal chain r in $[\hat{0}, x]$, we define a rooted interval $[x, y]_r$ which restricts the chain-edge labeling on $[x, y]$ to the labels given by the maximal chains containing r. With this we can make the following definition following [36].

Definition 3.2.1 *A* chain-lexicographic labeling, *or CL-labeling, of a bounded poset P is a chain-edge labeling such that in each closed rooted interval $[x, y]_r$ of P there is a unique strictly increasing maximal chain which is lexicographically less than all other maximal chains in the rooted interval $[x, y]_r$.*

The following result is similar to those in Section 3.1.

Theorem 3.2.2 *Let P and Q be two graded and bounded posets of rank n. If both P and Q are CL-shellable posets then $(P \star Q) \cup \{\hat{1}\}$ is also CL-shellable.*

Proof: First, we build a chain-edge labeling on $(P \star Q) \cup \{\hat{1}\}$ from the CL-labelings $\widetilde{\lambda}_P$ and $\widetilde{\lambda}_Q$ on P and Q. We want a labeling $\widetilde{\lambda}$ from the set of edges of maximal chains in $(P \star Q) \cup \{\hat{1}\}$, that is $\widetilde{\lambda} : \mathscr{E}^*((P \star Q) \cup \{\hat{1}\}) \longrightarrow \Lambda$ where $\mathscr{E}^*((P \star Q) \cup \{\hat{1}\}) = \{(c, x, y) : c \in \mathscr{M}((P \star Q) \cup \{\hat{1}\}), x, y \in c, x \prec y\}$. The set of maximal chains, $\mathscr{M}((P \star Q) \cup \{\hat{1}\})$, consists of augmented maximal chains in P paired with partial maximal chains in Q which allow repetition of the same element. We will label each $(c, x, y) = (c_P, \widehat{c_Q}, (a_1, b_1), (a_2, b_2))$ where c_P is the maximal in P and $\widehat{c_Q}$ is the maximal chain in Q corresponding the the chain in $(P \star Q) \cup \{\hat{1}\}$ with no repetitions allowed. If $y = \hat{1}$, then $\widetilde{\lambda}(c, x, \hat{1}) = (M, 0)$. Otherwise,

$$\widetilde{\lambda}(c, x, y) = \begin{cases} (\widetilde{\lambda}_P(c_P, a_1, a_2), \widetilde{\lambda}_Q(\widehat{c_Q}, b_1, b_2)) & \text{if } b_1 \neq b_2, \\ (\widetilde{\lambda}_P(c_P, a_1, a_2), 0) & \text{if } b_1 = b_2. \end{cases}$$

This is a chain-edge labeling. Any maximal chain in $(P \star Q) \cup \{\hat{1}\}$ can be split into a maximal chain in from P and a maximal chain from a lower order ideal of Q which may have been augmented by repeated elements. For any two maximal chains on $(P \star Q) \cup \{\hat{1}\}$ that agree on the first k elements, the first k elements from the corresponding chains in P and the corresponding augmented chains in Q also agree. Thus the first $k - 1$ labels on the maximal chain in $(P \star Q) \cup \{\hat{1}\}$ must agree because the labels on the chains in P and Q agree, as $\widetilde{\lambda}_P$ and $\widetilde{\lambda}_Q$ are chain-edge labelings.

We check that our chain-edge labeling is chain-lexicographic. This proof is similar to the proof of Theorem 3.1.3. In each rooted interval $[x, y]_r$, we find our unique maximal chain by using the unique maximal chain of the corresponding interval of P for the first coordinate and the unique maximal chain given by the corresponding interval in Q augmented at the beginning by a string of zeros for the second coordinate. One can check that this chain is lexicographically least. □

3.3 The Rees product of the face lattice of a polytope with the chain

In this section we generalize the results from Chapter 2. Given any polytope we can calculate its Möbius function and describe and explicit basis for the top homology group.

Let P be a polytope of dimension d with face lattice $\mathscr{L}$. The order complex of the lattice $\mathscr{L}$ is the barycentric subdivision of P and hence is homotopic to a $(d-1)$-dimensional sphere. Let

$$E_P = \text{Rees}(\mathscr{L}, C_d),$$

(Here, $\mathscr{L}$ has a minimal element $\hat{0}$ and a maximal element $\hat{1}$.) If $\mathscr{L}$ has an R-labeling or EL-labeling, we have an R-labeling on E_P. Using the same techniques created for the d-dimensional cube, we describe a basis for the top homology group of $\Delta(E_P)$.

Let $\widehat{\lambda}$ be the R-labeling for E_P given in Section 3.1, where λ_{C_n} is the labeling with 1 on every edge of the chain and $\lambda_{\mathscr{L}} : \mathscr{E}(\mathscr{L}) \longrightarrow \Lambda_{\mathscr{L}}$ is any R-labeling on $\mathscr{L}$. The labels on the edges of E_P have the form (a, b) where $a \in \Lambda_{\mathscr{L}}$ and $b \in \{0, 1\}$. As in the cubical case we can represent labeled maximal chains in E_P with (single) augmented skew diagrams composed of hooks of size greater than one labeled with elements from $\Lambda_{\mathscr{L}}$.

First create a set of barred "permutations" corresponding to labeled maximal chains in E_P. Write the first coordinates of the labeled chain in order. Place a bar over each first coordinate whose second coordinate is 1. These barred strings are in bijection with the set of skew diagrams composed of hooks of size greater than one, labeled with chains in $\mathscr{L}$, and augmented on the right with a block containing the element M.

Let $\delta = \delta_1\delta_2 \cdots \delta_k 1$ be an augmented, skew diagram composed of k hooks of size greater than 1 augmented at the end by a singleton block and labeled with chains of $\mathscr{L}$. We say δ is standard if the labels decrease from left to right and from top to bottom. Let F_P be the set of all such single augmented, standard labeled, skew diagrams δ.

Theorem 3.3.1 *The set $\{\rho_\sigma | \sigma \in F_P\}$ is a basis for $H(E_P)$.*

Proof: This follows very similarly to the proof for $\text{Rees}(\mathscr{C}_n, C_{n+1})$ relying heavily on the fact that $\mathscr{L}$ is the face lattice of a polytope. □

If we continue as we did for $\text{Rees}(\mathscr{C}_n, C_{n+1})$, the next step would be to look at the representation of $H(\Delta(C_P))$ over a group. In particular, if there is an action over the symmetric group, we would partition H_P into classes corresponding to the shape of the hook. Then, we would need a bijection into the Specht modules S^c for a shape c.

For the case where $\widetilde{P}$ is not the face lattice of a polytope, another method is needed to determine the generators of the top homology group $H(\Delta(C_P))$.

In general, for any graded, bounded poset P of rank n, not necessarily the face poset of a polytope, we can still determine the Möbius function of $\text{Rees}(P, C_n)$ by counting standard fillings of the augmented, skew diagrams with maximal labeled chains of P. Using this we have another proof for the case of the t-ary tree where $t = 1$ in Corollary 2.2.4. In this corollary we showed the Möbius functions of $\text{Rees}(P, C_n)$ and $\text{Rees}(P^*, C_n)$ coincide for any graded poset P. This result was motivated by the special case that P has an R-labeling.

Corollary 3.3.2 *Let P be a graded and bounded poset of dimension d with an R-labeling. Let P^* be the dual of P. Then the Möbius functions agree, that is,*

$$\mu(\text{Rees}(P, C_n)) = \mu(\text{Rees}(P^*, C_n)).$$

Proof: Recall if P has an R-labeling then P^* has an R-labeling simply by reversing the order relation on the set of labels on $\mathscr{E}(P)$. Recall that the Möbius function $|\mu(\text{Rees}(P, C_d))|$ is the number of standard labeled augmented skew diagrams with blocks of size greater than 1. We describe a bijection between labeled maximal chains in Rees(P, C_d) and labeled maximal chains in the dual Rees(P^*, C_d). Consider an unaugmented labeled skew diagram δ corresponding to a chain in Rees(P, C_d). Reorder the hooks of $\delta = \delta_1 \delta_2 \cdots \delta_k$ as $\hat{\delta} = \delta_k \cdots \delta_2 \delta_1$. Reverse the labels, $a_1, a_2, \ldots, a_n$, of δ to label $\hat{\delta}$ from left to right with $a_n, \ldots, a_2, a_1$. With respect to the partial order on the set of labels of P^*, the labeled $\hat{\delta}$ is standard. Finally, augment $\hat{\delta}$ with the single block labeled M. The labeled augmented $\hat{\delta}$ corresponds to a falling chain in Rees(P^*, C_d). Because this process can be reversed, every falling chain in Rees(P^*, C_d) can be written as a falling chain in Rees(P, C_d). Thus, thus number of labeled falling chains in Rees(P, C_d) equals the number of labeled falling chains in Rees(P^*, C_d) and we have proven the result. □

3.4 Some results on permanents

In Section 2.9, we posed the question of when does the permanent of a matrix equal, up to a sign, the Möbius function of a poset. For example, in the case of the identity matrix I, all rank n Eulerian posets satisfy $\mu(P) = (-1)^n \cdot \text{per}(I_n)$. Another example is Jonsson's result which states that the permanent of an $n \times n$ matrix with zeros on the diagonal and ones elsewhere is equal, up to a sign, to the Möbius function of Rees(B_n, C_n). For r a positive integer, we will consider the case where the matrix has $r-1$ in the diagonal entries and r in all other entries and also the case where the matrix has r is every entry.

Recall for r a positive integer, the r-cubical lattice $\mathscr{C}_n^r$ is the Cartesian product $(M_r \times \cdots \times M_r) \cup \{\hat{0}\}$ of n copies of the poset M_r where M_r is the r element antichain with all of the elements covered by a maximal element $\hat{1}$. As a special case, the cubical lattice corresponds to $r = 2$. See [23] for more details.

Theorem 3.4.1 *For $r \geq 2$ and $n \geq 0$,*

$$\mu(\mathscr{C}_n^r) = (-1)^{n+1} \,\text{per} \begin{bmatrix} r-1 & 0 & \cdots & 0 \\ 0 & r-1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r-1 \end{bmatrix},$$

where the matrix is $n \times n$.

This theorem is a straightforward application of the Möbius function of a Cartesian product of posets [48, Proposition 3.8.2].

Theorem 3.4.2 *For $r \geq 2$ and $n \geq 1$,*

$$\mu(\text{Rees}(\mathscr{C}_n^r, C_{n+1})) = (-1)^n \cdot (r-1) \cdot n \cdot \text{per} \begin{bmatrix} r-1 & r & \cdots & r \\ r & r-1 & \ddots & r \\ \vdots & \ddots & \ddots & \vdots \\ r & \cdots & r & r-1 \end{bmatrix},$$

where the matrix is $(n-1) \times (n-1)$.

Proof: The proof of Theorem 3.4.2 is similar to the cases where $r = 1$ or $r = 2$. We give a sketch here.

Let $\{\zeta^i : 1 \leq i \leq r\}$ be the set of the r signs found on the elements of $\{1, 2, \ldots, n\}$ in $\mathscr{C}_n^r$. Place a linear order on the set $[n] \times \{\zeta^1, \zeta^2, \ldots, \zeta^r\} \cup \{\hat{0}\}$ of labels on edges of $\mathrm{Rees}(\mathscr{C}_n^r, C_{n+1})$ so that only the edge labels of the form (i, r), $i = 1, \ldots, n$ are greater than zero. Then given a permutation in $\mathfrak{S}_{n-1}$ the fixed point free part of the permutation can be signed in r ways and the fixed points are signed in $r-1$ ways. For the permutation one can build a new set of fixed points $\{F_i\}$ in $(r-1) \cdot n$ ways where we either insert n or i as before and then we can sign this element in $r-1$ ways. Finally, there is only one way to order the elements within each block in a descending manner due to the given linear order of the edge labels. □

Let Π_n be the rank n partition lattice on n elements and let A^n be the rank 2 poset of an n element antichain adjoined with a minimal and a maximal element.

Theorem 3.4.3 *For $r \geq 1$ and $n \geq 1$*

$$\mu(\Pi_n \times A^{r^{n-1}+1}) = (-1)^{n+1} \cdot \mathrm{per} \begin{bmatrix} r & \cdots & r \\ \vdots & \ddots & \vdots \\ r & \cdots & r \end{bmatrix},$$

where the matrix is $(n-1) \times (n-1)$.

As in Theorem 3.4.2, this result follows directly from the fact that the Möbius function behaves multiplicatively with respect to the Cartesian product.

Question 3.4.4 *Is $\mu(\mathrm{Rees}(\Pi_n \times A^{r^{n-1}+1}, C_n)$ equal to the permanent of some matrix?*

3.5 Local h-vectors and Rees products

For Δ a finite $(d-1)$-dimensional abstract simplicial complex, the *h-vector* $h(\Delta) = (h_0, h_1, \ldots, h_d)$ is defined by

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i},$$

where f_i is the number of i-dimensional faces in Δ. The polynomial $h(\Delta, x) = h_0 + h_1 x + \cdots + h_d x^d$ is called the *h-polynomial* of Δ.

For a shellable simplicial complex, the h-polynomial corresponds to the numerator of the Hilbert series of the Stanley-Reisner ring of Δ. Here the Stanley-Reisner ring $k[\Delta]$ of a simplicial complex is defined by $k[\Delta] = k[x_1, \ldots, x_v]/I$ where there is one variable x_i where $i = 1, \ldots, v$ for each of the v vertices of Δ and I is the face ideal, that is, the ideal generated by the non-faces of Δ. See Stanley [53] for further information.

In [50] Stanley defines the local h-vector for any subdivision of a simplex. We do not require that these subdivisions be simplicial. We only need that each simplex in

n	$\ell(\Gamma, x)$	N
0	1	1
1	0	0
2	x	1
3	$x^2 + x$	2
4	$x^3 + 7x^2 + x$	9
5	$x^4 + 21x^3 + 21x^2 + x$	44
6	$x^5 + 51x^4 + 161x^3 + 51x^2 + x$	265

Table 3.1: Table of local h-vectors for the barycentric subdivision of the n-simplex.

the simplicial complex Δ is subdivided into a complex whose geometric realization is homeomorphic to a ball of the appropriate dimension with the boundary of the ball a subdivision of the boundary of the simplex.

First we consider the case when Δ is a simplex.

Definition 3.5.1 *Let Δ be a simplex with vertex set V of size d. For any subdivision Γ of Δ, the* local h-polynomial *of Γ is $\ell_V(\Gamma, x) = \ell_o + \ell_1 x + \cdots + \ell_d x^d$ where*

$$\ell_V(\Gamma, x) = \sum_{W \subseteq V} (-1)^{|V-W|} h(\Gamma_W, x)$$

and where Γ_W is the subcomplex of Γ generated by the vertices W. We call $\ell_V(\Gamma) := (\ell_0, \ell_1, \ldots, \ell_d)$ the local h-vector *of Γ.*

Of particular interest is when Γ_b is the first barycentric subdivision of the simplex. In this case the h-polynomial is

$$h(\Gamma_b, x) = \sum_{i=0}^{d-1} A_{d,i-1} x^i,$$

where $A_{d,i-1}$ is the Type A Eulerian number, that is, the number of permutations in the symmetric group $\mathfrak{S}_d$ having descent set of size $i-1$. In [48] Stanley gives a bijection proving

$$\sum_{\pi \in \mathfrak{S}_d} x^{\mathrm{des}(\pi)} = \sum_{\pi \in \mathfrak{S}_d} x^{\mathrm{exc}(\pi)}$$

where $\mathrm{des}(\pi) = \#\{i : \pi_i < \pi_{i+1}\}$ and $\mathrm{exc}(\pi) = \#\{i : \pi(i) > i\}$.

We want a similar expression for the local h-polynomial. We calculate the values for the local h-vector for the barycentric subdivision of the n-simplex. See Table 3.1. Notice the values in the last column are precisely the derangement numbers.

We find the following theorem in [50].

Theorem 3.5.2 (Stanley) *The local h-vector of the barycentric subdivision of the n-simplex Δ^n is given by*

$$\ell_{\Delta^n}(\Gamma_b, x) = \sum_{\pi \in \mathscr{D}_n} x^{\mathrm{exc}(\pi)}$$

Stanley generalized Definition 3.5.1 to Eulerian posets. For P an Eulerian poset, let P_t be the open interval $[\hat{0}, t)$ for any $t \in P$. In [49], Stanley defines two polynomials $f(P, x)$ and $g(P, x)$ by the following set of rules.

i. $f(\emptyset, x) = g(\emptyset, x) = 1$

ii. If P has rank $d+1 \geq 1$ and if $f(P, x) = h_0 + h_1 x + \cdots$ then for $m = [d/2]$ and $h_{-1} = 0$ we have $g(P, x) = \sum_{i=0}^{m} (h_i - h_{i-1}) x^i$.

iii. If P has rank $d+1 \geq 1$ then $f(P, x) = \sum_{t \in P} g(P_t, x)(x-1)^{d-\rho(t)}$.

The following example can be proven using induction on the rank of the poset.

Example 3.5.3 *For the Boolean algebra B_{d+1} of rank $d+1$, we have*

$$\begin{aligned} f(B_{d+1}, x) &= 1 + x + \cdots + x^d, \\ g(B_{d+1}, x) &= 1. \end{aligned}$$

Now we can define a local h-vector for a CW-regular subdivision of the face poset of a regular CW-complex. This is called a *CW-poset*. Any Eulerian poset with a minimal element $\hat{0}$ is a CW-poset because because the order complex of each open interval $(\hat{0}, t)$ is homotopic to a sphere.

Definition 3.5.4 (Stanley) *Let P be a CW-poset. A* CW-regular subdivision *is as a surjective function $\sigma : \Gamma \longrightarrow P$ where Γ is CW-poset with the properties:*

i. *For all $t \in \Gamma$, $\rho(t) \leq \rho(\sigma(t))$.*

ii. *For every $F \in P$, let $P_{\leq F} = \{G \in P : G \leq F\}$. Then $\Gamma_F := \sigma^{-1}(P_{\leq F})$ is an order ideal of Γ; i.e, if $t \in \Gamma_F$ and $s < t$ then $s \in \Gamma_F$. We say Γ_F is the restriction of Γ to F.*

iii. *For every $F \in P$, the poset $\Gamma_F - \{\hat{0}\}$ is homotopic to a ball and the interior of the poset Γ_F is given by $int(\Gamma_F) = \{p \in \Gamma : p \notin \delta\Gamma_F\}$ where $\delta\Gamma_F$ is the subcomplex of Γ generated by all codimension one faces of Γ_F which are contained in exactly one facet of Γ_F.*

See Stanley [50] for a generalization called a *formal subdivision* used with lower Eulerian posets.

Definition 3.5.5 *Let Γ be a CW-regular subdivision of a rank d Eulerian poset P. Then*

$$\ell_P(\Gamma, x) = \sum_{t \in P} h(\Gamma_t, x) \cdot \gamma^{-1}_{[t, \hat{1}]}(x),$$

where $\gamma_{[\hat{0}, t]} = g(P_t, x)$.

Equivalently, we have

$$h(\Gamma, x) = \sum_{t \in P} \ell_{[\hat{0},t]}(\Gamma_t, x)\gamma_{t,\hat{1}}(x).$$

The next most natural example to compute is barycentric subdivision an n-cube. Let Q_n denote this subdivision. The h-polynomials have interpretations similar to the simplex. In the case of the n-cube, the h-vector is given by

$$h(Q_n, x) = \sum_{i=0}^{d-1} B_{d,i} x^i,$$

where $B_{d,i}$ is the type B Eulerian number, that is the number of signed permutations in the signed symmetric group on d elements with descent set size $i-1$. In the signed case, let $\pi_0 = 0$. Then for $\pi \in \mathfrak{S}_d^{\pm}$, the number of descents of π is given by $\mathrm{des}^{\pm}(\pi) = \#\{i : \pi_i > \pi_{i+1},\ i = 0, 1, \ldots, n-1\}$. Now,

$$h(Q_n, x) = \sum_{\pi \in \mathfrak{S}_d^{\pm}} x^{\mathrm{des}^{\pm}(\pi)}.$$

We would like a signed analogue of the excedance statistic. One option is to define $\mathrm{exc}^{\pm}$ as follows. For $\pi \in \mathfrak{S}_n^{\pm}$ let $\mathrm{exc}_{|\pi|} = \#\{\pi_i : |\pi_i| > i\}$ and $\mathrm{neg}_{\pi} = \#\{\pi_i : \pi_i < 0\}$. Define $\mathrm{exc}^{\pm}$ as the ceiling of the quotient $\mathrm{exc}_{|\pi|}$ plus neg_{π} divided by two, that is,

$$\mathrm{exc}^{\pm}(\pi) = \left\lceil \frac{\mathrm{exc}_{|\pi|} + \mathrm{neg}_{\pi}}{2} \right\rceil.$$

Conjecture 3.5.6 *For the barycentric subdivision of the d-cube, Q_n,*

$$h(Q_n, x) = \sum_{\pi \in \mathfrak{S}_d^{\pm}} x^{\mathrm{exc}^{\pm}(\pi)}.$$

Let us compute the local h-polynomial. The face lattice of the n-dimensional cube, $\mathscr{C}_n$ is an Eulerian poset, so we can apply the new definitions once we know $g(\mathscr{C}_n, x) = \gamma_{\mathscr{C}_n(x)}$. From Example 3.5.3, we know for intervals $[s,t] \neq P$ that $\gamma_{[s,t]}(x) = 1$. For the interval $[\hat{0}, \hat{1}]$ we have the following theorem due to Gessel referenced in [49].

Theorem 3.5.7 (Gessel) *For the face lattice of the n-cube*

$$\gamma_{\mathscr{C}_n}(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{n-k+1} \binom{n}{k} \binom{2n}{2k} (x-1)^k.$$

We use Definition 3.5.5 and Theorem 3.5.7 to compute some values for the local h-polynomials of the barycentric subdivision of the cubical lattice Q_n. Observe if $t \neq \emptyset$ then $\gamma_{[t,\hat{1}]}(x) = 1$ because $[t, \hat{1}] \simeq B_{n+1-\rho(t)}$. If $t = \emptyset$ we use Theorem 3.5.7 to see that the h-vector $h(Q_n, x)$ is given by the type B Eulerian numbers [OEIS A060187]. The values for $\gamma_{\mathscr{C}_n}(x)$ and $h(Q_n, x)$ are found in Table 3.2.

We substitute these values into Definition 3.5.5. These resulting values appear Table 3.3. Unfortunately, the last column does not agree with the signed derangement numbers.

n	$\gamma_{\mathscr{C}_n}(x)$	$h(Q_n, x)$
-1	1	1
0	1	1
1	1	$x+1$
2	$x+1$	x^2+6x+1
3	$4x+1$	$x^3+23x^2+23x+1$
4	$2x^2+11x+1$	$x^4+76x^3+230x^2+76x+1$
5	$15x^2+26x+1$	$x^5+237x^4+1682x^3+1682x^2+237x+1$

Table 3.2: Table of values of $\gamma_{\mathscr{C}_n}(x)$ and $h(Q_n, x)$.

n	$\ell_{\mathscr{C}_n}(x)$	N
-1	1	1
0	0	0
1	x	1
2	x^2+x	2
3	x^3+17x^2+x	19
4	$x^4+68x^3+68x^2+x$	138
5	$x^5+227x^4+962x^3+227x^2+x$	1418

Table 3.3: Table of local h-vectors for the barycentric subdivision of the face lattice of the n-cube.

Conjecture 3.5.8 *There exists a subset L of permutations from the signed symmetric group $\mathfrak{S}_n^{\pm}$ such that*

i. $\displaystyle\sum_{\pi\in L} x^{\mathrm{des}(\pi)^{\pm}} = \ell_{\mathscr{C}_n}(\Gamma, x)$ *or*

ii. $\displaystyle\sum_{\pi\in L} x^{\mathrm{exc}(\pi)^{\pm}} = \ell_{\mathscr{C}_n}(\Gamma, x)$.

Question 3.5.9 *Find a family of posets P_n of rank n so that the sequence*

$$\{(-1)^{n+1}\mu(\mathrm{Rees}(P_n, C_n))\}_{n\geq 0}$$

agrees with the last column of Table 3.3.

3.6 Bijective proof of Lemma 2.3.1

Recall Lemma 2.3.1 is the identity

$$1+\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}k!(n-k+1)=0.$$

An alternate proof is as follows. This proof has the advantage of being bijective.

Proof: Let A be the set

$$A = \{(\tau, m) : \tau \text{ is a partial permutation of } [n] \text{ and } m \in [n] - \tau \cup \{*\}\} - \{(\emptyset, \star)\}.$$

Define sign : $A \longrightarrow \{-1, 1\}$ by $\text{sign}(\tau, m) = (-1)^{|\tau|+1}$. Let $\varphi : A \longrightarrow A$ be defined by

$$\varphi(\tau, m) = \begin{cases} (\tau_1 \cdots \tau_k m, *) & \text{if } m \neq *, \\ (\tau_1 \cdots \tau_{k-1}, \tau_k) & \text{if } m = *. \end{cases}$$

Observe that φ is an involution, that is, $\varphi(\varphi(\tau, m)) = (\tau, m)$ and φ is sign-reversing, that is, $\text{sign}(\varphi(\tau, m)) = -\text{sign}(\tau, m)$. Hence

$$\begin{aligned} 0 &= \sum_{(\tau,m)\in A} \text{sign}(\tau, m) \\ &= -n + \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} k!(n-k+1) \end{aligned}$$

which is equivalent to the statement of the lemma. □

Chapter 4 Inequalities for the Partition Lattice

The question of determining flag vector inequalities of posets has a long history. For polytopes, that is, the face lattice of a polytope, all the linear inequalities among the flag vector entries are known for polytopes up to dimension three. This is due to Steinitz [54] in 1906. For polytopes of dimension greater than or equal to 4 this question is wide open, though there are partial results. The case of the face lattice of the simplex, that is, the Boolean algebra is well-understood as there is a direct connection with descent sets of permutations in the symmetric group. See Section 4.2 for a more detailed history. The case of the face lattice of the cross-polytope was initiated in Readdy [39] and the more general r-cubical lattice by Ehrenborg and Readdy [24]. Work of Ehrenborg has shown how to lift known inequalities to polytopes of higher dimension [20]. These results have succinct formulations in terms of a noncommutative polynomial called the **cd**-index. For other results on polytopes, see [6, 8, 7, 19].

In this chapter we study flag vector inequalities for the partition lattice. Although the partition lattice is not the face lattice of a polytope, it is geometrically motivated as it is the intersection lattice of the hyperplane arrangement

$$\begin{aligned} x_i = x_j, &\qquad 1 \le i < j \le n, \\ x_i = 0, &\qquad 1 \le i \le n. \end{aligned}$$

Unlike the face lattice of a polytope, the partition lattice is not Eulerian. Hence its flag vector data, encoded using the **ab**-index, cannot be reduced to the more compact **cd**-index. The partition lattice does however enjoy many favorable properties, including being supersolvable and hence has an EL-labeling [45] and being Cohen-Macaulay, thus having only non-zero homology in the top dimension.

In Section 4.1 we review the notions of the descent set statistic, the **ab**-index and the flag f- and flag h-vectors. In Section 4.2 we review flag h-vector inequalities for the Boolean algebra which go back to work of Niven. We then focus on the technique of computing flag h-vector entries using the boustrophedon transform.

In Section 4.3 we show the descent set statistic of the partition lattice satisfies a weighted boustrophedon transform. In Section 4.4 we review the Dowling lattice, a family of posets which generalizes the partition lattice. Again we give a weighted boustrophedon transform for its flag h-vector.

In Section 4.5 we review Niven's determinantal formula for the descent set statistic. We generalize his results for posets all of whose lower intervals $[\hat{0}, x]$ for x of rank k (or dually, upper intervals $[x, \hat{1}]$) are isomorphic.

In Section 4.6 we return to the question of studying inequalities for the flag h-vector of the partition lattice Π_n. Ehrenborg and Readdy have conjectured in [23] that the maximum occurs at the alternating word $\mathbf{baba} \cdots \mathbf{bab}$ for n odd and the almost alternating word $\mathbf{baba} \cdots \mathbf{babb}$ for n even. Using direct permutation arguments and estimates, we show many inequalities satisfied by the coefficients of the **ab**-index of the partition lattice.

4.1 Background

Let P be a graded and bounded poset of rank n with an edge labeling λ. To each maximal labeled chain $m = \{\hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1}\}$ in P, we will associate a word $w(m) = w_1 w_2 \cdots w_{n-1}$ on the letters $\mathbf{a}$ and $\mathbf{b}$, where

$$w_i = \begin{cases} \mathbf{a} & \text{if } i \notin D(m), \\ \mathbf{b} & \text{if } i \in D(m). \end{cases}$$

Recall the descent set of a maximal chain m is $D(m) = \{i : \lambda(x_{i-1}, x_i) \not< \lambda(x_i, x_{i+1})\}$.

Definition 4.1.1 *For a poset P with an edge labeling the* descent set statistic *of an* ***ab****-word w is*

$$[w]_P = \#\{m : m \textit{ is a maximal chain in } P \textit{ and } w(m) = w\}$$

The set of descent words can be organized into a polynomial whose coefficients are the descent statistics. We follow [23] in the following definitions.

Definition 4.1.2 *Let P be a graded and bounded poset of rank n. For a subset $S \subseteq [n-1] = \{1, 2, \ldots, n-1\}$, let $\alpha(S) = \alpha_P(S)$ denote the number of maximal chains in the rank-selected subposet P_S and let the* beta invariant *$\beta(S) = \beta_P(S)$ be defined by*

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T).$$

For $S \subseteq \{1, \ldots, n-1\}$ we define a monomial in the noncommutative variables $\mathbf{a}$ and $\mathbf{b}$ by $u_S = u_1 \cdots u_{n-1}$ where u_i is $\mathbf{a}$ if $i \notin S$ and u_i is $\mathbf{b}$ if $i \in S$. Form a noncommutative polynomial of degree $n-1$ called the ***ab****-index* by

$$\Psi(P) = \sum_{S \subseteq [n-1]} \beta_P(S) \cdot u_S.$$

For the following theorem refer to [23].

Theorem 4.1.3 *The* ***ab****-index of a graded poset P with an R-labeling is given by*

$$\Psi(P) = \sum w(m)$$

where the sum is over all maximal chains m in P.

There are other interpretations of the descent set statistic. For example, up to sign it is equal to the Möbius function of the rank-selected subposet P_S where we select ranks associated to descents in a word w, that is,

$$[w]_P = (-1)^{\rho(P_S)} \mu(P_S) = (-1)^{|S|+1} \mu(P_S) \text{ where } S = \{i : w_i = \mathbf{b}\}.$$

S	f_S	w	$h_S = [w]_{B_4}$
$\emptyset$	1	**aaa**	1
1	4	**baa**	3
2	6	**aba**	5
3	4	**aab**	3
12	12	**bba**	3
13	12	**bab**	5
23	12	**abb**	3
123	24	**bbb**	1

Table 4.1: Flag f-vector and flag h-vector for the Boolean algebra B_4.

See [48, Section 3.12]. For polytopes another interpretation of the descent statistic uses the flag f-vector. A flag is chain of faces $F_1 \subset F_2 \subset \cdots \subset F_k$ in an n-dimensional polytope. For $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n]$ the flag f-vector is the 2^n values $(f_S)_{S \subseteq [n]}$ where the entry f_S is given by $f_S = \#\{F_{s_1} \subsetneq \cdots \subsetneq F_{s_k}\}$ and $\dim(F_{s_i}) = s_i - 1$, that is, $\rho(F_{s_i}) = s_i$ in the face lattice of the polytope for $i = 1, \ldots, k$.

The flag h-vector is defined by $h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T$. This relation in invertible, that is, $f_S = \sum_{T \subseteq S} h_T$. Furthermore, for polytopes Stanley [51] showed that $h_S = h_{\overline{S}}$ where $\overline{S}$ denotes the complement of S. Hence the flag h-vector stores the flag f-vector data more compactly. Furthermore, the **ab**-index of a polytope $\mathscr{P}$ is now $\Phi(\mathscr{P}) = \sum_{S \subseteq [n]} h_S \cdot u_s$. The descent set statistic is given by the flag h-vector, that is,

$$[w]_P = h_S \text{ where } S = \{i : w_i = \mathbf{b}\}.$$

The notions of flag f- and h-vectors extend to graded posets. One natural question to ask is which words maximize the descent statistic. In the next section, we will recall the classical example of the Boolean algebra.

4.2 The Boolean algebra and the boustrophedon transform

Recall the Boolean algebra on n elements B_n is the set of subsets of $[n]$ ordered by inclusion. Hence $A \prec B$ if and only if $A \subsetneq B$ and $|B| = |A| + 1$. An edge labeling on $\lambda : \mathscr{E}(B_n) \longrightarrow [n]$ is given by

$$\lambda(A \prec B) = i \text{ where } i \text{ is the unique element in } B - A.$$

Note that the set of edge labels on maximal chains of B_n corresponds bijectively with the set of permutations in $\mathfrak{S}_n$. See Table 4.1 for the flag f- and flag h-vectors of B_4.

For the Boolean algebra the alternating words **baba**$\cdots$ and **abab**$\cdots$ maximize the descent statistic.

Theorem 4.2.1 (Niven, de Bruijn) *For the Boolean algebra B_n, equivalently the symmetric group $\mathfrak{S}_n$ the alternating words* ***bab***$\cdots$ *and* ***aba***$\cdots$ *of length $n-1$ maximize the descent set statistic.*

This can be proven in many different ways. Niven [37] used a determinantal formula (see Section 4.5) and the following theorem [37, Theorem 5].

Definition 4.2.2 *For an* $\boldsymbol{ab}$*-word* $v = v_1 v_2 \cdots v_k$ *denote the word* $\overline{v} = \overline{v_1}\,\overline{v_2} \cdots \overline{v_k}$ *by*

$$\overline{v_i} = \begin{cases} \boldsymbol{a} & \text{if } v_i = \boldsymbol{b}, \\ \boldsymbol{b} & \text{if } v_i = \boldsymbol{a}, \end{cases}$$

that is, $\overline{v}$ *denotes the* $\boldsymbol{ab}$*-word* v *with the* $\boldsymbol{a}$*'s and* $\boldsymbol{b}$*'s uniformly exchanged.*

Observe that $[v]_{B_n} = [\overline{v}]_{B_n}$.

Theorem 4.2.3 (Niven) *For the Boolean algebra, equivalently the symmetric group* $\mathfrak{S}_n$*, the following inequality holds.*

$$[u\boldsymbol{aa}v]_{B_n} < [u\boldsymbol{ab}\overline{v}]_{B_n}.$$

Viennot [56] gives a recursive algorithm to construct the set of alternating permutations, Stanley [51, Corollary 2.9] and Readdy [39, Theorem 3.0.4] use a **cd**-index approach, Sagan–Yeh–Ziegler [42] consider families of subposets of B_n, and Ehrenborg–Levin–Readdy [21] used a probabilistic approach to determine non-linear inequalities on the flag h-vector. Another technique for computing the descent set statistic is the boustrophedon transform. It has been given in various forms by de Bruijn [17], Arnol'd [1], Millar–Sloane–Young [35], and Ehrenborg–Mahajan [22].

The *boustrophedon transformation* is an operation that takes a sequence $\{a_i\}_{i\geq 0}$ to another sequence $\{b_i\}_{i\geq 0}$ via a triangular array. We illustrate with an example.

In the following array, we start with a_1 at the top of the array. Then we alternate moving left to right along the $2k$th rows and right to left along the $(2k+1)$st rows of the array as we move down the array. The terms of the sequence $\{a_i\}_{i\geq 1}$ start each row either from the far right or the far left. As you follow the arrows across the array a new term is determined by adding the previous value in the same row and the previous value in the row directly above. The sequence $\{b_i\}$ is formed by the last element formed in each row.

Example 4.2.4 *Let* $\{a_i\}_{i\geq 1} = (1, 0, 0, 0, \ldots)$*. Then we have the array*

$$\begin{array}{ccccccccc}
 & & & & \textcircled{1} & & & & \\
 & & & \textcircled{0} & \longrightarrow & 1 & & & \\
 & & 1 & \longleftarrow & 1 & \longleftarrow & \textcircled{0} & & \\
 & \textcircled{0} & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 2 & \\
5 & \longleftarrow & 5 & \longleftarrow & 4 & \longleftarrow & 2 & \longleftarrow & \textcircled{0} \\
 & & & & \vdots & & & &
\end{array}$$

and $\{b_i\}_{i\geq 1} = (1, 1, 1, 2, 5, \ldots)$ *is the vector of elements at the end of each row and if you remove* b_1 *it is also the sequence of row sums.*

The word boustrophedon comes from the French for "ox-plowing". If you trace the progression of the sequence you can see it curls around like the path of an ox plowing a field.

The entries in this array can also be defined by a recurrence. If $\{S_{m,j}\}$ is the set of entries in the array where m denotes the row number reading from top to bottom ($m \geq 1$), and j gives the jth entry in row m reading along the arrows so the first element $S_{m,1}$ in row m is a_m, then

$$\begin{array}{lcll} S_{m,1} & = & a_m & (m \geq 1), \\ S_{m+1,j+1} & = & S_{m+1,j} + S_{m,m-j} & (m \geq 1, j \geq 1). \end{array}$$

When $(a_0, a_1, a_2, \ldots) = (1, 0, 0, \ldots)$, the entry $S_{m,j}$ equals the number of alternating permutations with descent word $\mathbf{baba}\cdots$ in the symmetric group $\mathfrak{S}_m$ beginning with the letter j. Hence the mth row of the array $\{S_{m,j}\}$ sums to the descent set statistic of the alternating word in B_m. Ehrenborg–Mahajan [22] generalize this transformation to calculate the descent statistic for any **ab**-word of the Boolean algebra. We redefine our array for any such word.

Given an **ab**-word $w = w_1 w_2 \cdots w_{n-1}$, define a triangular array with elements $s_{m,i}$, $1 \leq i \leq m$ by

$$s_{1,1} = 1$$

and

$$s_{m,i} = \begin{cases} \displaystyle\sum_{j=1}^{i-1} s_{m-1,j} & \text{if } w_{m-1} = \mathbf{b}, \\ \displaystyle\sum_{j=i}^{m-1} s_{m-1,j} & \text{if } w_{m-1} = \mathbf{a}, \end{cases}$$

where $m = 2, \ldots, n$ and $i = 1, \ldots, m$. In other words, if w ends with a $\mathbf{b}$, add all the elements from the $(n-1)$st row which are located above and to the left of $s_{n,i}$ and if w ends with an $\mathbf{a}$, add all the elements from the $(n-1)$st row which are above and to the right of $s_{n,i}$.

Lemma 4.2.5 (Ehrenborg–Mahajan) *If w is an **ab**-word of length $n-1$ then*

$$[w]_{B_n} = \sum_{j=1}^{n} s_{n,j}.$$

Furthermore, the entry $s_{n,j}$ equals the number of permutations in the symmetric group having descent word w beginning with the element n.

In [22] further generalizations are made for transformations to enumerate statistics for the subspace lattice, that is, the lattice of subspaces of an n-dimensional vector space over a finite field with q elements. Recall the subspace lattice can be viewed as a "q-analogue" of the Boolean algebra. We will be interested in defining a boustrophedon transformation for another class of lattices which generalize the partition lattice.

4.3 A boustrophedon transformation on the partition lattice

Definition 4.3.1 *The* partition lattice Π_n *is the poset of set partitions of* $\{1, 2, \ldots, n\}$ *with partial order* $\pi \leq \sigma$ *if every block of* π *is contained in a block of* σ*. Thus* $\pi \prec \sigma$ *if* σ *is formed by merging two blocks of* π*.*

Note that the rank of Π_n is $n-1$ and hence the degree of the **ab**-index of Π_n is $n-2$.

The partition lattice and many subposets of this lattice are well-studied. A standard EL-labeling $\lambda : \mathscr{E}(\Pi_n) \longrightarrow [n]$ for this lattice is given as follows. Suppose the two blocks B_1 and B_2 of a partition π are merged to form a partition σ. We have $\pi \prec \sigma$ and

$$\lambda(\pi \prec \sigma) = \max\{\min\{B_1, B_2\}\} - 1.$$

With respect to this labeling the maximal chains in Π_n are labeled with permutations from $\mathfrak{S}_{n-1}$. Unlike the Boolean algebra, different maximal chains can be labeled by the same permutation. For example in Π_3 the distinct maximal chains $1/2/3 \prec 13/2 \prec 123$ and $1/2/3 \prec 1/23 \prec 123$ are both labeled with the permutation $(2, 1)$.

In the partition lattice we need to reverse the order in which we build the descent word w. As we move down the array we move from right to left in the descent word, rather than from left to right as we did for the Boolean algebra. Given an **ab**-word $w_1 w_2 \cdots w_{n-1}$ define a new triangular array $\{t_{m,i}\}$ to count the **ab**-words in the partition lattice by

$$t_{1,1} = 1$$

and

$$t_{m,i} = \begin{cases} i \displaystyle\sum_{j=1}^{i-1} t_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{b}, \\ i \displaystyle\sum_{j=i}^{m-1} t_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{a}, \end{cases}$$

where $m = 2, \ldots, n$ and $i = 1, \ldots, m$. This is also a generalization of the boustrophedon transformation using $(a_1, a_2, a_3, \ldots) = (1, 0, 0, \ldots)$.

Example 4.3.2 *Let* $(a_1, a_2, a_3, a_4, a_5) = (1, 0, 0, 0, 0)$ *and* $w = \boldsymbol{babb}$*. In this case we have the triangular array*

$$\begin{array}{ccccccccc}
 & & & & \textcircled{1} & & & & \\
 & & & \textcircled{0} & \longrightarrow & 2 & & & \\
 & & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & & \\
 & 6 & \longleftarrow & 12 & \longleftarrow & 18 & \longleftarrow & \textcircled{0} & \\
\textcircled{0} & \longrightarrow & 12 & \longrightarrow & 54 & \longrightarrow & 144 & \longrightarrow & 180
\end{array}$$

and $(b_1, b_2, b_3, b_4, b_5) = (1, 2, 6, 6, 180)$*.*

Let $[w]_\Pi$ be the number of chains in Π_n which have descent word w, and set $[w|i]_\Pi$ to be to the number of chains in Π_n whose labels have descent word w and which begin with the element i.

Proposition 4.3.3 *The number of labeled maximal chains in* Π_{n+1} *having first label* i *and descent word* $w = w_1 w_2 \cdots w_{n-1}$ *is given by*

$$[w|i]_\Pi = t_{n,i}.$$

Proof: We proceed by induction on n. We have $t_{1,1} = 1$ corresponding to the one chain in Π_2 labeled with 1. Assume $t_{m,i} = [w|i]_\Pi$ for $1 \leq m < n$ There are two cases to consider, whether a word in Π_{n+1} begins with $\mathbf{a}$ or $\mathbf{b}$.

First consider the case $\mathbf{a} \cdot w_2 \cdots w_{n-1}$. We claim $t_{n,i} = i \sum_{j=i}^{n-1} t_{n-1,j}$. We determine how many atoms covering $\hat{0} = 1/2/\cdots/n{+}1$ have an edge label i. The singleton block $B = i+1$ can be merged with a singleton block B_2 containing an element less than $i+1$ to give the edge label i. There are i such singleton blocks. Consider the interval $[x, \hat{1}] \simeq \Pi_{n+1}$ where x is the element consisting of the merged block $\{i+1\} \cup B_2$ and all the remaining blocks are singleton blocks. By induction, $[w|j]_\Pi = t_{n-1,j}$ for chains in $[x, \hat{1}]$. We sum over $j = i, \ldots, n-1$ because $i \leq j$ creates an ascent and considering the chain in the whole poset Π_{n+1}, we have a chain whose labels have descent word $\mathbf{a} \cdot w$.

Similarly, consider the case $\mathbf{b} \cdot w$. We claim $t_{n,i} = i \sum_{j=1}^{i-1} t_{n-1,j}$. The multiple i is found in the same way by counting the number of edges from $\hat{0}$ with label i, and we also have the interval $[x, \hat{1}] \simeq \Pi_{n+1}$. This time we must sum over $j = 1, 2, \ldots, i-1$ so $i < j$ creates a descent word $\mathbf{b} \cdot w$. (Note in the isomorphism $[x, \hat{1}] \simeq \Pi_n$ labels greater than i in $[x, \hat{1}]$ are mapped to a label in Π_n which are one less.) $\square$

Corollary 4.3.4 *The descent statistic for an* ***ab****-word* w *in* Π_n *is given by*

$$[w]_\Pi = \sum_{i=1}^{n} t_{n,i}.$$

4.4 Generalization to Dowling lattices

The next natural family of lattices to consider are the Dowling lattices. Let ζ be a kth root of unity. Let H_n be the hyperplane arrangement in $\mathbb{C}^n$ consisting of the hyperplanes

$$\begin{array}{ll} z_i = \zeta^h z_j & \text{for } 1 \leq i < j \leq n \quad \text{and } 0 \leq h \leq k-1, \\ z_i = 0 & \text{for } 1 \leq i \leq n. \end{array}$$

Definition 4.4.1 *The* Dowling lattice L_n *is the intersection lattice of the hyperplane arrangement* H_n *ordered by reverse inclusion.*

The Dowling lattice is a generalization of the partition lattice since when $k = 1$ we have $L_n \cong \Pi_{n+1}$. We describe this isomorphism when $k = 1$. In this case the Dowling lattice L_n is the intersection lattice of the hyperplane arrangement consisting of the hyperplanes $z_i = z_j$ for $1 \leq i < j \leq n$ and $z_i = 0$ for $1 \leq i \leq n$. Let $\sigma = B_1 B_2 \cdots B_k$

be a partition in Π_{n+1} and assume the element $n+1$ is contained in block B_k. Then the partition σ is mapped to the intersection of hyperplanes given by

$$\begin{aligned} z_{b_{1,1}} &= z_{b_{1,2}} = \cdots = z_{b_{1,|B_1|}} \\ z_{b_{2,1}} &= z_{b_{2,2}} = \cdots = z_{b_{2,|B_2|}} \\ &\vdots \\ z_{b_{k-1,1}} &= z_{b_{k-1,2}} = \cdots = z_{b_{k-1,|B_{k-1}|}} \\ z_{b_{k,1}} &= z_{b_{k,2}} = \cdots = z_{b_{k,|B_k|}} = 0 \end{aligned}$$

where $B_i = \{b_{i,j}\}_{j=1,\ldots,|B_i|}$. We want to generalize the results of the previous section to this case, but first we need a comparable R-labeling for the Dowling lattice. Following [24], an *enriched block* is an ordered pair $\widetilde{B} = (B, f)$ where B is a nonempty subset of $[n]$ and f is a function from B to $\mathbb{Z}_k$. Two enriched blocks $\widetilde{B} = (B, f)$ and $\widetilde{C} = (C, g)$ are equivalent if $B = C$ and the functions f and g differ only by a constant. We have $k^{|B|-1}$ ways to enrich a block B. For a subset $E \subseteq [n]$ an *enriched partition* $\tilde{\pi} = (\widetilde{B_1}, \widetilde{B_2}, \ldots, \widetilde{B_k})$ is a partition $\pi = (B_1, B_2, \ldots, B_k)$ of E where each block B_i is enriched with a function f_i. Define

$$L_n^{'} = \{(\widetilde{\pi}, Z) : Z \subseteq \{1, \ldots, n\} \text{ and } \widetilde{\pi} \text{ is an enriched partition of } \overline{Z} = \{1, \ldots, n\} - Z\}.$$

Proposition 4.4.2 (Ehrenborg–Readdy) *The lattice $L_n^{'}$ is isomorphic to the Dowling lattice L_n.*

We will define an R-labeling $\lambda : \mathscr{E}(L_n^{'}) \to \mathbb{N}$ which is a simplification of the R-labeling in [24]. Let $x \prec y$ in $L_n^{'}$. If y is formed by merging two enriched blocks $\widetilde{B}_1$ and $\widetilde{B}_2$ of x then $\lambda(x, y) = \max\{\min(\widetilde{B}_1), \min(\widetilde{B}_2)\}$. If y is formed by merging a block $\widetilde{B}_1$ into the zero block Z, then $\lambda(x, y) = \min(\widetilde{B}_1)$.

Lemma 4.4.3 *The labeling λ is an R-labeling of the lattice L_n'.*

Proof: An increasing labeled chain on any interval $[x, \hat{1}] \in L_n^{'}$ is given by merging blocks with the zero block in increasing order according to the minimal element of each block. Let $x = (\{\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_m\}, Z)$ where $\min(\widetilde{B}_1) < \min(\widetilde{B}_2) < \cdots < \min(\widetilde{B}_m)$. The increasing labeled chain will be

$$x \prec (\{\widetilde{B}_2, \widetilde{B}_3, \ldots, \widetilde{B}_m\}, Z_1) \prec (\{\widetilde{B}_3, \widetilde{B}_4, \ldots, \widetilde{B}_m\}, Z_2) \prec \cdots \prec (\emptyset, Z_m) = \hat{1},$$

where $Z_i = Z \cup \widetilde{B}_1 \cup \widetilde{B}_2 \cup \cdots \cup \widetilde{B}_i$.

More generally, consider the interval $[x, y]$ for any $x, y \in L_n^{'}$. There exists a set of blocks $\{\widetilde{B}_i\}$ which are in the zero set Z of y, but not in the zero set of x. Also, there exists a set of blocks $\{\widetilde{B}_j\}$ in y where each is equal to a union of blocks of x, i.e., $\widetilde{B}_j = \cup \widetilde{B}_{k_j}$. Consider the set $S = \{\widetilde{B}_i, \widetilde{B}_{k_j}\}$. For each j, remove $\min\{\min(\widetilde{B}_{k_j})\}$ from the set S to form $\widehat{S}$. Order $\widehat{S}$ by the minimum element of each block so $\min(\widetilde{B}_{i_1}) < \min(\widetilde{B}_{i_2}) < \cdots < \min(\widetilde{B}_{i_m})$. The increasing labeled chain will be found by merging blocks with the zero set following the order given by this chain. These minimum elements will be the labels on the chain.

This increasing chain is unique. On the interval $[x, y]$, all chains have labels from the set $\widehat{S}$. We may refer to blocks in x by their minimum element, because any other elements in a block will never become edge labels in the interval $[x, y]$. Each of the minimum elements will become an edge label except the smallest of any merged, nonzero block of y. Suppose there is another increasing labeled chain on the interval $[x, y]$. This chain must have the same labels as the original increasing chain otherwise it will be decreasing at some position. Thus we have the same non-enriched blocks being merged, differing only by the associated functions on these blocks. Note that nonzero blocks $\widetilde{B}_j$ in y have each been enriched by a fixed function. This function determines which enrichment functions could occur on blocks being merged into $\widetilde{B}_j$. The functions associated with an enriched block of the partition $\widetilde{B}_{j_1} \cup \widetilde{B}_{j_2} \cup \cdots \cup \widetilde{B}_{j_l} / \widetilde{B}_{j_{l+1}} / \cdots / Bj_m$ must equal the enrichment function on $\widetilde{B}_j$ for the first block which has been merged blocks of x. The enrichment of the last $m - l$ blocks will be the same functions as those associated with enriched blocks in x. There is only one way to do this and thus only one chain with increasing labels.$\square$

Given an **ab** word $w = w_1 w_2 \cdots w_{n-1}$ of length $n - 1$, define a new triangular array with elements $t^k_{m,i}$ recursively by

$$t^k_{1,1} = 1$$

and

$$t^k_{m,i} = \begin{cases} (k(i-1)+1) \displaystyle\sum_{j=1}^{i-1} t^k_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{b}, \\ (k(i-1)+1) \displaystyle\sum_{j=i}^{m-1} t^k_{m-1,j} & \text{if } w_{n-m+1} = \mathbf{a}, \end{cases}$$

for $m = 2, \ldots, n$ and $i = 1, \ldots m$.

Example 4.4.4 *Let* $k = 3$, $(a_1, a_2, a_3, a_4, a_5) = (1, 0, 0, 0, 0)$ *and* $w = \boldsymbol{bbab}$. *In this case we have the triangular array*

$$\begin{array}{ccccccccc}
 & & & & \textcircled{1} & & & & \\
 & & & \textcircled{0} & \longrightarrow & 4 & & & \\
 & & 4 & \longleftarrow & 16 & \longleftarrow & \textcircled{0} & & \\
 & \textcircled{0} & \longrightarrow & 16 & \longrightarrow & 140 & \longrightarrow & 200 & \\
\textcircled{0} & \longrightarrow & 0 & \longrightarrow & 112 & \longrightarrow & 1560 & \longrightarrow & 4628
\end{array}$$

and $(b_1, b_2, b_3, b_4, b_5) = (1, 4, 4, 200, 4628)$.

Observe in this example the weights are now $1, 4, 7, 10, 13$ rather than $1, 2, 3, 4, 5$.

Let $[w]_L$ be the number of chains whose labels have descent word w in the kth Dowling lattice L_n and let $[w|i]_L$ be the number of such chains whose first label is i.

Proposition 4.4.5 *With respect to the R-labeling λ of the Dowling lattice, the number of maximal chains in the kth Dowling lattice beginning with the number i and having descent word w is given by*

$$[w|i]_L = t^k_{n,i}.$$

Proof: This proof is similar to the proof for the partition lattice when $k = 1$. We count the number of atoms x with the edge label of $\hat{0} \prec x$ equal to i. The non-enriched block containing i can be merged with any block containing a number smaller than i, that is, in $i - 1$ ways. For each of these there are k ways to enrich these merged blocks. Finally, the block containing i can be merged into the zero set in exactly one way. We have $k(i-1)+1$ elements labeled i on edges from $\hat{0}$.

In the Dowling lattice an interval $[a, \hat{1}]$ in L_n is isomorphic to L_{n-1} for any atom a. We induct on the rank n of the Dowling lattice. Notice when $n = 1$ there is only one way to merge the one element into the zero block, hence $t^k_{1,1} = 1$. For $n > 1$, consider the case $\mathbf{a} \cdot w$ where w is an **ab**-word of length $n - 1$. Then $t^k_{n+1,i} = (k(i-1)+1)\sum_{j=i+1}^{n=1} t_{n,j}$. $[w|j]_L = t^k_{n,j}$ for chains in $[x, \hat{1}]$ where x is an atom such that $\lambda(\hat{0}, x) = i$. We sum over $j = i+1, \ldots, n+1$ because $i < j$ creates an ascent and a chain whose labels have descent word $\mathbf{a} \cdot w$. The case $\mathbf{b} \cdot w$ is similar. □

Corollary 4.4.6 *The descent statistic for an **ab**-word w in the kth Dowling lattice is given by the row sum*

$$[w]_L = \sum_{i=1}^{n} t^k_{n,i}.$$

4.5 Determinantal formula and Whitney numbers

In this section we recall Niven's determinantal expression for the number of permutations in the symmetric group having a given descent set. We then generalize this result to a certain class of posets called upper or lower isomorphic.

Let $w = w_1 w_2 \cdots w_{n-1}$ be a descent word where $w_i \in \{a, b\}$ for all i. Let $\{k_1 < k_2 < \cdots < k_r\} \subseteq \{1, \ldots, n-1\}$ be the descent set of w, that is, the set of subscripts i where $w_i = b$.

Theorem 4.5.1 (Niven) *The number of permutations in the symmetric group $\mathfrak{S}_n$ with descent word w equals the determinant of the $(r+1) \times (r+1)$ matrix N with $n_{i,j} = \binom{k_{i+1}}{k_j}$ where the rows and columns of N are indexed by $i, j = 0, \ldots, r$ and it is understood that $k_0 = 0$, $k_{r+1} = n$, and that $\binom{m}{s} = 0$ if $m < s$.*

For a descent word w, denote this determinant by $[n; k_1, k_2, \ldots, k_r]$, that is,

$$[n; k_1, k_2, \ldots, k_r] = \begin{vmatrix} \binom{k_1}{0} & \binom{k_1}{k_1} & \cdots & \binom{k_1}{k_r} \\ \binom{k_2}{0} & \binom{k_2}{k_1} & \cdots & \binom{k_2}{k_r} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{k_1} & \cdots & \binom{n}{k_r} \end{vmatrix}.$$

In [37], Niven gives an inductive proof of this theorem. We wish to give a more direct proof and generalize to descent words for other posets.

If $w = w_1 w_2 \cdots w_{n-1}$ be an **ab**-word corresponding to maximal chains in the partition lattice Π_n, let w^* be the **ab**-word which is the dual of w, that is,

$$w^* = w_{n-1} \cdots w_2 w_1.$$

We give an analogous theorem to Theorem 4.5.1 for the partition lattice.

Theorem 4.5.2 *Let $w*$ be the dual of an **ab**-word w of length $n-1$ in the partition lattice Π_{n+1}. Let the descent set of $w*$, that is, $\{i' : {w^*}_i = b\}$ be given by $S = \{k_1, k_2, \ldots, k_r\}$ where $1 \leq k_1 < k_2 < \cdots < k_r \leq n-1$. The descent set statistic $[w]_\Pi$ equals the determinant of the $(r+1) \times (r+1)$ matrix N with $n_{i,j} = S(n+1-k_i, n+1-k_{j+1})$ where the rows and columns of N are indexed by $i, j = 0, \ldots, r$ and it is understood that $k_0 = 0$, $k_{r+1} = n$, and that $S(n+1-k_i, n+1-k_{j+1}) = 0$ if $i > j$.*

Proof: Following Stanley [48, Example 2.2.4] we define a function $f : [0, r+1] \times [0, r+1] \longrightarrow \mathbb{N}_{\geq 0}$ by $f(i,j) = S(n+1-k_i, n+1-k_j)$. We have $f(i,j) = 0$ whenever $i > j$. Therefore

$$A_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq r} (-1)^{r-j} f(0, i_1) f(i_1, i_2) \cdots f(i_j, r+1)$$

gives the non-terms of the expansion of the $(r+1) \times (r+1)$ determinant of the matrix N where $n_{i,j} = f(i, j+1)$. We rewrite the sum A_r as follows where the sum is over all subsets $\{k_1, k_2, \ldots, k_j\}$ of the descent set S.

$$\begin{aligned} A_r &= \sum_{k_{i_1} < k_{i_2} < \cdots < k_{i_j}} (-1)^{|S|-j} S(n+1, n+1-k_{i_1}) S(n+1-k_{i_1}, n+1-k_{i_2}) \\ &\quad \cdots S(n+1-k_{i_j}, n+1-n) \end{aligned}$$

Recall, the descent set statistic $[w]_\Pi$ of an **ab**-word w with descent set S is given by the flag h-vector h_S. We have

$$[w]_\Pi = h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T,$$

and because the number of elements of rank k in Π_{n+1} is equal to the Stirling number of the second kind $S(n+1, n-k)$ we have

$$f_T = S(n+1, n+1-t_1) \cdots S(n+1-t_{j-1}, n+1-t_j)$$

when $T = \{t_1, t_2, \ldots, t_j\}$ where $t_1 < t_2 < \cdots < t_j$. We sum over all T showing $A_r = [w]_\Pi$, and we have proven the claim. □

We can generalize Theorem 4.5.1 and Theorem 4.5.2. Any poset whose lower intervals $[\hat{0}, x]$ are isomorphic for elements x of the same rank or by duality whose upper intervals $[y, \hat{1}]$ are isomorphic for elements y of the same rank can have its descent words counted by these determinants. In particular, the partition lattice has isomorphic upper intervals $[y, \hat{0}]$ given by elements y of the same rank.

Definition 4.5.3 *We say a poset is upper isomorphic, respectively lower isomorphic, if all upper intervals $[y, \hat{1}]$, respectively lower intervals $[\hat{0}, x]$, of the same rank are isomorphic as posets.*

Let P be a rank n poset. Recall the Whitney number of the second kind $W_k(n)$ counts the number of rank k elements in P, that is,

$$W_k(n) = \sum_{\rho(x)=k} 1,$$

and the Whitney number of the first kind $w_{n-k}(n)$ sums the Möbius values of the rank k elements in P, that is,

$$w_{n-k}(n) = (-1)^k \sum_{\rho(x)=k} \mu_P(\hat{0}, x).$$

Following the convention for Stirling numbers of the first and second kind, let $\left\{ {n \atop k} \right\} = W_k(n)$ be the Whitney number of the second kind, and let $\left({n \atop n-k} \right) = (-1)^{n-k} w_{n-k}(n)$ be the signless Whitney number of the first kind.

Theorem 4.5.4 *If P is a lower isomorphic poset of rank n and $S = \{k_1 < k_2 < \ldots < k_r\} \subseteq [n-1]$, then the Möbius function of the rank-selected poset P_S is given by the $(r+1) \times (r+1)$ determinant of the matrix N where $n_{i,j} = W_{k_{j+1}}(k_i) = \left\{ {k_i \atop k_{j+1}} \right\}$ for $i, j = 0, 1, \ldots, r$ and we assume $k_0 = 0$, $k_{r+1} = n$, and $\left\{ {k_i \atop k_{j+1}} \right\} = 0$ when $i < j$, that is,*

$$(-1)^{|S|+1}\mu(P_S) = \begin{vmatrix} \left\{ {k_1 \atop 0} \right\} & \left\{ {k_1 \atop k_1} \right\} & \cdots & \left\{ {k_1 \atop k_r} \right\} \\ \left\{ {k_2 \atop 0} \right\} & \left\{ {k_2 \atop k_1} \right\} & \cdots & \left\{ {k_2 \atop k_r} \right\} \\ \vdots & \vdots & \ddots & \vdots \\ \left\{ {n \atop 0} \right\} & \left\{ {n \atop k_1} \right\} & \cdots & \left\{ {n \atop k_r} \right\} \end{vmatrix}.$$

In the case of upper isomorphic posets, we are working with upper order ideals so we must use the dual w^* of an **ab**-word w, that is, if $w = w_1 w_2 \cdots w_{n-1}$ then $w^* = w_{n-1} \cdots w_2 w_1$.

Theorem 4.5.5 *If P is an upper isomorphic poset of rank n and $S = \{k_1 < k_2 < \ldots < k_r\} \subseteq [n-1]$ then the Möbius function of the rank-selected subposet P_S is given by the determinant of the $(r+1) \times (r+1)$ matrix N where $n_{i,j} = W_{k_{j+1}}(n-k_i) = \left\{ {n-k_i \atop k_{j+1}} \right\}$ for $i, j = 0, 1, \ldots, r$ and we assume $k_0 = 0$, $k_{r+1} = n$, and $\left\{ {n-k_i \atop k_{j+1}} \right\} = 0$ when $i < j$, that is,*

$$(-1)^{|S|+1}\mu(P_S) = \begin{vmatrix} \left\{ {n-0 \atop k_1} \right\} & \left\{ {n-0 \atop k_2} \right\} & \cdots & \left\{ {n-0 \atop n} \right\} \\ \left\{ {n-k_1 \atop k_1} \right\} & \left\{ {n-k_1 \atop k_2} \right\} & \cdots & \left\{ {n-k_1 \atop n} \right\} \\ \vdots & \vdots & \ddots & \vdots \\ \left\{ {n-k_r \atop k_1} \right\} & \left\{ {n-k_r \atop k_2} \right\} & \cdots & \left\{ {n-k_r \atop n} \right\} \end{vmatrix}.$$

As the Boolean algebra is both lower and upper isomorphic, Niven's result is a corollary to both of the above theorems. The proof for Theorem 4.5.5 is analogous to the proof of the special case of the partition lattice in Theorem 4.5.2. The proof for Theorem 4.5.4 is dual.

For an upper isomorphic poset these Möbius values can also be represented with Whitney numbers of the first kind. We first look at the case of the partition lattice. The Whitney numbers of the second kind for the partition lattice are given by the signless Stirling numbers of the first kind, that is,

$$w_{n-k}(n) = (-1)^k \sum_{\rho(x)=k} \mu_{\Pi_{n+1}}(\hat{0}, x) = (-1)^k s(n+1, n-k+1).$$

Theorem 4.5.6 *Let $w*$ be the dual of an* ***ab****-word w of length $n-1$ in the partition lattice Π_{n+1}. Let the descent set of $w*$, that is, $\{i' : {w^*}_i = b\}$ be given by $S = \{k_1 < k_2 < \ldots < k_r\} \subseteq [n-1]$. Then set $T = \{t_1, t_2, \ldots t_{n-r}\} = [n-1] - S$ where $t_1 > t_2 > \cdots t_{n-r}$. The descent set statistic $[w]_\Pi$ equals the determinant of the $(n-r) \times (n-r)$ matrix N with $n_{i,j} = (-1)^{t_{i+1}-t_j} s(n - t_{i+1}, n - t_j)$ where the rows and columns of N are indexed by $i, j = 0, \ldots, n-r$ and it is understood that $t_0 = n$, $t_{n-r} = 0$, and that $s(n - t_{i+1}, n - t_j) = 0$ if $i + 1 < j$.*

Before we begin the proof of Theorem 4.5.6, recall the following theorem of Stanley [48, Lemma 3.14.4].

Theorem 4.5.7 (Stanley) *Let P be any poset of rank n and $S \subseteq [n-1]$. Then*

$$\mu(P_S) = \sum (-1)^k \mu_P(\hat{0}, x_1)\mu_P(x_1, x_2) \cdots \mu_P(x_k, \hat{1})$$

where the sum is over all chains $c : \hat{0} < x_1 < x_2 < \cdots < x_k < \hat{1}$ such that $\rho(x_i) \notin S$.

Proof of Theorem 4.5.6: Again, following Stanley [48, Example 2.4.4] we define a function $g : [0, n-r] \times [0, n-r] \longrightarrow \mathbb{N}_{\geq 0}$ by $g(i,j) = (-1)^{t_i - t_j} s(n+1-t_i, n+1-t_j)$. We have $g(i,j) = 0$ whenever $i < j$. Therefore

$$A_r = \sum_{n-r \geq i_1 > i_2 > \cdots > i_j \geq n-r} (-1)^{n-r-j} g(n-r+1, i_j) \cdots g(i_2, i_1) \cdots g(i_1, 0)$$

gives the non-terms of the expansion of the $(n-r) \times (n-r)$ determinant of the matrix N where $n_{i,j} = g(i+1, j)$. We rewrite the sum A_r as follows where the sum is over all subsets $\{t_1, t_2, \ldots, t_j\}$ of the $T = [n-1] - S$ the complement of the descent set S.

$$\begin{aligned} A_r &= \sum_{t_{i_1} > t_{i_2} > \cdots > t_{i_j}} (-1)^{n-|S|-j} (-1)^{t_{n-r} - t_{i_j}} s(n+1-t_{n-r}, n+1-t_{i_j}) \cdots \\ &\quad (-1)^{t_{i_2} - t_{i_1}} s(n+1-t_{i_2}, n+1-t_{i_1}) (-1)^{t_{i_1} - t_0} s(n+1-t_{i_1}, n+1-t_0) \\ &= \sum_{t_{i_1} > t_{i_2} > \cdots > t_{i_j}} (-1)^{|S|-j} \sum_{\rho(x)=t_{i_j}} \mu_{\Pi_{n+1}}(\hat{0}, x) \cdots \\ &\quad \sum_{\rho(x)=t_{i_2}} \mu_{\Pi_{n+1-t_{i_2}}}(\hat{0}, x) \sum_{\rho(x)=0} \mu_{\Pi_{n+1-t_{i_1}}}(\hat{0}, x) \end{aligned}$$

When $\rho(x) = t_i$ the interval $[0, x]$ in $\Pi_{n+1-t_{i+1}}$ is equal to an interval $[x', x]$ in Π_{n+1} where $\rho(x') = t_{i+1}$. Further recall the descent set statistic $[w]_\Pi$ of an **ab**-word w with descent set S is given by the flag h-vector entry h_S and the flag h-vector entry h_S is given by $(-1)^{|S|+1}\mu(P_S)$. We apply Theorem 4.5.7 to complete the proof. □

As in the case for Whitney number of the first kind, Theorem 4.5.6 can be generalized for any upper isomorphic poset.

Theorem 4.5.8 *If P is an upper isomorphic rank n poset and $S = \{k_1, k_2, \ldots, k_r\} \subseteq [n-1]$. Then set $T = \{t_1 > t_2 > \ldots > t_{n-r-1}\} = [n-1] - S$. The Möbius function of the rank selected subposet P_S is given by the $(n-r) \times (n-r)$ determinant of the matrix N where $n_{i,j} = w_{n-t_j}(n - t_{i+1}) = \binom{n-t_{i+1}}{n-k_{r-j+1}}$ for $i, j = 0, 1, \ldots, n-r$ and we assume $t_{n-r} = 0$, $t_0 = n$, and $\binom{n-t_{i+1}}{n-t_j} = 0$ when $i+1 < j$, that is,*

$$(-1)^{|S|+1}\mu(P_S) = \begin{vmatrix} \binom{n-t_1}{0} & \binom{n-t_1}{n-t_1} & \cdots & \binom{n-t_1}{n-t_{n-r-1}} \\ \binom{n-t_2}{0} & \binom{n-t_2}{n-t_1} & \cdots & \binom{n-t_2}{n-t_{n-r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{n-t_1} & \cdots & \binom{n}{n-t_{n-r-1}} \end{vmatrix},$$

where $\binom{n-t_{i+1}}{n-t_j}$ is the signless Whitney number of the first kind.

Theorem 4.5.8 is proven similarly to Theorem 4.5.6 replacing the Stirling numbers of the first kind with the Whitney numbers of the first kind.

We also have a similar result for lower isomorphic posets.

Corollary 4.5.9 *Let P be a lower isomorphic poset of rank n and let S be the subset $S = \{k_1, k_2, \ldots, k_r\} \subseteq [n-1]$. Then set $T = \{t_1 < t_2 < \ldots < t_{n-r-1}\} = [n-1] - S$ and let $\left\langle {n \atop k} \right\rangle$ denote the upper Whitney number of the first kind such that $\left\langle {n \atop k} \right\rangle = (-1)^{n-k}\sum_{\rho(x)=k}\mu_P(x, \hat{1})$ The Möbius function of the rank selected subposet P_S is given by the $(n-r) \times (n-r)$ determinant of the matrix N where $n_{i,j} = \left\langle {t_{i+1} \atop t_j} \right\rangle$ for $i, j = 0, 1, \ldots, n-r$ and we assume $t_0 = 0$, $t_{n-r} = n$, and $\left\langle {t_{i+1} \atop t_j} \right\rangle = 0$ when $i+1 < j$, that is,*

$$(-1)^{|S|+1}\mu(P_S) = \begin{vmatrix} \left\langle {t_1 \atop 0} \right\rangle & \left\langle {t_1 \atop t_1} \right\rangle & \cdots & \left\langle {t_1 \atop t_{n-r-1}} \right\rangle \\ \left\langle {t_2 \atop 0} \right\rangle & \left\langle {t_2 \atop t_1} \right\rangle & \cdots & \left\langle {t_2 \atop t_{n-r-1}} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle {n \atop 0} \right\rangle & \left\langle {n \atop t_1} \right\rangle & \cdots & \left\langle {n \atop t_{n-r-1}} \right\rangle \end{vmatrix}.$$

This can be proven by taking the dual of the poset and applying the proof of Theorem 4.5.8.

We now have an interesting relationship for Whitney numbers of the first and second kind for upper (or lower) isomorphic posets.

Corollary 4.5.10 *Given S and T as above, if P is an upper isomorphic poset of rank n then*

$$\begin{vmatrix} \binom{n-t_1}{0} & \binom{n-t_1}{n-t_1} & \cdots & \binom{n-t_1}{n-t_{n-r-1}} \\ \binom{n-t_2}{0} & \binom{n-t_2}{n-t_1} & \cdots & \binom{n-t_2}{n-t_{n-r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{n-t_1} & \cdots & \binom{n}{n-t_{n-r-1}} \end{vmatrix} = \begin{vmatrix} \left\{ {n-0 \atop k_1} \right\} & \left\{ {n-0 \atop k_2} \right\} & \cdots & \left\{ {n-0 \atop n} \right\} \\ \left\{ {n-k_1 \atop k_1} \right\} & \left\{ {n-k_1 \atop k_2} \right\} & \cdots & \left\{ {n-k_1 \atop n} \right\} \\ \vdots & \vdots & \ddots & \vdots \\ \left\{ {n-k_r \atop k_1} \right\} & \left\{ {n-k_r \atop k_2} \right\} & \cdots & \left\{ {n-k_r \atop n} \right\} \end{vmatrix},$$

If P is a lower isomorphic poset of rank n then

$$\begin{vmatrix} \left\{ {k_1 \atop 0} \right\} & \left\{ {k_1 \atop k_1} \right\} & \cdots & \left\{ {k_1 \atop k_r} \right\} \\ \left\{ {k_2 \atop 0} \right\} & \left\{ {k_2 \atop k_1} \right\} & \cdots & \left\{ {k_2 \atop k_r} \right\} \\ \vdots & \vdots & \ddots & \vdots \\ \left\{ {n \atop 0} \right\} & \left\{ {n \atop k_1} \right\} & \cdots & \left\{ {n \atop k_r} \right\} \end{vmatrix} = \begin{vmatrix} \left\langle {t_1 \atop 0} \right\rangle & \left\langle {t_1 \atop t_1} \right\rangle & \cdots & \left\langle {t_1 \atop t_{n-r-1}} \right\rangle \\ \left\langle {t_2 \atop 0} \right\rangle & \left\langle {t_2 \atop t_1} \right\rangle & \cdots & \left\langle {t_2 \atop t_{n-r-1}} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle {n \atop 0} \right\rangle & \left\langle {n \atop t_1} \right\rangle & \cdots & \left\langle {n \atop t_{n-r-1}} \right\rangle \end{vmatrix}.$$

4.6 Inequalities for the flag h-vector of the partition lattice

In this section we resume our study of determining inequalities for the flag h-vector of the partition lattice. In the paper [23, Theorem 4.9] a recurrence is given for the **ab**-index of the partition lattice. We will provide some partial results and conjectures restricting ourselves to the case of the partition lattice. As before, for w be a descent word of length $n-2$ let $[w]_{\Pi}$ denote $[w]_{\Pi_n}$.

For $\pi \in \mathfrak{S}_{n-1}$, let $m(\pi)$ be the number of maximal chains in Π_n labeled with π. For any fixed π we can compute $m(\pi)$ as follows.

Proposition 4.6.1 *Let $\pi = \pi_1\pi_2\cdots\pi_{n-1}$. Then*

$$m(\pi) = \widehat{\pi}_1 \cdot \widehat{\pi}_2 \cdots \widehat{\pi}_{n-1}$$

where $\widehat{\pi}_i = \pi_i - \#\{\pi_j : j < i \text{ and } \pi_j < \pi_i\}$.

Proof: We induct on the length of π. Let $\pi = \pi_1\pi_2\cdots\pi_{n-1}$ be a permutation of length $n-1$. We assume the result holds for the partition lattice with rank less than $n-1$. Observe for any atom $a \in \Pi_n$ the interval $[a, \hat{1}]$ in Π_n is isomorphic to Π_{n-1}. The partial permutation $\pi_2\cdots\pi_{n-1}$ can be uniquely identified with a permutation in $\tau \in \mathfrak{S}_{n-2}$ where $\tau_i = \pi_i$ if $\pi_i < \pi_1$ and $\tau_i = \pi_i - 1$ if $\pi_i > \pi_1$. There are $m(\tau)$ maximal chains in Π_{n-1} labeled with τ. Because we can identify the intervals $[a, \hat{1}] \in \Pi_n$ with the smaller partition lattice Π_{n-1}, there are $m(\tau)$ maximal chains labeled with $\pi_2\cdots\pi_{n-1}$. An edge $\hat{0} \prec a$ in Π_n from the minimal element to an atom a can be labeled with the element π_1 in exactly π_1 ways, that is, $\hat{0} \prec a$ is labeled with π_1 when the element $\pi_1 + 1$ is merged with any other element which is less than or equal to π_1. Thus we have $\pi_1 \cdot m(\tau)$ maximal chains in Π_n labeled with π and we have proven our claim. $\square$

Recall the following result.

Lemma 4.6.2 *For w and $\boldsymbol{ab}$-word of length $n-2$*

$$[w]_\Pi = \sum_\pi m(\pi),$$

where π ranges over all permutations in $\mathfrak{S}_{n-1}$ with descent word w.

We will use these facts in several of the following proofs.

Theorem 4.6.3 *The flag h-vector of the partition lattice satisfies the following inequalities:*

$$\begin{aligned} [\boldsymbol{aa}\cdot w]_\Pi &< [\boldsymbol{ab}\cdot w]_\Pi, \\ [\boldsymbol{bbb}\cdot w]_\Pi &< [\boldsymbol{bba}\cdot w]_\Pi, \\ [\boldsymbol{ba}\cdot \boldsymbol{a}^i]_\Pi &< [\boldsymbol{bb}\cdot \boldsymbol{a}^i]_\Pi \quad (i \geq 1). \end{aligned}$$

The current proofs of the inequalities follow from direct permutation counting arguments and estimates. For some notation before we begin, let π be a permutation in $\mathfrak{S}_{n-2}$. If a number $x \in [n-1]$ is added to the beginning of π, we form a new permutation $x\pi \in \mathfrak{S}_{n-1}$ by shifting all $\pi_i \geq x$ to $\pi_i + 1$ so every element in $[n-1]$ appears in the permutation. Let $m(\pi|i)$ be the number of maximal chains whose edges are labeled with π and where $\pi_1 = i$.

Proposition 4.6.4 *The flag h-vector of the partition lattice satisfies*

$$[\boldsymbol{aa}\cdot w]_\Pi < [\boldsymbol{ba}\cdot w]_\Pi,$$

for any $\boldsymbol{ab}$-word w.

Proof: Let $\sigma = xy\pi$ be any permutation with descent word $\mathbf{aa}\cdot w$. We must have $x < y < \pi_1$ so there are $x(y-1)\cdot m(\pi)$ maximal chains in Π_n labeled with σ. Set $\widetilde{\sigma} = yx\pi$, so $\widetilde{\sigma}$ has descent word $\mathbf{ba}\cdot w$. There are $xy\cdot m(\pi)$ chains labeled with $\widetilde{\sigma}$. Thus $xy\cdot m(\pi) > x(y-1)\cdot m(\pi)$ for all choices of σ. Note we have shown that each maximal chain with descent word $\mathbf{aab}\cdot w$ gets mapped to a larger number of maximal chains with descent word $\mathbf{ba}\cdot w$. Furthermore, there are chains in Π_n with descent word $\mathbf{ba}\cdot w$ which are not preimages of a chain with descent set $\mathbf{ab}\cdot w$, for instance, those chains $\sigma = \sigma_1 \cdots \sigma_{n-1}$ where $\sigma_1 = i+2$, $\sigma_2 = i$, and $\sigma_3 = i+1$. Hence the proposition holds. □

Proposition 4.6.5 *The flag h-vector of the partition lattice satisfies*

$$[\boldsymbol{bbb}\cdot w]_\Pi < [\boldsymbol{abb}\cdot w]_\Pi$$

for any $\boldsymbol{ab}$-word w.

Proof: Let $\sigma = yx\pi$ be a permutation with descent word $\mathbf{bbb} \cdot w$. Then we must have $y > x > \pi$, and there are $yx \cdot m(\pi)$ maximal chains labeled by σ. Let $\tilde{\sigma} = xy\pi$. Then $\tilde{\sigma}$ has descent word $\mathbf{abb} \cdot w$, and there are $x(y-1) \cdot m(\pi)$ maximal chains labeled by $\tilde{\sigma}$. Finally, let $\hat{\sigma} = \pi_1 yx\pi_2 \cdots \pi_{n-2}$. We see that $\hat{\sigma}$ also has word $\mathbf{abb} \cdot w$ because $x > \pi_1 > \pi_2$. There are $\pi_1(y-1) \cdot m(\pi)$ chains labeled by $\hat{\sigma}$. For a given σ we have at least $x(y-1) \cdot m(\pi) + \pi_1(y-1) \cdot m(\pi)$ chains with descent word $\mathbf{abb} \cdot w$ and exactly $xy \cdot m(\pi)$ chains with descent word $\mathbf{bbb} \cdot w$ words. We want to show

$$(x + \pi_1)(y-1) \cdot m(\pi) > xy \cdot m(\pi).$$

Because w begins with a descent, the element π_1 satisfies $\pi_1 \geq 2$, so it is sufficient to prove

$$(x+2)(y-1) > xy$$

or equivalently

$$2y > x - 2.$$

The last inequality holds because $y \geq x+1$ implies $2y \geq 2x+2 > x-2$. For every σ the number of $\mathbf{abb} \cdot w$ words is greater than the number of $\mathbf{bbb} \cdot w$ words, and we have proven the statement. □

Proposition 4.6.6 *The flag h-vector of the partition lattice satisfies*

$$[\boldsymbol{ab} \cdot (\boldsymbol{a})^i]_\Pi \leq [\boldsymbol{bb} \cdot (\boldsymbol{a})^i]_\Pi,$$

for $i \geq 0$.

Proof: In terms of flag vector entries, the number $[\mathbf{ab} \cdot (\mathbf{a})^i]_\Pi$ is given by $h_{\{2\}} = f_{\{2\}} - f_\emptyset$ and the number $[\mathbf{bb} \cdot (\mathbf{a})^i]_\Pi$ is given by $h_{\{1,2\}} = f_{\{1,2\}} - f_{\{1\}} - f_{\{2\}} + f_\emptyset$. We have

1. $f_\emptyset = 1$,
2. $f_{\{1\}} = S(n, n-1) = \binom{n}{2}$,
3. $f_{\{2\}} = S(n, n-2) = \binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$,
4. $f_{\{1,2\}} = S(n, n-1) \cdot S(n-1, n-2) = 3\binom{n}{3} + 2 \cdot \frac{1}{2}\binom{n}{2}\binom{n-2}{2}$.

Therefore we have the inequality

$$\begin{aligned}
[\mathbf{bb} \cdot (\mathbf{a})^i]_\Pi - [\mathbf{ab} \cdot (\mathbf{a})^i]_\Pi &= 3\binom{n}{3} + 2 \cdot \frac{1}{2}\binom{n}{2}\binom{n-2}{2} - \binom{n}{2} \\
&\quad -2\left(\binom{n}{3} + \frac{1}{2}\binom{n}{2}\binom{n-2}{2}\right) + 2 \\
&= 3\binom{n}{3} + 6\binom{n}{4} - \binom{n}{2} - 2\left(\binom{n}{3} + 3\binom{n}{4}\right) + 2 \\
&= \binom{n}{3} - \binom{n}{2} + 2 \\
&> 0
\end{aligned}$$

for $n \geq 5$. We check the cases $n = 3$ and $n = 4$ as follows.

$$\begin{array}{cccccc} & & & \textcircled{1} & & \\ \textbf{ab} & & \textcircled{0} & \longrightarrow & 2 & \\ & 2 & \longleftarrow & 4 & \longleftarrow & \textcircled{0} \end{array}$$

$$\begin{array}{cccccc} & & & \textcircled{1} & & \\ \textbf{bb} & & \textcircled{0} & \longrightarrow & 2 & \\ & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 \end{array}$$

$$\begin{array}{cccccccc} & & & & \textcircled{1} & & & \\ & & & 1 & \longleftarrow & \textcircled{0} & & \\ \textbf{aba} & & \textcircled{0} & \longrightarrow & 2 & \longrightarrow & 3 & \\ & 5 & \longleftarrow & 10 & \longleftarrow & 9 & \longleftarrow & \textcircled{0} \end{array}$$

$$\begin{array}{cccccccc} & & & & \textcircled{1} & & & \\ & & & 1 & \longleftarrow & \textcircled{0} & & \\ \textbf{bba} & & \textcircled{0} & \longrightarrow & 2 & \longrightarrow & 3 & \\ & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & \longrightarrow & 20 \end{array}$$

□

We can narrow down the maximum word a little more. The following lemma is a partition analogue of Lemma 3.3 in Ehrenborg–Mahajan [22]. For a word w of length $n - 1$ let $R(W)$ be the row vector $(t_{n,1}, t_{n,2}, \ldots, t_{n,n})$ given by the last row of the triangular array associated with w.

Lemma 4.6.7 *Suppose u and v are two $\boldsymbol{ab}$-words of length n having triangular arrays with their nth rows labeled $t_{n,1}, t_{n,2}, \ldots, t_{n,n}$ and $s_{n,1}, s_{n,2}, \ldots, s_{n,n}$, respectively. If $t_{n,i} \geq s_{n,i}$ for all $i = 1, 2, \ldots, n$ then*

$$[w \cdot u]_\Pi \geq [w \cdot v]_\Pi$$

for any $\boldsymbol{ab}$-word w. In this case, we say $R(u)$ is termwise *greater than $R(v)$ and write $R(u) \geq_{term} R(v)$.*

This is easy consequence of the fact that the exact linear combinations of the elements in each sequence are being used to create $[w \cdot u]_\Pi$ and $[w \cdot v]_\Pi$. We thus have the following theorem.

Theorem 4.6.8 *The flag h-vector of the partition lattice satisfies the following inequalities for any $\boldsymbol{ab}$-word w:*

1. $[w \cdot \boldsymbol{aa}]_\Pi < [w \cdot \boldsymbol{ab}]_\Pi$
2. $[w \cdot \boldsymbol{aab}]_\Pi < [w \cdot \boldsymbol{abb}]_\Pi$
3. $[w \cdot \boldsymbol{aba}]_\Pi < [w \cdot \boldsymbol{abb}]_\Pi$
4. $[w \cdot \boldsymbol{bba}]_\Pi < [w \cdot \boldsymbol{bab}]_\Pi$

5. $[w \cdot \boldsymbol{bbb}]_\Pi < [w \cdot \boldsymbol{bab}]_\Pi$

Proof: In each inequality, the words composed of the rightmost three letters, or two letters in the first case, are termwise less as can be seen by the following triangular arrays.

$$\textbf{aa}\quad \begin{array}{ccccc} & & \textcircled{1} & & \\ & 1 & \longleftarrow & \textcircled{0} & \\ 1 & \longrightarrow & 0 & \longrightarrow & \textcircled{0} \end{array}$$

$$\textbf{ab}\quad \begin{array}{ccccc} & & \textcircled{1} & & \\ & \textcircled{0} & \longrightarrow & 2 & \\ 2 & \longrightarrow & 4 & \longrightarrow & \textcircled{0} \end{array}$$

$$\textbf{aab}\quad \begin{array}{ccccccc} & & & \textcircled{1} & & & \\ & & \textcircled{0} & \longrightarrow & 2 & & \\ & 2 & \longrightarrow & 4 & \longrightarrow & \textcircled{0} & \\ 6 & \longleftarrow & 8 & \longleftarrow & 0 & \longleftarrow & \textcircled{0} \end{array}$$

$$\textbf{abb}\quad \begin{array}{ccccccc} & & & \textcircled{1} & & & \\ & & \textcircled{0} & \longrightarrow & 2 & & \\ & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & \\ 6 & \longleftarrow & 12 & \longleftarrow & 18 & \longleftarrow & \textcircled{0} \end{array}$$

$$\textbf{aba}\quad \begin{array}{ccccccc} & & & \textcircled{1} & & & \\ & & 1 & \longleftarrow & \textcircled{0} & & \\ & \textcircled{0} & \longrightarrow & 2 & \longrightarrow & 3 & \\ 5 & \longleftarrow & 10 & \longleftarrow & 9 & \longleftarrow & \textcircled{0} \end{array}$$

$$\textbf{bba}\quad \begin{array}{ccccccc} & & & \textcircled{1} & & & \\ & & 1 & \longleftarrow & \textcircled{0} & & \\ & \textcircled{0} & \longrightarrow & 2 & \longrightarrow & 3 & \\ \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & \longrightarrow & 20 \end{array}$$

$$\textbf{bab}\quad \begin{array}{ccccccc} & & & \textcircled{1} & & & \\ & & \textcircled{0} & \longrightarrow & 2 & & \\ & 2 & \longleftarrow & 4 & \longleftarrow & \textcircled{0} & \\ \textcircled{0} & \longrightarrow & 4 & \longrightarrow & 18 & \longrightarrow & 24 \end{array}$$

$$\textbf{bbb}\quad \begin{array}{ccccccc} & & & \textcircled{1} & & & \\ & & \textcircled{0} & \longrightarrow & 2 & & \\ & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & \\ \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 24 \end{array}$$

The theorem holds by Lemma 4.6.7. □

Observe that **ab**-words u and v of length n beginning with different letters will be termwise incomparable via the last row of the array because there will be a zero as the first term of one and as the last term of the other while the comparable terms are nonzero.

We have the following theorem.

Theorem 4.6.9 *For the partition lattice the maximum* ***ab*** *descent word with length at least three must end with either* ***bab*** *or* ***abb****.*

We observe similar termwise inequalities which extend inequality 3 in Theorem 4.6.8

Theorem 4.6.10 *The flag h-vector of the partition lattice satisfies the following inequalities for any* ***ab****-word w.*

1. $[w \cdot \boldsymbol{aabb}]_\Pi < [w \cdot \boldsymbol{abab}]_\Pi$
2. $[w \cdot \boldsymbol{aabab}]_\Pi < [w \cdot \boldsymbol{ababb}]_\Pi$

Proof: In each inequality, the words composed of the rightmost four or five letters are termwise less as can be seen by the following triangular arrays.

aabb

$$\begin{array}{ccccccccc}
 & & & & \textcircled{1} & & & & \\
 & & & \textcircled{0} & \longrightarrow & 2 & & & \\
 & & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & & \\
 & 6 & \longleftarrow & 12 & \longleftarrow & 18 & \longleftarrow & \textcircled{0} & \\
30 & \longleftarrow & 60 & \longleftarrow & 54 & \longleftarrow & 0 & \longleftarrow & \textcircled{0}
\end{array}$$

abab

$$\begin{array}{ccccccccc}
 & & & & \textcircled{1} & & & & \\
 & & & \textcircled{0} & \longrightarrow & 2 & & & \\
 & & 2 & \longleftarrow & 4 & \longleftarrow & \textcircled{0} & & \\
 & \textcircled{0} & \longrightarrow & 4 & \longrightarrow & 18 & \longrightarrow & 24 & \\
46 & \longleftarrow & 92 & \longleftarrow & 126 & \longleftarrow & 96 & \longleftarrow & \textcircled{0}
\end{array}$$

aabab

$$\begin{array}{ccccccccccc}
 & & & & & \textcircled{1} & & & & & \\
 & & & & \textcircled{0} & \longrightarrow & 2 & & & & \\
 & & & 2 & \longleftarrow & 4 & \longleftarrow & \textcircled{0} & & & \\
 & & \textcircled{0} & \longrightarrow & 4 & \longrightarrow & 18 & \longrightarrow & 24 & & \\
 & 46 & \longleftarrow & 92 & \longleftarrow & 126 & \longleftarrow & 96 & \longleftarrow & \textcircled{0} & \\
314 & \longleftarrow & 628 & \longleftarrow & 666 & \longleftarrow & 384 & \longleftarrow & 0 & \longleftarrow & \textcircled{0}
\end{array}$$

ababb

$$\begin{array}{ccccccccccc}
 & & & & & \textcircled{1} & & & & & \\
 & & & & \textcircled{0} & \longrightarrow & 2 & & & & \\
 & & & \textcircled{0} & \longrightarrow & 0 & \longrightarrow & 6 & & & \\
 & & 6 & \longleftarrow & 12 & \longleftarrow & 18 & \longleftarrow & \textcircled{0} & & \\
 & \textcircled{0} & \longrightarrow & 12 & \longrightarrow & 54 & \longrightarrow & 144 & \longrightarrow & 180 & \\
390 & \longleftarrow & 780 & \longleftarrow & 1134 & \longleftarrow & 1296 & \longleftarrow & 900 & \longleftarrow & \textcircled{0}
\end{array}$$

□

We conjecture the following termwise inequalities.

Conjecture 4.6.11 *In the partition lattice* Π_n*, the following termwise inequalities hold:*

$$\begin{array}{rclr} R(\boldsymbol{a}\cdot(\boldsymbol{ab})^k\cdot\boldsymbol{b}) & <_{term} & R(\boldsymbol{a}\cdot(\boldsymbol{ba})^k\cdot\boldsymbol{b}) & \text{for } n=2k+4, \\ R(\boldsymbol{a}\cdot(\boldsymbol{ab})^k\cdot\boldsymbol{ab}) & <_{term} & R(\boldsymbol{a}\cdot(\boldsymbol{ba})^k\cdot\boldsymbol{bb}) & \text{for } n=2k+5. \end{array}$$

We list some conjectures.

Conjecture 4.6.12 *For the partition lattice, we have the inequality*

$$[\boldsymbol{bb}\cdot w]_\Pi \leq [\boldsymbol{ab}\cdot w]_\Pi$$

when $w \neq (\boldsymbol{a})^i$ *for* $i \geq 1$*.*

Recall a sequence $\{c_i\}_{1\leq i\leq n}$ is said to be *unimodal* if there exists an index j such that

$$c_1 \leq c_2 \leq \cdots \leq c_j \geq c_{j+1} \geq \cdots \geq c_n.$$

Conjecture 4.6.13 *For the partition lattice* Π_n *the sequences* $\{[\boldsymbol{b}^i\boldsymbol{a}^{n-i-2}]_\Pi\}$ *and* $\{[\boldsymbol{a}^i\boldsymbol{b}^{n-i-2}]_\Pi\}$ *for* $i = 0,\ldots,n-2$ *are unimodal.*

For the partition lattice, we have to consider parity when maximizing the descent statistic. The following is a conjecture due to Ehrenborg and Readdy [23].

Conjecture 4.6.14 (Ehrenborg–Readdy) *For the partition lattice* Π_n*, the descent statistic is maximized when*

$$w = \begin{cases} (\boldsymbol{ba})^k\cdot\boldsymbol{b} & \text{when } n=2k+3, \\ (\boldsymbol{ba})^k\cdot\boldsymbol{bb} & \text{when } n=2k+4. \end{cases}$$

Our data supports the following conjecture for the Dowling lattice.

Conjecture 4.6.15 *For the Dowling lattice* L_n*, the descent statistic is maximized when*

$$w = \begin{cases} (\boldsymbol{ba})^k\cdot\boldsymbol{b} & \text{when } n=2k+2, \\ (\boldsymbol{ba})^k\cdot\boldsymbol{bb} & \text{when } n=2k+3. \end{cases}$$

Conjecture 4.6.16 *The descent number which is second to the largest in the partition lattice* Π_n *is of the form*

$$\begin{array}{rr} [(\boldsymbol{ab})^k\cdot\boldsymbol{b}]_\Pi & \text{for } n=2k+3, \\ [(\boldsymbol{ab})^k\cdot\boldsymbol{ab}]_\Pi & \text{for } n=2k+4. \end{array}$$

Ehrenborg and Mahajan have a similar result for the Boolean algebra in [22].

There are some strategies which may be used to prove these conjectures. If Conjecture 4.6.11 holds then the descent word with the maximal descent set statistic will not contain the string **aa**. To prove Conjecture 4.6.14 it would be left to show that the descent word $w = w_1 w_2 \cdots w_n$ with the maximal descent statistic does not contain the string **bb** except perhaps in the position $w_{n-1}w_n$ at the end of the word.

Another tactic may be to show that the descent word with the maximal descent set statistic of rank $n-2$ is contained in the descent word with the maximal descent set statistic of rank n. If this fact is true we apply Theorem 4.6.3 and Theorem 4.6.9 to help prove the maximization Conjecture 4.6.14. Once the Conjecture 4.6.14 for the partition lattice is proven, we hope to apply similar techniques to the case of the Dowling lattice to prove Conjecture 4.6.15.

Chapter 5 Future Research

The results of this dissertation open many avenues of research. We now describe a number of them.

I. Permanents and Möbius function connections

   Recall that the Möbius function of $\mathrm{Rees}(\mathscr{C}_n, C_{n+1})$ is given by n times the permanent of a matrix with 1's on the diagonal and 2's in the off-diagonal entries. See Chapter 2. It is tantalizing to ask if one can find other families of posets whose Möbius function is given by the permanent of a matrix. Perhaps there is some connection with posets having an R-labeling given by some family of permutations. It would be natural to conjecture that the Möbius function of the Rees product of such a poset with the chain would equal the number of derangements in this family of permutations. One such class of posets is the set of $\mathfrak{S}_n$-shellable posets [34].

II. Rees product of Cohen-Macaulay posets

   As was mentioned in the introductory section, Björner and Welker showed that the Cohen-Macaulay property is preserved under the Rees product. It would be interesting to look at the Rees product of other Cohen-Macaulay posets including the partition lattice and the face lattice of a convex polytope with the chain. See Section 2.9 for the Möbius values of the Rees product of the partition lattice with the chain. Is there a combinatorial interpretation of these values?

III. Poset properties preserved by the Rees product

   Besides the Cohen-Macaulay property, other properties the Rees product preserves include the property of being homotopically Cohen-Macaulay and poset formulations of shellability (EL-shellable, CL-shellable). Given a regular cell complex, one would like to simplify it to an easier-to-understand complex using elementary collapses. Forman's discrete Morse theory does precisely this [25]. Chari showed that given a generalized shelling there is a discrete Morse function [15]. The converse of this result is not known. Does the Rees product of two posets each having a discrete Morse function also have a discrete Morse function?

IV. Skew character on the Rees product of the cubical lattice with the chain

   For a given n, k, and i, Calderbank–Hanlon–Robinson study the character of the symmetric group $\mathfrak{S}_n$ acting on the homology groups of the order complex of the poset of partitions of n where each block size in congruent to i mod k [14]. They find the defining equations for the generating functions of the character values for this action. In the specific case $i = 0$ and $k = 2$ the authors prove a conjecture of Stanley that the restriction of the character in this case is a skew character. Can one find similar descriptions for the character of the action by the symmetric group on the top homology group of $\Delta(\text{Rees}(\mathscr{C}_n, C_{n+1}))$?

V. Inequalities for a q-analogue of the partition lattice Π_n

Ehrenborg and Mahajan [22] extended inequalities for **ab**-words in the Boolean algebra to its q-analogue, the subspace lattice. It is natural to see if one could find a q-analogue of the weighted boustrophedon transform of the partition lattice. The problem is there are a number of different q-analogues of the partition lattice. One such q-analogue is due to Hanlon, Hersh, and Shareshian [28]. Can one find q-inequalities for this analogue?

VI. Probabilistic approach to flag vector inequalities

Ehrenborg, Levin, and Readdy [21] use a probabilistic approach to study quadratic inequalities of the descent statistic. Is there a meaningful way to apply their techniques to weighted chains in the Dowling lattices to obtain new flag vector inequalities?

VII. Posets associated with weighted boustrophedon transforms

In this dissertation we have shown there are extensions of the boustrophedon transform which calculate the descent set statistics for the Boolean algebra, the partition lattice, and the Dowling lattice. These transforms all act on the vector $(1, 0, 0, \ldots)$. Are there any other transforms on this vector which give the descent set statistic for well-known posets? Can one determine the properties the transform must have to be associated to a poset?

VIII. Relaxation of the upper and lower isomorphic condition

A graded locally finite poset is called a *binomial poset* when the number of maximal chains in an interval depends only on the length of that interval. This number is given by the factorial function. It was an open question to determine if one can have two non-isomorphic posets which have the same factorial function. Backelin recently showed that this is the case [2]. Can the determinantal results on upper and lower isomorphic posets be generalized to binomial posets? If not, could one replace the upper and lower isomorphic condition with the condition that the rank-generating functions of every upper or lower interval coincide?

IX. Homotopy equivalence

For a poset P and its dual P^*, we have shown that although $\text{Rees}(P, T_{t,n+1})$ and $\text{Rees}(P^*, T_{t,n+1})$ are generally not isomorphic as posets, their Möbius function are equal, that is, $\mu(\text{Rees}(P, T_{t,n+1})) = \mu(\text{Rees}(P^*, T_{t,n+1}))$. In personal communication Stanley has asked if one can find a homotopy equivalence between the order complexes of these two Rees products. More interestingly would be to consider the case when the poset P, under consideration is not Cohen-Macaulay.

X. Extremal combinatorics for the Möbius function

   A philosophical question to ask is the following. Is the descent set statistic of a poset always maximized on some version of the alternating **ab**-word? If not, what properties does a poset need to have in order for this configuration to be extremal? For S-shellable posets having a **cd**-index the alternating **ab**-word is maximal [52].

XI. Implications to commutative algebra

   An ultimate goal is to take problems in commutative algebra, reformulate them in combinatorial terms, and solve them combinatorially. Conversely one should take results in combinatorics and see if their reformulation into commutative algebra gives new insights. Do the Rees products we have studied in this dissertation have any commutative algebraic implications?

We hope this dissertation motivates the reader to expand their mathematical horizons.

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