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HILBERT POLYNOMIALS AND STRONGLY STABLE IDEALS

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HILBERT POLYNOMIALS AND STRONGLY STABLE IDEALS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Dennis Moore
Lexington, Kentucky

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ABSTRACT OF DISSERTATION

HILBERT POLYNOMIALS AND STRONGLY STABLE IDEALS

Strongly stable ideals are important in algebraic geometry, commutative algebra, and combinatorics. Prompted, for example, by combinatorial approaches for studying Hilbert schemes and the existence of maximal total Betti numbers among saturated ideals with a given Hilbert polynomial, three algorithms are presented. Each of these algorithms produces all strongly stable ideals with some prescribed property: the saturated strongly stable ideals with a given Hilbert polynomial, the almost lexsegment ideals with a given Hilbert polynomial, and the saturated strongly stable ideals with a given Hilbert function. Bounds for the complexity of our algorithms are included. Also included are some applications for these algorithms and some estimates for counting strongly stable ideals with a fixed Hilbert polynomial.

KEYWORDS: Strongly Stable Ideals, Hilbert Functions, Hilbert Polynomials, Betti Numbers, Lexsegment Ideals

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Date: April 18, 2012

HILBERT POLYNOMIALS AND STRONGLY STABLE IDEALS

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Chapter 1 Introduction

Strongly stable monomial ideals arise naturally in algebraic geometry, commutative algebra, and combinatorics. In particular, Galligo, Bayer, and Stillman showed that the generic initial ideal of a homogeneous ideal is Borel-fixed. In characteristic zero, Borel-fixed ideals are strongly stable. Thus, strongly stable ideals play a prominent role in the structure of Hilbert schemes. A Hilbert scheme parametrizes the closed subschemes of a projective space with a fixed Hilbert polynomial. Its scheme structure is very complex. Strongly stable ideals are the basis for combinatorial approaches for studying Hilbert schemes. Strongly stable ideals also figure prominently in the algebraic approach to shifting. Shifting is a combinatorial technique that studies a given simplicial complex by modifying the given complex to a simpler one while preserving essential properties.

The main contribution of this dissertation, Algorithm 3.22, is an efficient algorithm for producing all saturated strongly stable ideals in a certain polynomial ring with a specified Hilbert polynomial. An upper bound for the complexity of this algorithm is given by the largest degree of a minimal generator in the saturated lexsegment ideal associated to the Hilbert polynomial (see Theorem 3.20); this bound can easily be determined from the Hilbert polynomial. This algorithm will be helpful for further study of Hilbert schemes. The algorithm can also be modified to suit other needs, as we illustrate.

Another highlight is the study of ideals with maximal total Betti numbers for a given Hilbert polynomial. In Proposition 4.5, we point out that, if a saturated ideal has as at least many minimal generators as any other ideal with the same Hilbert polynomial, then it will have maximal Betti numbers. In Examples 4.1 and 4.2, we demonstrate that the Hilbert functions of ideals with maximal Betti numbers are not always comparable. Algorithm 4.18 is a proposed construction for a saturated ideal whose Betti numbers are at least as large as the Betti numbers of any other saturated ideal with the same Hilbert polynomial *and* a specified initial degree. This construction would give a common generalization of work by Valla [33], who showed that there is an ideal with maximal Betti numbers for any constant Hilbert polynomial and fixed initial degree, and also of Caviglia and Murai [6], who showed that there is an ideal with maximal Betti numbers for any Hilbert polynomial.

Finally, we would like to emphasize an intriguing conjecture, an upper bound for the number of saturated strongly stable ideals with a given constant Hilbert polynomial stated in Conjecture 5.15.

We now give a more detailed description of the contents of the following pages.

In Chapter 2, we review some background material. In particular, we first define strongly stable ideals and lexsegment ideals and recall some basic properties. We then describe Hilbert functions, Hilbert polynomials, and Hilbert series, observing a connection between Hilbert polynomials and saturated lexsegment ideals. Next, we give a brief discussion of free resolutions and Betti numbers; this section includes formulas for the total Betti numbers, Hilbert polynomials, and Hilbert series of strongly stable ideals using the Eliahou–Kervaire resolution. We conclude by noting that the graded Betti numbers of a lexsegment ideal are maximal among all ideals with the same Hilbert function and a recent result of Caviglia and Murai that there is a saturated ideal whose total Betti numbers are maximal among all saturated ideals with the same Hilbert polynomial.

In Chapter 3, we develop several algorithms which produce certain saturated strongly stable ideals. We first introduce an operation, called an expansion, which replaces a minimal generator of a saturated strongly stable ideal with multiples of this generator so that the resulting ideal is another saturated strongly stable ideal. We then show how this procedure can be iterated to produce all saturated strongly stable ideals with a specified Hilbert polynomial. Next we show that we can modify this algorithm to produce the almost lexsegment ideals with a fixed Hilbert polynomial. A saturated strongly stable ideal is called almost lexsegment if it is a lexsegment ideal when considered in the polynomial ring where the last variable is dropped; almost lexsegment ideals play a crucial role in the study of ideals with maximal Betti numbers. In particular, the first algorithm may find several ideals with the same Hilbert series, while the latter will give exactly one ideal for each of these Hilbert series associated to the given Hilbert polynomial.

In Chapter 3, we also include an algorithm to generate all saturated strongly stable ideals with a given Hilbert series. These ideals form a subset of the ideals obtained by Algorithm 3.22; however, we present a more direct and efficient algorithm for computing them. We conclude the chapter with several consequences of the proof of the first algorithm. Specifically, we note that there is a fixed number of saturated strongly stable ideals in a polynomial ring with a sufficiently large number of variables which depends only on the given Hilbert polynomial—adding more variables will *not* increase the number of ideals. We also classify the saturated homogeneous ideals with the worst Castelnuovo–Mumford regularity.

In Chapter 4, we study saturated ideals whose Betti numbers are at least as large as the Betti numbers of any other saturated ideal with the same Hilbert polynomial. The major focus is the construction of a saturated ideal whose total Betti numbers are at least as large as the Betti numbers of any other saturated homogeneous ideal with the same Hilbert polynomial and initial degree. By a result of Bigatti, Hulett, and Pardue, it is sufficient to consider only almost lexsegment ideals. We give examples to illustrate that, in general, the Hilbert functions of ideals with maximal Betti numbers are not comparable. We observe that the construction can be reduced to choosing which monomial generators to expand and that we should favor generators which are only divisible by the first few variables. We demonstrate that a greedy algorithm will not be sufficient. We also include examples throughout the chapter showing that the construction cannot be much simpler.

In Chapter 5, we begin to study how many saturated strongly stable ideals, in a polynomial ring with a fixed number of variables, have a given Hilbert polynomial. We attack the problem by focusing on constant Hilbert polynomials. We identify the ideals with certain integer partitions, which can easily be described. This approach allows us to give a generating function when there are three variables. We also give a conjectured generating function when there are four variables. We conclude with a conjecture for a generating function giving an upper bound for the number of saturated strongly stable variables with a constant Hilbert polynomial in any number of variables.

The algorithms presented here have been implemented in the computer algebra system *Macaulay2*, [13]. This code has been included in an appendix.

Chapter 2 Preliminaries

In this chapter, we introduce some concepts and results which will play a fundamental role in the remaining chapters. In Section 1, we define strongly stable ideals and saturations. In Section 2, we introduce numerical invariants for modules, such as the Hilbert function. In Section 3, we describe Betti numbers and the extremal properties of lexsegment ideals. For more background, refer to [8], [16], [17], and [22].

Throughout this note we denote by R the polynomial ring, $K[x_0, \ldots, x_n]$, with $n+1$ variables over an arbitrary field K. At times, we will want to consider the polynomial rings where the last few variables are eliminated, so we denote by $R^{(1)}$ the polynomial ring $K[x_0, \ldots, x_{n-1}]$ where the last variable has been removed, and, more generally, $R^{(j)}$ is the polynomial ring $K[x_0, \ldots, x_{n-j}]$ where the last j variables have been removed.

We use multi-index notation for monomials: If $A = (a_0, \ldots, a_n)$ is an n-tuple of non-negative integers, then x^A is the monomial $x_0^{a_0} \cdots x_n^{a_n}$. Moreover, if $x^A \neq 1$, the *max index* of x^A is $\max\{i : a_i > 0\} = \max\{i : x_i | x^A\}$, and denoted $\max(x^A)$.

We will use the standard grading on R: the degree of a monomial x^A is the sum of the exponents, $a_0 + \cdots + a_n$. We also use the lexicographic order, $>_{lex}$, for comparing monomials of a given degree. Let $x^B = x_0^{b_0} x_1^{b_1} \cdots x_n^{b_n}$ and $x^C = x_0^{c_0} x_1^{c_1} \cdots x_n^{c_n}$ be two monomials of R of the same degree. Recall that $x^B >_{lex} x^C$, if the first nonzero entry of the vector $(b_0 - c_0,\ b_1 - c_1,\ \ldots,\ b_n - c_n)$ is positive.

If I is a monomial ideal, we denote by $G(I)$ the unique minimal monomial generators of I.

2.1 Strongly stable and lexsegment ideals

We begin by introducing an important class of monomial ideals, called strongly stable ideals. These ideals will be studied throughout the rest of this work. Thus, we make a few remarks about their significance. We also introduce a special type of strongly stable ideal called a lexsegment ideal.

Strongly stable ideals are a particular subset of a class of monomial ideals called stable ideals. A monomial ideal $I \subset R$ is a *stable ideal* if, for each monomial in a stable ideal, replacing the variable of largest index with a variable of smaller index produces another monomial in the ideal. The Hilbert polynomials and Betti numbers of stable ideals admit closed formulas because stable ideals have a simple free resolution—see Remark 2.35 and the preceding paragraph. We now introduce strongly stable ideals.

Definition 2.1. A monomial ideal $I \subset R$ is a *strongly stable ideal* if, for every monomial $x^A \in I$ and $x_j | x^A$, $(x_i/x_j) \cdot x^A \in I$ for all i between 0 and j.

In each monomial in a strongly stable ideal, we may replace any variable with a variable of smaller index to get another monomial in the ideal. When deciding whether or not an ideal is strongly stable, it is sufficient to check that the monomial generators satisfy the above condition. Clearly, strongly stable ideals are stable; however a stable ideal may not be strongly stable.

Example 2.2. The ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset K[x_0, x_1, x_2]$ is strongly stable.

The ideal $I_2 = (x_0^2, x_0x_1, x_1^2, x_1x_2) \subset K[x_0, x_1, x_2]$ is stable, but not strongly stable: if we replace x_1 with x_0 in the monomial $x_1x_2 \in I_2$ we get $x_0x_2 \notin I_2$.

Because of the structure, certain quotients of strongly stable ideals are easy to compute.

Proposition 2.3. (Bayer and Stillman) *If $I \subset R$ is a strongly stable ideal, then, for every $0 \le j \le n$ and $t \ge 0$,*

$$I : x_j^t = I : (x_0, \ldots, x_j)^t.$$

In particular,

$$I : x_j^\infty = I : (x_0, \ldots, x_j)^\infty.$$

Proof. Choose an integer t and monomial x^A so that $x^A x_j^t \in I$. Because I is strongly stable, every monomial $x^A x_0^{t_0} x_1^{t_1} \cdots x_j^{t_j} \in I$, if $t_0 + t_1 + \cdots + t_j = t$. Thus, $(I : x_j^t) \subset (I : (x_0, \ldots, x_j)^t)$. The reverse inclusion is clear because $x_j^t \in (x_0, \ldots, x_j)^t$. □

We next introduce the concept of the saturation of an ideal. Saturations play an important role in primary decompositions and determining projective closures. The homogeneous ideal of a closed subscheme is saturated.

Definition 2.4. The *saturation* of an ideal $I \subset R$ (with respect to the maximal ideal, $(x_0, \ldots, x_n)$),denoted $sat_{x_n}(I)$, is the ideal

$$I : (x_0, \ldots, x_n)^\infty = \{f \in R : f \cdot (x_0, \ldots, x_n)^d \subset I \text{ for some } d \in \mathbb{N}\}$$

We say an ideal is saturated if it is equal to its saturation. Because of the structure of strongly stable ideals, their saturations can easily be computed.

Remark 2.5. Let $I \subset R$ be a strongly stable ideal. The saturation of I is obtained from I by setting $x_n = 1$ in every monomial of I. An ideal I is saturated if the variable x_n does not appear in the minimal monomial generators of I.

Because $I : (x_0, \ldots, x_n)^\infty = I : x_n^\infty$ by Proposition 2.3, if $x^A x_n^t \in I$, then $x^A \in sat_{x_n}(I) = I : (x_0, \ldots, x_n)^\infty$.

Additionally, we will need the concept of a double saturation for a strongly stable ideal; doubly saturated ideals will play an important part in the algorithm for producing all strongly stable ideals with a given Hilbert polynomial.

Definition 2.6. The *double saturation* of a strongly stable ideal I is the extension ideal $sat_{x_{n-1},x_n}(I)$ in R of the saturation of $sat_{x_n}(I) \cap R^{(1)} \subset R^{(1)}$. This ideal is obtained from I by setting $x_n = x_{n-1} = 1$. An ideal is doubly saturated if the variables x_{n-1} and x_n do not appear in the minimal monomial generators of I.

Example 2.7. The ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset K[x_0, x_1, x_2]$ is a saturated strongly stable ideal.

The ideal $I_3 = (x_0, x_1^3, x_1^2x_2) \subset K[x_0, x_1, x_2]$ is not saturated. The saturation is $sat_{x_n}(I_3) = (x_0, x_1^2)$. The double saturation of I_3 is the entire ring $K[x_0, x_1, x_2]$.

We now explain the significance of strongly stable ideals. The first fact is that they are invariant under a particular group action.

Remark 2.8. If the characteristic of the field K is zero, then an ideal is fixed under the action of the Borel group (that is, the set of upper triangular matrices) if and only if the ideal is strongly stable. (Roughly, this is because each matrix in the Borel group corresponds to a change of coordinates where each variable, x_i, is mapped to a linear combination of the variables whose index is no larger, $x_0, \ldots, x_i$.) In positive characteristic, other ideals are also fixed by the Borel group.

In the literature, strongly stable ideals are sometimes called *Borel-fixed ideals* (or simply *Borel ideals*); however, this can be confusing because the set of ideals which is fixed by the Borel group depends on the characteristic of the field. Occasionally, the term *0-Borel* is used (which indicates the characteristic of the field). The latter term is much better than the former.

Often, questions about arbitrary ideals in R can be answered by considering monomial ideals. For instance, the Hilbert function of a homogeneous ideal, I, may be computed instead for the initial ideal of I (with respect to a fixed term order). Recall that an *initial ideal* is the ideal generated by largest terms (according to the term order) of the polynomials in the original ideal. Unfortunately, the initial ideal depends on the choice of coordinates; however, if a generic change of coordinates is made, this dependence is eliminated. The ideal produced in this manner is the *generic initial ideal*; see [8].

Generic initial ideals are combinatorial invariants, which contain information about the original ideal, such as the depth of the ideal. Generic initial ideals play a critical role in Hartshorne's proof of the connectedness of the Hilbert scheme—see [15]. Generic initial ideals are also used by Reeves to study the component structure of the Hilbert scheme—see [29]. For more information about generic initial ideals, see [14]. Generic initial ideals are, in fact, very particular monomial ideals.

Theorem 2.9. (Galligo, Bayer and Stillman) *For any term order on R such that $x_0 > x_1 > \cdots > x_n$, the generic initial ideal of a homogeneous ideal is Borel-fixed. In particular, if the characteristic of K is zero, generic initial ideals are strongly stable.*

The generic initial ideal depends only on the term order. Generally, the graded reverse lexicographic order is used in computations because it gives a much simpler ideal than the lexicographic order (that is, fewer generators and generators of smaller degree).

We conclude this section with the concept of a lexsegment ideal.

Definition 2.10. A monomial ideal $I \subset R$ is a *lexsegment ideal* if, for every integer t, $[I]_t$ is spanned by the first $\dim_K [I]_t$ monomials of $[R]_t$ in the lexicographic order.

In other words, for any monomial, x^B, in a lexsegment ideal in a given degree, any other monomial in the same degree, x^A, such that $x^A >_{lex} x^B$, is also in the ideal.

Example 2.11. The ideal $I_3 = (x_0, x_1^3, x_1^2 x_2) \subset K[x_0, x_1, x_2]$ is lexsegment.

The ideal $I_1 = (x_0^2, x_0 x_1, x_1^3) \subset K[x_0, x_1, x_2]$ is not lexsegment: $x_0 x_2^2 >_{lex} x_1^3$, $x_0 x_2^2 \notin I_1$, and $x_1^3 \in I_1$.

The example above illustrates that a lexsegment ideal may not be lexsegment when considered in a ring with more variables. We will be interested in ideals which are lexsegment in rings with more variables. Universal lexsegment ideals have been studied in [6] and [23].

Definition 2.12. A lexsegment ideal $I \subset R$ is a *universal lexsegment ideal* if the ideal $I \cdot R[x_{n+1}] \subset R[x_{n+1}]$ is lexsegment.

Example 2.13. The ideal $I_3 = (x_0, x_1^3, x_1^2 x_2) \subset K[x_0, x_1, x_2]$ is a universal lexsegment ideal.

The ideal $I_1 = (x_0^2, x_0 x_1, x_1^3) \subset K[x_0, x_1]$ is not a universal lexsegment as noted in the previous example.

Universal lexsegment ideals can easily be characterized.

Proposition 2.14. *A lexsegment ideal $I \subset R$ is a universal lexsegment ideal if and only if I has at most $n+1$ minimal generators.*

Proof. Suppose I is a lexsegment ideal and that $G(I) = \{x^{A_0}, x^{A_1}, \dots x^{A_t}\}$, where $\deg x^{A_i} \leq \deg x^{A_{i+1}}$ and $x^{A_i} >_{lex} x^{A_{i+1}}$ if $\deg x^{A_i} = \deg x^{A_{i+1}}$. For $1 \leq i \leq \min t, n$, x^{A_i} is of the form $x^{A_{i-1}} \cdot \frac{x_i^b}{x_{i-1}}$, where $b = 1 + \deg x^{A_i} - \deg x^{A_{i-1}}$, because this is the next monomial in the lexicographic order in any ring containing R in the correct degree. If $t > n$, then the monomial $x^B = x^{A_n} \cdot \frac{x_{n+1}^b}{x_n}$ is not in the ideal $I \cdot R[x_{n+1}] \subset R[x_{n+1}]$; however, the monomial $x^{A_{n+1}}$ is in $I \cdot R[x_{n+1}] \subset R[x_{n+1}]$ and $x^B >_{lex} x^A$. $\square$

Lexsegment ideals arise naturally in the study of Hilbert functions (see Remark 2.17) and have certain extremal properties (see Theorem 2.36). Note that lexsegment ideals are a special subset of strongly stable ideals: if $i < j$, then $(x_i/x_j) \cdot x^A >_{lex} x^A$.

2.2 Hilbert functions, polynomials, and series

We now define the Hilbert function, Hilbert polynomial, and Hilbert series. These numerical invariants give the size of a graded module in specific degrees. We also describe a correspondence between Hilbert polynomials and saturated lexsegment ideals.

Definition 2.15. If M is a finitely generated graded R-module, with grading by degree, then the *Hilbert function of M* is the numerical function

$$h_M(t) := \dim_K M_t.$$

Because M is finitely generated, each of these dimensions will be finite. The Hilbert function of R, in degree t, is the number of monomials in R of degree t.

Example 2.16. The Hilbert function of the polynomial ring, R, is

$$h_R(t) = \begin{cases} \binom{n+t}{n} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}.$$

The Hilbert function of the quotient of the ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset R = K[x_0, x_1, x_2]$ is

$$h_{R/I_1}(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t = 0 \\ 3 & \text{if } t = 1 \\ 4 & \text{if } t \geq 2 \end{cases}.$$

There is a natural correspondence between Hilbert functions and lexsegment ideals.

Remark 2.17. In 1927, Macaulay gave a complete characterization for Hilbert functions of modules of the form R/I for a homogeneous ideal I—see [32]. Macaulay also showed that each Hilbert function is attained by a unique lexsegment ideal.

One can think of the Hilbert function as being an infinite sequence. It turns out that most of the information from the Hilbert function is captured by a polynomial.

Theorem 2.18. (Hilbert) *If M is a finitely generated graded R-module, then $h_M(t)$ agrees with a polynomial of degree at most $n+1$ when $t \gg 0$ (that is, when t is sufficiently large).*

Definition 2.19. The polynomial in the theorem above is the *Hilbert polynomial of M*, denoted $p_M(t)$.

Theorem 2.18 can easily be proved by induction on the number of variables in the polynomial ring—see Theorem 1.11 in [8] or Theorem 7.5 in [16]. This also follows from Theorem 2.29. From the previous example, we can determine several Hilbert polynomials.

Example 2.20. The Hilbert polynomial of the polynomial ring, R, is

$$p_R(t) = \binom{n+t}{n} = \frac{(n+t)\cdots(t+1)}{n!}.$$

The Hilbert polynomial of the quotient of the ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset R = K[x_0, x_1, x_2]$ is

$$p_{R/I_1}(t) = 4.$$

At times we will abuse language and say that a homogeneous ideal, I, of R has Hilbert function h or Hilbert polynomial p if R/I has this Hilbert function or polynomial.

If M is a module of the form R/I for a homogeneous ideal I, then the possible Hilbert polynomials can easily be characterized terms of binomial coefficients. For more details, see [16].

Remark 2.21. Let $p \in \mathbb{Q}[t]$ be the Hilbert polynomial of a standard graded K-algebra of dimension $d+1 > 0$. Then there are unique integers $b_0 \geq b_1 \geq \ldots \geq b_d > 0$ such that

$$p(t) = \sum_{i=0}^{d} \left[\binom{t+i}{i+1} - \binom{t+i-b_i}{i+1} \right]. \tag{2.1}$$

Example 2.22. The Hilbert polynomial $p_1(t) = 2t^2 + t + 2$ can be written the form

$$\binom{t+2}{3} - \binom{t+2-4}{3} + \binom{t+1}{2} - \binom{t+1-5}{2} + \binom{t}{1} - \binom{t-8}{1}$$

so $b_0 = 8 \geq b_1 = 5 \geq b_2 = 4 > 0$.

The saturation of an ideal can be related to the Hilbert polynomial in the following way. By definition, the saturation of an ideal I contains I. If the ideals are not equal, then the saturation contains monomials for which some multiple is in the original ideal I; thus, in a large enough degree, the ideal and its saturation are the same. Specifically, I and the saturation of I have the same Hilbert polynomial. Furthermore, the saturation of I is the largest ideal, among all ideals containing I that have the same Hilbert polynomial as I.

There can be infinitely many Hilbert functions which have the same Hilbert polynomial. Specifying a Hilbert polynomial instead of a Hilbert function allows for a lot of choice about exactly which multiples of a particular monomial are in the ideal. By saturating the ideal, this choice disappears.

Just as there is a unique lexsegment ideal for each Hilbert function, there is a unique saturated lexsegment ideal associated to each Hilbert polynomial. In fact, the ideal is a universal lexsegment ideal. This particular ideal can easily be described with respect to the representation given in Remark 2.21. Some of the properties of this ideal have been studied by Bayer in [2]. For the convenience of the reader we provide short proofs for the results below.

Theorem 2.23. *Let $p \neq 0$ be a Hilbert polynomial of a quotient of R. Then there is a unique saturated lexsegment ideal $L_p \subset R$ such that the Hilbert polynomial of R/L_p is p. It is called the lexicographic ideal to p. The ideal L_p is generated by the set of monomials*

$$\begin{aligned}
\{&x_0, x_1, \ldots, x_{n-d-2}, x_{n-d-1}^{a_d+1}, x_{n-d-1}^{a_d} \cdot x_{n-d}^{a_{d-1}+1}, \\
&x_{n-d-1}^{a_d} \cdot x_{n-d}^{a_{d-1}} \cdot x_{n-d+1}^{a_{d-2}+1}, \ldots, \\
&x_{n-d-1}^{a_d} \cdot x_{n-d}^{a_{d-1}} \cdot x_{n-d+1}^{a_{d-2}} \cdot \ldots \cdot x_{n-3}^{a_2} \cdot x_{n-2}^{a_1+1}, \\
&x_{n-d-1}^{a_d} \cdot x_{n-d}^{a_{d-1}} \cdot x_{n-d+1}^{a_{d-2}} \cdot \ldots \cdot x_{n-2}^{a_1} \cdot x_{n-1}^{a_0}\},
\end{aligned}$$

where p is written as in Equation (2.1) and $a_d := b_d, a_{d-1} := b_{d-1} - b_d, \ldots, a_0 := b_0 - b_1$ (thus, $b_i = a_i + a_{i+1} + \ldots + a_d$, for $0 \leq i \leq d$).

Proof. Set $L(a_0, \dots, a_d) := L_p$. This ideal is clearly lexsegment. The ideal is also saturated by Remark 2.5. We use induction on $d \geq 0$ in order to compute the Hilbert polynomial of the quotient. If $d = 0$, then we have

$$R/L(a_0) = K[x_0, \dots, x_n]/(x_0, \dots, x_{n-2}, x_{n-1}^{a_0}) \cong K[x_{n-1}, x_n]/(x_{n-1}^{a_0})$$

and thus the Hilbert polynomial is

$$p_{R/L(a_0)}(t) = a_0 = \binom{t}{1} - \binom{t - a_0}{1} = p$$

as claimed.

Let $d > 0$. Then multiplication by $x_{n-d-1}^{a_d}$ provides the exact sequence

$$0 \to (R/L(a_0, \dots, a_{d-1}))(-a_d) \xrightarrow{x_{n-d-1}^{a_d}} R/L(a_0, \dots, a_d) \to$$

$$\to R/(x_0, \dots, x_{n-d-2}, x_{n-d-1}^{a_d}) \to 0.$$

Using the induction hypothesis we conclude that $p_{R/L(a_0,\dots,a_d)} = p$.

The uniqueness statement follows from the fact that L_p is a lexsegment ideal and saturated. $\square$

Note that the set of generators of the lexicographic ideal L_p given in Theorem 2.23 is not minimal when $a_0 = 0$.

Example 2.24. From Example 2.22, the b-values for the Hilbert polynomial $p_1(t) = 2t^2 + t + 2$ are $b_2 = 4$, $b_1 = 5$, and $b_0 = 8$, so the a-values are $a_2 = 4$, $a_1 = 1$, and $a_0 = 3$. Thus, the lexicographic ideal for $p_1(t) = 2t^2 + t + 2$ in $K[x_0, \dots, x_4]$ is

$$(x_0, x_1^5, x_1^4 x_2^2, x_1^4 x_2 x_3^3).$$

The ideal L_p has alternative characterizations.

Proposition 2.25. *(a) Let $L_h \subset R$ be a lexsegment ideal with Hilbert polynomial p (that is, if $t \gg 0$, then $p(t) = h(t)$). Then the saturation of L_h is the ideal $L_p \subset R$.*

(b) Let R/I be a graded quotient of R with Hilbert polynomial p. Then, for all integers t:

$$h_{R/I}(t) \geq h_{R/L_p}(t).$$

Proof. (a) Since L_h and L_p are both lexsegment ideals and $h(t) = p(t)$ whenever $t \gg 0$, we get

$$[L_h]_t = [L_p]_t,$$

whenever $t \gg 0$. As the ideal L_p is saturated, it follows that L_p is the saturation of L_h.

(b) Denote by h the Hilbert function of R/I. Then part (a) implies $L_h \subset L_p$, and the claim follows. $\square$

Finally, all of the information in the Hilbert function can be encoded in a generating function.

Definition 2.26. The *Hilbert series of M* (or *Hilbert-Poincaré series*) is the formal power series

$$H_M(t) := \sum_{j\in\mathbb{Z}} h_M(j)\cdot t^j.$$

If M is a module of the form R/I for a homogeneous ideal I, then the (non-reduced) Hilbert series has the form

$$H_{R/I}(t) = \frac{g(t)}{(1-t)^{n+1}}$$

for some polynomial $g(t) \in \mathbb{Z}[t]$. This in turn can be simplified to the reduced form $H_{R/I}(t) = \widetilde{g}(t)/(1-t)^e$, where $\widetilde{g}(t) \in \mathbb{Z}[t]$ and $\widetilde{g}(1) \neq 0$. (Note that e is the Krull dimension of M.)

From Example 2.16, we can determine several Hilbert series.

Example 2.27. The Hilbert series of the polynomial ring, R, is

$$H_R(t) = \frac{1}{(1-t)^{n+1}}.$$

The Hilbert series for the ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset R = K[x_0, x_1, x_2]$ is

$$\begin{aligned} H_{R/I_1}(t) &= 1 + 3t + 4t^2 + 4t^3 + 4t^4 + \cdots \\ &= \frac{1-2t^2+t^4}{(1-t)^3} \\ &= \frac{1+2t+t^2}{1-t}. \end{aligned}$$

(The middle line gives the non-reduced form, and the last line is the reduced form.)

Formulas for the Hilbert polynomial and Hilbert series for quotient rings R/I, where I is a stable ideal ideal, are given in the next section—see Remark 2.35.

2.3 Free resolutions and Betti numbers

We now want to define the total Betti numbers for a finitely generated graded module over R, so we will start with the concept of a resolution. Using Betti numbers, we can give formulas for the Hilbert function and Hilbert polynomial of stable ideals and state an extremal property of lexsegment ideals. For more details, see [8].

Definition 2.28. A *finite free resolution* (or simply *free resolution*) of M, a finitely generated graded R-module, is a complex

$$\mathcal{F}: 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

where the image of the map ϕ_i is the kernel of ϕ_{i-1} and each module F_i is a free R-module, for i between 1 and s. We say that s is the *length* of the resolution $\mathcal{F}$.

Fortunately, every module over the polynomial ring R has a finite free resolution. (This is not true for modules over an arbitrary ring.)

Theorem 2.29. (Hilbert) *Every finitely generated graded R-module has a finite graded free resolution of length at most $n+1$.*

Two different proofs of this result are provided in [8]; see chapters 15 and 19.

Because each map ϕ_i is a map between free R-modules, it can be represented as a matrix which acts on the standard basis by multiplication. In this setting, the number of columns in the matrix for ϕ_i is the rank of the module F_i, and the number of rows is the rank of the module F_{i-1}.

Example 2.30. A free resolution of the ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset R = K[x_0, x_1, x_2]$ is

$$0 \to R \xrightarrow{\begin{bmatrix} x \\ -1 \\ y^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} 0 & -y^3 & -y \\ -y^2 & 0 & x \\ x & x^2 & 0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R \to R/I_1 \to 0$$

There exist many free resolutions for a given module, but the resolution can be made unique (up to isomorphism). To do this, we need to consider graded free resolutions, which are resolutions over graded rings, where the free modules are graded and the maps are homogeneous maps of degree zero. (A homogeneous map is of degree zero if it takes elements of degree a to elements of degree a; in other words, it is degree-preserving.)

Definition 2.31. A graded free resolution

$$\mathcal{F} : 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

is *minimal* if $\phi_i(F_i) \subset (x_0, x_1, \ldots, x_n)F_{i-1}$ for all i between 1 and n.

A resolution is minimal if none of the entries of the matrices are units (that is, all entries are zero or a product of some of the variables). The resolution in Example 2.30 is not minimal because there is a unit in the first matrix. If we remove this row and the corresponding column in the next matrix, we obtain a minimal free resolution. From this example, it is clear that the "extra" column is a multiple of the of the other columns; we can eliminate this redundancy by removing this column. Thus, minimality ensures that the resolution is as short as possible and also contains free modules with the smallest ranks at every step.

Example 2.32. A minimal free resolution of the ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset R = K[x_0, x_1, x_2]$ is

$$0 \to R \xrightarrow{\begin{bmatrix} x \\ y^2 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} 0 & -y \\ -y^2 & x \\ x & 0 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} R \to R/I_1 \to 0$$

Clearly, the ranks for the free modules in a minimal free resolution are independent of the specific resolution, so we give them a name.

Definition 2.33. The i^{th} *(total) Betti number* of M, a finitely generated R-module, is the rank of the free module F_i (that is, $F_i = R^{\beta_i}$) in a minimal free resolution of M and is denoted $\beta_i(M)$.

The total Betti numbers give the size of the smallest resolution for a particular module. In addition to keeping track of the ranks of the free modules, one can also record the degrees of the homomorphisms using *graded Betti numbers.*

Example 2.34. The Betti numbers of the ideal $I_1 = (x_0^2, x_0x_1, x_1^3) \subset K[x_0, x_1, x_2]$ are

$$\beta_0(R/I) = 1 \qquad \beta_1(R/I) = 3 \qquad \beta_2(R/I) = 2.$$

The Betti numbers (or, consequently, the Hilbert polynomial or Hilbert series) for strongly stable ideals can be computed without finding a resolution. In [9], Eliahou and Kervaire describe a minimal resolution for a stable ideal. From this resolution, one can derive the following formulas for the Hilbert series, Hilbert series, and Betti numbers of quotients of stable ideals. Note that these invariants only depend on the max indices and the degrees of the minimal generators of the ideal.

Remark 2.35. If $I \subset R$ is a saturated strongly stable ideal with minimal monomial generators $\{x^{A_1}, \dots, x^{A_r}\}$, then let $l_i = \max(x^{A_i})$ and $d_i = \deg(x^{A_i})$, for all $1 \leq i \leq r$. The Hilbert polynomial and (nonreduced) Hilbert series of R/I are

$$p_{R/I}(z) = \binom{z+n}{n} - \sum_{i=1}^{r} \binom{z+n-d_i-l_i}{n-l_i} \tag{2.2}$$

and

$$H_{R/I}(t) = \left(1 - \sum_{i=1}^{r} (1-t)^{l_i} t^{d_i}\right)(1-t)^{-n-1}. \tag{2.3}$$

The total Betti numbers of the ideal I are

$$\beta_j(I) = \sum_{i=1}^{r} \binom{l_i}{j}. \tag{2.4}$$

Equations (2.3) and (2.4) follow from the Eliahou–Kervaire resolution for stable monomial ideals (see [9], p. 16); Equation (2.2) is a direct consequence of (2.3).

Because Betti numbers give an indication of the complexity of a resolution, it is natural to wonder if it is possible to find an upper bound for ideals of a prescribed size. In fact, lexsegment ideals have the largest Betti numbers among all ideals with a fixed Hilbert function.

Theorem 2.36. (Bigatti, Hulett, Pardue) *Let $I \subset R$ be a homogeneous ideal. If $L \subset R$ is the lexsegment ideal with the same Hilbert function, $h_{R/I} = h_{R/L}$, then $\beta_j(L) \geq \beta j(R/I)$ for all j.*

In fact, the inequality holds for all of the graded Betti numbers as well.

It has also been recently established that there are bounds for the Betti numbers of saturated ideals with a fixed Hilbert polynomial.

Theorem 2.37. (Caviglia and Murai) *Let $p(t)$ be the Hilbert polynomial of a homogeneous ideal of R. There exists a saturated homogeneous ideal $L \subset R$ such that $\beta_j(L) \geq \beta j(R/I)$ for all j for all saturated homogeneous ideals with Hilbert polynomial $p(t)$.*

The result was proved by constructing an ideal which achieves the bound. For more details, see [6].

Chapter 3 Algorithms for strongly stable ideals

In this chapter, we develop an algorithm for producing all saturated strongly stable ideals with a fixed Hilbert polynomial. An algorithm was first proposed by Alyson Reeves in an appendix to her Ph.D. thesis [28]. It was then corrected and simplified by Kai Gehrs in his Diploma Thesis [11]. Here the algorithm is simplified further so that it can be implemented more efficiently. As shown in Table 3.1, the number of ideals can be quite large. A similar algorithm has been developed independently by several researchers in Italy in [7]. For a comparison of the algorithms, see Remark 3.24.

We restrict ourselves to saturated ideals for two reasons. With respect to the reverse lexicographic order, the generic initial ideal of an ideal is saturated if and only if the ideal is saturated, and the homogeneous ideal of a closed subscheme is saturated. Moreover, there are an infinite number of strongly stable ideals with a given Hilbert polynomial.

In Section 3.1 we introduce certain algorithmic operations—contractions and expansions—on the set of minimal generators of strongly stable ideals. These operations were first proposed in [28] and also considered in [11]. For greater efficiency, we use suitable modifications of these operations, and we describe their effect on the Hilbert polynomial.

The theoretical core for our algorithms is provided by Theorem 3.20. It states that all saturated strongly stable ideals with the same Hilbert polynomial can be computed by using expansions of minimal monomial generators. The proof of this result is constructive and leads to a new algorithm for finding all saturated ideals having a prescribed Hilbert polynomial (see Algorithm 3.22). We also include a sharp estimate on the number of steps needed to generate a strongly stable ideal starting from a trivial ideal, based only on the Hilbert polynomial.

Algorithm 3.22 is modified in Section 3.3 in order to produce all almost lexsegment ideals to a given Hilbert polynomial (see Algorithm 3.35). These ideals represent all the Hilbert functions of saturated homogeneous ideals with the given Hilbert polynomial. We also present an algorithm for directly generating all saturated strongly stable ideals with a fixed Hilbert function (see Algorithm 3.37).

In Section 3.5 we discuss consequences of the complexity estimate in Theorem 3.20. In particular, we show that the number of saturated strongly stable ideals in a polynomial ring in n variables with a given Hilbert polynomial p depends only on p and not on n, when n is sufficiently large (see Proposition 3.40). Fixing the Hilbert polynomial, we also describe the ideals with the worst Castelnuovo–Mumford regularity (see Theorem 3.42).

Throughout this chapter, $I \subsetneq R = K[x_0, \ldots, x_n]$ always denotes a saturated strongly stable ideal, unless otherwise specified. $G(I)$ is the set of its (unique) minimal monomial generators. If $n \leq 1$, then saturated strongly stable ideals are principal; thus, we assume $n \geq 2$. At times, we will abuse terminology by saying that I has Hilbert polynomial p if p is actually the Hilbert polynomial of the quotient R/I.

3.1 Expansions and contractions of monomials

We first define left-shifts and right-shifts for monomials, and then use left-shifts and right-shifts to define contractions and expansions of monomials. We adapt Reeves's definitions for left-shifts and right-shifts of monomials and for contractions of monomials (see [28] and Remarks 3.3(iii) and 3.9 below). Expansions will play a central role in the algorithm to compute all saturated strongly stable ideals to a given Hilbert polynomial.

Definition 3.1. Let $x^A \in R$ be a monomial of positive degree.

(i) The set of *right-shifts* of x^A is

$$\mathcal{R}(x^A) := \left\{ x^A \frac{x_{i+1}}{x_i} : x_i | x^A, 0 \leq i < n-1 \right\}.$$

(ii) The set of *left-shifts* of x^A is

$$\mathcal{L}(x^A) := \left\{ x^A \frac{x_{i-1}}{x_i} : x_i | x^A, 0 < i \leq n-1 \right\}.$$

Example 3.2. Consider the monomial $x_1^2 x_3 \in K[x_0, \ldots, x_5]$. As its right-shifts we get

$$\mathcal{R}(x_1^2 x_3) = \left\{ x_1 x_2 x_3, x_1^2 x_4 \right\}.$$

For its left-shifts we obtain

$$\mathcal{L}(x_1^2 x_3) = \left\{ x_0 x_1 x_3, x_1^2 x_2 \right\}.$$

Remark 3.3. (i) Observe that all monomials in $\mathcal{L}(x^A)$ and $\mathcal{R}(x^A)$ have the same degree as x^A. Furthermore, every monomial in $\mathcal{L}(x^A)$ is larger than x^A in the lexicographic order, and every monomial in $\mathcal{R}(x^A)$ is less than x^A. In particular, $\mathcal{L}(x^A) \cap \mathcal{R}(x^A) = \varnothing$ and neither of the sets, $\mathcal{L}(x^A)$ nor $\mathcal{R}(x^A)$, contains the monomial x^A itself.

(ii) The set of left-shifts of any monomial of the form x_0^k is empty ($\mathcal{L}(x_0^k) = \varnothing$). This fact will be important below.

(iii) The definitions for left-shifts and right-shifts in [28] and [11] include redundant monomials. The above definitions provide the smallest sets which can be used to determine whether an ideal will continue to be strongly stable after adding or removing minimal monomial generators (see Lemma 3.7).

Next, we introduce expansion and contractions.

Definition 3.4. Let x^A be a monomial of R.

(i) If $x^A \neq 1$ is a minimal generator of I such that $G(I) \cap \mathcal{R}(x^A) = \varnothing$, then we call x^A *expandable in* I (or simply *expandable* if the ideal is understood). The *expansion of* x^A *in* I is defined to be the ideal $I^{\mathfrak{exp}}$ generated by the set

$$G(I^{\mathfrak{exp}}) := \left(G(I) \setminus \left\{x^A\right\}\right) \cup \left\{x^A \cdot x_r, x^A \cdot x_{r+1}, \ldots, x^A \cdot x_{n-1}\right\},$$

where $r = \max(x^A)$.

If $I = R$ and $x^A = 1$, then we set $I^{\mathfrak{exp}} := (x_0, \ldots, x_{n-1})$.

(ii) If $x^A \neq 1$ is a monomial in R such that $x^A \cdot x_{n-1} \in G(I)$ (so $x^A \notin I$) and $\mathcal{L}(x^A) \subset I$, then we call x^A *contractible in* I (or simply *contractible* if the ideal is understood). The *contraction of* x^A *in* I is defined to be the ideal $I^{\mathfrak{con}}$ generated by the set

$$G(I^{\mathfrak{con}}) := \left(G(I) \cup \left\{x^A\right\}\right) \setminus \left\{x^A \cdot x_r, x^A \cdot x_{r+1}, \ldots, x^A \cdot x_{n-1}\right\},$$

where $r = \max(x^A)$.

If $x_{n-1} \in G(I)$ and $x^A = 1$, then we set $I^{\mathfrak{con}} := (1) = R$.

We note that expandable monomials have been studied elsewhere.

Remark 3.5. The expandable monomials of a strongly stable ideal are exactly the Borel generators; compare our Definition 3.4(i) with Proposition 2.13 in [10].

Example 3.6. Consider the saturated strongly stable ideal $I := (x_0^3, x_0^2x_1, x_0^2x_2) \subset K[x_0, x_1, x_2, x_3]$. The monomial $x_0^2x_2$ is expandable in I because the monomial in $\mathcal{R}(x_0^2x_2) = \{x_0x_1x_2\}$ is not a minimal generator of I. The expansion of $x_0^2x_2$ in I is generated by

$$G(I^{\mathfrak{exp}}) = G(I) \setminus \{x_0^2x_2\} \cup \{x_0^2x_2^2\} = \{x_0^3, x_0^2x_1, x_0^2x_2^2\}.$$

Now the monomial $x_0^2x_2$ is contractible in $I^{\mathfrak{exp}} = (x_0^3, x_0^2x_1, x_0^2x_2^2)$ since it is not contained in $I^{\mathfrak{exp}}$ and $\mathcal{L}(x_0^2x_2) = \{x_0^2x_1\}$ is in $I^{\mathfrak{exp}}$. The contraction of $x_0^2x_2$ in $I^{\mathfrak{exp}}$ is the ideal I we started with.

Similarly, the monomial x_0^2 is contractible in I because it is not in the ideal, the monomial $x_0^2x_2$ is a minimal generator of I, and $\mathcal{L}(x_0^2) = \varnothing \subset I$. The contraction of x_0^2 in I is generated by

$$G(I^{\mathfrak{con}}) = G(I) \cup \{x_0^2\} \setminus \{x_0^3, x_0^2x_1, x_0^2x_2\} = \{x_0^2\}.$$

Now the monomial x_0^2 is expandable in $I^{\mathfrak{con}} = (x_0^2)$ since it is the only minimal generator (so the set of right-shifts is automatically disjoint from the set of minimal generators of the ideal). The expansion of x_0^2 in $I^{\mathfrak{con}}$ is the ideal I we started with.

Contractions and expansions are defined so that they will produce saturated strongly stable ideals. The proof is straightforward, but is included nonetheless.

Lemma 3.7. *If a monomial x^A is contractible or expandable in I, then $I^{\mathfrak{con}}$ or $I^{\mathfrak{exp}}$ is a saturated strongly stable ideal, respectively.*

Proof. Note that if I is saturated, then $I^{\mathfrak{con}}$ or $I^{\mathfrak{exp}}$ will by definition also be saturated.

Suppose that x^A is contractible. To prove that the ideal $I^{\mathfrak{con}}$ is strongly stable, we need only show that $(x_i/x_j) \cdot x^A \in I^{\mathfrak{con}}$ for all j such that $x_j|x^A$ and all $i < j$. Since x^A is contractible, $\mathcal{L}(x^A) \subset I$. Thus, for all j such that $x_j|x^A$, each monomial $(x_{j-1}/x_j) \cdot x^A \in I$ so the monomial is also in $I^{\mathfrak{con}}$. Because I is strongly stable, if $(x_{j-1}/x_j) \cdot x^A \in I$, then $(x_i/x_j) \cdot x^A \in I$ for all $i < j$, so $(x_i/x_j) \cdot x^A \in I^{\mathfrak{con}}$ for all $i < j$.

Suppose x^A is expandable. Now, we need to establish that we have a strongly stable ideal after removing the monomial x^A from $G(I)$. Consider a monomial x^B of the form $(x_k/x_j) \cdot x^A$ for some j such that $x_j|x^A$ and $k > j$. Then the monomial x^B is not in I, because $(x_{j+1}/x_j) \in \mathcal{R}(x^A)$, $\mathcal{R}(x^A)$ is disjoint from I, and I is strongly stable. Thus, the ideal $I^{\mathfrak{exp}}$ is strongly stable. □

As seen in Example 3.6, the contraction and expansion of a fixed monomial in a saturated strongly stable ideal are inverse operations. This will be a useful fact.

Lemma 3.8. *Let $x^A \in R$ be a monomial.*

(a) If x^A is expandable in I, then x^A is contractible in the resulting expansion $I^{\mathfrak{exp}}$. The contraction of x^A in $I^{\mathfrak{exp}}$ is I.

(b) If x^A is contractible in I, then x^A is expandable in the resulting contraction $I^{\mathfrak{con}}$. The expansion of x^A in $I^{\mathfrak{con}}$ is I.

Proof. (a) First we show that x^A is contractible in $I^{\mathfrak{exp}}$: By definition, $x^A \cdot x_{n-1} \in G(I^{\mathfrak{exp}})$. Because I is strongly stable, $\mathcal{L}(x^A) \subset I$. Thus, $\mathcal{L}(x^A) \subset I^{\mathfrak{exp}}$, since the left-shifts of x^A does not contain x^A and x^A is the only monomial removed in $I^{\mathfrak{exp}}$. To show that the contraction and expansion cancel, we note that, by definition, $G((I^{\mathfrak{exp}})^{\mathfrak{con}}) = G(I)$, so $(I^{\mathfrak{exp}})^{\mathfrak{con}} = I$.

(b) Clearly, $x^A \in G(I^{\mathfrak{con}})$, and $x^A \notin I$. Because I is strongly stable, $G(I) \cap \mathcal{R}(x^A) = \varnothing$. (If $x^A \cdot (x_j/x_i) \in I$ for $i < j$, then $x^A \in I$ or I is not strongly stable.) Thus, x^A is expandable in $I^{\mathfrak{con}}$. Similarly, by definition, $G((I^{\mathfrak{con}})^{\mathfrak{exp}}) = G(I)$, so $(I^{\mathfrak{con}})^{\mathfrak{exp}} = I$. □

Remark 3.9. Our Definition 3.4(ii) differs from Reeves's original definition in Appendix A.2 of [28] in two places as we insist on $x^A \cdot x_{n-1} \in G(I)$, but require only $\mathcal{L}(x^A) \subset I$ instead of $\mathcal{L}(x^A) \subset G(I)$. The first change is necessary for Lemma 3.8(b); the second is essential to establish Lemma 3.11 (see also Example 3.12).

In any saturated strongly stable ideal, there will always be expandable monomials. If the ideal is not doubly saturated, there will be contractible monomials. The particular expansions and contractions described in the following result form the basis for Section 3.3.

Lemma 3.10. *(a) In any fixed degree, the minimal monomial generator of I, which is smallest according to the lexicographic order, will be expandable.*

(b) *If the ideal I is not doubly saturated, then some minimal monomial generators will contain the variable x_{n-1}. In any fixed degree d, among the monomials x^A of degree $d-1$ such that $x^A x_{n-1}$ is a minimal monomial generator of the ideal, the monomial, which is largest according to the lexicographic order, will be contractible.*

Proof. (a) If $x^A \neq 1$, then we have to show that $\mathcal{R}(x^A) \cap G(I) = \varnothing$. To this end, let x^B be any monomial in $\mathcal{R}(x^A)$. By the choice of x^A, $x^A >_{lex} x^B$ implies that $x^B \notin G(I)$, and the claim follows. If 1 is a generator of I, then 1 is expandable in I.

(b) If $x^A \neq 1$, then we have to show that $\mathcal{L}(x^A) \subset I$. To this end, let x^B be any monomial in $\mathcal{L}(x^A)$. Since I is strongly stable, $x^A \cdot x_{n-1} \in I$ provides $x^B \cdot x_{n-1} \in I$. By the choice of $x^A \cdot x_{n-1}$, $x^B \cdot x_{n-1} >_{lex} x^A \cdot x_{n-1}$ implies that $x^B \cdot x_{n-1} \notin G(I)$. Hence, $x^B \in I$, and the claim follows. If x_{n-1} is a minimal generator of I, then 1 is contractible in I. □

Our aim is to use expansions to produce saturated strongly stable ideals from simpler ideals—ideals with fewer minimal generators or minimal generators of smaller degree. We start with the following result, which appears as Lemma 23 in [28]. We follow Reeves's argument with some suitable modifications.

Lemma 3.11. *There is a finite sequence of contractions taking the ideal I to its double saturation $sat_{x_{n-1},x_n}(I)$.*

Proof. Since I is saturated, no minimal generators are divisible by x_n. Consider the set M of monomials in $G(I)$ that are divisible by x_{n-1}. If $M = \varnothing$, then I is doubly saturated. Otherwise, choose the monomial $x^A \cdot x_{n-1}$ of least degree in M, which is largest with respect to the lexicographic order $>_{lex}$. As noted in Lemma 3.10, x^A is contractible in I.

Let I^{con} be the contraction of x^A in I. Note that contracting x^A replaces $x^A \cdot x_{n-1}$ (and possibly other monomials) by x^A. Thus, I^{con} has the same double saturation as I. After repeating the above step some finite number of times, we get an ideal whose minimal generators are not divisible by x_{n-1}. This is the double saturation of I. □

Example 3.12. We illustrate the last proof with the ideal $I = (x_0, x_1^2, x_1 x_2^3)$ in the ring $K[x_0, x_1, x_2, x_3]$.

- First we contract the monomial $x_1 x_2^2$ in I. (Note that $\mathcal{L}(x_1 x_2^2) = \{x_0 x_2^2, x_1^2 x_2\}$ is *not* a subset of the set of minimal generators of I. This shows that our modification of Reeves's definition of contraction in [28] is needed in the above argument.) The resulting ideal I_1 is generated minimally by

$$G(I_1) = G(I) \cup \{x_1 x_2^2\} \backslash \{x_1 x_2^3\} = \{x_0, x_1^2, x_1 x_2^2\}.$$

- Next, we contract $x_1 x_2^2$ and get the ideal

$$I_2 = (x_0, x_1^2, x_1 x_2).$$

- In the last step, contracting x_1 in I_2 gives the double saturation
$$I_3 = (x_0, x_1) = sat_{x_2,x_3}(I).$$

We now make the contractions necessary to get to the double saturation more explicit.

Remark 3.13. Assume that the ideal I is different from its double saturation. List the minimal generators of I that are divisible by x_{n-1},
$$x^{A_1}x_{n-1}^{e_1},\ x^{A_2}x_{n-1}^{e_2},\ \ldots,\ x^{A_s}x_{n-1}^{e_s},$$
where x^{A_i} is not divisible by x_{n-1}, so that $\deg x^{A_i}x_{n-1}^{e_i} \leq \deg x^{A_{i+1}}x_{n-1}^{e_{i+1}}$, and in case of equality $x^{A_i}x_{n-1}^{e_i} >_{lex} x^{A_{i+1}}x_{n-1}^{e_{i+1}}$. Then the contractions in the algorithm given in the proof of Lemma 3.11 use the following monomials
$$x^{A_1}x_{n-1}^{e_1-1},\ x^{A_1}x_{n-1}^{e_1-2}, \ldots, x^{A_1},\ x^{A_2}x_{n-1}^{e_2-1},\ \ldots,\ x^{A_2}, \ldots, x^{A_s}$$
in the stated order. Thus, we need $e_1 + e_2 + \ldots + e_s$ contractions to compute the double saturation of I.

Since this process is reversible, we can recover an ideal from its double saturation:

Corollary 3.14. *There is a finite sequence of expansions taking the double saturation of an ideal $sat_{x_{n-1},x_n}(I)$ to the ideal I. In particular, the necessary number of expansions can be determined by adding up the exponents of x_{n-1} in the minimal generators of I.*

Proof. The sequence of contractions described in Remark 3.13, which take I to its double saturation, can be reversed and considered as expansions by Lemma 3.8. □

We conclude this section by describing the change of the Hilbert function under contraction or expansion.

Lemma 3.15. *(a) Let $I^{\mathfrak{exp}}$ be the expansion of x^A in I. Then*
$$h_{R/I^{\mathfrak{exp}}}(j) = \begin{cases} h_{R/I}(j) & \text{if } j < \deg(x^A) \\ h_{R/I}(j) + 1 & \text{if } j \geq \deg(x^A) \end{cases}.$$

(b) Let $I^{\mathfrak{con}}$ be the contraction of x^B in I. Then
$$h_{R/I^{\mathfrak{con}}}(j) = \begin{cases} h_{R/I}(j) & \text{if } j < \deg(x^B) \\ h_{R/I}(j) - 1 & \text{if } j \geq \deg(x^B) \end{cases}.$$

Proof. (a) We have $I^{\mathfrak{exp}} \subset I$. Furthermore, if $j \geq \deg(x^A)$, then $x^A \cdot x_n^{j-\deg(x^A)}$ is the only monomial in $[I]_j \setminus [I^{\mathfrak{exp}}]_j$. The claim follows.

(b) Now, $I \subset I^{\mathfrak{con}}$, and $x^B \cdot x_n^{j-\deg(x^B)}$ is the only monomial in $[I^{\mathfrak{con}}]_j \setminus [I]_j$, provided $j \geq \deg(x^B)$. □

We can now determine the number of expansions to recover an ideal from its double saturation in a more abstract manner.

Corollary 3.16. *The number of expansions needed to take $J = sat_{x_{n-1},x_n}(I)$ to I is*
$$p_{R/I} - p_{R/J}.$$

3.2 Strongly stable ideals with a given Hilbert polynomial

In this section, we describe how to produce *all* saturated strongly stable ideals with a given Hilbert polynomial. We develop a few more tools, which culminate in Theorem 3.20 and Algorithm 3.22. We start with the simplest case, ideals with constant Hilbert polynomial:

Lemma 3.17. *Let $I \subset R$ be a saturated strongly stable ideal with constant Hilbert polynomial, say $p_{R/I} = c$. Then $sat_{x_{n-1},x_n}(I) = (1) = R$. Moreover, any saturated strongly stable ideal $J \subset R$ with $p_{R/J} = c$ can be obtained from the ideal (1) using c suitable expansions.*

Proof. If $x_{n-1}^k \in I$, then $sat_{x_{n-1},x_n}(I) = (1) = R$ by Definition 2.6. Assume that no power of x_{n-1} is in I. Let j be any positive integer. Since I is strongly stable, no monomial of the form $x_{n-1}^{j-i} \cdot x_n^i \in I$ for $0 \leq i \leq j$. Hence, there are at least $j+1$ monomials not contained in $[I]_j$ for every $j > 0$, which contradicts $p_{R/I}(z) = c$. Thus, some power of x_{n-1} is in I. The final claim is now a consequence of Corollaries 3.14 and 3.16. $\square$

Recall some previously introduced notation: $R^{(j)} := K[x_0, \ldots, x_{n-j}]$ is the polynomial ring where the last j variables of R have been removed. If $I \subset R$ is a saturated strongly stable ideal with Hilbert polynomial p, then the restriction of its double saturation $sat_{x_{n-1},x_n}(I)$ to $R^{(1)} := K[x_0, \ldots, x_{n-1}]$ is a saturated strongly stable ideal in $R^{(1)}$ with a Hilbert polynomial that can be computed from p:

Lemma 3.18. *If I is a saturated strongly stable ideal with Hilbert polynomial $p(z)$ and double saturation $J = sat_{x_{n-1},x_n}(I)$, then the Hilbert polynomial of $J^{(1)} := J \cdot R^{(1)} \subset R^{(1)}$ is $p_{R^{(1)}/J^{(1)}}(z) = \Delta p(z) := p(z) - p(z-1)$.*

Proof. Setting $I^{(1)} = I \cdot R^{(1)}$, multiplication by x_n induces the exact sequence

$$0 \longrightarrow R/I(-1) \xrightarrow{x_n} R/I \longrightarrow R^{(1)}/I^{(1)} \longrightarrow 0,$$

since x_n is not a zero divisor of R/I. Now, $p_{R^{(1)}/I^{(1)}}(z) = \Delta p(z)$. Passing to $J^{(1)}$, the saturation of $I^{(1)}$, does not change the Hilbert polynomial, so $p_{R^{(1)}/J^{(1)}}(z) = \Delta p(z)$. $\square$

This result can be extended. If $p(z)$ is a Hilbert polynomial of degree d, we set $\Delta^0 p(z) := p(z)$, and recursively define $\Delta^j p(z) := \Delta^{j-1}p(z) - \Delta^{j-1}p(z-1)$ for $1 \leq j \leq d$. Thus, $\Delta = \Delta^1$. Now, if I is a saturated strongly stable ideal, then, for $0 \leq j \leq d$, we denote by $I^{(j)} \subset R^{(j)}$, the saturated strongly stable ideal whose generating set is obtained by setting $x_{n-j} = \ldots = x_{n-1} = 1$ in the monomial generators of I. Note that the ideal $I^{(j+1)} \cdot R^{(j)}$ is the double saturation of $I^{(j)}$. Repeating the argument in Lemma 3.18 shows that $\Delta^j p(z)$ is the Hilbert polynomial of the ideal $I^{(j)}$:

Corollary 3.19. *If I is a saturated strongly stable ideal with Hilbert polynomial $p(z)$ of degree d, and $I^{(j)} \subset R^{(j)}$ is the ideal obtained by setting $x_{n-j} = \ldots = x_{n-1} = 1$ in the monomial generators of I, then the Hilbert polynomial of $I^{(j)}$ is $p_{R^{(j)}/I^{(j)}}(z) = \Delta^j p(z)$ for $0 \leq j \leq d$.*

We are now ready to prove the main result of this section.

Theorem 3.20. *Let $I \subsetneq R$ be a saturated strongly stable ideal with Hilbert polynomial $p(z)$ of degree d. Then there is a finite sequence of expansions (in the appropriate rings) that take the ideal $(1) = R^{(d)}$ to the ideal $I \subset R$.*

In particular, the number of expansions needed in $R^{(j)}$ to take $I^{(j+1)} \cdot R^{(j)}$ to $I^{(j)}$ is $\Delta^j p(z) - p_{R^{(j)}/I^{(j+1)}R^{(j)}}(z)$, which, in the notation of Theorem 2.23, is at most a_j, for $j = 0, \ldots, d$. The total number of expansions needed to take $(1) = R^{(d)}$ to the ideal $I \subset R$ is at most $b_0 = a_0 + \cdots + a_d$.

Proof. Let $p(z)$ be the Hilbert polynomial of R/I. We induct on the degree of $p(z)$. If $\deg p = 0$, then we are done by Lemma 3.17. Assume $\deg p > 0$. Since $\deg \Delta^1 p = \deg p - 1$, we conclude by the induction hypothesis that there is a finite sequence of expansions that takes the ideal $(1) \subset R^{(d)}$ to $J^{(1)} = sat_{x_{n-1},x_n}(I) \subset R^{(1)}$, the double saturation of I as an ideal in $R^{(1)}$. Considering the corresponding extension ideal in R, the ideal I can be obtained from $J^{(1)} \cdot R$ by Corollary 3.14 using a finite number of expansions.

The claim that the number of expansions needed in the ring $R^{(j)}$ to take the ideal $I^{(j+1)} \cdot R^{(j)}$ to the ideal $I^{(j)}$ is $\Delta^j p(z) - p_{R^{(j)}/I^{(j+1)}R^{(j)}}(z)$ follows from Corollaries 3.16 and 3.19. Thus it remains to show that

$$\Delta^j p(z) - p_{R^{(j)}/I^{(j+1)}R^{(j)}}(z) \leq a_j \tag{3.1}$$

because the final assertion then follows by recalling that $b_0 = a_0 + \cdots + a_d$.

In order to establish Inequality (3.1), write the given Hilbert polynomial as in Equation (2.1) as

$$p(z) = \sum_{i=0}^{d} \left[\binom{z+i}{i+1} - \binom{z+i-b_i}{i+1} \right].$$

By Lemma 3.18, the Hilbert polynomial of $I^{(j)}$ is

$$\begin{aligned} p_{R^{(j)}/I^{(j)}}(z) &= \Delta^j p(z) \\ &= \sum_{i=j}^{d} \left[\binom{z+i-j}{i+1-j} - \binom{z+i-b_i-j}{i+1-j} \right]. \end{aligned}$$

Using Theorem 2.23, it follows that exactly a_j expansions in $R^{(j)}$ are needed to take the lexicographic ideal $L_{\Delta^{j+1}p} R^{(j)}$ to the lexicographic ideal $L_{\Delta^j p}$ of $R^{(j)}$. Since $R^{(j+1)}/I^{(j+1)}$ and $R^{(j+1)}/L_{\Delta^{j+1}p}$ have the same Hilbert polynomial, namely $\Delta^{j+1}p$, Inequality (3.1) is equivalent to

$$p_{R^{(j)}/I^{(j+1)}R^{(j)}}(z) \geq p_{R^{(j)}/L_{\Delta^{j+1}p}R^{(j)}}(z). \tag{3.2}$$

(The difference of the two polynomials is a constant.) However, the latter estimate is a consequence of Proposition 2.25(b) because $L_{\Delta^{j+1}p} \subset R^{(j+1)}$ is the saturation of $L_{\Delta^j p} R^{(j+1)}$ in $R^{(j+1)}$, so, for all integers k,

$$h_{R^{(j+1)}/I^{(j+1)}}(k) \geq h_{R^{(j+1)}/L_{\Delta^{j+1}p}}(k).$$

Summing over k on both sides of this inequality, we get the Hilbert functions of $R^{(j)}/I^{(j+1)}R^{(j)}$ and $R^{(j)}/L_{\Delta^{j+1}p}R^{(j)}$, respectively. Now, Inequality 3.2 follows. $\square$

Note that the estimate on the number of needed expansions is sharp. This follows from Lemma 3.29 below.

The particular expansions leading to the lexicographic ideal L_p can be made explicit.

Remark 3.21. In Theorem 3.20, the lexicographic ideal will be obtained if, at each step, the minimal monomial generator to be expanded is of the highest degree, and is smallest according to the lexicographic order in that degree. This follows by Proposition 2.25(b) and Lemma 3.10.

After expanding the monomial x^A, x^A is no longer in the ideal. Thus, to produce a lexicographic ideal, the expanded monomials must be always be last lexicographically. They must in fact be in the highest degree because, if x^A is expanded, $x^A x_n^k$ is not in the ideal for any $k \geq 0$, and monomials of this form will precede other minimal generators in larger degrees.

Using Theorem 3.20 and its proof, we can now give the desired algorithm to compute all saturated strongly stable ideals with a prescribed Hilbert polynomial.

Algorithm 3.22. (*Generating all saturated strongly stable ideals with a given Hilbert polynomial*) Let $p(z)$ be a nonzero Hilbert polynomial of degree d of a graded quotient of R.

1. Compute the polynomials $\Delta^1 p(z)$, $\Delta^2 p(z)$, ..., $\Delta^d p(z)$. (Note that $\Delta^d p(z) = c$ for some $c \in \mathbb{N}$.) Set $\mathcal{S}^{(d)} = \cdots = \mathcal{S}^{(0)} = \varnothing$.

2. Generate $\mathcal{S}^{(d)}$, the set of all saturated strongly stable ideals I in $R^{(d)}$ with Hilbert polynomial $p_{R^{(d)}/I}(z) = \Delta^d p(z) = c$, using c successive expansions of monomial generators starting with the ideal $(1) = R^{(d)}$. Exhaust all choices for c successive expansions.

3. For $j = d-1,\ d-2,\ \ldots,\ 0$, repeat the following steps for each ideal $I \in \mathcal{S}^{(j+1)}$: Compute $p_{R^{(j)}/I}(z)$ (using Equation (2.2)). Let $a = \Delta^j p(z) - p_{R^{(j)}/I}(z)$.

   - If $a \geq 0$, then perform a successive expansions of monomial generators of I to obtain ideals with Hilbert polynomial $\Delta^j p(z)$. Exhaust all choices for a successive expansions. Add these ideals to $\mathcal{S}^{(j)}$.
   - If $a < 0$, then continue with the next ideal I in $\mathcal{S}^{(j+1)}$.

4. Return the set $\mathcal{S}^{(0)}$.

Proof. (*Correctness*) By Theorem 3.20, every saturated strongly stable ideal with Hilbert polynomial $p(z)$ will be generated by this algorithm, as long as every possible sequence of expansions is carried out at each step. Also, every ideal generated by this process will be saturated and strongly stable and have the desired Hilbert polynomial.

The algorithm terminates for any given Hilbert polynomial, since the number of steps performed in (3) is bounded by the degree of the Hilbert polynomial and the number of generators in each ideal computed in each loop is finite. $\square$

We include an example to illustrate this algorithm.

Example 3.23. Suppose we wish to find all saturated strongly stable ideals with Hilbert polynomial $p(z) = 2z^2 + z + 2 = \binom{z+2}{3} - \binom{z-2}{3} + \binom{z+1}{2} - \binom{z-4}{2} + \binom{z}{1} - \binom{z-8}{1}$ in $R = K[x_0, x_1, x_2, x_3, x_4]$.

- First we compute $\Delta^1 p(z)$ and $\Delta^2 p(z)$:

$$\Delta^1 p(z) = 4z - 1, \qquad \Delta^2 p(z) = 4$$

- Next we generate all ideals in $R^{(2)} = K[x_0, x_1, x_2]$ with Hilbert polynomial $\Delta^2 p(z) = 4$ using 4 successive expansions and starting from $(1) = R^{(2)}$. We get two ideals:

$$I = (x_0, x_1^4), \qquad J = (x_0^2, x_0x_1, x_1^3)$$

- Now we generate all ideals in $R^{(1)}$ with Hilbert polynomial $\Delta^1 p(z) = 4z - 1$. We compute the Hilbert polynomials of I and J in $R^{(1)}$:

$$p_{R^{(1)}/I}(z) = 4z - 2, \qquad p_{R^{(1)}/J}(z) = 4z$$

  We perform one expansion in I to obtain the following ideals:

$$I_1 = (x_0, x_1^5, x_1^4x_2), \qquad I_2 = (x_0^2, x_0x_1, x_0x_2, x_1^4)$$

  We discard the ideal J (as the Hilbert polynomial is too large).

- Finally we generate all ideals in R with Hilbert polynomial $p(z) = 2z^2 + z + 2$. We compute the Hilbert polynomials of I_1 and I_2:

$$p_{R/I_1}(z) = 2z^2 + z - 1, \qquad p_{R/I_2}(z) = 2z^2 + z + 2$$

  We perform three expansions in I_1 to obtain the following ideals:

$$(x_0^2, x_0x_1, x_0x_2^2, x_0x_2x_3, x_0x_3^2, x_1^5, x_1^4x_2), \qquad (x_0^2, x_0x_1, x_0x_2, x_0x_3^3, x_1^5, x_1^4x_2)$$

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3^2, x_1^5, x_1^4x_2^2, x_1^4x_2x_3), \qquad (x_0, x_1^5, x_1^4x_2^3, x_1^4x_2^2x_3, x_1^4x_2x_3^2)$$

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^5, x_1^4x_2^2, x_1^4x_2x_3^2), \qquad (x_0, x_1^6, x_1^5x_2, x_1^5x_3, x_1^4x_2^2, x_1^4x_2x_3^2)$$

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^6, x_1^5x_2, x_1^5x_3, x_1^4x_2^2, x_1^4x_2x_3), \qquad (x_0, x_1^5, x_1^4x_2^2, x_1^4x_2x_3^3)$$

  We perform no expansions in I_2 (as the Hilbert polynomial is correct).

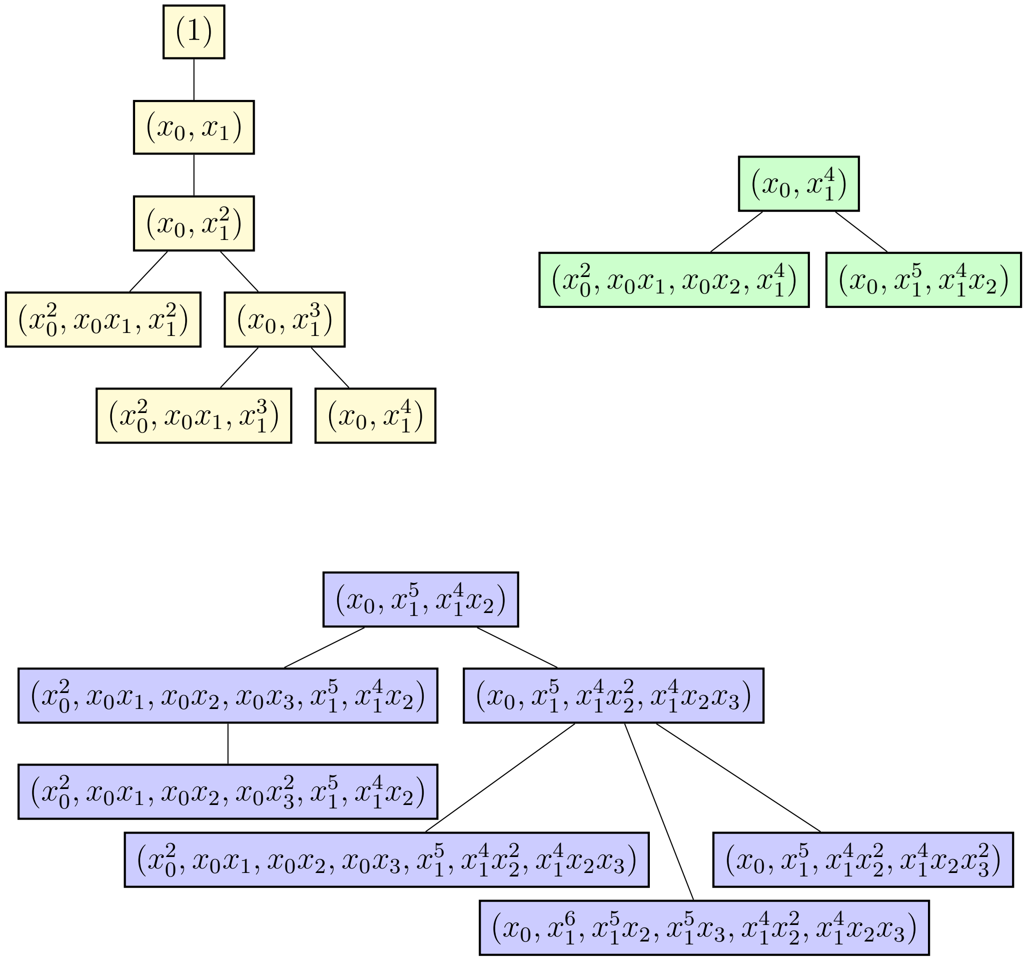

Figure 3.1: The intermediate ideals generated in Example 3.23 while finding all saturated strongly stable ideals with Hilbert polynomial $p(z) = 2z^2+z+2$ in $K[x_0, \ldots, x_4]$.

Thus, there are nine saturated strongly stable ideals in R with Hilbert polynomial $p(z) = 2z^2 + z + 2$. Note that $(x_0, x_1^5, x_1^4x_2^2, x_1^4x_2x_3^3)$ is the lexicographic ideal, as in Example 2.24.

The intermediate expansions are provided in Figure 3.1. The expansions in the first ring are shaded yellow, the expansions in the second ring are shaded green, and the expansions in the third ring are shaded blue.

We note that other algorithms have been proposed for this scenario. In an appendix to her thesis [28], Reeves presents an algorithm for generating saturated strongly stable ideals with a given Hilbert polynomial. Also, another algorithm was proposed independently in a recent paper [7] by Cioffi, Lella, Marinari, and Roggero. We thank the authors for kindly pointing this out to us after we submitted the first version of this paper. We briefly discuss the differences between these algorithms.

Remark 3.24. Algorithm 3.22 differs from the algorithm presented by Reeves in [28]: Her algorithm first computes all Hilbert series associated to the desired Hilbert polynomial by pairs of contractions and expansions and then generates all saturated strongly stable ideals for each Hilbert series. A single Hilbert series or ideal may be generated a number of times in each of these steps. On the other hand, our algorithm directly creates all ideals, each in a unique way, building them in larger and larger rings. We also give direct methods for producing all Hilbert series to a particular Hilbert polynomial in Algorithm 3.35 and all saturated strongly stable ideals with a particular Hilbert series in Algorithm 3.37 that appear more efficient.

Furthermore, Reeves uses special matrices to encode the set of monomial generators of a strongly stable ideal. On these matrices, a certain kind of elementary row operations is performed to compute other saturated strongly stable ideals with the same Hilbert series. One problem to be solved then is that the correspondence between such matrices encoding strongly stable ideals and the set of strongly stable ideals itself (in a fixed polynomial ring) is not a bijection. The elementary row operations used may produce matrices, which do not encode any saturated strongly stable ideal. Hence, one needs a special procedure within the algorithm to check whether or not a given matrix represents a saturated strongly stable ideal. To avoid this trial and error technique, we did not use these matrices.

The algorithm suggested in [7] is more similar to Algorithm 3.22 in that it is recursive in the number of variables (and the degree of the Hilbert polynomial). However, instead of increasing the degrees of the minimal generators to achieve the correct Hilbert polynomial, a number of new generators are added to make the Hilbert function as large as possible in a fixed degree. Certain generators are then removed in all possible combinations to produce the desired saturated strongly stable ideals.

Observe that our approach has the advantage of allowing us to estimate the number of steps to produce an ideal with a given Hilbert polynomial (see Theorem 3.20).

We conclude with some remarks about implementing Algorithm 3.22.

Remark 3.25. (i) When carrying out Algorithm 3.22, we can order the expansions so that each ideal is produced in a unique way. One natural ordering of minimal generators is to list the monomials first by degree in increasing order and then lexicographically in each degree. When expanding in some ring $R^{(j)}$, always pick monomials, which precede all other monomials that have been expanded in this ring (those monomials divisible by the variable x_{n-j-1}). (Thus, the expanded monomials, leading to a certain ideal, will be strictly increasing according to this order and, hence, unique.) This order for expansions is the reverse of the order for contractions discussed in Remark 3.13.

(ii) We can perform the expansions in a depth-first manner (as opposed to breadth-first) to minimize the amount of data stored.

(iii) Instead of computing and storing polynomials, we can evaluate the Hilbert polynomials at b_0 and find the difference of integers to get a in Step 3 of Algorithm 3.22. We evaluate at b_0 because, b_0 is a bound for the Castelnuovo–Mumford regularity of the ideals produced by the algorithm; this is a consequence of the last claim in Theorem 3.20.

(iv) Alternately, we could store the first b_0+1 values of the Hilbert function for each ideal, and use Lemma 3.15 to update the Hilbert function after each expansion. Then, when an ideal is considered in a ring with an extra variable, the old Hilbert function can be summed to obtain the new Hilbert function. By doing this, we no longer need all of the generators of the ideal. Instead, we could keep track of only the expandable monomials (since these are the Borel generators, as noted in Remark 3.5). In a polynomial ring with a large number of variables, the number of expandable monomials can be considerably smaller than the number of minimal generators.

We demonstrate the above remarks in the setting of Example 3.23.

Example 3.26. We again wish to produce all saturated strongly stable ideals with Hilbert polynomial $p(z) = 2z^2 + z + 2 = \binom{z+2}{3} - \binom{z-2}{3} + \binom{z+1}{2} - \binom{z-4}{2} + \binom{z}{1} - \binom{z-8}{1}$ in $R = K[x_0, x_1, x_2, x_3, x_4]$. We evaluate the Hilbert polynomials at $b_0 = 8$, and, for each ideal, we record (only) the expandable monomials and the first nine values of the Hilbert function in the current polynomial ring. To indicate that only some of the generators are shown, we use angle brackets when displaying ideals, $\langle \cdots \rangle$.

- First we compute $\Delta^0 p(b_0)$, $\Delta^1 p(b_0)$, and $\Delta^2 p(b_0)$:

$$\Delta^0 p(8) = 2(8)^2 + (8) + 2 = 138, \quad \Delta^1 p(8) = 4(8) - 1 = 31, \quad \Delta^2 p(8) = 4$$

- Next we generate ideals in $R^{(2)} = K[x_0, x_1, x_2]$ whose Hilbert function in degree eight is 4. We get two ideals:

$$I = \langle x_0, x_1^4 \rangle, \quad J = \langle x_0 x_1, x_1^3 \rangle$$

  with (truncated) Hilbert functions:

$$h_{R^{(2)}/I} = \{1, 2, 3, 4, 4, 4, 4, 4, 4\}, \quad h_{R^{(2)}/J} = \{1, 3, 4, 4, 4, 4, 4, 4, 4\}$$

  (because I is expanded in degrees 0, 1, 2, and 3 and J is expanded in degrees 0, 1, 1, and 2).

- Now we generate ideals in $R^{(1)} = K[x_0, x_1, x_2, x_3]$ whose Hilbert function in degree eight is 31. We compute the Hilbert functions of I and J in $R^{(1)}$ in degree k by summing up the first $k+1$ terms in the previous Hilbert functions:

$$h_{R^{(1)}/I} = \{1, 3, 6, 10, 14, 18, 22, 26, 30\}, \quad h_{R^{(1)}/J} = \{1, 4, 8, 12, 16, 20, 24, 28, 32\}$$

  We perform one expansion in I to obtain the following ideals:

$$I_1 = \langle x_0, x_1^5, x_1^4 x_2 \rangle, \quad I_2 = \langle x_0 x_2, x_1^4 \rangle$$

  with Hilbert functions:

$$h_{R^{(1)}/I_1} = \{1, 3, 6, 10, 15, 19, 23, 27, 31\}$$

$$h_{R^{(1)}/I_2} = \{1, 4, 7, 11, 15, 19, 23, 27, 31\}$$

  We discard the ideal J (as the Hilbert function in degree eight is too large).

- Finally we generate ideals in R whose Hilbert function in degree eight is 138. We compute the Hilbert functions of I_1 and I_2 in R:

$$h_{R/I_1} = \{1, 4, 10, 20, 35, 54, 77, 104, 135\}$$

$$h_{R/I_2} = \{1, 5, 12, 23, 38, 57, 80, 107, 138\}$$

  We perform three expansions in I_1 to obtain the following ideals:

$$\langle x_0x_1, x_0x_3^2, x_1^4x_2\rangle, \quad \langle x_0x_2, x_0x_3^3, x_1^4x_2\rangle$$

$$\langle x_0x_2, x_0x_3^2, x_1^5, x_1^4x_2x_3\rangle, \quad \langle x_0, x_1^5, x_1^4x_2x_3^2\rangle$$
$$\langle x_0x_3, x_1^5, x_1^4x_2^2, x_1^4x_2x_3^2\rangle, \quad \langle x_0, x_1^5x_3, x_1^4x_2^2, x_1^4x_2x_3^2\rangle$$
$$\langle x_0x_3, x_1^4x_2x_3\rangle, \quad \langle x_0, x_1^5, x_1^4x_2^2, x_1^4x_2x_3^3\rangle$$

  We perform no expansions in I_2 (as the Hilbert function in degree eight is correct).

We produce the same nine saturated strongly stable ideals in R. The intermediate expansions are provided in Figure 3.2. The expansions in the first ring are shaded yellow, the expansions in the second ring are shaded green, and the expansions in the third ring are shaded blue. In particular, compare the number of Borel generators to the number of minimal generators for each of the ideals.

3.3 Almost lexsegment ideals with a given Hilbert polynomial

The algorithm in the previous section produces all saturated strongly stable ideals with a given Hilbert polynomial. These ideals may be sorted into classes of ideals with the same Hilbert function. In this section, we develop an algorithm for producing a unique ideal in all of these classes of ideals associated to particular Hilbert functions simultaneously. This algorithm is useful when studying ideals with a fixed Hilbert polynomial, which have maximal Betti numbers.

We begin by introducing the class of strongly stable ideals in which we are now interested. If a strongly stable ideal is saturated, then no minimal monomial generators contain the last variable x_n. Thus, the ideal can be considered in the polynomial ring $R^{(1)}$, where the variable x_n is omitted. This class of ideals is characterized by the fact that they are lexsegment ideals when viewed in the smaller ring $R^{(1)}$.

Definition 3.27. A saturated strongly stable ideal $I \subset R$ is called *almost lexsegment* if $I \cdot R^{(1)}$ is a lexsegment ideal.

Example 3.28. Consider the saturated strongly stable ideals

$$I_1 = (x_0^2, x_0x_1, x_0x_2, x_1^2), \qquad I_2 = (x_0^2, x_0x_1, x_0x_2, x_1^3, x_1^2x_2),$$

$$I_3 = (x_0^2, x_0x_1, x_0x_2^2, x_1^3, x_1x_2^2)$$

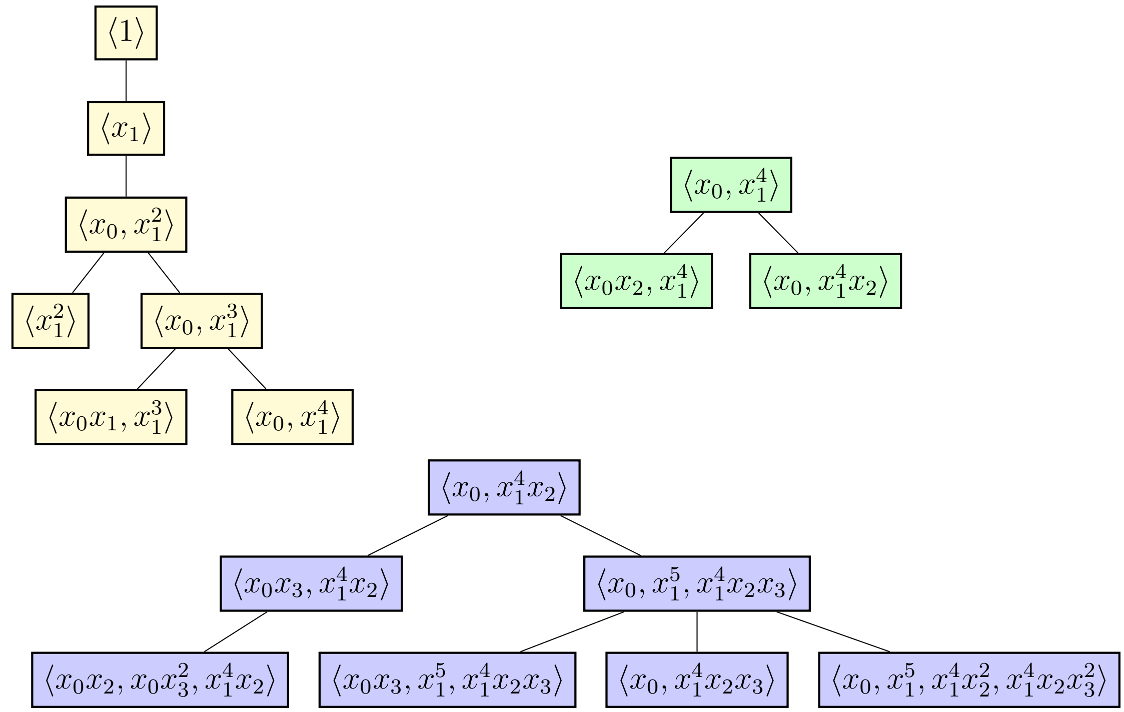

Figure 3.2: The intermediate ideals generated in Example 3.26 while finding all saturated strongly stable ideals with Hilbert polynomial $p(z) = 2z^2+z+2$ in $K[x_0, \dots, x_4]$. Only expandable monomials are listed.

in $R = K[x_0, x_1, x_2, x_3]$. I_1, I_2, and I_3 are almost lexsegment ideals. I_1 is generated by the first four monomials of $R^{(1)} = K[x_0, x_1, x_2]$ in degree two. I_2 contains the first three monomials of $R^{(1)}$ in degree two, the first seven monomials of $R^{(1)}$ in degree three, etc; I_3 contains the first two monomials of $R^{(1)}$ in degree two, the first eight monomials of $R^{(1)}$ in degree three, etc.

We will now focus on characterizing how to generate almost lexsegment ideals. The process will be similar to the previous algorithm, except for two simplifications: all of the almost lexsegment ideals have the same double saturation and only certain expansions need to be performed.

Recall the lexicographic ideal L_p and the nonnegative integers a_i introduced earlier in Theorem 2.23, which are associated to each Hilbert polynomial.

Lemma 3.29. *Every almost lexsegment ideal with Hilbert polynomial $p(z)$ has the same double saturation, namely $L_{\widetilde{p}}$, where*

$$\widetilde{p}(z) = p(z) - a_0.$$

Proof. Using the definition of $\widetilde{p}$, we see that the ideal $L_{\widetilde{p}}$ is doubly saturated by Theorem 2.23 (because no minimal generator will be divisible by x_{n-1}). Thus, the ideal $L_{\widetilde{p}} \cdot R^{(1)} \subset R^{(1)}$ is the unique saturated lexsegment ideal of $R^{(1)}$ with Hilbert polynomial $\Delta p(z)$ by Lemma 3.18.

The double saturation of an almost lexsegment ideal $I \subset R$ with Hilbert polynomial $p(z)$ will also be a saturated lexsegment ideal in $R^{(1)}$ with Hilbert polynomial $\Delta p(z)$. Thus, the double saturation must be $L_{\widetilde{p}}$. □

Note that the uniqueness statement of the double saturation in Lemma 3.29 is equivalent to Proposition 2.3 in [6]. The explicit description of the double saturation is new.

We give a name to the special expansions and contractions that were noted earlier in Lemma 3.10.

Definition 3.30. Let $I \subset R$ be an almost lexsegment ideal.

(i) In any fixed degree, an expansion of the minimal monomial generator of I, which is last according to the lexicographic order, is called a *lex expansion*.

(ii) In any fixed degree, a contraction of the monomial x^A, which, among minimal generators of the form $x^A \cdot x_{n-1}$, is first according to the lexicographic order, is called a *lex contraction*.

Note that lex expansions and lex contractions are inverse operations.

Lemma 3.31. *If a lex expansion of a monomial x^A is performed in an ideal, then a lex contraction of x^A can be performed in the resulting ideal which yields the original ideal. Conversely, if a lex contraction of a monomial x^A is performed in an ideal, then a lex expansion of x^A can be performed in the resulting ideal which again yields the original ideal.*

Proof. These facts follow from Lemma 3.8, and the observation that, if x^A is the last minimal generator in a certain degree in an almost lexsegment ideal I, then $x^A \cdot x_{n-1}$ will be the first minimal generator divisible by x_{n-1} in the resulting ideal $I^{\mathfrak{exp}}$, so it will be lex contractible. Similarly, if $x^A \cdot x_{n-1}$ is the first minimal generator divisible by x_{n-1} in an almost lexsegment ideal I, then x^A will be the last minimal generator in that degree in the resulting ideal $I^{\mathfrak{con}}$, so it will be lex expandable. (Otherwise, there would be "missing" generators contradicting the fact that $I \cdot R^{(1)}$ is a lexsegment ideal.) □

Lex expansions and lex contractions are the only tools needed to produce almost lexsegment ideals:

Lemma 3.32. *If I is an almost lexsegment ideal, then applying a lex expansion or a lex contraction to I will produce another almost lexsegment ideal.*

In fact, the only expansions of almost lexsegment ideals which produce almost lexsegment ideals are the lex expansions, and, similarly, the only contractions of almost lexsegment ideals which produce almost lexsegment ideals are the lex contractions.

Proof. Assume $I \subset R$ is an almost lexsegment ideal.

Expanding a monomial x^A of degree d only changes the ideal $I \cdot R^{(1)}$ in degree d (by removing the monomial x^A from I). If x^A is the smallest minimal monomial generator of I in degree d according to the lexicographic order, then the expansion of x^A will be an almost lexsegment ideal. Expanding a monomial of degree d which comes before x^A in the lexicographic order will produce an ideal which is not almost lexsegment.

Similarly, contracting a monomial x^A of degree d only changes the ideal $I \cdot R^{(1)}$ in degree d (by adding the monomial x^A). If x^A is the largest minimal monomial generator of I in degree d according to the lexicographic order, then the contraction of x^A will be an almost lexsegment ideal. Contracting a monomial of degree d which comes after x^A in the lexicographic order will produce an ideal which is not almost lexsegment. $\square$

We illustrate the last lemma with an example.

Example 3.33. Consider again the almost lexsegment ideals $I_1 = (x_0^2, x_0x_1, x_0x_2, x_1^2)$, $I_2 = (x_0^2, x_0x_1, x_0x_2, x_1^3, x_1^2x_2)$, and $I_3 = (x_0^2, x_0x_1, x_0x_2^2, x_1^3, x_1x_2^2)$ in $K[x_0, x_1, x_2, x_3]$ from Example 3.28.

Observe that the smallest monomial generator in I_1 of degree two, according to the lexicographic order, is x_1^2. This monomial is expandable, and expanding it produces the almost lexsegment ideal I_2. The monomial x_0x_2 is also expandable in I_1, but expanding it produces an ideal, $J = (x_0^2, x_0x_1, x_0x_2^2, x_1^2)$, which is not almost lexsegment (since x_0x_2 is not in J, x_1^2 is in J, and $x_0x_2 >_{lex} x_1^2$).

Observe that there are two contractible monomials in I_3: x_0x_2 and x_1^2. As x_0x_2 is greater than x_1^2 in the lexicographic order, contracting x_0x_2 produces the almost lexsegment ideal I_2, while contracting x_1^2 produces the ideal J, which is not almost lexsegment.

We summarize the above results:

Corollary 3.34. *Each almost lexsegment ideal with Hilbert polynomial $p(z)$ can be obtained from its double saturation $L_{\widetilde{p}}$ through a sequence of a_0 lex expansions (exclusively) through almost lexsegment ideals.*

Proof. If an almost lexsegment ideal is not doubly saturated, then, by Lemma 3.32, we can perform a lex contraction to produce another almost lexsegment ideal with the same double saturation. Repeating a finite number of times will yield the double saturation. By Lemma 3.31, lex expansions and lex contractions are inverse operations so we can go the other direction. The number of needed expansions is a_0 by Lemma 3.29 and Corollary 3.16. $\square$

Combining Corollaries 3.16 and 3.34 yields the following procedure.

Algorithm 3.35. (*Generating all almost lexsegment ideals with a given Hilbert polynomial*) Let $p(z)$ be a nonzero Hilbert polynomial of some graded quotient of R.

1. Compute a_0 from $p(z)$ and the double saturation of the lexicographic ideal, $L_{\widetilde{p}}$ (as in Theorem 2.23), where $\widetilde{p}(z) = p(z) - a_0$.

2. Perform a_0 successive lex expansions of monomial generators of $L_{\widetilde{p}}$. Exhaust all choices for a_0 successive lex expansions.

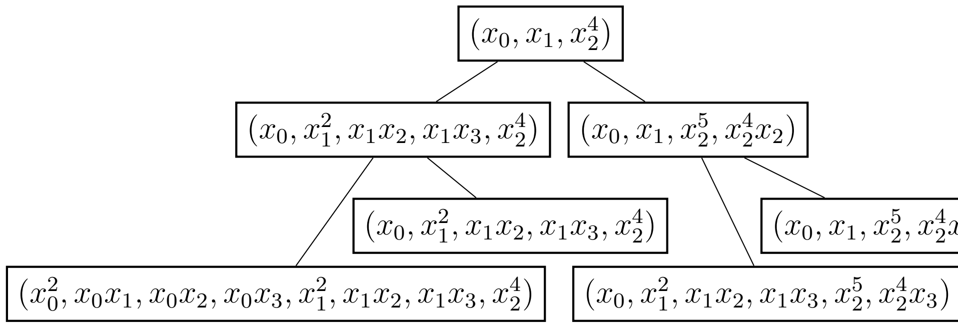

Figure 3.3: The intermediate ideals generated in Example 3.36 while finding all almost lexsegment ideals with Hilbert polynomial $p(z) = 4z + 1$ in $K[x_0, \dots, x_4]$.

The following example illustrates this process.

Example 3.36. Suppose we wish to find all almost lexsegment ideals with Hilbert polynomial $p(z) = 4z + 1$ in $R = K[x_0, x_1, x_2, x_3, x_4]$.

- First we compute the double saturation of the lexicographic ideal for p and a_0. The Hilbert polynomial can be written as

$$\binom{z+1}{2} - \binom{z-3}{2} + \binom{z}{1} - \binom{z-7}{1}.$$

  Thus, the lexicographic ideal for $p(z)$ is $(x_0, x_1, x_2^5, x_2^4x_3^3)$ so

$$L_{\widetilde{p}} = (x_0, x_1, x_2^4)$$

  and $a_0 = 3$.

- Next we make three lex expansions in all possible ways, starting from the ideal $L_{\widetilde{p}}$, to produce the following eight almost lexsegment ideals with the desired Hilbert polynomial:

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3^2, x_2^4), \qquad (x_0, x_1^2, x_1x_2^2, x_1x_3^2, x_2^4)$$

$$(x_0, x_1^2, x_1x_2, x_1x_3^3, x_2^4), \qquad (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3, x_2^5, x_2^4x_3)$$

$$(x_0, x_1^2, x_1x_2, x_1x_3^2, x_2^5, x_2^4x_3^2), \qquad (x_0, x_1^2, x_1x_2, x_1x_3, x_2^5, x_2^4x_3^2)$$

$$(x_0, x_1, x_2^6, x_2^5x_3, x_2^4x_3^2), \qquad (x_0, x_1, x_2^5, x_2^4x_3^3)$$

  By comparison, there are twelve saturated strongly stable ideals in R with Hilbert polynomial $p(z) = 4z + 1$. The intermediate expansions are provided in Figure 3.3.

As with Algorithm 3.22, several things should be noted to simplify the implementation for Algorithm 3.35. In particular, parts (i) and (iv) of Remark 3.25 apply in this case: the expansions can be uniquely ordered, and only the (lex) expandable monomials need to be stored.

In Chapter 4, we study ideals with maximal Betti numbers among all ideals with a fixed Hilbert polynomial. By the Bigatti–Hulett–Pardue Theorem (2.36), it is sufficient to only consider almost lexsegment ideals. Thus, we can use Algorithm 3.35. In fact, we can modify the algorithm so that we consider fewer ideals: we can start by expanding *all* monomial of least degree in the ideal as many times as possible. For more details, see Chapter 4.

3.4 Strongly stable ideals with a given Hilbert series

We first introduced an algorithm for producing all saturated strongly stable ideals with a prescribed Hilbert polynomial, Algorithm 3.22. We then presented an algorithm for finding a unique ideal corresponding to the distinct Hilbert series, Algorithm 3.35. We conclude with an algorithm for generating all saturated strongly stable ideals with fixed Hilbert series, Algorithm 3.37.

The process we employ for creating these saturated strongly stable ideals is similar to the procedure for producing the lexsegment ideal for a prescribed Hilbert series. In the latter procedure, one simply adds monomial generators, in the appropriate degree, according to the lexicographic order until the desired Hilbert series is obtained. We adapt this strategy by adding any monomial generator, in the appropriate degree, which yields another saturated strongly stable ideal. However, we make several observations to simplify this process and to make it easier to implement.

Monomial generators will be added to an ideal in order of increasing degree: generators in lowest degree will be added first, starting with a power of the variable x_0 (because if a principal ideal is strongly stable, it must be generated by a power of x_0) and ending with the generators of highest degree. To ensure that each saturated strongly stable ideal is created in a unique way, monomial generators are added lexicographically.

For each saturated strongly stable ideal I, we maintain a list, L_I, of the monomials which can be added to the generators of I, so that the resulting ideal is strongly stable. We also record the "remaining portion" of the numerator of the Hilbert series, $f_I(t) = \sum_{i=0}^{r} C_i t^i$, using Equation 2.3. We always add monomials generators in the smallest degree for which there is a non-zero coefficient in $f_I(t)$, which we denote by

$$sd(f_I) = \min\{i : C_i \neq 0\}.$$

We use the notation $l_A = \max(x^A)$ for the max index of the monomial x^A and $d_A = \deg(x^A)$ for the degree of the monomial x^A.

Algorithm 3.37. (*Generating all saturated strongly stable ideals with a given Hilbert series*) Let $g(t)/(1-t)^{n+1}$ be the non-reduced Hilbert series of a graded quotient of R with $g(t) \neq 1$.

1. Set $\mathcal{S} = \mathcal{M} = \varnothing$. Compute $f_{(0)}(t) = 1 - g(t)$ and $sd(f_{(0)})$. Add the ideal $I = (x_0^{sd(f_{(0)})})$ to $\mathcal{M}$. Update $f_I(t)$ to $f_{(0)}(t) - t^{sd(f_{(0)})}$, compute $sd(f_I)$, and set

L_I to $\{x_0^{sd(f_{(0)})-1}x_1^{sd(f_I)-sd(f_{(0)})+1}\}$.

2. Repeat until $\mathcal{M}$ is empty. Choose an ideal $I \in \mathcal{M}$. Do one of the following:
   - If $f_I(t) = 0$, remove the ideal I from $\mathcal{M}$ and add it to $\mathcal{S}$.
   - If $L_I = \varnothing$, remove I from $\mathcal{M}$ and continue with the next ideal in $\mathcal{M}$.
   - If $f_I(t) \neq 0$ and $L_I \neq \varnothing$, remove I from $\mathcal{M}$ and replace it with the $|L_I|$ ideals obtained by adding a single monomial x^B from L_I to the generators of I. For each ideal J_B added to $\mathcal{M}$, which is generated by $G(I) \cup \{x^B\}$: update $f_{J_B}(t)$ to $f_I(t) - (1-t)^{l_B}t^{d_B}$, compute $sd(f_{J_B})$, and set L_{J_B} to $\{x^A x_{l_A}^{sd(f_{J_B})-d_A} : x^A \in L_I, x^B >_{lex} x^A\}$. Do the following:
     - If $l_B < n-1$ and $\mathcal{L}(\frac{x_{l_B+1}}{x_{l_B}}x^B) \subset I$, include $x^B x_{l_B}^{sd(f_J)-d_B-1}x_{l_B+1}$ in L_{J_B}.
     - If $x_{l_B-1}|x^B$ and $\mathcal{L}(\frac{x_{l_B}}{x_{l_B-1}}x^B) \subset I$, include $x^B x_{l_B-1}^{-1}x_{l_B}^{sd(f_J)-d_B+1}$ in L_{J_B}.

3. Return the set of ideals $\mathcal{S}$.

Proof. (*Correctness*) Certainly, any ideal produced by the above process will be strongly stable (because we check that the ideal generated by $G(I) \cup \{x^B\}$ is strongly stable before adding the monomial x^B to L_I) and saturated (because no monomials added to the set of generators will be divisible by the variable x_n), and it will have the desired Hilbert series (because the ideal is added to $\mathcal{S}$ when the Hilbert series is correct).

We need to show that every saturated strongly stable is produced: specifically, for each ideal I produced in the algorithm, L_I contains every monomial x^B which can be added (in lexicographic order) to the ideal I to produce a saturated strongly stable ideal, say J, generated by $G(I) \cup \{x^B\}$. Suppose that the ideal J is strongly stable; then

$$\left\{\frac{x_i}{x_j}x^B : x_j|x^B, i < j\right\} \subset I, \text{ so, in particular, } x^A = \frac{x_{l_B-1}}{x_{l_B}}x^B \in I.$$

Note that the monomial x^A is the smallest monomial in the lexicographic order (in degree d_B), which must be contained in the ideal I if J is strongly stable. Turning this around, at most two new monomials, say x^E and x^F, can be added to the generators of I after the monomial x^A:

$$x^E = \frac{x_{l_A+1}}{x_{l_A}}x^A \text{ (if } l_A < n-1\text{) and } x^F = \frac{x_{l_A}}{x_{l_A-1}}x^A \text{ (if } x_{l_A-1}|x^A).$$

These monomials, x^E and x^F, are precisely those which are included in L_I. The monomials x^E and x^F are added to L_I, provided that $\mathcal{L}(x^E) \subset I$ or $\mathcal{L}(x^F) \subset I$ so that the ideals generated by $G(I) \cup \{x^E\}$ and $G(I) \cup \{x^F\}$ are saturated and strongly stable. Thus, every monomial x^B, which can be added to the generators of an ideal I to produce a saturated strongly stable ideal, appears in L_I, so the algorithm will generate all of the desired ideals.

The algorithm terminates for any given Hilbert series because each list L_I is finite and, by Theorem 3.20, there is an upper bound for the largest degree of a minimal generator of a saturated strongly stable ideal that depends only on its Hilbert polynomial, which in turn is determined by the Hilbert series. □

We include an example to illustrate this algorithm.

Example 3.38. Suppose we wish to find all saturated strongly stable ideals in $R = K[x_0, x_1, x_2, x_3, x_4]$ with Hilbert series $H_{R/I}(t) = (1 - 6t^2 + 8t^3 - 3t^4)/(1 - t)^5$. Thus, the numerator of the Hilbert series is $1 - 6t^2 + 8t^3 - 3t^4$, and $n = 4$.

- We begin with the zero ideal. We compute $f_{(0)}(t) = 6t^2 - 8t^3 + 3t^4$ and $sd(f_{(0)}) = 2$ (because $6t^2$ is the smallest nonzero term in $f_{(0)}$). We add $I_1 = (x_0^2)$ to $\mathcal{M}$, update $f_{I_1}(t)$ to $f_{(0)}(t) - t^2 = 5t^2 - 8t^3 + 3t^4$, record $sd(f_{I_1}) = 2$, and set L_{I_1} to $\{x_0x_1\}$.
- We replace I_1 in $\mathcal{M}$ with a new ideal $I_2 = (x_0^2, x_0x_1)$. We update f_{I_2} to $f_{I_1}(t) - (1 - t)t^2 = 4t^2 - 7t^3 + 3t^4$, record $sd(f_{I_2}) = 2$, and set L_{I_2} to $\{x_0x_2, x_1^2\}$.
- We replace I_2 in $\mathcal{M}$ with the ideals $I_3 = (x_0^2, x_0x_1, x_0x_2)$ and $I_4 = (x_0^2, x_0x_1, x_1^2)$.
  - $f_{I_3} = f_{I_2}(t) - (1 - t)^2t^2 = 3t^2 - 5t^3 + 2t^4$, $sd(f_{I_3}) = 2$, and $L_{I_3} = \{x_0x_3, x_1^2\}$
  - $f_{I_4} = f_{I_2}(t) - (1 - t)t^2 = 3t^2 - 6t^3 + 3t^4$, $sd(f_{I_4}) = 2$, and $L_{I_4} = \varnothing$ (because $x_0x_2 >_{lex} x_1^2$ and $x_0x_2 \notin I_4$ so x_1x_2 cannot be added to I_4)
- We replace I_3 in $\mathcal{M}$ with the two ideals $I_5 = (x_0^2, x_0x_1, x_0x_2, x_0x_3)$ and $I_6 = (x_0^2, x_0x_1, x_0x_2, x_1^2)$. We ignore I_4 (because $L_{I_4} = \varnothing$).
  - $f_{I_5} = f_{I_3}(t) - (1 - t)^3t^2 = 2t^2 - 2t^3 - t^4 + t^5$, $sd(f_{I_5}) = 2$, and $L_{I_5} = \{x_1^2\}$
  - $f_{I_6} = f_{I_3}(t) - (1 - t)t^2 = 2t^2 - 4t^3 + 2t^4$, $sd(f_{I_6}) = 2$, and $L_{I_6} = \{x_1x_2\}$
- We replace I_5 in $\mathcal{M}$ with the ideal $I_7 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2)$, and we replace I_6 with the ideal $I_8 = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2)$.
  - $f_{I_7} = f_{I_5}(t) - (1 - t)t^2 = t^2 - t^3 - t^4 + t^5$, $sd(f_{I_7}) = 2$, and $L_{I_7} = \{x_1x_2\}$
  - $f_{I_8} = f_{I_6}(t) - (1 - t)^2t^2 = t^2 - 2t^3 + t^4$, $sd(f_{I_8}) = 2$, and $L_{I_8} = \{x_2^2\}$
- We replace I_7 in $\mathcal{M}$ with the ideal $I_9 = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2)$, and we replace I_8 with the ideal $I_{10} = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$.
  - $f_{I_9} = f_{I_7}(t) - (1 - t)^2t^2 = t^3 - 2t^4 + t^5$, $sd(f_{I_9}) = 3$, and $L_{I_9} = \{x_1x_3^2, x_2^3\}$ (because we need to add monomials of degree 3)
  - $f_{I_{10}} = f_{I_8}(t) - (1 - t)^2t^2 = 0$ (We do not need $sd(f_{I_{10}})$ or $L_{I_{10}}$.)
- We add I_{10} to $\mathcal{S}$, and we replace I_9 in $\mathcal{M}$ with the two ideals $I_{11} = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3^2)$ and $I_{12} = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_2^3)$.
  - $f_{I_{11}} = f_{I_9}(t) - (1 - t)^3t^3 = t^4 - 2t^5 + t^6$, $sd(f_{I_{11}}) = 4$, and $L_{I_{11}} = \{x_2^4\}$
  - $f_{I_{12}} = f_{I_9}(t) - (1 - t)^2t^3 = 0$ (We do not need $sd(f_{I_{12}})$ or $L_{I_{12}}$.)

- We add I_{12} to $\mathcal{S}$, and we replace I_{11} in $\mathcal{M}$ with the ideal $I_{13} = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3^2, x_2^4)$.
  - $f_{I_{13}} = f_{I_{11}}(t) - (1-t)^2t^4 = 0$
- We add I_{13} to $\mathcal{S}$.

Thus, there are three saturated strongly stable ideals with the given Hilbert series:

$$(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$$

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_2^3)$$

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3^2, x_2^4)$$

3.5 Consequences and related questions

We conclude by discussing some questions that, we believe, deserve further investigation along with some initial results.

It is well known that saturated strongly stable ideals figure prominently in the combinatorial structure of the Hilbert scheme. This motivates the following problem.

Question 3.39. What is the number of saturated strongly stable ideals in R with a given Hilbert polynomial p?

Is there an explicit formula or a generating function for this number that depends only on p and the number of variables in R?

In Chapter 5, we identify saturated strongly stable ideals with (generalized) partitions in order to attack this question, at least in the case of constant Hilbert polynomials.

In Table 1 we present some experimental results for the number of strongly stable ideals with a given Hilbert polynomial in a given polynomial ring. Recall that the Hilbert polynomial is actually the Hilbert polynomial of the quotient by the ideal.

Table 1 illustrates that, fixing the Hilbert polynomial, the number of strongly stable ideals in a polynomial ring with $n+1$ variables having this Hilbert polynomial increases with n initially until it becomes stable and independent of n. This is indicated by the rightmost column in the table.

Our next result explains this observation.

Proposition 3.40. *If $p(z)$ is a Hilbert polynomial, written as in Equation* (2.1)*, then the number of saturated strongly stable ideals with Hilbert polynomial $p(z)$ in $R = K[x_0, \ldots, x_n]$ is the same whenever $n \geq b_0 + d - 1$.*

Proof. The first expansion, the expansion of 1 in $R^{(d)}$, gives $(x_0, \ldots, x_{n-d-1})$, an ideal with $n-d$ variables. By Theorem 3.20, the number of the remaining expansions will be at most $b_0 - 1$ (and depends upon how the expansions are chosen). It follows that the max index of any expanded monomial is at least $n - d - b_0 + 1$. Hence, if $b_0 - 1 \leq n - d$, then the number of saturated strongly stable ideals generated is not constrained by the number of variables. $\square$

$p(z)$	a_0, a_1, a_2	$n = 3$	$n = 6$	$n = 9$	$n = 12$	$n \gg 0$
4	$4, 0, 0$	3	3	3	3	3
8	$8, 0, 0$	12	19	20	20	20
12	$12, 0, 0$	44	104	117	119	119
16	$16, 0, 0$	143	504	617	640	644
20	$20, 0, 0$	425	2262	3034	3223	3271
24	$24, 0, 0$	1193	9578	14140	15425	15818
$4z + 2$	$4, 4, 0$	14	28	28	28	28
$4z + 6$	$8, 4, 0$	94	394	433	434	434
$4z + 10$	$12, 4, 0$	469	3702	4536	4627	4632
$4z + 14$	$16, 4, 0$	1939	27486	37792	39462	39677
$8z - 16$	$4, 8, 0$	10	18	18	18	18
$8z - 12$	$8, 8, 0$	66	213	232	233	233
$8z - 8$	$12, 8, 0$	347	1911	2268	2310	2313
$8z - 4$	$16, 8, 0$	1576	14490	18812	19510	19607
$2z^2 + 6$	$4, 0, 4$	3	18	19	19	19
$2z^2 + 10$	$8, 0, 4$	12	224	268	271	271
$2z^2 + 14$	$12, 0, 4$	44	2073	2835	2930	2938
$2z^2 + 18$	$16, 0, 4$	143	15883	24927	26468	26687
$2z^2 + 4z - 12$	$4, 4, 4$	14	45	46	46	46
$2z^2 + 4z - 8$	$8, 4, 4$	94	776	868	872	872
$2z^2 + 4z - 4$	$12, 4, 4$	469	9165	11417	11636	11649
$2z^2 + 8z - 46$	$4, 8, 4$	10	37	38	38	38
$2z^2 + 8z - 42$	$8, 8, 4$	66	588	667	671	671
$2z^2 + 8z - 38$	$12, 8, 4$	347	6535	8281	8464	8476
$4z^2 - 16z + 40$	$4, 0, 8$	3	18	19	19	19
$4z^2 - 16z + 44$	$8, 0, 8$	12	224	268	271	271
$4z^2 - 16z + 48$	$12, 0, 8$	44	2073	2835	2930	2938
$4z^2 - 12z + 6$	$4, 4, 8$	14	45	46	46	46
$4z^2 - 12z + 10$	$8, 4, 8$	94	761	853	857	857
$4z^2 - 12z + 14$	$12, 4, 8$	469	8662	10851	11069	11082
$4z^2 - 8z - 44$	$4, 8, 8$	10	37	38	38	38
$4z^2 - 8z - 40$	$8, 8, 8$	66	588	667	671	671
$4z^2 - 8z - 36$	$12, 8, 8$	347	6523	8269	8452	8464

Table 3.1: The number of saturated strongly stable ideals with a given Hilbert polynomial, $p(z)$, in $K[x_0, \ldots, x_n]$ for several values of n

The bound on the number of variables given in the last result is optimal in some cases.

Example 3.41. (i) Fix integers $d \geq 0$ and $b_0 \geq 1$. Consider the saturated strongly stable ideals I of $R = K[x_0, \ldots, x_n]$ with Hilbert polynomial

$$p(z) = \binom{z+d}{d} + b_0 - 1.$$

Then, using the notation of Theorem 2.23, $a_0 = b_0 - 1$, $a_1 = \cdots = a_{d-1} = 0$, and $a_d = 1$. Following Algorithm 3.22, the first expansion will produce the ideal $I^{(d)} = (x_0, \ldots, x_{n-d-1}) \subset R^{(d)}$. The remaining $b_0 - 1$ expansions all occur in R. If $n = b_0 + d - 1$, then expanding all of the $n - d = b_0 - 1$ variables will produce a saturated strongly stable ideal with the desired Hilbert polynomial that is generated by quadrics. However, if $n \leq b_0 + d - 2$, then any $b_0 - 1$ expansions of $I^{(d)}$ will produce an ideal having a minimal generator whose degree is at least 3. Hence the bound on n in Proposition 3.40 is optimal for this Hilbert polynomial.

(ii) Not every Hilbert polynomial will achieve this bound. Consider $p(z) = 3z = \binom{z+1}{2} - \binom{z-2}{2} + \binom{z}{1} - \binom{z-3}{1}$. If $n \geq 2$, there is exactly one saturated strongly stable ideal for this Hilbert polynomial even though $b_0 + d - 1 = 3$. (The Hilbert polynomial of the ideal generated by $(x_0, \ldots, x_{n-3}, x_{n-2}^3)$ is $p(z) = 3z$, while the Hilbert polynomial of the ideal $(x_0, \ldots, x_{n-4}, x_{n-3}^2, x_{n-3}x_{n-2}, x_{n-2}^2)$ is $3z + 1 \neq p(z)$)

It is known that the lexicographic ideal has the worst Castelnuovo–Mumford regularity among all saturated ideals with a fixed Hilbert polynomial (see [2], [12], and [29]). Theorem 3.20 provides a quick new argument. It also allows us to discuss the extremal ideals. We denote by $\operatorname{gin} I$ the generic initial ideal of the ideal I with respect to the reverse lexicographic order.

Theorem 3.42. *Let $I \neq R$ be a saturated homogeneous ideal of R. Write the Hilbert polynomial, p, of R/I as in Equation* (2.1)*. Then the Castelnuovo–Mumford regularity of I satisfies*

$$\operatorname{reg} I \leq b_0.$$

Furthermore, if I is strongly stable, then equality is true if and only if $I = L_p$.

Moreover, if I is any saturated homogeneous ideal and $\operatorname{char} K = 0$*, then* $\operatorname{reg} I = b_0$ *if and only if* $\operatorname{gin} I = L_P$ *and I is of the form*

$$I = (l_0, \ldots, l_{n-d-2}, f_d l_{n-d-1}, f_d f_{d-1} l_{n-d}, \ldots, f_d \ldots f_{t+1} l_{n-t-2}, f_d \ldots f_t) \tag{3.3}$$

where $0 \leq t \leq d$, every $f_i \neq 0$ is a homogeneous polynomial of degree $a_i \geq 0$, $a_n, a_t \geq 1$, every l_i is a linear form, and I has (as indicated) $n + 1 - t$ minimal generators. (Note that when $n = t$ the ideal I is simply defined as $I = (f_d)$.)

Proof. First, we show the claims when I is a strongly stable ideal. The Eliahou–Kervaire resolution shows that the regularity of I is the maximal degree of a minimal generator of I. By Theorem 3.20 we know that I can be obtained from the ideal $(1) = R^{(d)}$ by at most b_0 expansions. Since each expansion replaces a monomial by monomials whose degree is one more, it follows immediately that the degrees of the minimal generators of I are at most b_0.

In order to characterize equality we use induction on $b_0 \geq 1$. If $b_0 = 1$, then I is generated by linear forms, and the claim follows. Let $b_0 > 1$, and assume that I has a minimal generator of degree b_0. Then, by the above argument, I must have been obtained from the ideal $(1) = R^{(d)}$ by exactly b_0 expansions. Denote by J' the ideal obtained by the first $b_0 - 1$ expansions, and put $J = J'R$. Then J must have a minimal generator of degree $b_0 - 1$. Write the Hilbert polynomial of R/J as

$$p'(z) = \sum_{i=0}^{d} \left[\binom{z+i}{i+1} - \binom{z+i-b'_i}{i+1} \right].$$

Then $b'_0 = b_0 - 1$. Hence, the induction hypothesis provides that $J = L_{p'}$. It follows that among the minimal generators of J' having degree $b_0 - 1$ only the smallest one in the lexicographic order is expandable. Expanding it, we get $I = L_p$ (see Remark 3.21).

Second, let I be an arbitrary saturated homogeneous ideal with the given Hilbert polynomial. Passing from I to the almost lexsegment ideal I^* with the same Hilbert function as I can only increase the regularity by a result of Bigatti, Hulett, and Pardue (see [5], [18], [27]). Since almost lexsegment ideals are strongly stable we get $\operatorname{reg} I \leq \operatorname{reg} I^* \leq b_0$.

Finally, assume that the base field K has characteristic zero. Then $\operatorname{gin} I$ is strongly stable and has the same regularity as I by [3]. Hence, by the first part of the proof, $\operatorname{reg} I = b_0$ if and only if $\operatorname{gin} I = L_p$. The claimed description of I in this case now follows by Theorem 4.4 and Lemma 3.4 in [25]. □

Combined with the main result of Murai and Hibi in [24], we obtain the following consequence. We would like to thank Jeff Mermin for pointing this out.

Recall that a homogeneous ideal I of $R = K[x_0, \ldots, x_n]$ is a Gotzmann ideal if it has as many minimal generators as the lexsegment ideal $L_h \subset R$ corresponding to the Hilbert function of I. Notice that an ideal I of R is saturated if it has at most n minimal generators.

Corollary 3.43. *Let $I \subset R$ be a saturated homogeneous ideal, where* $\operatorname{char} K = 0$. *Write the Hilbert polynomial of R/I as in Equation* (2.1). *Then the following conditions are equivalent:*

(a) $\operatorname{reg} I = b_0$*;*

(b) $\operatorname{gin} I$ *is a lexicographic ideal;*

(c) I is a Gotzmann ideal with at most n minimal generators;

(d) I is an ideal of the form as specified in Equation (3.3).

Proof. Conditions (a), (b), and (d) are equivalent by Theorem 3.42. The equivalence to Condition (c) follows by Theorem 1.1 in [24] because (d) shows that I is a canonical critical ideal up to a coordinate transformation. □

We conclude with a crude estimate on the number of strongly stable ideals with a given Hilbert polynomial.

Corollary 3.44. *Let p be the Hilbert polynomial of a graded quotient of R. Using the notation of Theorem 2.23, put $c = \min\{n, b_0 + d - 1\}$. Then the number of saturated strongly stable ideals in R with Hilbert polynomial p is at most*

$$\binom{\binom{c-d+b_0-1}{b_0-1}+1}{a_d}\binom{\binom{c-(d-1)+b_0-1}{b_0-1}+1}{a_{d-1}}\cdot\ldots\cdot\binom{\binom{c+b_0-1}{b_0-1}+1}{a_0}.$$

Proof. Assume first that $n \leq b_0 + d - 1$, that is, $c = n$.

Using the notation of Theorem 3.20, it takes at most a_j expansions to take $I^{(j+1)} \cdot R^{(j)}$ to $I^{(j)}$. By Theorem 3.42, the degree of each expanded monomial is at most $b_0 - 1$. Moreover, we expand only monomials in $K[x_0, \ldots, x_{n-j-1}]$. There are $N_j = \binom{n-j+b_0-1}{b_0-1}$ such monomials whose degree is at most $b_0 - 1$. For expanding at most a_j of them, there are at most

$$\binom{N_j}{0} + \binom{N_j}{1} + \cdots + \binom{N_j}{a_j} = \binom{N_j+1}{a_j}$$

possibilities. Since we take $I^{(j+1)}$ to $I^{(j)}$ for $j = d, d-1, \ldots, 0$, the claim follows in this case.

Second, if $n \geq b_0 + d - 1$, then the number of strongly stable ideals is the same as for $n = b_0 + d - 1$ by Proposition 3.40. This concludes the argument. $\square$

Chapter 4 Ideals with maximal Betti numbers

In this chapter we explore an application of the Algorithm 3.35: constructing an ideal with maximal Betti numbers among all ideals with a fixed Hilbert polynomial. We will use the concepts of almost lexsegment ideals and and lex expansions which were discussed in Section 3.3.

We first start with some background, which will motivate what follows. Recall that the Betti numbers are the ranks of the free modules in minimal free resolutions. Thus, these invariants are a measure of complexity so it is only natural to wonder, if one specifies a size (that is, a Hilbert function or Hilbert polynomial) for an ideal, whether there are bounds on the Betti numbers. Bigatti, Hulett, and Pardue proved that, among ideals with a given Hilbert function, the lexicographic ideal has the largest Betti numbers. Using this result, Valla, [33], was able to show that, among saturated ideals with a given constant Hilbert polynomial, there is an ideal with maximal Betti numbers. In fact, the almost lexsegment ideal corresponding to the maximal Hilbert function achieves the desired bound. Valla further proved that the result still holds if an initial degree is specified for the ideals (in addition to the constant Hilbert polynomial). Note, however, that Valla phrases his result in terms of perfect ideals in a regular local ring with fixed multiplicity and height.

Recently, Caviglia and Murai, [6], extended Valla's first result by showing that there is a saturated ideal which achieves maximal Betti numbers among all ideals with any given Hilbert polynomial. Caviglia and Murai note in their paper [6] that their proof "is very long and complicated" and their construction "is not easy to understand." Thus, we seek an alternate construction and proof. Ideally, we want to produce an ideal with maximal Betti numbers among all saturated ideals with a fixed Hilbert polynomial and specified initial degree, a full generalization of Valla's theorem.

Examples 4.1 and 4.2 show that there can be more than one ideal with maximal Betti numbers for a given Hilbert polynomial. It is possible to find all of these ideals using only tools that we have developed so far. By the result of Bigatti, Hulett and Pardue, it is sufficient to consider almost lexsegment ideals when looking for ideals with maximal Betti numbers. Using Algorithm 3.35, one can determine all of the almost lexsegment ideals and then simply select those with the largest Betti numbers. Unfortunately, the number of almost lexsegment ideals with a particular Hilbert polynomial can be quite large. As mentioned at the end of Section 3.3, the number of ideals to consider can be reduced by first expanding *all* of the monomials in the lowest degree as many times as possible. Still, this approach is generally impractical, though it works fine in small cases.

A simple construction and proof could perhaps be found by choosing a particular family of ideals. This motivates the questions: How many ideals attain maximal Betti numbers and how can they be distinguished?

One idea is to consider the Hilbert function of the ideals in question. Because the ideals are almost lexsegment, their Hilbert functions will be distinct. One might hope that among all ideals with maximal Betti numbers, there is one which has a Hilbert function which is either larger in all degrees than the other Hilbert functions, or which is smaller in all degrees. As noted above, in the case of constant Hilbert polynomials, it suffices to pick the almost lexsegment ideal with maximal Hilbert function. Unfortunately, the following two examples show that, in general, there may not be a maximal or minimal Hilbert function among the ideals with maximal Betti numbers.

Example 4.1. In the polynomial ring $K[x_0, x_1, x_2, x_3, x_4]$, there are 509 saturated strongly stable ideals with Hilbert polynomial $p(z) = z^2 + 5z + 3$. Of these, 129 are almost lexsegment ideals, and four ideals attain maximal Betti numbers. All four ideals are obtained by making two lex expansions in the ideal

$$(x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3, x_0x_3^2, x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2^3).$$

(The ideal above is obtained by repeatedly expanding monomials in the initial degree, starting with the doubly saturated lexicographic ideal $(x_0, x_1^3, x_1^2x_2^3)$.)

To maximize the Hilbert function, we want to expand in the smallest degree possible, but to achieve maximal Betti numbers we need to expand a monomial whose last variable is x_2. We have two choices: either we expand $x_0x_3^2$ and $x_1^2x_2^3$ (to maximize the Hilbert function in degree three) to obtain the ideal

$$\begin{aligned}(x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3,&\\ x_0x_3^3, x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2^4, x_1^2x_2^3x_3),&\end{aligned}$$

or we expand $x_1^3x_2$ and $x_1^3x_3$ (to maximize the Hilbert function in degree four) to obtain

$$\begin{aligned}(x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3, x_0x_3^2,&\\ x_1^4, x_1^3x_2^2, x_1^3x_2x_3, x_1^3x_3^2, x_1^2x_2^3).&\end{aligned}$$

The Hilbert functions of these two ideals are incomparable.

Minimal Hilbert functions among the ideals with maximal Betti numbers do not exist even in the case of a constant Hilbert polynomial.

Example 4.2. In the polynomial ring $K[x_0, x_1, x_2, x_3]$, there are 6,481 saturated strongly stable ideals with Hilbert polynomial $p(z) = 31$. Of these, 2,649 are almost lexsegment ideals, and five ideals attain maximal Betti numbers. All five ideals are obtained by making eleven lex expansions in the ideal

$$(x_0, x_1, x_2)^4.$$

To minimize the Hilbert function, we want to expand in the largest degree possible, but we have two choices: either we expand the last nine monomials in degree four and expand the last monomial in the largest degree twice more (to minimize the Hilbert function in degree four) to obtain the ideal

$$\begin{aligned}(x_0^4, x_0^3x_1, x_0^3x_2, x_0^2x_1^2, x_0^2x_1x_2, x_0^2x_2^2, x_0x_1^4, x_0x_1^3x_2, x_0x_1^2x_2^2, x_0x_1x_2^3,&\\ x_0x_2^4, x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4, x_2^7),&\end{aligned}$$

or we expand the last six monomials in degree four and the last five monomials in degree six (to minimize the Hilbert function in degree five) to obtain

$$(x_0^4, x_0^3x_1, x_0^3x_2, x_0^2x_1^2, x_0^2x_1x_2, x_0^2x_2^2, x_0x_1^3, x_0x_1^2x_2, x_0x_1x_2^2, x_0x_2^3,\\ x_1^6, x_1^5x_2, x_1^4x_2^2, x_1^3x_2^3, x_1^2x_2^4, x_1x_2^5, x_2^6).$$

The Hilbert functions of these two ideals are incomparable.

4.1 A reduction

We now introduce a new vector, through which we can compare the Betti numbers of a set of almost lexsegment ideals, all of which are produced by making a fixed number of expansions in some doubly saturated universal lexsegment ideal. By Equation (2.4), the Betti numbers of strongly stable ideals only depend on the last variables in the minimal generators. Expanding a monomial whose last variable is x_i replaces the monomial ending in x_i with monomials ending in x_i, ..., x_{n-2}, and x_{n-1}. Thus, for an expanded monomial ending in x_i, we should record the indices which are at least as big as i.

We are ultimately making observations about generators which are expanded to produce almost lexsegment ideals. These ideals are, by definition, saturated strongly stable ideals so, by Remark 2.5, the last variable, x_n, will not appear in the generators. Thus, instead of working with almost lexsegment ideals in $K[x_0, \ldots, x_n]$, we can focus on lexsegment ideals in $S = K[x_0, \ldots, x_{n-1}]$, the polynomial ring where the last variable has been omitted. This will make notation simpler in what follows.

If E is a finite set, we denote the cardinality of E by $\#E$. We recall the max index for a monomial $x^A = x_0^{a_0} \cdots x_n^{a_n}$:

$$\max(x^A) = \max\{i : a_i > 0\}.$$

We declare

$$\max(1) = 0.$$

When giving the total Betti numbers for a lexsegment ideal I, we write them as a vector

$$\beta(I) = (\beta_0(I), \beta_1(I), \ldots, \beta_{n-1}(I)),$$

which implicitly asserts that $\beta_j(I) = 0$ for $j \geq n$; again, this is because we are considering lexsegment ideals in $S = K[x_0, \ldots, x_{n-1}]$.

Definition 4.3. If E is a finite set of monomials in S, we define the (trailing) *m-vector*

$$m(E) = (m_{\leq 0}(E), m_{\leq 1}(E), \ldots, m_{\leq n-1}(E)) \in \mathbb{Z}^n,$$

where

$$m_{\leq j}(E) = \#\{x^A \in E : \max(x^A) \leq j\}.$$

Observe that if E is a set of expanded monomials, the last entry in $m(E)$ is the total number of expansions, $\#E$. The sum of all but the last entry in $m(E)$ is the number of generators that have been added by expanding the monomials in E.

We also note that instead of specifying a Hilbert polynomial, we can specify a universal lexsegment ideal (which is saturated in S) and a number of expansions. (This is Corollary 3.34.) By Proposition 2.14, saturated lexsegment ideals in S are automatically universal lexsegment ideals. From now on, we will specify either a Hilbert polynomial or a saturated lexsegment ideal and a number of expansions.

We can define a partial order, $\succeq$, on the set of vectors in $\mathbb{Z}^n$:

$$(a_1, \ldots, a_n) \succeq (b_1, \ldots, b_n) \quad \text{if } a_i \geq b_i \text{ for all } i.$$

If I and J are two lexsegment ideals obtained by making expansions in a saturated (universal) lexsegment ideal, U, we let $\mathcal{M}_I$ and $\mathcal{M}_J$ be the set of monomials in $U \setminus I$ and $U \setminus J$, respectively; (these are the expanded monomials). We can compare the Betti numbers of I and J by comparing $m(\mathcal{M}_I)$ and $m(\mathcal{M}_J)$, with respect to this partial order. This result is Corollary 3.6 in [6].

Proposition 4.4. *Let I and J be lexsegment ideals obtained by making the same number of expansions in a saturated lexsegment ideal, U. Let $\mathcal{M}_I = \{x^A \in U \setminus I\}$ and $\mathcal{M}_J = \{x^A \in U \setminus J\}$. If $m(\mathcal{M}_I) \succeq m(\mathcal{M}_J)$, then, for all i,*

$$\beta_i(I) \geq \beta_i(J).$$

Proof. Expanding a monomial whose last variable is x_i replaces the monomial with monomials ending in x_i, …, x_{n-2}, and x_{n-1}. For each expansion in I or J, we record the index of the largest variables for the added generators. (The largest index of the generators which are not expanded do not matter and can be ignored.) Thus, $m_{\leq i}(\mathcal{M}_I) \geq m_{\leq i}(\mathcal{M}_J)$ exactly when the ideal I has at least as many generators ending in x_i as J does. If $m(\mathcal{M}_I) \succeq m(\mathcal{M}_J)$, then, by Equation (2.4), the Betti numbers of I are all at least as big as the corresponding Betti numbers of J. $\square$

Thus, to find an ideal with maximal Betti numbers, we just need to maximize the trailing m-vector for the expanded monomials. In fact, it is sufficient to find an ideal with the most generators, that is, the ideal whose first Betti number, β_1, is largest.

Proposition 4.5. *Let I be a lexsegment ideal obtained by making some fixed number of expansions, say c, in a saturated lexsegment ideal, U. If $\beta_1(I) \geq \beta_1(J)$ for any other lexsegment ideal J produced by making c expansions in U, then, for all i,*

$$\beta_i(I) \geq \beta_i(J).$$

Proof. The first Betti number of an ideal gives the number of minimal generators. As noted above,

$$\beta_1(I) = \beta_1(U) + \sum_{k=0}^{n-2} m_{\leq k}(\mathcal{M}_I),$$

where $\mathcal{M}_I$ is the set of monomials in $U \setminus I$. By Proposition 3.9 in [6], there is an ideal I such that $m(\mathcal{M}_I) \succeq m(\mathcal{M}_J)$ for all J. Thus, $\sum_{k=0}^{n-2} m_{\leq k}(\mathcal{M}_I) \geq \sum_{k=0}^{n-2} m_{\leq k}(\mathcal{M}_J)$. By Proposition 4.4, $\beta_i(I) \geq \beta_i(J)$ for all i. $\square$

We always start by making expansions in some saturated universal lexsegment ideal. In order to find an ideal with maximal Betti numbers, this assumption is crucial. If we start with an arbitrary lexsegment ideal, the Betti numbers of the resulting ideals may not be comparable.

Example 4.6. Consider the saturated lexsegment ideal $L \subset S = K[x_0, x_1, x_2, x_3]$ with minimal generators

$$(x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^3, x_1^2x_2, x_1^2x_3^2, x_1x_2^3, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_3^4, x_2^6).$$

Note that L is not a universal lexsegment (because it has more than four generators). If we look for maximal Betti numbers among the ideals obtained by making four expansions in L, then we have two choices.

Let $I \subset S$ be the lexsegment ideal obtained by expanding the four generators of L of degree two ($\{x_0^2, x_0x_1, x_0x_2, x_0x_3\}$), and let $J \subset S$ be the lexsegment ideal obtained by expanding the two generators of L of degree three and the last monomial of degrees four and six ($\{x_1^3, x_1^2x_2, x_1x_2^3, x_2^6\}$). Now, the trailing m-vectors are incomparable:

$$m(\mathcal{M}_I) = (1, 2, 3, 4) \qquad m(\mathcal{M}_J) = (0, 1, 4, 4).$$

The Betti numbers for I and J are also incomparable:

$$\beta(I) = (1, 18, 39, 30, 8) \qquad \beta(J) = (1, 17, 39, 32, 9).$$

Thus, there is no maximal set of Betti numbers among these ideals.

4.2 A greedy algorithm will not work

We now turn our attention to how we should choose the expansions to produce a lexsegment ideal in S with maximal Betti numbers. By Proposition 4.4, it is sufficient to maximize the trailing m-vector for the expanded monomials. The simplest approach would be to choose the expansions one at a time, and each time expand a monomial whose last variable has the smallest index. (This is a greedy algorithm.) Unfortunately, this does not always produce an ideal with maximal Betti numbers.

Example 4.7. Suppose we wish to make four expansions in the saturated universal lexsegment ideal $L = (x_0, x_1^3) \subset K[x_0, x_1, x_2]$ to produce an ideal with maximal Betti numbers.

We first use the greedy strategy outlined in the previous paragraph. We expand the generator x_0 to produce the ideal

$$I = (x_0^2, x_0x_1, x_0x_2, x_1^3).$$

Next, we expand the generator x_1^3 to produce the ideal

$$(x_0^2, x_0x_1, x_0x_2, x_1^4, x_1^3x_2).$$

We now expand either the last two generators in degree two or in degree four. Both choices give the m-vector

$$(1, 3, 4).$$

Alternatively, we can always expand in the lowest degree. We first expand x_0 to produce the ideal I. Then we expand the three generators of degree two. This yields a larger m-vector:

$$(2, 3, 4).$$

By Proposition 4.4, the latter m-vector corresponds to larger Betti numbers.

Clearly, picking the expansions one at a time will not work; however, the previous example suggests a new strategy. We should look ahead to see how well we can do if we choose several expansions at once. Instead of choosing expansions one monomial at a time, perhaps we choose several monomials at a time so that we can ultimately expand a monomial with the smallest max index possible. In particular, we try the following: Determine the smallest max index of a monomial which may be expanded using (some of) the remaining expansions, say i. Then, among the generators with largest variable x_i, select the generator which may be expanded using the fewest steps. Perform the selected expansions. Repeat this process until no expansions remain. Unfortunately, this method also fails to produce an ideal with maximal Betti numbers.

Example 4.8. Suppose we wish to make fifteen expansions in the saturated universal lexsegment ideal $L = (x_0, x_1^3, x_1^2x_2^2) \subset K[x_0, x_1, x_2, x_3]$ to produce an ideal with maximal Betti numbers.

We first follow strategy outlined in the previous paragraph, so we expand the five monomials in the lowest degree to produce the ideal

$$(x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3, x_0x_3^2, x_1^3, x_1^2x_2^2).$$

We now expand the monomial x_1^3 (because we cannot expand x_0^3) to produce the ideal

$$(x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3, x_0x_3^2, x_1^4, x_1^3x_2, x_1^3x_3, x_1^2x_2^2).$$

We now expand the four monomials in degree four (instead of the last six monomials of degree three) to produce the ideal

$$\begin{aligned}(&x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3, x_0x_3^2,\\ &x_1^5, x_1^4x_2, x_1^4x_3, x_1^3x_2^2, x_1^3x_2x_3, x_1^3x_3^2, x_1^2x_2^3, x_1^2x_2^2x_3).\end{aligned}$$

With the five remaining expansions we can expand two monomials which end in x_2 (but none which end in x_1), so the m-vector of the expanded monomials is

$$(2, 5, 10, 15).$$

Alternatively, we can always expand in the lowest degree. This provides a larger m-vector:

$$(2, 6, 10, 15).$$

Obviously, we need to be a little more clever in how we pick these expansions. Again, the previous example suggests a new tactic: Instead of just selecting a single monomial with the smallest max index, say i, we should select as many as we can at once. There is one major problem with this strategy: while we are expanding monomials ending in x_i, we need to know how many monomials ending in x_i we can ultimately expand based on the intervening expansions. (In other words, we need to know how the choice of expansions will affect the future choices.)

One way to avoid this dilemma is to select monomials in a particular degree, where we pick the degree to maximize the number of monomials ending in the desired variable x_i. Specifically, among the degrees containing generators with max index i, we select the degree containing the most monomials ending in the variable x_i. If there are ties (that is, if two degrees contain the same number of such monomials), then we expand the smaller set. If there are still ties, we can favor the smaller degree. Unfortunately, this method also fails to generate an ideal with the largest Betti numbers (whether we favor smaller or larger degrees in breaking ties).

Example 4.9. Suppose we wish to make twelve expansions in the saturated universal lexsegment ideal $L = (x_0, x_1^3, x_1^2x_2^2) \subset K[x_0, x_1, x_2, x_3]$ to produce an ideal with maximal Betti numbers.

We first adopt the strategy outlined in the previous paragraph, so we expand the five monomials in the lowest degree to produce the ideal

$$I = (x_0^3, x_0^2x_1, x_0^2x_2, x_0^2x_3, x_0x_1^2, x_0x_1x_2, x_0x_1x_3, x_0x_2^2, x_0x_2x_3, x_0x_3^2, x_1^3, x_1^2x_2^2).$$

We now expand the last seven monomials in degree three (because this includes two monomials ending in x_1), so the m-vector of the expanded monomials is

$$(2, 5, 8, 12).$$

Alternatively, starting with the ideal I above, we could expand the last generator of degree three (x_1^3), the last four generators of degree four (x_1^4, $x_1^3x_2$, $x_1^3x_3$, and $x_1^2x_2^2$), and the last two generators of degree five ($x_1^2x_2^3$ and $x_1^2x_2^2x_3$). This results in a larger m-vector:

$$(2, 5, 9, 12).$$

4.3 Examining structures

It should be clear from the previous examples that our strategy for expanding monomials to produce an ideal with maximal Betti numbers must be refined enough to account for the monomials which could be expanded after certain intermediate steps. In order to simplify such a process, we introduce some new terminology.

Recall that we are concentrating on lexsegment ideals in a polynomial ring with n variables, $S = K[x_0, x_1, \ldots, x_{n-1}]$. We let $\widehat{S}$ denote the polynomial ring where the first variable is omitted;

$$\widehat{S} = K[x_1, \ldots, x_{n-1}].$$

More generally, we let $S^{[i]}$ denote the polynomial ring where the first i variables are omitted; specifically,

$$S^{[i]} = K[x_i, \ldots, x_{n-1}].$$

(Note that we now drop the *first* few variables in $S^{[i]}$—not the last few as in $R^{(i)}$.)

We first turn our attention to the structure of a universal lexsegment ideal. All the monomials in the ideal are multiples of (at least) one of the minimal generators, so we can sort the monomials in the ideal into different sets based on the first minimal generator by which they are divisible.

Definition 4.10. If $U \subset S = K[x_0, \ldots, x_{n-1}]$ is a saturated (universal) lexsegment ideal with minimal generators $\{x^{A_0}, x^{A_1}, \ldots, x^{A_t}\}$, then U has the following unique decomposition into disjoint K-vector spaces:

$$U = x^{A_0}S^{[0]} \oplus x^{A_1}S^{[1]} \oplus \ldots \oplus x^{A_t}S^{[t]}.$$

We call each piece $x^{A_i}S^{[i]}$ of this decomposition a *cone*.

We can now view the expanded monomials in a lexsegment ideal as falling into distinct cones. When distinguishing ideals with a fixed Hilbert polynomial that attain maximal Betti numbers, one possibility is to examine how the expanded monomials are arranged in these cones.

In many instances, among the ideals with maximal Betti numbers, there is (at least) one ideal where all but one of the cones is flat: in a given degree, the multiples of the generator are either all expanded or none are expanded. For instance, consider the lexsegment ideal $L \subset S = K[x_0, x_1, x_2, x_3]$ with minimal generators

$$\{x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^3, x_1^2x_2, x_1^2x_3^2, x_1x_2^3, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_3^4, x_2^6\}$$

(as in Example 4.6, except we change the variables); the corresponding universal lexsegment ideal is

$$x_0S^{[0]} \oplus x_1S^{[1]} \oplus x_2^6S^{[2]}.$$

The first and third cones are flat because all of the generators that are multiples of x_0 have degree 2 and there is only one generator that is a multiple of x_2^6; however, the second cone is not flat because the generators that are multiples of x_1 (but not multiples of x_0) have degree three, four, and five.

One may wonder if there exists an ideal with maximal Betti numbers for any Hilbert polynomial, where at most one cone is not flat. In general there may not be such an ideal.

Example 4.11. There is only one almost lexsegment ideal, I, with Hilbert polynomial $p(z) = 3z^2 - 6z + 175$ with maximal Betti numbers. The corresponding Betti numbers are

$$\beta(I) = (1, 151, 510, 662, 389, 87).$$

(Because there are 151 generators, we do not describe the ideal.) The first and third cones in this ideal are not flat.

We now make some observations about monomials in a polynomial ring under the lex order. We can decompose the set of degree d monomials in S as

$$S_d = \bigoplus_{k=0}^{d} x_0^k [\widehat{S}]_{\leq d} \tag{4.1}$$

by setting $x_0 = 1$. This decomposition suggests the following term order, so that the ordering of terms in $\oplus_{k=0}^{d} x_0^k [\widehat{S}]_{\leq d}$ corresponds to the lex order in S_d. This term order is adopted in [6].

Definition 4.12. The *opposite degree lex order* on a set of monomials in S (or $S^{[i]}$) is defined by $x^A >_{olex} x^B$ if $\deg x^A < \deg x^B$ or $\deg x^A = \deg x^B$ and $x^A >_{lex} x^B$.

For example, if we set $x_0 = 1$ in the six monomials of $K[x_0, x_1, x_2]$ of degree 2 (listed in the lex order):

$$\{x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2\},$$

we get the six monomials of degree at most 3 in $K[x_1, x_2]$ (listed in the opposite degree lex order):

$$\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}.$$

We give a name to sets of monomials which are consecutive under the opposite degree lex order.

Definition 4.13. For monomials $x^A, x^B \in S$ (or $S^{[i]}$), the (closed) *interval* $[x^A, x^B]$ is

$$[x^A, x^B] = \{x^C \in R : x^A \geq_{olex} x^C \geq_{olex} x^B\}.$$

The half open intervals $(x^A, x^B]$ and $[x^A, x^B)$ are defined analogously but omitting the monomial x^A or x^B, respectively.

From the discussion above, $\{x_2, x_1^2, x_1x_2\}$ is an interval of length three in $K[x_1, x_2]$, which can be specified as $[x_2, x_1x_2]$ or $[x_2, x_2^2)$; however, $\{x_1, x_1^2, x_2^2\}$ is not an interval because the monomials x_2 and x_1x_2 are missing.

The decomposition in Equation (4.1) illustrates a subtle point about S: when d is large, most of the generators in S_d end in x_{n-1}, and the smaller the monomial, according to the lex order, the more likely it is to end in x_{n-1}. Thus, when we are expanding monomials, if we are only making a few expansions, it is probably better to expand in a cone with fewer monomials, so that we expand the fewest monomials ending in x_{n-1}. If we are making quite a few expansions, then we should favor the current cone, assuming we can get to a generator with a small max index. We need to know exactly when we should switch cones.

We can now explain a condition for deciding whether we should make expansions in a particular cone or move to the next cone. We will switch cones if there are not enough expansions to 'fill the remaining cones.' Suppose we have a saturated lexsegment ideal with minimal generators $\{x^{A_0}, x^{A_1}, \ldots, x^{A_t}\}$ with degrees d_0, d_1, $\ldots d_t$, respectively, and we are expanding multiples of x^{A_j} of degree d. Consider the set of monomials which are multiples of x^{A_i} whose degree is between $d - d_i$ and $d - d_j$

for i from $j+1$ to t. We call this set of monomials the *ceiling*. If there are more remaining expansions than monomials in the ceiling, then we expand the multiples of x^{A_j}; if there are fewer expansions, we increase j by 1.

4.4 A proposed algorithm for producing a saturated ideal with maximal Betti numbers among all ideals with a fixed Hilbert polynomial and initial degree

We are now ready to offer some algorithms. We will use the terminology introduced above. The algorithms will be more complicated than the last few; thus, we will write them out in detail, rather than just describe them heuristically. Recall that $S^{[i]} = K[x_i, \ldots, x_{n-1}]$, and that specifying a Hilbert polynomial is equivalent to giving a doubly saturated lexsegment ideal and the number of expansions to be performed.

The basic idea for the construction is that we make expansions in a given cone until we have reached some stopping point, then we move to the second cone where we make more expansions until we reach another stopping point. We continue until the expansions have been exhausted or we reach the last cone, in which case, we perform the remaining expansions. The stopping point is the ceiling, which was previously described.

Algorithm 4.14. Let $L = (x^{A_0}, x^{A_1}, \ldots, x^{A_t}) \subset R$ be a doubly saturated universal lexsegment ideal. Let $d_i = \deg x^{A_i}$ for $i = 0, 1, \ldots, t$. Let $c > 0$.

1. Set $d = d_0$, $j = 0$, and $J = L$.
2. If $c = 0$ go to step 5; else do one of the following:
   - If $c \geq \dim_K [J]_d$, then: update c to $c - \dim_K [J]_d$; expand the monomials in $[J]_d$; update d to $d+1$. Go to step 2.
   - If $c < \dim_K [J]_d$ and $j < t$, then: set C to
   $$\left(\sum_{i=j+1}^{t-1} \sum_{k=d-d_i}^{d-\min\{d_i, 1+d_j\}} \#[S^{[i]}]_k \right) + \sum_{k=d-d_t}^{d-d_j} \#[S^{[t]}]_k.$$
   Go to step 3.
   - Else: expand the last c monomials in $[J]_d$. Go to step 5.
3.
   - If $c < C$ then do the following: update j to $j+1$; update d to $\max\{d+1, d_j\}$. Go to step 2.
   - Else expand the last C monomials in $[J]_d$; update c to $c - C$. If $c = 0$, go to step 5; else go to step 4.
4. Denote by x^V the monomial last monomial in $[J]_d$ which has not been expanded; set D to $\deg x^{A_j} + a_{j,n-1}$, where $a_{j,n-1}$ is the last exponent in A_j; update C to
$$\left(\sum_{i=j+1}^{t-1} \sum_{k=1+d-d_i}^{D-\min\{d_i, 1+d_j\}} \#[S^{[i]}]_k \right) + \sum_{k=1+d-d_t}^{D-d_j} \#[S^{[t]}]_k.$$

If $C > 0$ go to step 3; else expand the last c monomials in $[J]_d$; go to step 5.

5. Return J.

Unfortunately, this method also fails to produce an ideal with maximal Betti numbers.

Example 4.15. Suppose we wish to make 48 expansions in the doubly saturated universal lexsegment ideal $L = (x_0, x_1, x_2^4) \subset K[x_0, x_1, x_2, x_3, x_4]$ to produce an ideal with maximal Betti numbers.

We follow Algorithm 4.14. First we make 25 expansions in the initial degree to produce the ideal

$$(a(a, b, c, d)^3, b(b, c, d)^3, c^4).$$

We compute C to be 20. (There are 10 monomials in the second cone and 10 monomials in the third cone.) We expand the twenty monomials in degree four and determine that $x^V = x_0x_1^3$ so $D = 1$. The new C is zero so we expand the last three monomials, $x_0^2x_2x_3$, $x_0^2x_3^2$, and $x_0x_1^3$.

Alternatively, instead of expanding the last two monomials, $x_0^2x_2x_3$ and $x_0^2x_3^2$, we could expand x_2^5 and $x_2^4x_3$ (in the last cone). Clearly this will give a larger m-vector.

We can slightly modify Algorithm 4.14 so that we only expand in the current cone after reaching the ceiling, until we reach a monomial with a suitably large power of x_{n-1}. The monomial that follows a monomial with a large power of x_{n-1} will have a small max index.

Algorithm 4.16. (*Generating a saturated ideal with maximal Betti numbers among all ideals with a given Hilbert polynomial*) Let $L = (x^{A_0}, x^{A_1}, \ldots, x^{A_t}) \subset R$ be a doubly saturated universal lexsegment ideal. Let $d_i = \deg x^{A_i}$ for $i = 0, 1, \ldots, t$. Let $c > 0$.

1. Set $d = d_0$, $j = 0$, and $J = L$.

2. If $c = 0$ go to step 5; else do one of the following:

   - If $c \geq \dim_K[J]_d$, then: update c to $c - \dim_K[J]_d$; expand the monomials in $[J]_d$; update d to $d + 1$. Go to step 2.
   - If $c < \dim_K[J]_d$ and $j < t$, then: set C to

   $$\left(\sum_{i=j+1}^{t-1} \sum_{k=d-d_i}^{d-\min\{d_i,1+d_j\}} \#[S^{[i]}]_k\right) + \sum_{k=d-d_t}^{d-d_j} \#[S^{[t]}]_k.$$

     Go to step 3.
   - Else: expand the last c monomials in $[J]_d$. Go to step 5.

3.
   - If $c < C$ then do the following: update j to $j+1$; update d to $\max\{d+1, d_j\}$. Go to step 2.

- Else expand the last C monomials in $[J]_d$; update c to $c - C$. If $c = 0$, go to step 5; else go to step 4.

4. Let x^V be the last monomial in $[J]_d$ which has not been expanded; set D to $d_j + a_{j,n-1}$, where $a_{j,n-1}$ is the exponent of x_{n-1} in x^{A_j}; update C to
$$\left(\sum_{i=j+1}^{t-1} \sum_{k=1+d-d_i}^{D-\min\{d_i,1+d_j\}} \#[S^{[i]}]_k \right) + \sum_{k=1+d-d_t}^{D-d_j} \#[S^{[t]}]_k.$$
If $C > 0$ go to step 3; else determine if there are monomials before x^V whose last exponent is at least $d + 1 - d_t$.
   - If there are such monomials, let x^W be the last. Set $\widetilde{C}$ to be the number of monomials after x^W, that is $\min\{c, \#\left(x^W, x^V\right]\}$; expand the last $\widetilde{C}$ monomials in $[J]_d$; update c to $c - \widetilde{C}$. If $c = 0$ go to step 5; else update C using x^W instead of x^V and go to step 3.
   - Else set $\widetilde{C}$ to $\min\{c, \#[J]_d\}$; expand the last $\widetilde{C}$ monomials in $[J]_d$; update c to $c - \widetilde{C}$; go to step 2.

5. Return J.

After testing Algorithm 4.16 for over 30,000 different Hilbert polynomials (linear, quadratic, and cubic), it appears that our construction always produces the same ideal as the construction given by Caviglia and Murai in [6]. If so, the ideal will have maximal Betti numbers among all saturated ideals with the given Hilbert polynomial.

We demonstrate that this algorithm produces the correct ideal in the setting of Example 4.15.

Example 4.17. Suppose we wish to make 48 expansions in the doubly saturated universal lexsegment ideal $L = (x_0, x_1, x_2^4) \subset K[x_0, x_1, x_2, x_3, x_4]$ to produce an ideal with maximal Betti numbers.

We follow Algorithm 4.16. First we make 25 expansions in the initial degree to produce the ideal
$$(x_0(x_0, x_1, x_2, x_3)^3, x_1(x_1, x_2, x_3)^3, x_2^4).$$
We compute C to be 20. (There are 10 monomials in the second cone and 10 monomials in the third cone.) We expand the twenty monomials in degree four and determine that $x^V = x_0 x_1^3$ so $D = 1$. The new C is zero so we now determine that $x^W = x_0^2 x_3^2$ and $\widetilde{C} = 1$. We expand $x_0 x_1^3$ and compute $D = 3$ so the new C is 9. We now move to the next cone where $C = 14$, so we move to the final cone. We now expand the last two monomials, $x_0^2 x_2 x_3$ and $x_0^2 x_3^2$. This ideal has maximal Betti numbers (as stated in Example 4.15).

As noted at the beginning of this section, Valla proved that, among all ideals with a fixed constant Hilbert polynomial and any possible initial degree, there exists one with maximal Betti numbers. It is clear from step 2 in Algorithm 4.16 that the ideal that is created will have the largest possible initial degree. We suggest the construction above because we can generalize to smaller initial degrees. The only change in the following algorithm is that, if we want to produce an ideal with initial degree e, we skip the expansion of x_0^e.

Algorithm 4.18. (*Generating a saturated ideal with maximal Betti numbers among all ideals with a given Hilbert polynomial and fixed initial degree*)

Let $L = (x^{A_0}, x^{A_1}, \ldots, x^{A_t}) \subset R$ be a doubly saturated universal lexsegment ideal. Let $d_i = \deg x^{A_i}$ for $i = 0, 1, \ldots, t$. Let $c > 0$, and let $e \geq d_0$ be a bound for the initial degree of the resulting ideal.

1. Set $d = d_0$, $j = 0$, and $J = L$.

2. If $c = 0$ go to step 5; else do one of the following:

   - If $j = 0$, $d = e$, and $c \geq \dim_K [J]_d - 1$, then: update c to $c - \dim_K [J]_d + 1$; expand the monomials in $[J]_d \setminus \{x_0^e\}$; update d to $d + 1$. Go to step 2.
   - If $j > 0$ or $d \neq e$ and $c \geq \dim_K [J]_d$, then: update c to $c - \dim_K [J]_d$; expand the monomials in $[J]_d$; update d to $d + 1$. Go to step 2.
   - If $c < \dim_K [J]_d$ and $j < t$, then: set C to

   $$\left(\sum_{i=j+1}^{t-1} \sum_{k=d-d_i}^{d-\min\{d_i, 1+d_j\}} \#[S^{[i]}]_k \right) + \sum_{k=d-d_t}^{d-d_j} \#[S^{[t]}]_k.$$

     Go to step 3.
   - Else: expand the last c monomials in $[J]_d$. Go to step 5.

3. - If $c < C$ then do the following: update j to $j+1$; update d to $\max\{d+1, d_j\}$. Go to step 2.
   - Else expand the last C monomials in $[J]_d$; update c to $c - C$. If $c = 0$, go to step 5; else go to step 4.

4. Let x^V be the last monomial in $[J]_d$ which has not been expanded; set D to $d_j + a_{j,n-1}$, where $a_{j,n-1}$ is the exponent of x_{n-1} in x^{A_j}; update C to

   $$\left(\sum_{i=j+1}^{t-1} \sum_{k=1+d-d_i}^{D-\min\{d_i, 1+d_j\}} \#[S^{[i]}]_k \right) + \sum_{k=1+d-d_t}^{D-d_j} \#[S^{[t]}]_k.$$

   If $C > 0$ go to step 3; else determine if there are monomials before x^V whose last exponent is at least $d + 1 - d_t$.

   - If there are such monomials, let x^W be the last. Set $\widetilde{C}$ to be the number of monomials after x^W, that is $\min\{c, \#\left(x^W, x^V\right]\}$; expand the last $\widetilde{C}$ monomials in $[J]_d$; update c to $c - \widetilde{C}$. If $c = 0$ go to step 5; else update C using x^W instead of x^V and go to step 3.

- Else set $\widetilde{C}$ to $\min\{c, \#[J]_d\}$; expand the last $\widetilde{C}$ monomials in $[J]_d$; update c to $c - \widetilde{C}$; go to step 2.

5. Return J.

Algorithm 4.18 produces an ideal with an initial degree of at most e because the monomial x_0^e will never be expanded. Experimentation suggests that the resulting ideal has maximal Betti numbers among all saturated ideals with the given Hilbert polynomial. We include an example to demonstrate this construction and compare it to the construction of Caviglia and Murai.

Example 4.19. Suppose we wish to make 6 expansions in the doubly saturated universal lexsegment ideal $L = (x_0, x_1^3, x_1^2 x_2) \subset K[x_0, x_1, x_2, x_3, x_4]$ to produce an ideal with initial degree 2 and maximal Betti numbers.

We follow Algorithm 4.18. First we make an expansion in the initial degree to produce the ideal

$$J = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^3, x_1^2x_2).$$

To keep the initial degree at two, we expand the last three monomials of degree two (as opposed to all four). We then expand the last two monomials of degree three. The resulting ideal has maximal Betti numbers among all saturated ideals with the same Hilbert polynomial and an initial degree of two.

If we were to attempt the construction of Caviglia and Murai, it is not clear what to do expand in the ideal J given above. The only "admissible element" in the first cone is x_0^2, but we cannot expand this and maintain the desired initial degree. If we expand in the second and third cones, we cannot match the Betti numbers of the ideal described above.

It remains to be shown that Algorithm 4.18 will always produce an ideal with maximal Betti numbers and, if no initial degree is specified, that the ideal produced is the same as the ideal constructed in [6]. To do this, one could adapt the strategy used by Caviglia and Murai in [6]. In particular, the Interval Lemma can be generalized so that multiples of some power of x_0 are ignored as is the case when the specified initial degree is reached. It may also be possible to give a more intuitive construction.

Chapter 5 Counting strongly stable ideals

In this chapter, we take up Question 3.39: How many saturated strongly stable ideals are there with a given Hilbert polynomial in a polynomial ring with $n+1$ variables? The main contribution of this chapter is an intriguing conjecture for an upper bound for the number of saturated strongly stable ideals with a constant Hilbert polynomial.

We first introduce integer partitions. We then discuss a geometric way to identify partitions with monomial ideals using lattice points in an n dimensional space. We focus on saturated strongly stable ideals with constant Hilbert polynomials, where we can identify the ideals with certain shifted $n-1$ dimensional integer partitions. We note that this can be generalized to get a correspondence between generalized partitions and saturated strongly ideals with any Hilbert polynomial. This culminates with generating functions for the number of these ideals with a constant Hilbert polynomial when n is two or three, and a generating function for a conjectured upper bound if n is greater than three.

5.1 Integer partitions

The study of integer partitions goes back at least as far as Euler who proved that partitions into odd parts are in bijection with partitions into distinct parts. Here we briefly introduce some terminology and pictures that will be useful in the next sections. For a good introduction to partitions, see [1]; for a survey on plane partitions, see [31].

A partition is simply a way of expressing a positive integer as a sum of smaller positive integers. Typically, the integers are arranged in decreasing order and the addition signs are omitted.

Definition 5.1. A *partition* of $n \in \mathbb{N}$ is a finite, non-increasing sequence of positive integers,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r,$$

for some $r \in \mathbb{N}$, which sum to n. Each λ_j is a *part* of n.

A *distinct partition* is a partition in which no part is repeated, that is, the sequence

$$\lambda_1 > \lambda_2 > \ldots > \lambda_r$$

is strictly decreasing.

For example,

$$5 \quad 3 \quad 3 \quad 3 \quad 2 \quad 1 \quad 1 \quad 1 \quad 1$$

is a partition of 20, and

$$12 \quad 5 \quad 2 \quad 1$$

is a distinct partition of 20.

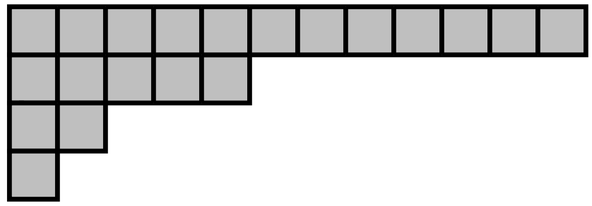

Figure 5.1: A Ferrers diagram for a distinct partition

We can illustrate partitions graphically using a Ferrers diagram. For example, Figure 5.1 is the Ferrers diagram for the distinct integer partition

$$12 \quad 5 \quad 2 \quad 1.$$

We can encode more information by arranging the parts into a two dimensional array. In this case, the integers decrease along each row and column.

Definition 5.2. A *plane partition* of $n \in \mathbb{N}$ is a finite, two dimensional array of positive integers $(\lambda_{i,j})$, which sum to n, such that $\lambda_{i,j} \geq \lambda_{i+1,j}$ and $\lambda_{i,j} \geq \lambda_{i,j+1}$ for all i and j.

A plane partition is *shifted* if the entries in each row begin along the diagonal. In particular, the lengths of the rows must be decreasing. Alternatively, a plane partition is shifted if the parts weakly decrease along the anti-diagonals of the array, that is, $\lambda_{i,j+1} \geq \lambda_{i+1,j}$ (for all i and j so that $\lambda_{i,j+1}$ and $\lambda_{i+1,j}$ are nonzero).

A plane partition is *row-strict* if the entries in each row are strictly decreasing, that is, $\lambda_{i,j} > \lambda_{i,j+1}$ (for all i and j so that $\lambda_{i,j}$ and $\lambda_{i,j+1}$ are nonzero).

Consequently, a row-strict shifted plane partition is a shifted plane partition in which the entries in each row are strictly decreasing.

A row-strict plane partition is a stack of distinct partitions, where the entries along the columns are weakly decreasing. A shifted row-strict plane partition is a shifted stack of distinct partitions with weak decrease along the columns.

For example,

$$\begin{array}{cccc} 5 & 3 & 2 & 1 \\ 3 & 3 & 1 & \\ 1 & & & \end{array}$$

is a plane partition of 20,

$$\begin{array}{cccc} 5 & 3 & 2 & 1 \\ 3 & 1 & & \\ 3 & 1 & & \end{array}$$

is a row-strict plane partition of 10, and

$$\begin{array}{cccc} 5 & 4 & 3 & 1 \\ & 3 & 2 & 1 \\ & & 1 & \end{array} \qquad \text{or} \qquad \begin{array}{cccc} 5 & 4 & 3 & 1 \\ 3 & 2 & 1 & \\ 1 & & & \end{array}$$

is a shifted row-strict plane partition of 20.

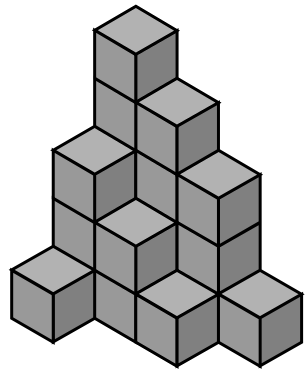

Figure 5.2: A row-strict shifted plane partition represented as stacks of boxes

We can illustrate plane partitions graphically with stacks of boxes in a corner. Each part is interpreted as a height. For example, Figure 5.2 represents the shifted row-strict plane partition

$$\begin{array}{cccc} 5 & 4 & 3 & 1 \\ 3 & 2 & 1 & \\ 1 & & & \end{array}$$

We can think of a plane partition as a Ferrers diagram where each spot is assigned a height (a positive integer) so that the heights are weakly decreasing. Similarly, in a graphical representation for a plane partition, we can assign each box a weight (a positive integer) so that in each direction the weights weakly decrease as you move away from the wall.

In other words, we can arrange the parts of a partition into a three dimensional array.

Definition 5.3. A *solid partition* of $n \in \mathbb{N}$ is a finite, three dimensional array of positive integers, $(\lambda_{i,j,k})$, which sum to n, such that the entries decrease weakly in each column, row, and stack of the array.

A solid partition is *row-strict* if the entries in each row are strictly decreasing, that is, $\lambda_{i,j,k} > \lambda_{i,j+1,k}$ (for all i and j so that $\lambda_{i,j,k}$ and $\lambda_{i,j+1,k}$ are nonzero).

A solid partition is *doubly-shifted* if the entries weakly decrease along the anti-diagonals; specifically, $\lambda_{i,j+1,k} \geq \lambda_{i+1,j,k}$ and $\lambda_{i,j,k+1} \geq \lambda_{i,j+1,k}$ (for all corresponding nonzero $\lambda_{r,s,t}$; consequently, $\lambda_{i,j,k+1} \geq \lambda_{i+1,j,k}$).

The term doubly-shifted is not standard, but it is descriptive. One can think of a plane partition as a stack of certain integer partitions and a solid partition as a stack of plane partitions. A plane partition is shifted when the stacks are offset; a solid partition is doubly-shifted when the stacks of shifted plane partitions are offset.

For example, if we layer the row-strict shifted plane partitions

$$\begin{array}{ccc} 5 & 3 & 2 \\ & 3 & 1 \end{array}, \qquad \begin{array}{cc} 3 & 1 \\ & 1 \end{array}, \qquad \text{and} \qquad 1$$

with an offset, we can form a doubly-shifted row-strict solid partition of 20.

More generally, we can define partitions of higher dimensions in an analogous manner; however, little research has been devoted to anything beyond solid partitions. One reason for this is that as the dimension of the partition is increased, the difficulty in establishing generating functions grows considerably. Specifically, generating functions are known for many classes of integer partitions and for quite a few classes of plane partitions, though few are known for solid partitions.

We conclude the section by introducing a transform which will simplify a later claim. This transform will turn a sequence of exponents into a sequence of coefficients in a generating function.

Many generating functions for classes of partitions and plane partitions can be written in the form

$$1+\sum_{t=1}^{\infty} c_t x^t = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots \tag{5.1}$$

$$= \prod_{k=1}^{\infty} \frac{1}{(1-x^k)^{s_k}}, \tag{5.2}$$

for some sequence of nonnegative integers $\{s_k\}$. Unfortunately, the sequence of exponents $\{s_k\}$ in the generating functions for solid partitions often contain negative numbers which makes combinatorial proofs considerably more difficult.

We say that the sequence $\{c_t\}$ of coefficients in (5.1) is the *Euler transform* of the sequence $\{s_k\}$ of exponents in (5.2); the process of recovering the sequence $\{s_k\}$ from the coefficients, $\{c_t\}$, of the generating function is an inverse Euler transform.

For example, the t^{th} term in the Euler transform of $\{1,1,1,1,1,1,\dots\}$ gives the number of partitions of t, and the t^{th} term in the Euler transform of $\{1,2,3,4,5,6,\dots\}$ gives the number of plane partitions of t. Alternatively, the t^{th} term in the Euler transform of $\{1,3,6,10,15,21,\dots\}$ does not give the number of solid partitions of t if $t>5$. (The sixth term in the Euler transform is 141, but there are only 140 solid partitions of six.)

5.2 Identifying ideals with partitions

In this section, we will identify certain classes of monomial ideals with partitions. In particular, we will describe the partitions associated to strongly stable ideals in a polynomial ring with a fixed number of variables whose Hilbert polynomials are constant.

We assume throughout the remainder of this chapter that ideals are proper subsets of the polynomial ring under consideration. In order to easily fix the number of variables in a polynomial ring and to be consistent, we will denote by R_n the polynomial ring in $n+1$ variables; we will also use variables without subscripts.

We first focus on artinian monomial ideals in the polynomial ring $R_1 = K[x,y]$. We identify the monomials in R_1 with lattice points in a quadrant of the plane, where one coordinate gives the exponent of x and the other coordinate give the exponent of y, as illustrated in Figure 5.3. Now, for an artinian monomial ideal $I \subset R_1$, we consider

1	y	y^2	y^3	$\cdots$
x	xy	xy^2	xy^3	$\cdots$
x^2	x^2y	x^2y^2	x^2y^3	$\cdots$
x^3	x^3y	x^3y^2	x^3y^3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 5.3: Monomials in $K[x, y]$ as lattice points in a quadrant

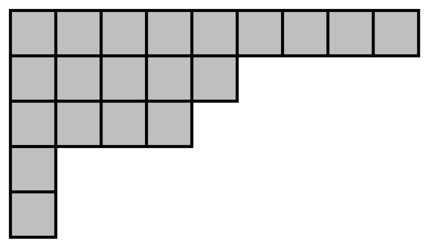

Figure 5.4: The Ferrers diagram for an artinian monomial ideal

the monomials in R_1/I as a Ferrers diagram for a partition. Since an ideal $I \subset R_1$ is artinian if R_1/I has finite dimension, artinian monomial ideals correspond to integer partitions. For example, the artinian monomial ideal $(x^5, x^3y, x^2y^4, xy^5, y^9) \subset R_1$ is represented by the Ferrers diagram in Figure 5.4; the twenty monomials in the complement of the ideal correspond to the shaded boxes, following the convention in Figure 5.3.

We can characterize the integer partitions that correspond to strongly stable ideals in R_1.

Proposition 5.4. *Artinian strongly stable ideals in $R_1 = K[x, y]$ correspond to partitions into distinct parts.*

Proof. By definition, a monomial ideal I is strongly stable, if for any monomial x^iy^j in I, the monomial $x^{i+1}y^{j-1}$ is also in I. If I is artinian, $y^j \in I$ for some j. Let j_0 be the smallest integer such that $y^{j_0} \in I$; j_0 will be the largest part of the corresponding partition. Since I is strongly stable, $xy^{j_0-1} \in I$ so there is some minimal generator $xy^{j_1} \in I$, where $j_1 \leq j_0 - 1 < j_0$. The integer j_1 is the second part of the partition. We can repeat this argument to get the set of minimal generators of the ideal $\{x^iy^{j_i}\}$, where $j_i < j_{i-1}$. Thus, the parts of the partition are distinct.

(If I is not artinian, then no power of y is in the ideal, so the largest part of the corresponding "partition" is infinite; hence, we do not have an actual partition. This case will be explored in the next section.) $\square$

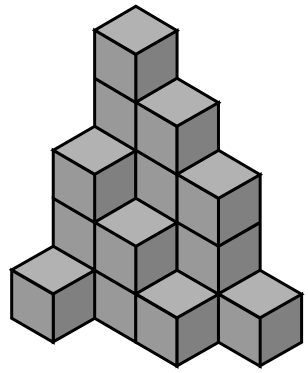

Figure 5.5: Stacks of boxes representing an ideal in $K[x, y, z]$

Because the sum of the partition corresponding to an ideal $I \subset R_1$ tells us the dimension of R_1/I, we can relate this to the Hilbert polynomial of I when considered as an ideal of $R_2 = K[x, y, z]$. If $I \subset R_1$ is a strongly stable ideal, then $I \cdot R_2 \subset R_2$ will be a saturated strongly stable ideal, since the new variable will not appear in the minimal generators of I. Accordingly, we can sum the Hilbert function of I to get the Hilbert function of $I \cdot R_2 \subset R_2$.

Corollary 5.5. *The saturated strongly stable ideals $I \subset R_2 = K[x, y, z]$ with Hilbert polynomial $p_{R_2/I} = t$ correspond to the distinct partitions of t.*

In fact, if $x^i y^j$ is a minimal generator of I and $j > 0$, then j is the $i + 1^{st}$ part of the partition.

We now focus on artinian monomial ideals in the polynomial ring $R_2 = K[x, y, z]$. We can identify the monomials in R_2 with lattice points in an octant of three dimensional space, where one coordinate gives the exponent of x, another gives the exponent of y, and the other coordinate give the exponent of z. Now, for an artinian monomial ideal $I \subset R_2$, we consider the monomials in R_2/I as stacks of boxes arranged in a corner. Since an ideal I in R_2 is artinian if R_2/I has finite dimension, artinian monomial ideals correspond to plane partitions.

For example, the artinian monomial ideal

$$(x^3, x^2y, x^2z, xy^3, xy^2z, xyz^2, xz^3, y^4, y^3z, y^2z^3, yz^4, z^5) \subset R_2$$

is represented by the stack of boxes in Figure 5.5. The twenty boxes are the monomials in the complement of the ideal: the left-most box is the monomial x^2, the right-most box is the monomial y^3, and the top-most box is the monomial z^4. (Remember that the monomial 1 will be the box in the corner.)

We can characterize the plane partitions that correspond to artinian strongly stable ideals in R_2.

Proposition 5.6. *Artinian strongly stable ideals in $R_2 = K[x, y, z]$ correspond to row-strict shifted plane partitions.*

Proof. By definition, a monomial ideal I is strongly stable, if for any monomial $x^h y^i z^j$ in I, we have three other monomials $x^h y^{i+1} z^{j-1}$, $x^{h+1} y^i z^{j-1}$, and $x^{h+1} y^{i-1} z^j$ in I. If we think of these monomials in terms of replacements of variables, the second is a consequence of the first and third; specifically, if we can replace a z with a y and a y with an x, then by extension we can replace a z with an x.

Let $x^h y^i z^j$ be a minimal generator of I. Since I is strongly stable, $x^h y^{i+1} z^{j-1}$ and $x^{h+1} y^{i-1} z^j$ are also in I, though they may not be minimal generators. Thus, there exist minimal generators $x^h y^{i+1} z^{j_{h,i+1}}$ and $x^{h+1} y^{i-1} z^{j_{h+1,i-1}}$ where $j_{h+1,i-1} \leq j_{h,i}$ and $j_{h,i+1} \leq j_{h,i} - 1 < j_{h,i}$. We can do this for every minimal generator. The resulting set of integers $(j_{h,i})$ is a row-strict shifted plane partition. □

We now recognize the sum of the parts of the plane partition for an artinian strongly stable ideal I in three variables as the Hilbert polynomial of the saturated strongly stable ideal with the same generators in a polynomial ring with four variables.

Corollary 5.7. *The saturated strongly stable ideals $I \subset R_3 = K[x, y, z, w]$ with Hilbert polynomial $p_{R_3/I} = t$ correspond to the row-strict shifted plane partitions of t.*

In fact, if $x^h y^i z^j$ is a minimal generator of I and $j > 0$, then j is the part with indices $(h + 1, i + 1)$.

We now focus on artinian monomial ideals in the polynomial ring R_3, $K[x, y, z, w]$. We can proceed as before by identifying an artinian monomial ideal with a solid partition. Now, the monomials in R_3 are identified with the lattice points with nonnegative coordinates in four dimensional space; each coordinate for a lattice point will give the exponent of one of the variables of the corresponding monomials.

We can characterize the solid partitions that correspond to artinian strongly stable ideals in R_3.

Proposition 5.8. *Artinian strongly stable ideals in $R_3 = K[x, y, z, w]$ correspond to row-strict doubly-shifted solid partitions.*

Proof. By definition, a monomial ideal I is strongly stable, if for any monomial $x^g y^h z^i w^j$ in I, we have three other monomials $x^g y^h z^{i+1} w^{j-1}$, $x^g y^{h+1} z^{i-1} w^j$, and $x^{g+1} y^{h-1} z^i w^j$ in I. (The other monomials which are guaranteed by strong stability will give redundant conditions below.)

Let $x^g y^h z^i w^j$ be a minimal generator of I. Since I is strongly stable, $x^g y^h z^{i+1} w^{j-1}$, $x^g y^{h+1} z^{i-1} w^j$, and $x^{g+1} y^{h-1} z^i w^j$ are also in I, though they may not be minimal generators. Thus, there exist minimal generators $x^g y^h z^{i+1} w^{j_{g,h,i+1}}$, $x^g y^{h+1} z^{i-1} w^{j_{g,h+1,i-1}}$, and $x^{g+1} y^{h-1} z^i w^{j_{g+1,h-1,i}}$ where $j_{g,h,i+1} \leq j_{g,h,i} - 1 < j_{g,h,i}$, $j_{g,h+1,i-1} \leq j_{g,h,i}$, and $j_{g+1,h-1,i} \leq j_{g,h,i}$.

The resulting set of integers $(j_{g,h,i})$ forms a row-strict doubly-shifted solid partition. □

We now recognize the sum of the parts of the solid partition for an artinian strongly stable ideal I in four variables as the Hilbert polynomial of the saturated strongly stable ideal with the same generators in a polynomial ring with five variables.

Corollary 5.9. *The saturated strongly stable ideals $I \subset R_4$ with Hilbert polynomial $p_{R_4/I} = t$ correspond to the row-strict doubly-shifted solid partitions of t.*

In fact, if $x^g y^h z^i w^j$ is a minimal generator of I and $j > 0$, then j is the part with indices $(g+1, h+1, i+1)$.

We now identify the partitions associated to saturated strongly stable ideals with constant Hilbert polynomials in a large number variables. We say that a partition has dimension l if the parts are arranged in an l dimensional array.

Proposition 5.10. *The saturated strongly stable ideals I in R_n (a polynomial ring with $n+1$ variables) with Hilbert polynomial $p_{R_n/I} = t$ correspond to the $n-1$ dimensional partitions of t whose parts $(j_{i_1,i_2,\ldots,i_{n-2},i_{n-1}})$ are subject to the constraints*

$$j_{i_1,i_2,\ldots,i_{n-2},i_{n-1}+1} < j_{i_1,i_2,\ldots,i_{n-2},i_{n-1}}$$

$$j_{i_1,i_2,\ldots,i_{n-2}+1,i_{n-1}-1} \leq j_{i_1,i_2,\ldots,i_{n-2},i_{n-1}}$$

$$\vdots$$

$$j_{i_1+1,i_2-1,\ldots,i_{n-2},i_{n-1}} \leq j_{i_1,i_2,\ldots,i_{n-2},i_{n-1}}$$

for the appropriate values of $\{i_1, i_2, \ldots, i_{n-2}, i_{n-1}\}$.

(The first condition makes the rows strictly decrease; the other conditions give shifts in the layers of the partitions.)

Proof. By definition, a monomial ideal I is strongly stable, if, for any monomial $\prod_{h=1}^{n} x_h^{j_h}$ in I, where $j_i > 0$, we can replace the variable x_i with the variable x_{i-1} and obtain another monomial in the ideal. Each of these replacements in a minimal generator of I which is divisible by x_{n-1} gives a multiple of another minimal generator of I. The possible decrease in the last exponent corresponds to the inequalities above. $\square$

We can generalize the concept of n dimensional partitions to get a correspondence for saturated strongly stable ideals with arbitrary Hilbert polynomials, by allowing the parts in a partition to be infinite. These ideas are explored in [20] and [21]: the authors give a characterization for stable, strongly stable, and lexsegment ideals in terms of these generalized partitions which they call ε-vectors.

If we wish to identify a monomial ideal $I \subset R_n$, which is not artinian, with an $n-1$ dimensional partition, then we must allow some of the parts to be infinite because the complement of the ideal, R_n/I, will have infinite dimension.

Geometrically, if we think of the monomials as lattice points in an n dimensional array, then (at least) one entire ray will be in the complement. In this case, we can use the symbol ∞ in the partition. It could also happen that an entire quarter plane is contained in the complement, prompting the use of the symbol ∞^2. In general if an entire r dimensional array is contained in the complement, the symbol ∞^r is used.

In [21], the authors observe that if the ideal is stable, then the ε-vector can be derived from the set of minimal generators and vice versa. The location of each part gives the exponents of all the variables but the last in a minimal generator and the part itself gives the last exponent. If the part is infinite, then it is skipped (so it does not correspond to a generator). The / characters are dealt with in a similar manner. For details, see Remark 3.6 in [20].

Unfortunately, it is not clear how to identify the partitions corresponding to saturated strongly stable ideals with a fixed Hilbert polynomial of positive degree without following a recursive algorithm such as Algorithm 3.22. If the Hilbert polynomial has positive degree, there can be several double saturations, which means that the ∞^r symbols will be arranged differently. The coefficients of the Hilbert polynomial, especially the constant term, can vary dramatically by rearranging these symbols. Thus, for a particular Hilbert polynomial, one must determine the different double saturations and the number of expansions performed in the last ring for each double saturation. Because of these difficulties, we will focus only on saturated strongly stable ideals with a constant Hilbert polynomial.

5.3 Generating functions for saturated strongly stable ideals with constant Hilbert polynomials

In this section, we examine the generating functions for saturated strongly stable ideals in a polynomial ring with $n+1$ variables with a constant Hilbert polynomial. We first describe the generating function when n is two; we then give a conjecture for when n is three. We conclude with a conjectured bound for any n; this bound is much better than the one in Corollary 3.44, though it is still an overestimate if $n > 3$ and the Hilbert polynomial is at least 18.

Recall that by Corollary 5.5, saturated strongly stable ideals in R_2 with constant Hilbert polynomial correspond to distinct partitions. The number of partitions of t into distinct parts, q_t, has been well studied. The generating function can be written in several ways:

$$1 + \sum_{t=1}^{\infty} q_t x^t = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + \dots \tag{5.3}$$

$$= \prod_{k=1}^{\infty}(1 + x^k) \tag{5.4}$$

$$= \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k-1}}. \tag{5.5}$$

In (5.4), the coefficient q_t clearly counts the number of ways that t can be written as a sum of distinct positive integers. In (5.5), the coefficient q_t counts the number of ways that t can be written as a sum of odd positive integers. Because of the well known bijection between partitions into distinct parts and partitions into odd parts, the generating functions share the same coefficients. From (5.5), it is clear that the sequence of coefficients in (5.3) is the Euler transform of the sequence $\{1, 0, 1, 0, 1, 0, 1, 0, \dots\}$. This sequence $\{s_k\}$ describes the number of partitions of $k-1$ into parts of size 2. There is a unique partition for even values of $k-1$ and no partition for odd values.

Recall that by Corollary 5.7, saturated strongly stable ideals in R_3 with constant Hilbert polynomial correspond to row-strict shifted partitions. The number of row-strict shifted plane partitions of t, r_t, does not appear to have been studied: as of February 2012, the sequence is not listed in the online encyclopedia of integer sequences, [30]. The generating function is

$$1+\sum_{t=1}^{\infty} r_t x^t = 1 + x + x^2 + 2x^3 + 3x^4 + 4x^5 + 6x^6 + 9x^7 + 12x^8 + 17x^9 + \dots \quad (5.6)$$

It appears that the generating function (5.6) can be written in the form of (5.2) for a sequence of integers $\{s_k\}$. In fact $\{s_k\}$ should be sequence A103221 in [30], the number of partitions of $k-1$ into parts of size 2 and 3.

Conjecture 5.11. *The number of row-strict shifted plane partitions of t, r_t, is*

$$1+\sum_{t=1}^{\infty} r_t x^t = \prod_{k=1}^{\infty} (1-x^k)^{-s_k},$$

where $\{s_k\} = \{1,0,1,1,1,1,2,1,2,2,2,2,3,2,3,3,3,3,4,3,4,\dots\}$.

Thus, r_t appears to be the Euler transform of the sequence A103221. In particular, the two sequences agree for the first fifty terms.

Similar generating functions for classes of row-strict plane partitions are given in [4] using a correspondence between Young tableaux and symmetric matrices of nonnegative integers. Determinantal formulas for row-strict shifted plane partitions are examined in [19] and [26]; however, these formulas are for shifted plane partitions of a given shape and first column.

Before moving on, we highlight a connection between the number q_t of distinct partitions of t and the number r_t of row-strict shifted plane partitions of t. Up to now, we have not mentioned the 1 at the beginning of Equation (5.1). Clearly the first term in (5.2) is a 1, but it does not make sense to talk about partitions of zero, so we add this 1 to the generating function.

Remark 5.12. (i) The sequence $\{q_t\}$ can be generated by taking the Euler transform of the sequence $\{0,1,0,0,0,0,\dots\}$, prepending a 1, and taking another Euler transform.

(ii) The sequence $\{r_t\}$ can be generated by taking the Euler transform of the sequence $\{0,1,1,0,0,0,\dots\}$, prepending a 1, and taking another Euler transform.

Recall that by Corollary 5.9, saturated strongly stable ideals in R_4 with constant Hilbert polynomial correspond to row-strict doubly shifted solid partitions. The number of row-strict doubly-shifted solid partitions of t, ss_t, also does not appear to have been studied—it is not listed in [30]. The generating function is

$$1+\sum_{t=1}^{\infty} ss_t x^t = 1 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + 7x^6 + 11x^7 + 16x^8 + 24x^9 + \dots \quad (5.7)$$

Upon examining the first few terms of (5.7), one may suppose that the generating function can be obtained in the manner of Remark 5.12; however, this is not the case.

Remark 5.13. The sequence obtained by taking the Euler transform of the sequence $\{0, 1, 1, 1, 0, 0, \ldots\}$, prepending a 1, and taking another Euler transform is not the same as ss_t if $t \geq 18$. The 18^{th} term in the former is 561, while the 18^{th} term in the latter is 560.

The generating function (5.7) can *not* be written in the form of (5.2) for a sequence $\{s_k\}$ of positive integers. This is not very surprising, as the formulas for many generating functions of classes of plane partitions can be proved combinatorially while the generating functions for solid partitions cannot.

Remark 5.14. The number of row-strict doubly-shifted solid partitions of t, ss_t, is not the Euler transform of a sequence $\{s_k\}$ of positive integers. If (at least) the first 47 terms of ss_t are computed and an inverse Euler transform is applied to the sequence, then the 47^{th} term will be negative.

Remarks 5.12 and 5.13 suggest a bound for the number of saturated strongly stable ideals with a constant Hilbert polynomial in a fixed number of variables.

Conjecture 5.15. *Fix $n \geq 4$. Let $e_n = \{0, 1, \ldots, 1, 0, 0, 0, \ldots\}$ be the sequence of a single zero, followed by $n - 1$ ones, and ending with infinitely more zeros. Consider the sequence $\{s_k\}$ obtained by taking the Euler transform of e_n, prepending a 1, and taking another Euler transform.*

The t^{th} term in $\{s_k\}$ is an upper bound for the number of saturated strongly stable ideals I in R_n with Hilbert polynomial $p_{R_n/I} = t$.

Experimentation shows that if $t < 18$ the bound in Conjecture 5.15 is exact, but, if $n \geq 5$ and $t \geq 18$, the bound above is an overestimate. This is illustrated in the following tables: Table 5.1 lists the number of ideals and Table 5.2 provides the bounds described in Conjecture 5.15. The tables were generated using *Macaulay2* [13] and the computer code provided in the appendix.

$p_{R_n/I}$	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1
3	2	2	2	2	2	2	2	2
4	2	3	3	3	3	3	3	3
5	3	4	5	5	5	5	5	5
6	4	6	7	8	8	8	8	8
7	5	9	11	12	13	13	13	13
8	6	12	16	18	19	20	20	20
9	8	17	24	28	30	31	32	32
10	10	24	35	42	46	48	49	50
11	12	32	50	62	69	73	75	76
12	15	44	72	92	104	111	115	117
13	18	60	103	135	156	168	175	179
14	22	80	146	197	231	252	264	271
15	27	107	206	287	342	377	398	410
16	32	143	289	415	504	561	596	617
17	38	188	403	596	737	830	888	923
18	46	248	560	855	1076	1225	1320	1378
19	54	326	775	1219	1564	1800	1953	2049
20	64	425	1068	1732	2262	2635	2879	3034

Table 5.1: The number of saturated strongly stable ideals in a polynomial ring with n variables with constant Hilbert polynomial for several values of n

$p_{R_n/I}$	2	3	4	5	6	7	8	9
16	32	143	289	415	504	561	596	617
17	38	188	403	596	737	830	888	923
18	46	248	561	856	1077	1226	1321	1379
19	54	326	776	1221	1566	1802	1955	2051
20	64	425	1070	1736	2267	2640	2884	3039

Table 5.2: A bound for the number of saturated strongly stable ideals in a polynomial ring with n variables with constant Hilbert polynomial for several values of n computed using Euler transforms

Appendix: Computer code

In this appendix we provide some computer code which was written for the computer algebra system *Macaulay2* [13]; the code was written for version 1.4. Preceding each function are a few lines of comments which briefly describe the purpose of the method.

Code for strongly stable ideals

The following code corresponds to the material in Chapters 2 through 4. It is sorted into two categories: the implementations of the Algorithms in Chapters 3 and 4 and everything else.

We provide a brief guide for the methods corresponding to the algorithms in the previous chapters:

Algorithm 3.35 is *aLIdealsWithHilbPoly(p,n)*;
Algorithm 4.18 is *newMaxBettiWithHilbPoly(p,n,InitialDegree=>e)*;
Algorithm 3.22 is *sSSIdealsWithHilbPoly(p,n)*; and
Algorithm 3.37 is *sSSIdealsWithHilbSeries(g,n)*.

```
-- -- -- -- -- % main functions % -- -- -- -- --

--finds every almost lexsegment ideal with a given Hilbert polynomial
aLIdealsWithHilbPoly = method(TypicalValue=>List,
                          Options=>{IndexedVars=>true});
aLIdealsWithHilbPoly(ZZ,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     apply(doLastExpansions(apply(n,i->toList
           join(i:0,1:1,n-i-1:0)),p-1,n,false),
              J->monomialIdeal apply(J,m->R_m))
);
aLIdealsWithHilbPoly(RingElement,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     AVals:=aValues p;
     apply(doLastExpansions(lexicIdeal(prepend(0,drop(AVals,1)),n),
              sub(AVals#0,ZZ),n,false),
                 J->monomialIdeal apply(J,m->R_m))
);

--finds all almost lex ideals with max Betti numbers
--  with a given Hilbert polynomial (and specified initial degree)
allMaxBettiWithHilbPoly = method(TypicalValue=>List,
                             Options=>{InitialDegree=>0,
                             IndexedVars=>true});
allMaxBettiWithHilbPoly(ZZ,ZZ) := opts -> (p,n) -> (
```

```
     R:=makeRing(n,opts.IndexedVars);
     FirstIdeal:=apply(n,i->toList join(i:0,1:1,n-i-1:0));
     num:=p-1;
     LengthDeg:=n;
     Deg:=1;
     flag:=(opts.InitialDegree==Deg);
     while num>=LengthDeg do
     (  if flag then LengthDeg=LengthDeg-1;
        FirstIdeal=makeExpands(FirstIdeal,LengthDeg,Deg,n);
        num=num-LengthDeg;
        Deg=Deg+1;
        LengthDeg=#select(FirstIdeal,g->sum g==Deg);
        flag=(opts.InitialDegree==Deg);  );
     flag=(opts.InitialDegree!=0 and opts.InitialDegree<=Deg);
     AllIdeals:=doLastExpansions(FirstIdeal,num,n,flag);
     if opts.InitialDegree>0 then AllIdeals=select(AllIdeals,
                         J->sum J_0<=opts.InitialDegree);
     NGs:=apply(AllIdeals,I->#I);
     MaxNGs:=max NGs;
     Posn:=positions(NGs,g->g==MaxNGs);
     apply(apply(Posn,i->AllIdeals#i),
           J->monomialIdeal apply(J,m->R_m))
);
allMaxBettiWithHilbPoly(RingElement,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     AVals:=aValues p;
     num:=AVals#0;
     FirstIdeal:=lexicIdeal(prepend(0,drop(AVals,1)),n);
     Deg:=sum FirstIdeal_0;
     LengthDeg:=#select(FirstIdeal,g->sum g==Deg);
     flag:=(opts.InitialDegree==Deg);
     while num>=LengthDeg do
     (  if flag then LengthDeg=LengthDeg-1;
        FirstIdeal=makeExpands(FirstIdeal,LengthDeg,Deg,n);
        num=num-LengthDeg;
        Deg=Deg+1;
        LengthDeg=#select(FirstIdeal,g->sum g==Deg);
        flag=(opts.InitialDegree==Deg);  );
     flag=(opts.InitialDegree!=0 and opts.InitialDegree<=Deg);
     AllIdeals:=doLastExpansions(FirstIdeal,num,n,flag);
     if opts.InitialDegree>0 then AllIdeals=select(AllIdeals,
                         J->sum J_0<=opts.InitialDegree);
     NGs:=apply(AllIdeals,I->#I);
     MaxNGs:=max NGs;
     Posn:=positions(NGs,g->g==MaxNGs);
```

```
     apply(apply(Posn,i->AllIdeals#i),
            J->monomialIdeal apply(J,m->R_m))
);

--counts the number of saturated strongly stable ideals in ring
--  with n+1 variables for a given Hilbert polynomial
--this method does not store the ideals
countSSSIdealsWithHilbPoly = method(TypicalValue=>ZZ);
countSSSIdealsWithHilbPoly(ZZ,ZZ) := (p,n) -> (
     #doExpansions(apply(n,i->toList join(i:0,1:1,n-i-1:0)),p-1,n)
);
countSSSIdealsWithHilbPoly(RingElement,ZZ) := (p,n) -> (
     AVals:=aValues p;
     countExpansions(apply(n+1-#AVals,i->
                    toList join(i:0,1:1,n-i-1:0)),0,AVals,n)
);

--finds an almost lex ideal with max Betti numbers among all ideals
--  with a given Hilbert polynomial
--this is the construction of Caviglia and Murai
maxBettiWithHilbPoly = method(TypicalValue=>MonomialIdeal,
                          Options=>{IndexedVars=>true});
maxBettiWithHilbPoly(ZZ,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     J:=apply(n,i->toList join(i:0,1:1,n-i-1:0));
     num:=p-1;
     d:=1;
     while num!=0 do
     (  ExpMon:=take(rsort select(J,g->sum g==d),-num);
        uze:=#ExpMon;
        J=makeExpands(J,uze,d,n);
        d=d+1;
        num=num-uze;  );
     monomialIdeal apply(J,m->R_m)
);
maxBettiWithHilbPoly(RingElement,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     AVals:=aValues p;
     J:=lexicIdeal(prepend(0,drop(AVals,1)),n);
     Ugens:=J;
     t:=#J;
     num:=sub(AVals#0,ZZ);
     d:=sum first J;
     while num!=0 do
     (  ExpMon:=take(rsort select(J,g->sum g==d),-num);
```

```
        Psn:=position(ExpMon,m->isAdmissible(m,Ugens));
        if Psn=!=null then
        (  uze:=#ExpMon-Psn;
           J=makeExpands(J,uze,d,n);
           num=num-uze;  );
        d=d+1;  );
     monomialIdeal apply(J,m->R_m)
);

--finds an ideal with max Betti numbers and fixed Hilbert polynomial
--includes an option for specifying the initial degree of the ideal
newMaxBettiWithHilbPoly = method(TypicalValue=>MonomialIdeal,
                             Options=>{IndexedVars=>true,
                                       InitialDegree=>0});
newMaxBettiWithHilbPoly(RingElement,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     AVals:=aValues p;
     Ugens:=lexicIdeal(prepend(0,drop(AVals,1)),n);
     t:=#Ugens-1;
     H:=new MutableHashTable from
           {J=>Ugens,K=>sub(AVals#0,ZZ),d=>sum Ugens_0,
            j=>0,C=>0,Udegs=>apply(Ugens,g->sum g)};
     while H#K>0 do (
        SND:=#select(H#J,g->sum g==H#d and
                     all(H#j,l->min(g-Ugens#l)<0));
        if (H#j==0 and opts.InitialDegree==H#d) then SND=SND-1;
        if H#K>=SND then step2(H,SND,n)
          else if H#j==t then (H#J=makeExpands(H#J,H#K,H#d,n);
            H#K=0;)
          else (D:=H#d;
            H#C=sum(D-H#Udegs#-1..D-H#Udegs#(H#j),
                       k->binomial(n-t-1+k,k))+
                sum(H#j+1..t-1,i->
                   sum(D-H#Udegs#i..D-min(H#Udegs#i,1+H#Udegs#(H#j)),
                       k->binomial(n-i-1+k,k)));
            step3(H,n);  );  );
     monomialIdeal apply(H#J,m->R_m)
);

--finds all saturated strongly stable ideals with a given
--  Hilbert polynomial
sSSIdealsWithHilbPoly = method(TypicalValue=>List,
                           Options=>{IndexedVars=>true});
sSSIdealsWithHilbPoly(ZZ,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
```

```
     apply(doExpansions(
          apply(n,i->toList join(i:0,1:1,n-i-1:0)),p-1,n),
               I->monomialIdeal apply(I,m->R_m))
);
sSSIdealsWithHilbPoly(RingElement,ZZ) := opts -> (p,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     AVals:=aValues p;        --compute a-values for p
     m:=n+1-#AVals;
     IdealList:=doExpansions(apply(m,i->toList
                         join(i:0,1:1,n-i-1:0)), last AVals-1,m);
     for j from 1 to n-m do
        IdealList=flatten for J in IdealList list
        (  Diff:=AVals#(-j-1)-first aValues hilbertPoly(J,m+j);
           if Diff<0 then continue
             else doExpansions(J,sub(Diff,ZZ),m+j)  );
     apply(IdealList,I->monomialIdeal apply(I,m->R_m))
);

--finds all saturated strongly stable ideals with a given
--  non-reduced Hilbert series
--  (actually, only n and the numerator of the series are given)
sSSIdealsWithHilbSeries = method(TypicalValue=>List,
                                Options=>{IndexedVars=>true});
sSSIdealsWithHilbSeries(Divide) := opts -> (H) -> (
     sSSIdealsWithHilbSeries(numerator H,(last last denominator H)-1,
          IndexedVars=>opts.IndexedVars)
);
sSSIdealsWithHilbSeries(RingElement,ZZ) := opts -> (g,n) -> (
     R:=makeRing(n,opts.IndexedVars);
     if #support g>1 then error "Hilbert series should be univariate";
     t:=first support g;        --take variable of g
     beginf:=1-g;
     sdf:=min flatten exponents beginf;
     nextf:=beginf-t^sdf;
     nextsdf:=min flatten exponents nextf;
     GoodIdeals:={};
     if nextsdf==infinity then GoodIdeals={{toList join(1:sdf,n-1:0)}}
      else (  nextmons:={toList join(1:sdf-1,1:1+nextsdf-sdf,n-2:0)};
        IdealList:={new HashTable from
           {ideal=>{toList join(1:sdf,n-1:0)},numerator=>nextf,
                monomials=>nextmons}};
        while #IdealList!=0 do
        (  IdealList=flatten for H in IdealList list
           (  sdf=min flatten exponents H#numerator;
              for i to #H#monomials-1 list
```

```
                (  m:=(H#monomials)#i;
                   J:=append(H#ideal,m);
                   nextf=H#numerator-(1-t)^(maxIndex m)*t^sdf;
                   nextsdf=min flatten exponents nextf;
                   if nextsdf==infinity then
                   (  GoodIdeals=append(GoodIdeals,J);
                      continue  )
                     else
                      (nextmons=apply(rsort join(drop(H#monomials,i+1),
                           select(modRightShift m,k->all(leftShift k,g->
                                  any(J,i->min(g-i)>=0)))),l->l+toList
                                  join(maxIndex l:0,1:(nextsdf-sdf),
                                       (n-maxIndex l-1):0));
                        if #nextmons==0 then continue else
                          new HashTable from {ideal=>J,numerator=>nextf,
                                              monomials=>nextmons}  )
                      )  )
        );  );
     apply(GoodIdeals,I->monomialIdeal apply(I,m->R_m))
);

-- -- -- -- -- % other functions % -- -- -- -- --

--finds values for a special representation of a Hilbert polynomial
--use for finding almost lexsegment ideals
aValues = method(TypicalValue=>List);
aValues(RingElement) := (q) -> (
     deg:=first degree q;
     z:=first gens ring q;
     BVals:=for i from 0 to deg list
     (  b:=(deg-i)!*leadCoefficient q;
        q=q-binom(z+deg-i,deg-i+1)+binom(z+deg-i-b,deg-i+1);
        b  );
     apply(append(apply(deg,j->BVals#(deg-j)-BVals#(deg-j-1)),
           BVals#0),a->lift(a,ZZ))
);

--computes polynomials from binomial coefficient description
--use for finding a particular description of Hilbert polynomials
binom = method(TypicalValue=>RingElement);
binom(RingElement,ZZ) := (t,k) -> (
     if k<0 then 0
       else product(k,i->(t-i))/k!
);
```

```
--counts the number of possible expansions in a strongly stable
--  ideal without storing all of the ideals
--use for counting strongly stable ideals
countExpansions = method(TypicalValue=>ZZ);
countExpansions(List,ZZ,List,ZZ) := (J,j,AVals,n)->(
     m:=n+1-#AVals;
     if j>n-m then return 1;
     Diff:=AVals#(-j-1)-first aValues hilbertPoly(J,m+j);
     if Diff<0 then return 0;
     sum(apply(doExpansions(J,sub(Diff,ZZ),m+j),J'->
               countExpansions(J',j+1,AVals,n)))
);

--returns a list of the ideals obtained by num expansions of
--  generators of an ideal J in a ring with n+1 variables
--exhausts all possible combinations
--use for computing all ideals with a given Hilbert polynomial
doExpansions = method(TypicalValue=>List);
doExpansions(List,ZZ,ZZ) := (J,num,n) -> (
     IdealList:={J};
     for i to num-1 do
        IdealList=flatten apply(IdealList,I->apply(
           select(I,g->testExpand(I,g,n)),m->makeExpand(I,m,n)));
     IdealList
);

--returns the ideals obtained by num expansions of the last monomial
--  generators in each degree for each ideal in a ring with n+1
--  variables in a list in all possible combinations.
--use for computing every almost lex ideal
doLastExpansions = method(TypicalValue=>List);
doLastExpansions(List,ZZ,ZZ,Boolean) := (J,num,n,flag) -> (
     IdealList:={new HashTable from
                  {ideal=>J,monomials=>
                   lastMonomials(J,InitialDegree=>flag)}};
     apply(num,i->IdealList=flatten apply(IdealList,H->
        apply(H#monomials,m->new HashTable from
           {ideal=>I=makeExpand(H#ideal,m,n),
            monomials=>select(lastMonomials I,g->sum g>=sum m)})));
     apply(IdealList,H->H#ideal)
);

--finds the first difference for a given univariate polynomial
--    [p(z)-p(z-1)]
--use for computing all ideals with a given Hilbert polynomial
```

```
findDiff = method(TypicalValue=>List);
findDiff(RingElement) := (p) -> (
     if #support p==1 then
      (z:=first support p;   --take variable of p
       p-sub(p,z=>z-1))
     else 0
);

--finds a Hilbert polynomial from a list of nonnegative integers
--the integers correspond to the exponent of the last generator
--  in the associated lex ideal
findPolyA = method(TypicalValue=>RingElement);
findPolyA(List) := (AVals) -> (
     r:=#AVals;
     aVals:=apply(r,j->sum(drop(AVals,j)));
     if min(AVals)>=0 then
       (PolyRing:=QQ[z];
        sum(r,i->(binom(z+i,i+1)-binom(z+i-aVals#i,i+1)))  )
     else error("Not a valid Hilbert polynomial")
);

--computes the Hilbert polynomial for a stable ideal in a ring
--  with n+1 variables
--use for computing all strongly stable ideals
--  with given Hilbert polynomial
hilbertPoly = method(TypicalValue=>RingElement);
hilbertPoly(List,ZZ) := (I,n) -> (
     RFP:=QQ(monoid[local z]);
     binom(RFP_0+n,n)-sum(I,g->binom(RFP_0+n-sum g-maxIndex g,
                                     n-maxIndex g))
);

--determines whether or not to expand a monomial
--use for finding an ideal with a given Hilbert polynomial
--  with max Betti numbers
isAdmissible = method(TypicalValue=>Boolean);
isAdmissible(List,List) := (m,Ugens) -> (
     i:=position(Ugens,g->min(m-g)>=0);
     e:=sum m;
     t:=#Ugens-1;
     MMV:=apply(t-i,j->m=moveMap(m,Ugens));
     all(t-i-1,j->(sum (MMV#j)<e+2) or (MMV#j==Ugens#(j+i+1)))
       and ((m==Ugens#t) or sum m<e+1 or (e+1-sum Ugens#t>=0
             and m==Ugens#t+toList(join(t:0,1:(e+1-sum Ugens#t),
                                        (#m-t-1):0))))
```

```
);

--picks the last monomial generators in each degree in lexicographic
--  order in a monomial ideal
--use for finding all ideals with max Betti numbers
lastMonomials = method(TypicalValue=>List,
                   Options=>{InitialDegree=>false});
lastMonomials(List) := opts-> (I) -> (
     LM:=flatten apply(rsort values partition(sum,sort I),
                        L->take(L,1));
     if opts.InitialDegree then drop(LM,1) else LM
);

--determines the left-shifts of a monomial k
--use for testing whether a monomial can be added to the generators
--  of a strongly stable ideal while preserving stability
leftShift = method(TypicalValue=>List);
leftShift(List) := (k) -> (
     apply(positions(drop(k,1),i->i>0),
           p->k+toList join(p:0,1:1,1:-1,#k-p-2:0))
);

--finds the list of indices for the generators of the lexicographic
--  ideal for given AVals in a polynomial ring with n+1 variables
--n must be at least the length of AVals!
--use for finding almost lex ideals with a given Hilbert polynomial
lexicIdeal = method(TypicalValue=>List);
lexicIdeal(List,ZZ) := (AVals,n) -> (
     l:=#AVals;
     ERAVals:=apply(n-1,j->take(join(toList((n-l):0),
                    reverse AVals),j+1));
     Gens:=append(apply(n-2,j->ERAVals#j+append(toList(j:0),1)),
                  ERAVals#-1);
     apply(n-1,i->join(Gens#i,n-i-1:0))
);

--computes the expansion of the monomial k in a given ideal I
--use for computing all expansions of monomial generators
--assumes working with ring R(d)=QQ[x_0..x_(n-d)]
makeExpand = method(TypicalValue=>MonomialIdeal);
makeExpand(List,List,ZZ) := (I,k,n) -> (
     rsort (select(I,g->g!=k) | apply(n-maxIndex k,
               i->k+toList join(n-i-1:0,1:1,i+#k-n:0)))
          --remove k from I and add required multiples of k to I
);
```

```
--computes the expansion of the last c monomials in degree deg in a
--  given ideal I in the ring QQ[x_0..x_n]
--use for computing ideal with max Bettis and fixed Hilbert polynomial
makeExpands = method(TypicalValue=>MonomialIdeal);
makeExpands(List,ZZ,ZZ,ZZ) := (J,c,deg,n) -> (
     MonList:=select(c,sort J,g->sum g==deg);
     for i to c-1 do J=makeExpand(J,MonList#i,n);
     J
);

--creates a polynomial ring QQ[x_0..x_n] with n+1 variables
--  with option for specifying whether variables are indexed
makeRing = method(TypicalValue=>Ring);
makeRing(ZZ,Boolean) :=  (n,flag) -> (
     if flag then x:=local x;
     R:=if not flag then QQ(monoid[vars(0..n),MonomialOrder=>Lex])
       else QQ(monoid[(symbol x)_0..(symbol x)_n,MonomialOrder=>Lex])
);

--determines the max index of a monomial given the exponent vector
maxIndex = method(TypicalValue=>ZZ);
maxIndex(List) := (L) -> (
     position(L,i->i>0,Reverse=>true)
);

--determines up to two elements in the right-shift of a monomial k
--  a ring with n+1 variables
--use for finding saturated strongly stable ideals with given
--  nonreduced Hilbert series
modRightShift = method(TypicalValue=>List);
modRightShift(List) := (k) -> (
     l:=maxIndex k;
     MRS:={};
     if l<#k-1 then MRS={k+toList join(l:0,1:-1,1:1,#k-l-2:0)};
     if (l>0 and k#(l-1)>0)
        then MRS=append(MRS,k+toList join(l-1:0,1:-1,1:1,#k-l-1:0));
     MRS
);

--moves a monomial in almost lex ideal to the "next cone"
--use for checking whether monomial is admissible
moveMap = method(TypicalValue=>RingElement);
moveMap(List,List) := (f,Ugens) -> (
     i:=position(Ugens,g->min(f-g)>=0);
```

```
     Ugens#(i+1)+join(toList(i+1:0),drop(f-Ugens#i,i+1))
);

--determines the right-shifts of a monomial k
--use for testing whether a monomial is expandable in an ideal
rightShift = method(TypicalValue=>List);
rightShift(List,ZZ) := (k,n) -> (
     apply(select(n-1,i->k#i>0),
           v->k+toList join(v:0,1:-1,1:1,#k-v-2:0))
);

--use for newMaxBettiWithHilbPoly
step2 = (H,SND,n) -> (
    H#J=makeExpands(H#J,SND,H#d,n);
    H#K=H#K-SND;
    H#d=H#d+1;
);

--use for newMaxBettiWithHilbPoly
step3 = (H,n) -> (
    if H#K<H#C then (H#j=H#j+1;
        H#d=max{H#d,H#Udegs#(H#j)};)
      else (H#J=makeExpands(H#J,H#C,H#d,n);
        H#K=H#K-H#C;
        if H#K>0 then step4(H,n);  );
);

--use for newMaxBettiWithHilbPoly
step4 = (H,n) -> (
    t:=#(H#Udegs)-1;
    u:=first sort select(H#J,g->sum g==H#d);
    D:=H#Udegs#(H#j)+u#(n-1);
    H#C=sum(H#d+1-H#Udegs#-1..D-H#Udegs#(H#j),k->
            binomial(n-t-1+k,k))+
        sum(H#j+1..t-1,i->sum(H#d+1-H#Udegs#i..D-min(
            H#Udegs#i,1+H#Udegs#(H#j)),
                k->binomial(n-i-1+k,k)));
    if H#C>0 then step3(H,n)
      else (L:=select(H#J,g->sum g==H#d and
                    g#(n-1)>=H#d+1-H#Udegs#-1);
        if #L>0 then (v:=first sort L;
            c1:=#select(H#J,g->sum g==H#d and v>g and g>=u);
            c1=min{c1,H#K};
            H#J=makeExpands(H#J,c1,H#d,n);
            H#K=H#K-c1;
```

```
            D=H#Udegs#(H#j)+v#(n-1);
            H#C=sum(H#d+1-H#Udegs#-1..D-H#Udegs#(H#j),k->
                    binomial(n-t-1+k,k))+
                sum(H#j+1..t-1,i->sum(H#d+1-H#Udegs#i..D-min(
                    H#Udegs#i,1+H#Udegs#(H#j)),
                        k->binomial(n-i-1+k,k)));
            if H#C>0 then step3(H,n);  )
          else (c2:=#select(H#J,g->sum g==H#d);
            c2=min{c2,H#K};
            H#J=makeExpands(H#J,c2,H#d,n);
            H#K=H#K-c2;);  );
);

--tests whether a monomial k can be expanded in an ideal I
--use for computing all expansions of monomial generators
testExpand = method(TypicalValue=>Boolean);
testExpand(List,List,ZZ) := (Igens,k,n) -> (
     all(take(Igens,position(Igens,g->g==k)),h->h#(n-1)==0)
        and #(set rightShift(k,n)*set Igens)==0
);
```

Code for integer partitions and generating functions

This code corresponds to the material in Chapter 5. It is sorted into two categories: methods for generating partitions and methods for manipulating generating functions.

```
-- -- -- -- -- % methods for creating partitions % -- -- -- -- --

--generates all distinct partitions of n
distinctParts = method(TypicalValue=>List);
distinctParts(ZZ,ZZ) := (n,m) -> (
    if n==0 then {{}} else
    if m==0 then {} else
    flatten(for i from floor sqrt (2*n) to min{m,n}
        list apply(distinctParts(n-i,i-1),l->prepend(i,l)))
);
distinctParts(ZZ) := (n) -> (distinctParts(n,n));

--generates all row-strict shifted plane partitions of n
rssPlanePartitions = method(TypicalValue=>List);
rssPlanePartitions(List,ZZ) := (l,n) -> (
    if n==0 then {{}} else
    flatten for i from 1 to min{n,sum l}
        list(L=select(distinctParts(i,l_0),m->#m<=#l and
                      all(#m,i->m_i<=l_i));
```

```
          flatten for m in L list apply(
                 rssPlanePartitions(drop(m,1),n-i),t->prepend(m,t)))
);
rssPlanePartitions(ZZ) := (n) -> (rssPlanePartitions(toList(n:n),n));

--generates all row-strict doubly shifted solid partitions of n
rssSolidPartitions = method(TypicalValue=>List);
rssSolidPartitions(List,ZZ) := (p,n) -> (
    if n==0 then {{}} else
    flatten for i from 1 to min{n,sum(apply(p,l->sum l))}
        list(L=select(rssPlanePartitions(p_0,i),m->#m<=#p
                                        and all(#m,i->m_i<=p_i));
          flatten for m in L list apply(
             rssSolidPartitions(drop(m,1),n-i),t->prepend(m,t)))
);
rssSolidPartitions(ZZ) := (n) ->
                  (rssSolidPartitions(toList(n:toList(n:n)),n));

--converts a plane partition into a matrix for ease of reading
pp2matrix = method(TypicalValue=>Matrix);
pp2matrix List := p -> (
    n:=#p;
    c:=apply(n,i->i+#(p#i));
    m:=max c;
    matrix apply(n,i->join(toList(i:0),p#i,toList(m-c#i:0)))
);

--converts a list of plane partitions into a list of matrices
pps2matrix = method(TypicalValue=>Matrix);
pps2matrix List := l -> (
    apply(l,p->pp2matrix p)
);

-- -- -- -- -- % methods for generating functions % -- -- -- -- --

--computes the Euler transform for a list of integers A
--  the list will be the same length as A
eulerTransform = method(TypicalValue=>List);
eulerTransform(List) := (A) -> (
    C:=apply(1..#A,i->sum(select(1..#A,d->i%d==0),j->j*A#(j-1)));
    B:=new MutableList from C;
    scan(2..#A,i->B#(i-1)=(C#(i-1)+
                  sum(1..i-1,j->C#(j-1)*B#(i-j-1)))//i);
    toList B
);
```

```
--computes the inverse Euler transform for a list of integers B
--  the list will be the same length as B
eulerUntransform = method(TypicalValue=>List);
eulerUntransform(List) := (B) -> (
    C:=new MutableList from B;
    scan(1..#B,i->C#(i-1)=i*B#(i-1)-sum(0..i-2,j->C#j*B#(i-2-j)));
    toList apply(1..#C,i->sum(select(1..i,d->i%d==0),
                 j->moebius(i//j)*C#(j-1))//i)
);

moebius = (n) -> (
    F:=factor n;
    if all(F,f->f#1==1) then return (-1)^#F else return 0;
);
```

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