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Boij-Söderberg Decompositions, Cellular Resolutions, and Polytopes

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Boij-Söderberg Decompositions, Cellular Resolutions, and Polytopes

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
Stephen Sturgeon
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ABSTRACT OF DISSERTATION

Boij-Söderberg Decompositions, Cellular Resolutions, and Polytopes

Boij-Söderberg theory shows that the Betti table of a graded module can be written as a linear combination of pure diagrams with integer coefficients. In chapter 2 using Ferrers hypergraphs and simplicial polytopes, we provide interpretations of these coefficients for ideals with a d-linear resolution, their quotient rings, and for Gorenstein rings whose resolution has essentially at most two linear strands. We also establish a structural result on the decomposition in the case of quasi-Gorenstein modules. These results are published in the Journal of Algebra, see [25].

In chapter 3 we provide some further results about Boij-Söderberg decompositions. We show how truncation of a pure diagram impacts the decomposition. We also prove constructively that every integer multiple of a pure diagram of codimension 2 can be realized as the Betti table of a module.

In chapter 4 we introduce the idea of a c-polar self-dual polytope. We prove in 4.3.7 that in dimension 2 only the odd n-gons have an embedding which is polar self-dual. We also define the family of Ferrers polytopes in 4.4.1. In 4.4.8 we prove that the Ferrers polytope in dimension d is d-polar self-dual hence establishing a nontrivial example of a polar self-dual polytope in all dimension. In 4.5.6 we prove that the Ferrers polytope in dimension d supports a cellular resolution of the Stanley-Reisner ring of the $(d+3)$-gon.

KEYWORDS: Boij-Söderberg Decomposition, Cellular Resolutions, Polytopes, Posets, Gorenstein

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Chapter 1 Introduction

The study of free resolutions has its roots in understanding solutions to systems of polynomial equations. Given a system of polynomial equations we know that in general there are many solutions. A standard question to ask is what is the "best" solution in some sense. To answer this question it is first important to understand what the set of solutions looks like. One of the first natural questions is what is the dimension of the solution space? The theory of free resolutions answers this questions and many others.

The idea of a system of equations can be reformulated in the language of an "ideal". Another simple, but surprisingly useful step is to transform a standard ideal into a graded ideal and consider solutions to the ideal in projective space. This simplification motivates the study of graded free resolutions which were first introduced by David Hilbert, perhaps the most influential mathematician of the late nineteenth century and early twentieth century.

Although systems of equations provide a sufficient reason for the study of free resolutions, they also arise in many other areas of study. One such area is the study of combinatorial structures. Throughout this paper we exploit the many connections between combinatorial objects such as graphs and polytopes and their corresponding ideals.

Along with giving the dimension of a ring, free resolutions provide many numerical invariants of a ring, known as the Betti numbers. One convenient way to display the numerical information in a free resolution is called the Betti table. This is essentially just a matrix with integer entries. One natural question is to ask if a given integer matrix is actually the Betti table of a module.

In 2009 Eisenbud and Schreyer proved the existence of a Boij-Söderberg decomposition for the Betti table of any module over the polynomial ring $k[x_1, ..., x_n]$. This theory gives a simple algorithm for decomposing the Betti table into a sequence of "pure diagrams" with positive integer coefficients. This algorithm gives us a partial way to answer the question posed in the last paragraph. If this algorithm fails then we know the integer matrix we were considering was not the Betti table of a module. On the other hand if the algorithm succeeds then there is an integer multiple of this table which is the Betti table of a module. However, to date there is no known significance to the particular "pure diagrams" or their coefficients.

Chapter 2 seeks to restrict the study of the Boij-Söderberg decompositions to rings where the "pure diagrams" are easy to predict, and then find meaning in the integer coefficients. These results are published in the Journal of Algebra, see [25].

In section 2.2 we introduce the necessary background for our results in Boij-Söderberg theory. In section 2.3 we describe the Boij-Söderberg decompositions of ideals with linear resolutions in terms of their associated Ferrers ideals. In particular Theorem 2.3.6 characterizes the decompositions of ideals with linear resolutions.

In section 2.4 we show that Boij-Söderberg decompositions of Quasi-Gorenstein modules have symmetric decompositions. The main theorem of this section is Theorem 2.4.4 which presents the symmetry arguement.

In section 2.5 we explicitly describe the decompositions of some Gorenstein rings. In particular we point out in 2.5.6 that if R/I is the quotient ring associated with the Stanley-Reisner ring of a stacked polytope, then the coefficients in the decomposition are either the dimension of the polytope, or the dimension minus one.

Chapter 3 is a collection of smaller disconnected results in Boij-Söderberg theory. Considering Theorems 2.3.2 and 2.3.9 we might try to classify how the Boij-Söderberg decomposition changes when we pass from a quotient ring to the corresponding ideal. This corresponds to understanding how Boij-Söderberg decompositions change when we truncate a Betti table. We provide some results in this area in section 3.2.

Another important question in Boij-Söderberg theory is an integer combination of "pure diagrams" not actually the Betti table of a module? Considering all possible linear combinations of "pure diagrams" with positive coefficients we get a cone in which the "pure diagrams" form the extremal rays. Then we show in this section 3.3 that every integer point which lies along one of these extremal rays of codimension 2 is actually the Betti table of a module.

The theory of cellular resolutions was developed in [7] and [8]. A cellular resolution is a way of encoding the minimal graded free resolution of a monomial ideal in a labeled cell complex. Once each face is labeled by a monomial the maps in the associated free resolution become:

$$\partial_i(e_P) = \sum_{Q \text{ facet of } P} \epsilon(Q,P)\frac{m_P}{m_Q}\, e_Q,$$

This theory allows us to associate the Betti numbers of the free resolution with the face vector of our cell complex. The biggest question in the theory of cellular resolutions is which monomial ideals have a cellular resolution? In [27] Velasco shows that there are monomial ideals whose resolutions are not supported on a CW-complex.

In chapter 4 we seek to construct a cellular resolution of the Stanley-Reisner ring of the n-gon. In order to do this we first introduce several of the basic notions of cellular resolutions and cite a few particular cases in 4.2.

In section 4.3 we introduce the idea of a c-polar self-dual polytope. We prove that the set of polar self-dual polytopes is strictly smaller than the set of self-dual polytopes in Theorem 4.3.7. In order to do this we prove Lemma 4.3.6, which provides some necessary conditions on the face-poset of a polytope in any dimension in order for it to have a polar self-dual embedding. We conclude the section by giving some results about polar self-dual simplices and defining the overlap between polar self-dual an reflexive polytopes.

In section 4.4 we construct the Ferrers polytopes 4.4.1. We prove that the d dimensional Ferrers polytope is actually d-polar self-dual in Theorem 4.4.8 thus establishing a nontrivial example of polar self-dual polytopes in all dimensions.

In section 4.5 we label the Ferrers polytope in 4.5.2, hence turning the Ferrers polytope into a labeled polytope. In 4.5.6 we show that the labeled Ferrers polytope in dimension

d supports a minimal free resolution of the Stanley-Reisner ring of the $(d+3)$-gon. We extend this result to show that a different labeling of the Ferrers polytope supports a minimal free resolution of the Stanley-Reisner ring of some stacked polytopes in Corollary 4.5.9.

Chapter 2 Combinatorial Interpretations of some Boij-Söderberg Decompositions

2.1 Introduction

Boij-Söderberg theory classifies all Betti tables of graded modules over a polynomial ring R up to a rational multiple. This is achieved by writing the Betti table of such a module as a unique linear combination of pure diagrams whose coefficients are positive integers (see Section 2.2 for details). The purpose of this chapter is to demonstrate combinatorial significance of these coefficients in a few cases. Each of these cases is related to ideals that are derived from some combinatorial objects. Moreover, we show that the self-duality of the minimal free resolution of a quasi-Gorenstein module is reflected in its Boij-Söderberg decomposition. This includes all standard graded Gorenstein algebras.

In Section 2.3 we first consider the Boij-Söderberg decompositions of ideals with a d-linear resolution. The Ferrers ideals associated to d-uniform Ferrers hypergraphs provide examples of such ideals. Their resolutions are well-understood thanks to results in [11], [12], and [23]. We show that each Betti table of an ideal with a d-linear resolution corresponds to the Betti table of a suitable Ferrers hypergraph. This allows us to give a combinatorial interpretation of the Boij-Söderberg coefficients. In particular, they must form an O-sequence. This result (see Theorem 2.3.6) provides a characterization of the Betti numbers of ideals with a d-linear resolution that complements the recent characterization obtained in [18].

Then we consider the Boij-Söderberg decomposition of certain quotient rings R/I. The Betti tables of the ideal I and R/I are closely related. However, their Boij-Söderberg decompositions are very different in general and the precise relationship is not known. In the case where I has a d-linear resolution, we obtain an interpretation of the Boij-Söderberg coefficients of R/I (see Theorem 2.3.9). Again it relies on a suitable Ferrers hypergraph though the coefficients are extracted by counting different subsets in the hypergraph this time.

Quasi-Gorenstein modules were introduced in [22]. They are important in the liaison theory of modules. Gorenstein rings are examples of cyclic such modules. Quasi-Gorenstein modules have a self-dual minimal free resolution.

In Section 2.4 we consider the Boij-Söderberg decompositions of such modules. We show that their Betti table can be rewritten as a linear combination of self-dual diagrams, where each summand is the sum of at most two pure diagrams (see Theorem 2.4.4). Specific instances of such decompositions are derived in Section 2.5. We consider the Betti tables of Gorenstein rings with few linear strands. Such Betti tables arise naturally. In particular, they can be obtained from the resolutions of Stanley-Reisner rings corresponding to boundary complexes of simplicial polytopes. The Boij-Söderberg decomposition are described in Theorem 2.5.3. The coefficients admit a very transparent interpretation in the case of stacked polytopes (see Corollary 2.5.6).

We review basic facts on Boij-Söderberg decompositions and Ferrers hypergraphs in Section 2.2. Furthermore, given any strongly stable monomial ideal I whose generators have degree d, an explicit construction of a d-uniform Ferrers hypergraph with the same graded Betti numbers as I is provided in Remarks 2.2.5 and 2.2.6.

2.2 Boij-Söderberg decomposition, O-sequences, and Ferrers hypergraphs

We recall some results and concepts that are needed in subsequent sections.

We work over a polynomial ring $R = K[x_1, \ldots, x_n]$ in n variables, where K is any field. All modules are assumed to be graded finitely generated R-modules. We denote the graded Betti numbers of M (as an R-module) by

$$\beta_{i,j}(M) = \dim_K [\mathrm{Tor}_i^R(M, K)]_j.$$

The numerical information of the minimal free resolution of M is captured in the Betti table $\beta(M) = (\beta_{i,j}(M))$ of M.

Definition 2.2.1. *Given an increasing sequence of integers $\sigma = (d_0, d_1, \ldots, d_s)$ we denote by π_σ the matrix with entries $\beta_{i,j}$, where*

$$\beta_{i,j} = \begin{cases} \displaystyle\prod_{k=0, k\neq i}^{s} \frac{1}{|d_i - d_k|} & \text{if } j = d_i \\ 0 & \text{otherwise.} \end{cases}$$

It is called the pure diagram *to the* degree sequence *σ.*

Note that this is the convention used in [10], which differs from the original proposal in [9]. We point out that $\beta_{i,j} = 0$ in any Betti table when $j < i$. For this reason we will compress our Betti tables so that the first column entry is $be_{i,i}$. For example:

$$\pi_{(0,2,3,5)} = \begin{array}{c|cccc} \beta_{i,j} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{30} & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 2 & 0 & 0 & 0 & \frac{1}{30} \end{array}$$

Every pure diagram is a rational multiple of the Betti table of a Cohen-Macaulay module. Due to the seminal results by Eisenbud and Schreyer in [14], much more is true. Define a partial order on the set of pure diagrams by setting $\pi_\sigma \le \pi_\tau$, where $\sigma = (d_0, d_1, \ldots, d_s)$ and $\tau = (d'_0, d'_1, \ldots, d'_t)$ are degree sequences, if $s \ge t$ and $d_i \le d'_i$ for all $i = 0, 1, \ldots, t$. Boij-Söderberg theory as developed by Boij and Söderberg in [9], [10] and Eisenbud and Schreyer in [14] classifies all Betti tables of graded R-modules up to a rational multiple. More precisely, one has (see [10, Theorem 2]):

Theorem 2.2.2. *For every graded, finitely generated R-module M, there are unique pure diagrams $\pi_{\sigma_1} < \pi_{\sigma_2} < \cdots < \pi_{\sigma_t}$ and positive integers $a_1, \ldots, a_t$ such that*

$$\beta(M) = \sum_{i=1}^{t} a_i \pi_{\sigma_i}, \tag{2.2.1}$$

We call the right-hand side in Equation (2.2.1) the *Boij-Söderberg decomposition* of the Betti table of M, the pure diagrams π_{σ_i} its *summands* and the integers a_i the *Boij-Söderberg coefficients* of M.

Next, we recall Macaulay's characterization of Hilbert functions of graded K-algebras. Given positive integers b and d, there are unique integers $m_d > m_{d-1} > m_s \geq s \geq 1$ such that

$$b = \binom{m_d}{d} + \binom{m_{d-1}}{d-1} + \ldots + \binom{m_s}{s}.$$

Then define

$$b^{\langle d \rangle} := \binom{m_d + 1}{d+1} + \binom{m_{d-1}+1}{d} + \ldots + \binom{m_s + 1}{s+1}$$

and $b^{\langle d \rangle} := 0$ if $b = 0$. A sequence of non-negative integers $(h_j)_{j \geq 0}$ is called an *O-sequence* if $h_0 = 1$ and $h_{j+1} \leq h_j^{\langle j \rangle}$ for all $j \geq 1$. Macaulay (see, e.g., [5, Theorem 4.2.10]) showed that, for a numerical function $h : \mathbb{Z} \to \mathbb{Z}$, the following conditions are equivalent:

(a) h is the Hilbert function of a standard graded K-algebra R/I, that is, $\dim_K [R/I]_j = h(j)$ for all integers j;

(b) $h(j) = 0$ if $j < 0$ and $\{h(j)\}_{j \geq 0}$ is an O-sequence.

Finally, we consider Ferrers hypergraphs. Ferrers graphs are parametrized by partitions and form an important class of bipartite graphs. Their edge ideals admit an explicit minimal free resolution (see [11]). They can be specialized to the edge ideals of threshold graphs (see [12]). These results have been extended to d-partite hypergraphs, $d \geq 2$, in [23].

Definition 2.2.3. *A* Ferrers hypergraph *is a d-partite d-uniform hypergraph F on a vertex set $X^{(1)} \sqcup \ldots \sqcup X^{(d)}$ such that there is a linear ordering on each $X^{(j)}$ and whenever $(i_1, \ldots, i_d) \in F$ and $(i'_1, \ldots, i'_d)$ satisfies $i'_j \leq i_j$ in $X^{(j)}$ for all j, one also has $(i'_1, \ldots, i'_d) \in F$. In other words, F is an order ideal in the componentwise partial ordering on $X^{(1)} \times \ldots \times X^{(d)}$.*

The ideal $I(F)$ generated by all the monomials $x_{i_1}^{(1)} \cdots x_{i_d}^{(d)}$, where $(i_1, \ldots, i_d) \in F$, is called a (generalized) Ferrers ideal*.*

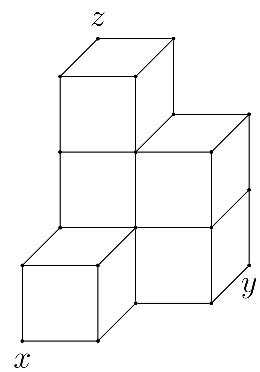
We may assume that the sets $X^{(j)}$ consist of consecutive positive integers $1, 2, \ldots, n_j$.

Example 2.2.4. *Monomial ideals generated by variables correspond to 1-uniform Ferrers hypergraphs. Ferrers tableaux describe 2-uniform Ferrers graphs, whereas 3-uniform Ferrers hypergraphs correspond to cubical stackings. For example, consider the stacking Using variables $x_1, x_2, \ldots$, $y_1, y_2, \ldots$, and $z_1, z_2, \ldots$ to avoid super scripts, the associated Ferrers ideal is*

$$I(F) = (x_1y_1z_1, x_1y_1z_2, x_1y_1z_3, x_1y_2z_1, x_1y_2z_2, x_2y_1z_1).$$

Recall that a monomial ideal $I \subset R$ is said to be *strongly stable* if, for any monomial $u \in S$, the conditions $u \in I$ and x_i divides u imply that $x_j \cdot \frac{u}{x_i}$ is in I whenever $j \leq i$. A squarefree monomial ideal $I \subset R$ is said to be *squarefree strongly stable* if, for any squarefree monomial $u \in R$, the conditions $u \in I$, x_i divides u, and x_j does not divide u imply that $x_j \cdot \frac{u}{x_i}$ is in I whenever $j \leq i$.

Figure 2.1: 3-uniform Ferrers Hypergraph



Remark 2.2.5. *It is well-known how to associate to a given strongly stable ideal a squarefree strongly stable ideal in a ring with enough variables that has the same graded Betti numbers. Indeed, define a map*

$$\varphi : \{monomials\} \longrightarrow \{squarefree\ monomials\}$$

by

$$x_{i_1}x_{i_2}\cdots x_{i_j} \mapsto x_{i_1}x_{i_2+1}\cdots x_{i_j+j-1}, \quad where\ 1 \le i_1 \le i_2 \cdots \le i_j.$$

If I is a strongly stable ideal with minimal generators $u_1, ..., u_t$, then $J = (\varphi(u_1), ..., \varphi(u_t))$ is a squarefree strongly stable ideals with the same graded Betti numbers as I by Lemmas 1.2 and 2.2 in [1].

According to [23, Propostion 3.7], every Ferrers hypergraph is isomorphic to a skew squarefree strongly stable hypergraph. However, here we need a different construction.

Remark 2.2.6. *Using new variables $x_i^{(j)}$, where $i \ge 1$ and $1 \le j \le d$, consider the map ψ defined by*

$$x_{i_1}x_{i_2}\cdots x_{i_d} \mapsto x_{i_1}^{(1)}x_{i_2-i_1}^{(2)}\cdots x_{i_d-i_{d-1}}^{(d)} \quad if\ 1 \le i_1 \le i_2 \cdots \le i_j.$$

If J is a squarefree strongly stable ideal whose minimal generators $v_1, \ldots, v_t$ all have degree d, then one checks that the ideal generated by $\psi(v_1), \ldots, \psi(v_t)$ is the Ferrers ideal $I(F)$ of a d-uniform Ferrers graph F. Moreover, the ideals J and $I(F)$ have the same graded Betti numbers by [23, Theorem 3.13] as their minimal free resolutions can be described by using isomorphic cell complexes.

We illustrate the passage from a strongly stable ideal to a Ferrers ideal.

Example 2.2.7. *Consider the strongly stable ideal $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_1x_2x_3, x_1^2x_3)$. Applying the above map φ to each of its generators we get:*

$$I = (x_1^3, x_1^2x_2, x_1x_2^2, x_1x_2x_3, x_1^2x_3) \mapsto (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_1x_3x_5, x_1x_2x_5)$$

Using the above map ψ we get a Ferrers ideal. However, to avoid superscripts we use variables $y_i = x_i^{(2)}$ and $z_i = x_i^{(3)}$. We obtain:

$$(x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_1x_3x_5, x_1x_2x_5) \mapsto I(F) = (x_1y_1z_1, x_1y_1z_2, x_1y_2z_1, x_1y_2z_2, x_1y_1z_3).$$

2.3 Ideals with d-linear resolutions

In this section, our goal is to describe the Boij-Söderberg decompositions of the Betti tables of ideals with a d-linear resolution and of the tables of their quotient rings.

Notation 2.3.1. *We use $\pi_{d-lin;k}$ to denote the pure diagram representing a d-linear resolution, that is, $\pi_{d-lin;k} = \pi_\sigma$, where $\sigma = (d, d+1, \dots, d+k)$.*

We begin by considering Ferrers ideals. Notice that the graded Betti numbers of an ideal I over R are the same as the ones of the extension ideal $IR[t]$ over $R[t]$, where t is a new variable. Thus, we may drop the reference to the polynomial ring R.

Proposition 2.3.2. *Let F be a d-uniform Ferrers hypergraph. Then the Boij-Söderberg decomposition of the associated Ferrers ideal $I(F)$ is*

$$\beta(I(F)) = \sum_{k\geq 0} \alpha_k(F)\, k!\ \pi_{d-lin;k},$$

where:

$$\alpha_k(F) := \#\{(i_1, .., i_d) \in F : \sum_j i_j = k + d\}.$$

Proof. Observe that the non-zero entry in homological degree i of $\pi_{d-lin;k}$ is $\frac{1}{i!(k-i)!}$. Hence, the i-th non-zero entry in the diagram $\sum_{k\geq 0} \alpha_k(F)\, k!\ \pi_{d-lin;k}$ is

$$\left[\sum_{k\geq 0} \alpha_k(F)\, k!\ \pi_{d-lin;k}\right]_i = \sum_{k\geq 0} \alpha_k(F) \binom{k}{i}.$$

Our claim follows because

$$\beta_i(I(F)) = \sum_{k\geq 0} \alpha_k(F) \binom{k}{i}$$

by Corollary 3.14 in [23]. □

Example 2.3.3. *Consider the 3-uniform Ferrers hypergraph*

$$F = \{(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 1), (1, 2, 2), (2, 1, 1)\}.$$

Its Ferrers ideal (see Example 2.2.4)

$$I(F) = (x_1y_1z_1, x_1y_1z_2, x_1y_1z_3, x_1y_2z_1, x_1y_2z_2, x_2y_1z_1)$$

has Betti table

$$\beta(I(F)) = \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 6 & 7 & 2 \end{array}\ .$$

Since $\alpha_2(F) = 2$, $\alpha_1(F) = 3$, and $\alpha_0(F) = 1$, by Theorem 2.3.2, its Boij-Söderberg decomposition is

$$\begin{aligned} \beta(I(F)) &= \alpha_2(F) \cdot 2! \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & \frac{1}{2} & 1 & \frac{1}{2} \end{array} + \alpha_1(F) \cdot 1! \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 1 & 1 & \cdot \end{array} \\ &\quad + \alpha_0(F) \cdot 0! \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 1 & \cdot & \cdot \end{array} \\ &= 4 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & \frac{1}{2} & 1 & \frac{1}{2} \end{array} + 3 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 1 & 1 & \cdot \end{array} + \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 1 & \cdot & \cdot \end{array}\ . \end{aligned}$$

The following result characterizes the α-sequences of Ferrers graphs as defined in Proposition 2.3.2. Recall that an O-sequence $h_0, h_1, \ldots$ is a sequence of non-negative integers such that $h_0 = 1$ and $h_{i+1} \leq h_i^{<i>}$ for all $i \geq 1$ (see Section 2.2). This following result follows from [19, Proposition 3.8] and the fact that $\alpha_{k+1}(F) = l_k^*(F(I))$ in the notation of that paper. We provide a direct proof by construction.

Proposition 2.3.4. *Let $(h_0, ..., h_s)$ be a sequence of non-negative integers. Then the following conditions are equivalent:*

(a) The given sequence is the α-sequence of a d-uniform Ferrers hypergraph, that is, there is such a graph F such that $\alpha_i(F) = h_i$ whenever $0 \leq i \leq s$ and $\alpha_i = 0$ if $i > s$.

(b) The given sequence is an O-sequence with $h_1 \leq d$.

Proof. Denote by M the set of monomials in the polynomial ring $R = K[x_1, ..., x_d]$ and consider the map

$$\varphi : M \longrightarrow S := K[x_i^{(j)} | 1 \leq j \leq d], \quad x^a = x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} \mapsto x_{a_1+1}^{(1)} x_{a_2+1}^{(2)} \cdots x_{a_d+1}^{(d)}$$

First, we show that (b) implies (a). By Macaulay's theorem, Assumption (b) provides that there is a lexsegment ideal I of R such that its Hilbert function satisfies

$$\dim_K [R/I]_j = \begin{cases} h_j & \text{if } 0 \leq j \leq s \\ 0 & \text{if } s < j. \end{cases}$$

Denote by L_j the monomials of degree j in $M \setminus I$. Let $J \subset S$ be the ideal that is generated by $\varphi(L_0) \cup \ldots \cup \varphi(L_s)$. Note that all minimal generators of J have degree d. We claim that J is a Ferrers ideal. Indeed, if $\varphi(x^a)$ is a minimal generator of J, then $x^a \notin I$. Thus, $\frac{x^a}{x_i} \notin I$ for each variable x_i, so $\varphi(\frac{x^a}{x_i}) \in J$.

Let F be the d-uniform Ferrers graph such that $J = I(F)$. Then $\alpha_i(F) = h_i$ follows from the construction of F.

Second, we assume (a) and show (b). Let $L \subset R$ be the set of monomials consisting of the preimages under φ of the minimal generators of the Ferrers ideal $I(F)$. It consists of monomials whose degree is at most s. Let $L_j \subset L$ be the subset of monomials having degree j. Then the cardinality of L_j is $\alpha_j(F)$ by construction. Moreover, observe that L is an order ideal of M with respect to the partial order given by divisibility because F is a Ferrers hypergraph.

Let $I \subset R$ be the ideal that is generated by all the monomials that are not in L. Then I is an artinian ideal whose inverse system is the order ideal L. Thus, we get

$$\dim_K [R/I]_j = \# L_j = \alpha_j(F) = h_j.$$

Hence Macaulay's characterization of Hilbert functions implies that $(h_0, ..., h_s)$ is an O-sequence. ∎

Remark 2.3.5. *Although we construct a Ferrers hypergraph for each O-sequence this is not in general the unique Ferrers hypergraph for that sequence. For example consider Ferrers ideals $I = (x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2)$ and $J = (x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_3y_1)$. These hypergraphs are clearly not isomorphic and yet their Boij-Söderberg decompositions give the same O-sequence.*

We are ready for the first main result of this section. We use Notation 2.3.1 in our characterization of the Betti numbers of ideals with a d-linear resolution.

Theorem 2.3.6. *Let $R = K[x_1 \ldots, x_n]$ and consider the diagram*

$$\beta = \sum_{k=0}^{v} \alpha_k \, k! \; \pi_{d-lin;k},$$

where $\alpha_0, \ldots, \alpha_v$ are rational numbers and $v \leq n$. Then the following conditions are equivalent:

(a) β is the Betti table of an ideal of R with a d-linear resolution.

(b) β is the Betti table of a strongly stable ideal I whose minimal generators have degree d.

(c) β is the Betti table of the ideal to a d-uniform Ferrers hypergraph F with

$$\alpha_i(F) = \begin{cases} \alpha_i & \text{if } 0 \leq i \leq v \\ 0 & \text{if } v < i. \end{cases}$$

(d) $(\alpha_0, \alpha_1, ..., \alpha_v)$ is an O-sequence with $\alpha_1 \leq d$.

Proof. (a) ⇒ (b): Let I be an ideal with Betti table β. Then the generic initial ideal of I with respect to the reverse lexicographic order has the same graded Betti numbers as I (see [16, Corollary 4.3.18] and [16, Corollary 6.1.5]). Furthermore, if K has characteristic zero, then $\operatorname{gin} I$ is strongly stable, and we are done. If the characteristic of K is positive, then $\operatorname{gin} I$ is at least a stable monomial ideal. This follows, for example, by [24, Theorem 2.5]. Consider now $\operatorname{gin} I$ as an ideal in a polynomial ring whose base field, L, has characteristic zero. Since the minimal free resolution of $\operatorname{gin} I$, as described by Eliahou and Kervaire, does not depend on the characteristic, the Betti numbers of $\operatorname{gin} I$ remain the same when considered over L. Passing now to the generic initial ideal with respect to the reverse lexicographic order gives the desired strongly stable ideal.

(b) ⇒ (c): Remarks 2.2.5 and 2.2.6 provide to each strongly stable ideal whose generators have degree d a Ferrers ideal of a d-uniform hypergraph with the same graded Betti numbers.

(c) ⇒ (a): This is true by Proposition 2.3.2.

Conditions (c) and (d) are equivalent by Proposition 2.3.4. □

Remark 2.3.7. *A related, but compared to Theorem 2.3.6 different characterization of the Betti tables of ideals with a d-linear resolution has been established by Herzog, Sharifan, and Varbaro in [18, Theorem 3.2]. It uses combinatorial information on the generators of a strongly stable monomial ideal. The proposition by Murai in [19, Proposition 3.8] gives a similar characterization of strongly stable ideals based on the existence of a simplicial complex whose Stanley-Reisner ideal has the desired O-sequence. The author would like to thank Isabella Novik and Alexander Engström for pointing out related work.*

Since the Betti numbers of the quotient ring R/I are determined by the Betti numbers of the ideal I, one might expect the decompositions of the Betti tables to be similar or, at least, related. However, in general the precise relationship is not known. We solve this problem if the ideal I has a d-linear resolution. By the previous result, we may assume that I is a Ferrers ideal. In this case we show that the decomposition of the quotient ring can be found by counting the same set that defined the numbers $\alpha_k(F)$, just in a different fashion.

In order to state the result, we need some notation.

Notation 2.3.8. *We use $\pi_{0,d-lin;k}$ to denote the pure diagram $\pi_{\underline{d}}$, where the degree sequence is $\underline{d} = (0, d, d+1, \ldots, d+k)$.*

In the following result we exclude the case $d = 1$ in which the Boij-Söderberg decomposition is trivial. It has only one summand.

Theorem 2.3.9. *Let F be a d-uniform Ferrers hypergraph on the vertex set $X^{(1)} \sqcup \ldots \sqcup X^{(d)}$, where $d \geq 2$. Then the Boij-Söderberg decomposition of the quotient ring $R/I(F)$ is*

$$\beta(R/I(F)) = \sum_{j=1}^{d} \sum_{S \in F_j} n_S \cdot k_S! \cdot \pi_{0,d-lin;k_S},$$

where F_j is the Ferrers hypergraph

$$F_j := \{(i_1, \ldots, \widehat{i_j}, \ldots, i_d) : \text{There is some } i_j \in X^{(j)} \text{ such that } (i_1, \ldots, i_j, \ldots, i_d) \in F\},$$
$$n_S := \max\{i_j \in X^{(j)} : (i_1, \ldots, i_j, \ldots, i_d) \in F\} \quad \text{if } S = (i_1, \ldots, \widehat{i_j}, \ldots, i_d) \in F_j, \text{ and}$$
$$k_S := n_S - d + \sum_{p=1, p \neq j}^{d} i_p.$$

Proof. Note that the non-zero entries in $\pi_{0,d-lin;k_S}$ are

$$\beta_i(\pi_{0,d-lin;k_S}) = \begin{cases} \dfrac{(d-1)!}{(d+k_S)!} & \text{if } i = 0 \\ \dfrac{1}{(i-1)! \cdot (k_S - i + 1)! \cdot (d+i-1)} & \text{if } 1 \leq i \leq k_S + 1. \end{cases} \tag{2.3.1}$$

Thus, the entry of $\sum_{j=1}^{d} \sum_{S \in F_j} n_S \cdot k_S! \cdot \pi_{0,d-lin;k_S}$ in homological degree i is

$$\beta_i \left(\sum_{j=1}^{d} \sum_{S \in F_j} n_S \cdot k_S! \cdot \pi_{0,d-lin;k_S} \right) = \frac{1}{d+i-1} \sum_{j=1}^{d} \sum_{S \in F_j} n_S \cdot \binom{k_S}{i-1}.$$

Consider first the Betti numbers with positive index, i.e., assume that $i \geq 1$. Then [23, Corollary 3.14] gives that

$$\beta_i(R/I(F)) = \sum_{(i_1, \ldots, i_d) \in F} \binom{\sum_p i_p - d}{i-1}.$$

It follows that we have to show the identity

$$(d+i-1) \sum_{(i_1, \ldots, i_d) \in F} \binom{\sum_p i_p - d}{i-1} = \sum_{j=1}^{d} \sum_{S \in F_j} n_S \cdot \binom{k_S}{i-1}. \tag{2.3.2}$$

To this end denote by N the number of possibilities for choosing pairs (X, y), where $y \in X$ and X is a subset of $X^{(1)} \sqcup \ldots \sqcup X^{(d)}$ with cardinality $d+i-1$ and, for each p, maxima $m_p = \max(X \cap X^{(p)})$ in $X^{(p)}$ such that $(m_1, \ldots, m_d) \in F$. We establish Identity (2.3.2) by determining N in two different ways.

Approach 1.: We classify the possible subsets X according to their maxima in each set $X^{(p)}$.

Fix $(m_1, \ldots, m_d) \in F$. To extend $\{m_1, \ldots, m_d\}$ to a subset X with maxima $m_1, \ldots, m_d$, we can choose $i-1$ numbers among any of the first $m_p - 1$ elements in each $X^{(p)}$. There are $\binom{\sum_p m_p - d}{i-1}$ such choices. Taking into account the number of choices for $y \in X$, we conclude that

$$N = (d+i-1) \sum_{(m_1,\ldots,m_d)\in F} \binom{\sum_p m_p - d}{i-1}. \tag{2.3.3}$$

Approach 2.: This time we classify the possibilities for choosing (X, y) according to the number j such that $y \in X^{(j)}$ and the maxima of X in all $X^{(p)}$, except $X^{(j)}$.

Fix $j \in \{1, \ldots, d\}$ and $S = (m_1, \ldots, \widehat{m_j}, \ldots, m_d) \in F_j$. We want to pick y in $S^{(j)}$ and extend $\{m_1, \ldots, \widehat{m_j}, \ldots, m_d, y\}$ to a subset X with $d+i-1$ elements and maxima vector $(m_1, \ldots, \max(X \cap X^{(j)}), \ldots, m_d)$ in F. In order to ensure the latter condition, all elements in $X \cap X^{(j)}$ have to be among the first n_S elements of $X^{(j)}$ by definition of n_S and using the defining property of a Ferrers hypergraph. Thus, there are n_S choices for y in $S^{(j)}$. The other $i-1$ numbers in X can be chosen among any of the first $m_p - 1$ elements in each $X^{(p)}$ if $p \neq j$ and among the first n_S elements in $X^{(j)}$, except y. We conclude that

$$N = \sum_{j=1}^{d} \sum_{S\in F_j} n_S \cdot \binom{n_S - d + \sum_{p=1,p\neq j}^{d} i_p}{i-1} = \sum_{j=1}^{d} \sum_{S\in F_j} n_S \cdot \binom{k_S}{i-1}. \tag{2.3.4}$$

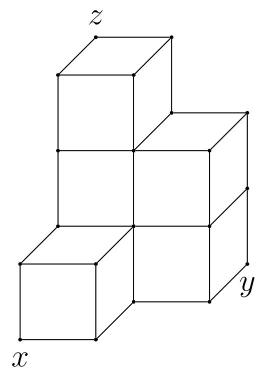
Comparing Equations (2.3.3) and (2.3.4), we obtain the desired Identity (2.3.2).

It remains to consider the 0-th Betti number. However, the alternating sum of the total Betti numbers in a minimal free resolution is zero. Hence our claim for $i = 0$ follows from our results for $i \geq 1$. $\square$

Remark 2.3.10. *Theorem 2.3.9 extends the conclusions of group 10.2 (E. Celikbas, D. Linsay, S. Sanyal, S. Sturgeon, K. Yu) at the MSRI summer workshop in commutative algebra 2011. In their report they show the conclusion in the case $d = 2$.*

We illustrate the last result in case $d = 3$.

Example 2.3.11. *Consider again the ideal $I(F) = (x_1y_1z_1, x_1y_1z_2, x_1y_1z_3, x_1y_2z_1, x_1y_2z_2, x_2y_1z_1)$, corresponding to the cubical stacking*



The Betti table of $R/I(F)$ is

$$\beta(R/I(F)) = \begin{array}{c|cccc} \beta_{i,j} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & 6 & 7 & 2 \end{array}.$$

Abusing notation by, for example, identifying $(i,j,k) \in F$ with the monomial $x_i y_j z_k$, we get the following data for the Ferrers graphs F_1, F_2, and F_3:

$$F_1 : \quad \begin{array}{c|ccccc} S & y_1z_1 & y_1z_2 & y_1z_3 & y_2z_1 & y_2z_2 \\ \hline n_S & 2 & 1 & 1 & 1 & 1 \\ k_S & 1 & 1 & 2 & 1 & 2 \end{array},$$

$$F_2 : \quad \begin{array}{c|cccc} S & x_1z_1 & x_1z_2 & x_1z_3 & x_2z_1 \\ \hline n_S & 2 & 2 & 1 & 1 \\ k_S & 1 & 2 & 2 & 1 \end{array},$$

and

$$F_3 : \quad \begin{array}{c|ccc} S & x_1y_1 & x_1y_2 & x_2y_1 \\ \hline n_S & 3 & 2 & 1 \\ k_S & 2 & 2 & 1 \end{array}.$$

Since $20 = (n_{y_1z_3} + n_{y_2z_2} + n_{x_1z_2} + n_{x_1z_3} + n_{x_1y_1} + n_{x_1y_2}) \cdot 2!$ *and* $8 = (n_{y_1z_1} + n_{y_1z_2} + n_{y_2z_1} + n_{x_1z_1} + n_{x_2z_1} + n_{x_2y_1}) \cdot 1!$*, Theorem 2.3.9 yields the Boij-Söderberg decomposition*

$$\beta(R/I(F)) = 20 \cdot \begin{array}{c|cccc} \beta_{i,j} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{60} & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \frac{1}{6} & \frac{1}{4} & \frac{1}{10} \end{array} + 8 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{12} & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \frac{1}{3} & \frac{1}{4} \end{array}.$$

Notice that the Boij-Söderberg decomposition of the Betti table of $I(F)$ has three summands (see Example 2.3.3), whereas the one of $R/I(F)$ has only two summands.

Theorem 2.3.9 provides a curious identity for each Ferrers hypergraph.

Corollary 2.3.12. *Let F be a d-uniform Ferrers hypergraph and adopt the notation of Theorem 2.3.9. Then*

$$d = \sum_{j=1}^{d} \sum_{S \in F_j} \frac{n_S}{\binom{d+k_S}{d}}.$$

Proof. Considering the 0-th Betti numbers in Theorem 2.3.9 and using Equation (2.3.1) we get the identity

$$1 = \sum_{j=1}^{d} \sum_{S \in F_j} n_S \cdot k_S! \cdot \frac{(d-1)!}{(d+k_S)!}.$$

Our claim follows. □

Our argument for Corollary 2.3.12 is rather indirect. There also is a more direct argument to establish this identity using induction on the number of vertices in the Ferrers hypergraph.

2.4 Quasi-Gorenstein modules

Quasi-Gorenstein modules were introduced in [22] as the graded perfect R-modules that are isomorphic to a degree shift of their canonical module. A cyclic module R/I is quasi-Gorenstein if and only if I is a Gorenstein ideal. In module liaison theory quasi-Gorenstein modules assume the role Gorenstein ideals play in Gorenstein liaison theory. These modules have a self-dual minimal free resolution. The goal of this section is to show that this self-duality is reflected in the Boij-Söderberg decomposition of the Betti table.

Let M be a finitely generated graded module over $R = K[x_1, \ldots, x_n]$. We denote its R-dual $\operatorname{Hom}_R(M, R)$ by M^*. It also is a graded module. We call $c = \dim R - \dim M = n - \dim M$ the codimension of M. The canonical module of M is $K_M = \operatorname{Ext}^c_R(M, R)(-n)$. If M is Cohen-Macaulay and there is an integer t such that $M \cong K_M(t)$, then M is said to be a *quasi-Gorenstein module*.

Let now M be a Cohen-Macaulay R-module of codimension c with minimal free resolution

$$0 \longrightarrow F_c \xrightarrow{\varphi_c} F_{c-1} \longrightarrow \cdots \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0.$$

Dualizing with respect to R we get the minimal free resolution

$$0 \longrightarrow F_0^* \xrightarrow{\varphi_1^*} F_1^* \longrightarrow \cdots \xrightarrow{\varphi_c^*} F_c^* \longrightarrow \operatorname{Ext}^c_R(M, R) \longrightarrow 0 \qquad (2.4.1)$$

because $\operatorname{Ext}^i_R(M, R) = 0$ whenever $i \neq c$ as M is Cohen-Macaulay. Hence, if M is a quasi-Gorenstein module, then the two free resolutions are isomorphic as exact sequences, up to a degree shift. It follows that the free modules F_i and F^*_{c-i} are isomorphic, up to a degree shift that is independent of i. The resulting self-duality of the free resolution means in particular that, for all integers i and j,

$$\beta_{i,j}(M) = \beta_{c-i,m-j}(\operatorname{Ext}^c_R(M, R)), \qquad (2.4.2)$$

where $m = \operatorname{reg} M + c + a(M)$. Here $a(M)$ denotes the least degree of a minimal generator of M and $\operatorname{reg} M = -c + \max\{j : \beta_{c,j}(M) \neq 0\}$ its Castelnuovo-Mumford regularity. In order to capture this self-duality of the free resolution of M in the Boij-Söderberg decomposition, we introduce.

Definition 2.4.1. *Consider the pure diagram π_σ to the degree sequence $\sigma = (d_0, d_1, \ldots, d_c)$. Then its* dual pure diagram *is the pure diagram π_{σ^*}, where $\sigma^* := (-d_c, \ldots, -d_1, -d_0)$.*

Moreover, for any integer m, we denote by $\pi_{m+\sigma}$ the pure diagram to the degree sequence $m + \sigma := (m + d_0, m + d_1, \ldots, m + d_c)$.

We note that the new pure Betti diagrams have the following properties.

Lemma 2.4.2. *(a) For each i,* $\beta_i(\pi_\sigma) = \beta_{c-i}(\pi_{\sigma^*})$.

(b) For each integer m and each i, $\beta_i(\pi_\sigma) = \beta_i(\pi_{m+\sigma})$.

Proof. Both claims follow directly from the definition of the pure Betti diagrams. $\square$

We will refer to any Betti diagram of the form $\pi_\sigma + \pi_{m+\sigma^*}$ as a self-dual Betti diagram. This is justified by comparing Equation (2.4.2) with the following observation.

Corollary 2.4.3. *For each integers i and j, the entries of the diagram $\pi_\sigma + \pi_{m+\sigma^*}$ satisfy*

$$\beta_{i,j} = \beta_{c-i,m-j}$$

Proof. This is a consequence of Lemma 2.4.2. $\square$

We are ready for the main result of this section.

Theorem 2.4.4. *Let M be a quasi-Gorenstein module of codimension c. Set $m = \operatorname{reg} M + c + a(M)$. Then the Boij-Söderberg decomposition of M is an integer linear combination of self-dual Betti diagrams of the form $\pi_\sigma + \pi_{m+\sigma^*}$ and, in case the number of Boij-Söderberg summands of M is odd, a pure diagram π_σ such that $\pi_\sigma = \pi_{m+\sigma^*}$.*

Proof. Consider the Boij-Söderberg decomposition of the Betti table of M

$$\beta(M) = \sum_{i=1}^{t} a_i \pi_{\sigma_i},$$

where $\pi_{\sigma_1} < \pi_{\sigma_2} < \cdots < \pi_{\sigma_t}$.

Observe that, for any pure diagrams π_σ and π_τ, the relation $\pi_\sigma < \pi_\tau$ implies $\pi_{\tau^*} < \pi_{\sigma^*}$. It follows (see Sequence (2.4.1)) that the Betti table of $\operatorname{Ext}_R^c(M, R)$ has the Boij-Söderberg decomposition

$$\beta(\operatorname{Ext}_R^c(M, R)) = \sum_{i=1}^{t} a_i \pi_{\sigma_i^*},$$

where $\pi_{\sigma_1^*} > \pi_{\sigma_2^*} > \cdots > \pi_{\sigma_t^*}$.

By assumption, M is quasi-Gorenstein, and thus $\operatorname{Ext}_R^c(M, R)(-m) \cong M$ as graded R-modules. Hence, comparing the above decompositions and using the uniqueness of the Boij-Söderberg decomposition, we conclude that, for all i,

$$\pi_{\sigma_i} = \pi_{m+\sigma_{t+1-i}^*}.$$

Our claim follows. $\square$

A referee of the journal version of this work asked the question, if the number of summands is odd then is the coefficient on the middle summand even? We are not aware of an example where this is not the case.

If M is a Cohen-Macaulay module, then $M \oplus K_M(j)$ is a quasi-Gorenstein module for each integer j (see [22, Remark 2.5(iii)]). Its Boij-Söderberg decomposition is determined by the one of M. Notice that the number of summands in the decomposition of $M \oplus K_M(j)$

is always even. This is not true for arbitrary quasi-Gorenstein modules. In the next section, we will exhibit explicit examples in the case of cyclic quasi-Gorenstein modules. Such a cyclic module is isomorphic to a Gorenstein ring, up to a degree shift. Here we give an example arising in the birational geometry of surfaces.

Example 2.4.5. *Let S be a regular surface of general type such that the canonical map is a birational morphism onto its image $Y \subset \mathbb{P}^4$. If the geometric genus of S is five and $K_S^2 = 11$, then the canonical ring $M = \oplus_{m \geq 0} H^0(S, \mathcal{O}_S(mK_S))$ is a quasi-Gorenstein module over the coordinate ring R of $\mathbb{P}^4$ with minimal free resolution of the form (see [3, Theorem 1.5])*

$$0 \to R(-6) \oplus R^2(-4) \to R^6(-3) \to R \oplus R^2(-2) \to M \to 0.$$

The Boij-Söderberg decomposition of its Betti table is

$$\beta(M) = \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 0 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 2 & 2 & 6 & 2 \\ 3 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & 1 \end{array}$$

$$= 8 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{12} & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \frac{1}{3} & \frac{1}{4} \\ 3 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot \end{array} + 6 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 0 & \frac{1}{18} & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 2 & \cdot & \frac{1}{9} & \cdot \\ 3 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \frac{1}{18} \end{array} + 8 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 0 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ 2 & \frac{1}{4} & \frac{1}{3} & \cdot \\ 3 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \frac{1}{12} \end{array}.$$

Observe that the third summand is $\pi_{(2,3,6)} = \pi_{6+(0,3,4)^*}$, where $\pi_{(0,3,4)}$ is the first summand, and that the second summand satisfies $\pi_{(0,3,6)} = \pi_{6+(0,3,6)^*}$, as predicted by Theorem 2.4.4.

2.5 Gorenstein rings

In this section we consider Gorenstein rings whose minimal free resolutions have at most two linear strands of maximal length when one ignores the first and the last homological degree. Gorenstein rings with such resolutions occur naturally. In fact, any such resolution is the minimal free resolution of the Stanley-Reisner ring associated to the boundary complex of a simplicial polytope, see [20].

The following result describes the Betti tables whose Boij-Söderberg decomposition we derive in this section. It follows from [20, Theorem 8.13].

Lemma 2.5.1. *Let s, t, c be positive integers such that $s \geq 2t$ and $c \leq n$. Then there is a homogeneous Gorenstein ideal $I \subset R$ of codimension c such that the graded minimal* *resolution of R/I has the shape*

$$0 \longrightarrow R(-s-c) \longrightarrow \begin{array}{c} R^{a_{c-1}}(-t-c+1) \\ \oplus \\ R^{a_1}(-s+t-c+1) \end{array} \longrightarrow \cdots$$

$$\longrightarrow \begin{array}{c} R^{a_2}(-t-2) \\ \oplus \\ R^{a_{c-2}}(-s+t-2) \end{array} \longrightarrow \begin{array}{c} R^{a_1}(-t-1) \\ \oplus \\ R^{a_{c-1}}(-s+t-1) \end{array} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where, for $i = 1, \ldots, c-1$,

$$a_i = \binom{c+t-1}{i+t}\binom{t-1+i}{t}.$$

We would like to point out that the ideals with this minimal free resolution arise in various ways. Recall that the Hilbert series of any graded K-algebra R/I can be uniquely written as

$$H_{R/I}(t) := \sum_{j\geq 0} \dim_K [R/I]_j = \frac{h(t)}{(1-t)^d},$$

where $h(1) \neq 0$, $h(t) = h_0 + h_1 t + \cdots h_r t^r \in \mathbb{Z}[t]$, and $d = \dim R/I$. The coefficient vector $(h_0, h_1, \ldots, h_r)$ is called the *h-vector* of R/I.

Remark 2.5.2. *(i) Assume R/I is a Gorenstein ring with a free resolution as in Lemma 2.5.1. Then its h-vector $h = (h_0, ..., h_s)$ is given by*

$$h_i = \begin{cases} \binom{c-1+i}{c-1} & \text{if } 0 \leq i \leq t; \\ \binom{c-1+t}{c-1} & \text{if } t \leq i \leq s-t; \\ \binom{s-i+c-1}{c-1} & \text{if } s-t \leq i \leq s. \end{cases}$$

(ii) Conversely, Gorenstein rings with this h-vector are often forced to have a free resolution as described in Lemma 2.5.1. To this end recall that a graded Gorenstein algebra R/I of dimension d has the weak Lefschetz property *if there are linear forms $\ell, \ell_1, \ldots, \ell_d$ such that $A = R/(I, \ell_1, \ldots, \ell_d)$ has dimension zero and, for each j, the multiplication map*

$$\times\ell : [A]_{j-1} \longrightarrow [A]_j, \quad a \mapsto \ell a,$$

has maximal rank, that is, it is injective or surjective. [20, Corollary 8.14] shows:

Let c, s, t be positive integers, where either $s = 2t$ or $s \geq 2t+2$. Let R/I be a Gorenstein algebra of dimension $d = n - c$ with an h-vector as in (i) above. If R/I has the weak Lefschetz property, then R/I has a minimal free resolution as in Lemma 2.5.1.

Furthermore, it follows by [20, Theorem 9.6] that each of the resolutions described in Lemma 2.5.1 occurs as the minimal free resolution of the Stanley-Reisner ring associated to the boundary complex of a simplicial polytope. Our main result in this section describes the Boij-Söderberg decomposition of the corresponding Betti table.

Theorem 2.5.3. *Let R/I be a Gorenstein ring with a free resolution as in Lemma 2.5.1. Then the Boij-Söderberg decomposition of R/I is*

$$\beta(R/I) = a \cdot [\pi_{\sigma_1} + \pi_{\sigma_c}] + b \cdot \sum_{j=2}^{c-1} \pi_{\sigma_j}, \tag{2.5.1}$$

where

$$\begin{aligned} a =& (s+1-t)\frac{(t+c-1)!}{t!}, \\ b =& (s+1-2t)\frac{(t+c-1)!}{t!}, \end{aligned}$$

and

$$\sigma_j = (0, d_{j,1}, \ldots, d_{j,c-1}, s+c)$$

with

$$d_{j,k} = \begin{cases} t+k & \text{if } 1 \leq k \leq c-j \\ s-t+k & \text{if } c-j+1 \leq k \leq c-1 \end{cases}.$$

As preparation for its proof, we derive the following identity.

Lemma 2.5.4. *If a, b, m are positive integers such that $m \leq a$, then*

$$\sum_{j=1}^{m} \frac{\binom{a}{j}}{\binom{a+b}{j}} = -\frac{(a+b+1-m)\binom{a}{m+1}}{(b+1)\binom{a+b}{m+1}} + \frac{a}{b+1}.$$

Proof. Define

$$\mu(j) = -\frac{(a+b-j+1)\binom{a}{j}}{(b+1)\binom{a+b}{j}}.$$

Then one checks that

$$\frac{\binom{a}{j}}{\binom{a+b}{j}} = \mu(j+1) - \mu(j).$$

Hence, we get the telescope sum

$$\begin{aligned} \sum_{j=1}^{m} \frac{\binom{a}{j}}{\binom{a+b}{j}} &= \sum_{j=1}^{m} \mu(j+1) - \mu(j) \\ &= \mu(m+1) - \mu(1) \\ &= -\frac{(a+b+1-m)\binom{a}{m+1}}{(b+1)\binom{a+b}{m+1}} + \frac{a}{b+1}, \end{aligned}$$

as claimed. □

Proof of Theorem 2.5.3. Let $1 \leq i \leq c-1$, and consider the graded Betti number $\beta_{i,t+i} = a_i$. According to Formula (2.5.1), we claim that precisely the pure Betti tables $\pi_{\sigma_1}, \ldots, \pi_{\sigma_{c-i}}$ contribute to this Betti number. For $j = 1, \ldots, c-i$, the degree sequence σ_j is

$$\sigma_j = (0, t+1, \ldots, t+i, \ldots, t+c-j, s-t+c-j, \ldots, s-t+c-1, c+s).$$

Hence, we get for the non-zero entry in homological degree i of π_{σ_j}

$$\beta_i(\pi_{\sigma_j}) = \frac{(s-2t+c-i-j)!}{(t+i)(s+c-t-i)(i-1)!(c-j-i)!(s-2t+c-1-i)!}.$$

Using that $a = \frac{s+1-t}{s+1-2t} \cdot b$, our claim for $\beta_{i,t+i}$ is equivalent to

$$\frac{a_i}{b} = \frac{s+1-t}{s+1-2t} \cdot \beta_i(\pi_{\sigma_1}) + \sum_{j=2}^{c-i} \beta_i(\pi_{\sigma_j}).$$

Simplify $\frac{a_i}{b}$ by multiplying by $(t+i)(s+c-t-i)(i-1)!(c-1-i)!$, this means that we have to show:

$$\begin{aligned}
&\frac{1}{(t+i)(s+1-2t)(i-1)!(c-j-i)!} = \\
&\qquad\qquad \frac{s+1-t}{s+1-2t} \cdot \frac{1}{(t+i)(s+c-t-i)(i-1)!(c-1-i)!} \\
&\qquad\qquad + \sum_{j=2}^{c-i} \frac{(s-2t+c-i-j)!}{(t+i)(s+c-t-i)(i-1)!(c-j-i)!(s-2t+c-1-i)!}.
\end{aligned}$$

The latter is equivalent to

$$\begin{aligned}
\frac{s+c-i-t}{s+1-2t} &= \frac{s+1-t}{s+1-2t} + \sum_{j=2}^{c-i} \frac{(s-2t+c-i-j)!(c-1-i)!}{(s-2t+c-1-i)!(c-j-i)!} \\
&= \frac{s+1-t}{s+1-2t} + \sum_{j=2}^{c-i} \frac{\binom{c-1-i}{j-1}}{\binom{s-2t+c-1-i}{j-1}} \\
&= \frac{s+1-t}{s+1-2t} + \frac{c-1-i}{s+1-2t},
\end{aligned}$$

which is certainly true and where we used Lemma 2.5.4 with $a = c-1-i$ and $b = s-2t$ to establish the last equality.

Using the symmetry on both sides of Identity (2.5.1), it only remains to check the claim for the Betti number $\beta_0(R/I) = 1$. To this end note that, for each $j = 1, \ldots, c$,

$$\beta_0(\pi_{\sigma_j}) = \frac{t!\,(s-t+c-j)!}{(s+c)\,(t+c-j)!\,(s-t+c-1)!}.$$

Thus, we have to show the identity

$$\begin{aligned}
1 =& \frac{(t+c-1)!}{t!} \Bigg[(s+1-t) \left(\frac{t!}{(s+c)\,(t+c-1)!} + \frac{(s-t)!}{(s+c)\,(s-t+c-1)!} \right) \\
&+ (s+1-2t) \sum_{j=2}^{c-1} \frac{t!\,(s-t+c-j)!}{(s+c)\,(t+c-j)!\,(s-t+c-1)!} \Bigg].
\end{aligned}$$

It is equivalent to

$$\frac{s+c}{s+1-2t} = \frac{s+1-t}{s+1-2t}\left[1+\frac{(s-t)!\,(t+c-1)!}{(s-t+c-1)!\,t!}\right] + \sum_{j=2}^{c-1}\frac{(t+c-1)!\,(s-t+c-j)!}{(t+c-j)!\,(s-t+c-1)!}$$

$$= \frac{s+1-t}{s+1-2t}\left[1+\frac{\binom{t+c-1}{c-1}}{\binom{s-t+c-1}{c-1}}\right] + \sum_{j=2}^{c-1}\frac{\binom{t+c-1}{j-1}}{\binom{s-t+c-1}{j-1}}$$

$$= \frac{s+1-t}{s+1-2t}\left[1+\frac{\binom{t+c-1}{c-1}}{\binom{s-t+c-1}{c-1}}\right] - \frac{(s+1-t)\binom{t+c-1}{c-1}}{(s+1-2t)\binom{s-t+c-1}{c-1}} + \frac{t+c-1}{s+1-2t},$$

which is certainly true and where we used Lemma 2.5.4 with $a = t+c-1$ and $b = s+1-2t$ to establish the last equality. This completes the argument. □

Remark 2.5.5. *Notice that the summands appearing in the Boij-Söderberg decomposition in Theorem 2.5.3 satisfy*

$$\pi_{\sigma_i} = \pi_{s+c+\sigma^*_{c+1-i}}.$$

This is in accordance with Theorem 2.4.4.

We illustrate the last result in the case of stacked polytopes. Recall that a d-dimensional simplicial polytope is stacked if it admits a triangulation Γ which is a $(d-1)$-tree, that is, Γ is a shellable $(d-1)$-dimensional simplicial complex with h-vector $(1, c-1)$. For example, such a polytope is obtained by pairwise gluing of d-simplices along a facet. The following result shows how the decomposition of the Betti table of its boundary complex reflects the data that determine the polytope.

Corollary 2.5.6. *Let Δ be the boundary complex of a stacked polytope with $n = c+d$ vertices that is obtained by stacking c simplices of dimension d. Then the Betti table of its Stanley-Reisner ring $K[\Delta]$ has the Boij-Söderberg decomposition*

$$\beta(K[\Delta]) = d\cdot c!\cdot[\pi_{\sigma_1}+\pi_{\sigma_c}] + (d-1)\cdot c!\cdot\sum_{j=2}^{c-1}\pi_{\sigma_j},$$

where

$$\sigma_j = (0, 2, .., c-j+1, n-1-j, .., n-2, n).$$

Proof. The graded Betti numbers of $K[\Delta]$ are given by Lemma 2.5.1 with $s = d = n-c$ and $t = 1$. Hence Theorem 2.5.3 yields the claim. □

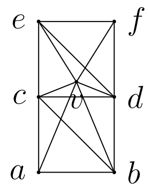
Remark 2.5.7. *In the special case of 3-dimensional stacked polytopes ($d = 3$), Corollary 2.5.6 establishes a conjecture made in the report of group 10.1 (V. Kalyankar, V. Lorman, S. Seo, M. Stamps, Z. Yang) at the MSRI summer workshop in commutative algebra 2011.*

We conclude by considering a specific instance of the last result.

Example 2.5.8. *Consider a 3-dimensional polytope on the seven vertices $a, \ldots, f, v$, obtained by stacking four 3-simplices with common vertex v. Denote by Δ its boundary complex. Its Stanley-Reisner ideal is*

$$I_\Delta = (ad, ae, af, be, bf, cf, bcv, cdv, dev).$$

Figure 2.2: Example Stacked Polytope



The Betti table of the Stanley-Reisner ring $K[\Delta]$ is

$$\beta(K[\Delta]) = \begin{array}{c|ccccc} \beta_{i,j} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & 6 & 8 & 3 & \cdot \\ 2 & \cdot & 3 & 8 & 6 & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & 1 \end{array}$$

It has the following Boij-Söderberg decomposition:

$$\beta(K[\Delta]) = 3 \cdot 4! \begin{array}{c|ccccc} \beta_{i,j} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \frac{1}{168} & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \frac{1}{20} & \frac{1}{12} & \frac{1}{24} & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \frac{1}{420} \end{array} + 2 \cdot 4! \begin{array}{c|ccccc} \beta_{i,j} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \frac{1}{210} & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \frac{1}{30} & \frac{1}{24} & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \frac{1}{60} & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \frac{1}{280} \end{array}$$

$$+ 2 \cdot 4! \begin{array}{c|ccccc} \beta_{i,j} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \frac{1}{280} & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \frac{1}{60} & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \frac{1}{24} & \frac{1}{30} & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \frac{1}{210} \end{array} + 3 \cdot 4! \begin{array}{c|ccccc} \beta_{i,j} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \frac{1}{420} & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \frac{1}{24} & \frac{1}{12} & \frac{1}{20} & \cdot \\ 3 & \cdot & \cdot & \cdot & \cdot & \frac{1}{168} \end{array}$$

Notice how one can read off the dimension and number of stacked simplices in the polytope from the coefficients in the decomposition.

Chapter 3 Further Observations in Boij-Söderberg Theory

3.1 Introduction

In this section we consider several complementary results on three different topics in Boij-Söderberg theory.

The change in complexity of the decompositions given in Theorem 2.3.2 to those of the quotient ring in 2.3.9 lead us to hope we might classify these changes. The change in the Betti table of a quotient ring to the corresponding ideal is just truncation of the Betti table. In section 3.2 we consider how the Boij-Söderberg decomposition changes when we truncate a pure diagram. The main result of this section is 3.2.2. We point out example 3.2.4 as an example of how the Boij-Söderberg decomposition of the quotient ring can be derived from the Boij-Söderberg decomposition of the ideal.

In [9, Remark 3.2] Boij and Söderberg showed how to construct a module of codimension 2 whose Betti table is the first integer multiple of a pure diagram when the shifts in the degree sequence are relatively prime. In section 3.3 we extend their result to the case where the degree sequences are not relatively prime. This shows that *every* integer table which is a multiple of pure diagram is actually the Betti table of a module in codimension 2. Since we know of integer multiples of pure diagrams which are not the Betti table of a pure module in codimension 3 this is as far as we can hope to extend this result by dimension.

3.2 Truncation of Pure Diagrams

We wish to describe the change in a decomposition when we pass from considering the graded free resolution of a quotient ring $k[x_1, ..., x_m]/I$ to considering the resolution of the ideal I as a module itself. As far as the Betti tables are concerned this transition is completely understood.

In the free resolution of the quotient ring the first free module is simply a copy of the ring, and hence the Betti table has a 1 in the zero column with zero shift. When we pass to the resolution of the ideal I as a module in itself the free resolution is the same as that of $k[x_1, ..., x_m]/I$ except that the ith free module in the resolution of I is the $(i + 1)$th free module in the resolution of $k[x_1, ..., x_m]/I$. This leads us to consider how the Boij-Söderberg decomposition changes when we consider the abstract diagram derived from removing the first column of a Betti table.

Definition 3.2.1. *Let $\pi_{\bar{d}}$ be the pure diagram with degree sequence $\bar{d} = (d_0, d_1, ..., d_n)$. Then define $\pi^*_{\bar{d}}$ to be the truncated diagram such that $\beta_{i,j}(\pi^*_{\bar{d}}) = \beta_{i+1,j}(\pi_{\bar{d}})$ for $i \geq 0$.*

It is important to notice $\pi^*_{\bar{d}} \neq \pi_{(d_1,...,d_n)}$. We derive the following result based on the definition above.

Theorem 3.2.2. *Let $\pi_{\bar{d}}$ be a pure diagram with degree sequence $\bar{d} = (d_0, d_1, ..., d_n)$, and let $\pi^*_{\bar{d}}$ be its truncated diagram. Let $l = [\pi_{\bar{d}}]_0 = \frac{1}{\prod_{i=1}^{n}(d_i - d_0)}$. Then*

$$\pi^*_{\bar{d}} = l \prod_{i=1}^{n-1}(d_i - d_0)\pi_{(d_1,d_2,...,d_n)} + ... + l \cdot (d_1 - d_0)\pi_{(d_1,d_2)} + l\pi_{(d_1)} \tag{3.2.1}$$

Proof. We consider the truncated diagram and decomposition at a particular position. For simplicity we may assume that $d_0 = 0$. Then let $[\pi^*_{\bar{d}}]_i$ be the entry corresponding to the i^{th} Betti number in $\pi_{\bar{d}}$. Hence it suffices to show:

$$[\pi^*_{\bar{d}}]_i = l\left([\pi_{(d_1)}]_i + d_1[\pi_{(d_1,d_2)}]_i + ... + \prod_{i=1}^{n-1} d_i[\pi_{(d_1,d_2,...,d_n)}]_i\right) \tag{3.2.2}$$

We notice that $[\pi_{(d_1,d_2,...,d_k)}]_i = 0$ if $k < i$. Then we collect all the terms.

$$[\pi^*_{\bar{d}}]_i = \prod_{j=1}^{n}\frac{1}{d_j}\sum_{k=i}^{n}\prod_{j=1}^{k-1} d_j \prod_{j=1,j\neq i}^{k}\frac{1}{|d_j - d_i|} \tag{3.2.3}$$

We write out the left hand side and reduce the right to get:

$$\prod_{j=0,j\neq i}^{n}\frac{1}{|d_i - d_j|} = \sum_{k=i}^{n}\prod_{j=k}^{n}\frac{1}{d_j}\prod_{j=1,j\neq i}^{k}\frac{1}{|d_j - d_i|} \tag{3.2.4}$$

Then we notice that if $j < i$ and $j \neq 0$ then $\frac{1}{d_i - d_j}$ appears in each term.

$$\frac{1}{d_i}\prod_{j=i+1}^{n}\frac{1}{d_j - d_i} = \sum_{k=i}^{n}\prod_{j=k}^{n}\frac{1}{d_j}\prod_{j=i+1}^{k}\frac{1}{d_j - d_i} \tag{3.2.5}$$

Then by finding a common denominator we get the following.

$$\frac{\prod_{j=i+1}^{n} d_j}{d_i \prod_{j=i+1}^{n} d_j(d_j - d_i)} = \frac{\sum_{k=i}^{n}\prod_{j=k+1}^{n}(d_j - d_i)\prod_{j=i}^{k-1} d_j}{d_i \prod_{j=i+1}^{n} d_j(d_j - d_i)} \tag{3.2.6}$$

Then we just need to show the numerators are equal. The left hand side counts the volume of the rectangular solid in $\mathbb{R}^{n-i}$ whose interior points satisfy $0 \leq x_{j-i} \leq d_j$ for $i+1 \leq j \leq n$. Then consider a single term on the right hand side re-written.

$$\prod_{j=k+1}^{n}(d_j - d_i)\prod_{j=i}^{k-1} d_j = \prod_{j=k+1}^{n}(d_j - d_i)\prod_{j=i+1}^{k-1} d_j \cdot d_i \tag{3.2.7}$$

This is the volume of the rectangular solid in $\mathbb{R}^{n-i}$ whose interior points $(x_1, ..., x_{n-i})$ satisfy $0 \leq x_{j-i} \leq d_j$ for $i+1 \leq j \leq k-1$, $0 \leq x_{k-i} \leq d_i$, and $d_j - d_i \leq x_{j-i} \leq d_j$ for $k+1 \leq j \leq n$. Given any point $(x_1, ..., x_{n-i})$ in the first rectangular solid we can find which of the right hand side blocks this point is in by looking for the last entry x_k such that $x_k \leq d_i$. The requirement that we look for the last such k also gives us that each of these blocks intersect only on their boundary and hence this is a subdivision of the first rectangular solid. Since this final equation is equivalent to the original expression this proves the decomposition. □

Example 3.2.3. *Consider the pure Betti table arising from the degree sequence* $(0, 3, 4, 5)$

$$60 \cdot \beta = \begin{array}{c|cccc} \beta_{i,j} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & 10 & 15 & 6 \end{array}$$

Then the decomposition of the truncation is below

$$60 \cdot \beta^* = \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 10 & 15 & 6 \end{array} = 12 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & \frac{1}{2} & 1 & \frac{1}{2} \end{array} + 3 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 1 & 1 & \cdot \end{array} + 1 \cdot \begin{array}{c|ccc} \beta_{i,j} & 0 & 1 & 2 \\ \hline 3 & 1 & \cdot & \cdot \end{array}$$

Since $l = 1$ *and* $d_1 = 3$,$d_2 = 4$.

It is interesting to note that we can reverse this process in the case of Betti tables of pure modules since we know the order of the pure diagrams. Given the decomposition of I we can recover the decomposition of $k[x_1, ..., x_n]/I$ by creating the longest pure diagrams first, and then working sequentially. This process is shown in the following linear example.

Example 3.2.4. *Consider the ideal* $I(F) = (x_1y_1z_1, x_1y_1z_2, x_1y_1z_3, x_1y_2z_1, x_1y_2z_2, x_2y_1z_1)$ *as in 2.3.3. We know that*

$$\beta(I(F)) = 4\pi_{(3,4,5)} + 3\pi_{(3,4)} + 1\pi_{(3)}$$

Since $I(F)$ *is an ideal we know this is the truncation of some quotient:*

$$4\pi_{(3,4,5)} + 3\pi_{(3,4)} + 1\pi_{(3)} = a\pi^*_{(0,3,4,5)} + b\pi^*_{(0,3,4)} + c\pi^*_{(0,3)} + d\pi^*_{(0)}$$

All that remains is to solve for the coefficients on the right. We know the leading coefficient of $\pi_{(0,3,4,5)}$ *is* $\frac{1}{3 \cdot 4 \cdot 5}$, *hence using the truncation formula we get:*

$$4\pi_{(3,4,5)} + 1\pi_{(3,4)} + \frac{1}{3}\pi_{(3)} = 20\left(\frac{1}{5}\pi_{(3,4,5)} + \frac{1}{20}\pi_{(3,4)} + \frac{1}{60}\pi_{(3)}\right) = 20\pi^*_{(0,3,4,5)}$$

$$2\pi_{(3,4)} + \frac{2}{3}\pi_{(3)} = 8\left(\frac{1}{4}\pi_{(3,4)} + \frac{1}{12}\pi_{(3)}\right) = 8\pi^*_{(0,3,4)}$$

Hence the Boij-Söderberg decomposition of $R/I(F)$ *is:*

$$\beta(R/I(F)) = 20\pi_{(0,3,4,5)} + 8\pi_{(0,3,4)}$$

as seen in 2.3.11

3.3 Pure Codimension 2 Betti Tables

If $\pi_{\bar{d}}$ is a pure diagram then some integer multiple of that pure diagram is actually the Betti table of a module, see [14]. However in general the construction in [14] does not give the first integer multiple. This leads us to ask if all integer multiples of a pure diagram are actually the Betti table of a pure module. This is not in fact true as is easily seen by considering the first integer multiple of the pure diagram $\pi_{(0,1,3,4)}$.

In this section we consider the question for pure diagrams with degree sequences of length 3. These correspond to the codimension 2 modules. We show by explicit construction that the first integer multiple of each of these is in fact the Betti table of a module. Boij and Söderberg showed this for degree sequences with no common factors in [9, Remark 3.2] and we extend their construction to those with common factors. We also provide some constructions which give us pure diagrams with longer lengths.

Theorem 3.3.1. *Let $\pi_{(d_0,d_1,d_2)}$ be a pure diagram. Then every table of integers that is a multiple of that pure diagram is actually the Betti table of a module.*

Proof. Without loss of generality let $d_0 = 0$. Then if $gcd(d_1, d_2 - d_1) = 1$ then let $I = (x^{d_1}, y^{d_1})^{d_2-d_1-1}$ and $J = (x^{d_2-d_1}, y^{d_2-d_1})^{d_1}$. By [9, Remark 3.2] we note $\beta(I/J(d_1(d_2-d_1-1)) = d_1 \cdot d_2 \cdot (d_2-d_1)\pi_{(0,d_1,d_2)}$. Note this is the first integer multiple of $\pi_{(0,d_1,d_2)}$ which has entries $\frac{1}{d_1 d_2}$, $\frac{1}{d_1(d_2-d_1)}$, and $\frac{1}{d_2(d_2-d_1)}$. Also note that any later integer multiple must be a multiple of this Betti table since we must at least clear the denominators of the pure diagram to get an integer multiple which means we must have that the coefficient is a multiple of $d_1 \cdot d_2 \cdot (d_2 - d_1)$.

If $gcd(d_1, d_2 - d_1) = d$ then we can generate this Betti table by considering the module generated in step one with degree sequence $(0, \frac{d_1}{d}, \frac{d_2}{d})$, call this module M where $M = I/J(s)$. Then we notice that if $F_\bullet$ is the free resolution of M then if we use the map $f : k[x, y] \longrightarrow k[x, y]$ such that $f(x) = x^d$ and $f(y) = y^d$ then $f(F_\bullet)$ is a free resolution of $f(M)$. We consider the module $f(M)((d-1)\cdot s)$. Since the resolution $F_\bullet$ was pure, then the resolution $f(F_\bullet)$ is pure and has degree sequence $(0, d_1, d_2)$. Notice that this Betti table is the smallest integer table along this ray and every integer point along this ray is again an integer multiple of this Betti table. □

Example 3.3.2. *We wish to construct a Betti table with degree sequence* $(0, 6, 15)$*. First we construct a module with degree sequence* $(0, 2, 5)$*. Let* $I = (x^2, y^2)^2$ *and* $J = (x^3, y^3)^2$*. Then the resolution of* I/J *is below:*

$$0 \longrightarrow R^2(-5) \xrightarrow{\varphi_2} R^5(-2) \xrightarrow{\varphi_1} R^3 \xrightarrow{\varphi_0} I/J(-4) \longrightarrow 0, \tag{3.3.1}$$

where

$$\varphi_0 = \begin{pmatrix} x^4 & x^2y^2 & y^4 \end{pmatrix}, \varphi_1 = \begin{pmatrix} x^2 & y^2 & 0 & 0 & 0 \\ 0 & -x^2 & y^2 & 0 & xy \\ 0 & 0 & -x^2 & y^2 & 0 \end{pmatrix}, \varphi_2 = \begin{pmatrix} -y^3 & 0 \\ x^2y & 0 \\ 0 & xy^2 \\ 0 & x^3 \\ x^3 & -y^3 \end{pmatrix}$$

Then using the map $f(x) = x^3$ *and* $f(y) = y^3$ *and shifting the degrees by multiplying the degree shift by 3 we get*

$$0 \longrightarrow R^2(-15) \xrightarrow{\varphi_2} R^5(-6) \xrightarrow{\varphi_1} R^3 \xrightarrow{\varphi_0} I/J(-12) \longrightarrow 0, \tag{3.3.2}$$

where

$$\varphi_0 = \begin{pmatrix} x^{12} & x^6y^6 & y^{12} \end{pmatrix}, \varphi_1 = \begin{pmatrix} x^6 & y^6 & 0 & 0 & 0 \\ 0 & -x^6 & y^6 & 0 & x^3y^3 \\ 0 & 0 & -x^6 & y^6 & 0 \end{pmatrix}, \varphi_2 = \begin{pmatrix} -y^9 & 0 \\ x^6y^3 & 0 \\ 0 & x^3y^6 \\ 0 & x^9 \\ x^9 & -y^9 \end{pmatrix}$$

Theorem 3.3.3. *Given a monomial ideal with pure resolution and* $\beta_{i,d_i} \neq 0$ *only in degrees* $0 = d_0, d_1, ..., d_n$ *where* $gcd(d_1, d_2 - d_1, ..., d_n - d_{n-1}) = 1$ *we can apply the map* $f : k[x_i] \longrightarrow k[x_i]$ *such that* $f(x_i) = x_i^d$ *to get a monomial ideal with pure resolution and degree sequence* $(0, d \cdot d_1, d \cdot d_2, ..., d \cdot d_n)$.

Proof. Let $(m_1, ..., m_k)$ be the generators of I. Then let L_I be the lcm lattice of $(m_1, ..., m_k)$ then $L_I = L_{f(I)}$ so by [15, Lemma 2.2] the total Betti numbers of these two modules are the same. □

Corollary 3.3.4. *Let* $\bar{d} = (0, d, d + k, d + 2k, ..., d + ck)$ *where* k *divides* d. *Then there exist an quotient ring whose Betti table is the smallest integer point on the Boij-Söderberg ray* $\pi_{\bar{d}}$.

Proof. Let $\frac{d}{k} = l$, then $\frac{\bar{d}}{k} = (0, l, l + 1, l + 2, ..., l + c)$. Let $m = (x_1, x_2, ..., x_c)$, then it is well known that the Betti table of R/m^l is the first multiple of the pure diagram $\pi_{\bar{d}}$. Then applying the map $x_i \mapsto x_i^k$ gives us the desired result. □

Chapter 4 A Cellular Resolution and Stacked Polytopes

4.1 Introduction

Cellular resolutions have been a topic of great interest in commutative algebra for the past decade (see [7], [2], [27], [11], [12], [15], [23] [21], [5]). We use the theory of cellular resolutions and polyhedral cell complexes as developed in [7] and [8]. The general concept is to interpret the maps in the free resolution of a monomial ideal as the boundary maps of some labeled cell complex. This gives some combinatorial meaning to the Betti numbers as the ith Betti numbers are now the $i-1$ dimensional faces of the cell complex.

There are a few general constructions such as the Taylor, Hull, and Scarf resolutions (see [21, 5] for details) but these are not always minimal, and the ideals for which these are minimal have been classified. Outside of these constructions a number of results have been published for specific families of ideals, most of which have linear resolutions. In [2] Biermann gives a recursive construction of a cell complex which supports a cellular resolution of the n-gon. Our construction is not the same beyond dimension 2. We also note that our construction is more explicit and we show it is in fact the boundary of a polytope.

In this chapter we construct a cellular resolution of the Stanley-Reisner ring of the n-gon in 4.4.1 and 4.5.2. This case is interesting because this is a Gorenstein ring, and hence its resolution is nonlinear.

In section 4.2 we introduce some of the basic notions of cellular resolutions and the family of Ferrers and specialized Ferrers ideals along with their resolutions given in [11] and [12].

In section 4.3 we introduce the notion of a c-polar self dual polytope. There are several definitions of what it means for a polytope in $\mathbb{R}^d$ to be self-dual. Our definition is stricter than all these and hence polar self-dual polytopes are a subset of self-dual polytopes in general. However we prove some results which show that this restriction is actually helpful in constructing polytopes with a desired face structure.

In Lemma 4.3.6 we give some necessary conditions on the face-poset of a polytope for such a polytope to have a polar self-dual embedding. Using this lemma we prove Theorem 4.3.7 which shows that the family of polar self-dual polytopes is strictly smaller than the family of self-dual polytopes. In particular we show that all the odd n-gons have polar self-dual embeddings and none of the even n-gons have such an embedding. In Lemma 4.3.13 we show that a c-polar self-dual polytope is reflexive if and only if $c = 1$.

In section 4.4 we explicitly construct the family of Ferrers polytopes 4.4.1. In 4.4.8 we prove that the Ferrers polytope in dimension d is d-polar self-dual. In 4.4.10 we establish an pairing of faces in the Ferrers polytope which we show in the next section models Gorenstein duality in the cellular resolution.

In section 4.5 we label the Ferrers polytope by monomials, see 4.5.2, to make it a labeled polytope. Then we show in 4.5.6 that this labeled polytope supports the minimal free resolution of the Stanley-Reisner ring of the n-gon. We extend this result in 4.5.9 to a cellular resolution of certain stacked polytope. Since the Betti numbers of all stacked d-polytopes with a fixed number of vertices are the same we can interpret the Betti numbers of Stanley-Reisner rings of all stacked polytopes as the f-vector of the appropriate Ferrers polytope.

4.2 Preliminaries

A *finite regular cell complex* X is a finite collection $\mathcal{C}$ of sets, called *cells*, such that their union is a non-empty topological space and

(i) $\emptyset \in \mathcal{C}$;

(ii) The cells in $\mathcal{C}$ are pairwise disjoint; and

(iii) For each non-empty cell $P \in \mathcal{C}$, there is a homeomorphism from a closed i-dimensional unit ball $B \in \mathbb{R}^i$ onto the closure of P that maps the interior of B onto P.

The number i in (iii) is uniquely determined by P and called the dimension of the cell P. The empty set has dimension -1. The *dimension* of $\mathcal{C}$ is the maximum dimension of its cells. A cell Q is a *face* of a cell P if Q is in the closure of P.

Each d-dimensional convex polytope $P \subset \mathbb{R}^d$ determines a d-dimensional regular cell complex. Its cells are the relative interiors of the faces of P.

A (finite) simplicial complex Δ on a finite vertex set V is a collection of subsets of V, including the empty set, that is closed under taking subsets. It is common to call the elements of Δ faces. It may be naturally identified with a regular cell complex whose open cells correspond the faces of Δ. The dimension of a face is its cardinality minus one. If Δ consists of all subsets of V, then it is the face complex of the (abstract) simplex V.

Definition 4.2.1. *Let Δ_1 and Δ_2 be two simplicial complexes on disjoint vertex sets V_1 and V_2, respectively. Then their* product *is the simplicial complex $\Delta_1 \times \Delta_2$ on the vertex set $V_1 \cup V_2$ defined by*

$$\Delta_1 \times \Delta_2 = \{A \cup B \mid A \in \Delta_1,\ B \in \Delta_2\}.$$

Note that $\Delta_1 \times \Delta_2$ is the face complex of a convex polytope P if Δ_1 and Δ_2 are face complexes of simplices. In fact, if $P_1 \subset \mathbb{R}^{d_1}$ and $P_2 \subset R^{d_2}$ are convex hulls of $d_1 + 1$ and $d_2 + 1$ linearly independent points, then $P \subset \mathbb{R}^{d_1+d_2}$ is a convex hull of $d_1 + d_2 + 2$ points of dimension $d_1 + d_2$. Its boundary is the set

$$\{A \cup V_2 \mid A \text{ a facet of } \Delta_1\} \cup \{V_1 \cup B \mid B \text{ a facet of } \Delta_2\}.$$

Each regular cell complex X admits an incidence function ε, where $\varepsilon(Q, P) \in \{1, -1\}$ if Q is a codimension one face of $P \in X$. It is determined by an orientation of the cells. We use it to define a complex of free modules.

Definition 4.2.2. *Let X be a finite regular cell complex of dimension d with an incidence function ε. Given a labeling of the vertices, i.e. 0-dimensional cells, of X by monomials in a polynomial ring $R = K[x_1, \ldots, x_N]$, one defines a labeling of each face $P \in X$ by the least common multiple m_P of the monomials that label the vertices of P. The* cellular complex $\mathcal{F}_X$ supported on X *is the complex of free $\mathbb{Z}^N$-graded R-modules*

$$\mathcal{F}_X\colon \quad 0 \to F_d \xrightarrow{\partial_d} F_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} = R \to 0,$$

where F_i is a free R-module with basis elements e_P, indexed by the i-dimensional cells P of X, and e_P has the same multidegree as the monomial label m_P. The differentials are the $\mathbb{Z}^N$-graded R-module homomorphisms defined by

$$\partial_i(e_P) = \sum_{Q \text{ facet of } P} \epsilon(Q,P)\frac{m_P}{m_Q}\, e_Q,$$

where $i = \dim P$.

If the complex $\mathcal{F}_X$ is acyclic, then it provides a free $\mathbb{Z}^N$-graded resolution of the image I of ∂_0, that is, of the ideal generated by the labels of the vertices of X. In this case, $\mathcal{F}_X$ is called a cellular resolution of I (or R/I), and one says that the cell complex X supports a free resolution of I.

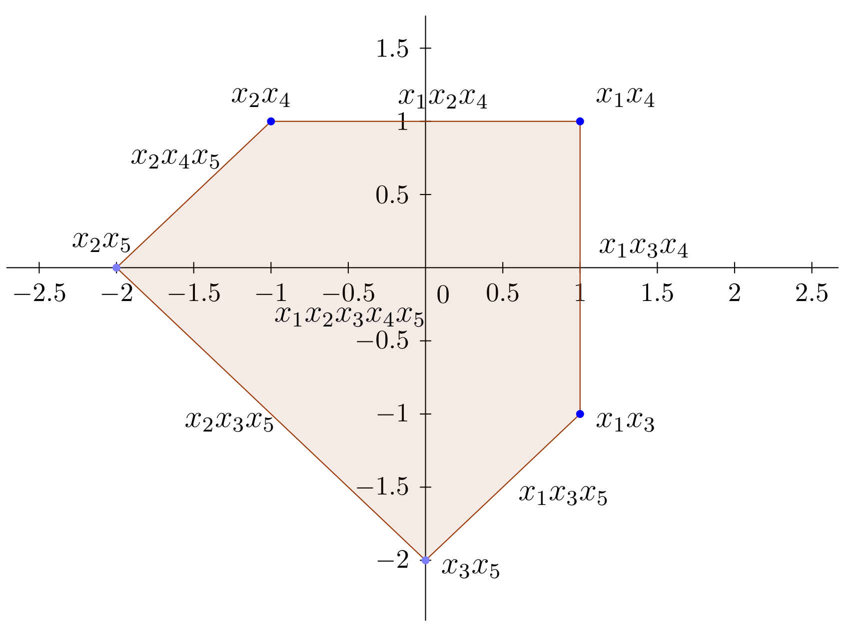
Each monomial ideal I admits a cellular resolution by a result of Taylor (see [13]). If I has s minimal monomial generators, then the Taylor resolution of I is the cellular resolution supported by a simplex with s vertices labelled by the generators of I. However, this resolution is only minimal if I is generated by an R-regular sequence.

For ease of notation, we typically write down a cellular resolution with its $\mathbb{Z}$-grading. If $M = \oplus_{i\in\mathbb{Z}} M_i = \oplus_{i\in\mathbb{Z}}[M]_i$ is a $\mathbb{Z}$-graded module, then $M(j)$ is obtained by just shifting the degrees so that $[M(j)]_i = M_{i+j}$.

We illustrate the above definition by some specific examples. They concern ideals with few generators that belong to classes of ideals that we study in this paper.

Example 4.2.3. *Let $I = (x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5)$ and consider the labeled cell complex X of the polytope below:*

Figure 4.1: Example Cellular Resolution



The associated cellular complex $\mathcal{F}_X$ is:

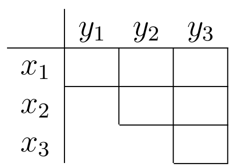
$$0 \to R(-5) \xrightarrow[\begin{bmatrix} x_2x_5 \\ x_3x_5 \\ x_1x_3 \\ x_1x_4 \\ x_2x_4 \end{bmatrix}]{} R^5(-3) \xrightarrow[\begin{bmatrix} x_4 & 0 & 0 & 0 & -x_5 \\ -x_3 & x_2 & 0 & 0 & 0 \\ 0 & -x_1 & x_5 & 0 & 0 \\ 0 & 0 & -x_4 & x_3 & 0 \\ 0 & 0 & 0 & -x_2 & x_1 \end{bmatrix}]{} R^5(-2) \xrightarrow[\begin{bmatrix} x_1x_3 & x_1x_4 & x_2x_4 & x_2x_5 & x_3x_5 \end{bmatrix}]{} I \to 0.$$

It is a graded minimal free resolution of the ideal I.

Recall that a partition $\lambda = (\lambda_1, ..., \lambda_n)$ is a sequence of weakly decreasing non-negative integers. To every partition λ we associate a *Ferrers diagram* F_λ. We here define F_λ to be the right-justified diagram of boxes in which there are λ_i boxes in the i^{th} row. We will display F_λ throughout this paper as an upper right adjusted diagram. We give the boxes a label by the ordered pair (row, column). We label the columns left to right beginning with one, so the right-most column has number λ_1. The *Ferrers ideal* to a partition λ is

$$I_\lambda = \{x_iy_j \mid (i,j) \in F_\lambda\} = (x_iy_j \mid \lambda_1 - \lambda_j + 1 \le i \le \lambda_1).$$

Example 4.2.4. *Given the partition $\lambda = (3, 2, 1)$, its Ferrers diagram is*



The corresponding Ferrers ideals is

$$I_\lambda = (x_1y_1, x_1y_2, x_1y_3, x_2y_2, x_2y_3, x_3y_3).$$

Every Ferrers ideal admits a minimal cellular resolution. In fact, Corso and Nagel defined a polyhedral cell complex X_λ which supports a minimal free resolution of I_λ for each partition λ (see [11, Theorem 3.2]). The faces of this cell complex are in bijection with the disjoint union of subsets of rows $r = \{i_1, \dots, i_k\}$ and columns $c = \{j_1, \dots, j_l\}$ such that $(i, j) \in F_\lambda$ for all $i \in r$ and all $j \in c$. The face with these rows and columns can be realized geometrically as the product of two simplices with k and l vertices, respectively. In fact, the cell complex X_λ can be realized as a subcomplex of a face complex of a product of two simplices.

Example 4.2.5. *The labelled polyhedral cell complex X_λ for the partition $\lambda = (3, 2, 1)$ is show in figure 4.2:*

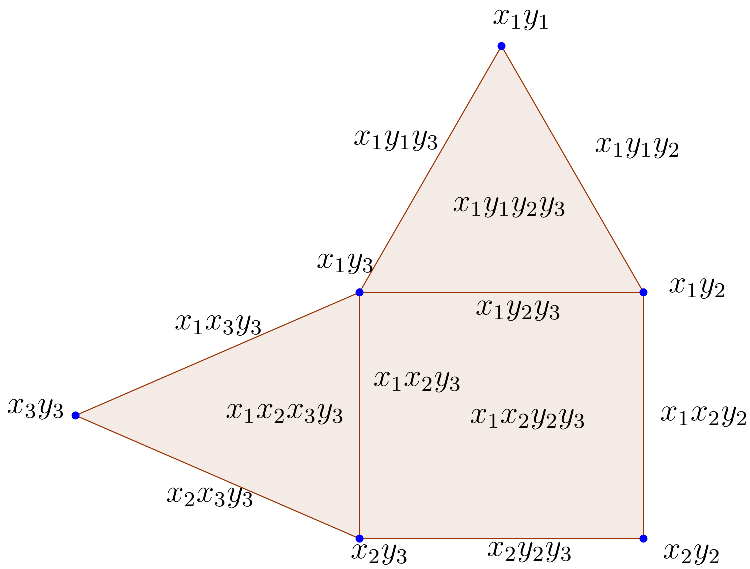
In [12] Corso and Nagel consider partitions λ such that their Ferrers diagram satisfies $i < j$ whenever $(i, j) \in F_\lambda$. Replacing the variables y_j by x_j one obtains from the Ferrers ideal I_λ the *specialized Ferrers ideal*

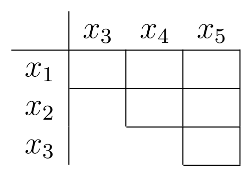
$$J_\lambda = (x_ix_j \mid (i,j) \in F_\lambda) = (x_ix_j \mid \lambda_1 - \lambda_j + 1 \le i \le \lambda_1).$$

By [12, Theorem 3.12], the cell complex X_λ also supports a minimal cellular resolution of J_λ, once the labeling is adjusted accordingly.

Example 4.2.6. *For the partition $\lambda = (3, 2, 1)$, the specialized Ferrers diagram is*

Figure 4.2: Cellular Resolution of a Ferrers Ideal





The corresponding specialized Ferrers ideal

$$J_\lambda = (x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5, x_3x_5)$$

has a minimal cellular resolution supported on the labeled cell complex in figure 4.3:

Every cell complex has a corresponding graded face-poset. Then if this cell complex is labeled with monomial labels, this corresponds to labeling the poset elements by the same monomial labels. Hence when we consider the map used in a cellular resolution:

$$\partial(G) = \sum_{\text{F facet of G}} \epsilon(G,F)\frac{m_G}{m_F}e_F$$

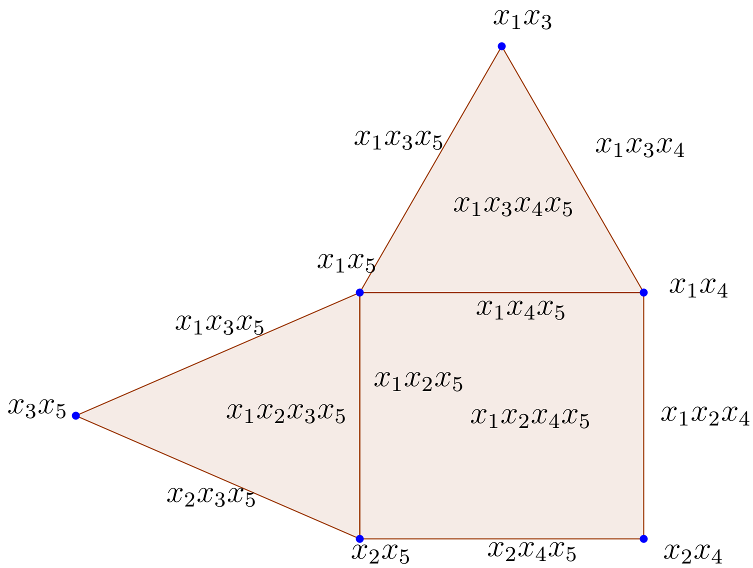
Where G, F are faces in the cell complex ,m_G, m_F are their corresponding monomial label, and $\epsilon(G,F)$ is an incidence function. Since this map depends only on the incidence function, the monomial labels, and the facets of a given face, then all this information is already stored in the corresponding face-poset. Hence we can rewrite the formula as:

$$\partial(G) = \sum_{\text{F} < \text{G}} \epsilon(G,F)\frac{m_G}{m_F}e_F \tag{4.2.1}$$

Where G, F are elements in the face-poset, m_G, m_F are their corresponding monomial labels, and $\epsilon(G,F)$ is an incidence function.

Hence to give a resolution by a poset it is sufficient to create a resolving poset with the diamond property. The diamond property is simply the property that poset interval

Figure 4.3: Cellular Resolutions of a Specialized Ferrers Ideal



$[x, y]$ where $dim(x) = n$ and $dim(y) = n + 2$ should be isomorphic to B_2 (the boolean algebra on 2 elements). If this property does not hold then either the mappings given by this formula do not give a complex, or the incidence function may fail to take values in the set $\{1, -1\}$.

This view of resolving by poset is actually natural given the construction of a resolution in general. If we consider the maps in a resolution as matrices then we can think of the columns as giving us the $dim = n$ elements of the poset and the rows giving us the $dim = n + 1$ elements of the poset. Then we can consider a $dim = n$ element as a parent of a $dim = n+1$ element whenever there is a nonzero entry in that position in the matrix.

Then every cellular resolution can be translated into a poset resolution through its face-poset.

Theorem 4.2.7. *Every resolution of a monomial ideal is a poset resolution.*

Proof. First consider that a monomial ideal is naturally finely graded ($\mathbb{Z}^n$ graded) and the resolution is finely graded. Then by considering the free modules in the poset we can construct a graded poset where the covering relations are given by $G \lessdot F$ if $im(\phi_n)_{e_G} \neq 0$. Given this fine grading if $R(-\bar{a})$ appears in the resolution we can label the corresponding poset element with the monomial $x^{\bar{a}}$. Then the map below is well defined.

$$\varphi_i(e_F) = \sum_{G \lessdot F} [G : F]\mathbf{x}^{a_F - a_G} e_G$$

Where $[G : F]$ is the coefficient that appears in the matrix representation of ϕ_n in the column and row associated with e_F and e_G. $\square$

Remark 4.2.8. *Every monomial ideal has a unique maximal poset resolution in the sense that there is a unique poset with the most possible relations and every poset resolution of that monomial ideal is a sub-poset of the maximal poset in the sense that fewer elements* *are related. This poset is the poset we get by considering the multigraded Betti numbers as a set of elements labeled by the multigrade. Then every monomial appearing in the resolution is greater than every element whose monomial label divides its own. This poset then has the most possible connections since we cannot connect two monomials if the lower one does not divide the higher.*

Construction 4.2.9. *Begin with any minimal poset resolution P of I. Then let v_{m_i} be the vector in the resolution corresponding to the i^{th} monomial m in the multigraded minimal free resolution. Let $v_{m_1}, ..., v_{m_l}$ be the set of all vectors that correspond to the monomial m in the multigraded minimal free resolution. Then choose $\beta_1, ..., \beta_l$ to be sufficiently general coefficients. If we replace*

$$v_{m_i} \mapsto v_{m_i} + \sum_{k \neq i} \beta_k v_{m_k}$$

then this changes the resolving poset P to a new poset P' where P' is the graded poset with the same graded elements as P in each grade and P' has covering relations $x \leq y$ for any two poset elements if $m_x | m_y$ and $gr(x) = gr(y) - 1$.

4.3 Polar Self-Dual Polytopes

The study of self-dual polytopes is an attractive topic. Here we introduce a subclass of these polytopes that we call polar self-dual polytopes. Let us first recall the standard concept.

Definition 4.3.1. *An (abstract) polytope P is said to be* self-dual *if its face-poset is isomorphic to its dual face-poset.*

For an embedded polytope, there is a stronger condition. We use the standard inner product of $\mathbb{R}^n$.

Definition 4.3.2. *Let P be a polytope in $\mathbb{R}^d$. Then*

$$P^* = \{x \in \mathbb{R}^d \mid x \cdot y \geq -1 \text{ for all } y \in P\}$$

is the dual polyhedron *of P. We note that when the origin is interior to P then P^* is a polytope and we call it the* dual polytope.

Remark 4.3.3. *These two definitions align with the use of the word dual since the face-poset of P^* is the dual of the face-poset of P when P^* is a polytope. If P is self-dual then we usually write $P \cong P^*$. However, P in general is not equal to P^* as a subset of $\mathbb{R}^d$.*

We now relax the condition $P = P^*$ somewhat.

Definition 4.3.4. *Let P be a polytope in $\mathbb{R}^d$, and let $c > 0$ be a real number. Then we define the c-*dual *of P as*

$$P^*_c = \{x \in \mathbb{R}^d \mid x \cdot y \geq -c \text{ for all } y \in P\}.$$

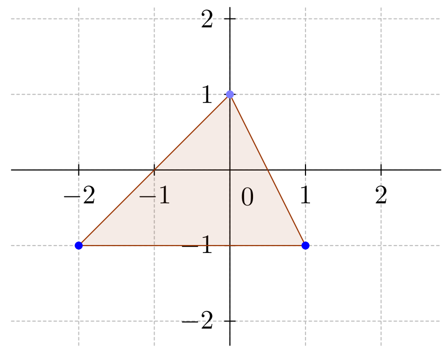
*The polytope P is said to be c-*polar self-dual *if $P = P^*_c$. We simply say P is* polar self-dual *if c is understood or not relevant.*

Note that this definition is dependent on the particular embedding of P. Observe also that a polytope is c-polar self-dual if some scalar multiple of it is equal to its dual. More precisely, of $P \subset \mathbb{R}^d$ is c-polar self-dual, then $(\frac{1}{\sqrt{c}}P)^* = \frac{1}{\sqrt{c}}P$.

We present several examples of polar self-dual polytopes. The first one is very simple.

Example 4.3.5. *The triangle conv*$\{(0,1),(1,-1),(-2,-1)\}$ *is 1-polar self-dual.*

Figure 4.4: Example of a Polar Self-Dual Polytope



We wish to point out that not all self-dual polytopes have a polar self-dual embedding. The following result gives a necessary condition on the face poset of polar self-dual polytopes.

Lemma 4.3.6. *Let* $\mathcal{P} = (\mathcal{P}, <)$ *be the face poset of a c-polar self-dual polytope, partially ordered by inclusion. Then there is an order reversing automorphism* $\iota : \mathcal{P} \to \mathcal{P}$ *such that the following conditions hold:*

1. $\iota \circ \iota = id_{\mathcal{P}}$; *and*

2. *Each face* $F \in \mathcal{P}$ *satisfies* $F \cap \iota(F) = \emptyset$.

Proof. The existence of ι and Condition (1) are an immediate consequence of polar self-duality.

To see the second condition, consider any vertex a of a face $F \in \mathcal{P}$. Then $a \cdot b = -c < 0$ for each point b of $\iota(F)$. Since $a \cdot a > 0$ we get $a \notin \iota(F)$. $\square$

There are 2-dimensional polytopes whose face posets do not admit such an involution. The following result characterizes these.

Theorem 4.3.7. *Let* $P \subset \mathbb{R}^2$ *be a 2-dimensional convex polytope with* n *vertices. Then there is a polar self-dual polytope in* $\mathbb{R}^2$ *whose face poset is combinatorially equivalent to the one of* P *if and only if* n *is odd.*

Proof. Let $P \subset \mathbb{R}^2$ be a 2-dimensional polar self-dual polytope. For this argument, it is more convenient to enumerate the vertices of P from zero to $n-1$. Each vertex v_i determines a dual line $\{x \in \mathbb{R}^2 \mid x \cdot v_i = -c\}$. We denote this line by ℓ_i. Then there is the following easy, but useful observation. Two lines ℓ_i and ℓ_j meet in a vertex v_k of P if and only if i and j are the vertices of the edge that is dual to v_k.

Assume first that n is odd. Fix any $c > 0$. We will recursively construct certain polyhedra. The last one will be the desired c-polar self-dual polytope.

Write $n = 2m + 1$. Because of Example 4.3.5, we may assume $n \geq 5$. Choose as vertices v_0 and v_1 any points in the plane other than the origin such that the line through v_0 and v_1 does not pass through the origin. The dual lines ℓ_0 and ℓ_1 meet in a point. This is the vertex v_{m+1}. Thus, ℓ_{m+1} is the line through v_0 and v_1. Let P_1 be the intersection of the half spaces determined by the lines ℓ_0, ℓ_1, and ℓ_{m+1} that contain the origin. We continue to construct polyhedra $P_2, \ldots, P_m$ such that P_i is the intersection of P_{i-1} and two other half-spaces and P_i has vertices $v_0, \ldots, v_i, v_{m+1}, \ldots, v_{m+i}$.

In order to describe the additional half-spaces, assume the polyhedra $P_1, \ldots, P_{i-1}$ have been constructed, where $i \leq m$. Let H be the half-space bounded by the line through vertices v_{m+1} and v_{i-1} that does not contain the origin. If $i < m$, then pick the vertex v_i in the interior of the polyhedron $P_{i-1} \cap H$. If $i = m$, then pick the vertex v_m in the open interior of the edge of P_{m-1} that is supported on the line ℓ_0.

With this choice of v_i, the vertex v_{m+i} is the intersection of the lines ℓ_{i-1} and ℓ_i. Thus, ℓ_{m+i} is the line through v_{i-1} and v_i. The polyhedron P_i is now defined as the intersection of P_{i-1} and the half-spaces bounded by the lines ℓ_i and ℓ_{m+i} that contain the origin.

It is not difficult to see that P_m is the desired c-polar self-dual polyhedron. We leave the details to the reader.

Second, assume that $P \subset \mathbb{R}^2$ is a polar self-dual polytope with n vertices, where n is even. Label its vertices clock-wise so that the vertices $v_0, v_1, \ldots, v_{n-1}$ are adjacent. Let $(j, j+1)$ be the edge that is dual to the vertex v_0. It is adjacent to the dual edge of v_1 by the observation above. There are two cases.

Assume first $(j+1, j+2)$ is the edge dual to vertex v_1. It then follows that, for each vertex v_i, its dual edge is $(\overline{j+i}, \overline{j+i+1})$, where $\overline{k}$ denote the remainder of an integer k on dividing by n. In particular, the edges $(n-1, 0)$ and $(0, 1)$ are dual to the vertices v_{n-j-1} and v_{n-j}. It follows that the the edge $(n-j-1, n-j)$ must be dual to the vertex v_0. However, by assumption its dual edge is $(j, j+1)$. This forces $n = 2j+1$, which is impossible since n is even.

In the second case, where the edge $(j-1, j)$ is dual to v_1 we get that, for each i, the vertex v_i is dual to the edge $(\overline{j-i}, \overline{j-i+1})$. Applying this for $i = \lceil \frac{j}{2} \rceil$, it follows that the vertex $v_{\lceil \frac{j}{2} \rceil}$ is a vertex of its dual edge. This is impossible (see Lemma 4.3.6). $\square$

Since every 2-dimensional convex polytope is self-dual, the above theorem shows that the class of polar self-dual polytopes is strictly smaller than that of self-dual polytopes.

Now we observe that the face complex of each simplex can be realized as the face complex of a polar-self-dual polytope.

Proposition 4.3.8. *There is a polar self-dual embedding of the simplex in each dimension.*

Proof. Choose vertices $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$ recursively as follows:

1. Place $v_1 \neq 0$.

2. Assume $v_1, \ldots, v_{i-1}$ are chosen. Then place v_i outside of the affine span of $v_1, \ldots, v_{i-1}$ such that $v_k \cdot v_i = -c$ for all $k = 1, \ldots, i-1$.

The convex hull of $v_1, \ldots, v_{n+1}$ is clearly a c-polar self-dual polytope of dimension n. $\square$

Remark 4.3.9. *The two previous constructions call for solutions to linear equations and linear inequalities. Hence, we can make these polytopes rational. In fact, we give explicit examples of simplices with integer coordinates below.*

Example 4.3.10. *Let $c > 0$ be a real number, and let d be a positive integer. Then define points $v_1, \dots, v_{d+1} \in \mathbb{R}^d$ by*

$$v_k = (1, \dots, 1, -c-k+1, 0, \dots,),$$

where the entry $-c-k+1$ is in position k. In particular,

$$v_1 = (-c, 0, \dots, 0) \text{ and } v_{d+1} = (1, \dots, 1).$$

Then one checks easily that $v_i \cdot v_j = -c$ for all $i \neq j$. Hence the convex hull of the points $v_1, \dots, v_{d+1} \in \mathbb{R}^d$ is a c-polar self-dual d-dimensional simplex because of the following observation.

Lemma 4.3.11. *If points $v_1, \dots, v_{d+1} \in \mathbb{R}^d$ satisfy $v_i \cdot v_j = -c$ for all $i \neq j$, then their convex hull is a c-polar self-dual d-simplex.*

Proof. For the convenience of the reader we provide the argument.

Assume there are real numbers $\lambda_1, \dots, \lambda_{d+1}$ such that

$$\sum_i \lambda_i v_i = 0, \quad \sum_i \lambda_i = 0, \text{ and } \lambda_j \neq 0.$$

Then

$$0 = v_j \cdot \left(\sum_i \lambda_i v_i \right) = \lambda_j (v_j \cdot v_j) - c \sum_{i \neq j} \lambda_i = \lambda_j (v_j \cdot v_j + c),$$

which is a contradiction because $v_j \cdot v_j > 0$. □

Remark 4.3.12. *Using the notation of Example 4.3.10, observe that the d-dimensional 1-polar self-dual simplex $\mathrm{conv}\{v_1, \dots, v_{d+1}\} \subset \mathbb{R}^d$ has normalized volume $d+1$. This follows by evaluating the determinant with row vectors $v_i - v_1$, where $i = 2, \dots, d+1$.*

We close this section by pointing out a relation to an important class of polytopes. Recall that a d-polytope with interior point $(0, ..., 0)$ is *reflexive* if it is a lattice polytope whose supporting hyperplanes can be written in the form $\sum_{i=1}^d h_i x_i = 1$ where the $h_i \in \mathbb{Z}$ and have no common factors. Reflexive polytopes were introduced by Batyrev in [3, Definition 4.1.4,4.1.5], in order to construct mirror families as predicted by mirror symmetry.

Lemma 4.3.13. *A c-polar self-dual lattice polytope is reflexive if and only if $c = 1$.*

Proof. If $P = P_1^*$ and it is a lattice polytope, then it is reflexive by definition.

If P is reflexive and $P = P_c^*$, then each defining hyperplane supported on a facet is of the form $\sum_{i=1}^d h_i x_i = c$, where $(h_1, ..., h_d)$ is a vertex of P. The defining hyperplanes of P_1^* are $\sum_{i=1}^d h_i x_i = 1$. By [3, Theorem 4.1.6] if P^* is reflexive then P_1^* is also reflexive.

If the h_i have no common factor, then the supporting hyperplanes of P are not of the correct form, and hence P is not reflexive. If the h_i have common factor a, then we can rewrite the hyperplane $\sum_{i=1}^d h_i x_i = 1$ as $\sum_{i=1}^d \frac{h_i}{a} x_i = \frac{1}{a}$, and hence P_1^* is not reflexive. Hence P is reflexive only when $c = 1$. □

In this case the two mirror partners are in fact equal.

4.4 A Family of d-Polar Self-Dual Polytopes

In this section we explicitly construct, for each positive integer d, a d-polar self-dual polytope of dimension d. This family shows that the set of polar self-dual polytopes is richer than just a set of simplices. We show in the next section that his particular family support a cellular resolution of the $d+3$-gon.

Construction 4.4.1. *Fix an integer $d \geq 1$. We construct a d-dimensional polytope as the convex hull of specified points. Let T_d be the Ferrers tableau associated with the partition $\lambda = (d, d-1, \ldots, 1)$. Note that there is a box in row n and column m of T_d if and only if $1 \leq n \leq m \leq d$. We define two kinds of points that, as we will show, are the vertices of the polytope.*

1. *For $(n, m) \in T_d$, define the point $Q_{n,m}$ by*

$$Q_{n,m} = (0, \ldots, 0, -d+n-1, 1, \ldots, 1, -m, 0, \ldots, 0) \in \mathbb{R}^d,$$

where the first 1 is in position n and the last 1 is in position m. In particular, we get

$$Q_{1,m} = (1, ..., 1, -m, 0, \ldots, 0) \quad \text{if } n = 1 \leq m < d,$$
$$Q_{n,d} = (0, , 0, -d+n-1, 1, \ldots, 1) \quad \text{if } 2 \leq n \leq m = d,$$

and

$$Q_{1,d} = (1, \ldots, 1).$$

Observe that each of these $\binom{d+1}{2}$ points has at most two negative coordinates and each positive coordinate equals one.

2. *We also use the d points $L_1 = (-d, 0, \ldots, 0), L_2 = (0, -d, 0, \ldots, 0), \ldots, L_d = (0, \ldots, 0, -d)$ in $\mathbb{R}^d$.*

Let $\mathrm{V}_d = \{Q_{n,m} \mid (n,m) \in T_d\} \cup \{L_i \mid 1 \leq i \leq d\}$. We call the convex hull of V the Ferrers polytope *$\mathcal{P} = \mathcal{P}_d \subset \mathbb{R}^d$. It has dimension d because the convex hull of the points $L_1, \ldots, L_d$, and $Q_{1,d}$ is a d-dimensional simplex. (Theorem 4.4.8 below shows that its vertex set is in fact V_d.)*

Note that the 2-dimensional Ferrers polytope $\mathcal{P}_2$ is depicted in Example 4.2.3.

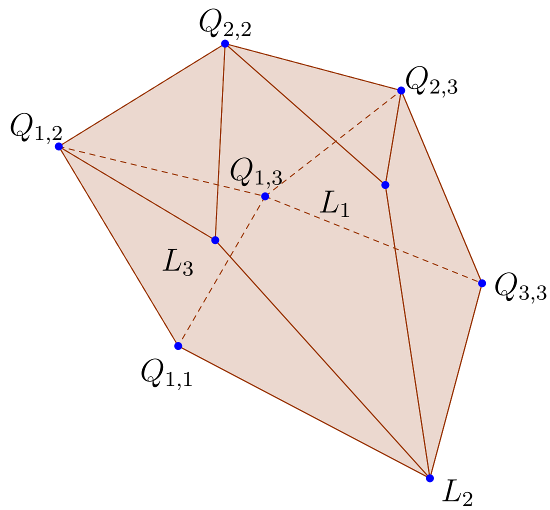
Recall that the cell complex X_λ described above Example 4.2.5 provides a cellular resolution of the Ferrers ideal I_λ. Using the points $Q_{n,m}$, we provide a geometric realization of this cell complex in $\mathbb{R}^d$ if $\lambda = (d, d-1, \ldots, 1)$ (see Proposition 4.4.5 below). Its faces are products of simplices. This fact becomes more apparent if we arrange the points in a Ferrers tableau. We illustrate this by two examples.

Example 4.4.2. *Let $d = 3$. Below we give the coordinates of the vertices in the Ferrers cell complex X_λ by presenting them in a Ferrers tableau. To the right of it we list the points L_1, L_2, and L_3.*

$(1,-1,0)$	$(1,1,-2)$	$(1,1,1)$	$(-3,0,0)$
	$(-2,1,-2)$	$(-2,1,1)$	$(0,-3,0)$
		$(0,-1,1)$	$(0,0,-3)$

The convex hull of these nine points is our 3-dimensional Ferrers polytope $\mathcal{P}_3$. This polytope is combinatorially equivalent to the one pictured below.

Figure 4.5: The 3-Dimensional Ferrers Polytope



Example 4.4.3. *Consider the case $d = 4$. Again we present the points $Q_{n,m}$ in a Ferrers tableau and the points $L_1, \ldots, L_4$ to its right.*

$(1,-1,0,0)$	$(1,1,-2,0)$	$(1,1,1,-3)$	$(1,1,1,1)$	$(-4,0,0,0)$
	$(-3,1,-2,0)$	$(-3,1,1,-3)$	$(-3,1,1,1)$	$(0,-4,0,0)$
		$(0,-2,1,-3)$	$(0,-2,1,1)$	$(0,0,-4,0)$
			$(0,0,-1,1)$	$(0,0,0,-4)$

The convex hull of these points is the 4-dimensional Ferrers polytope $\mathcal{P}_4$.

We will show that the following polytopes generate a polyhedral cell complex.

Definition 4.4.4. *Fix $d \geq 1$ and adopt the notation of Construction 4.4.1. For $k = 1, \ldots, d$, define the polytopes*

$$\mathcal{F}_k = conv\{Q_{n,m} \mid 1 \leq n \leq k \leq m \leq d\}.$$

Proposition 4.4.5. *The polytopes $\mathcal{F}_1, \ldots, \mathcal{F}_d$ form the facets of a polyhedral cell complex Y_d that we call the $(d-1)$-dimensional* Ferrers complex. *Each facet has dimension $d-1$ and is the product of two simplices. If $i \leq j$, then*

$$\mathcal{F}_i \cap \mathcal{F}_j = conv\{Q_{n,m} \mid 1 \leq n \leq i \text{ and } j \leq m \leq d\},$$

which is a $(d+i-j-1)$-dimensional face of $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively.

Furthermore, the polytope $\mathcal{F}_k$ is contained in the hyperplane defined by $x_k = 1$.

Proof. First note that $\mathcal{F}_1$ and $\mathcal{F}_d$ are $(d-1)$-simplices. Indeed, projecting the points $Q_{1,1}, \ldots, Q_{1,d}$ onto the hyperplane $\{x_1 = 0\}$, we get the vertices of a 1-polar self-dual simplex by Lemma 4.3.11. We argue similarly for $\mathcal{F}_d$ by projecting the points $Q_{1,d}, \ldots, Q_{d,d}$ onto the hyperplane $\{x_d = 1\}$.

Second, fix k such that $2 \leq k < d$. Then the last $d-k$ coordinates of the points $Q_{1,k}, \ldots, Q_{1,d}$ give the vertices of a k-polar self-dual $(d-k)$-simplex (see Lemma 4.3.11). Similarly, the first $(k-1)$ coordinates of the points $Q_{1,d}, \ldots, Q_{k,d}$ determine the vertices of a $(d-k+1)$-polar self-dual $(k-1)$-simplex. The product of these two simplices is $\mathcal{F}_k$, considered as a polytope in $\{x_k = 1\} \cong \mathbb{R}^{d-1}$.

Finally, notice that $\mathcal{F}_i \cap \mathcal{F}_j = \mathcal{F}_i \cap \{x_j = 1\} = \mathcal{F}_j \cap \{x_i = 1\}$. □

This result implies that Y_d is combinatorially equivalent to the complex X_λ, where $\lambda = (d, d-1, \ldots, 1)$. Thus, the Ferrers complex Y_d is an embedded realization of X_λ. We will use this when establishing properties of the Ferrers polytopes.

Corollary 4.4.6. *The Ferrers complex Y_d is homeomorphic to a ball of dimension $d-1$.*

Proof. This is true because, for each $i \leq d$, the complex Z_i with facets $\mathcal{F}_1, \ldots, \mathcal{F}_i$ is homeomorphic to a $(d-1)$-dimensional ball. Indeed, Proposition 4.4.5 shows that $Z_i \cap \mathcal{F}_{i+1}$ is a $(d-2)$-dimensional polytope, which implies our claim. □

Proposition 4.4.5 also provides the following technical observation that turns out useful.

Corollary 4.4.7. *Fix $k \in \{1, \ldots, d\}$, and let i, j be any integers such that $1 \leq i \leq k \leq j \leq d$. Then the coordinate vectors of the d points $Q_{i,k}, Q_{i,k+1}, \ldots, Q_{i,d}, Q_{1,j}, Q_{2,j}, \ldots, Q_{k,j}$ are linearly independent. (The point $Q_{i,j}$ is listed twice for convenience.)*

Proof. Observe that the last $d-k$ coordinates of $Q_{i,m}$ and $Q_{1,m}$ agree if $k \leq m \leq d$ and that the first $k-1$ coordinates of $Q_{n,j}$ and $Q_{n,d}$ are the same if $1 \leq n \leq k$. Hence, the points $Q_{i,k}, Q_{i,k+1}, \ldots, Q_{i,d}$ are the vertices of a $(d-k)$-simplex, and $Q_{1,j}, Q_{2,j}, \ldots, Q_{k,j}$ are the vertices of a $(k-1)$-simplex. It follows that the projections of the d considered points onto the hyperplane $\{x_k = 1\}$ are the vertices of a $(d-1)$-dimensional polytope in this hyperplane. Hence, their coordinate vectors are linearly independent in $\mathbb{R}^d$. □

Note that the d points considered in the above result are the vertices of the polytope $\mathcal{F}_k$ that are in row i and column j of the Ferrers tableau presentation of all the points $Q_{n,m}$.

We are ready for the main result of this section.

Theorem 4.4.8. *Each Ferrers polytope $\mathcal{P}_d \subset \mathbb{R}^d$ (see Construction 4.4.1) is d-polar self-dual. Its vertex set is V_d.*

Proof. Fix $d > 0$, and consider the Ferrers polytope $\mathcal{P} = \mathcal{P}_d$ as well as the set $V = V_d$ of points. We have to show that $\mathcal{P} = \{x \in \mathbb{R}^d \mid x \cdot y \geq -d \text{ for all } y \in \mathcal{P}\}$. We divide the argument in to several steps.

1. First we show that all the points in V are vertices of P. To this end we prove that, for each point $v \in V$, there is a linear functional that is maximized at v.

   It $n \neq 1$ and $m \neq d$, then define $H_{Q_{n,m}} = (0, , 0, -\frac{1}{d}, 1, ..., 1, -\frac{1}{d}, 0, , 0) \in \mathbb{R}^d$, where the first 1 is in position n and the last 1 is in position m. If $n = 1$, then set $H_{Q_{1,m}} = (1, ..., 1, -\frac{1}{d}, 0, , 0)$, and if $m = d$ we put $H_{Q_{n,d}} = (0, , 0, -\frac{1}{d}, 1, ..., 1)$. Furthermore, set $H_{Q_{1,1}} = (1, \ldots, 1) \in \mathbb{R}^d$, and let $H_{L_1} = (-d, 0, \ldots, 0), H_{L_2} =$

$(0, -d, 0, \dots, 0), \dots, H_{L_d} = (0, \dots, 0, -d)$. Using the structure of the coordinates of the points in V, one sees that, for each $v \in V$,

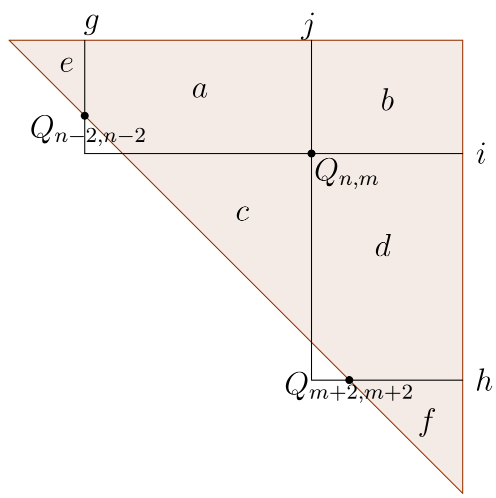
$$H_v \cdot v > H_v \cdot w \quad \text{for all } w \in V \setminus \{v\}.$$

Hence, each point in V is in fact a vertex of $\mathcal{P}$, and so V is the vertex set of $\mathcal{P}$.

2. We determine the dot product of each pair of vertices v, w of $\mathcal{P}$. We show that $v \cdot w \geq -d$ and record the cases, where this is in fact an equality.

   Fix a vertex $Q_{n,m}$. Then we consider ten different cases for choosing a point $Q_{k,l}$. The corresponding ten regions are shown in the picture below. Depending on n and m, some of these regions can be empty.

   Figure 4.6: Duality Regions in the Ferrers Polytope



   a) If $k < n$ and $n-2 < l < m$, then $Q_{n,m} \cdot Q_{k,l} = -d+n-1+l-n+1-l = -d$.
   b) If $k < n$ and $m < l$, $l \leq d$, then $Q_{n,m} \cdot Q_{k,l} = -d+n-1+m-n+1-m = -d$.
   c) If $n < k$ and $l < m$, $k < l$, then $Q_{n,m} \cdot Q_{k,l} = -d+k-1+l-k+1-l = -d$.
   d) If $n < k < m+2$ and $m < l$, then $Q_{n,m} \cdot Q_{k,l} = -d+k-1+m-k+1-m = -d$.
   e) If $l < n-2$, then $Q_{n,m} \cdot Q_{k,l} = 0 > -d$.
   f) If $m+2 < k$, then $Q_{n,m} \cdot Q_{k,l} = 0 > -d$.
   g) If $l = n-2$, then $Q_{n,m} \cdot Q_{k,l} = (-l)(-d+n-1) > 0 > -d$.
   h) If $k = m+2$, then $Q_{n,m} \cdot Q_{k,l} = (-m)(-d+k-1) > 0 > -d$.
   i) Assume $k = n$. If $l = m$, then clearly $Q_{n,m} \cdot Q_{k,l} > 0$. Otherwise, we may assume $l < m$. Then

$$Q_{n,m} \cdot Q_{k,l} = \begin{cases} (-d+n-1)^2 + l - n + 1 - l & \text{if } 1 < n \\ 0 & \text{if } 1 = n. \end{cases}$$

Since $(-d+n-1)^2-n+1 \geq -d+2 > -d$, we set that always $Q_{n,m} \cdot Q_{k,l} > -d$ in this case.

j) If $l = m$ and $n < k$, then

$$Q_{n,m} \cdot Q_{k,l} = \begin{cases} -d+k-1+m-k+1+m^2 & \text{if } m < d \\ 0 & \text{if } m = d. \end{cases}$$

It follows that $Q_{n,m} \cdot Q_{k,l} > -d$.

These considerations show that, for all points $Q_{n,m}, Q_{k,l} \in V$,

$$Q_{n,m} \cdot Q_{k,l} \geq -d,$$

and equality is true if and only if one of the conditions (a) - (d) is satisfied.

It remains to consider the points $L_1, \ldots, L_d$. Notice $L_i \cdot L_j = 0$ if $i \neq j$.

Furthermore, for each point $Q_{k,l}$, we get $L_j \cdot Q_{k,l} \geq -d$, where equality is true if and only if the j-th coordinate of $Q_{k,l}$ is 1. This yields

$$L_j \cdot Q_{k,l} = -d \ \text{ if and only if } k \leq j \leq l. \tag{4.4.1}$$

3. For each point $v \in V$, define a half-space

$$D_v^+ := \{x \in \mathbb{R}^n \mid x \cdot v \geq -d\},$$

and denote by D_v its boundary hyperplane. Simplifying notation, denote the d-dual of $\mathcal{P}$ by $\mathcal{P}^*$. It is

$$\mathcal{P}^* = \bigcap_{v \in V} D_v^+.$$

Hence, Step (2) implies $\mathcal{P} \subset \mathcal{P}^*$.

4. Now we show that each vertex of $\mathcal{P}$ is the intersection of d supporting hyperplanes of $\mathcal{P}^*$ whose normal vectors are linearly independent. Since $\mathcal{P} \subset \mathcal{P}^*$, it follows in particular that V is a subset of the vertices of $\mathcal{P}^*$.

To see this let us simplify our notation and set $D_{i,j} := D_{Q_{i,j}}$ and $D_{i,j}^+ := D_{Q_{i,j}}^+$.

We begin by observing that each point L_j is indeed a vertex of $\mathcal{P}^*$ because

$$L_j \in \bigcap_{j<m\leq d} D_{j,m} \cap \bigcap_{1\leq n\leq j} D_{n,j}$$

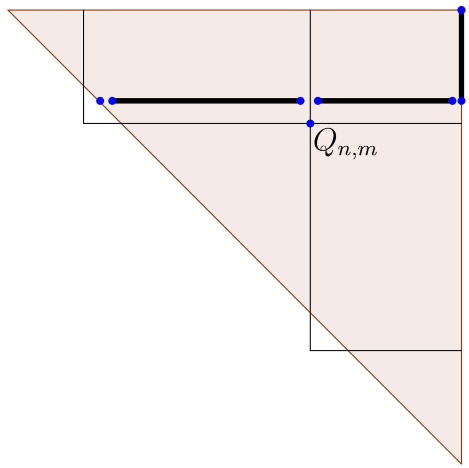
and the normal vectors of these d hyperplanes are linearly independent by Corollary 4.4.7.

Let now $Q_{n,m} \in \mathbb{R}^d$ be any point among the other vertices of $\mathcal{P}$. We consider four cases.

Case 1: Assume $1 < n$ and $m < d$. Then Step (2) shows that

$$Q_{n,m} \in \bigcap_{n-1\leq j<m} D_{1,j} \ \cap \bigcap_{m<j\leq d} D_{1,j} \ \cap \bigcap_{1\leq i<n} D_{i,d} \ \cap \ D_{L_n}.$$

Figure 4.7: Hyperplane Choices for a Vertex



The following picture illustrates the choice of the normal vectors of the first $d-1$ hyperplanes. These normal vectors are linearly independent by Corollary 4.4.7. Moreover, they are all contained in the hyperplane $x_{n-1} = 0$. Since this hyperplane does not contain L_n we conclude that the above intersection of d hyperplanes is the point $Q_{n,m}$, as desired.

Case 2: Assume $1 = n$ and $m < d$. Then we argue similarly to see that

$$\{Q_{1,m}\} \in \bigcap_{m<j\leq d} D_{2,j} \;\cap\; \bigcap_{3\leq i\leq m+1} D_{i,d} \;\cap\; D_{L_m}.$$

Case 3: Assume $1 < n$ and $m = d$. Then we get

$$\{Q_{n,d}\} = \bigcap_{n-1\leq j<d} D_{1,j} \;\cap\; \bigcap_{2\leq i<n} D_{i,d-1} \;\cap\; D_{L_n}.$$

Case 4: Assume $1 = n$ and $m = d$. Then Step (2) provides

$$\{Q_{1,d}\} = \bigcap_{1\leq j\leq d} D_{L_j}.$$

5. Fix integers k and j satisfying $2 \leq k, j \leq d$. If $k \leq j$, then the point $Q_{k,j}$ is a vertex of $\mathcal{P}$ and satisfies $Q_{k,j} = Q_{1,j} + Q_{k,d} - Q_{1,d}$ by Proposition 4.4.5. If $k > j$, then we define a point $Q_{k,j}$ by setting

$$Q_{k,j} := Q_{1,j} + Q_{k,d} - Q_{1,d}.$$

Since the positive entries of $Q_{1,j}$ and $Q_{k,d}$ are in disjoint positions and are equal to one, it follows that the entries of $Q_{k,j}$ are not positive.

Observe also that the definition implies that whenever $i < k$ and $l < j$

$$Q_{k,j} + Q_{i,l} = Q_{i,j} + Q_{k,l}. \tag{4.4.2}$$

Furthermore, using $Q_{1,j} \cdot Q_{1,d} = 0 = Q_{k,d} \cdot Q_{1,d}$, we conclude

$$Q_{k,j} \cdot Q_{1,d} = -Q_{1,d} \cdot Q_{1,d} = -d \quad \text{if } k > j, \tag{4.4.3}$$

that is, the sum of the entries of $Q_{k,j}$ is $-d$.

6. Let $Q_{i,j}, Q_{k,l}$ be vertices of $\mathcal{P}$ (so $i \leq j$ and $k \leq l$). Then we claim that

$$D_{i,j} \cap D_{k,l} \cap \mathcal{P}^* = D_{i,j} \cap D_{i,l} \cap D_{k,j} \cap D_{k,l} \cap \mathcal{P}^*.$$

   Furthermore, if one of the points $Q_{i,l}$ and $Q_{k,j}$ is not a vertex of $\mathcal{P}$, then there is some integer $m \in \{1, \dots, d\}$ such that the m-th coordinate of each point in the above set equals one.

   To show this let $T = (t_1, \dots, t_d)$ be a point in $D_{i,j} \cap D_{k,l} \cap P^*$ and consider two cases.

   First, assume that $i < k$ and $j > l$. By Equation (4.4.2), we can write $Q_{i,j} = Q$, $Q_{i,l} = Q + v$, $Q_{k,j} = Q + w$, and $Q_{k,l} = Q + v + w$. By the choice of T we know

$$Q \cdot T = (Q + v + w) \cdot T = -d.$$

   Moreover, $Q + v$ and $Q + w$ are vertices of $\mathcal{P}$, so $T \in \mathcal{P}^*$ gives

$$(Q + v) \cdot T \geq -d \quad \text{and} \quad (Q + w) \cdot T \geq -d.$$

   The last four relations imply that the latter two of them are in fact equalities, so T is in $D_{i,l} \cap D_{k,j}$, as required.

   Second, assume $i < k$ and $j < l$. Then $Q_{i,l}$ is a vertex of $\mathcal{P}$. Suppose that $Q_{k,j}$ is not a vertex of $\mathcal{P}$, that is, $k > j$. By Step (5), the entries of $Q_{k,j}$ are not positive and their sum is $-d$. Moreover, $T \in \mathcal{P}^*$ implies that all its coordinates are at most one. It follows that

$$Q_{k,j} \cdot T \geq -d.$$

   This inequality is also true if $k \leq j$ because then $Q_{k,j}$ is a vertex of $\mathcal{P}$. Now we argue as in the first case to conclude $T \in D_{i,l} \cap D_{k,j}$.

   Moreover, if $k > j$, then $Q_{k,j} \cdot T = -d$ forces each entry of T to equal one if the entry of $Q_{k,j}$ at the corresponding position is not zero.

7. Now we show that the polytope $\mathcal{P}^*$ has no edge that connects any vertex L_i to a vertex that is not in V, the vertex set of $\mathcal{P}$.

   Consider all the supporting hyperplanes of $\mathcal{P}^*$ that contain L_i. By Equation (4.4.1), these are precisely the hyperplanes $D_{n,m}$ satisfying $n \leq i \leq m$. By Proposition 4.4.5, the convex hull of the dual vertices $Q_{n,m}$ is the $(d-1)$-dimensional polytope $\mathcal{F}_i$ that is a product of two simplices. It follows that the supporting lines of edges in $\mathcal{P}^*$ through

L_i are of the form $\bigcap_{n\neq k, n\leq i\leq m} D_{n,m}$ for some $k \in \{1,\ldots,i\}$ or $\bigcap_{m\neq k, n\leq i\leq m} D_{n,m}$ for some $k \in \{i,\ldots,d\}$.

In the first case, this line contains the vertex $Q_{k,i-1}$ if $k < i$. If $k = i$, this line contains L_{i-1}. In the second case, the line contains the vertex $Q_{i+1,k}$, provided $k > i$, and the vertex L_{i+1} if $k = i$. Hence each of these lines through L_i supports an edge of $\mathcal{P}^*$ whose other vertex is a vertex of $\mathcal{P}$, as claimed.

8. Since $Y_d \subset \mathcal{P} \subset \mathcal{P}^*$, the polyhedron $\bigcap_{n\leq m} D^+_{n,m}$ contains the Ferrers complex Y_d (as introduced in Proposition 4.4.5). Denote the boundary of the polyhedron by $\partial\left(\bigcap_{n\leq m} D^+_{n,m}\right)$. We define the boundary of Y_d as the set ∂Y_d of faces of Y_d that are properly contained in exactly one of its facets $\mathcal{F}_1,\ldots,\mathcal{F}_d$. We claim

$$Y_d \cap \partial\left(\bigcap_{n\leq m} D^+_{n,m}\right) = \partial Y_d.$$

Indeed, the left-hand side certainly contains ∂Y_d. It remains to show the opposite inclusion.

Proposition 4.4.5 implies that the $(d-2)$-dimensional faces of Y_d in its boundary are of the form $conv\{Q_{n,m} \mid n \leq i \leq m, n \neq k\}$ for some integers $k < i$ or $conv\{Q_{n,m} \mid n \leq i \leq m, m \neq k\}$ for some integers $k > i$, where in both cases k and i are in in $\{1,\ldots,d\}$. Notice that

$$conv\{Q_{n,m} \mid n \leq i \leq m, n \neq k\} = D_{k,i-1} \cap \mathcal{F}_i$$

and

$$conv\{Q_{n,m} \mid n \leq i \leq m, m \neq k\} = D_{i+1,k} \cap \mathcal{F}_i.$$

The claimed equality follows.

9. Our next goal is to show that, for each integer $i = 1,\ldots,d$,

$$\mathcal{P}^* \cap D_{L_i} = Y_d \cap D_{L_i}.$$

Indeed, since $Y_d \cap D_{L_i} = \mathcal{F}_i \subset \mathcal{P}^*$ it remains to prove $\mathcal{P}^* \cap D_{L_i} \subset \mathcal{F}_i$. Assume there is some point $T \in (\mathcal{P}^* \cap D_{L_i}) \setminus \mathcal{F}_i$. Then consider any line $\mathcal{L}$ through T and a point P in $\mathcal{F}_i$. Since the affine hull of $\mathcal{F}_i$ is D_{L_i}, the line $\mathcal{L}$ meets $\partial\mathcal{F}_i$ in a point. If this point is in ∂Y_d, then $\mathcal{L}$ intersects $\partial\left(\bigcap_{n\leq m} D^+_{n,m}\right)$ properly by Step (8). It follows that T lies outside $\mathcal{P}^*$, a contradiction.

Hence the line $\mathcal{L}$ intersects $\mathcal{F}_i\cap\mathcal{F}_{i-1}$ or $\mathcal{F}_i\cap\mathcal{F}_{i+1}$ in a point. Since $P \in D^+_{L_{i-1}} \cap D^+_{L_{i+1}}$, the points P and T are in opposite half-spaces determined by $D_{L_{i-1}}$ in the first case and $D_{L_{i+1}}$ in the second case, respectively. It follows that $T \cdot L_{i-1} < -d$ or $T \cdot L_{i+1} < -d$. In either case this gives $T \notin P^*$, a contradiction to our original assumption.

10. Finally, we show that there is no edge in $\mathcal{P}^*$ that connects a vertex $Q_{n,m}$ and a vertex T of P^* that is not a vertex of $\mathcal{P}$. Together with Step (7) this proves that the polytopes $\mathcal{P}^*$ and $\mathcal{P}$ have the same vertex set, and thus $\mathcal{P}^* = \mathcal{P}$, as required.

    By Step (9) we know that T cannot lie on any hyperplane $D_{L_i}.Hencetheedgeof\mathrm{P}^*$ with vertices $Q_{n,m}$ and T is supported on a line $\mathcal{L}$ that is the intersection of hyperplanes of the form $D_{i,j}$ with $i \leq j$.

    Assume T is in $D_{i,j} \cap D_{k,l}$, where $i \leq j$ and $k \leq l$. If one of the points $Q_{i,l}$ and $Q_{k,j}$ is not a vertex of $\mathcal{P}$, then Step (6) shows that one of the entries of T is equal to one. But then T is in D_{L_q} for some integer q, a contradiction. Therefore, Step (6) gives that $\mathcal{L}$ is the intersection of hyperplanes $D_{i,j}$ with $i \leq j$ such that the convex hull of the vertices $Q_{i,j}$ is a $(d-2)$-dimensional product of simplices. By Step (2), it follows that $\mathcal{L}$ is $\bigcap_{i \leq n-1 \leq j, j \neq m} D_{i,j}$ or $\bigcap_{i \leq m+1 \leq j, i \neq n} D_{i,j}$. In the first case the point L_{n-1} is on $\mathcal{L}$, which forces $T = L_{n-1}$, and in the second case the point L_{m+1} is on $\mathcal{L}$, which yields $T = L_{m+1}$. This completes the argument.

□

In order to state an important consequence, we make the following definition.

Definition 4.4.9. *Let F be a face of the Ferrers polytope $\mathcal{P}_d$. Then its* dual face *F^* is*

$$F^* = \mathcal{P}_d \cap \bigcap_v D_v,$$

where the intersection is taken over the vertices of F.

Note that $\dim F^* = d - 1 - \dim F$, where the empty set is defined to have dimension -1.

Now we can state the following property of Ferrers polytopes.

Corollary 4.4.10. *Let F be a face of the Ferrers polytope $\mathcal{P}_d$. Then:*

(a) If F is a face of Y_d, then F^* is not a face of Y_d.

(b) If F is not a face of Y_d, then there is a face G of Y_d such $F = G^*$.

Proof. Consider a facet $\mathcal{F}_i$ of Y_d. By Step (9) of the above proof, it is a face of $\mathcal{P}_d$. Equation (4.4.1) shows that $\mathcal{F}_i^* = L_i$, which is not a vertex of Y_d. Now Claim (a) follows.

In order to show (b), consider a face F of $\mathcal{P}_d$ that is not a face of Y_d. Then

$$F = \mathcal{P}_d \cap \bigcap_v D_v,$$

where the intersection is taken over the vertices of F^*. Using again Step (9) of the above proof, it follows that none of the points $L_1, \ldots, L_d$ can be a vertex of F^*. In other words, the vertices of F^* are in $\{Q_{i,j} \mid 1 \leq i \leq j \leq d\}$. We now consider two cases.

First, assume that the vertex set V of F^* has the following property: Whenever $Q_{i,j}$ and $Q_{k,l}$ are in V, then so are the points $Q_{i,l}$ and $Q_{j,k}$. In this case, Proposition 4.4.5 implies that F^* is a face of Y_d, and we are done.

Second, assume that $Q_{i,j}$ and $Q_{k,l}$ are vertices of F^*, but one of the $Q_{i,l}$ and $Q_{j,k}$ is not in V. Then Step (6) of the above proof gives that there is some integer $m \in \{1, \ldots, d\}$ such that the m-th coordinate of each vertex of F equals one. However, then the point L_m is a vertex of F^*, a contradiction to our initial observation. □

It also allows us to determined the face vector of a Ferrers polytope.

Corollary 4.4.11. *Let d be a positive integer. Then:*

(a) The face vector of the Ferrers complex Y_d is given by

$$f_i(Y_d) = (i+1)\binom{d+1}{i+2} \quad (0 \le i \le d-1).$$

(b) The face vector of the Ferrers polytope $\mathcal{P}_d$ is given by $f_d(\mathcal{P}_d) = 1$ and

$$f_i(\mathcal{P}_d) = (i+1)\binom{d+1}{i+2} + (d-i)\binom{d+1}{i} \quad \text{if } 0 \le i \le d-1.$$

Proof. According to Proposition 4.4.5, an i-dimensional face of Y_d is given by choosing k rows and $i+2-k$ suitable columns in the Ferrers tableau T_d, where $1 \le k \le i+1$.

First consider the faces of Y_d obtained by using k rows. Assume the largest index of a choice of k rows is r. Fixing r, there are $\binom{r-1}{k-1}$ choices for the other columns and $\binom{d+1-r}{i+2-k}$ choices for picking $i+2-k$ column indices in $\{r, r+1, \ldots, d\}$. Varying r, we see that Y_d has $\binom{d+1}{i+2}$ i-dimensional faces that use k columns. Since k can be any integer between 1 and $i+1$, the required formula for Y_d follows.

Second, Corollary 4.4.10 gives $f_i(\mathcal{P}_d) = f_i(Y_d) + f_{d-1-i}(Y_d)$ if $0 \le i \le d-1$. Thus, Claim (b) follows from (a). □

4.5 A Cellular Resolution of the n-gon

In this section we consider the simplicial complex Δ_n of a convex n-gon, where $n \ge 4$. Our goal is to establish that the $(n-3)$-dimensional Ferrers polytope $\mathcal{P}_{n-3}$ (see Construction 4.4.1) supports a minimal graded free resolution of the Stanley-Reisner ring on Δ_n. In particular, this provides a combinatorial interpretation of the graded Betti numbers of stacked polytopes.

We begin by enumerating the vertices of the n-gon consecutively from 1 to n. Then the Stanley-Reisner ideal of its face complex in the polynomial ring $R = K[x_1, \ldots, x_n]$ is

$$I_{\Delta_n} = (x_1 x_{n-1}, x_i x_j \mid 3 \le i+2 \le j \le n-1) + x_n(x_2, \ldots, x_{n-2}).$$

Its generators correspond to the diagonals.

Using the diagonals through the vertex n, we get a triangulation of the polytope enclosed by the n-gon. Its Stanley-Reisner ideal is

$$I_n = (x_i x_j \mid 3 \le i+2 \le j \le n-1).$$

Hence we get

$$I_{\Delta_n} = I_n + J_n, \tag{4.5.1}$$

where

$$J_n = x_n(x_2, \ldots, x_{n-2}).$$

We consider two gradings on R: the standard $\mathbb{Z}$-grading and the fine $\mathbb{Z}^n$-grading, where the degree of x_i is the i-th vector in the standard basis of $\mathbb{R}^n$.

Now we use the Ferrers complex Y_{n-3} (see Proposition 4.4.5).

Lemma 4.5.1. *The polyhedral complex Y_{n-3} supports a minimal $\mathbb{Z}^n$-graded free resolution of R/I_n:*

$$\mathbf{F}_\bullet : 0 \to F_{n-3} \to \cdots \to F_1 \to F_0 \to R/I_n \to 0.$$

Proof. Consider the Ferrers tableau to the partition $\lambda = (n-2, n-3, \ldots, 2, 1)$, where the rows are labeled by the variables $x_1, x_2, ..., x_{n-3}$ and the columns by $x_3, x_4, ..., x_{n-1}$. Its boxes correspond to the generators of the ideal I_n. Hence, I_n is a specialization of the Ferrers ideal to λ. Labeling each vertex $Q_{i,j}$ of Y_{n-3} by $x_i x_{j+2}$, the claim follows from [12, Theorem 3.12]. □

We have seen that the complex Y_{n-3} is part of the face complex of the Ferrers polytope $\mathcal{P}_{n-3}$. We use the following labels on its face complex.

Definition 4.5.2. *For an integer $n \geq 4$, label the vertices $Q_{i,j}$ $(1 \leq i \leq j \leq n-3)$ of the Ferrers polytope $\mathcal{P}_{n-3}$ by $x_i x_{j+2}$ (as above) and the vertices L_i $(1 \leq i \leq n-3)$ by $x_{i+1}x_n$. The labeled polyhedral cell complex $X(\mathcal{P}_{n-3})$ associated to the Ferrers polytope $\mathcal{P}_{n-3}$ is the complex whose faces are the faces of $\mathcal{P}_{n-3}$ and whose labeling is induced by the specified labels of its vertices.*

We need some preliminary result as preparation for the main result of this section. A key observation is a relation between the labels of a face G of $\mathcal{P}_{n-3}$ and its dual face G^* (see Definition 4.4.9). We denote the label of G in $X(\mathcal{P}_{n-3})$ by m_G. It is the least common multiple of the label of the vertices of G.

Lemma 4.5.3. *Let G be any face of the Ferrers polytope $\mathcal{P}_{n-3}$. Then the labels of G and G^* satisfy*

$$m_G \cdot m_{G^*} = x_1 x_2 \cdots x_n.$$

Proof. By Corollary 4.4.10, it suffices to show this for a face G in the Ferrers complex Y_{n-3}. Thus, none of the points $L_1, \ldots, L_{n-3}$ is a vertex of G.

Observe that the label of each face of $\mathcal{P}_{n-3}$ is a squarefree monomial because the vertices have squarefree labels. We also use the computations in Step (2) of the proof of Theorem 4.4.8.

Consider now a vertex $Q_{i,j}$ of Y_{n-3}. The mentioned computations show that a point $Q_{k,l}$ is in $Q^*_{i,j}$ if and only if

$$k \in \{1, \ldots, n-3\} \setminus [\{i\} \cup \{p \mid j+2 \leq p \leq n-3\}]$$

and

$$l \in \{1, \ldots, n-3\} \setminus [\{j\} \cup \{p \mid 1 \leq p \leq i-2].$$

This implies that the labels of $Q_{i,j}$ and $Q^*_{i,j}$ are relatively prime and that their product is divisible by $x_1, \ldots, x_{n-1}$.

In order to extend this observation, denote the set of row indices of the vertices $Q_{i,j}$ of G by $R = \{i_1, \ldots, i_m\}$, where $i_1 < \cdots < i_m$. Similarly, let $C = \{j_1, \ldots, j_r\}$ be the set of the column indices of these vertices, where $j_1 < \cdots < j_r$. Note that $i_m \leq j_1$ and $m + r \leq n-2$ by the description of faces in Proposition 4.4.5. Step (6) of the proof of Theorem 4.4.8 provides

$$G^* = Q^*_{i_1,j_1} \cap Q^*_{i_1,j_2} \cap \ldots \cap Q^*_{i_1,j_r} \cap Q^*_{i_2,j_r} \cap \ldots \cap Q^*_{i_m,j_r}.$$

It follows that the labels of G and G^* are relatively prime and that their product is divisible by $x_1, \ldots, x_{n-1}$.

Finally, since G is in Y_{n-3}, it must be contained in one of its facets, say in $\mathcal{F}_q$. Then the point L_q is a vertex of G^*. Since its label is $x_{q+1}x_n$, the label m_{G^*} is divisible by x_n, and we get $m_G \cdot m_{G^*} = x_1 \cdots x_n$, as required. □

Corollary 4.5.4. *Let G be any face of the Ferrers polytope $\mathcal{P}_{n-3}$. Then the variable x_n divides the label m_G if and only if G is not a face of Y_{n-3}.*

Proof. If G is in Y_{n-3}, then x_n does not divide m_G because the only vertices of $\mathcal{P}_{n-3}$ whose labels are divisible by x_n are the points $L_1, \ldots, L_{n-3}$. These points are not in Y_{n-3}.

If G is not in Y_{n-3}, then $G = F^*$ for some face F of Y_{n-3} by Corollary 4.4.10. As we just saw, x_n does not divide m_F. Thus, it must divide $m_{F^*} = m_G$ by Lemma 4.5.3. □

Recall that the *canonical module* of an R-module M, denoted by ω_M, is the K-dual of the local cohomology module $H^{\dim M}(M)$, where $= (x_1, \ldots, x_n)$. Using the fine $\mathbb{Z}^n$-grading of R, the canonical module of R is $\omega_R \cong R(-1, \ldots, -1)$. If M is a finely graded R-module, then so is its canonical module and local duality gives an isomorphism of finely graded modules $\omega_M \cong \operatorname{Ext}^c_R(M, \omega_R)$, where $c = \dim R - \dim M = n - \dim M$. For Cohen-Macaulay modules, this extends to a relation among the free resolutions of M and ω_M.

Lemma 4.5.5. *Let I be a squarefree monomial ideal in R with minimal free $\mathbb{Z}^n$-graded resolution $F_\bullet$. If R/I is Cohen Macaulay, then $\operatorname{Hom}_R(F_\bullet, \omega_R)$ is a minimal free $\mathbb{Z}^n$-graded resolution of $\omega_{R/I}$.*

Proof. This follows by local duality (see, for example, [5, Corollary 3.3.9]). □

We are ready to establish the goal of this section.

Theorem 4.5.6. *The cellular complex supported on the labeled cell complex $X(\mathcal{P}_{n-3})$ to the $(n-3)$-dimensional Ferrers polytope $\mathcal{P}_{n-3}$ is a minimal $\mathbb{Z}^n$-graded free resolution of the Stanley-Reisner ring R/I_{Δ_n} of the n-gon Δ_n.*

Proof. We use the notation introduced at the beginning of this section. The triangulation of the polytope enclosed by the n-gon at vertex n is a disk whose boundary is the n-gon. Since $I_{\Delta_n} = I_n + J_n$ by Equation (4.5.1), a result of Hochster (see [5, Theorem 5.7.2]) gives the following exact sequence of $\mathbb{Z}^n$-graded modules

$$0 \to \omega_{R/I_n} \to R/I_n \to K[\Delta_n] \to 0, \tag{4.5.2}$$

where $K[\Delta_n] = R/I_{\Delta_n}$ is the Stanley-Reisner ring of Δ_n.

By Lemma 4.5.1, R/I_n has a minimal $\mathbb{Z}^n$-graded free resolution

$$\mathbf{F}_\bullet : \qquad 0 \to F_{n-3} \xrightarrow{\alpha_{n-3}} \cdots \xrightarrow{\alpha_2} F_1 \xrightarrow{\alpha_1} F_0 \to R/I_n \to 0,$$

where the basis elements e_G of F_i are indexed by the $(i-1)$-dimensional faces G of Y_{n-3} and have degree equal to $\deg m_G$. Therefore Lemma 4.5.5 shows that the minimal free $\mathbb{Z}^n$-graded resolution of ω_{R/I_n} has the form

$$0 \to G_{n-3} \xrightarrow{\beta_{n-3}} \cdots \xrightarrow{\beta_2} G_1 \xrightarrow{\beta_1} G_0 \to \omega_{R/I_n} \to 0,$$

where $G_i = \operatorname{Hom}_R(F_{n-3-i}, \omega_R)$. Hence

$$G_i = \bigoplus_{G \in Y_{n-3}} R(\deg m_G - (1, \ldots, 1)) = \bigoplus_{G \in Y_{n-3}} e_G^* R,$$

where the sum is taken over the faces G of Y_{n-3} with dimension $n-4-i$ and the dual basis elements have degree $\deg e_G^* = (1, \ldots, 1) - \deg e_G = (1, \ldots, 1) - \deg m_G$.

Let ε be an incidence function on the cell complex $X(\mathcal{P}_{n-3})$. It induces an incidence function on its subcomplex Y_{n-3}. We assume that the cellular resolution $\mathbf{F}_\bullet$ is constructed by using this induced incidence function.

Consider now an $(n-4-i)$-dimensional face G of Y_{n-3}. Its dual face G^* has dimension i and is not a face of Y_{n-3} by Corollary 4.4.10. Moreover, Lemma 4.5.3 shows that its label has degree

$$\deg m_{G^*} = (1, \ldots, 1) - \deg m_G = \deg e_G^*.$$

This is the degree of the generator e_{G^*} to the face G^* of $\mathcal{P}_{n-3}$ at homological degree $i+1$ in the cellular complex supported by $X(\mathcal{P}_{n-3})$. Using again Corollary 4.4.10, it follows that we can rewrite G_i as

$$G_i = \bigoplus_{F \notin Y_{n-3}} e_F R,$$

where now the sum is taken over the i-dimensional faces F of $\mathcal{P}_{n-3}$ that are not in Y_{n-3}. Thus, we can use the incidence function on $X(\mathcal{P}_{n-3})$ to define a map $\beta_i : G_i \to G_{i-1}$ on the generators of G_i by

$$\beta_i(e_F) := \sum_{P \text{ facet of } F,\, P \notin Y_{n-3}} -\varepsilon(P, F)\, \frac{m_F}{m_P}\, e_P.$$

Comparing with the maps α_j in the cellular resolution supported on Y_{n-3}, we obtain

$$\beta_i = -\operatorname{Hom}_R(\alpha_{n-2-i}, \omega_R).$$

Thus, our definition of the maps β_i gives indeed a minimal free $\mathbb{Z}^n$-graded resolution of ω_{R/I_n}, as required.

In order to apply the mapping cone procedure we need comparison maps $\varphi_i : G_i \to F_i$. We define these on the generators by

$$\varphi_i(e_F) = \sum_{Q \text{ facet of } F,\, Q \in Y_{n-3}} \varepsilon(Q, F)\, \frac{m_F}{m_Q}\, e_Q.$$

Note that the right-hand side is indeed in F_i because the facets Q have dimension $i-1$. Furthermore, since F is not in Y_{n-3}, but Q is, the coefficient $\frac{m_F}{m_Q}$ is not a constant because it is divisible by x_n, due to Corollary 4.5.4. It follows that each map φ_i is minimal.

Next, we want to show that the following diagram commutes if $i \geq 1$:

$$\begin{array}{ccc} G_i & \xrightarrow{\varphi_i} & F_i \\ \Big\downarrow{\scriptstyle \alpha_i} & & \Big\downarrow{\scriptstyle \beta_i} \\ G_{i-1} & \xrightarrow{\varphi_{i-1}} & F_{i-1}. \end{array}$$

It is enough to check this on the generators of G_i. Let e_F be such a generator, i.e., F is an i-dimensional face of $\mathcal{P}_{n-3}$, but not of Y_{n-3}. Then $\beta_i(\varphi_i(e_F))$ as well as $\varphi_{i-1}(\alpha_i(e_F))$ is a linear combination of generators e_G, where each G is an $(i-2)$-dimensional face of G that is in Y_{n-3}.

Consider any codimension two face G of F. Since $\mathcal{P}_{n-3}$ is a polytope there are precisely two facets P and Q of F that contain G. Since ε is an incidence function there is the following relation

$$\varepsilon(G, P) \cdot \varepsilon(P, F) + \varepsilon(G, Q) \cdot \varepsilon(Q, F) = 0.$$

Assume now that G is in Y_{n-3}. There are two cases.

First, assume P and Q are both either in Y_{n-3} or both not in Y_{n-3}. Then the coefficient of e_G in $\beta_i(\varphi_i(e_F))$ as well as in $\varphi_{i-1}(\alpha_i(e_F))$ is zero.

Second, assume that one of P and Q is in Y_{n-3}, but the other is not. Say, $Q \in Y_{n-3}$. Then e_G occurs in $\beta_i(\varphi_i(e_F))$ and $\varphi_{i-1}(\alpha_i(e_F))$ with the same coefficient, namely $\varepsilon(G, Q) \cdot \varepsilon(Q, F) \cdot \frac{m_F}{m_G} = -\varepsilon(G, P) \cdot \varepsilon(P, F) \cdot \frac{m_F}{m_G}$.

These considerations establish the desired commutativity of diagrams. It follows that the mapping cone procedure gives a $\mathbb{Z}^n$-graded free resolution of $K[\Delta_n]$, which is exactly the cellular resolution supported on $X(\mathcal{P}_{n-3})$.

Moreover, by Lemma 4.5.1 the maps α_i are minimal if $1 \leq i \leq n-3$. Thus, so are the maps $\beta_i = -\operatorname{Hom}_R(\alpha_{n-2-i}, \omega_R)$. As pointed out above, each comparison map φ_i is also minimal. Hence, the mapping cone procedure yields a minimal resolution of $K[\Delta_n]$, which completes the argument. □

We would like to point out that 4.2.3 is the cellular resolution supported on $X(\mathcal{P}_2)$.

The face complex of a Ferrers polytope $\mathcal{P}_{n-3}$ supports also cellular resolutions of boundary complexes of other simplicial polytopes if we adjust the labeling of the vertices $L_1, \ldots, L_{n-3}$ suitably. We use the following construction.

Construction 4.5.7. *Let $\mathcal{P}$ be a convex simplicial polytope on n vertices. Triangulate $\mathcal{P}$ at the vertex n, and let $\mathcal{P}'$ be a cone over this triangulation with a new vertex $(n+1)$.*

There is the following extension of Theorem 4.5.6.

Proposition 4.5.8. *Starting with a two-dimensional convex simplicial polytope on n vertices, apply Construction 4.5.7 $t \geq 0$ times, and denote by $\mathcal{P}$ the resulting polytope on $n+t$ vertices of dimension $t+2$. Let $X'(\mathcal{P}_{n-3})$ be the labelled cell complex obtained from $X(\mathcal{P}_{n-3})$ (see Definition 4.5.2) by only relabelling the vertices $L_1, \ldots, L_{n-3}$ so that L_i has label $x_{i+1}x_nx_{n+1}\cdots x_{n+t}$. Then $X'(\mathcal{P}_{n-3})$ supports a minimal $\mathbb{Z}^{n+t}$-graded free resolution of the Stanley-Reisner ring of the boundary complex $\partial\mathcal{P}$ of $\mathcal{P}$ over $R = K[x_1, \ldots, x_{n+t}]$.*

In particular, a minimal $\mathbb{Z}$-graded free resolution of $K[\partial\mathcal{P}]$ has the form

$$0 \longrightarrow R(-n-t) \longrightarrow \begin{array}{c} R^{f_{n-4}}(-n+2) \\ \oplus \\ R^{f_0}(-n-t+2) \end{array} \longrightarrow \cdots$$

$$\longrightarrow \begin{array}{c} R^{f_1}(-3) \\ \oplus \\ R^{f_{n-3}}(-3-t) \end{array} \longrightarrow \begin{array}{c} R^{f_0}(-2) \\ \oplus \\ R^{f_{n-4}}(-2-t) \end{array} \longrightarrow R \longrightarrow K[\partial\mathcal{P}] \longrightarrow 0,$$

where

$$f_i = (i+1)\binom{n-2}{i+2} = f_i(Y_{n-3}).$$

Proof. Notice that the Stanley-Reisner ideal of the boundary complex $\partial\mathcal{P}$ is

$$I_{\partial\mathcal{P}} = I_n + x_n x_{n+1} \cdots x_{n+t} \cdot (x_2, \ldots, x_{n-2}).$$

Thus, we get again an exact sequence

$$0 \to \omega_{R/I_n} \to R/I_n \to K[\Delta_n] \to 0,$$

but this time of $\mathbb{Z}^{n+t}$-graded modules.

Each i-dimensional face G of Y_{n-3} has still a label of $\mathbb{Z}$-degree $i+2$, but the $\mathbb{Z}$-degree of the label to its dual face is $n+t-i-2$ because

$$m_G \cdot m_{G^*} = x_1 x_2 \cdots x_{n+t}.$$

Now we conclude as in the proof of Theorem 4.5.6. □

Recall that a d-dimensional simplicial polytope is stacked if it admits a triangulation Γ which is a $(d-1)$-tree, that is, Γ is a shellable $(d-1)$-dimensional simplicial complex with h-vector $(1, c-1)$. For example, such a polytope is obtained by pairwise gluing of d-simplices along a facet. The Betti numbers of these rings are the same as the Betti numbers of some stacked polytopes and these have been calculated previously in several places, including [26], [17], [20], and [6]. We offer the following interpretation of their Betti numbers.

Corollary 4.5.9. *Let $\mathcal{P}$ be a d-dimensional stacked polytope on v vertices. The i-th total Betti number of the Stanley-Reisner ring to the boundary complex of $\mathcal{P}$ equals $f_{i-1}(\mathcal{P}_{v-d-1})$, the number of i-dimensional faces of the Ferrers polytope of dimension $v-d-1$.*

Proof. For fixed d and v, the boundary complexes of all such polytopes have the same graded Betti numbers, see [26, Theorem 1.1]. Thus, it is enough to consider a polytope $\mathcal{P}$ that is obtained from a 2-dimensional convex simplicial polytope on $v-d+2$ vertices by applying Construction 4.5.7 (d_2) times. It is stacked. Now the claim follows from Proposition 4.5.8. □

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