



University of Kentucky
UKnowledge

University of Kentucky Doctoral Dissertations

Graduate School

2009

Eigenvalue Inequalities for a Family of Spherically Symmetric Riemannian Manifolds

Julie Miker

University of Kentucky, jmiker@ms.uky.edu

[Right click to open a feedback form in a new tab to let us know how this document benefits you.](#)

Recommended Citation

Miker, Julie, "Eigenvalue Inequalities for a Family of Spherically Symmetric Riemannian Manifolds" (2009). *University of Kentucky Doctoral Dissertations*. 783.
https://uknowledge.uky.edu/gradschool_diss/783

This Dissertation is brought to you for free and open access by the Graduate School at UKnowledge. It has been accepted for inclusion in University of Kentucky Doctoral Dissertations by an authorized administrator of UKnowledge. For more information, please contact UKnowledge@lsv.uky.edu.

ABSTRACT OF DISSERTATION

Julie Miker

The Graduate School
University of Kentucky
2009

Eigenvalue Inequalities for a Family of Spherically Symmetric Riemannian Manifolds

ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Julie Miker
Lexington, Kentucky

Director: Dr. Peter Hislop, Professor of Mathematics
Lexington, Kentucky 2009

Copyright© Julie Miker 2009

ABSTRACT OF DISSERTATION

Eigenvalue Inequalities for a Family of Spherically Symmetric Riemannian Manifolds

This thesis considers two isoperimetric inequalities for the eigenvalues of the Laplacian on a family of spherically symmetric Riemannian manifolds. The Payne-Pólya-Weinberger Conjecture (PPW) states that for a bounded domain Ω in Euclidean space \mathbb{R}^n , the ratio $\lambda_1(\Omega)/\lambda_0(\Omega)$ of the first two eigenvalues of the Dirichlet Laplacian is bounded by the corresponding eigenvalue ratio for the Dirichlet Laplacian on the ball B_Ω of equal volume. The Szegő-Weinberger inequality states that for a bounded domain Ω in Euclidean space \mathbb{R}^n , the first nonzero eigenvalue of the Neumann Laplacian $\mu_1(\Omega)$ is maximized on the ball B_Ω of the same volume. In the first three chapters we will look at the known work for the manifolds \mathbb{R}^n and \mathbb{H}^n . Then we will take a family of spherically symmetric manifolds given by \mathbb{R}^n with a spherically symmetric metric determined by a radially symmetric function f . We will then give a PPW-type upper bound for the eigenvalue gap, $\lambda_1(\Omega) - \lambda_0(\Omega)$, and the ratio $\lambda_1(\Omega)/\lambda_0(\Omega)$ on a family of symmetric bounded domains in this space. Finally, we prove the Szegő-Weinberger inequality for this same class of domains.

KEYWORDS: eigenvalues of the Laplacian, isoperimetric inequalities, Payne-Pólya-Weinberger conjecture, Szegő-Weinberger inequality, Faber-Krahn inequality

Author's signature: Julie Miker

Date: December 11, 2009

Eigenvalue Inequalities for a Family of Spherically Symmetric Riemannian Manifolds

By
Julie Miker

Director of Dissertation: Peter Hislop

Director of Graduate Studies: Qiang Ye

Date: December 11, 2009

RULES FOR THE USE OF DISSERTATIONS

Unpublished dissertations submitted for the Doctor's degree and deposited in the University of Kentucky Library are as a rule open for inspection, but are to be used only with due regard to the rights of the authors. Bibliographical references may be noted, but quotations or summaries of parts may be published only with the permission of the author, and with the usual scholarly acknowledgments.

Extensive copying or publication of the dissertation in whole or in part also requires the consent of the Dean of the Graduate School of the University of Kentucky.

A library that borrows this dissertation for use by its patrons is expected to secure the signature of each user.

Name

Date

DISSERTATION

Julie Miker

The Graduate School
University of Kentucky
2009

Eigenvalue Inequalities for a Family of Spherically Symmetric Riemannian Manifolds

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Julie Miker
Lexington, Kentucky

Director: Dr. Peter Hislop, Professor of Mathematics
Lexington, Kentucky 2009

Copyright© Julie Miker 2009

Dedicated to the Lord, for in Him all things are possible.
May He receive all the glory.

ACKNOWLEDGMENTS

The writing of this dissertation would not have been possible without the help and support of several people. First and foremost, I would like to thank my adviser, Dr. Peter Hislop, for all of the advice, support, and encouragement that he provided throughout this process. I am thankful for the many hours he sacrificed and for his continual interest in helping me to learn and become a better mathematician. I would also like to thank Dr. Russell Brown, Dr. Peter Perry, Dr. Changyou Wang, and Dr. Zhongwei Shen. Due to their excellent teaching, helpfulness, and willingness to try to answer any question I had, I have learned much from them. In addition, I want to thank Dr. Perry and Dr. Brown for serving on my advisory committee, reading this dissertation, and providing helpful comments.

Next, I would like to thank my parents, Steve and Joyce Miker, and my family for believing in me all the time. Thank you for always encouraging me and giving me the strength to keep going even when I wanted to stop. I appreciate how you have always supported me and helped me through every step as I continue to move forward in what the Lord has called me to do. I would also like to thank my church family in Lexington for being my home away from home.

Finally, I would like to thank all of my fellow graduate students that have helped to make graduate school a great experience. Thank you so much for your friendship, encouragement, and all of the fun times we have had in between all of the hard work. I especially want to thank Joel Kilty and Daniel Kiteck. Without your prayers, constant encouragement, good advice, and steadfastness, I might never have made it this far.

TABLE OF CONTENTS

Acknowledgments	iii
List of Figures	vi
Chapter 1 Introduction	1
Chapter 2 Some Preliminaries	9
2.1 The Dirichlet and Neumann Eigenvalues	10
2.2 Symmetric Rearrangements	11
2.3 Faber-Krahn Inequality	16
2.4 The Gap Inequality for the Eigenvalues of the Dirichlet Laplacian	18
Chapter 3 Payne-Pólya-Weinberger Inequality in Euclidean space	20
3.1 Center of Mass Theorem for \mathbb{R}^n	24
Chapter 4 Payne-Pólya-Weinberger Inequality in Hyperbolic Space	26
4.1 The Geometry of Hyperbolic Space	26
4.2 Eigenvalue properties	27
4.3 The Payne-Pólya-Weinberger Inequality	29
4.4 The Center of Mass Argument in Hyperbolic Space	31
4.5 Chiti's Comparison Argument in \mathbb{H}^n	34
Chapter 5 PPW-type Eigenvalue Inequalities for \mathbb{F}^n	39
5.1 The Geometry of \mathbb{F}^n	39
5.2 A PPW-type Inequality for the Eigenvalues of the Dirichlet Laplacian	41
5.3 Monotonicity Properties	53
5.4 Properties of q	56
5.5 Center of Mass Theorem in \mathbb{F}^n	59
5.6 Chiti's Comparison Argument	60
5.7 Eigenvalue Properties	64
5.8 Another Gap Inequality for \mathbb{F}^n	66
Chapter 6 Szegő-Weinberger Inequality	69
6.1 Szegő-Weinberger in \mathbb{R}^n	69
6.2 Szegő-Weinberger in Hyperbolic Space	74
6.3 Szegő-Weinberger for \mathbb{F}^n	77
6.4 Szegő-Weinberger for More General Spherically Symmetric Manifolds	87
Chapter 7 Vector Fields on Spheres	89
Bibliography	92

Vita 94

LIST OF FIGURES

6.1	$f'(r)$ for $\alpha = 2$ on $(.5,1.5)$	81
6.2	$f'(r)$ for $\alpha = 3$ on $(.5,1.5)$	81
6.3	$f'(r)$ for $\alpha = 4$ on $(.5,1.5)$	82
6.4	$f'(r)$ for $\alpha = 5$ on $(1,2)$	82
6.5	$f'(r)$ for $\alpha = 6$ on $(1,2)$	83
6.6	$f'(r)$ for $\alpha = 7$ on $(1,2)$	83
6.7	$f'(r)$ for $\alpha = 8$ on $(1,2)$	84
6.8	$f'(r)$ for $\alpha = 9$ on $(1,2)$	84

Chapter 1 Introduction

In this thesis we will consider problems involving the eigenvalues of the Laplacian in various settings. First, we will consider the case of Dirichlet boundary conditions. Here we will look at the Payne-Pólya-Weinberger conjecture, which states that the ratio of the first two Dirichlet eigenvalues for a domain Ω is bounded above by the same eigenvalue ratio for a ball having the same volume. Then we will move on to the case of Neumann boundary conditions and look at the Szegő-Weinberger inequality for the first nonzero Neumann eigenvalue. In both cases we will first look at known results in Euclidean space, \mathbb{R}^n , and then hyperbolic space, \mathbb{H}^n . We give the generalization of these results to a certain family of spherically symmetric Riemannian manifolds.

Ashbaugh and Benguria first proved the Payne-Pólya-Weinberger conjecture for bounded domains in Euclidean space in [2], and then extended this result for domains contained in a hemisphere of \mathbb{S}^n in [3]. The original conjecture considered the problem of bounding eigenvalue ratios for the homogeneous membrane problem

$$\begin{cases} -\Delta u = \lambda u & \text{on } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

for a bounded domain Ω in the plane. In this work, a domain is an open, connected set. In 1955 and 1956, Payne, Pólya, and Weinberger were able to show that the ratio of the first two eigenvalues for this problem satisfies

$$\frac{\lambda_1}{\lambda_0} \leq 3 \quad (1.2)$$

and further conjectured that

$$\frac{\lambda_1}{\lambda_0} \leq \frac{\lambda_1}{\lambda_0}|_{\Omega=\text{disk}} \approx 2.539. \quad (1.3)$$

Later, Thompson [26] conjectured an n -dimensional extension of this result, namely

$$\frac{\lambda_1}{\lambda_0} \leq \frac{\lambda_1}{\lambda_0}|_{\Omega=n\text{-dimensional ball}} = (j_{n/2,1}/j_{n/2-1,1})^2, \quad (1.4)$$

where $j_{p,k}$ denotes the k^{th} positive zero of the Bessel function $J_p(x)$.

This was proved in the following result by Ashbaugh and Benguria in [2].

Theorem 1. *The ratio of the first two Dirichlet eigenvalues of the Laplacian on a domain $\Omega \subset \mathbb{R}^n$ satisfies*

$$\frac{\lambda_1}{\lambda_0} \leq \frac{\lambda_1}{\lambda_0} \Big|_{\Omega=\text{unit } n\text{-dimensional ball}} = \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \quad (1.5)$$

Moreover, equality is obtained if and only if Ω is a ball.

Observe that in Euclidean space, the ratio $\frac{\lambda_1}{\lambda_0}$ is independent of the radius of the ball. Later Benguria and Linde gave an analogous result as it applied to hyperbolic space in [8], stated as follows.

Theorem 2. *Let $\Omega \subset \mathbb{H}^n$ be an open bounded domain in the hyperbolic space of constant negative curvature and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . If $S_1 \subset \mathbb{H}^n$ is a geodesic ball such that $\lambda_0(\Omega) = \lambda_0(S_1)$ then*

$$\lambda_1(\Omega) \leq \lambda_1(S_1) \quad (1.6)$$

with equality if and only if Ω is a geodesic ball.

They had to add the extra condition of $\lambda_0(\Omega) = \lambda_0(S_1)$ since, in hyperbolic space, the ratio of the first two eigenvalues is a decreasing function of the radius of the ball where as in Euclidean space it is independent of the radius of the ball. In the next several chapters we will present the proofs for these cases.

In the final chapter we move to the Neumann case and consider the Szegő-Weinberger inequality. In 1952, Weinberger [27] proved the following result.

Theorem 3. *Let $\mu_1(\Omega)$ be the first nonzero Neumann eigenvalue for a bounded domain $\Omega \subset \mathbb{R}^n$. Then*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*)$$

with equality if and only if $\Omega = \Omega^$. Here Ω^* is the symmetric rearrangement of Ω , so it has the same volume.*

In 1992, Ashbaugh and Benguria [4] extended this result to hyperbolic space and proved the following theorem.

Theorem 4. *Let Ω be a bounded domain in a space of constant negative sectional curvature. Then its first nonzero Neumann eigenvalue $\mu_1(\Omega)$ satisfies*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^ is a geodesic ball in the same space having the same n -volume as Ω . Equality occurs if and only if Ω is a geodesic ball.*

We seek to generalize these results under the following hypotheses. In general, we want to work in a space defined by the following metric

$$ds^2 = dr^2 + f(r)^2|d\omega|^2. \tag{1.7}$$

Here $d\omega$ is the standard spherical portion of the metric on \mathbb{S}^{n-1} . Observe that the spaces \mathbb{R}^n and \mathbb{H}^n also have metrics of this form. For \mathbb{R}^n , we would take $f(r) = r$ and for \mathbb{H}^n , $f(r) = \sinh r$, as is outlined in section 4.1. To generate our family of spherically symmetric manifolds, we take f so that $f > 0$, $f(0) = 0$, and $f'(0) = 1$. Henceforth, we will refer to this space as \mathbb{F}^n . Typically, in this paper, we will take

$$f(r) = \begin{cases} \sim r, & \text{as } r \rightarrow 0 \\ \sim r^\alpha, & \text{as } r \rightarrow \infty \end{cases} \tag{1.8}$$

so that f satisfies our above conditions, $f \in C^2(\Omega)$, $r \in [0, \infty)$ and $\alpha \geq 2$. Two examples of these functions are $f(r) = r + \beta r^\alpha$ and $f(r)$ given by a polynomial

interpolation as described below in (1.20). So from this we see that by separation of variables the eigenvalues and eigenfunctions of $-\Delta$ on a geodesic ball of radius \tilde{r} are determined by the differential equation

$$-z''(r) - \frac{(n-1)f'}{f}z'(r) + \frac{l(l+n-2)}{f^2}z(r) = \lambda z(r), \quad (1.9)$$

where $l = 0, 1, 2, \dots$ with the boundary conditions $z'(0) = 0$ for $l = 0$ or $z(r) \sim r^l$ as $r \downarrow 0$ for $l > 0$ and $z(\tilde{r}) = 0$. We will use the convention that $z_l(r) > 0$ for $r \downarrow 0$.

In the later sections we will state and give the proof of our theorems and the subsequent sections will give the remaining details needed for the proof. In these sections we will obtain necessary monotonicity properties and rearrangement inequalities as in previous works. Also, we look at a generalized center of mass argument and Chiti comparison result.

Here we will set up and state the our results for the family of spherically symmetric manifolds. As mentioned above, we refer to the set of manifolds generated by the functions f chosen to satisfy the above properties as \mathbb{F}^n . We first seek to generalize the PPW result. However, we see that in the space \mathbb{F}^n , we no longer have the property that the Laplacian commutes with translations. Thus, the center of mass theorem of Weinberger that allows us to show the orthogonality of our optimal test functions no longer holds. Hence we obtain an error term in addition to the expected upper bound of the eigenvalue gap on the geodesic disk having the same volume as our domain.

Theorem 5. (A PPW-like result for \mathbb{F}^n) Let $\Omega \subset \mathbb{F}^n$ be an open bounded domain and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . Denote by Ω^* the geodesic ball in \mathbb{F}^n having the same volume as Ω and \tilde{r} its geodesic radius. Also, we denote by $\vec{\nu}$ the vector associated with the center of mass theorem for Ω in \mathbb{F}^n . So, if we have that the angle between \vec{x} and $\vec{\nu}$ is between $-\pi/2$ and $\pi/2$ for all $x \in \Omega$, then

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \left\{ \sup_{x \in \Omega} \frac{B(|x + \nu|)}{B(|x|)} \right\} [1 + 2\epsilon^2 + b_\Omega(\nu, \epsilon)] (\lambda_1(\Omega^*) - \lambda_0(\Omega^*)) + \mathcal{F}(\nu, \epsilon), \quad (1.10)$$

where

$$B(r) = g'(r)^2 + (n-1)f^{-2}(r)g^2(r), \quad (1.11)$$

$$g(r) = \begin{cases} \frac{z_1(r)}{z_0(r)} & r \in [0, \tilde{r}), \\ \lim_{r \uparrow \tilde{r}} g(r) & r \geq \tilde{r}, \end{cases} \quad (1.12)$$

$$\mathcal{F}(\nu, \epsilon) = \left(1 + \epsilon^2 + \frac{1}{\epsilon^2}\right) \sup_{x \in \Omega + \nu} \left\{ \sum_k \left[\left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} \right) |(x - \nu)_k l(x - \nu) + 1| \right. \right. \\ \left. \left. \left| \frac{\partial J(-\nu, x - \nu)}{\partial x_k} J(\nu, x) \right| \right] \right\}, \quad (1.13)$$

and

$$b_\Omega(\nu, \epsilon) \equiv \max_{k=1, \dots, n} \sup_{x \in \Omega + \nu} \left\{ \left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} \right) \left(\frac{f(|x|)^2}{f'(|x|) - 1} \right) \right. \\ \left[|\nu_k l(\nu) - x_k l(\nu) - \nu_k l(x)| + \frac{|F(x, \nu)| f(|x - \nu|)^2}{f'(|x - \nu|) - 1} |x_k l(x)| \right] \\ \left. \left[\frac{f'(|x|) - 1}{f(|x|)^2} |x_k l(x)| + 1 \right]^{-1} \right\} \left(1 + \frac{2}{\epsilon^2}\right), \quad (1.14)$$

where

$$F(x, \nu) = \left[\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} - \frac{f'(|x|) - 1}{f(|x|)^2} \right]. \quad (1.15)$$

Next, we would like to look at a certain set of domains having additional symmetry properties. We will take $\mathcal{S} \subset \mathbb{F}^n$ to be the set of domains Ω such that Ω contains the origin, is rotationally symmetric to the x_i axes for $i = 3, \dots, n$, and is symmetric

with respect to the $x_1 - x_2$ plane. This subset of domains allows us to choose our test functions without the need for shifting the domains as in the previous result. So in this case, we obtain the following theorem.

Theorem 6. *Let $\Omega \subset \mathcal{S}$ be an open bounded domain in \mathbb{F}^n and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . Take Ω^* to be the geodesic ball in \mathbb{F}^n having the same volume as Ω . Then,*

1)

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \lambda_1(\Omega^*) - \lambda_0(\Omega^*). \quad (1.16)$$

2) *If $S_1 \subset \mathbb{F}^n$ is a geodesic ball such that $\lambda_0(\Omega) = \lambda_0(S_1)$ then*

$$\lambda_1(\Omega) \leq \lambda_1(S_1). \quad (1.17)$$

Also, we present a result based on using Gram-Schmidt orthogonalization to find a suitable test function for the Rayleigh-Ritz inequality.

Theorem 7. *Let $\Omega \subset \mathbb{F}^n$ be an open bounded domain and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω , u_i the i -th eigenfunction. Take $g(r)$ to be a continuous, differentiable, positive function on $[0, \infty)$. Then for*

$$P_j(x) = \frac{x_j}{f(r)} g(r), \quad (1.18)$$

$j = 1, \dots, n$, we have

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2 \left(g'(r)^2 + (n-1) \left(\frac{g(r)}{f(r)} \right)^2 \right) dV}{\int_{\Omega} g(r)^2 u_0^2 dV - \sum_j |\langle P_j u_0, u_0 \rangle|^2}. \quad (1.19)$$

Finally, we state the Szegő-Weinberger theorem. In order to construct suitable test functions we must restrict ourselves to domains Ω containing the origin.

Theorem 8. *Let $\Omega \subset \mathcal{S}$ be a bounded open set in \mathbb{F}^n . Then its first nonzero Neumann eigenvalue $\mu_1(\Omega)$ satisfies*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^* is a geodesic ball in the same space having the same n -volume as Ω . Equality occurs if and only if Ω is a geodesic ball.

Also, for the Szegő-Weinberger theorem we will show some examples of functions f that satisfy the properties we have claimed above. In general, it appears that several functions that satisfy our properties can be generated in the following way.

$$f(r) = \begin{cases} r & r \leq \tilde{r}_1 \\ p(r) & \tilde{r}_1 < r < \tilde{r}_2 \\ r^\alpha & r \geq \tilde{r}_2 \end{cases} \quad (1.20)$$

so that f satisfies our above conditions for \mathbb{F}^n , $f \in C^2(\Omega)$, $r \in [0, \infty)$ and $\alpha \geq 2, \alpha \in \mathbb{Z}$. In order to find $p(r)$, we use the data from f at \tilde{r}_1 and \tilde{r}_2 and its first and second derivatives to generate a C^2 fifth degree polynomial on the interval $[\tilde{r}_1, \tilde{r}_2]$. In the proof of the theorem, we see that \tilde{r}_1, \tilde{r}_2 depend on α and it turns out that, for example,

$$\begin{aligned} \tilde{r}_1 = .5, \tilde{r}_2 = 1.5 & \quad \text{for } 2 \leq \alpha \leq 4 \\ \tilde{r}_1 = 1, \tilde{r}_2 = 2 & \quad \text{for } 5 \leq \alpha \leq 9. \end{aligned} \quad (1.21)$$

We can compute $p(r)$ for each of our intervals, and so it turns out that for $\alpha = 2, 3, 4$,

$$\begin{aligned} p(r) = & \left(-6 - \frac{1}{2}\alpha 1.5^{\alpha-2} - 3\alpha 1.5^{\alpha-1} + 6 \cdot 1.5^\alpha + .5\alpha^2 1.5^{\alpha-2}\right)r^5 \\ & + (30.5 + 2.25\alpha 1.5^{\alpha-2} + 14.5\alpha 1.5^{\alpha-1} - 30 \cdot 1.5^\alpha - 2.25\alpha^2 1.5^{\alpha-2})r^4 \\ & + (-57 - 3.75\alpha 1.5^{\alpha-2} - 25.5\alpha 1.5^{\alpha-1} + 55 \cdot 1.5^\alpha + 3.75\alpha^2 1.5^{\alpha-2})r^3 \\ & + (47.25 + 2.875\alpha 1.5^{\alpha-2} + 20.25\alpha 1.5^{\alpha-1} - 45 \cdot 1.5^\alpha - 2.875\alpha^2 1.5^{\alpha-2})r^2 \\ & + (-16.875 - 1.03125\alpha 1.5^{\alpha-2} - 7.4375\alpha 1.5^{\alpha-1} + 16.875 \cdot 1.5^\alpha + 1.03125\alpha^2 1.5^{\alpha-2})r \\ & + (2.53125 - .140625\alpha 1.5^{\alpha-2} + 1.03125\alpha 1.5^{\alpha-1} - 2.375 \cdot 1.5^\alpha - .140625\alpha^2 1.5^{\alpha-2}) \end{aligned} \quad (1.22)$$

and for $5 \leq \alpha \leq 9$,

$$\begin{aligned}
p(r) = & \left(-9 - \frac{1}{2}\alpha 2^{\alpha-2} - 3\alpha 2^{\alpha-1} + 6 \cdot 2^\alpha + .5\alpha^2 2^{\alpha-2}\right)r^5 \\
& + \left(68 + \frac{7}{2}\alpha 2^{\alpha-2} + 22\alpha 2^{\alpha-1} - 45 \cdot 2^\alpha - 3.5\alpha^2 2^{\alpha-2}\right)r^4 \\
& + \left(-198 - \frac{19}{2}\alpha 2^{\alpha-2} - 62\alpha 2^{\alpha-1} + 130 \cdot 2^\alpha + 9.5\alpha^2 2^{\alpha-2}\right)r^3 \\
& + \left(276 + \frac{25}{2}\alpha 2^{\alpha-2} + 84\alpha 2^{\alpha-1} - 180 \cdot 2^\alpha - 12.5\alpha^2 2^{\alpha-2}\right)r^2 \\
& + \left(-184 - 8\alpha 2^{\alpha-2} - 55\alpha 2^{\alpha-1} + 120 \cdot 2^\alpha + 8\alpha^2 2^{\alpha-2}\right)r \\
& + \left(48 + 2\alpha 2^{\alpha-2} + 14\alpha 2^{\alpha-1} - 31 \cdot 2^\alpha - 2\alpha^2 2^{\alpha-2}\right).
\end{aligned} \tag{1.23}$$

In the remainder of the thesis, we will refer to these examples and show specific cases to see that we can find functions that satisfy the properties we have claimed.

Chapter 2 Some Preliminaries

In this section we will outline some of the basic background material and tools that are used in the proofs of the various cases for the Payne-Pólya-Weinberger conjecture and for the Szegő-Weinberger result. We begin with stating the classical isoperimetric inequality for domains in \mathbb{R}^n . Let V denote n -dimensional Lebesgue measure and A denote $(n - 1)$ -dimensional Lebesgue measure. Take \mathbb{B}^n to be the unit ball in \mathbb{R}^n , and \mathbb{S}^{n-1} to be the unit sphere in \mathbb{R}^n .

Theorem 9 (Isoperimetric inequality). *Let Ω be any bounded domain in \mathbb{R}^n and $\partial\Omega$ its boundary. Then we have*

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq \frac{A(\mathbb{S}^{n-1})}{V(\mathbb{B}^n)^{1-1/n}}.$$

Also, equality is achieved if and only if Ω is an n -ball.

So from this we see that for a fixed volume in \mathbb{R}^n , the ball has the least area of all domains enclosing this volume. Now we would like to relate this to an eigenvalue problem defined on domains in \mathbb{R}^n , \mathbb{H}^n , and \mathbb{F}^n .

In general, we would like to see how changing the shape of the domain affects the first nonzero eigenvalue of the Laplacian, both with Dirichlet and Neumann boundary conditions and then also the ratio of the first two Dirichlet eigenvalues. The Faber-Krahn inequality looks at how the lowest eigenvalue for the Dirichlet Laplacian is affected by the shape of a domain with fixed volume. We see that it is minimized when the domain is a ball. We prove this using an approach involving the symmetric rearrangement of a function. This result is extended by Chavel to domains of constant positive and constant negative sectional curvature. In the final chapter, we move to the Neumann case and look at the result of Szegő and Weinberger and we see here that the first eigenvalue is maximized when the domain is a ball.

2.1 The Dirichlet and Neumann Eigenvalues

This thesis focuses on studying the properties of some of the eigenvalues of both the Dirichlet and Neumann Laplacian. Here we outline the basic set-up of these problems. See [25] for a discussion of this topic.

Definition (Dirichlet Problem). Let Ω be a bounded domain in \mathbb{R}^n , \mathbb{H}^n , or \mathbb{F}^n . Find a unique function u such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We define the eigenvalues of this problem in the following way.

Definition (Rayleigh-Ritz Formulation). For any open set $\Omega \in \mathbb{R}^n$ and $\phi \in H_0^1(\Omega)$, one considers the functional

$$F[\phi] = \frac{\|\nabla\phi\|^2}{\|\phi\|^2} \quad (2.1)$$

and the associated infimum

$$\lambda_0(\Omega) = \inf_{\substack{\phi \in H_0^1(\Omega) \\ 0 \neq \phi}} F[\phi]. \quad (2.2)$$

Here $\|\cdot\|$ is the $L^2(\Omega)$ norm. We refer to $\lambda_0(\Omega)$ as the fundamental tone of Ω . $\lambda_0(\Omega)$ is the infimum of the spectrum of the Laplacian $-\Delta$ on Ω , subject to Dirichlet boundary conditions. If Ω has compact closure and C^∞ boundary, then $\lambda_0(\Omega)$ is an eigenvalue of $-\Delta$ on Ω , i.e. there exists $\Phi \in C^\infty(\Omega)$ which satisfies $\Delta\Phi + \lambda_0(\Omega)\Phi = 0$ with $\Phi|_{\partial\Omega} = 0$.

Next we move to the Neumann Laplacian. Here the problem is stated as follows.

Definition (Neumann Problem). Let Ω be a bounded domain in \mathbb{R}^n , \mathbb{H}^n , or \mathbb{F}^n . Let $\frac{\partial}{\partial n}$ denote the outward normal derivative. Find a unique function u such that

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

We may also describe this problem variationally, as follows.

$$F[\phi] = \frac{\|\nabla\phi\|^2}{\|\phi\|^2} \tag{2.3}$$

and the associated infimum

$$\mu_1(\Omega) = \inf_{\substack{\phi \in H^1(\Omega) \\ 0 \neq \phi \perp 1}} \frac{\int_{\Omega} |\nabla\phi|^2}{\int_{\Omega} \phi^2}. \tag{2.4}$$

In the case of the Neumann problem, we recognize that we must be able to define the outward normal on the boundary to be able to solve this problem and this imposes a restriction on the boundary. This restriction is lifted in the variational set-up and so we see that the eigenvalues generated need not necessarily be the same. In the case of nice regions that obey what Reed and Simon [25] call the segment property, which is satisfied if the boundary is either a smooth curve or is bounded by finitely many smooth curves that meet at nonzero angles, the eigenvalues in these two formulations agree.

2.2 Symmetric Rearrangements

Now we move to the concept of symmetric rearrangements to define the framework to relate the isoperimetric inequality to the eigenvalue problem.

Definition. Let D be a Borel set of finite measure, i.e.

$$\mu(D) = \int_D \chi_D(r, \psi) f(r)^{n-1} dr d\omega(\psi) < \infty. \tag{2.5}$$

Note that μ is a Borel measure, and μ is absolutely continuous with respect to Lebesgue measure since $\frac{f(r)}{r}$ is a bounded function on bounded sets. Also, note that $r = |x|$ is the geodesic distance in our space.

Definition. We define D^* , the symmetric rearrangement of the set D , to be the open geodesic ball centered at the origin whose volume is that of D .

$$D^* = \{x : |x| < r\} \text{ with } A(\mathbb{S}^{n-1}) \int_0^r f(s)^{n-1} ds = |D|$$

This definition allows us to define the symmetric decreasing rearrangement, f^* , of a function f as follows:

Definition. Let $\chi_{D^*} = \chi_{D^*}$. If $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a Borel measurable function vanishing at infinity in the sense that $\mu(\{x : |f(x)| > t\})$ is finite for all $t > 0$, we define

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}}^*(x) dt \quad (2.6)$$

Note that f^* is nonnegative. Also, the level sets of f^* are simply the rearrangements of the level sets of $|f|$, i.e.

$$\{x : f^*(x) > t\} = \{x : |f(x)| > t\}^*$$

so

$$\mu(\{x : f^*(x) > t\}) = \mu(\{x : |f(x)| > t\}^*) = \mu(\{x : |f(x)| > t\})$$

for all $t > 0$.

If $f : [0, \infty) \rightarrow \mathbb{R}^+$ is a monotone decreasing function then $f^* = f$ since $\{x : |f(x)| > t\}^* = \{x : f(x) > t\}$.

We can see that $\|f\|_p = \|f^*\|_p$ by combining this with the layer cake representation given in Lieb-Loss [22]. Let ν be a measure on the Borel sets of $[0, \infty)$ such that

$$\Phi(t) := \nu([0, t])$$

is finite for every $t > 0$. Let (Ω, Σ, μ) be a measure space and f any nonnegative measurable function on Ω . Then

$$\int_\Omega \Phi(f(x)) \mu(dx) = \int_0^\infty \mu(\{x : f(x) > t\}) \nu(dt)$$

and by choosing $\nu(dt) = pt^{p-1} dt$ for $p > 0$ we have

$$\begin{aligned} \int_{\Omega} |f(x)|^p \mu(dx) &= p \int_0^{\infty} t^{p-1} \mu(\{x : |f(x)| > t\}) dt = \\ &= p \int_0^{\infty} t^{p-1} \mu(\{x : f^*(x) > t\}) dt = \int_{\Omega} (f^*)^p \mu(dx) \end{aligned}$$

So $\|f\|_p = \|f^*\|_p$.

This fact and the following rearrangement inequality of Riesz are the key steps of the first proof of the Faber-Krahn inequality for \mathbb{R}^n to show that the norm of the gradient decreases under symmetric rearrangement.

Theorem 10 (Riesz's rearrangement inequality). *Let f, g , and h be three nonnegative functions on \mathbb{R}^n . Then, with*

$$I(f, g, h) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(x-y)h(y) dx dy, \text{ we have}$$

$$I(f, g, h) \leq I(f^*, g^*, h^*),$$

with the understanding that $I(f^*, g^*, h^*) = \infty$ if $I(f, g, h) = \infty$.

Lemma 1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonnegative measurable function that vanishes at infinity, i.e. $V(\{x : |f(x)| > t\})$ is finite for all $t > 0$, and let f^* denote its symmetric decreasing rearrangement. Assume that ∇f , in the sense of distributions, is a function that satisfies $\|\nabla f\|_2 < \infty$. Then ∇f^* has the same property and*

$$\|\nabla f^*\|_2 \leq \|\nabla f\|_2.$$

Proof. This proof is taken from [22].

Step 1: First, we show that it suffices to prove the lemma for $f \in L^2(\mathbb{R}^n)$. Define

$$f_c(x) = \min[\max(f(x) - c, 0), \frac{1}{c}] \text{ for } c > 0.$$

Note that for

$$c > f(x) \text{ we have } f_c(x) = 0,$$

for

$$c < f(x) < \frac{1}{c} + c \text{ we have } f_c(x) = f(x) - c,$$

and for

$$\frac{1}{c} + c < f(x) \text{ we have } f_c(x) = \frac{1}{c}.$$

So $\nabla f_c = 0$ when $x \in \mathbb{R}^n$ is such that $f(x) \notin (c, \frac{1}{c} + c)$ and for $x \in \mathbb{R}^n$ such that $c < f(x) < \frac{1}{c} + c$, we have $\nabla f_c(x) = \nabla f(x)$.

Also, by definition of the rearrangement, $(f_c)^* = (f^*)_c$ and since f vanishes at infinity, $f_c \in L^2(\mathbb{R}^n)$. So by the monotone convergence theorem,

$$\lim_{c \rightarrow 0} \|\nabla f_c\|_2 = \|\nabla f\|_2.$$

Likewise,

$$\lim_{c \rightarrow 0} \|\nabla (f_c)^*\|_2 = \lim_{c \rightarrow 0} \|\nabla (f^*)_c\|_2 = \|\nabla f^*\|_2.$$

Step 2: Now, define

$$I^t(f) = t^{-1}[(f, f) - (f, e^{\Delta t} f)]$$

for $f \in \mathcal{H}^1(\mathbb{R}^n)$ where

$$e^{\Delta t} f = \int_{\mathbb{R}^n} e^{t\Delta}(x, y) f(y) dy = \int_{\mathbb{R}^n} e^{t\Delta}(x - y) f(y) dy$$

and the heat kernel is

$$e^{t\Delta}(x, y) = (4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}.$$

We would like to show that

$$\lim_{t \rightarrow 0} I^t(f) = \|\nabla f\|_2^2.$$

Let \hat{f} denote the Fourier transform of f defined as

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i(k, x)} f(x) dx$$

where

$$(k, x) := \sum_{i=1}^n k_i x_i.$$

So for $f \in \mathcal{L}^2$ we also have

$$\widehat{\nabla f}(k) = 2\pi i k \hat{f}(k).$$

Observe that

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\hat{f}(k)|^2 (1 + 4\pi^2 |k|^2) dk.$$

So by Plancherel's Theorem we have

$$I^t(f) = \frac{1}{t} \int_{\mathbb{R}^n} [1 - e^{-4\pi^2 |k|^2 t}] |\hat{f}(k)|^2 dk.$$

Note that $y^{-1}(1 - e^{-y})$ is a decreasing function for $y > 0$. So if we look at the following limit, we see

$$\lim_{t \rightarrow 0^+} \frac{1 - e^{-4\pi^2 |k|^2 t}}{t} = \lim_{t \rightarrow 0^+} 4\pi^2 |k|^2 e^{-4\pi^2 |k|^2 t} = 4\pi^2 |k|^2 = |2\pi k|^2$$

by L'Hospital's rule.

So since $[1 - e^{-4\pi^2 |k|^2 t}]$ converges monotonically to $|2\pi k|^2$, we see that $I^t(f)$ is uniformly bounded so the monotone convergence theorem gives

$$\lim_{t \rightarrow 0^+} I^t(f) = \int_{\mathbb{R}^n} |2\pi k|^2 |\hat{f}(k)|^2 dk = \|\widehat{\nabla f}\|_2^2 = \|\nabla f\|_2^2.$$

Step 3: Observe that

$$(f, e^{\Delta t} f) = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} e^{t\Delta}(x-y) f(y) dy dx$$

so by using Fubini's theorem, the Riesz rearrangement inequality, Theorem 10, and the fact that $(f, f) = (f^*, f^*)$ we see that

$$\begin{aligned} (f, e^{\Delta t} f) &\leq (f^*, e^{\Delta t} f^*) && \Rightarrow \\ (f^*, f^*) - (f^*, e^{\Delta t} f^*) &\leq (f, f) - (f, e^{\Delta t} f) && \Rightarrow \\ I^t(f^*) &\leq I^t(f) \end{aligned}$$

and we also have that $I^t(f^*) \rightarrow \|\nabla f^*\|_2^2$ as $t \rightarrow 0^+$. Hence

$$\|\nabla f^*\|_2 \leq \|\nabla f\|_2.$$

□

2.3 Faber-Krahn Inequality

One of the earliest isoperimetric inequalities for an eigenvalue is the Faber-Krahn inequality which bounds the first eigenvalue of the Dirichlet Laplacian. This inequality was conjectured by Rayleigh [24] in 1877 and was later proved (independently) by Faber [19] and Krahn [21] in the 1920's. We state the result and give its proof in this section.

Theorem 11 (Faber-Krahn). *Let Ω be a bounded domain in \mathbb{R}^n , and let Ω^* be the open ball in \mathbb{R}^n satisfying $V(\Omega) = V(\Omega^*)$, where V denotes n -dimensional Lebesgue measure. Then*

$$\lambda_0(\Omega^*) \leq \lambda_0(\Omega).$$

If Ω also has C^∞ boundary, then one has equality if and only if Ω is isometric to Ω^ .*

Proof of theorem. Here we follow the argument given in Ashbaugh's paper [1]. Let u_0 be the first real eigenfunction for Ω where u_0 satisfies

$$\Delta u_0 + \lambda_0(\Omega)u_0 = 0 \tag{2.7}$$

and normalize it so that

$$\int_{\Omega} u_0^2 = 1.$$

Multiplying (2.7) by u_0 and integrating gives us

$$\lambda_0(\Omega) = \int_{\Omega} u_0 \lambda_0(\Omega) u_0 = \int_{\Omega} -u_0 \Delta u_0 = \int_{\Omega} |\nabla u_0|^2$$

where the last equality comes from Green's identity and the fact that we have Dirichlet boundary conditions. So we get the following, where the last inequality comes from the Rayleigh-Ritz inequality and the fact that $u_0 \in \mathcal{H}_0^1(\Omega)$ implies that $u_0^* \in \mathcal{H}_0^1(\Omega^*)$. Thus u_0^* is an admissible test function.

$$\lambda_0(\Omega) = \int_{\Omega} |\nabla u_0|^2 \geq \int_{\Omega^*} |\nabla u_0^*|^2 \geq \lambda_0(\Omega^*).$$

□

We also have the Faber-Krahn inequality for hyperbolic space, which is stated as follows.

Theorem 12 (Faber-Krahn for \mathbb{H}^n). *Let $\Omega \subset \mathbb{H}^n$ be a bounded domain with smooth boundary and $\Omega^* \subset \mathbb{H}^n$ an open geodesic ball of the same measure. Denote by $\lambda_0(\Omega)$ and $\lambda_0(\Omega^*)$ the lowest eigenvalue of the Dirichlet-Laplace operator on the respective domain. Then*

$$\lambda_0(\Omega^*) \leq \lambda_0(\Omega) \tag{2.8}$$

with equality only if Ω itself is a geodesic ball.

For the proof of this theorem, we refer to Chavel's book, [12], where an analog of this result for general Riemannian manifolds is presented. We also state it here as it would apply to the space \mathbb{F}^n .

Theorem 13 (Faber-Krahn for \mathbb{F}^n). *Let $\Omega \subset \mathbb{F}^n$ be a bounded domain with smooth boundary and $\Omega^* \subset \mathbb{F}^n$ an open geodesic ball of the same measure. Denote by $\lambda_0(\Omega)$ and $\lambda_0(\Omega^*)$ the lowest eigenvalue of the Dirichlet-Laplace operator on the respective domain. Then*

$$\lambda_0(\Omega^*) \leq \lambda_0(\Omega) \tag{2.9}$$

with equality only if Ω itself is a geodesic ball.

2.4 The Gap Inequality for the Eigenvalues of the Dirichlet Laplacian

Here we will define the Dirichlet Laplacian $-\Delta$ on some bounded open set Ω , where Ω will be contained in either \mathbb{R}^n , \mathbb{H}^n , or \mathbb{F}^n depending on the situation considered. The operator $-\Delta$ will be defined by the quadratic form

$$D[u] = (\nabla u, \nabla u), \quad (2.10)$$

whose domain is the completion of $C_0^\infty(\Omega)$ with respect to the norm induced by $D[u]^{\frac{1}{2}}$. Here we define ∇ on a space having the metric (g_{ij}) and local coordinates x_j as follows,

$$(\nabla u)_k = \sum_l (g^{kl} \frac{\partial u}{\partial x_l}). \quad (2.11)$$

Here g^{ij} is the $(i, j)^{th}$ element of g^{-1} . The operator $-\Delta$ is a positive self-adjoint operator having positive eigenvalues with finite multiplicity. We will use λ_i to denote its i -th eigenvalue, where the distinct eigenvalues are labeled in increasing order, $\lambda_i < \lambda_{i+1}$. Also we use $\langle \cdot, \cdot \rangle$ to denote the usual inner product on $L^2(\Omega)$.

Thus we have the usual Rayleigh-Ritz characterization of the eigenvalues λ_i of $-\Delta$ as in [25]: if $u \in L^2(\Omega)$ is some function in the domain of D and if u is orthogonal to the first $k - 1$ eigenfunctions of $-\Delta$, then

$$\lambda_k \leq \frac{D[u]}{\langle u, u \rangle}. \quad (2.12)$$

We can then use this inequality to derive the following gap inequality for the first two eigenvalues. First, we are going to use a function P so that $Pu_0 \perp u_0$ for our trial

function in the Rayleigh-Ritz inequality, which gives

$$\begin{aligned}
\lambda_1(\Omega) &\leq \frac{\int_{\Omega} \langle \nabla P u_0, \nabla P u_0 \rangle dV}{\int_{\Omega} P^2 u_0^2 dV} & (2.13) \\
&= \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle + 2P u_0 \langle \nabla P, \nabla u_0 \rangle + P^2 \langle \nabla u_0, \nabla u_0 \rangle dV}{\int_{\Omega} P^2 u_0^2 dV} \\
&= \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle dV + \int_{\Omega} \langle \nabla (P^2 u_0), \nabla u_0 \rangle dV}{\int_{\Omega} P^2 u_0^2 dV} \\
&= \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle dV + \int_{\Omega} P^2 u_0 (-\Delta u_0) dV}{\int_{\Omega} P^2 u_0^2 dV} \\
&= \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle dV + \lambda_0(\Omega) \int_{\Omega} P^2 u_0^2 dV}{\int_{\Omega} P^2 u_0^2 dV}
\end{aligned}$$

where the next to last step follows from integration by parts and the last step from our initial set-up of the Dirichlet problem. Hence we obtain the gap inequality

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle dV}{\int_{\Omega} P^2 u_0^2 dV}. \quad (2.14)$$

Chapter 3 Payne-Pólya-Weinberger Inequality in Euclidean space

In this chapter we present the proof given in [2] for the PPW conjecture in \mathbb{R}^n .

Theorem 14. *The ratio of the first two Dirichlet eigenvalues of the Laplacian on a domain $\Omega \subset \mathbb{R}^n$ satisfies*

$$\frac{\lambda_1}{\lambda_0} \leq \frac{\lambda_1}{\lambda_0} \Big|_{\Omega=\text{n-dimensional ball}} = \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} \quad (3.1)$$

Moreover, equality is obtained if and only if Ω is a disk.

Proof. Let $\lambda_0, \lambda_1, \lambda_2, \dots$ denote the eigenvalues of problem (1.1) and u_0, u_1, u_2, \dots denote an orthonormal sequence of corresponding eigenfunctions, assumed to be positive. Also recall that $j_{p,k}$ denotes the k^{th} positive zero of the Bessel function $J_p(x)$. We want to make use of the Rayleigh-Ritz inequality for λ_1 , so we take a set of n trial functions $P = P_i$ for $i = 1, \dots, n$ defined by

$$P_i(x) = g(r) \frac{x_i}{r}, \quad i = 1, \dots, n, \quad (3.2)$$

where $g(r)$ is a nonnegative and nontrivial function of the radial variable $r = |x|$ and the x_i 's are the standard cartesian variables. The function g will be chosen to be continuous, differentiable, and bounded on $(0, \infty)$. So we have that

$$\lambda_1 - \lambda_0 \leq \frac{\int_{\Omega} |\nabla P|^2 u_0^2 dx}{\int_{\Omega} P^2 u_0^2 dx} \quad (3.3)$$

provided that

$$\int_{\Omega} P u_0^2 dx = 0 \quad \text{and} \quad P \neq 0, \quad (3.4)$$

as is shown in section 2.4. Here dx denotes the standard Lebesgue measure in \mathbb{R}^n .

So now we apply Weinberger's center of mass argument from [27] to see that we may choose our origin so that

$$\int_{\Omega} P_i(x) u_0^2 dx = 0 \quad \text{for } i = 1, \dots, n. \quad (3.5)$$

The proof of this theorem is given at the end of this chapter. Thus, rewriting (3.1) as

$$(\lambda_1 - \lambda_0) \int_{\Omega} P_i^2 u_0^2 dx \leq \int_{\Omega} |\nabla P_i|^2 u_0^2 dx \quad (3.6)$$

and summing on i for $i = 1, \dots, n$ gives us that

$$\lambda_1 - \lambda_0 \leq \frac{\int_{\Omega} (\sum_{i=1}^n |\nabla P_i|^2) u_0^2 dx}{\int_{\Omega} (\sum_{i=1}^n P_i^2) u_0^2 dx}. \quad (3.7)$$

So from (3.2) it is easy to see that

$$\sum_{i=1}^n P_i^2(x) = g(r)^2. \quad (3.8)$$

Also, we can compute that

$$\nabla P_i = \sum_{j=1}^n e_j \left[\left(\frac{g(r)}{r} \right)' \frac{x_i x_j}{r} + \delta_{ij} \frac{g(r)}{r} \right] \quad (3.9)$$

which gives

$$|\nabla P_i|^2 = \left(\frac{g'}{r} \right)^2 x_i^2 + \frac{g^2}{r^2} - \frac{g^2}{r^4} x_i^2 \quad (3.10)$$

and then finally

$$\sum_{i=1}^n |\nabla P_i|^2 = (g')^2 + (n-1) \frac{g^2}{r^2}. \quad (3.11)$$

Thus our basic gap inequality becomes

$$\lambda_1 - \lambda_0 \leq \frac{\int_{\Omega} \left[(g')^2 + (n-1) \frac{g^2}{r^2} \right] u_0^2 dx}{\int_{\Omega} g(r)^2 u_0^2 dx}. \quad (3.12)$$

So now we want to pick a trial function $g(r)$ as a ratio of Bessel functions so that (3.12) is an equality if Ω is an n -dimensional ball. Hence we take

$$g(r) = \tilde{g}(\gamma r), \quad (3.13)$$

where

$$\tilde{g}(x) \equiv \begin{cases} \frac{J_{n/2}(\beta x)}{J_{n/2-1}(\alpha x)} & \text{for } 0 \leq x < 1, \\ \tilde{g}(1) \equiv \lim_{x \rightarrow 1^-} \tilde{g}(x) & \text{for } x \geq 1, \end{cases} \quad (3.14)$$

with $\alpha = j_{n/2-1,1}$, $\beta = j_{n/2,1}$ and $\gamma = \sqrt{\lambda_0}/\alpha$. Substituting this into (3.12) gives us that

$$\lambda_1 - \lambda_0 \leq \frac{\lambda_0 \int_{\Omega} B(\gamma r) u_0^2 dx}{\alpha^2 \int_{\Omega} \tilde{g}(\gamma r)^2 u_0^2 dx}, \quad (3.15)$$

where

$$B(x) \equiv \tilde{g}'(x)^2 + (n-1) \frac{\tilde{g}(x)^2}{x^2}. \quad (3.16)$$

It is shown in [2] that $\tilde{g}(x)$ is increasing and $B(x)$ is decreasing, since both $\tilde{g}'(x)$ and $\tilde{g}(x)/x$ are positive and decreasing. It should be noted that this is the difficult step of the proof, but is omitted here for brevity. Now let f^* denote the spherical decreasing rearrangement of f and f_* denote the spherical increasing arrangement and we obtain

$$\int_{\Omega} B(\gamma r) u_0^2 dx \leq \int_{\Omega^*} B(\gamma r)^* u_0^{*2} dx \leq \int_{\Omega^*} B(\gamma r) u_0^{*2} dx \leq \int_{S_1} B(\gamma r) z^2 dx \quad (3.17)$$

and

$$\int_{\Omega} \tilde{g}(\gamma r)^2 u_0^2 dx \geq \int_{\Omega^*} \tilde{g}(\gamma r)_*^2 u_0^{*2} dx \geq \int_{\Omega^*} \tilde{g}(\gamma r)^2 u_0^{*2} dx \geq \int_{S_1} \tilde{g}(\gamma r)^2 z^2 dx. \quad (3.18)$$

Here S_1 is the n -dimensional ball so that

$$\begin{cases} -\Delta z = \lambda z & \text{on } S_1, \\ z = 0 & \text{on } \partial S_1, \end{cases} \quad (3.19)$$

has λ_0 as its first eigenvalue. Then we see that

$$z = cr^{1-n/2} J_{n/2-1}(\sqrt{\lambda_0}r), \quad (3.20)$$

where c is a nonzero constant. Chiti's comparison result [14] gives us that if c is chosen so that

$$\int_{\Omega} u_0^2 dx = \int_{\Omega^*} u_0^{*2} dx = \int_{S_1} z^2 dx, \quad (3.21)$$

then there exists a point $r_1 \in (0, 1/\gamma)$ such that

$$\begin{cases} u_0^*(r) \leq z(r) & \text{for } 0 \leq r \leq r_1, \\ u_0^*(r) \geq z(r) & \text{for } r_1 \leq r \leq 1/\gamma. \end{cases} \quad (3.22)$$

The last of the inequalities in (3.17) and (3.18) follows from the fact that

$$\int_{\Omega^*} f(r)u_0^{*2} dx \geq \int_{S_1} f(r)z^2 dx \quad \text{if } f \text{ is increasing} \quad (3.23)$$

and

$$\int_{\Omega^*} f(r)u_0^{*2} dx \leq \int_{S_1} f(r)z^2 dx \quad \text{if } f \text{ is decreasing.} \quad (3.24)$$

We prove (3.24) as follows: Let f be an increasing function and let r^* be the radius of Ω^* . Then we have

$$\begin{aligned} \int_{S_1} f(r)z^2 dx - \int_{\Omega^*} f(r)u_0^{*2} dx &= nC_n \left[\int_0^{r_1} f(r)(z^2 - u_0^{*2})r^{n-1} dr \right. \\ &\quad \left. + \int_{r_1}^{1/\gamma} f(r)(z^2 - u_0^{*2})r^{n-1} dr \right. \\ &\quad \left. - \int_{1/\gamma}^{r^*} f(r)u_0^{*2}r^{n-1} dr \right] \\ &\leq nC_n \left[f(r_1) \int_0^{r_1} (z^2 - u_0^{*2})r^{n-1} dr \right. \\ &\quad \left. + f(r_1) \int_{r_1}^{1/\gamma} (z^2 - u_0^{*2})r^{n-1} dr \right. \\ &\quad \left. - f(r_1) \int_{1/\gamma}^{r^*} u_0^{*2}r^{n-1} dr \right] \\ &= f(r_1) \left[\int_{S_1} z^2 dx - \int_{\Omega^*} u_0^{*2} dx \right] \\ &= 0 \end{aligned}$$

by equation (3.21). Here C_n is the volume of the unit ball in n dimensions; nC_n represents its surface area. Also, an analogous result proves the reverse inequality for f decreasing.

Now that we have shown (3.17) and (3.18), we combine them with (3.15) to obtain

$$\begin{aligned} \lambda_1 - \lambda_0 &\leq \frac{\lambda_0 \int_{S_1} B(\gamma r)z^2 dx}{\alpha^2 \int_{S_1} \tilde{g}(\gamma r)^2 z^2 dx} = \frac{\lambda_0 \int_0^1 B(r)J_{n/2-1}^2(\alpha r)r dr}{\alpha^2 \int_0^1 \tilde{g}(r)^2 J_{n/2-1}^2(\alpha r)r dr} \\ &= \frac{\lambda_0}{\alpha^2} [(\lambda_1 - \lambda_0) \text{for the ball of radius } 1] \\ &= \frac{\lambda_0}{\alpha^2} (\beta^2 - \alpha^2); \end{aligned} \quad (3.25)$$

which immediately gives

$$\lambda_1/\lambda_0 \leq \beta^2/\alpha^2 = j_{n/2,1}^2/j_{n/2-1,1}^2. \quad (3.26)$$

The identification of $\lambda_1 - \lambda_0$ for the ball of radius 1 follows by observing that (3.12) is an equality if Ω is a ball of radius 1 and $g(r)$ is chosen to be $\tilde{g}(r) = J_{n/2}(\beta r)/J_{n/2-1}(\alpha r)$ for $0 \leq r \leq 1$.

Finally, we look at when we have equality. It is easy to see that equality is obtained when Ω is an n -dimensional ball. To see that this is the only case for which equality holds we need only note that the last two inequalities in (3.17) and (3.18) are strict unless Ω is a ball. \square

3.1 Center of Mass Theorem for \mathbb{R}^n

Here we present the proof of the center of mass theorem that allows us to choose our coordinate system to satisfy the condition needed for the proof of the PPW conjecture. This can be found in [9].

Lemma 2 (Center of Mass Theorem). *The origin may be chosen in such a way so that*

$$\int_{\Omega} \frac{G(r)x_i}{r} dV = 0, \quad (3.27)$$

where $G(r)$ is a continuous, differentiable, positive function on $[0, \infty)$.

Proof. Consider the n-vector

$$\vec{v}(\vec{a}) = \int_{\Omega+\vec{a}} \frac{G(r)\vec{x}}{r} dV \quad (3.28)$$

where

$$\vec{v}_i(\vec{a}) = \int_{\Omega+\vec{a}} \frac{G(r)\vec{x}_i}{r} dV \quad (3.29)$$

as a function of the origin of the x_i coordinates and note that it is a continuous vector field. Take $B \subset \mathbb{R}^n$ to be the ball containing Ω such that the origin of the

x_i coordinates is the center of B . Now take $\vec{a} \in \partial B$. It is a known theorem in topology that any nonvanishing vector field on a ball must point directly inward at some point. See Theorem 29 for a proof of this in the appendix. Hence, if we can show that $\vec{a} \cdot \vec{v}(\vec{a}) > 0$, then this vector field is always pointing outwards. Thus, it must vanish at some point $\vec{a}_0 \in B$. We will take this point to be our origin.

To show this, we calculate

$$\begin{aligned} \vec{a} \cdot \vec{v}(\vec{a}) &= \int_{\Omega + \vec{a}} \frac{G(r)\vec{x} \cdot \vec{a}}{r} dV \\ &= \int_{\Omega} \frac{G(|x + a|)(\vec{x} + \vec{a}) \cdot \vec{a}}{|x + a|} dV. \end{aligned} \tag{3.30}$$

Since $G > 0$, we must now show that

$$\frac{(\vec{x} + \vec{a}) \cdot \vec{a}}{|x + a|} > 0. \tag{3.31}$$

To do this, we recognize

$$\frac{(\vec{x} + \vec{a}) \cdot \vec{a}}{|x + a|} = \frac{|a|^2 + \vec{a} \cdot \vec{x}}{|x + a|} \tag{3.32}$$

$$\geq \frac{|a|^2 - |a||x|}{|x + a|} \tag{3.33}$$

$$= \frac{|a|(|a| - |x|)}{|x + a|} > 0, \tag{3.34}$$

where (3.33) follows from the definition of the dot product and (3.34) follows from the fact that $\vec{x} \in \Omega$, $\vec{a} \in \partial B$, and $\Omega \subset B$. Hence our vector field here would always be outward pointing and must vanish at some point.

Remark. Observe also from this that if Ω is already a ball, then the origin must be located in the center of the ball, for no matter the radius of the ball, the above argument would show that the vector field would always point outward. Hence we could take smaller and smaller concentric balls to see that the vanishing point must be the center of the ball. □

Chapter 4 Payne-Pólya-Weinberger Inequality in Hyperbolic Space

In this section we will discuss the result obtained by Benguria and Linde in [8] which is the extension of the Payne-Pólya-Weinberger Inequality in hyperbolic space. First, we give some preliminaries of the space in which we are working, then move on to the proof of the main result.

4.1 The Geometry of Hyperbolic Space

We define hyperbolic space \mathbb{H}^n as the unique simply connected n -dimensional manifold of constant negative sectional curvature, which is normalized to -1 . The ball model of this space is given by

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

with the Riemannian metric

$$ds^2 = \frac{4|dx|^2}{(1 - |x|^2)^2}.$$

If we define spherical coordinates about $x = 0$ by

$$x = |x|\xi \quad |x| = \tanh(r/2),$$

where $|x| \in [0, 1)$, $r \in [0, \infty)$, $\xi \in \mathbb{S}^{n-1}$, then we obtain

$$dx = \frac{\operatorname{sech}^2(\frac{r}{2})}{2}(dr)\xi + \tanh(r/2)d\xi$$

and

$$|dx|^2 = \frac{\operatorname{sech}^4(\frac{r}{2})}{4}(dr)^2 + \tanh^2(r/2)|d\xi|^2.$$

So this gives us

$$ds^2 = (dr)^2 + \sinh^2(r)|d\xi|^2$$

for our metric on $[0, \infty) \times \mathbb{S}^{n-1}$ and the Riemannian measure is

$$dV = \rho^{n-1} \sinh^{n-1}(r) dr d\xi,$$

where $d\xi$ is the measure on \mathbb{S}^n .

Another way to view this space is through the hyperboloid model. This model is useful for our center of mass argument in hyperbolic space. Here \mathbb{H}^n is realized as a subset of \mathbb{R}^{n+1} determined by the indefinite quadratic form $q : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R} \cup \{0\}$ defined as

$$q(x, y) \equiv x_{n+1}y_{n+1} - \sum_{i=1}^n x_i y_i.$$

We wish to find the group of linear transformations that preserve q . Denote the matrix representation of our metric by g_M . Then g_M is a diagonal matrix with $(1, \dots, 1, -1)$ as the diagonal entries, so that this group is the group of transformations that satisfy $R^T g_M R = I$. We denote this group by $O(n, 1)$. Now we consider the surface

$$Q_n^+ \equiv \{x \in \mathbb{R}^{n+1} | q(x, x) = 1 \text{ and } x_{n+1} > 0\}$$

which is the positive sheet of the two-sheeted hyperboloid in \mathbb{R}^{n+1} determined by $q(x, x) = 1$. The metric on Q_n^+ is given by

$$ds^2 = \sum_{i=1}^n dx_i^2 - dx_{n+1}^2.$$

4.2 Eigenvalue properties

The standard separation of variables for the Laplacian gives us that the eigenvalues and eigenfunctions of $-\Delta$ on a geodesic ball in \mathbb{H}^n of radius \tilde{r} are determined by the differential equation

$$-z''(r) - \frac{(n-1)}{\tanh r} z'(r) + \frac{l(l+n-2)}{\sinh^2 r} z(r) = \lambda z(r), \quad (4.1)$$

where $l = 0, 1, 2, \dots$, with the boundary conditions $z'(0) = 0$ (for $l = 0$) or $z(r) \sim r^l$ as $r \downarrow 0$ (for $l > 0$) and $z(\tilde{r}) = 0$. Here we use the convention that $z_l(r) > 0$ for $r \downarrow 0$.

Let h_l denote the operator applied to z on the left hand side of (4.1) with the given boundary conditions. Then it is easy to see that $h_{l'} > h_l$ in the sense of quadratic forms if $l' > l$. Thus we know that the lowest eigenvalue on the geodesic ball is given by $\lambda_0(h_0)$. Next we show that the second eigenvalue must be $\lambda_0(h_1)$.

Lemma 3. *The first eigenvalue of the Dirichlet Laplacian on a geodesic ball in \mathbb{H}^n is the first eigenvalue of (4.1) with $l = 0$, while the second eigenvalue on the geodesic ball is the first eigenvalue of (4.1) with $l = 1$. The second eigenvalue is n -fold degenerate.*

Proof. Assume that z_l solves (4.1) for some fixed λ . Then it is not hard to show that if you replace l with $l + 1$, then

$$-z_l' - l \coth r z_l$$

satisfies (4.1) and for l replaced by $l - 1$

$$z_l' + (l + n - 2) \coth r z_l$$

satisfies (4.1). So then we obtain that

$$z_{l+1} = -z_l' - l \coth r z_l \tag{4.2}$$

$$z_{l-1} = z_l' + (l + n - 2) \coth r z_l \tag{4.3}$$

Setting $l = 0$ in (4.2) we get

$$z_1 = -z_0'. \tag{4.4}$$

Letting $l = 1$ in (4.3) and multiplying both sides by $\sinh^{n-1} r$ we obtain

$$\sinh^{n-1} r z_0 = (\sinh^{n-1} r z_1)'. \tag{4.5}$$

Rolle's Theorem tells us that between any two zeros of z_0 there is a zero of z_0' . Also, since we have (4.4) it is clear z_1 will also have a zero between any two zeros of z_0 . Similarly, between any two zeros of $\sinh^{n-1} r z_1$ there is a zero of its derivative. From (4.5) we also get that z_0 will have a zero here. Thus for fixed $\lambda > 0$ the zeros of z_0

and z_1 on $[0, \infty)$ interlace. Now look at z_0 and z_1 for $\lambda = \lambda_0(h_1)$. Then we know from the boundary conditions that the first positive zero of z_1 is equal to the radius \tilde{r} of the geodesic ball. From what we have just shown we have that z_0 has exactly one zero in $[0, \tilde{r}]$ and the second zero of z_0 is greater than \tilde{r} . Combining this with the fact that the positive zeros of any z_l are decreasing functions of λ shows that $\lambda_1(h_0) > \lambda_0(h_1)$. Hence we have that the second eigenvalue of $-\Delta$ on the geodesic ball is given by $\lambda_0(h_1)$.

The degeneracy of the second eigenvalue on the geodesic ball follows from the separation of variables. □

4.3 The Payne-Pólya-Weinberger Inequality

Theorem 15. *Let $\Omega \subset \mathbb{H}^n$ be an open bounded domain in the hyperbolic space of constant negative curvature and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . If $S_1 \subset \mathbb{H}^n$ is a geodesic ball such that $\lambda_0(\Omega) = \lambda_0(S_1)$ then*

$$\lambda_1(\Omega) \leq \lambda_1(S_1) \tag{4.6}$$

with equality if and only if Ω is a geodesic ball.

Proof. We proceed with the proof of the main theorem. Let u_0 be the first eigenfunction of $-\Delta$ on Ω . Take S_1 to be a geodesic ball such that $\lambda_0(S_1) = \lambda_0(\Omega)$ and take z_0 to be the corresponding eigenfunction. From Lemma 3, the second eigenvalue $\lambda_1(S_1)$ is n -fold degenerate and the corresponding eigenspace is spanned by the function $z_1(r)\chi_k$, where z_1 is the solution of (1.9) for $l = 1$ and χ_k is the k -th spherical coordinate.

Let $P \neq 0$ be a function on Ω such that Pu_0 is in the domain of the quadratic form D where

$$D[u] = (\nabla u, \nabla u) \tag{4.7}$$

and

$$\int_{\Omega} P u_0^2 dV = 0. \quad (4.8)$$

By the Rayleigh-Ritz theorem we have the estimate

$$\begin{aligned} \lambda_1(\Omega) - \lambda_0(\Omega) &= \frac{D[Pu_0]}{(Pu_0, Pu_0)} \\ &= (Pu_0, Pu_0)^{-1} \int_{\Omega} (\langle \nabla Pu_0, \nabla Pu_0 \rangle - \lambda_0 P^2 u_0^2) dV \end{aligned}$$

Using (4.7) and the argument in section 2.4 we can find the gap inequality

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle dV}{\int_{\Omega} P^2 u_0^2 dV}. \quad (4.9)$$

We choose the following set of n functions,

$$P_i(r, \vec{\chi}) := g(r) \chi_i \quad (4.10)$$

where χ_i is the i -th spherical coordinate and with

$$g(r) = \begin{cases} \frac{z_1(r)}{z_0(r)} & r \in [0, \tilde{r}), \\ \lim_{r \uparrow \tilde{r}} g(r) & r \geq \tilde{r}. \end{cases} \quad (4.11)$$

Recall the \tilde{r} is the geodesic radius of S_1 , and that by convention z_0 and z_1 are positive.

With our choice of P_i and the center of mass theorem, we may always shift Ω such that (4.8) is satisfied.

We may then calculate, following [12], that

$$\sum_{i=1}^n P_i^2(r, \vec{\chi}) = g^2(r), \quad (4.12)$$

$$\sum_{i=1}^n \langle \nabla P_i, \nabla P_i \rangle = \left(\frac{\partial g}{\partial r} \right)^2 + (n-1) \sinh^{-2}(r) g^2(r). \quad (4.13)$$

Then multiply (4.9) by $\int_{\Omega} P^2 u_0^2 dV$ and then sum over $i = 1, \dots, n$ to obtain

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2(r, \vec{\chi}) B(r) dV}{\int_{\Omega} u_0^2(r, \vec{\chi}) g^2(r) dV} \quad (4.14)$$

where

$$B(r) = g'(r)^2 + (n-1) \sinh^{-2}(r) g^2(r). \quad (4.15)$$

To finish the proof, we need the following inequalities to hold:

$$\begin{aligned}
\int_{\Omega} u_0^2(r, \vec{\chi}) B(r) dV &\leq \int_{\Omega^*} u_0^*(r)^2 B^*(r) dV & (4.16) \\
&\leq \int_{\Omega^*} u_0^*(r)^2 B(r) dV \\
&\leq \int_{S_1} z_0^2(r) B(r) dV
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} u_0^2(r, \vec{\chi}) g(r)^2 dV &\geq \int_{\Omega^*} u_0^*(r)^2 g_*(r)^2 dV & (4.17) \\
&\geq \int_{\Omega^*} u_0^*(r)^2 g(r)^2 dV \\
&\geq \int_{S_1} z_0^2(r) g(r)^2 dV.
\end{aligned}$$

Here we assume that z_0 is normalized such that $\int_{\Omega} u_0^2 dV = \int_{S_1} z_0^2 dV$. Also, f^* is used to denote the spherical decreasing rearrangement of f , and f_* the spherical increasing rearrangement. In each of (4.16) and (4.17), the first inequality follows from the properties of rearrangements. The second inequality follows from the monotonicity properties of g and B , which will be proven below. Finally, the third inequality follows from a modified version of Chiti's comparison result and also from the monotonicity properties of g and B . This will also be proven later. Finally, from (4.14), (4.16), and (4.17) we have

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{S_1} z_0^2(r) B(r) dV}{\int_{S_1} z_0^2(r) g^2(r) dV} = \lambda_1(S_1) - \lambda_0(S_1). \quad (4.18)$$

Since we have chosen $\lambda_0(\Omega) = \lambda_0(S_1)$ we have that

$$\lambda_1(\Omega) \leq \lambda_1(S_1), \quad (4.19)$$

which proves our theorem. □

4.4 The Center of Mass Argument in Hyperbolic Space

Recall that we need a center of mass argument to aid in selecting appropriate test functions for the Rayleigh-Ritz variational inequality, so here we have the version of

this argument for \mathbb{H}^n . First let's take \mathbb{H}^n to be hyperbolic space in the blown-up ball model and $\tilde{\mathbb{H}}^n$ to be hyperbolic space in the hyperboloid model. Then

$$I : \mathbb{H}^n \rightarrow \tilde{\mathbb{H}}^n, |x| = \tanh(r/2)\xi \rightarrow y = ((\sinh r)\xi, \cosh r)$$

is an isometry between the two spaces.

Now observe that for the space $\tilde{\mathbb{H}}^n$ with $y = ((\sinh r)\xi, \cosh r)$ we have that

$$\begin{aligned} q(y, y) &\equiv y_{n+1}y_{n+1} - \sum_{i=1}^n y_i y_i \\ &= \cosh^2(r) - \sinh^2(r)(\xi_1^2 + \xi_2^2 + \dots + \xi_n^2) = 1, \end{aligned}$$

hence this is a valid quadratic form for our space.

Also, each Lorentz transformation in \mathbb{R}^{n+1} induces an isometry of $\tilde{\mathbb{H}}^n$ onto itself. This group of transformations has the transitivity property, so for any two points $p_1, p_2 \in \tilde{\mathbb{H}}^n$, there exists a Lorentz transformation that maps p_1 on p_2 . So then we have that $I^{-1}RI$ is an isometry on \mathbb{H}^n . So now we are ready to state and prove our theorem.

Theorem 16. *Let Ω be a bounded domain in \mathbb{H}^n and $\xi \in \mathbb{S}^{n-1}$. Take $h(r)$ to be a positive continuous function on $[0, \infty)$ and $P_i(\xi, r) = \xi_i h(r)$. Then one can shift Ω such that*

$$\int_{\Omega} P_i(r, \xi) u_0^2(x) dV = 0 \quad \text{for all } i = 1, \dots, n. \quad (4.20)$$

Proof. Let $\tilde{\Omega} = I(\Omega)$ and $\tilde{P}_i(y) = P_i(I^{-1}(y))$. Then

$$\int_{\Omega} P_i(x) dV = \int_{\tilde{\Omega}} \tilde{P}_i(y) d\tilde{V}. \quad (4.21)$$

Because it doesn't matter whether we shift Ω or P_i (changing variables shows these are equivalent), we need only to show that there is some Lorentz transformation R such that

$$\int_{\tilde{\Omega}} \tilde{P}_i(Ry) d\tilde{V} = 0. \quad (4.22)$$

Take $\Theta(z, y)$ to be the $(n + 1)$ coordinate of y after a Lorentz transformation that maps z to $e = (0, \dots, 0, 1)$ and then define the vector field

$$v(z) = \int_{\tilde{\Omega}} \frac{y}{\sinh \Theta(z, y)} h(\Theta(z, y)) d\tilde{V},$$

where y is the variable of integration. Also, let

$$\Pi y = (y_1, \dots, y_n) \quad \text{for } y \in \tilde{\mathbb{H}}^n. \quad (4.23)$$

With this definition we see that

$$I^{-1}(y) = (\cosh^{-1}(y_{n+1}), \frac{\Pi y}{\sinh r}). \quad (4.24)$$

Now we assume that there exists some $z_0 \in \tilde{\mathbb{H}}^n$ and $\alpha \in \mathbb{R}$ such that

$$v(z_0) = \alpha z_0. \quad (4.25)$$

Under this assumption we choose R to be a Lorentz transformation that maps z_0 to $(0, \dots, 0, 1)$. Then the r -coordinate of Ry is $\Theta(z, y)$ and the ξ -coordinate is $\Pi Ry / \sinh \Theta(z_0, y)$, such that

$$\begin{aligned} \int_{\tilde{\Omega}} \tilde{P}_i(Ry) d\tilde{V} &= \int_{\tilde{\Omega}} \frac{(\Pi Ry)_i}{\sinh \Theta(z_0, y)} h(\Theta(z_0, y)) d\tilde{V} \\ &= (\Pi Rv(z_0))_i \\ &= \alpha (\Pi Rz_0)_i = 0 \end{aligned}$$

So now we must show that such a z_0 exists. Note that the projection Π has a well defined inverse

$$\Pi^{-1}\xi = (\xi_1, \dots, \xi_n, \sqrt{1 + \xi_1^2 + \dots + \xi_n^2}) \quad \text{for } \xi \in \mathbb{R}^n. \quad (4.26)$$

Also $\Pi\tilde{\Omega} \subset \mathbb{R}^n$ is a bounded domain since Ω is bounded so we can find a ball $B_R \subset \mathbb{R}^n$, that is centered at the origin of Euclidean radius R , such that $\Pi\tilde{\Omega}$ is contained in B_R .

On B_R we define the vector field $w : B_R \rightarrow \mathbb{R}^n$ with

$$w(x) = \Pi \left(v(\Pi^{-1}x) - \frac{[v(\Pi^{-1}x)]_{n+1}}{[\Pi^{-1}x]_{n+1}} \Pi^{-1}x \right). \quad (4.27)$$

From our definition of Π^{-1} we see that the set $\Pi^{-1}B_R$ and the origin of \mathbb{R}^{n+1} span the cone

$$C = \left\{ y \in \mathbb{R}^{n+1} \mid y_{n+1} \geq 0 \text{ and } \frac{y_1^2 + \dots + y_n^2}{y_{n+1}^2} \leq \frac{R^2}{R^2 + 1} \right\}$$

and $\tilde{\Omega}$ is contained in C . Therefore, since $v(z)$ is an integral over vectors in $\tilde{\Omega}$ with positive coefficients it also lies in this cone for every $z \in \tilde{\mathbb{H}}^n$. From this we have the estimate

$$\frac{\|\Pi v(z)\|}{[v(z)]_{n+1}} \leq \frac{R}{\sqrt{R^2 + 1}}. \quad (4.28)$$

Now if we apply the Cauchy-Schwartz inequality and the above inequality we obtain for any $x \in \partial B_R$ the following inequality,

$$\begin{aligned} w(x) \cdot x &= \Pi v(\Pi^{-1}x) \cdot x - \frac{[v(\Pi^{-1}x)]_{n+1}}{[\Pi^{-1}x]_{n+1}} \|x\|^2 \\ &\leq \|\Pi v(\Pi^{-1}x)\| R - \frac{[v(\Pi^{-1}x)]_{n+1}}{\sqrt{R^2 + 1}} R^2 \\ &\leq \frac{[v(\Pi^{-1}x)]_{n+1}}{\sqrt{R^2 + 1}} R^2 - \frac{[v(\Pi^{-1}x)]_{n+1}}{\sqrt{R^2 + 1}} R^2 = 0 \end{aligned} \quad (4.29)$$

So we see that $w(x)$ cannot point directly outward at any point of ∂B_R . So as a consequence of the Brouwer Fixed Point Theorem, the vector field must vanish at some point. So there exists some $x_0 \in B_R$ such that $w(x_0) = 0$. Thus from our definition of w and the fact that $\Pi^{-1}(0) = 0$ we have that

$$v(\Pi^{-1}x_0) = \frac{[v(\Pi^{-1}x_0)]_{n+1}}{[\Pi^{-1}x_0]_{n+1}} \Pi^{-1}x_0. \quad (4.30)$$

So by setting $z_0 = \Pi^{-1}x_0$ and $\alpha = \frac{[v(\Pi^{-1}x_0)]_{n+1}}{[\Pi^{-1}x_0]_{n+1}}$ we see the proof is complete. \square

4.5 Chiti's Comparison Argument in \mathbb{H}^n

This section will give the justification for the last step in the chains of inequalities (4.16) and (4.17). Here we take Ω^* to be the symmetric rearrangement of Ω , which is the geodesic ball centered at the origin having the same n -dimensional volume of Ω . Define $\Omega_t = \{x \in \Omega \mid u_0(x) > t\}$ and $\partial\Omega_t = \{x \in \Omega \mid u_0(x) = t\}$. Let $\mu(t) = |\Omega_t|$ and

$|\partial\Omega_t| = H_{n-1}(\partial\Omega_t)$, where H_{n-1} denotes the $(n-1)$ -dimensional measure on \mathbb{H}^n . For any function $f : \Omega \rightarrow \mathbb{R}^+$ we define the decreasing rearrangement, f^\sharp to be

$$f^\sharp(s) = \inf\{t \geq 0 \mid \mu(t) < s\}. \quad (4.31)$$

Also, we still take $f^*(r, \vec{\chi}) = f^*(r)$ to be the symmetric decreasing rearrangement. The former is a decreasing function from $[0, |\Omega|]$ to \mathbb{R}^+ and is equimeasurable with f . The latter is defined on Ω^* , spherically symmetric, equimeasurable with f , and is decreasing in r . The relationship between f^* and f^\sharp is given by

$$f^*(r, \vec{\chi}) = f^\sharp(A(r)), \quad (4.32)$$

where

$$A(r) = nC_n \int_0^r \sinh^{n-1} \bar{r} d\bar{r} \quad (4.33)$$

is the volume of a geodesic ball in \mathbb{H}^n with radius r . Here nC_n is the surface area of the $(n-1)$ -dimensional unit sphere in Euclidean space. We analogously define f_\sharp and f_* to be the increasing rearrangements of f .

Lemma 4. (*Chiti comparison result*) *Let $u_0(r, \vec{\chi})$ be the first Dirichlet eigenfunction of $-\Delta$ on Ω and $z_0(r)$ the first eigenfunction of $-\Delta$ on S_1 , normalized such that*

$$\int_{\Omega} u_0^2 dV = \int_{S_1} z_0^2 dV. \quad (4.34)$$

Then there exists some $r_0 \in (0, \tilde{r})$ such that

$$\begin{aligned} z_0(r) &\geq u_0^*(r) \quad \text{for } r \in (0, r_0) \text{ and} \\ z_0(r) &\leq u_0^*(r) \quad \text{for } r \in (r_0, \tilde{r}). \end{aligned}$$

Proof. Observe that the co-area formula gives the following

$$-\mu'(t) = \int_{\partial\Omega_t} \frac{1}{|\nabla u_0|} dH_{n-1} \quad (4.35)$$

(see [12], p. 86). Also, applying Gauss' Theorem (see [12], p. 7) to $-\Delta u_0 = \lambda_0 u_0$ we obtain

$$\int_{\Omega_t} \lambda_0 u_0 dV = \int_{\partial\Omega_t} |\nabla u_0| dH_{n-1}, \quad (4.36)$$

since the outward normal to Ω_t is $-\nabla u_0/|\nabla u_0|$. Using the Cauchy-Schwarz inequality and equations (4.35) and (4.36) we find that

$$(H_{n-1}(\partial\Omega_t))^2 = \left(\int_{\partial\Omega_t} dH_{n-1} \right)^2 \leq -\mu'(t)\lambda_0 \int_{\Omega_t} u_0 dV. \quad (4.37)$$

In \mathbb{H}^n we also have that the classical isoperimetric inequality holds, so we have

$$H_{n-1}(\partial\Omega_t) \geq H_{n-1}(\partial(\Omega_t^*)). \quad (4.38)$$

Recall definition (4.33) and let A^{-1} be the inverse function of A . Then the $(n-1)$ -dimensional measure of $\partial(\Omega_t^*)$ can be written as

$$H_{n-1}(\partial(\Omega_t^*)) = nC_n \sinh^{n-1} r A^{-1}(|\Omega_t^*|) = A'(A^{-1}(|\Omega_t^*|)). \quad (4.39)$$

Hence, substituting into (4.38) yields

$$H_{n-1}(\partial\Omega_t) \geq A'(A^{-1}(|\Omega_t^*|)) \quad (4.40)$$

and (4.37) can be written as

$$\lambda_0 \int_{\Omega_t} u_0 dV \geq -\frac{1}{\mu'(t)} A'(A^{-1}(|\Omega_t^*|))^2. \quad (4.41)$$

Then we use the fact that

$$\int_{\Omega_t} u_0 dV = \int_0^{\mu(t)} u_0^\sharp(s) ds, \quad (4.42)$$

which follows from the definition of u_0^\sharp . Since it is not hard to see that $u_0^\sharp(s)$ is the inverse function of $\mu(t)$, we have that

$$-\frac{du_0^\sharp}{ds} = -\frac{1}{\mu'(t)}, \quad (4.43)$$

which, when combined with (4.41) and (4.42), gives that

$$-\frac{du_0^\sharp}{ds} \leq \lambda_0 A'(A^{-1}(s))^{-2} \int_0^s u_0^\sharp(s') ds'. \quad (4.44)$$

Also, one can check that for Ω replaced by Ω^* and u_0 replaced by z_0 then equality holds in all of the steps leading to the previous equation, so we also have

$$-\frac{dz_0^\sharp}{ds} \leq \lambda_0 A'(A^{-1}(s))^{-2} \int_0^s z_0^\sharp(s') ds'. \quad (4.45)$$

Using these two relations and recalling the assumed normalization, we will show that the functions u_0^\sharp and z_0^\sharp are either identical or they cross each other exactly once on the interval $[0, |\Omega_t|]$. In the following, we make use of the fact that u_0^\sharp and z_0^\sharp are continuous. By the definition of the decreasing rearrangement, both functions are decreasing and we know that $z_0^\sharp(|\Omega_t|) = u_0^\sharp(|\Omega|) = 0$. Recall that from the Rayleigh-Faber-Krahn inequality and since we took $\lambda_0(S_1) = \lambda_0(\Omega)$ it follows that $|S_1| \leq |\Omega|$. From the normalization, it is clear that z_0^\sharp and u_0^\sharp are either identical or cross at least once on $[0, |S_1|]$. To show that they cross exactly once, we assume that they cross at least twice and obtain a contradiction. Under this assumption, there are two points $0 \leq s_1 < s_2 < |S_1|$ where $u_0^\sharp(s) > z_0^\sharp(s)$ for $s \in (s_1, s_2)$, $u_0^\sharp(s_2) = z_0^\sharp(s_2)$ and either $u_0^\sharp(s_1) = z_0^\sharp(s_1)$ or $s_1 = 0$. Now we define the following function

$$v(s) = \begin{cases} u_0^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_0^\sharp(s) ds > \int_0^{s_1} z_0^\sharp(s) ds, \\ z_0^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_0^\sharp(s) ds \leq \int_0^{s_1} z_0^\sharp(s) ds, \\ u_0^\sharp(s) & \text{on } [s_1, s_2], \\ z_1^\sharp(s) & \text{on } [s_2, |S_1|]. \end{cases} \quad (4.46)$$

Then by substituting $v(s)$ into (4.44) and (4.45) we see that

$$-\frac{dv}{ds} \leq \lambda_0 A'(A^{-1}(s))^{-2} \int_0^s v(s') ds' \quad (4.47)$$

for all $s \in [0, |S_1|]$.

Finally, we define the test function $\Psi(r, \vec{\chi}) = v(A(r))$. If z_0 and u_0 are not identical, using the Rayleigh-Ritz characterization of λ_0 , (4.47), and integration by parts, we

obtain

$$\begin{aligned}
\lambda_0 \int_{S_1} \Psi^2 dV &< \int_{S_1} |\nabla \Psi|^2 dV && (4.48) \\
&= \int_0^{\bar{r}} (A'(r)v'(A(r)))^2 A'(r) dr \\
&\leq - \int_0^{\bar{r}} A'(r)v'(A(r))\lambda_0 \int_0^{A(r)} v(s') dr \\
&= \lambda_0 \int_0^{|S_1|} v(s)^2 ds \\
&= \lambda_0 \int_{S_1} \Psi^2 dV
\end{aligned}$$

So we see that we obtain a contradiction to the assumption that u_0^\sharp and z_0^\sharp intersect twice, so the lemma is proved. \square

Chapter 5 Payne-Pólya-Weinberger-like Inequalities for a Family of Spherically Symmetric Riemannian Manifolds

In this chapter we will set up and state our results for our family of spherically symmetric Riemannian manifolds, \mathbb{F}^n . In the first section we present an overview of the geometry of this space. We then go on to state and prove our main result, which is a PPW-type inequality for the gap of the first two Dirichlet eigenvalues. In the subsequent sections we present several lemmas and theorems needed for the proof of the main theorem, including a modified center of mass theorem and Chiti comparison argument. Finally, we look at another approach to finding test function for the Rayleigh-Ritz inequality and present one more gap inequality based on Gram-Schmidt orthogonalization.

5.1 The Geometry of \mathbb{F}^n

In this section we will talk more about the space \mathbb{F}^n , which serves as an interpolation space between Euclidean space and hyperbolic space. As is mentioned in the introduction, in general we take a function f so that $f > 0$, $f(0) = 0$, and $f'(0) = 1$. Specifically, we use

$$f(r) = r + \beta r^\alpha \tag{5.1}$$

so that f satisfies our above conditions, $f \in C^2(\Omega)$, $r \in [0, \infty)$, $\alpha \geq 2$, and $\beta > 0$ is a constant.

Using this f , we take

$$ds^2 = dr^2 + f(r)^2 |d\omega|^2. \tag{5.2}$$

to be the metric of this space, relative to a fixed origin. Note that the straight lines from this origin are geodesics in \mathbb{F}^n . For any $x \in \mathbb{R}^n$, $r(x)$ or $|x|$ denotes the geodesic distance from 0 to x . We can also write the metric in the following way with

$(r, \theta_1, \theta_2, \dots, \theta_{n-2}, \phi)$ as our coordinates. Also, the off-diagonal elements are zero in this case. So then the metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 & & & & & & \\ 0 & f(r)^2 & & & & & & \\ & 0 & f(r)^2 \sin^2 \theta_1 & & & & & \\ & & 0 & f(r)^2 \sin^2 \theta_1 \sin^2 \theta_2 & & & & \\ & & & 0 & f(r)^2 \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \theta_3 & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & f(r)^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} \end{pmatrix}.$$

We also occasionally need to use the modified Cartesian coordinates for this space, which are given as

$$\begin{aligned} x_1 &= f(r) \sin \theta_{n-2} \cdots \sin \theta_1 \cos \phi & (5.3) \\ x_2 &= f(r) \sin \theta_{n-2} \cdots \sin \theta_1 \sin \phi \\ &\vdots \\ x_{n-1} &= f(r) \sin \theta_{n-2} \cos \theta_{n-3} \\ x_n &= f(r) \cos \theta_{n-2}. \end{aligned}$$

We can compute the gradient in the spherical system to obtain

$$\nabla v = \frac{\partial v}{\partial r} + \frac{1}{f(r)} \frac{\partial v}{\partial \theta_1} + \cdots + \frac{1}{f(r) \sin \theta_1 \cdots \sin \theta_{n-2}} \frac{\partial v}{\partial \phi}, \quad (5.4)$$

and in (x_1, \dots, x_n) to obtain

$$\nabla v = \sum_{k=1}^n \hat{e}_k \left[\left(\frac{f'(r) - 1}{f^2(r)} \right) x_k l(x) + 1 \right] \frac{\partial v}{\partial x_k}, \quad (5.5)$$

where

$$l(x) = \sum_{m=1}^n x_m. \quad (5.6)$$

Also,

$$\Delta v = \frac{1}{f(r)^{n-1}} \frac{\partial}{\partial r} \left(f(r)^{n-1} \frac{\partial v}{\partial r} \right) + \frac{1}{f(r)^2} \Delta_{\mathbb{S}^{n-1}} v, \quad (5.7)$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the spherical Laplacian. Finally, the volume form dV is given by the following formula,

$$dV = f(r)^{n-1} \sin^{n-2} \theta_1 \cdots \sin \theta_{n-2} dr d\theta_1 \cdots d\theta_{n-2}. \quad (5.8)$$

Under this definition, we can find that the sectional curvature of \mathbb{F}^n with respect to the spherical coordinates is given by

$$K_{ii} = 0 \quad \text{for } 1 \leq i \leq n \quad (5.9)$$

$$K_{1i} = K_{i1} = \frac{-f''}{f} \quad \text{for } 2 \leq i \leq n \quad (5.10)$$

$$K_{ij} = \frac{1 - (f')^2}{f^2} \quad \text{for } 2 \leq i, j \leq n, i \neq j. \quad (5.11)$$

Also, the Ricci curvature is given by

$$Ric(x_1) = \frac{-f''}{f} \quad (5.12)$$

$$Ric(x_i) = \frac{\frac{-f''}{f} + \frac{(n-2)(1-(f')^2)}{f^2}}{n-1}, \text{ for } 2 \leq i \leq n \quad (5.13)$$

and the scalar curvature by

$$\kappa = \frac{1}{n} \left(\frac{-f''}{f} + \frac{\frac{-f''}{f} + \frac{(n-2)(1-(f')^2)}{f^2}}{n-1} \right). \quad (5.14)$$

From these calculations we see that the metric would be degenerate for $f = r^\alpha$ when $r = 0$, hence the reason we add the linear term so that the manifold agrees with Euclidean space near the origin. Also, as r grows large, this manifold grows closer to Euclidean space. We remark that for \mathbb{F}^n , $R = \int_0^\infty \frac{ds}{f(s)}$ is finite. This means that \mathbb{F}^n is conformal to an open ball in \mathbb{R}^n of radius R . For a discussion of this see [17].

5.2 A PPW-type Inequality for the Eigenvalues of the Dirichlet Laplacian

In this section, we find a bound for difference of the first two Dirichlet eigenvalues of a bounded domain contained in \mathbb{F}^n . For our main result, we obtain a version of the

Payne-Pólya-Weinberger inequality for our set of manifolds. The optimal result is not obtained here because the Laplacian does not commute with translations in \mathbb{F}^n . To account for this, we must now work on shifted domains in our manifold. So first define

$$T_\nu : L^2(\Omega) \rightarrow L^2(\Omega + \nu), \quad (5.15)$$

and

$$-\Delta_\nu = T_\nu(-\Delta_\Omega)T_\nu^{-1} \quad (5.16)$$

on $L^2(\Omega + \nu)$. From this definition we see that

$$-\Delta_\nu(T_\nu u_0)(x) = \lambda_0(\Omega)T_\nu u_0(x). \quad (5.17)$$

Also,

$$T_\nu u_0(x) = J(\nu, x)u_0(x - \nu), \quad (5.18)$$

and we would like to find the Jacobian term $J(\nu, x)$. We will show this computation in three dimensions, and note that this method will work for higher dimensions as well. First, note that for $n = 3$,

$$dV = f(r)^2 \sin \theta \, dr \, d\theta \, d\phi, \quad (5.19)$$

and

$$x_1 = f(r) \sin \theta \cos \phi$$

$$x_2 = f(r) \sin \theta \sin \phi$$

$$x_3 = f(r) \cos \theta.$$

So then we find

$$\begin{aligned} dx_1 &= f'(r) \sin \theta \cos \theta \, dr + f(r) \cos \theta \cos \phi \, d\theta - f(r) \sin \theta \sin \phi \, d\phi \\ dx_2 &= f'(r) \sin \theta \sin \phi \, dr + f(r) \cos \theta \sin \phi \, d\phi + f(r) \sin \theta \cos \phi \, d\theta \\ dx_3 &= f'(r) \cos \theta \, dr - f(r) \sin \theta \, d\theta. \end{aligned} \quad (5.20)$$

Next, we compute that

$$dx_1 \wedge dx_2 \wedge dx_3 = f(r)^2 f'(r) \sin \theta dr \wedge d\theta \wedge d\phi = dV(x) f'(|x|). \quad (5.21)$$

So the Jacobian term $J(\nu, x)$ is given by

$$J(\nu, x)^2 = \left[\frac{dV(x - \nu)}{dV(x)} \right] = \frac{f'(|x|)}{f'(|x - \nu|)}. \quad (5.22)$$

Thus, there is an extra error term along with the upper bound of the eigenvalues on the geodesic disk having the same volume as our original domain. Now we state our bound based on the PPW inequality.

Theorem 17. *(A PPW-like result for \mathbb{F}^n) Let $\Omega \subset \mathbb{F}^n$ be an open bounded domain and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . Denote by Ω^* the geodesic ball in \mathbb{F}^n having the same volume as Ω and \tilde{r} its geodesic radius. Also, we denote by $\vec{\nu}$ the vector associated with the center of mass theorem for Ω in \mathbb{F}^n . So, if we have that the angle between \vec{x} and $\vec{\nu}$ is between $-\pi/2$ and $\pi/2$ for all $x \in \Omega$, then*

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \left\{ \sup_{x \in \Omega} \frac{B(|x + \nu|)}{B(|x|)} \right\} [1 + 2\epsilon^2 + b_\Omega(\nu, \epsilon)] (\lambda_1(\Omega^*) - \lambda_0(\Omega^*)) + \mathcal{F}(\nu, \epsilon), \quad (5.23)$$

where

$$B(r) = g'(r)^2 + (n - 1)f^{-2}(r)g^2(r), \quad (5.24)$$

$$g(r) = \begin{cases} \frac{z_1(r)}{z_0(r)} & r \in [0, \tilde{r}), \\ \lim_{r \uparrow \tilde{r}} g(r) & r \geq \tilde{r}, \end{cases} \quad (5.25)$$

$$\mathcal{F}(\nu, \epsilon) = \left(1 + \epsilon^2 + \frac{1}{\epsilon^2} \right) \sup_{x \in \Omega + \nu} \left\{ \sum_k \left[\left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} \right) |(x - \nu)_k l(x - \nu) + 1| \right. \right. \\ \left. \left. \left| \frac{\partial J(-\nu, x - \nu)}{\partial x_k} J(\nu, x) \right| \right] \right\}, \quad (5.26)$$

and

$$\begin{aligned}
b_{\Omega}(\nu, \epsilon) \equiv & \max_{k=1, \dots, n} \sup_{x \in \Omega + \nu} \left\{ \left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} \right) \left(\frac{f(|x|)^2}{f'(|x|) - 1} \right) \right. \\
& \left[|\nu_k l(\nu) - x_k l(\nu) - \nu_k l(x)| + \frac{|F(x, \nu)| f(|x - \nu|)^2}{f'(|x - \nu|) - 1} |x_k l(x)| \right] \\
& \left. \left[\frac{f'(|x|) - 1}{f(|x|)^2} |x_k l(x)| + 1 \right]^{-1} \right\} \left(1 + \frac{2}{\epsilon^2} \right), \tag{5.27}
\end{aligned}$$

where

$$F(x, \nu) = \left[\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} - \frac{f'(|x|) - 1}{f(|x|)^2} \right]. \tag{5.28}$$

Proof. Let u_0 be the first eigenfunction of $-\Delta$ on Ω . We again wish to find suitable test functions to substitute into the Rayleigh-Ritz inequality. Using our modified center of mass argument, Theorem 19, we may choose the following test function,

$$P_j T_{\nu} u_0 = h_j(\psi) g(r) T_{\nu} u_0, \tag{5.29}$$

where

$$h_j(\psi) = \frac{x_j}{f(r)}. \tag{5.30}$$

Thus, we get the following upper bound

$$\lambda_1(\Omega) \leq \frac{\int_{\Omega + \nu} |\nabla^{\nu} P_j T_{\nu} u_0|^2 dV}{\int_{\Omega + \nu} |P_j T_{\nu} u_0|^2 dV}, \tag{5.31}$$

where $\nabla^{\nu} = T_{\nu} \nabla T_{\nu}^{-1}$. The next step is to compute the numerator of this bound, so observe that

$$\begin{aligned}
|\nabla^{\nu} P_j T_{\nu} u_0|^2 &= |(\nabla^{\nu} P_j) T_{\nu} u_0 + P_j (\nabla^{\nu} T_{\nu} u_0)|^2 \\
&= |\nabla^{\nu} P_j|^2 |T_{\nu} u_0|^2 + 2 \nabla^{\nu} P_j \cdot \nabla^{\nu} (T_{\nu} u_0) P_j T_{\nu} u_0 \\
&+ P_j^2 |\nabla^{\nu} T_{\nu} u_0|^2.
\end{aligned} \tag{5.32}$$

Then recognize that

$$\nabla^{\nu} (P_j^2 T_{\nu} u_0) = 2 P_j (\nabla^{\nu} P_j) T_{\nu} u_0 + P_j^2 (\nabla^{\nu} T_{\nu} u_0) \tag{5.33}$$

and so we may substitute to see that

$$\begin{aligned} |\nabla^\nu P_j T_\nu u_0|^2 &= |\nabla^\nu P_j|^2 |T_\nu u_0|^2 + \nabla^\nu (P_j^2 T_\nu u_0) \cdot \nabla^\nu (T_\nu u_0) \\ &- P_j^2 |\nabla^\nu T_\nu u_0|^2 + P_j^2 |\nabla^\nu T_\nu u_0|^2. \end{aligned} \quad (5.34)$$

Finally, integrating by parts allows us to find

$$\int_{\Omega+\nu} |\nabla^\nu P_j T_\nu u_0|^2 dV = \int_{\Omega+\nu} |\nabla^\nu P_j|^2 |T_\nu u_0|^2 dV + \lambda_0(\Omega) \int_{\Omega+\nu} P_j^2 |T_\nu u_0|^2 dV. \quad (5.35)$$

So, substituting into (5.31) gives us that

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega+\nu} |\nabla^\nu P_j|^2 |T_\nu u_0|^2 dV}{\int_{\Omega+\nu} |P_j T_\nu u_0|^2 dV}. \quad (5.36)$$

Next, we again want to calculate this numerator, so

$$\begin{aligned} (\nabla^\nu P_j)(x) &= (T_\nu \nabla T_\nu^{-1} P_j)(x) \\ &= J(\nu, x) (\nabla T_\nu^{-1} P_j)(x - \nu) \\ &= J(\nu, x) J(-\nu, x - \nu) \left\{ \sum_k \hat{e}_k \left[\left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|^2)} \right) \right. \right. \\ &\quad \left. \left. (x - \nu)_k l(x - \nu) + 1 \right] \frac{\partial P_j(x)}{\partial x_k} \right\} \\ &+ J(\nu, x) \left\{ \sum_k \hat{e}_k \left[\left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|^2)} \right) (x - \nu)_k l(x - \nu) + 1 \right] \right. \\ &\quad \left. \left[\frac{\partial J(-\nu, x - \nu)}{\partial x_k} \right] \right\} P_j(x). \end{aligned} \quad (5.37)$$

Observe that

$$J(\nu, x) J(-\nu, x - \nu) = 1, \quad (5.38)$$

and now write

$$(\nabla^\nu P_j)(x) = \nabla P_j(x) + L_g(\nu) P_j(x) + \left(\sum_k \hat{e}_k \mathcal{J}_k(\nu, x) \right) P_j(x), \quad (5.39)$$

where

$$\begin{aligned} L_g(\nu) &= - \sum_k \hat{e}_k \left[\left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|^2)} \right) [x_k l(\nu) - \nu_k l(x) - \nu_k l(\nu)] \right. \\ &\quad \left. + F(x, \nu) [x_k l(x) + 1] \right] \frac{\partial}{\partial x_k}, \end{aligned} \quad (5.40)$$

$$F(x, \nu) = \left[\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} - \frac{f'(|x|) - 1}{f(|x|)^2} \right], \quad (5.41)$$

and

$$\mathcal{J}_k(\nu, x) = \sum_k \left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} \right) [(x - \nu)_k l(x - \nu) + 1] \left(\frac{\partial J(-\nu, x - \nu)}{\partial x_k} J(\nu, x) \right). \quad (5.42)$$

We will now use the notation that

$$(\nabla^\nu P_j)(x) = a + b + c \quad (5.43)$$

to see that

$$\begin{aligned} |\nabla^\nu P_j(x)|^2 &= |a|^2 + |b|^2 + |c|^2 + 2\langle a, b \rangle + 2\langle a, c \rangle + 2\langle b, c \rangle \\ &\leq (1 + 2\epsilon^2) |a|^2 + \left(1 + \frac{2}{\epsilon^2}\right) |b|^2 + \left(1 + \epsilon^2 + \frac{1}{\epsilon^2}\right) |c|^2 \\ &= (1 + 2\epsilon^2) |\nabla P_j(x)|^2 + \left(1 + \frac{2}{\epsilon^2}\right) |L_g(\nu) P_j(x)|^2 \\ &\quad + \left(1 + \epsilon^2 + \frac{1}{\epsilon^2}\right) g(r)^2 \mathcal{F}(\nu), \end{aligned} \quad (5.44)$$

where

$$\begin{aligned} \mathcal{F}(\nu, \epsilon) &= \left(1 + \epsilon^2 + \frac{1}{\epsilon^2}\right) \sup_{x \in \Omega + \nu} \left\{ \sum_k \left[\left(\frac{f'(|x - \nu|) - 1}{f(|x - \nu|)^2} \right) |(x - \nu)_k l(x - \nu) + 1| \right. \right. \\ &\quad \left. \left. \left| \frac{\partial J(-\nu, x - \nu)}{\partial x_k} J(\nu, x) \right| \right] \right\}^2. \end{aligned} \quad (5.45)$$

Also we compute, using the formula (5.5) for the gradient in \mathbb{F}^n ,

$$\nabla P_j(x) = \sum_{k=1}^n \hat{e}_k \left[\left(\frac{f'(r) - 1}{f^2(r)} \right) x_k l(x) + 1 \right] \frac{\partial P_j(x)}{\partial x_k}, \quad (5.46)$$

and

$$\begin{aligned} \sum_j |\nabla P_j|^2 &= \sum_j g'(r)^2 h_j(\psi)^2 + \sum_j g(r)^2 |\nabla h_j(\psi)|^2 \\ &= g'(r)^2 + g(r)^2 \sum_j |\nabla h_j|^2 \\ &= g'(r)^2 + (n - 1) \left(\frac{g(r)}{f(r)} \right)^2. \end{aligned} \quad (5.47)$$

Recall the definition that

$$B(r) = g'(r)^2 + (n-1)f^{-2}(r)g^2(r), \quad (5.48)$$

so substituting into (5.36) gives that

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega+\nu} (1+2\epsilon^2) B(r) |T_\nu u_0|^2 dV + \mathcal{E}(\nu, u_0) + g(r)^2 |T_\nu u_0|^2 \mathcal{F}(\nu, \epsilon)}{\int_{\Omega+\nu} \sum_j |P_j T_\nu u_0|^2 dV}, \quad (5.49)$$

where

$$\mathcal{E}(\nu, u_0) = \sum_j \int_{\Omega+\nu} |T_\nu u_0|^2 |L_g(\nu) P_j(x)|^2 dV. \quad (5.50)$$

We now would like to obtain an upper bound for this error term, so observe that

$$|b|^2 = |(L_g(\nu) P_j)(x)|^2 = b_\Omega(\nu) |\nabla P_j(x)|^2, \quad (5.51)$$

where

$$\begin{aligned} b_\Omega(\nu) \equiv & \max_{k=1, \dots, n} \sup_{x \in \Omega+\nu} \left\{ \left(\frac{f'(|x-\nu|) - 1}{f(|x-\nu|)^2} \right) \left(\frac{f(|x|)^2}{f'(|x|) - 1} \right) \right. \\ & \left[|\nu_k l(\nu) - x_k l(\nu) - \nu_k l(x)| + \frac{|F(x, \nu)| f(|x-\nu|)^2}{f'(|x-\nu|) - 1} |x_k l(x)| \right] \\ & \left. \left[\frac{f'(|x|) - 1}{f(|x|)^2} |x_k l(x)| + 1 \right]^{-1} \right\}, \end{aligned} \quad (5.52)$$

and

$$F(x, \nu) = \left[\frac{f'(|x-\nu|) - 1}{f(|x-\nu|)^2} - \frac{f'(|x|) - 1}{f(|x|)^2} \right]. \quad (5.53)$$

So now combining (5.49) and (5.51) gives

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq [1 + 2\epsilon^2 + b_\Omega(\nu, \epsilon)] \frac{\int_{\Omega+\nu} B(r) |T_\nu u_0|^2 dV}{\int_{\Omega+\nu} g(r)^2 |T_\nu u_0|^2 dV} + \mathcal{F}(\nu, \epsilon). \quad (5.54)$$

Now we want to make use of the tools of symmetric rearrangement as well as a modified version of the Chiti comparison result that is proved later in this chapter.

We also will later prove that $B(r)$ is a decreasing function of r , while $g(r)$ is increasing.

Let us use f^* to denote the spherical decreasing rearrangement of f , and f_* the

spherical increasing rearrangement. Thus

$$\begin{aligned}
\int_{\Omega+\nu} |(T_\nu u_0)(x)|^2 B(r) dV &= \int_{\Omega} |u_0(x)|^2 B(|x + \nu|) dV & (5.55) \\
&\leq \int_{\Omega^*} |u_0^*(x)|^2 B^*(|x + \nu|) dV \\
&\leq \left\{ \sup_{x \in \Omega} \frac{B(|x + \nu|)^*}{B(|x|)^*} \right\} \int_{\Omega^*} |u_0^*(x)|^2 B(r) dV \\
&\leq \left\{ \sup_{x \in \Omega} \frac{B(|x + \nu|)}{B(|x|)} \right\} \int_{\Omega^*} z_0^2(r) B(r) dV.
\end{aligned}$$

Here the first inequality follows from the properties of rearrangements, the second inequality from the monotonicity properties of g and B , and the third inequality from the modified Chiti comparison argument proved in section 5.6 and the fact that $B^* = B$ since B is decreasing.

We also need a lower bound for the denominator, so we now make use of the fact the the angle between \vec{x} and $\vec{\nu}$ is between $-\pi/2$ and $\pi/2$ to see that

$$|x + \nu|^2 = |x|^2 + |\nu|^2 + 2|x||\nu| \cos \varphi \geq |x|^2. \quad (5.56)$$

Thus we apply a similar argument as for the numerator to see that

$$\begin{aligned}
\sum_j \int_{\Omega+\nu} P_j(x)^2 |(T_\nu u_0)(x)|^2 dV &= \int_{\Omega+\nu} g(r)^2 |(T_\nu u_0)(x)|^2 dV & (5.57) \\
&= \int_{\Omega} g(|x + \nu|)^2 |u_0(x)|^2 dV \\
&\geq \int_{\Omega} g(|x|)^2 |u_0(x)|^2 dV \\
&\geq \int_{\Omega^*} g_*(|x|)^2 |u_0^*(x)|^2 dV \\
&\geq \int_{\Omega^*} g(|x|)^2 |u_0^*(x)|^2 dV \\
&\geq \int_{\Omega^*} g(r)^2 z_0^2(r) dV.
\end{aligned}$$

Also, we prove that the first two eigenvalues of the Laplacian on a geodesic disk are the first eigenvalues of (1.9) with $l = 0$ and $l = 1$, respectively. So now we can apply these inequalities to (5.49) and make use of the fact that

$$\lambda_1(\Omega^*) - \lambda_0(\Omega^*) = \frac{\int_{\Omega^*} z_0^2(r) B(r) dV}{\int_{\Omega^*} g(r)^2 z_0^2(r) dV} \quad (5.58)$$

to see that

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \left\{ \sup_{x \in \Omega} \frac{B(|x + \nu|)}{B(|x|)} \right\} [1 + 2\epsilon^2 + b_\Omega(\nu, \epsilon)] (\lambda_1(\Omega^*) - \lambda_0(\Omega^*)) + \mathcal{F}(\nu, \epsilon), \quad (5.59)$$

which is our desired result. \square

Remark. Observe that

$$\lim_{\nu \rightarrow 0} b_\Omega(\nu, \epsilon) = 0, \quad (5.60)$$

and

$$\lim_{\nu \rightarrow 0} \left\{ \sup_{x \in \Omega} \frac{B(|x + \nu|)}{B(|x|)} \right\} = 1. \quad (5.61)$$

Also, $\mathcal{F}(\nu, \epsilon)$ goes to zero as ν goes to zero. Thus, we see that as $\nu \rightarrow 0$ and $\epsilon \rightarrow 0$, we recover the expected result as in Euclidean space. In the case where $\nu \rightarrow 0$ and $\epsilon \rightarrow 0$, we still impose the condition that $\lambda_0(\Omega) = \lambda_0(S_1)$, where S_1 is a geodesic ball as in the hyperbolic case. Then we would obtain the analogous result for \mathbb{F}^n and the modified Chiti comparison theorem would apply in a similar manner.

In addition, we can bound the maximum size of the translation ν . Let B_Ω be the smallest ball centered at the origin such that $\Omega \subset B_\Omega$. If $R = \sup_{x, y \in B_\Omega} d_g(x, y)$, where d_g is the distance for the metric g , then $|\nu| \leq R/2$. This follows from the center of mass argument since we take $\nu \in \partial B_\Omega$.

We also obtain the following corollary by slightly modifying our lower bound of the denominator in the Rayleigh-Ritz inequality. The result is stated below.

Corollary 1. *Let $\Omega \subset \mathbb{F}^n$ be an open bounded domain and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . Denote by Ω^* the geodesic disk in \mathbb{F}^n having the same volume as Ω and \tilde{r} its geodesic radius. Also, we denote by $\vec{\nu}$ the vector associated with the center of mass theorem for \mathbb{F}^n . So, if we have that $\left\{ \inf_{x \in \Omega} \frac{g(|x + \nu|)}{g(|x|)} \right\} > 0$, then*

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq C(\Omega, \nu) [1 + \epsilon^2 + b_\Omega(\nu, \epsilon)] (\lambda_1(\Omega^*) - \lambda_0(\Omega^*)) + \mathcal{F}(\nu, \epsilon), \quad (5.62)$$

where

$$C(\Omega, \nu) = \frac{\left\{ \sup_{x \in \Omega} \frac{B(|x+\nu|)}{B(|x|)} \right\}}{\left\{ \inf_{x \in \Omega} \frac{g(|x+\nu|)}{g(|x|)} \right\}}, \quad (5.63)$$

and the other terms are defined as in the statement of the main theorem.

Proof. The proof follows just as in the main theorem, only here we replace the first inequality in (5.57) with

$$\int_{\Omega} g(|x + \nu|)^2 |u_0(x)|^2 dV \geq \left\{ \inf_{x \in \Omega} \frac{g(|x + \nu|)}{g(|x|)} \right\} \int_{\Omega} g(|x|)^2 |u_0(x)|^2 dV, \quad (5.64)$$

so that

$$\int_{\Omega} g(|x + \nu|)^2 |u_0(x)|^2 dV \geq \left\{ \inf_{x \in \Omega} \frac{g(|x + \nu|)}{g(|x|)} \right\} \int_{\Omega^*} g(r)^2 z_0^2(r) dV. \quad (5.65)$$

□

We now would like to look at a certain set of domains having additional symmetry properties. We will take $\mathcal{S} \subset \mathbb{F}^n$ to be the set of domains Ω such that Ω contains the origin, is rotationally symmetric to the x_i axes for $i = 3, \dots, n$, and is symmetric with respect to the $x_1 - x_2$ plane. This subset of domains allows us to choose our test functions without the need for shifting the domains as in the previous result. So in this case, we obtain the following theorem.

Theorem 18. *Let $\Omega \subset \mathcal{S}$ be an open bounded domain in \mathbb{F}^n and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω . Take Ω^* to be the geodesic ball in \mathbb{F}^n having the same volume as Ω . Then,*

1)

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \lambda_1(\Omega^*) - \lambda_0(\Omega^*). \quad (5.66)$$

2) *If $S_1 \subset \mathbb{F}^n$ is a geodesic ball such that $\lambda_0(\Omega) = \lambda_0(S_1)$ then*

$$\lambda_1(\Omega) \leq \lambda_1(S_1). \quad (5.67)$$

Proof. 1) We see here that if Ω has the assumed symmetries, then u_0 has the same symmetries and hence the center of mass theorem would give

$$\int_{\Omega} P_j u_0(x)^2 dV = 0. \quad (5.68)$$

Thus we would have $\nu = 0$, and as is mentioned in the remarks after the main theorem, we can now recover that 1)

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \lambda_1(\Omega^*) - \lambda_0(\Omega^*). \quad (5.69)$$

Essentially, choosing our domains having the specified symmetries forces the center of mass to lie at the origin. Hence we can take $P_j u_0$ to be our trial functions.

2) Now let u_0 be the first eigenfunction of $-\Delta$ on Ω . Take S_1 to be a geodesic ball such that $\lambda_0(S_1) = \lambda_0(\Omega)$ and take z_0 to be the corresponding eigenfunction. We first observe that it is possible to choose S_1 such that $\lambda_0(S_1) = \lambda_0(\Omega)$ using the idea of domain monotonicity. Since if $|\Omega| \leq |S_1|$, we have that $\lambda_0(S_1) \leq \lambda_0(\Omega)$ and if $|S_1| \leq |\Omega|$, then $\lambda_0(\Omega) \leq \lambda_0(S_1)$. Since $\lambda_0(\Omega)$ is a continuous function, we can apply the intermediate value theorem to see that we must be able to find a domain S_1 to satisfy our condition.

Also we can find the gap inequality

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2 \langle \nabla P, \nabla P \rangle dV}{\int_{\Omega} P^2 u_0^2 dV}. \quad (5.70)$$

Using our observation about the center of mass theorem we choose the following set of n functions,

$$P_j(x) := g(r) h_j(\psi) \quad (5.71)$$

where

$$g(r) = \begin{cases} \frac{z_1(r)}{z_0(r)} & r \in [0, \tilde{r}), \\ \lim_{r \uparrow \tilde{r}} g(\tilde{r}) & r \geq \tilde{r}. \end{cases} \quad (5.72)$$

Recall that \tilde{r} is the geodesic radius of S_1 , and that by convention z_0 and z_1 are positive.

We may then calculate, following [12], that

$$\sum_{j=1}^n P_j^2(x) = g^2(r), \quad (5.73)$$

$$\sum_{j=1}^n \langle \nabla P_j, \nabla P_j \rangle = \left(\frac{\partial g}{\partial r} \right)^2 + (n-1)f^{-2}(r)g^2(r). \quad (5.74)$$

Then multiply (5.70) by $\int_{\Omega} P^2 u_0^2 dV$ and then sum over $i = 1, \dots, n$ to obtain

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2(r, \vec{\chi}) B(r) dV}{\int_{\Omega} u_0^2(r, \vec{\chi}) g^2(r) dV} \quad (5.75)$$

where

$$B(r) = g'(r)^2 + (n-1)f^{-2}(r)g^2(r). \quad (5.76)$$

To finish the proof, we need the following inequalities to hold:

$$\begin{aligned} \int_{\Omega} u_0^2(x) B(r) dV &\leq \int_{\Omega^*} u_0^*(r)^2 B^*(r) dV \\ &\leq \int_{\Omega^*} u_0^*(r)^2 B(r) dV \\ &\leq \int_{S_1} z_0^2(r) B(r) dV \end{aligned} \quad (5.77)$$

and

$$\begin{aligned} \int_{\Omega} u_0^2(x) g(r)^2 dV &\geq \int_{\Omega^*} u_0^*(r)^2 g_*(r)^2 dV \\ &\geq \int_{\Omega^*} u_0^*(r)^2 g(r)^2 dV \\ &\geq \int_{S_1} z_0^2(r) g(r)^2 dV. \end{aligned} \quad (5.78)$$

Here we assume that z_0 is normalized such that $\int_{\Omega} u_0^2 dV = \int_{S_1} z_0^2 dV$. Also, f^* is used to denote the spherical decreasing rearrangement of f , and f_* the spherical increasing rearrangement. In each of (5.77) and (5.78), the first inequality follows from the properties of rearrangements. The second inequality follows from the fact that g is increasing in r and B is decreasing in r , which will be proven below. Finally, the third inequality follows from a modified version of Chiti's comparison result and

also from the monotonicity properties of g and B . This will also be proven later. Finally, from (5.75), (5.77), and (5.78) we have

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{S_1} z_0^2(r) B(r) dV}{\int_{S_1} z_0^2(r) g^2(r) dV} = \lambda_1(\Omega) - \lambda_0(S_1). \quad (5.79)$$

Since we have chosen $\lambda_0(\Omega) = \lambda_0(S_1)$ we have that

$$\lambda_1(\Omega) \leq \lambda_1(S_1), \quad (5.80)$$

which proves our theorem. \square

5.3 Monotonicity Properties

In this section we will establish that g is an increasing function of r and that B is a decreasing function of r . Here we follow the approach of Benguria and Linde from [9]. We will prove the necessary monotonicity properties of g and B by analyzing the function

$$q(r) = f(r) \frac{g'(r)}{g(r)} = f(r) \left(\frac{z_1'}{z_1} - \frac{z_0'}{z_0} \right). \quad (5.81)$$

Recall that in our space \mathbb{F}^n , the eigenfunctions are determined by a function f . We will take this function to be

$$f(r) = r + \beta r^\alpha. \quad (5.82)$$

By showing that

$$q(r) \geq 0, \quad (5.83)$$

$$q(r) \leq f'(r), \quad (5.84)$$

$$q'(r) \leq 0, \quad (5.85)$$

we may prove the desired properties of g and B as follows. If q satisfies the above inequalities, then from (5.83), the definition of q and the fact that $g \geq 0$ we can easily see that $g' \geq 0$ and thus g is increasing.

Next we show that

$$B(r) = g'(r)^2 + \frac{(n-1)g^2(r)}{f^2(r)}$$

is decreasing in r . First, if we solve (5.81) for g' and differentiate both sides, substituting (5.81) again, we obtain

$$g''(r) = \frac{g(r)}{f(r)^2} (f(r)q' + q(q - f'(r))) \quad (5.86)$$

So from (5.83), (5.84), and (5.85) we see that g' is decreasing.

Thus, the first summand of $B(r)$ is decreasing. Additionally, since $g(0) = 0$ and g' is decreasing we have the estimate

$$g(r) = \int_0^r g'(\tau) d\tau \geq \int_0^r g'(r) d\tau = g'(r)r. \quad (5.87)$$

Thus we see that

$$\begin{aligned} \left(\frac{g(r)}{f(r)}\right)' &= \frac{1}{f(r)} \left(g'(r) - \frac{g(r)f'(r)}{f(r)}\right) \\ &\leq \frac{1}{f(r)} \left(g'(r) - \frac{rg'(r)f'(r)}{f(r)}\right) \\ &= \frac{g'(r)}{f(r)} \left(1 - \frac{rf'(r)}{f(r)}\right) \end{aligned} \quad (5.88)$$

Then we compute

$$f'(r) = 1 + \alpha\beta r^{\alpha-1} \quad (5.89)$$

$$\begin{aligned} r\frac{f'(r)}{f(r)} &= r\frac{1 + \alpha\beta r^{\alpha-1}}{r(1 + \beta r^{\alpha-1})} = \frac{1 + \beta r^{\alpha-1} + (\alpha-1)\beta r^{\alpha-1}}{1 + \beta r^{\alpha-1}} \\ &= 1 + \frac{(\alpha-1)\beta r^{\alpha-1}}{1 + \beta r^{\alpha-1}}. \end{aligned} \quad (5.90)$$

Hence we obtain

$$\left(\frac{g(r)}{f(r)}\right)' \leq \frac{g'(r)}{f(r)} \left[-\frac{(\alpha-1)\beta r^{\alpha-1}}{1 + \beta r^{\alpha-1}}\right] < 0 \quad (5.91)$$

for $\alpha \geq 2$ and so g/f is decreasing. We now need to show the properties of q that we have claimed are true. First, note that these properties clearly hold for $r \geq \tilde{r}$ since g is constant there and hence $q(r) = 0$. Thus, we now look at $0 \leq r < \tilde{r}$.

Also, we can quickly see from the definition of q that

$$q(0) = 1 \text{ and } q'(0) = 0. \quad (5.92)$$

Differentiating (5.81) and replacing the second derivatives of z_0 and z_1 according to the differential equations they fulfill (see equation (1.9) for $l = 0$ and $l = 1$) gives the equation

$$q' = \frac{[(f'(r)(2-n) - q)q + (n-1)]}{f(r)} - f(r)(\lambda_1 - \lambda_0) - 2sq \quad (5.93)$$

where

$$s(r) = \frac{z_0'(r)}{z_0(r)}. \quad (5.94)$$

From equation (5.93) we establish

$$q''(0) = \frac{2}{n+2} \left[\lambda_0(\Omega) \left(1 + \frac{2}{n} \right) - \lambda_1(\Omega) \right]. \quad (5.95)$$

Similarly, we can establish the following equation for s and obtain

$$s' = -s^2 - \frac{(n-1)sf'}{f} - \lambda_0. \quad (5.96)$$

From these equations we see that

$$s(0) = 0 \quad \text{and} \quad s'(0) = \frac{-\lambda_0(\Omega)}{n}. \quad (5.97)$$

Next we make the definitions

$$\varepsilon = \lambda_1(\Omega) - \lambda_0(\Omega) > 0 \quad (5.98)$$

$$N_q = q^2 - n + 1 \quad (5.99)$$

and so our formula for q' becomes

$$T(r, q) = -\varepsilon f(r) - \left[\frac{f'(r)(n-2)q + N_q}{f(r)} \right] - 2sq \quad (5.100)$$

and so we find

$$\frac{\partial T}{\partial r} = -\varepsilon f'(r) - 2s'q - \left[\frac{f(r)f''(r) - f'(r)^2}{f(r)^2} \right] (n-2)q + \frac{f'(r)}{f(r)^2} N_q. \quad (5.101)$$

We are now interested in points (r, q) where $T = 0$, so we first solve (5.93) to see

$$s|_{T=0} = \frac{-\varepsilon f(r)}{2q} - \frac{f'(r)(n-2)}{2f(r)} - \frac{N_q}{2qf(r)}, \quad (5.102)$$

and then from (5.96) and (5.102) we find that

$$\begin{aligned} s'|_{T=0} &= -\lambda_0(\Omega) + \frac{(n-1)f'(r)}{f(r)} \left(\frac{\varepsilon f(r)}{2q} + \frac{f'(r)(n-2)}{2f(r)} + \frac{N_q}{2qf(r)} \right) \\ &\quad - \left(\frac{\varepsilon f(r)}{2q} + \frac{f'(r)(n-2)}{2f(r)} + \frac{N_q}{2qf(r)} \right)^2. \end{aligned} \quad (5.103)$$

Now we substitute into (5.101) to obtain

$$\frac{\partial T}{\partial r}|_{T=0} = \frac{M_q}{f(r)^2} + (\lambda_1(\Omega) - \lambda_0(\Omega))^2 \frac{f(r)^2}{2q} + Q_q \quad (5.104)$$

where

$$M_q = -\frac{(n-2)^2}{2} q f'(r)^2 + \frac{N_q^2}{2q}, \quad (5.105)$$

$$Q_q = (\lambda_1(\Omega) - \lambda_0(\Omega)) \left(\frac{N_q}{q} - 2f'(r) \right) - \frac{f''(r)}{f(r)} (n-2)q + 2q\lambda_0(\Omega). \quad (5.106)$$

5.4 Properties of q

In this section, we now proceed to prove equations (5.83), (5.84), and (5.85), which will establish the desired monotonicity properties of $g(r)$ and $B(r)$.

Lemma 5. *For $0 \leq r \leq \tilde{r}$, we have $q(r) \geq 0$.*

Proof. Assume this is not true. Since $q(0) = 1, q'(0) = 0$, and $q(\tilde{r}) = 0$, this means there must exist two points, s_1, s_2 , such that $0 < s_1 < s_2 \leq \tilde{r}$ and $q(s_1) = q(s_2) = 0$ with $q'(s_1) \leq 0$ and $q'(s_2) \leq 0$. If $s_2 < \tilde{r}$ we see from (5.93) that

$$q'(s_1) = -\varepsilon f(s_1) + \frac{n-1}{f(s_1)} \leq 0 \quad (5.107)$$

$$q'(s_2) = -\varepsilon f(s_2) + \frac{n-1}{f(s_2)} \geq 0. \quad (5.108)$$

Also, recall that f is monotone increasing, so $f(s_2) > f(s_1)$ and $\varepsilon > 0$ and so $-\varepsilon f(s_2) < -\varepsilon f(s_1)$. Thus we obtain

$$0 \geq q'(s_1) = -\varepsilon f(s_1) + \frac{n-1}{f(s_1)} > -\varepsilon f(s_2) + \frac{n-1}{f(s_2)} = q'(s_2) \geq 0 \quad (5.109)$$

and hence arrive at a contradiction. Now supposed $s_2 = \tilde{r}$. Then, similarly we see

$$0 \geq q'(s_1) = -\varepsilon f(s_1) + \frac{n-1}{f(s_1)} > -\varepsilon f(\tilde{r}) + \frac{n-1}{f(\tilde{r})} = q'(\tilde{r}) \geq 0 \quad (5.110)$$

and arrive at the same contradiction. \square

Lemma 6. *There exists some $r_0 > 0$ such that $q(r) \leq 1$ for all $r \in (0, r_0)$ and $q(r_0) < 1$.*

Proof. Again, suppose the contrary, i.e., that $q(r)$ first increases away from $r = 0$. Then, since $q(0) = 1$ and $q(\tilde{r}) = 0$ and q is continuous and differentiable, we find two points $s_1 < s_2$ so that $\hat{q} := q(s_1) = q(s_2) > 1$ and $q'(s_1) > 0 > q'(s_2)$. Also, we may choose these points so that \hat{q} is arbitrarily close to one. So we take $\hat{q} = 1 + \delta$ and then calculate from (5.106) that

$$Q_{1+\delta} = Q_1 + \delta n \left(\lambda_1(\Omega) - \left(\frac{n-2}{n} \right) \left(\lambda_0(\Omega) + \frac{f''(r)}{f(r)} \right) \right) + \mathcal{O}(\delta^2). \quad (5.111)$$

Also, we can easily see that

$$\lambda_1(\Omega) - \left(\frac{n-2}{n} \right) \lambda_0(\Omega) \geq \lambda_1(\Omega) - \lambda_0(\Omega) > 0. \quad (5.112)$$

We may also assume that $Q_1 > 0$, for otherwise we see that $q''(0) < 0$ so that q is concave down. Note that in this case, we are done. Thus, we may choose s_1 and s_2 so that δ is sufficiently small, ensuring that $Q_{\hat{q}} > 0$. Next we want to consider $T(r, \hat{q})$ as a function of r for our fixed \hat{q} . From the definition of T and our original assumptions, we have that $T(s_1, \hat{q}) > 0 > T(s_2, \hat{q})$. Also, we see that

$$\lim_{r \rightarrow 0^+} T(r, \hat{q}) = -\infty. \quad (5.113)$$

Thus, it is apparent that $T(r, \hat{q})$ must change sign at least twice on $[0, \tilde{r}]$. So we can find two zeros $0 < \hat{s}_1 < \hat{s}_2 < \tilde{r}$ of $T(r, \hat{q})$ such that

$$T'(\hat{s}_1, \hat{q}) \geq 0 \quad \text{and} \quad T'(\hat{s}_2, \hat{q}) \leq 0. \quad (5.114)$$

Recalling our formula (5.104), we see that

$$T'(r, \hat{q}) = \left[\frac{N_{\hat{q}}^2}{2\hat{q}f(r)^2} - \frac{(n-2)^2}{2} \left(\frac{f'(r)}{f(r)} \right)^2 \hat{q} \right] + \frac{\varepsilon^2 f(r)^2}{2\hat{q}} + Q_{\hat{q}}. \quad (5.115)$$

We now wish to inspect the term in brackets. We find

$$\begin{aligned} N_{\hat{q}}^2 &= ((1+\delta)^2 - n + 1)^2 = (2 - n + 2\delta + \delta^2)^2 \\ &= (2 - n)^2 + 4\delta(2 - n) + \mathcal{O}(\delta^2) \end{aligned} \quad (5.116)$$

and

$$\left(\frac{f'(r)}{f(r)} \right)' = \frac{-2\beta\alpha r^{\alpha-1} + \beta^2\alpha^2 r^{2(\alpha-1)}}{(r + \beta r^\alpha)^2}. \quad (5.117)$$

So we see that the right hand side of (5.115) is either positive or increasing. So we cannot have that $T'(\hat{s}_2, \hat{q}) \leq 0$. Hence, we arrive at contradiction and so our lemma is proved. \square

Lemma 7. *For all $0 \leq r \leq \tilde{r}$, $q'(r) \leq 0$.*

Proof. We again proceed by contradiction. So because of our boundary conditions $q(0) = 1$ and $q(\tilde{r}) = 0$, there must exist three points $s_1 < s_2 < s_3$ in $(0, \tilde{r})$ with $0 < \hat{q} := q(s_1) = q(s_2) = q(s_3) < 1$ and $q'(s_1) < 1, q'(s_2) > 0, q'(s_3) < 0$. We again look at the function $T(r, \hat{q})$ and make the observation that for $i = 1, 2, 3$ we have

$$T(s_i, \hat{q}) = q'(s_i). \quad (5.118)$$

Taking (5.113) into account, we see $T(r, \hat{q})$ must have at least three sign changes. Hence, $T(r, \hat{q})$ has at least three zeros $\hat{s}_1 < \hat{s}_2 < \hat{s}_3$ so that

$$T'(\hat{s}_1, \hat{q}) \leq 0, \quad T'(\hat{s}_2, \hat{q}) \geq 0, \quad T'(\hat{s}_3, \hat{q}). \quad (5.119)$$

But we see again as in the previous lemma that this is impossible. Hence we arrive at our contradiction. \square

5.5 Center of Mass Theorem in \mathbb{F}^n

Now we generalize the center of mass theorem to the space \mathbb{F}^n . As before, we use this result to properly choose our coordinate system to apply the gap estimate for eigenvalues.

Theorem 19. *Let $g(r)$ be a positive continuous function on $[0, \infty)$ and $P_j = \frac{x_j}{f(r)}g(r)$.*

Then one can shift $\Omega \subset \mathbb{F}^n$ such that

$$\int_{\Omega+\nu} P_j(x) T_\nu u_0(x)^2 dV(x) = 0 \quad (5.120)$$

for all $j = 1, \dots, n$.

Proof. Here we will use the technique of Weinberger to construct a vector field which must vanish, and hence our test function will satisfy the necessary orthogonality condition. So we make the definition

$$\nu \rightarrow \langle P_j T_\nu u_0, T_\nu u_0 \rangle_{\Omega+\nu} \quad (5.121)$$

and then consider the n-vector

$$\begin{aligned} \vec{v}(\vec{\nu}) &= \int_{\Omega+\vec{\nu}} \frac{T_\nu u_0(\vec{x}) g(r) \vec{x}}{f(r)} T_\nu u_0(\vec{x})^2 dV(\vec{x}) \\ &= \int_{\Omega} u_0(\vec{u} - \vec{\nu}) \left(T_\nu^{-1} \frac{g(r) \vec{x}}{f(r)} T_\nu \right) u_0(\vec{u} - \vec{\nu}) dV(\vec{x}) \\ &= \int_{\Omega} (\vec{u} + \vec{\nu}) \frac{g(|\vec{u} + \vec{\nu}|)}{f(|\vec{u} + \vec{\nu}|)} u_0^2(\vec{u} - \vec{\nu}) dV(\vec{u}) \end{aligned} \quad (5.122)$$

as a function of the origin of the x_i coordinates and note that it is a continuous vector field. Take $B \subset \mathbb{F}^n$ to be the ball containing Ω such that the origin of the x_i coordinates is the center of B . Now take $\vec{\nu} \in \partial B$. It is a known theorem in topology that any nonvanishing vector field on a ball must point directly inward at some point. Hence, if we can show that $\vec{\nu} \cdot \vec{v}(\vec{\nu}) > 0$, then we have that this vector field is always pointing outwards. Thus, it must vanish at some point $\vec{\nu}_0 \in B$. We will take this point to be our origin.

To show this we calculate

$$\vec{\nu} \cdot \vec{\nu}(\vec{\nu}) = \int_{\Omega} \vec{\nu} \cdot (\vec{u} + \vec{\nu}) \frac{g(|\vec{u} + \vec{\nu}|)}{f(|\vec{u} + \vec{\nu}|)} u_0^2(\vec{u} - \vec{\nu}) dV(\vec{u}). \quad (5.123)$$

Since we know that $g > 0$ and $f > 0$, we must now show that

$$(\vec{u} + \vec{\nu}) \cdot \vec{\nu} > 0. \quad (5.124)$$

To do this we recognize

$$(\vec{u} + \vec{\nu}) \cdot \vec{\nu} = |\nu|^2 + \vec{\nu} \cdot \vec{u} \quad (5.125)$$

$$\geq |\nu|^2 - |\nu||u| \quad (5.126)$$

$$= |\nu|(|\nu| - |u|) > 0, \quad (5.127)$$

where (5.126) follows from the definition of the dot product and (5.127) follows from the fact that $\vec{u} \in \Omega$, $\vec{\nu} \in \partial B$, and $\Omega \subset B$. Hence our vector field here would always be outward pointing and must vanish at some point. \square

5.6 Chiti's Comparison Argument

This section will give the justification for the last step in the chains of inequalities (5.55) and (5.57). Here we take Ω^* to be the symmetric rearrangement of Ω , which is the geodesic ball centered at the origin having the same n -dimensional volume of Ω . Define $\Omega_t = \{x \in \Omega | u_0(x) > t\}$ and $\partial\Omega_t = \{x \in \Omega | u_0(x) = t\}$. Let $\mu(t) = |\Omega_t|$ and $|\partial\Omega_t| = V_{n-1}(\partial\Omega_t)$, where V_{n-1} denotes the $(n-1)$ -dimensional measure. For any function $f : \Omega \rightarrow \mathbb{R}^+$ we define the decreasing rearrangement, $f^\#$ to be

$$f^\#(s) = \inf\{t \geq 0 | \mu(t) < s\}. \quad (5.128)$$

Also, we still take $f^*(r, \vec{\chi}) = f^*(r)$ to be the symmetric decreasing rearrangement. The former is a decreasing function from $[0, |\Omega|]$ to \mathbb{R}^+ and is equimeasurable with f . The latter is defined on Ω^* , spherically symmetric, equimeasurable with f , and is

decreasing in r . The relationship between f^* and f^\sharp is given by

$$f^*(r, \vec{\chi}) = f^\sharp(A(r)), \quad (5.129)$$

where

$$A(r) = nC_n \int_0^r f^{n-1}(\bar{r}) d\bar{r} \quad (5.130)$$

is the volume of a geodesic ball with radius r . Here nC_n is the surface area of the $(n-1)$ -dimensional unit sphere in Euclidean space. We analogously define f_\sharp and f_* to be the increasing rearrangements of f .

Lemma 8. (*Chiti comparison result*) *Let $u_0(r, \vec{\chi})$ be the first Dirichlet eigenfunction of $-\Delta$ on Ω and $z_0(r)$ the first eigenfunction of $-\Delta$ on S_1 , normalized such that*

$$\int_{\Omega} u_0^2 dV = \int_{S_1} z_0^2 dV. \quad (5.131)$$

Then there exists some $r_0 \in (0, \tilde{r})$ such that

$$\begin{aligned} z_0(r) &\geq u_0^*(r) \quad \text{for } r \in (0, r_0) \text{ and} \\ z_0(r) &\leq u_0^*(r) \quad \text{for } r \in (r_0, \tilde{r}). \end{aligned}$$

Proof. Observe that the co-area formula gives the following

$$-\mu'(t) = \int_{\partial\Omega_t} \frac{1}{|\nabla u_0|} dV_{n-1} \quad (5.132)$$

(see [12], p. 86). Also, applying Gauss' Theorem (see [12], p. 7) to $-\Delta u_0 = \lambda_0 u_0$ we obtain

$$\int_{\Omega_t} \lambda_0 u_0 dV = \int_{\partial\Omega_t} |\nabla u_0| dV_{n-1}, \quad (5.133)$$

since the outward normal to Ω_t is $-\nabla u_0/|\nabla u_0|$. Using the Cauchy-Schwarz inequality and equations (5.132) and (5.133) we find that

$$(V_{n-1}(\partial\Omega_t))^2 = \left(\int_{\partial\Omega_t} dV_{n-1} \right)^2 \leq -\mu'(t) \lambda_0 \int_{\Omega_t} u_0 dV. \quad (5.134)$$

On our manifold, we also have that the classical isoperimetric inequality holds, so we have

$$V_{n-1}(\partial\Omega_t) \geq V_{n-1}(\partial(\Omega_t^*)). \quad (5.135)$$

Recall definition (5.130) and let A^{-1} be the inverse function of A . Then the $(n-1)$ -dimensional measure of $\partial(\Omega_t^*)$ can be written as

$$V_{n-1}(\partial(\Omega_t^*)) = nC_n f^{n-1} r A^{-1}(|\Omega_t^*|) = A'(A^{-1}(|\Omega_t^*|)). \quad (5.136)$$

Hence, substituting into (5.135) yields

$$V_{n-1}(\partial\Omega_t) \geq A'(A^{-1}(|\Omega_t^*|)) \quad (5.137)$$

and (5.134) can be written as

$$\lambda_0 \int_{\Omega_t} u_0 dV \geq -\frac{1}{\mu'(t)} A'(A^{-1}(|\Omega_t^*|))^2. \quad (5.138)$$

Then we use the fact that

$$\int_{\Omega_t} u_0 dV = \int_0^{\mu(t)} u_0^\sharp(s) ds, \quad (5.139)$$

which follows from the definition of u_0^\sharp . Since it is not hard to see that $u_0^\sharp(s)$ is the inverse function of $\mu(t)$, we have that

$$-\frac{du_0^\sharp}{ds} = -\frac{1}{\mu'(t)}, \quad (5.140)$$

which, when combined with (5.138) and (5.139), gives that

$$-\frac{du_0^\sharp}{ds} \leq \lambda_0 A'(A^{-1}(s))^{-2} \int_0^s u_0^\sharp(s') ds'. \quad (5.141)$$

Also, one can check that for Ω replaced by Ω^* and u_0 replaced by z_0 then equality holds in all of the steps leading to the previous equation, so we also have

$$-\frac{dz_0^\sharp}{ds} \leq \lambda_0 A'(A^{-1}(s))^{-2} \int_0^s z_0^\sharp(s') ds'. \quad (5.142)$$

Using these two relations and recalling the assumed normalization, we will show that the functions u_0^\sharp and z_0^\sharp are either identical or they cross each other exactly once on the interval $[0, |\Omega_t|]$. In the following, we make use of the fact that u_0^\sharp and z_0^\sharp are continuous. By the definition of the decreasing rearrangement, both functions are decreasing and we know that $z_0^\sharp(|\Omega_t|) = u_0^\sharp(|\Omega|) = 0$. Recall that from the Rayleigh-Faber-Krahn inequality and since we took $\lambda_0(S_1) = \lambda_0(\Omega)$ it follows that $|S_1| \leq |\Omega|$. From the normalization, it is clear that z_0^\sharp and u_0^\sharp are either identical or cross at least once on $[0, |S_1|]$. To show that they cross exactly once, we assume that they cross at least twice and obtain a contradiction. Under this assumption, there are two points $0 \leq s_1 < s_2 < |S_1|$ where $u_0^\sharp(s) > z_0^\sharp(s)$ for $s \in (s_1, s_2)$, $u_0^\sharp(s_2) = z_0^\sharp(s_2)$ and either $u_0^\sharp(s_1) = z_0^\sharp(s_1)$ or $s_1 = 0$. Now we define the following function

$$v(s) = \begin{cases} u_0^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_0^\sharp(s) ds > \int_0^{s_1} z_0^\sharp(s) ds, \\ z_0^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_0^\sharp(s) ds \leq \int_0^{s_1} z_0^\sharp(s) ds, \\ u_0^\sharp(s) & \text{on } [s_1, s_2], \\ z_1^\sharp(s) & \text{on } [s_2, |S_1|]. \end{cases} \quad (5.143)$$

Then by substituting $v(s)$ into (5.141) and (5.142) we see that

$$-\frac{dv}{ds} \leq \lambda_0 A'(A^{-1}(s))^{-2} \int_0^s v(s') ds' \quad (5.144)$$

for all $s \in [0, |S_1|]$.

Finally, we define the test function $\Psi(r, \vec{\chi}) = v(A(r))$. If z_0 and u_0 are not identical, using the Rayleigh-Ritz characterization of λ_0 , (5.144), and integration by parts, we

obtain

$$\begin{aligned}
\lambda_0 \int_{S_1} \Psi^2 dV &< \int_{S_1} |\nabla \Psi|^2 dV & (5.145) \\
&= \int_0^{\bar{r}} (A'(r)v'(A(r)))^2 A'(r) dr \\
&\leq - \int_0^{\bar{r}} A'(r)v'(A(r))\lambda_0 \int_0^{A(r)} v(s') dr \\
&= \lambda_0 \int_0^{|S_1|} v(s)^2 ds \\
&= \lambda_0 \int_{S_1} \Psi^2 dV
\end{aligned}$$

So we see that we obtain a contradiction to the assumption that u_0^\sharp and z_0^\sharp intersect twice, so the lemma is proved. □

5.7 Eigenvalue Properties

In this section we see that, in \mathbb{F}^n , the first eigenvalue is a decreasing function of the geodesic radius. Also, we prove that the first two eigenvalues of the Laplacian on a geodesic disk are the first eigenvalues of (1.9) with $l = 0$ and $l = 1$, respectively.

Lemma 9. *Take a geodesic ball $B \subset \mathbb{F}^n$ of radius r_0 . Then for any $c > 0$ we have*

$$\lambda_0(cr_0) = c^{-2}\lambda_0(r_0). \quad (5.146)$$

That is, the first eigenvalue is a decreasing function of the radius of the geodesic ball.

Proof. The proof is straightforward. Observe that, from our differential equation,

$$-z_0''(r_0) - \frac{(n-1)f'(r_0)}{f(r_0)}z_0'(r_0) = \lambda z_0(r_0). \quad (5.147)$$

Then if we make the change of variables $r_0 \rightarrow r_0/c$ we obtain

$$-\frac{z_0''(r_0/c)}{c^2} - \frac{(n-1)f'(r_0/c)}{c^2 f(r_0/c)}z_0'(r_0/c) = \lambda z_0(r_0/c). \quad (5.148)$$

Thus it is easy to see we have the desired relationship. □

Lemma 10. *The first eigenvalue of the Dirichlet Laplacian on a geodesic ball in \mathbb{F}^n is the first eigenvalue of (1.9) with $l = 0$, while the second eigenvalue on the geodesic ball is the first eigenvalue of (1.9) with $l = 1$.*

Proof. First, we define

$$h_l z = -z_l''(r) - \frac{(n-1)f'}{f} z_l'(r) + \frac{l(l+n-2)}{f^2} z_l(r) = \lambda_{z_l}(r),$$

as the operator h_l applied to z (with the boundary conditions as given in the introduction). Recall that z_0 and z_1 are the eigenfunctions corresponding to $l = 0$ and $l = 1$, respectively, in the above equation. We then have that $h_{l'} > h_l$ in the sense of quadratic forms if $l' > l$ (see [25]). Thus the lowest eigenvalue of the Dirichlet Laplacian on a geodesic ball is $\lambda_0(h_0)$.

The next step is to determine whether the second eigenvalue is $\lambda_0(h_1)$ or $\lambda_1(h_0)$. We will show that it is the former. To do so, we will make use of the following result from Coddington and Levinson, [15].

Theorem 20. *Suppose φ is a real solution on (a, b) of*

$$(px')' + g_1 x = 0 \tag{5.149}$$

and ψ is a real solution on (a, b) of

$$(px')' + g_2 x = 0 \tag{5.150}$$

Let $g_2(t) > g_1(t)$ on (a, b) . If t_1 and t_2 are successive zeros of φ on (a, b) , then ψ must vanish at some point of (t_1, t_2) .

Now observe that we may rewrite (1.9) for $l = 0$ as

$$-(f^{n-1}z')' - \lambda f^{n-1}z = 0 \tag{5.151}$$

and for $l = 1$ as

$$-(f^{n-1}z')' - (\lambda f^{n-1} - f^{n-3}(n-1))z = 0 \tag{5.152}$$

If we take $g_1 = \lambda f^{n-1} - f^{n-3}(n-1)$ and $g_2 = \lambda f^{n-1}$ we see we have that $g_2 > g_1$. Hence, the comparison result gives us that between any two zeros of z_1 there is a zero of z_0 .

Next we consider z_0 and z_1 for $\lambda = \lambda_0(h_1)$. We know that for $\lambda = \lambda_0(h_1)$, $z_1(0) = 0$ and $z_1(\tilde{r}) = 0$ and there are no other zeros of z_1 on this interval. The first consequence we obtain from the comparison theorem is that the first zero of z_0 must be strictly positive. Also, there cannot be two positive zeros of z_0 on $(0, \tilde{r})$, else we would arrive at a contradiction to the interlacing of zeros on this interval. Now we need only show that as we force the second zero of z_0 to be at \tilde{r} , we must increase the energy. So we must show that the positive zeros of any z_l are decreasing functions of λ . To do this, we may apply the comparison theorem to general z_l , here taking

$$g_i = \lambda_i f^{n-1} - f^{n-3}l(l+n-2), \quad (5.153)$$

and

$$g_j = \lambda_j f^{n-1} - f^{n-3}l(l+n-2). \quad (5.154)$$

Thus we see if $\lambda_i > \lambda_j$ then $g_i > g_j$ so the positive zeros of any z_l are decreasing functions of λ . Thus it must be the case that $\lambda_1(h_0) > \lambda_0(h_1)$. This gives us that the second eigenvalue must be $\lambda_0(h_1)$. \square

5.8 Another Gap Inequality for \mathbb{F}^n

In this final section, we present a result based on using Gram-Schmidt orthogonalization to find a suitable test function for the Rayleigh-Ritz inequality.

Theorem 21. *Let $\Omega \subset \mathbb{F}^n$ be an open bounded domain and call $\lambda_i(\Omega)$ the i -th Dirichlet eigenvalue on Ω , u_i the i -th eigenfunction. Take $g(r)$ to be a continuous, differentiable, positive function on $[0, \infty)$. Then for*

$$P_j(x) = \frac{x_j}{f(r)}g(r), \quad (5.155)$$

$j = 1, \dots, n$, we have

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2 \left(g'(r)^2 + (n-1) \left(\frac{g(r)}{f(r)} \right)^2 \right) dV}{\int_{\Omega} g(r)^2 u_0^2 dV - \sum_j |\langle P_j u_0, u_0 \rangle|^2}. \quad (5.156)$$

Proof. Our goal here is to find a suitable trial function for the Rayleigh-Ritz formula, so that we may obtain an upper bound on the gap of the first two Dirichlet eigenvalues.

First we recognize that

$$P_j(x) = \frac{x_j}{f(r)} g(r) = g(r) h_j(\psi), \quad (5.157)$$

where $h_j(\psi)$ are the spherical coordinates on the unit sphere, \mathbb{S}^n . Hence we have that

$$\sum_j |h_j(\psi)|^2 = 1. \quad (5.158)$$

So we now compute, using the formula for the gradient in \mathbb{F}^n ,

$$\nabla P_j(x) = \sum_{k=1}^n \hat{e}_k \left[\left(\frac{f'(r) - 1}{f^2(r)} \right) x_j l(x) + 1 \right] \frac{\partial P_j(x)}{\partial x_k}, \quad (5.159)$$

and

$$\begin{aligned} \sum_j |\nabla P_j|^2 &= \sum_j g'(r)^2 h_j(\psi)^2 + \sum_j g(r)^2 |\nabla h_j(\psi)|^2 \\ &= g'(r)^2 + g(r)^2 \sum_j |\nabla h_j|^2 \\ &= g'(r)^2 + (n-1) \left(\frac{g(r)}{f(r)} \right)^2. \end{aligned} \quad (5.160)$$

Let us now take for a trial function

$$\Phi_j u_0 = (P_j - \langle P_j u_0, u_0 \rangle) u_0, \quad (5.161)$$

under the assumption that $\|u_0\| = 1$. Thus we easily see that

$$\langle \Phi_j u_0, u_0 \rangle = 0, \quad (5.162)$$

and so we have a suitable test function. Hence, we are able to substitute into the Rayleigh-Ritz gap formula to see that

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} |\nabla \Phi_j|^2 u_0^2 dV}{\int_{\Omega} \Phi_j^2 u_0^2 dV}. \quad (5.163)$$

Also, we immediately see that

$$\nabla \Phi_j = \nabla P_j, \quad (5.164)$$

which we know from (5.159). So now we compute that

$$\begin{aligned} \int_{\Omega} \Phi_j^2 u_0^2 &= \int_{\Omega} ((P_j - \langle P_j u_0, u_0 \rangle) u_0) (\Phi_j u_0) dV \\ &= \int_{\Omega} u_0 P_j (P_j - \langle P_j u_0, u_0 \rangle) u_0 dV \\ &= \int_{\Omega} P_j^2 u_0^2 dV - |\langle P_j u_0, u_0 \rangle|^2, \end{aligned} \quad (5.165)$$

which allows us to see that

$$\sum_j \int_{\Omega} \Phi_j^2 u_0^2 dV = \int_{\Omega} g(r)^2 u_0^2 dV - \sum_j |\langle P_j u_0, u_0 \rangle|^2. \quad (5.166)$$

Thus we may now substitute into (5.163) to see that

$$\lambda_1(\Omega) - \lambda_0(\Omega) \leq \frac{\int_{\Omega} u_0^2 \left(g'(r)^2 + (n-1) \left(\frac{g(r)}{f(r)} \right)^2 \right) dV}{\int_{\Omega} g(r)^2 u_0^2 dV - \sum_j |\langle P_j u_0, u_0 \rangle|^2}. \quad (5.167)$$

□

Chapter 6 Szegö-Weinberger Inequality

We now find an eigenvalue inequality for the first nonzero eigenvalue of

$$-\Delta u = \mu u \quad \text{on } \Omega$$

with the boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Here Ω is a bounded domain, and $\partial/\partial n$ denotes the outward normal derivative. However this places a restriction on our boundary in that we must be able to define the normal derivative there. Another way to formulate this problem is in the variational sense where we consider the functional

$$F[\phi] = \frac{\|\nabla\phi\|^2}{\|\phi\|^2} \tag{6.1}$$

and the associated infimum

$$\mu_1(\Omega) = \inf_{\substack{\phi \in H^1(\Omega) \\ 0 \neq \phi \perp 1}} \frac{\int_{\Omega} |\nabla\phi|^2}{\int_{\Omega} \phi^2}. \tag{6.2}$$

We observe that in the case of a “nice” boundary as is outlined in section 2.1, we do get that this eigenvalue agrees with our first set-up.

In this chapter we will look at this problem for the spaces $\mathbb{R}^n, \mathbb{H}^n, \mathbb{F}^n$, and for more general manifolds as outlined in the final section. The proofs in the first two sections are due to Weinberger [27] and Ashbaugh and Benguria [4]. The first result is stated in the following theorem.

6.1 Szegö-Weinberger in \mathbb{R}^n

Theorem 22. *Let $\mu_1(\Omega)$ be the first nonzero Neumann eigenvalue for a bounded domain $\Omega \subset \mathbb{R}^n$. Then*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*)$$

with equality if and only if $\Omega = \Omega^*$. Here Ω^* is the symmetric rearrangement of Ω , so it has the same volume.

Proof. We follow the proof given in [27]. The general method involves finding suitable test functions to put in the Rayleigh-Ritz formula to obtain an upper bound for the first nonzero eigenvalue. We first look at the eigenfunctions on the sphere as potential candidates. The key property we will observe is that the eigenfunctions satisfy the orthogonality condition required for valid test functions, as is shown in the Center of Mass Theorem below.

1. Let R be the radius of the n -sphere of a given volume, V and let $\mu_1(\Omega^*)$ be the first nonzero Neumann eigenvalue for the sphere. This eigenvalue has multiplicity n and corresponding eigenfunctions

$$\frac{g(r)x_i}{r}, \quad i = 1, \dots, n. \quad (6.3)$$

In this case, r is the distance from the origin, the x_i are Cartesian coordinates, and $g(r)$ satisfies the following differential equation

$$g'' + \frac{n-1}{r}g' + \left(\mu_1(\Omega^*) - \frac{n-1}{r^2} \right) g = 0 \quad (6.4)$$

for $0 < r < R$ and vanishes at $r = 0$. Note also that the first zero of the outward normal derivative, $g'(r)$, occurs at $r = R$.

Define the function $G(r)$ as follows

$$G(r) = \begin{cases} g(r) & r \leq R \\ g(R) & r > R \end{cases} \quad (6.5)$$

and then substitute into the eigenfunctions to obtain the functions

$$f_i = \frac{G(r)x_i}{r}. \quad (6.6)$$

We fix our origin at the point so that

$$\langle f_i, 1 \rangle_{L^2(\Omega)} = \int_{\Omega} \frac{G(r)x_i}{r} dV = 0, \quad (6.7)$$

using the center of mass theorem from Chapter 3 and the fact that the Laplacian commutes with translations. Then by substituting f into the inequality given by the variational principle, we obtain that

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} [G'^2 x_i x_i / r^2 + G^2 (1 - x_i x_i / r^2) / r^2] dV}{\int_{\Omega} [G^2 x_i x_i / r^2] dV}. \quad (6.8)$$

Now by summing and simplifying, we get

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} [G'^2(r) + (n-1)G^2(r)/r^2] dV}{\int_{\Omega} G^2(r) dV}. \quad (6.9)$$

2. Take Ω^* to be the ball of radius R centered at the origin. Let Ω_1 be the intersection of Ω^* and Ω . Since R is the first zero of $g'(r)$, $G(r)$ is non-decreasing for $r > 0$. We now would like to obtain a lower bound for the denominator in our variational inequality.

Thus,

$$\begin{aligned} \int_{\Omega} G^2(r) dV &= \int_{\Omega_1} G^2(r) dV + \int_{\Omega - \Omega_1} G^2(r) dV \\ &\geq \int_{\Omega_1} G^2(r) dV + G^2(R) \int_{\Omega - \Omega_1} dV, \end{aligned} \quad (6.10)$$

and

$$\begin{aligned} \int_{\Omega^*} G^2(r) dV &= \int_{\Omega_1} G^2(r) dV + \int_{\Omega^* - \Omega_1} G^2(r) dV \\ &\leq \int_{\Omega_1} G^2(r) dV + G^2(R) \int_{\Omega^* - \Omega_1} dV. \end{aligned} \quad (6.11)$$

Thus

$$\int_{\Omega_1} G^2(r) dV \geq \int_{\Omega^*} G^2(r) dV - G^2(R) |\Omega^* - \Omega_1|, \quad (6.12)$$

where $|\cdot|$ is the standard Lebesgue measure on \mathbb{R}^n . Note that by definition, Ω^* and Ω have the same volume. This gives us that

$$\int_{\Omega - \Omega_1} G^2(R) dV = \int_{\Omega^* - \Omega_1} G^2(R) dV, \quad (6.13)$$

so substituting into the above inequalities gives

$$\int_{\Omega} G^2(r) dV \geq \int_{\Omega^*} G^2(r) dV = \int_{\Omega^*} g^2(r) dV. \quad (6.14)$$

3. Next, we would like to obtain an upper bound for the numerator in our variational inequality. Hence, we differentiate to get that

$$\frac{d}{dr} \left[G'^2 + (n-1) \frac{G^2}{r^2} \right] = 2G'G'' + 2(n-1)(rGG' - G^2)/r^3, \quad (6.15)$$

which is clearly negative for $r > R$ since G is constant there. If $r \leq R$ observe that $g(r) \equiv G(r)$ and from (6.4) we see that

$$G'' = -\frac{n-1}{r}G' + \left(\frac{n-1}{r^2} - \mu_1(\Omega^*) \right) G. \quad (6.16)$$

Thus we can substitute into (6.15) to obtain

$$\begin{aligned} \frac{d}{dr} \left[G'^2 + (n-1) \frac{G^2}{r^2} \right] &= 2G' \left[-\frac{n-1}{r}G' + \left(\frac{n-1}{r^2} - \mu_1(\Omega^*) \right) G \right] \\ &\quad + 2(n-1) \frac{(rGG' - G^2)}{r^3} \\ &= -2 \left[\mu_1(\Omega^*)GG' + \frac{(n-1)(G')^2}{r} - \frac{(n-1)GG'}{r^2} \right] \\ &\quad + 2(n-1) \frac{(rGG' - G^2)}{r^3} \\ &= -2 \left[\mu_1(\Omega^*)GG' + (n-1) \left(\frac{(G')^2}{r} - 2\frac{GG'}{r^2} + \frac{G^2}{r^3} \right) \right] \\ &= -2 \left[\mu_1(\Omega^*)GG' + (n-1)(rG' - G)^2/r^3 \right] < 0. \end{aligned} \quad (6.17)$$

So since $g'(r) \geq 0$ for $0 \leq r \leq R$, the integrand in the numerator is decreasing for $r > 0$.

4. Let us define the following function,

$$B(r) = G'^2 + (n-1) \frac{G^2}{r^2}. \quad (6.18)$$

Then we may apply a method similar to that used for the denominator to show that

$$\int_{\Omega} B(r) dV = \int_{\Omega_1} B(r) dV + \int_{\Omega - \Omega_1} B(r) dV$$

$$\leq \int_{\Omega_1} B(r) dV + B(R) \int_{\Omega - \Omega_1} dV, \quad (6.19)$$

and

$$\begin{aligned} \int_{\Omega^*} B(r) dV &= \int_{\Omega_1} B(r) dV + \int_{\Omega^* - \Omega_1} B(r) dV \\ &\geq \int_{\Omega_1} B(r) dV + B(R) \int_{\Omega^* - \Omega_1} dV. \end{aligned} \quad (6.20)$$

Again by definition, Ω^* and Ω have the same volume. This gives us that

$$\int_{\Omega - \Omega_1} B(R) dV = \int_{\Omega^* - \Omega_1} B(R) dV, \quad (6.21)$$

so substituting into (6.19) and (6.20) gives

$$\int_{\Omega} \left[G'^2 + (n-1) \frac{G^2}{r^2} \right] dV \leq \int_{\Omega^*} \left[g'^2 + (n-1) \frac{g^2}{r^2} \right] dV. \quad (6.22)$$

This is only equality if Ω is a sphere (except for a set a measure zero). So we have shown that

$$\mu_1(\Omega) \leq \frac{\int_{\Omega^*} B(r) dV}{\int_{\Omega^*} g^2(r) dV}. \quad (6.23)$$

Thus we need only now show

$$\frac{\int_{\Omega^*} B(r) dV}{\int_{\Omega^*} g^2(r) dV} = \mu_1(\Omega^*). \quad (6.24)$$

Now if we apply polar coordinates and then integration by parts, we obtain

$$\int_{\Omega^*} (g')^2 dV = \int_0^R \int_{\mathbb{S}^{n-1}} (g')^2 r^{n-1} d\sigma dr = \omega_n \int_0^R (g')^2 r^{n-1} dr \quad (6.25)$$

$$= -\omega_n \int_0^R \left[g''g + \frac{(n-1)}{r} g' \right] r^{n-1} dr = - \int_{\Omega^*} g''g + \frac{(n-1)}{r} g' dV. \quad (6.26)$$

So if we recall (6.4), we see this gives

$$\int_{\Omega^*} \left[(g')^2 + \frac{n-1}{r^2} g^2 \right] dV = \mu_1(\Omega^*) \int_{\Omega^*} g^2 dV. \quad (6.27)$$

Thus we have the desired result

$$\mu_1(\Omega) \leq \mu_1(\Omega^*). \quad (6.28)$$

□

6.2 Szegő-Weinberger in Hyperbolic Space

We now consider this problem for domains contained in the space of constant negative sectional curvature. This proof was outlined in [4], and we give the details here. We again wish to show that the first non-zero Neumann eigenvalue is maximized on the geodesic ball of a given hyperbolic volume. So now, given the new metric as is outlined in section 4.1, our differential equation becomes

$$h_1(g) = \frac{-1}{\sinh^{n-1} r} \frac{d}{dr} \sinh^{n-1} r \frac{dg}{dr} + \frac{n-1}{\sinh^2 r} g = \mu g, \quad (6.29)$$

and in the same manner as before we obtain that the first nonzero Neumann eigenvalue for the geodesic ball of radius R on \mathbb{H}^n , $\mu_1(R)$ is the first eigenvalue of this equation. Next we see that g is an increasing function on $[0, R]$ via the following lemma.

Lemma 11. *If $0 < R$, then $g' > 0$ on $[0, R)$, where g is the eigenfunction for the first nonzero eigenvalue of*

$$\frac{-1}{\sinh^{n-1} r} \frac{d}{dr} \sinh^{n-1} r \frac{dg}{dr} + \frac{n-1}{\sinh^2 r} g = \mu g,$$

(with $g > 0$ on $(0, R]$ assumed). Thus g is strictly increasing on $[0, R]$. In addition, the first eigenvalue μ_1 satisfies

$$\mu_1 > \frac{n-1}{\sinh^2 R}.$$

Proof. Define the following function

$$N(r) \equiv (\sinh r)^{n-1} g'(r). \quad (6.30)$$

Then we see that $N(0) = 0, N(R) = 0$ and by differentiation and substituting into the differential equation

$$g'' = -\mu_1 g + \frac{n-1}{\sinh^2 r} g - (n-1)g' \coth r \quad (6.31)$$

one obtains

$$N'(r) = \left[\frac{n-1}{\sinh^2 r} - \mu_1 \right] (\sinh r)^{n-1} g(r). \quad (6.32)$$

Next we notice that our function N first increases from 0 and then decreases to 0 as we move along the interval $[0, R]$. First note that for small r we have that $N'(r)$ must be positive. Next observe since $\frac{n-1}{\sinh^2 r}$ is monotonic decreasing, this implies that $(n-1)\operatorname{csch}^2 r - \mu_1$ changes sign at most once. However, also note that since N decreases to 0, $(n-1)\operatorname{csch}^2 r - \mu_1$ must change sign at least once. Hence $(n-1)\operatorname{csch}^2 r - \mu_1$ changes sign once in $(0, R)$ and from this we see that $\mu_1 > \frac{n-1}{\sinh^2 R}$. Thus we see $N > 0$ on $(0, R)$ and so it follows that $g' > 0$. \square

Similarly, our function $B(r)$ becomes $g'(r)^2 + (n-1)g(r)^2/\sinh^2(r)$. So we need to see that $B(r)$ is decreasing on $[0, R]$. Writing the differential equation as

$$g'' = -(n-1)g' \coth r + (n-1)g \operatorname{csch}^2 r - \mu_1 g,$$

we can easily compute that

$$B' = -2[\mu_1 g g' + (n-1)\{\cosh r(\sinh r g' - g)^2 + 2(\cosh r - 1) \sinh r g g'\}]/\sinh^3 r.$$

Recall that we have $g' > 0, g > 0, \mu_1 > 0$, and $(\cosh r - 1) > 0$ so we see that $B' < 0$. Finally, we arrive at our main theorem.

Theorem 23. *Let Ω be a bounded domain in a space of constant negative sectional curvature $\kappa = -1$. Then its first nonzero Neumann eigenvalue $\mu_1(\Omega)$ satisfies*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^ is a geodesic ball in the same space having the same n -volume as Ω . Equality occurs if and only if Ω is a geodesic ball.*

Proof. The proof follows in a manner very similar to the Weinberger case substituting the equations for g and B involving $\sinh r$. We follow the sketch of this proof given in [4] and fill in the necessary details. First we apply separation of variables and compare the eigenfunctions of the resulting family of equations to see that the first

eigenvalue of

$$\frac{-1}{\sinh^{n-1} r} \frac{d}{dr} \sinh^{n-1} r \frac{dg}{dr} + \frac{n-1}{\sinh^2 r} g = \mu g, \quad (6.33)$$

is the first nonzero Neumann eigenvalue of the Laplacian. We use the same argument as in the case of constant positive curvature and note that we have the result for all $r \in [0, \infty)$ since $\coth r > 0$ here. Also we have that $g(0)$ is finite. Now we must pick trial functions for the Rayleigh-Ritz variational inequality which gives

$$\mu_1 \leq \frac{\int_{\Omega} |\nabla \phi|^2 dV}{\int_{\Omega} \phi^2 dV}, \quad \text{where } \phi \in H^1(\Omega) \setminus \{0\} \quad \text{and} \quad \int_{\Omega} \phi dV = 0. \quad (6.34)$$

So we then apply the center of mass argument in Theorem 16 to Ω taking $P_i(r) = G(r)x_i/\sinh r$ to get a choice of coordinates as in section 4.1 so that

$$\int_{\Omega} G(r)(x_i/\sinh r) dV = 0 \quad (6.35)$$

for $i = 1, 2, \dots, n$, where

$$G(r) = \begin{cases} g(r) & 0 \leq r \leq R \\ g(R) & R \leq r \end{cases}$$

We now compute an upper bound on μ_1 . Substituting $P_i(r)$ into (6.34) gives us the n inequalities

$$\mu_1 \int_{\Omega} G(r)^2 (x_i/\sinh r)^2 d\omega \leq \int_{\Omega} |\nabla(G(r)x_i/\sinh r)|^2 d\omega \quad \text{for } 1 \leq i \leq n.$$

Then we apply the formula from page 51 of [12] and use the metric given in section 4.1 to see that

$$|\nabla(G(r)x_i/\sinh r)|^2 = G'(r)^2 \left(\frac{x_i}{\sinh r}\right)^2 + \left(\frac{G(r)}{\sinh r}\right)^2 \sum_{\alpha, \beta} \frac{\partial x_i}{\partial u^\alpha} g^{\alpha\beta} \frac{\partial x_i}{\partial u^\beta}. \quad (6.36)$$

Thus by summing the inequalities we obtain

$$\mu_1 \leq \frac{\int_{\Omega} [G'(r)^2 + (n-1)G(r)^2/\sinh^2 r] d\omega}{\int_{\Omega} G(r)^2 d\omega}.$$

Now we make the definition that

$$G'(r)^2 + (n-1)G(r)^2/\sinh^2 r = \begin{cases} B(r) & 0 \leq r \leq R \\ (n-1)g(R)^2/\sinh^2 R & R \leq r. \end{cases} \quad (6.37)$$

As noted above, $B' < 0$ and so it is a decreasing function. So our inequality becomes

$$\mu_1 \leq \frac{\int_{\Omega} B(r) dV}{\int_{\Omega} G(r)^2 dV}. \quad (6.38)$$

Now we may apply rearrangement results to the following integrals

$$\int_{\Omega} B(r) dV = \int_{\Omega^*} B(r)^* dV \leq \int_{\Omega^*} B(r) dV \quad (6.39)$$

and

$$\int_{\Omega} G(r)^2 dV = \int_{\Omega^*} G(r)^2_* dV \geq \int_{\Omega^*} G(r)^2 dV = \int_{\Omega^*} g(r)^2 dV, \quad (6.40)$$

where Ω^* denotes the geodesic ball in \mathbb{H}^n having the same n -volume as Ω . Thus we see that

$$\mu_1 \leq \frac{\int_{\Omega^*} B(r) dV}{\int_{\Omega^*} g(r)^2 dV} = \mu_1(\Omega^*). \quad (6.41)$$

We also note that if we had started with $\Omega = D$ then we would arrive at equality since the functions $g(r)x_i/\sinh r$ for $0 \leq r \leq R$ are all exact eigenfunctions corresponding to μ_1 in that case. In addition, if Ω is not a geodesic ball we see that the inequalities must be strict, hence our proof is completed. \square

6.3 Szegő-Weinberger for \mathbb{F}^n

Finally, we consider the Neumann problem for our set of spherically symmetric Riemannian manifolds. Here we will follow a method similar to the previous two cases, only now our metric is the one given in section 5.1. We wish to show this theorem holds for the set of f we have chosen to define the manifolds.

So now, given the new metric our differential equation for the first nonzero Neumann eigenvalue becomes

$$h_1(g) = \frac{-1}{f^{n-1}} \frac{d}{dr} f^{n-1} \frac{dg}{dr} + \frac{n-1}{f^2} g = \mu g. \quad (6.42)$$

The first nonzero Neumann eigenvalue for the geodesic ball of radius \tilde{r} on \mathbb{F}^n , $\mu_1(\tilde{r})$ is the first solution of this equation, as is shown in the following lemma. Next we see that g is an increasing function on $[0, \tilde{r}]$ via the following lemma.

Lemma 12. *Let $\mu_1(\tilde{r})$ be the first nonzero Neumann eigenvalue for the geodesic ball of radius \tilde{r} on \mathbb{F}^n . If $r \in [0, \infty)$, then $\mu_1(\tilde{r})$ is given by the first eigenvalue of*

$$\frac{-1}{f^{n-1}(r)} \frac{d}{dr} f^{n-1}(r) \frac{dg}{dr} + \frac{n-1}{f^2(r)} g = \mu g. \quad (6.43)$$

Proof. By separation of variables in spherical coordinates, we can see that the eigenvalues of the Neumann Laplacian for a geodesic ball in \mathbb{F}^n of radius \tilde{r} are the eigenvalues of the one-dimensional problems

$$h_l(y) \equiv \frac{-1}{f^{n-1}(r)} \frac{d}{dr} \left(f^{n-1}(r) \frac{dy}{dr} \right) + \frac{l(l+n-2)}{f^2(r)} y = \mu y \quad (6.44)$$

on $(0, \tilde{r})$ for $l = 0, 1, 2, \dots$, where y must satisfy the boundary conditions $y(0)$ is finite and $y'(\tilde{r}) = 0$. It is easy to see that $h_{l'} > h_l$ in a quadratic form sense if $l' > l$ since we see for fixed n , the only change is in the term involving l . Thus $\mu_1(\tilde{r})$ is either the second eigenvalue of h_0 (the first eigenvalue is 0) or the first eigenvalue of h_1 . We show that the latter is the case. We proceed by comparing the first eigenvalue μ_1 of

$$h_{1,g} = \frac{-1}{f^{n-1}(r)} \frac{d}{dr} f^{n-1}(r) \frac{dg}{dr} + \frac{n-1}{f^2(r)} g = \mu g, \quad (6.45)$$

where $g(0)$ is finite and $g'(\tilde{r}) = 0$ with the second eigenvalue τ_2 for

$$h_{0,v} = \frac{-1}{f^{n-1}(r)} \frac{d}{dr} f^{n-1}(r) \frac{dv}{dr} = \tau v, \quad (6.46)$$

where $v(0)$ is finite and $v'(\tilde{r}) = 0$. Note that g is a first eigenfunction we may assume that $g > 0$ in $(0, \tilde{r}]$.

Now take $h(r)$ to be a nonsingular and nontrivial solution to the equation

$$-h'' - \frac{(n-1)f'}{f} h' = \mu_1 h \quad \text{on } (0, \tilde{r}]. \quad (6.47)$$

We see by differentiating the operator for $l = 0$ that g and h' must be proportional so we may assume that $g = h'$. Also we see by differentiation that v and h satisfy the same equation except for the associated eigenparameters, we can compute that

$$\frac{d}{dr}[f^{n-1}(vh' - v'h)] = (\tau_2 - \mu_1)vhf^{n-1}. \quad (6.48)$$

Since v is a second eigenfunction, it must change signs in $(0, \tilde{r})$ else we would not have that v is orthogonal to the constant function. Thus we can pick $a \in (0, \tilde{r})$ such that $v(a) = 0$ and $v'(a) < 0$. Now integrate (6.48) from 0 to a to get

$$(\tau_2 - \mu_1) \int_0^a vhf^{n-1} dr = [f^{n-1}(vh' - v'h)]|_0^a = (f(a))^{n-1}(-v'(a))h(a). \quad (6.49)$$

From this we see that the inequality $\mu_1 < \tau_2$ will follow if we show that $h < 0$ on $(0, \tilde{r})$. Since $h' = g > 0$ on $(0, \tilde{r}]$ we have that h is increasing on this interval so we need only show that $h(\tilde{r}) \leq 0$. To see this rewrite (6.47) as

$$\mu_1 h(\tilde{r}) = -g'(\tilde{r}) - \frac{(n-1)f'(\tilde{r})}{f(\tilde{r})}g(\tilde{r}) = -\frac{(n-1)f'(\tilde{r})}{f(\tilde{r})}g(\tilde{r}).$$

Thus we see that $h(\tilde{r}) \leq 0$ if $\tilde{r} \in (0, \infty]$, so it follows that $h(a) < 0$. \square

Now we want to show that the function g associated with μ_1 is strictly increasing on $[0, \tilde{r}]$ since this function will appear in the denominator of the ratio we will obtain as an upper bound for the first nonzero Neumann eigenvalue from the Rayleigh-Ritz characterization.

Lemma 13. *We assume $g > 0$ on $(0, \tilde{r}]$, then $g' > 0$ on $[0, \tilde{r})$, where g is the first eigenfunction for (6.42).*

Thus g is strictly increasing on $[0, \tilde{r}]$. In addition, the first eigenvalue μ_1 satisfies

$$\mu_1 > \frac{n-1}{f^2(\tilde{r})}. \quad (6.50)$$

Proof. Define the following function

$$N(r) \equiv (f(r))^{n-1}g'(r).$$

Then we see that $N(0) = 0, N(\tilde{r}) = 0$ and by differentiation and substituting into the differential equation one obtains

$$N'(r) = \left[\frac{n-1}{f^2(r)} - \mu_1 \right] (f(r))^{n-1} g(r).$$

Next we notice that our function N first increases from 0 and then decreases to 0 as we move along the interval $[0, \tilde{r}]$. First note that since $\frac{n-1}{f^2(r)}$ is monotonic decreasing, this implies that $\frac{n-1}{f^2(r)} - \mu_1$ changes sign at most once. However, also note that since N decreases to 0, it must change sign at least once. Hence $\frac{n-1}{f^2(r)} - \mu_1$ changes sign once in $(0, \tilde{r})$ and from this we see that $\mu_1 > \frac{n-1}{f^2(\tilde{r})}$. Also, since we see $N > 0$ on $(0, \tilde{r})$ it follows that $g' > 0$. \square

Lemma 14. *We have that $f' \geq 1$ for the f associated with the metric on \mathbb{F}^n .*

Proof. Clearly

$$f'(r) = 1 + \alpha r^{\alpha-1} \geq 1 \tag{6.51}$$

for $r \in [0, \infty), \alpha \geq 2$. Now for our second set of potential f , observe that for $r \in [0, \tilde{r}_1]$, $f'(r) = 1$ and for $r \geq \tilde{r}_2$, $f'(r) = \alpha r^{\alpha-1}$. Hence here $f' \geq 1$ since $\alpha \geq 2$ and $\tilde{r}_2 \geq 1$. Also observe that by definition, we have that $f(\tilde{r}_1) = 1$. As before, we appeal to numerical data to show that our examples satisfy this condition. In the following figures we will present the graphs of these functions for $2 \leq \alpha \leq 9$ to see that f has the desired property here. \square

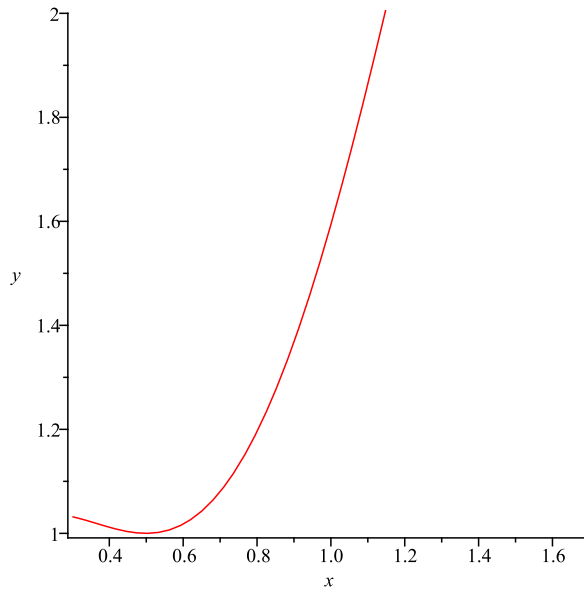


Figure 6.1: $f'(r)$ for $\alpha = 2$ on $[.5, 1.5]$

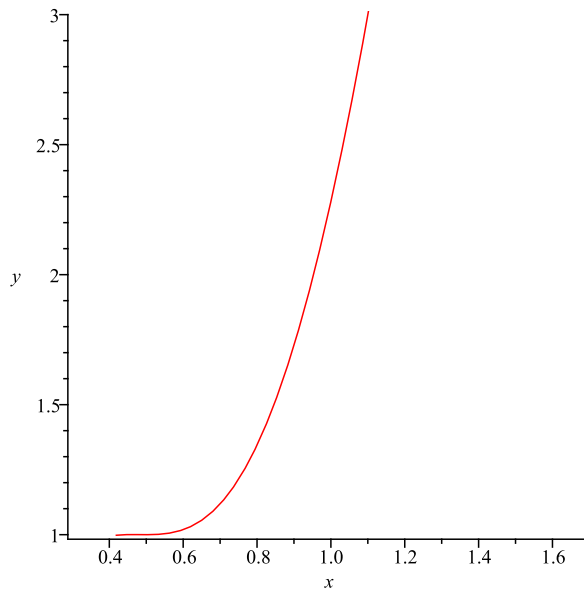


Figure 6.2: $f'(r)$ for $\alpha = 3$ on $[.5, 1.5]$

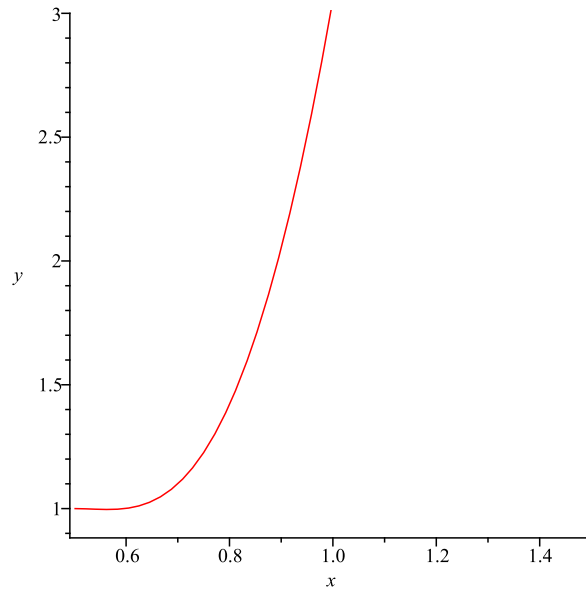


Figure 6.3: $f'(r)$ for $\alpha = 4$ on $[.5, 1.5]$

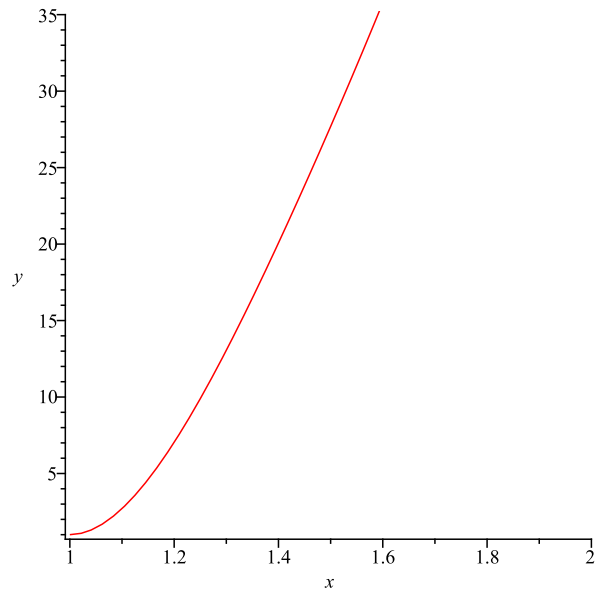


Figure 6.4: $f'(r)$ for $\alpha = 5$ on $[1, 2]$

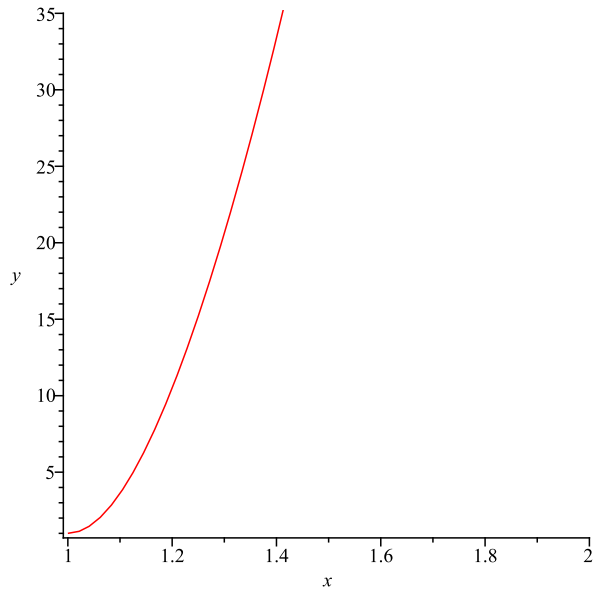


Figure 6.5: $f'(r)$ for $\alpha = 6$ on $[1,2]$

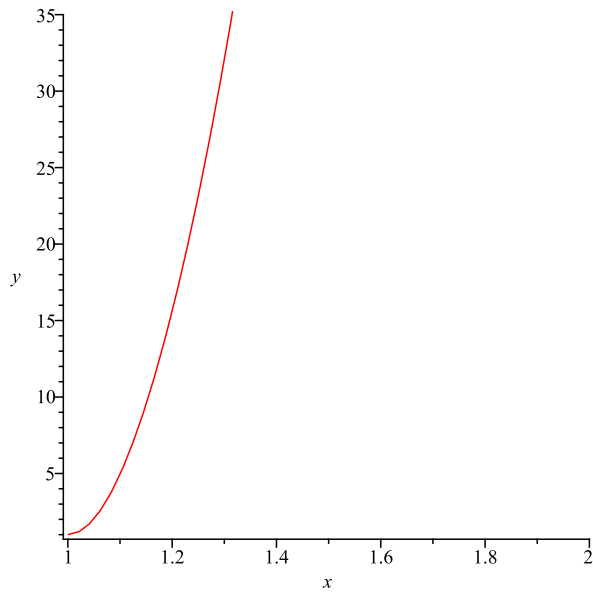


Figure 6.6: $f'(r)$ for $\alpha = 7$ on $[1,2]$

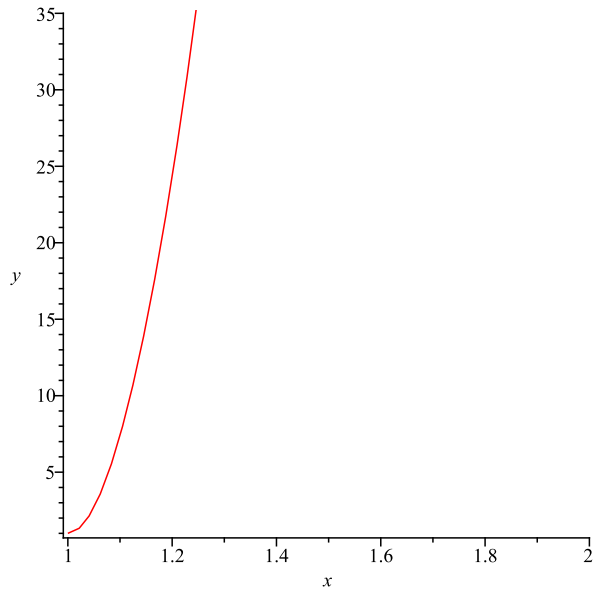


Figure 6.7: $f'(r)$ for $\alpha = 8$ on $[1,2]$

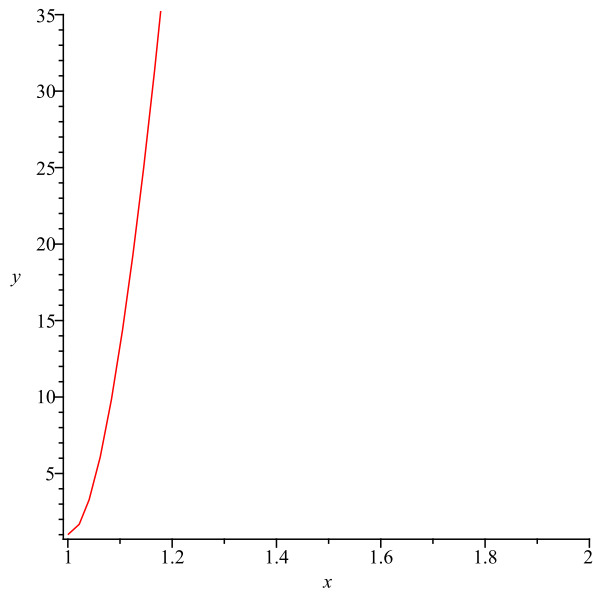


Figure 6.8: $f'(r)$ for $\alpha = 9$ on $[1,2]$

Finally, we arrive at our main theorem. Here we also need the extra symmetry conditions on our domain Ω as in Theorem 18 so that we may choose the appropriate test functions.

Theorem 24. *Let $\Omega \subset \mathcal{S}$ be a bounded open set in \mathbb{F}^n . Then its first nonzero Neumann eigenvalue $\mu_1(\Omega)$ satisfies*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^* is a geodesic ball in the same space having the same n -volume as Ω . Equality occurs if and only if Ω is a geodesic ball.

Proof. The proof follows in a manner very similar to the Weinberger case substituting the equations for g and B involving f . First we apply separation of variables and compare the eigenfunctions of the resulting family of equations to see that the first eigenvalue of

$$\frac{-1}{f^{n-1}} \frac{d}{dr} f^{n-1} \frac{dg}{dr} + \frac{n-1}{f^2} g = \mu g,$$

is the first nonzero Neumann eigenvalue of the Laplacian. We use the same argument as in the case of constant curvature and note that we have the result for all $r \in [0, \infty)$ since $\frac{f'}{f} > 0$ here. Also we have that $g(0)$ is finite. Now we must pick trial functions for the Rayleigh-Ritz variational inequality which gives

$$\mu_1 \leq \frac{\int_{\Omega} |\nabla \phi|^2 d\omega}{\int_{\Omega} \phi^2 d\omega}, \quad \text{where } \phi \in H^1(\Omega) \setminus \{0\} \quad \text{and} \quad \int_{\Omega} \phi d\omega = 0.$$

So we then take $P_i(r) = G(r)x_i/f(r)$ to be our trial functions and get a choice of coordinates so that $\int_{\Omega} G(r)(x_i/f(r)) dV = 0$ for $i = 1, 2, \dots, n$, which is possible as in Theorem 18, where

$$G(r) = \begin{cases} g(r) & 0 \leq r \leq \tilde{r} \\ g(\tilde{r}) & \tilde{r} \leq r. \end{cases}$$

Thus we have an upper bound on μ_1 . This gives us the n inequalities

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} |\nabla(G(r)x_i/f(r))|^2 dV}{\int_{\Omega} G(r)^2(x_i/f(r))^2 dV} \quad \text{for } 1 \leq i \leq n. \quad (6.52)$$

So we now want to compute $|\nabla(G(r)x_i/f(r))|^2$. We now make use of our metric given in section 5.1 and we write

$$P_i(r, x_i) = \frac{G(r)}{f(r)}x_i \quad (6.53)$$

and take the x_i to be the spherical coordinates in our space and

$$|d\omega|^2 = \sum_{\alpha, \beta} g_{\alpha\beta} du^\alpha du^\beta.$$

Then we apply the formula from page 51 of [12] to see that

$$|\nabla(G(r)x_i/f(r))|^2 = G'(r)^2 \left(\frac{x_i}{f(r)}\right)^2 + \left(\frac{G(r)}{f(r)}\right)^2 \sum_{\alpha, \beta} \frac{\partial x_i}{\partial u^\alpha} g^{\alpha\beta} \frac{\partial x_i}{\partial u^\beta}. \quad (6.54)$$

Thus by summing the inequalities we obtain

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} [G'(r)^2 + (n-1)G(r)^2/f(r)^2] dV}{\int_{\Omega} G(r)^2 dV}. \quad (6.55)$$

Now we make the definition that

$$G'(r)^2 + (n-1)G(r)^2/f(r)^2 = \begin{cases} B(r) & 0 \leq r \leq \tilde{r} \\ (n-1)g(\tilde{r})^2/f(\tilde{r})^2 & \tilde{r} \leq r, \end{cases} \quad (6.56)$$

Now we need the function in the numerator of the bound we obtain from the Rayleigh-Ritz characterization of the first nonzero eigenvalue to be decreasing. So we show that $B(r)$ is decreasing on $[0, \tilde{r}]$. Writing the differential equation as

$$g'' = -\frac{n-1}{f}g' + \frac{n-1}{f^2}g - \mu_1 g \quad (6.57)$$

we can easily compute that

$$B' = -2[\mu_1 g g' + (n-1)\{f'(f g' - g)^2 + 2(f' - 1) f g g'\}/f^3]. \quad (6.58)$$

Recall that we have $g' > 0$, $g > 0$, $\mu_1 > 0$, and $(f' - 1) > 0$ so we see that $B' < 0$. So our inequality becomes

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} B(r) dV}{\int_{\Omega} G(r)^2 dV}. \quad (6.59)$$

Now we may apply rearrangement results to the following integrals

$$\int_{\Omega} B(r) dV = \int_{\Omega^*} B(r)^* dV \leq \int_{\Omega^*} B(r) dV \quad (6.60)$$

and

$$\int_{\Omega} G(r)^2 dV = \int_{\Omega^*} G(r)_*^2 dV \geq \int_{\Omega^*} G(r)^2 dV = \int_{\Omega^*} g(r)^2 dV, \quad (6.61)$$

where Ω^* denotes the geodesic ball in \mathbb{F}^n having the same n -volume as Ω . Thus we see that

$$\mu_1(\Omega) \leq \frac{\int_{\Omega^*} B(r) dV}{\int_{\Omega^*} g(r)^2 dV} = \mu_1(\Omega^*).$$

We also note that if we had started with $\Omega = \Omega^*$ then we would arrive at equality since the functions $g(r)x_i/f(r)$ for $0 \leq r \leq \tilde{r}$ are all exact eigenfunctions corresponding to μ_1 in that case. In addition, if Ω is not a geodesic ball we see that the inequalities must be strict, hence our proof is completed. \square

6.4 Szegő-Weinberger for More General Spherically Symmetric Manifolds

In this final section, we observe that the theorem of Szegő-Weinberger will hold in a more general set of spherically symmetric Riemannian manifolds than the set \mathbb{F}^n . We again take

$$ds^2 = dr^2 + f(r)^2|d\omega|^2. \quad (6.62)$$

to be the metric of this space. However, we only need the following restrictions on our function f : $f(r) > 0$, $f(0) = 0$, $f'(0) = 1$, $f'(r) \geq 1$, and $f \in C^2(\Omega)$ for $r \in [0, \infty)$. So if we take a bounded domain Ω contained in this set of manifolds, we again have that the Szegő-Weinberger Theorem holds. For example, we could take the functions $f(r) = r + r^\alpha$ or $f(r) = \frac{1}{2}r(1 + e^r)$. We will call this more general set of manifolds $\tilde{\mathbb{F}}^n$. So then the theorem becomes

Theorem 25. *Let $\Omega \subset \mathcal{S}$ be a bounded open set in $\tilde{\mathbb{F}}^n$. Then its first nonzero Neumann eigenvalue $\mu_1(\Omega)$ satisfies*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where Ω^ is a geodesic ball in the same space having the same n -volume as Ω . Equality occurs if and only if Ω is a geodesic ball.*

Chapter 7 Vector Fields on Spheres

In order to prove the various center of mass theorems, we need the following theorems from topology. First we will introduce some definitions and notation.

Definition. If $A \subset X$, a retraction of X onto A is a continuous map $r : X \rightarrow A$ such that $r|_A$ is the identity map of A . If such a map r exists, we say that A is a retract of X .

We will denote the n -th reduced homology group of X by $\tilde{\mathcal{H}}_n$ and the n -th homology group of X by $H_n(X)$.

Definition. For a map $f : X \rightarrow Y$, there is an induced homomorphism

$$f_* : H_n(X) \rightarrow H_n(Y)$$

defined by

$$f_*([c]) = [f \circ c].$$

Theorem 26. *For each n , there is no retraction $r : B^n \rightarrow \mathbb{S}^{n-1}$.*

Proof. The proof is found on page 186 of [11]. If $r : B^{n+1} \rightarrow \mathbb{S}^n$ is a retraction map, and $i : \mathbb{S}^{n-1} \rightarrow B^n$ is the inclusion, then $r \circ i = 1$. Thus the composition

$$G = \tilde{\mathcal{H}}_{n-1}(\mathbb{S}^{n-1}) \xrightarrow{i_*} \tilde{\mathcal{H}}_{n-1}(B^n) \xrightarrow{r_*} \tilde{\mathcal{H}}_{n-1}(\mathbb{S}^{n-1}) = G \quad (7.1)$$

factors the identity map $1 = r_* \circ i_* : G \rightarrow G$ through the middle group which is 0. This implies that the coefficient group G is zero. However, we know that $\tilde{\mathcal{H}}_{n-1}(\mathbb{S}^{n-1}) = \mathbb{Z}$. Thus we arrive at a contradiction. \square

Theorem 27. *The inclusion map $j : \mathbb{S}^{n-1} \rightarrow B^n - 0$ is not nullhomotopic.*

Proof. There is a retraction of $\mathbb{R}^n - 0$ onto \mathbb{S}^{n+1} given by the equation $r(x) = \frac{x}{\|x\|}$. Therefore, j_* is injective, and hence nontrivial. \square

The center of mass theorems all use a result that is related to the Brouwer fixed-point theorem, as is referred to in Weinberger's paper. Here we state the theorem and then state and prove the related result that is used for our proof of the center of mass theorem.

Theorem 28 (Brouwer's fixed-point theorem). *Let $B \subset \mathbb{R}^n$ be the unit ball for $n \geq 0$. If $f : B \rightarrow B$ is continuous then f has a fixed point, i.e., there is some $x \in B$ such that $f(x) = x$.*

The proof of this theorem is found in many topology texts, for example, [23].

Theorem 29. *Every nonvanishing vector field on B^n points directly outward at some point of \mathbb{S}^{n-1} , and directly inward at some point of \mathbb{S}^{n-1} .*

Proof. Here we follow [23]. A vector field on B^n is an ordered pair $(x, v(x))$, where x is in B^n and v is a continuous map of B^n into \mathbb{R}^n . To say that a vector field is nonvanishing means that $v(x) \neq 0$ for every x ; in such a case $v : B^n \rightarrow \mathbb{R}^n - 0$.

We suppose first that $v(x)$ does not point directly inward at any point x of \mathbb{S}^{n-1} and derive a contradiction. Consider the map $v : B^n \rightarrow \mathbb{R}^n - 0$; let w be its restriction to \mathbb{S}^{n-1} . Because B^n is contractible and the map w extends to a map of B^n into $\mathbb{R}^n - 0$ it is nulhomotopic.

On the other hand, w is homotopic to the inclusion map $j : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n - 0$. Now we define the homotopy

$$F(x, t) = tx - (1 - t)w(x), \tag{7.2}$$

for $x \in \mathbb{S}^{n-1}$. We must show that $F(x, t) \neq 0$. Clearly, $F(x, t) \neq 0$ for $t = 0$ and $t = 1$. If $F(x, t) = 0$ for some t with $0 < t < 1$, then $tx + (1 - t)w(x) = 0$, so that $w(x)$ equals a negative scalar multiple of x . But this means that $w(x)$ points directly

inward at x . Hence F maps $\mathbb{S}^{n-1} \times [0, 1]$ into $\mathbb{R}^n - 0$ as desired. It follows that j is nulhomotopic, but this contradicts the preceding theorem.

To show that v points directly outward at some point of \mathbb{S}^{n-1} , we apply the result just proved to the vector field $(x, -v(x))$. □

Bibliography

- [1] Ashbaugh, M., **Isoperimetric and Universal Inequalities for Eigenvalues**, Spectral theory and geometry (Edinburgh, 1998), London Math. Soc. Lecture Note Ser., **273**, Cambridge Univ. Press, Cambridge, (1999), 95-139.
- [2] Ashbaugh, M. and R. Benguria, **A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions**, Annals of Mathematics, **135** (1992), 601-628.
- [3] Ashbaugh, M. and R. Benguria, **A sharp bound for the ratio of the first two Dirichlet eigenvalues of a domain in a hemisphere of \mathbb{S}^n** , Trans. Amer. Math. Soc., **353** (2001), 1055-1087.
- [4] Ashbaugh, M. and R. Benguria, **Sharp upper bound to the first nonzero Neumann eigenvalue for bounded domains in spaces of constant curvature**, J. London Math. Soc. (2), **52** (1995), 402-416.
- [5] Ashbaugh, M. and R. Benguria, **Proof of the Payne-Pólya-Weinberger conjecture**, Bull. Amer. Math. Soc., **25** (1991), 19-29.
- [6] Ashbaugh, M. and R. Benguria, **A second proof of the Payne-Pólya-Weinberger conjecture**, Comm. Math. Phys., **147** (1992), 181-190.
- [7] Bandle, C., **Isoperimetric Inequalities and Applications**, Pitman Monographs and Studies in Mathematics, **7** Pitman, Boston, 1980.
- [8] Benguria, R. and H. Linde, **A Second Eigenvalue Bound for the Dirichlet Laplacian in Hyperbolic Space**, Duke Math J., **140** (2007), 245-279.
- [9] Benguria, R. and H. Linde, **Isoperimetric Inequalities for Eigenvalues of the Laplace Operator**, Contemporary Mathematics Series, **476** American Mathematical Society, Providence, 2008.
- [10] Birkhoff, G. and G.C. Rota, **Ordinary Differential Equations**, Wiley, New York, 1978.
- [11] Bredon, G., **Topology and Geometry**, Springer, New York, 1993.
- [12] Chavel, I., **Eigenvalues in Riemannian Geometry**, Academic Press, Inc., 1984.
- [13] Chavel, I., **Isoperimetric Inequalities**, Cambridge University Press, Cambridge, 2001.
- [14] Chiti, G., **A bound for the ratio of the first two eigenvalues of a membrane**, SIAM J. Math. Anal, **14** (1983), 1163 - 1167.

- [15] Coddington, E. and N. Levinson, **Theory of Ordinary Differential Equations**, McGraw-Hill, New York, 1955.
- [16] do Carmo, M., **Riemannian Geometry**, Birkhauser, Boston, 1992.
- [17] Dodziuk, J., L^2 **Harmonic Forms on Rotationally Symmetric Riemannian Manifolds**, Proc. Amer. Math. Soc., **77** (1979), 395-400.
- [18] Evans, R., and R. Gariepy, **Measure Theory and Fine Properties of Functions**, CRC Press, Ann Arbor, 1992.
- [19] Faber, G., **Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt**, Sitzungsber. Bayr. Akad. Wiss. München, Math-Phys. Kl., (1923), 169-172.
- [20] Federer, H., **Geometric Measure Theory**, Springer-Verlag, New York, 1969.
- [21] Krahn, E., **Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises**, Math. Ann., **94** (1925), 97-100.
- [22] Lieb, E. H., and M. Loss, **Analysis**, Graduate Studies in Mathematics, vol. **14**, American Mathematical Society, Providence, Rhode Island, 1997.
- [23] Munkres, J., **Topology**, Prentice-Hall, NJ, 1975.
- [24] Rayleigh, J. W. S., **The Theory of Sound**, second edition, Dover Publications, New York, 1945 (republication of 1894 edition).
- [25] Reed, M. and B. Simon, **Methods of Modern Mathematical Physics, IV: Analysis of Operators**, Academic Press, New York, 1978.
- [26] Thompson, C. J., **On the Ratio of Consecutive Eigenvalues in N-dimensions**, Stud. Appl. Math., **48** 1969.
- [27] Weinberger, H. F., **An Isoperimetric Inequality for the n -dimensional Free Membrane Problem**, J. Rational Mech. Anal., **5** (1956), 633-637.

Vita

- **Personal Information**

- Born May 21, 1982 in Fairmont, West Virginia

- **Education**

- 2009 (Expected), Ph.D., University of Kentucky

- 2006, M.A., University of Kentucky

- 2004, B.S., Grove City College

- **Scholastic and Professional Honors**

- AY 2008-2009, Provost's Award for Outstanding Teaching

- AY 2008-2009, Algebra Cubed Fellowship

- AY 2007-2008, Outstanding Teaching Assistant in the College of Arts and Sciences

- Summer 2009, 2008, Summer Research Award

Author's signature: _____ Julie Miker

Date: _____ December 11, 2009