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Decay Estimates on Trace Norms of Localized Functions of Schrödinger Operators

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DECAY ESTIMATES ON TRACE NORMS OF LOCALIZED FUNCTIONS OF SCHRÖDINGER OPERATORS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
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By
Aaron D. Saxton
Lexington, Kentucky

Director: Dr. Peter D. Hislop, Professor of Mathematics
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ABSTRACT OF DISSERTATION

DECAY ESTIMATES ON TRACE NORMS OF LOCALIZED FUNCTIONS OF SCHRÖDINGER OPERATORS

In 1973, Combes and Thomas discovered a general technique for showing exponential decay of eigenfunctions. The technique involved proving the exponential decay of the resolvent of the Schrödinger operator localized between two distant regions. Since then, the technique has been been applied to several types of Schrödinger operators. This dissertation will show that the Combes–Thomas method works well with trace, Hilbert–Schmidt and other trace-type norms. The first result we prove shows exponential decay on trace-type norms of a resolvent of a Schrödinger operator localized between two distant regions. We build on this result by applying the Combes–Thomas method again to prove polynomial and sub-exponential decay estimates on functions of Schrödinger operators localized between two distant regions.

KEYWORDS: Schrödinger Operator, Combes–Thomas Method, Trace Ideal, Trace Norm, Magnetic Schrödinger Operator

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Dedicated to my Grampa

Joseph J. Horzich

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Chapter 1 Introduction

We start our discussion not by dazzling the reader with the most general version of our results, but rather by introducing the most basic form of a Schrödinger operator. Define the Schrödinger operator as,

$$H = -\Delta + V$$

acting on the space of functions $L^2(\mathbb{R}^d)$, an infinite dimensional linear vector space. The Laplacian, $\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$, is a second order partial differential operator, and $V : \mathbb{R}^d \to \mathbb{R}$ is an operator that acts by pointwise multiplication. The Hamiltonian H represents the total energy of a quantum mechanical system with $-\Delta$ representing the kinetic energy, and $V(x)$ represents the potential energy. The eigenvalues of H represent real energy states in quantum mechanics. We require that H be self-adjoint on its domain to guarantee real energies. The Laplacian is formally self-adjoint because of integration by parts, but what condition on V is there to make sure H is self-adjoint? A theorem by T. Kato and F. Rellich[9] gives us this condition.

Theorem 1.1 (Kato–Rellich)**.** *Let A be a self-adjoint operator on its domain $D(A)$, and B be a symmetric operator with $D(A) \subset D(B)$. If there exist positive real constants a, b and $a < 1$ such that*

$$\|Bu\| \leq a\|Au\| + b\|u\| \quad \textit{for } u \textit{ in the domain of } A$$

then $A + B$ is self-adjoint on the domain of A.

If the above inequality is true for some constants a, b we say that B is relatively A bounded with relative bound a.

Because of the Kato–Rellich theorem, we require that the potential V be relatively Laplacian bounded. Two common examples of this kind of potential are a Coulomb potential or a bounded potential. If V is relatively Laplacian bounded with relative bound less than 1, then H is an unbounded self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^d)$.

For linear operators with infinite dimensional domains, the eigenvalue problem becomes more complicated. Typically we ask, for what complex number z is the operator $H - z$ not invertible. However, it can be that $H - z$ is invertible, but the inverse is not a bounded operator. We say that z is in the *spectrum* of H if $H - z$ does not have a bounded inverse. The set of all $z \in \mathbb{C}$ such that z is in the spectrum of H is a closed set that we denote as $\sigma(H) \subset \mathbb{C}$. Since H is self-adjoint, the spectrum is in fact a closed subset of the real line. As a complementary definition, the *resolvent set* is defined as the set of all complex numbers z for which the operator $H - z$ has a bounded inverse. This open set is denoted as $\rho(H)$. And for each $z \in \rho(H)$, the operator $(H - z)^{-1}$ is called the *resolvent* of H and is denoted $R_H(z)$. For a given $z \in \rho(H)$ we use η_z to denote the distance of z to $\sigma(H)$.

In the Combes–Thomas method, they introduce another operator that acts by pointwise multiplication, $U_\alpha : \mathbb{R}^d \to \mathbb{C}$ defined as $U_\alpha(x) = e^{i\alpha \cdot x}$ with $\alpha \in \mathbb{R}^d$. We use U_α by considering $U_\alpha H U_\alpha^{-1}$. This operator is denoted H_α. The Combes-Thomas method asserts that H_α is an analytic family of operators in α when we extend α from merely being vector in $\mathbb{R}^d$ to an open subset of $\mathbb{C}^d$. Since H_α is an analytic family, the resolvent of H_α is also bounded holomorphic for $|\alpha|$ small enough.

To define a trace norm on an operator that acts on an infinite dimensional vector space we must wrestle with three restrictions. First, does the operator have any eigenvalues? The prototypical operator that has eigenvalues is one of *finite rank*. What's more, finite rank operators can approximate the class of *compact* operators which have discrete eigenvalue except possibly at 0. As for the next two problems, there are many non zero operators that have a trace of zero, and the trace may not even be real. To fix these for some bounded operator A, we choose an associated operator that is both symmetric and positive definite. We choose A^*A, where A^* is the adjoint of A. The operator A^*A is bounded self-adjoint and always has positive real eigenvalues. If A is compact, the n^{th} *singular* value of an operator is $\mu_n(A) := \sqrt{\lambda_n(A^*A)}$, where $\lambda_n(A^*A)$ is the n^{th} eigenvalue of A^*A. For $1 \leq p < \infty$ the p^{th}

trace norm is defined as,

$$\|A\|_p := \left[\sum_k \mu_k(A)^p\right]^{1/p}$$

When $p = 1$, the norm is called the trace norm. When $p = 2$, it is called the Hilbert-Schmidt norm. To localize the resolvent, let $\chi : \mathbb{R}^d \to \mathbb{R}$ act as an operator by pointwise multiplication. Also, let χ have compact support inside a ball of radius one centered at the origin. Let the sup norm of χ be less than 1. And let χ be infinitely differentiable. Furthermore, for $p, q, w \in \mathbb{R}^d$, let $\chi_x(w) = \chi(w - x)$ and $\chi_y(w) = \chi(w - y)$.

In this thesis, we prove the following basic result that gives *exponential decay* in trace norm extending the operator norm result of Combes–Thomas.

Theorem 1.2 (Basic Result)**.** *Let V be a real valued potential that is relatively Laplacian bounded with relative bound less than 1 and $z \in \rho(H)$. There exists positive integers k, m and d such that $k > \frac{d}{2m}$, and there exists real positive finite constants, C and c, that depend on m, d, k, z, χ and V so that,*

$$\|\chi_x R_H^k(z)\chi_y\|_m \leq Ce^{-c\sqrt{\eta_z}\|x-y\|}$$

Functions of Schrödinger operators

The resolvent may be thought of as a function of a Schrödinger operator. That is, let $G_z(x) = (x - z)^{-1}$. Then $R_H(z) = G_z(H)$. The exponential decay in Theorem 1.2 is allowed because of the particular form of the resolvent illustrated by $G_z(x)$. If we want to generalize Theorem 1.2 to functions of Schrödinger operators, we will very likely lose the exponential decay. To this end, define a set of functions called *slowly decreasing smooth functions* by choosing $f : \mathbb{R} \to \mathbb{C}$ to be infinitely differentiable functions such that,

$$|f^{(n)}(x)| \leq \frac{c_n}{\langle x\rangle^{1+n}} \quad \text{where} \quad \langle x\rangle := \sqrt{x^2+1}$$

for some $c_n < \infty$ and all $x \in \mathbb{R}$ and all $n = 0, 1, 2, ...$ When the imaginary part of z is not 0, then $G_z(x)$ is a slowly decreasing smooth functions.

A paper by F. Germinet and A. Klein [6] studied functions of Schrödinger operator by using what is known as the Helffer–Sjöstrand formula. I follow their approach but instead apply a p^{th} trace norm. In the following theorem, the price we pay for generalizing Theorem 1.2 is trading exponential decay for polynomial decay. In the next two theorems a basic quantity we will need is M a real number such that $-M$ is below the infimum of the spectrum of H.

Theorem 1.3. *Let k and m be an integer such that $k > \frac{d}{2m}$. Let f be such that $f(x)(x+M)^k$ is a slowly decreasing smooth function for $x > -1$, then $\chi_p f(H)\chi_q$ has a finite m^{th} trace norm. And there exists a constant C that depends on d, k, V, f and f's derivatives such that*

$$\|\chi_p f(H)\chi_q\|_m \leq \frac{C}{\langle p - q\rangle^k}$$

The next result I prove explores what happens when f is close to an analytic function.

In a paper by J. Bouclet, F. Germinet and A. Klein [3] the L^1-*Gevrey class of order* $a \geq 1$ is used with the Helffer–Sjöstrand formula to show sub-exponential decay in the operator norm. We say a function is *Gevrey class of order* $a \geq 1$ if for each compact subset $K \subset \mathbb{R}$ there are constants, C, that depend on K such that,

$$|f^{(n)}(x)| \leq C(C(n+1)^a)^n \quad \text{with} \quad x \in K \,\text{and}\, n = 0, 1, 2, ...$$

Recall that a function is real analytic if and only if there exists a positive real constant C that depends on the compact subset $K \subset \mathbb{R}$ such that for each n and every $x \in K$,

$$|f^{(n)}(x)| \leq C^{n+1} n!$$

We see, after using Stirling's formula on $n!$, that when $a = 1$ the Gevrey class of functions are analytic. Mixing the idea of slowly decreasing functions with the Gevrey class, L^1-Gevrey is defined as follows.

Definition 1.4. Let I be an open interval and $a \geq 1$, $f \in C^\infty(\mathbb{R})$. The function f is L^1-*Gevrey of class* a *on* I if for all $k = 0, 1, 2, ...$ there exists a constant $C_{f,I}$ greater than one that depends on f and I such that

$$\int_I |f^{(k)}(u)| \langle u \rangle^{k-1} du \leq C_{f,I}(C_{f,I}(k+1)^a)^k$$

In my research, I apply the techniques of Bouclet–Germinet–Klein[3] and use the p^{th} trace norm of an operator to obtain sub-exponential decay in the following result.

Theorem 1.5. *Let f be a function such that $f(x)(x+M)^k$ is L^1-Gevrey Class of order $a \geq 1$ when $x > -M$. Let d, m and k be integers such that $k > d/2m$. There exists real positive constants, C and c, that depends on d, m, k, a, V and f and a constant $\gamma \in (0, 1]$ that depends on a, such that*

$$\|\chi_p f(H) \chi_q\|_m \leq C e^{c\|p-q\|^{-\gamma}}$$

The constant $\gamma = 1$ corresponds to when $a = 1$. Therefore the above theorem gives us back exponential decay when $a = 1$.

Chapter 2 Trace Ideals

2.1 Brief Introduction to Operators

Good references for an introduction to functional analysis are, [12] [14] and the appendices of [8]. Everything stated in this section can be found in one or more of these sources.

We start our discussion by defining the objects that we are operating on. A *Banach* space is a complete, normed, linear vector space over a field. The field we will exclusively use is the complex numbers. Let X denote a Banach space. An *inner product* on X is a complex valued function on $X \times X$ with the following properties. For every $x, y, z \in X$ and $\alpha \in \mathbb{C}$,

- $\langle x, x\rangle \geq 0$
- $\langle x, x\rangle = 0$ if and only if $x = 0$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$

Remark 2.2. The above choice of axioms for the inner product make it linear in the first argument, and conjugate linear in the second. That is, for $\beta \in \mathbb{C}$, $\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$

A Hilbert space is a Banach space with an inner product that is complete in the induced norm. The induced norm on $\mathcal{H}$ is $\|f\|_{\mathcal{H}} := \sqrt{\langle f, f, \rangle}$. All Hilbert spaces in this document are *separable*. See [12] for more details.

An *operator* is a linear map on a Hilbert space into another. We will usually work with operators that are maps on a Hilbert space into itself. To fully define an operator one must take care, for it may be that the operator is not "compatible" with every element of its Hilbert space. Take for example the derivative operating on $L^2(\mathbb{R}^d)$. There are plenty of elements of $L^2(\mathbb{R}^2)$ that are not differentiable. But, there is a linear subspace that is compatible, namely $L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$. Notice that $L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$. When one defines an operator, they must define the map, the Hilbert space it operates on, and a linear subspace that the operator is defined on. The linear subspace for which an operator is defined is call the *domain*. Let A be an operator form $\mathcal{H}$ into itself. We denote the domain of A as $D(A)$. The domain of all operators in this document will be dense subspaces.

Next we define some properties that combine an operator with the inner product. Let A and B be some operators on the Hilbert space $\mathcal{H}$. Fixing A we define B in the following way. Let the set $D(B)$ be all $g \in \mathcal{H}$ for which there exists $h \in \mathcal{H}$ such that

$$\langle Af, g \rangle = \langle f, h \rangle \quad \text{for all} \quad f \in D(A)$$

and define the action of B as $Bg := h$. We call B the *adjoint* of A. And denote the adjoint of A as $A^* := B$. In compact notation, the definition of adjoint is written as

$$\langle Af, g \rangle = \langle f, A^* g \rangle \quad \text{for all} \quad f \in D(A) \quad \text{and} \quad g \in D(A^*)$$

Remark 2.3. One may think that the above definition of adjoint is overly complicated because the Riesz representation Theorem tells us that the h always exists. But we have not distinguished between bounded and unbounded operators yet. The Riesz representation Theorem does not apply if A is an unbounded operator.

The utility of the adjoint will become apparent later. Now we define some properties of an operator that involve the adjoint. An operator, A, is *symmetric* if

$$\langle Af, g \rangle = \langle f, Ag \rangle \quad \text{for all} \quad f, g \in D(A)$$

Remark 2.4. Notice that this implies that $D(A) \subset D(A^*)$ because A^* will agree with A on $D(A)$, thus $A = A^*$ on $D(A)$. But, A^* may still be defined outside of $D(A)$.

Taking the definition of symmetric one step further, an operator is *self-adjoint* if it is symmetric and $D(A) = D(A^*)$.

Given a fixed Hilbert space, the set of all operators is divided into two classes, bounded and unbounded. A *bounded operator* is one that is defined on all of $\mathcal{H}$ and there exist a positive real constant C such that,

$$\|Af\|_{\mathcal{H}} \le C\|f\|_{\mathcal{H}} \qquad \text{for all} \quad f \in \mathcal{H}$$

The *operator norm* is defined as,

$$\|A\| := \inf\{C \in [0,\infty) \,|\, \|Af\|_{\mathcal{H}} \le C\|f\|_{\mathcal{H}}, \text{ for all } f \in \mathcal{H}\}$$

An operator is called *semi-bounded* if there exists a positive real number C such that,

$$-C\langle f, f\rangle \le \langle Af, f\rangle \qquad \text{for all} \quad f \in \mathcal{H}$$

We say A is a *positive operator* if $0 \le \langle f, Af\rangle$ for every f in the domain of A.

Example 2.5. Consider the Hilbert space of C^1 complex valued differentiable functions on a closed interval $[0,1] \subset \mathbb{R}$. For $f, g \in C^1([0,1])$, define the inner product on this space as

$$\langle f, g\rangle := \int_0^1 f(x)\bar{g}(x)dx$$

Since the space that an operator acts on and the domain of an operator can be different, the question of invertibility becomes a bit more complicated. The *inverse* of an operator is the map, A^{-1}, such that AA^{-1} is the identity in the linear subspace that is the range of A. And, $A^{-1}A$ is the identity in $D(A)$. However, if one is able to find an inverse map, that does not mean that an operator is invertible. An operator is *invertible* if an inverse for A exists, so A is injective, and it is a bounded operator on the range of A that extends to a bounded operator on $L^2(\mathbb{R}^d)$.

The set of all $z \in \mathbb{C}$ such that $A - z$ is invertible is called the *resolvent set* of A. To reiterate, that means that an inverse exists and it is a bounded operator. We denote the resolvent set as $\rho(A) \subset \mathbb{C}$. The *resolvent* of A is denoted as

$$R_A(z) := (A - z)^{-1}$$

The resolvent has a couple of interesting properties that will be of great use.

Lemma 2.6 (First resolvent formula)**.** *Let A be a linear operator on a Banach space X. Then for $\mu, \lambda \in \rho(A)$,*

$$R_A(\lambda) - R_A(\mu) = (\lambda - \mu)R_A(\lambda)R_A(\mu)$$

Proof. Since μ and λ are two complex numbers that are in the resolvent set. Then, by definition, $A-\mu$ and $A-\lambda$ are invertible. Then we can write,

$$\begin{aligned} R_A(\lambda) - R_A(\mu) &= R_A(\lambda)[(A-\mu)R_A(\mu)] - [R_A(\lambda)(A-\lambda)]R_A(\mu) \\ &= R_A(\lambda)[(A-\mu)-(A-\lambda)]R_A(\mu) \end{aligned}$$

□

This first resolvent formula can be used to prove two convenient results; that $R_A(\lambda)$ and $R_A(\mu)$ commute, and $\rho(A)$ is open. Another useful result, and one we will be using frequently, is

Lemma 2.7 (Second resolvent formula)**.** *Let A and B be two linear operators on a Banach space X. Then for $z \in \rho(A) \cap \rho(B)$,*

$$R_A(z) - R_B(z) = R_A(\lambda)(B-A)R_B(\mu)$$

Proof. This proof is very similar to the *first resolvent formula's* proof.

$$\begin{aligned} R_A(z) - R_B(z) &= R_A(z)[(B-z)R_B(z)] - [R_A(z)(A-z)]R_B(z) \\ &= R_A(z)[(B-z)-(A-z)]R_B(z) \\ &= R_A(z)[B-A]R_B(z) \end{aligned}$$

□

Compact Operators

In a certain sense, compact operators are operators that most resemble matrices operating on a finite dimensional vector space.

Let $\{f_n\}$ be a sequence in $\mathcal{H}$. Then $\{f_n\}$ is said to *converge strongly* to f in $\mathcal{H}$ if $\|f_n - f\|_{\mathcal{H}} \to 0$. And $\{f_n\}$ is said to *converge weakly* to f in $\mathcal{H}$ if $\langle f_n, \phi\rangle \to \langle f, \phi\rangle$ for each $\phi \in \mathcal{H}$. There are many equivalent definitions of compact operators. The one we choose to begin our description of compact operators with is the following.

Definition 2.8. A bounded operator A on a Hilbert space is compact if for every weakly convergent sequence $\{f_n\}$ in $\mathcal{H}$, then Af_n is a strongly convergent sequence.

The set of all bounded linear operators, $\mathcal{L}(\mathcal{H})$, forms an algebra. An ideal is a subring such that if $y \in \mathcal{I}_\infty$ then $xy \in \mathcal{I}_\infty$ and $yx \in \mathcal{I}_\infty$ for all $x \in \mathcal{L}(\mathcal{H})$. Let $\mathcal{I}_\infty$ be the set of all compact operators. Then $\mathcal{I}_\infty$ is an ideal inside of $\mathcal{L}(\mathcal{H})$.

Singular Values and Trace Ideals

Let S be a compact operator acting on a Hilbert space. We define the trace ideals and trace norms as they are described in Simon's book *Trace Ideals and Their Applications* [14]. We summarize the definitions in the following list.

- Denote $\lambda_n(S)$ as the n'th *eigenvalue* of S.
- Define $\mu_n(S) := \sqrt{\lambda(S^*S)}$ as the n'th *singular value* of S.
- Define $\|S\|_1 := \sum_{n=1} \mu_n(S)$ to be the trace norm of S.
- For any real $p \geq 0$, Define the p'th trace norm of S as $\|S\|_p := \{\sum_{n=1} \mu_n(S)^p\}^{1/p}$.
- In the special case when $p = 2$, we call $\|S\|_2$ the *Hilbert Schmidt* norm of S.
- Denote $\mathcal{I}_p$ as the set of all operators such that $\|S\|_p < \infty$.

Theorem 2.9. *If $p > 0$ is a real number then $\mathcal{I}_p$ is an ideal in the algebra of bounded operators $\mathcal{L}(\mathcal{H})$.*

In Simon's book [14] he goes on to describe the complete structure of ideals in $\mathcal{L}(\mathcal{H})$. Three interesting features of $\mathcal{I}_p$ are the following. First, $\mathcal{I}_p \subset \mathcal{I}_\infty$ for every $p > 0$. It turns out that when $\mathcal{H}$ is infinite dimensional then $\mathcal{I}_\infty$ is the largest proper ideal in $\mathcal{L}(\mathcal{H})$. Second, if $1 \leq p < q$ then $\mathcal{I}_p \subset \mathcal{I}_q$. And third, if A and B are in some ideal for a large p, then the product AB is in some smaller ideal. Formally, if we raise A to higher and higher powers, then A^m will be in smaller and smaller ideals. This third fact is precisely stated in the next theorem which can be referenced in [14, p21].

Theorem 2.10 (Abstract Hölder's inequality)**.** *Suppose that $A \in \mathcal{I}_p$ and $B \in \mathcal{I}_q$. Let p, q and r be positive real number such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

Also suppose that $A \in \mathcal{I}_p$ and $B \in \mathcal{I}_q$. Then $AB \in \mathcal{I}_r$ and

$$\|AB\|_r \leq \|A\|_p \|B\|_q$$

Compact Operators of the Form $f(x)g(-i\nabla)$

The typical Schrödinger operator, $H = -\Delta + V$, is not bounded, much less compact. Therefore the object we will be studying is the resolvent of H, $R_H(z)$. Unfortunately $R_H(z)$ still has a problem, it is not compact. Many of the techniques we will use localize the resolvent by multiplying it by some bounded function that has compact support. So the general form of many operators we will be working with is $f(x)g(-i\nabla)$ where f is chosen to be some localization function. Once we localize the resolvent, it will be important to know which trace ideals it belongs to. Chapter 4 of [14] is dedicated to this question. A key result from that chapter is,

Theorem 2.11. *If $f, g \in L^p(\mathbb{R}^d)$ for $2 \leq p < \infty$, then $f(x)g(-i\nabla) \in \mathcal{I}_p$ and,*

$$\|f(x)g(-i\nabla)\|_p \leq (2\pi)^{-d/p} \|f\|_p \|g\|_p \tag{2.1}$$

2.12 Magnetic Schrödinger Operators

From Chapter 1 we know that a Schrödinger operator is the negative Laplacian plus a scalar potential V. The scalar potential allows us to study many interesting phenomena in electrostatics using the Coulomb potential. If we want to study magnetism, we must modify H. Let $\mathbf{a}(x)$ be a vector potential from $\mathbb{R}^d \to \mathbb{R}^d$. Then the *magnetic Schrödinger operator* is,

$$H_{\mathbf{a}} = (-i\nabla - \mathbf{a})^2 + V$$

Once again, we are challenged with the question of what trace ideal is $\chi R_{H_{\mathbf{a}}}(z)$ in? To answer this, we start by defining a *pointwise bound* between two compact operators and state the diamagnetic inequality [14, p. 24] and [1, p. 850].

Definition 2.13. Let A and B be bounded operators on $L^2(\mathbb{R}^d)$ then we write $A \lessdot B$ if and only if

$$|(Ah)(x)| \leq (B|h|)(x)$$

for all $h \in L^2(\mathbb{R}^d)$.

Theorem 2.14 (Diamagnetic Inequality)**.** *If* $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$, $H_0 = -\Delta$ *and* $H_{\mathbf{a}_0} = (-i\nabla - \mathbf{a})^2$ *then*

$$e^{-H_{\mathbf{a}_0}} \lessdot e^{-H_0}$$

What is more, from Simon's [15] Theorem 1 and Theorem 3 and the Diamagnetic Inequality above, we can prove that

$$e^{-H_{\mathbf{a}_0}+V} \lessdot e^{-H_0+V}$$

This result will let us prove the next technical lemma.

Lemma 2.15. *If* $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$, $H = -\Delta + V$ *and* $H_{\mathbf{a}} = (-i\nabla - \mathbf{a})^2 + V$ *and* χ *is a positive bounded function with compact support then,*

$$\chi R^m_{H_{\mathbf{a}}}(z) \lessdot \chi R^m_H(z)$$

for $z \in \rho(H) \cap \rho(H_{\mathbf{a}})$ *and* $Re(z) < 0$.

Proof. Notice that we can write,

$$\chi R^m_{H_{\mathbf{a}}}(z) = \chi \int_0^\infty t^{m-1} e^{-t(H_{\mathbf{a}_0}+V)} e^{-t(-z)} dt$$

and

$$\chi R^m_H(z) = \chi \int_0^\infty t^{m-1} e^{-t(H_0+V)} e^{-t(-z)} dt$$

Then for $\phi \in L^2(\mathbb{R}^d)$ and $\mathrm{Re}(z) < 0$, we can write the first expression as

$$|\chi R^m_{H_{\mathbf{a}}}(z)\phi| \leq \chi \int_0^\infty t^{m-1} |e^{-t(H_{\mathbf{a}_0}+V)}\phi| e^{-t(-\mathrm{Re}(z))} dt$$

Then use the pointwise bound $e^{-H_{\mathbf{a}_0}+V} \lessdot e^{-H_0+V}$ to show,

$$|\chi R^m_{H_{\mathbf{a}}}(z)\phi| \leq \chi \int_0^\infty t^{m-1} e^{-tH_0+V} |\phi| e^{-t(-\mathrm{Re}(z))} dt$$

So,

$$|\chi R^m_{H_{\mathbf{a}_0+V}}(z)\phi| \quad \leq \quad \chi R^m_{H_0+V}(z)|\phi| \quad \text{when} \quad \mathrm{Re}(z) < 0$$

$\square$

The next result is from [14, p. 24] and tells us that if $\chi R^m_H(z)$ is in a certain trace ideal then so is $\chi R^m_{H_\mathbf{a}}(z)$.

Theorem 2.16. *If $n \geq 1$ is an integer, $A \underset{\cdot}{\overset{<}{-}} B$, and $B \in \mathcal{I}_{2p}$ then $A \in \mathcal{I}_{2p}$ and $\|A\|_{2n} \leq \|B\|_{2n}$*

Therefore, if we can show that $\chi R^m_H(z)$ is in a particular trace ideal, then $\chi R^m_{H_{\mathbf{a}_0}}(z)$ will be also for $\mathrm{Re}(z) < 0$. Lastly we can use the first resolvent formula, $R_A(\zeta) = R_A(z) + (z-\zeta)R_A(z)R_A(\zeta)$, to show $\chi R^m_{H_{\mathbf{a}_0}}(z)$ is in a particular trace ideal for every $z \in \rho(H) \cap \rho(H_\mathbf{a})$.

Chapter 3 Combes-Thomas Method With Trace Norms

In this chapter we generalize Theorem 1.2 from the introduction. That is, H is no longer the simple $H = -\Delta + V$ acting on $L^2(\mathbb{R}^d)$ but rather we use the *magnetic Schrödinger operator* as it was defined in the previous chapter,

$$H := (-i\nabla - \mathbf{a})^2 + V$$

acting on $L^2(\mathbb{R}^d)$, and with $\mathbf{a} \in L^2_{\text{loc}}(\mathbb{R}^d)$. This result will also expose a more precise rate of decay than just $\sqrt{\eta_z}$ that was stated in Theorem 1.2. To this end, define the constants γ and ν_z as follows. Choose γ be a real number such that $0 < \gamma < 1$. Let ν_z be a positive real constant that is a function of z and also depends on γ, the potential V, and and vector potential $\mathbf{a}$, then

$$\nu_z := \frac{3\eta_z \gamma \sqrt{1-a}}{8\sqrt{\eta_z + |z| + b}} \quad \text{with} \quad \eta_z = \operatorname{dist}\{z, \sigma(H)\}$$

Theorem 3.1. *Let s and p be positive real numbers such that $p \geq 1$ and $s > \frac{d}{2p}$. Let $0 < \gamma < 1$. Let $x, y \in \mathbb{R}^d$, $\chi_x, \chi_y : \mathbb{R}^d \to \mathbb{R}$ be functions with compact support, bounded by 1, and the support of χ_x, χ_y contain $x, y \in \mathbb{R}^d$ respectively. Then $\chi_x R_H^s(z) \chi_y$ is in the p^{th} trace ideal for $z \in \rho(H)$. And there exists a constant that depends on γ, d, p, s, V, z such that,*

$$\|\chi_x R_H^s(z) \chi_y\|_p \leq C_{\gamma,V,z}^{(s,d)} e^{-\nu_z \|x-y\|}$$

Proof of Theorem 3.1

Because of Lemma 2.15 we can assume $H = -\Delta + V$ with out loss of generality.

As with Germinet and Klein's paper [6], and Combes, Thomas paper before that [4], we consider the operator $U_\alpha = e^{i\alpha \cdot x}$ that acts by multiplication on a function $f \in \mathcal{H}^{(n)}$ as $(U_\alpha f)(x) = e^{i\alpha \cdot x} f(x)$. If $\alpha \in \mathbb{R}^d$ then U_α is a unitary operator on $\mathcal{H}^{(n)}$. The operator U_α will be the source of exponential decay in this result. We follow the

Combes–Thomas method that uses U_α with a Schrödinger operator H to build the family of operators $U_\alpha H U_\alpha^{-1}$ parameterized by α. Then analytically continue α so that $(U_\alpha H U_\alpha^{-1} - z)^{-1}$ remains a bounded operator for $z \in \rho(H)$.

To this end we define $H(\alpha) := U_\alpha H U_\alpha^{-1}$ and first write,

$$H(\alpha) = -\left(U_\alpha \nabla U_\alpha^{-1}\right)^2 + V$$

Computing $U_\alpha \nabla U_\alpha^{-1}$ gives

$$U_\alpha \nabla U_\alpha^{-1} = \nabla - i\alpha$$

Then $H(\alpha)$ becomes,

$$H(\alpha) = -\Delta + 2i\alpha \cdot \nabla + |\alpha|^2 + V$$

Then we have,

$$\begin{aligned} & H(\alpha) - z \\ = \quad & H - z + 2i\alpha \cdot \nabla + |\alpha|^2 \\ = \quad & \left[1 + 2i\alpha \cdot \nabla R_H(z) + |\alpha|^2 R_H(z)\right](H - z) \end{aligned} \tag{3.1}$$

We want to know for what α is the operator $R_{H(\alpha)}(z)$ is bounded on $L^2(\mathbb{R}^d)$. Or in other words, when is $H(\alpha) - z$ invertible. We assumed $z \in \rho(H)$, therefore $(H - z)$ is invertible. So we concern ourselves with the first multiplicative term on (3.1). Let

$$B(\alpha, z) := 2i\alpha \cdot \nabla R_H(z) + |\alpha|^2 R_H(z)$$

We establish when the first term is invertible through the Neumann series

$$(1 - B)^{-1} = \sum_{m=0}^{\infty} B^m$$

This series converges if and only if $\|B\| < 1$. We want to find the α's for which $\|B\| < 1$. To bound the first term of B we prove the following technical lemma.

Lemma 3.2. *Let V be Δ bounded with relative bound $a < 1$, and let $z \in \rho(H)$.*

Then

$$\|\nabla R_H(z)\| \leq \frac{1}{\eta_z} \sqrt{\frac{\eta_z + |z| + b}{1 - a}} \tag{3.2}$$

where b is defined in Theorem 1.1

Proof. See appendix. $\square$

Next we introduce another parameter γ that α will depend on and $0 < \gamma < 1$. This parameter is the radius of a closed ball in the algebra of bounded operators on which the Neumann series converges. Therefore we are searching for an α such that,

$$\|B(\alpha, z)\| \leq \gamma < 1 \tag{3.3}$$

There is a constant, ν_z that will depend on γ, z, the potential, the coefficient matrices $\mathcal{K}, \mathcal{R}$ and will bound $|\alpha|$. It is defined as,

$$\nu_z := \frac{3\eta_z\gamma\sqrt{1-a}}{8\sqrt{\eta_z + |z| + b}}$$

Lemma 3.3. *Let $\alpha \in \mathbb{C}^d$, $z \in \rho(H)$, and V be a potential that is relatively Δ bounded with relative $a < 1$. Then $R_{H(\alpha)}(z)$ is a bounded operator on $L^2(\mathbb{R}^d)$ when $|\alpha| \leq \nu_z$*

Proof. See appendix. $\square$

For $\alpha \in \mathbb{R}^d$, let $R_{H(\alpha)} = R_{H(\alpha)}(z)$. Next, we apply the Combes–Thomas method to $\chi_x R_H^s(z)\chi_y$.

$$\chi_x R_H^s(z)\chi_y = \chi_x U_\alpha R_{H(\alpha)}^s U_\alpha^{-1}\chi_y$$

Without loss of generality, we may assume s is an even integer. If s is odd use $R_{H(\alpha)}^s = R_{H(\alpha)}^{\frac{s-1}{2}} R_{H(\alpha)} R_{H(\alpha)}^{\frac{s-1}{2}}$ below. So we assume that $s \in \mathbb{N}$ is even. Because the trace norm has a Hölder type inequality it will be useful to write,

$$\chi_x R_H^s(z)\chi_y \quad = \quad \chi_x U_\alpha R_{H(\alpha)}^{s/2} R_{H(\alpha)}^{s/2} U_\alpha^{-1}\chi_y$$

Since $R_{H(\alpha)}(z)$ is a bounded, but not necessarily a compact operator our strategy centers around getting functions with compact support to act on the resolvent. We choose,

$$J(x), \bar{J}(x) \in C_0^\infty(\mathbb{R}^d)$$

such that,

$$J(x)\chi_x(x) = \chi_x(x) \quad \text{and} \quad \bar{J}(x)\chi_y(x) = \chi_y(x)$$

Then we write,

$$\chi_x R_H^s(z)\chi_y = \chi_x U_\alpha J^{s/2} R_{H(\alpha)}^{s/2} R_{H(\alpha)}^{s/2} \bar{J}^{s/2} U_\alpha^{-1}\chi_y \tag{3.4}$$

The next thing to do is commute the J's past the resolvents so that $J^{s/2}R^{s/2}$ become a product of $s/2$ JR's. A technical result found in S. Nakamura's paper [11, Appendix A] will help us do just that. This lemma allows us to exchange the power $s/2$ from $J^{s/2}R_{H(\alpha)}^{s/2}$ into a product of $s/2$ $JR_{H(\alpha)}Q$'s, where Q is a bounded operator. In the following proof, Q will come from commuting J past $R_{H(\alpha)}$. This will produce an extra resolvent and gradient. The proof will show how resolvents and gradients combine so that we indeed do get a bounded operator Q.

The bounded operator Q will be made up of the following parts. Choose a $J \in C_0^\infty(\mathbb{R}^d)$. Let $\mathcal{J}$ be a linear subspace of $C^\infty(\mathbb{R}^d)$ generated by linear combinations of $\{\partial_i^j J\}$, derivatives on J. Define Q_i, $i = 1,2,3,4$ to be the following $\{Q_1 = R_{H(\alpha)}, Q_2 = \nabla R_{R(\alpha)}, Q_3 = R_{H(\alpha)}\nabla, Q_4 = \nabla R_{H(\alpha)}\nabla\}$. Let $R = R_{H(\alpha)}(z)$ for $z \in \rho(H)$, $|\alpha| \leq \nu_z$. Define an algebra $\mathcal{Q}$ generated by polynomials of Q_1, Q_2, Q_3, Q_4 and the identity with coefficients from $\mathcal{J} \cup \mathbb{C}$. Note that the elements of $\mathcal{Q}$ depend on z.

Lemma 3.4. *For an integer $s \geq 2$ there exists $J_{\gamma\delta} \in \mathcal{J}$ and $Q_{\gamma\delta} \in \mathcal{Q}$ such that,*

$$J^s R_{H(\alpha)}^s = \sum_{\delta=1}^{N} \prod_{\beta=1}^{s} J_{\beta\delta} R_{H(\alpha)} Q_{\beta\delta} \tag{3.5}$$

Formally that is, one may exchange powers on J^sR^s for a product of s many JRQ terms.

As an example, we compute J^2R^2, where $R = R_{H(\alpha)}(z)$, to understand where $Q_{\gamma\delta}$ comes from. With the Hamiltonian $H = -\Delta + V$, start by commuting J past R,

$$\begin{aligned} JR &= RJ + [J,R] \\ &= RJ + R[J,H(\alpha)]R \end{aligned}$$

Notice that $[J,H(\alpha)]$ is a first order differential operator that is localized on the right.

$$\begin{aligned} [J,H(\alpha)] &= 2\nabla \cdot \nabla J + \Delta J - 2\alpha \cdot \nabla J \\ &= \nabla \cdot J_1 + J_2 - \alpha \cdot J_1 \end{aligned} \tag{3.6}$$

The first term has a gradient that must be combined with a resolvent on the right to make a bounded operator. J_1 in the first term must be commuted past another resolvent to do this.

$$
\begin{aligned}
[J,R] &= R\nabla\cdot J_1R + R(J_2 - i\alpha\cdot J_1)R \\
&= R\nabla R\cdot J_1 + R\nabla\cdot[J_1,R] + R(J_2 - i\alpha\cdot J_1)R \\
&= R\nabla R\cdot J_1 \\
&\quad +R\nabla\cdot R(\nabla\cdot J_2 + J_3 - i\alpha\cdot J_2)R \\
&\quad +RJ_2R - Ri\alpha\cdot J_1R \\
&= RQ_2\cdot J_1 \\
&\quad +RQ_4J_2R + RQ_2\cdot J_3R + RJ_2R \\
&\quad -RQ_2\cdot i\alpha J_2R - Ri\alpha\cdot J_1R
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\quad \text{(recall } Q_1 = R\text{)} \\
JJRR &= JRJR + J[J,R]R \\
&= JRJR + JRQ_2\cdot J_1R \\
&\quad +JRQ_4J_2RQ_1 + JRQ_2\cdot J_3RQ_1 + JRJ_2RQ_1 \\
&\quad -JRQ_2\cdot i\alpha J_2RQ_1 - JRi\alpha\cdot J_1RQ_1 \qquad (3.8) \\
J^2R^2 &= \sum_{\delta=0}^{6}\prod_{\gamma=0}^{2} J_{\delta\gamma}RQ_{\delta\gamma}
\end{aligned}
$$

With $Q_{11} = I, Q_{12} = I, Q_{21} = Q_2, Q_{22} = I, Q_{31} = Q_4, Q_{32} = Q_2$ etc.

Continuing with the general case, the next lemma establishes that Q_1, Q_2, Q_3, Q_4 are bounded operators, and thus $\mathcal{Q}$ is indeed an algebra of bounded operators.

Lemma 3.5. *Let γ be as defined above in* (3.3)*, then*

$$
\|R_{H(\alpha)}(z)\| \leq \frac{1}{\eta_z(1-\gamma)} \tag{3.9}
$$

and

$$
\|\nabla R_{H(\alpha)}(z)\| \leq \frac{1}{\eta_z(1-\gamma)}\sqrt{\frac{\eta_z + |z| + b}{1-a}} \tag{3.10}
$$

Let ∇V be relatively bounded with respect to Δ with bounding constants $\tilde{a}, \tilde{b}$. Then,

$$\|\nabla R_H(z)\nabla\| \;\leq\; \frac{\eta_z + |z| + b}{(1-a)\eta_z} + \frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\left[\frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\tilde{a} + \frac{\tilde{b}}{\eta_z}\right]$$

and

$$\|\nabla R_{H(\alpha)}\nabla\| \;\leq\; \frac{9}{16}\frac{\gamma^2}{1-\gamma} + \left(1 + \frac{3}{4}\frac{\eta_z\gamma}{\sqrt{\eta_z + |z| + b}}\right)\|AR_H A\|.$$

Proof. See appendix. □

With the previous result we can rewrite (3.4) as,

$$\begin{aligned}&\chi_x U_\alpha J^{s/2} R^{s/2}_{H(\alpha)} R^{s/2}_{H(\alpha)} \bar{J}^{s/2} U_\alpha \chi_y \\ &= \chi_x U_\alpha \left[\sum_{\delta=1}^{N}\prod_{\gamma=1}^{s/2} J_{\gamma\delta} R_{H(\alpha)} Q_{\gamma\delta}\right]\left[\sum_{\delta=1}^{N}\prod_{\gamma=1}^{s/2} \bar{Q}_{\gamma\delta} R_{H(\alpha)} \bar{J}_{\gamma\delta}\right] U_\alpha \chi_y\end{aligned}$$

where $\bar{Q}_{\gamma\delta} \in \mathcal{Q}$ is the result of commuting $\bar{J}$ past R. Then using Hölders inequality s times we do the following. Let p be such that $ps > \frac{d}{2}$. Then,

$$\begin{aligned}&\|\chi_x R^s_H \chi_y\|_p \\ &\leq \|\chi_x U_\alpha\| \sum_{\delta=1}^{N}\left\|\prod_{\gamma=1}^{s/2} J_{\gamma\delta} R_{H(\alpha)} Q_{\gamma\delta}\right\|_{p/2} \sum_{\delta=1}^{N}\left\|\prod_{\gamma=1}^{s/2} \bar{Q}_{\gamma\delta} R_{H(\alpha)} \bar{J}_{\gamma\delta}\right\|_{p/2} \|U_\alpha \chi_y\|\end{aligned}$$

Then consider just one of the products,

$$\left\|\prod_{\gamma=1}^{s/2} J_{\gamma\delta} R_{H(\alpha)} Q_{\gamma\delta}\right\|_{p/2} \;\leq\; \left\|J_{1\delta} R_{H(\alpha)} Q_{1\delta}\right\|_{p/s} \left\|\prod_{\gamma=2}^{s/2} J_{\gamma\delta} R_{H(\alpha)} Q_{\gamma\delta}\right\|_{\frac{p(s-2)}{2s}} \tag{3.11}$$

If one continues to iterate Hölders inequality through the above product, we see how to construct the following inequality.

$$\begin{aligned}&\|\chi_x R^s_H \chi_y\|_p \\ &\leq \|\chi_x U_\alpha\| \left[\sum_{\delta=1}^{N}\prod_{\gamma=1}^{s/2} \|J_{\gamma\delta} R_{H(\alpha)}\|_{p/s}\|Q_{\gamma\delta}\|\right]\left[\sum_{\delta=1}^{N}\prod_{\gamma=1}^{s/2} \|\bar{Q}_{\gamma\delta}\|\|R_{H(\alpha)}\bar{J}_{\gamma\delta}\|_{p/s}\right]\|U_\alpha \chi_y\|\end{aligned}$$

From the above inequality, the next task is to study which trace classes that each $J_{\gamma\delta}R_{H(\alpha)}$ is in. The main tool we have to study an operators of this form is result 2.11 from B. Simon's book [14, Chapter 4]. We apply the Theorem 2.11 by letting $f(x) = J(x)$ and $g(x) = g_z(x) := \dfrac{1}{x^2 - z}$. Let's investigate the $L^p(\mathbb{R}^d)$ norm $\|g_z\|_p^p$

Lemma 3.6. *Suppose $x \in \mathbb{R}^d$, $z \in \mathbb{C} - [0, \infty)$, and $p > d/2$. Then, $g_{z,s}(x) \in L^p(\mathbb{R}^d)$.*

Furthermore, if $Re(z) \geq 0$ there exists a constant, $\tilde{C}_{d,p}$, that depends on d, p, such that,

$$\|g_z\|_p^p \leq \frac{\tilde{C}_{d,p}|z|^{d/2}}{|Im(z)|^p} \tag{3.12}$$

If $Re(z) < 0$ there exists a constant, $C_{d,p}$, that depends on d, p, such that,

$$\|g_z\|_p^p \leq \frac{C_{d,p}}{|z|^{p-d/2}} \tag{3.13}$$

The above result illustrates the character of the singularity of g_z, which further reflects the spectrum of $-\Delta$. Meaning, $-\Delta$ has a spectrum that is the positive real axis. When z gets close to the spectrum of $-\Delta$, the estimates get larger. Lemma 3.6 is what causes the behavior in the following result.

Corollary 3.7. *Let $p \geq 2$ and $p > d/2$. Then χR_0 is in the p-th trace ideal.*

Also there is a constant, $C_{d,p}$, that depends on d and p such that,

$$\|\chi R_0(z)\|_p \leq \frac{C_{d,p}}{|z|^{1-d/2p}} \tag{3.14}$$

when $Re(z) < 0$.

And there is a constant, $\tilde{C}_{d,p}$, that depends on d and p such that,

$$\|\chi R_0(z)\|_p \leq \frac{\tilde{C}_{d,p}|z|^{d/2p}}{|Im(z)|} \tag{3.15}$$

when $Re(z) \geq 0$

The Hamiltonian we want to work with is more complicated than just $-\Delta$, so we wish to extend the trace ideal results to $\chi R_H(z)$ and $\chi R_{H(\alpha)}(z)$.

Corollary 3.8. *Assume the hypothesis from Lemma 3.3 and Corollary 3.7. Then for $\alpha \leq \nu_z$, $\chi R_H(z)$ and, $\chi R_{H(\alpha)}(z)$ are all in the p^{th} trace ideal for $p > d/2$ and $p \geq 2$. And they have the following estimates.*

$$\|\chi R_H(z)\|_p \leq \frac{C_{d,p}}{|z|^{1-d/2p}} \left[1 + \frac{a(\eta_z + |z|) + b}{\eta_z(1-a)}\right] \qquad Re(z) < 0 \tag{3.16}$$

$$\|\chi R_H(z)\|_p \leq \frac{\tilde{C}_{d,p}|z|^{d/2p}}{|Im(z)|} \left[1 + \frac{a(\eta_z + |z|) + b}{\eta_z(1-a)}\right] \qquad Re(z) \geq 0 \tag{3.17}$$

$$\|\chi R_{H(\alpha)}(z)\|_p \leq \frac{C_{d,p}(1+\gamma)}{|z|^{1-d/2p}} D_{z,\gamma,V} \qquad Re(z) < 0 \tag{3.18}$$

$$\|\chi R_{H(\alpha)}(z)\|_p \leq \frac{\tilde{C}_{d,p}|z|^{d/2p}}{|Im(z)|} D_{z,\gamma,V} \qquad Re(z) \geq 0 \tag{3.19}$$

With,

$$D_{z,\gamma,V} = \left(1 + \frac{a(\eta_z + |z|) + b)}{\eta_z(1-a)}\right)\left(1 + \frac{2\gamma}{\eta_z(1-\gamma)}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\right) \tag{3.20}$$

Corollary 3.7 and 3.8 tells us precisely which trace ideals a cutoff function acting on a resolvent will be in. And furthermore how they depend on $z \in \rho(H)$.

The next result we state for convenience in proving later results and to summarize what we have shown so far. It can be proven by looking at how we iterated Hlöders inequality in 3.11 and using Corollary 3.8.

Lemma 3.9. *Recall $\alpha \in \mathbb{C}^d$ and that s and r are integers such that $r > d/s$. Let $C^{(s,d,r)}_{\gamma,V,z}$ be a positive constant that depends on γ, the potential V, s, d, z and r. Then*

$$\|\chi_x U_\alpha R^{s/2}_{H(\alpha)}\|_r \leq C^{(s,d,r)}_{\gamma,V,z}|e^{i\alpha\cdot x}| \tag{3.21}$$

and

$$\|R^{s/2}_{H(\alpha)} U^{-1}_\alpha \chi_y\|_r \leq C^{(s,d,r)}_{\gamma,V,z}|e^{-i\alpha\cdot y}|. \tag{3.22}$$

Let $\hat{n} = \frac{x-y}{\|x-y\|}$. From lemma 3.3, let $\alpha \in \mathbb{C}^d$ with $\alpha = i\nu_z\hat{n}$. Then using lemma 3.9 and Hölder's inequality write,

$$\begin{aligned}
\|\chi_x R_H^s \chi_y\|_p &\leq \|\chi_x U_\alpha R_{H(\alpha)}^{s/2}\|_{2p}\|R_{H(\alpha)}^{m/2} U_\alpha^{-1}\chi_q\|_{2p} \\
&\leq C_{\gamma,V,z}^{(m,d,p)}|e^{i\alpha\cdot x}||e^{-i\alpha\cdot y}| \\
&\leq C_{\gamma,V,z}^{(m,d,p)} e^{-\nu_z\hat{n}\cdot x} e^{\nu_z\hat{n}\cdot y} \\
&\leq C_{\gamma,V,z}^{(m,d,p)} e^{-\nu_z\|y-x\|}
\end{aligned}$$

□

Chapter 4 Polynomial Decay

In the previous result a lot of effort was put into analytically continuing $R_{H(\alpha)}(z)$ in α. When we allowed α to be complex then $\|\chi_x U_\alpha\|$ gives us exponential decay. The price we payed was that there was an additional U_α^{-1} that had to combine with another operator, namely the resolvent of H, and still stay bounded when we analytically continued in α. In the next two chapters we generalize the previous main result and replace the resolvent of H with a function of H. To illustrate this we consider only the most basic ingredients.

Let $H = -\Delta$ and $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a $L^p(\mathbb{R})$ function with compact support. Define $f(x) := \tilde{f}(x^2)$. Then formally, $\tilde{f}(H) = f(-i\nabla)$. We apply the Combes–Thomas method to $\chi_x f(-i\nabla)\chi_y$ in the following way,

$$\chi_x f(-i\nabla)\chi_y = \chi_x U_\alpha U_\alpha^{-1} f(-i\nabla) U_\alpha U_\alpha^{-1} \chi_y$$

And so,

$$\|\chi_x f(-i\nabla)\chi_y\|_s \leq \|\chi_x U_\alpha\| \|U_\alpha^{-1} f(-i\nabla) U_\alpha\|_s \|U_\alpha^{-1}\chi_y\|$$

Now, when we analytically continue in α, say $\alpha = i\frac{y-x}{|x-y|}$ then

$$\begin{aligned}
\|\chi_x f(-i\nabla)\chi_y\|_s &\leq Ce^{\frac{(y-x)\cdot x}{|x-y|}} \|U_\alpha^{-1} f(-i\nabla) U_\alpha\|_s e^{-\frac{(y-x)\cdot y}{|x-y|}} \\
&\leq Ce^{-|x-y|} \|U_\alpha^{-1} f(-i\nabla) U_\alpha\|_s
\end{aligned}$$

We still need to know if $U_\alpha^{-1} f(-i\nabla) U_\alpha$ has an analytic extension in α and is in the right trace class. For that we use Theorem 2.11 again.

The resolvent may be thought of as a function of a Schrödinger operator. That is, let $G_z(x) = (x-z)^{-1}$. Then formally $R_H(z) = G_z(H)$. As it was described in the previous chapter, the Combes–Thomas method is about analytically continuing the family of operators $R_{H(\alpha)}$ from $\alpha \in \mathbb{R}^d$ into $\alpha \in \mathbb{C}^d$. This was done by using a Neumann series. Therefore, the reason we were able to analytically continue $R_{H(\alpha)}$ is because the form of $G_z(x)$ allowed us to use a Neumann series. The next result replaces $G_z(x)$ with a more general class of functions. Fortunately, this class of functions is tailored so that we may use the Helffer-Sjöstrand formula as found in [5]. As a result we will still be able to apply the Combes–Thomas method but we will not be able to get exponential decay. To this end, we define a set of functions called *slowing decreasing smooth functions*. Just as in [6], slowly decreasing smooth functions will be used with the Helffer-Sjöstrand formula as found in [5].

Definition 4.1.

- For a function $f : \mathbb{R} \to \mathbb{C}$, define the following norms,

$$|||f|||_n := \sum_{r=0}^{n} \int_{\mathbb{R}} |f^{(r)}(u)| \langle u \rangle^{r-1} du, \quad n = 1, 2, ... \tag{4.1}$$

- We say $f : \mathbb{R} \to \mathbb{C}$ is a *slowly decreasing smooth function* if it is infinitely differentiable and $|||f|||_n$ is finite for every n.

In order to extend the domain of a slowly decreasing smooth function to the complex plane, we use the following extention.

Definition 4.2. Let $z = u + iv$ and $\tau(t)$ be some even function on $\mathbb{R}$ that is equal to 1 when $t < 1/2$ and 0 when $t > 1$. Define an *almost analytic extension* as,

$$\tilde{f}(z) := \left\{ \sum_{r=0}^{n} f^{(r)}(u)(iv)^r / r! \right\} \sigma(u,v), \quad \sigma(u,v) := \tau(v/\langle u \rangle) \tag{4.2}$$

The derivative of an almost analytic extension is,

$$\begin{aligned} \frac{\partial \tilde{f}(z)}{\partial \bar{z}} &:= \frac{1}{2} \left\{ \frac{\partial \tilde{f}}{\partial u} + i \frac{\partial \tilde{f}}{\partial v} \right\} \\ &= \frac{1}{2} \left\{ \sum_{r=0}^{n} f^{(r)}(u)(iv)^r / r! \right\} \{\sigma_u(u,v) + i\sigma_v(u,v)\} \\ &\quad + \frac{1}{2} f^{n+1}(u)(iv)^n (n!)^{-1} \sigma(u,v) \end{aligned} \tag{4.3}$$

In the coming proofs, an important feature of $\tilde{f}(z)$ and $\frac{\partial \tilde{f}(z)}{\partial \bar{z}}$ is their support. In Figure 4.1 the support of $\sigma(u,v)$ is the entire colored region. The support of $\frac{\partial \sigma(u,v)}{\partial \bar{z}}$ are the two blue bands above and below the real axis.

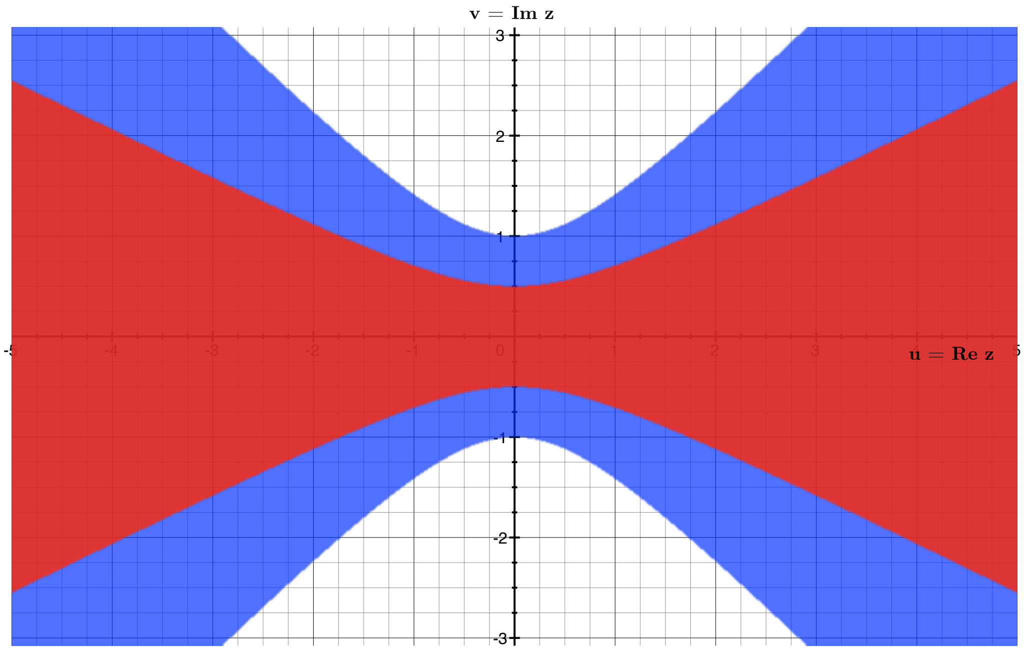

Figure 4.1: Illustrates the support of $\sigma(u,z)$, the union of the red and blue regions, and $\frac{\partial\sigma(u,v)}{\partial\bar{z}}$, the blue region, when f has non-compact support.

For $z = u + iv$, the *Helffer-Sjöstrand formula* is,

$$f(H) := \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}(z)}{\partial\bar{z}}(H-z)^{-1}dvdu \tag{4.4}$$

Theorem 4.3. *Let k and p be positive real numbers such that $k > d/2p$. Let V be relatively $-\Delta$ bounded with relative bound $a < 1$ and M be the lower semi-bound of H such that $-\infty < -M \leq H$. Let f be such that $f(u)(u+M)^k$ is a slowly decreasing function, then $\chi_x f(H)\chi_y$ is in the p trace ideal. There exists a constant C that depends on d, k, p, V and M such that*

$$\|\chi_x f(H)\chi_y\|_p \leq C\frac{|||f(u)(u+M)^k|||_{k+2}}{\langle x-y\rangle^k} \tag{4.5}$$

Remark 4.4. If f has compact support, then the integral in (4.6) is over a bounded set in $\mathbb{C}$ and $\hat{\nu}$ is bounded. If one skips approximating $e^{-\hat{\nu}\|x-y\|}$ and performs the calculations up to (4.10) and (4.11), then one would discover that $\|\chi_x f(H)\chi_y\|_p \leq Ce^{-c\|x-y\|}$ for some positive nonzero constants C, c.

Proof of Theorem 4.3

Since V is relatively $-\Delta$ bounded, H has a lower semi-bound, $-M \in \mathbb{R}$. Let f be a slowly decaying function. Let m be an integer greater than $d/2p$. Now if m is even, insert the operator $H + M$ into $\chi_x f(H)\chi_y$ in the following way,

$$\chi_x f(H)\chi_y = \chi_x(H+M)^{-m/2}(H+M)^{m/2}f(H)(H+M)^{m/2}(H+M)^{-m/2}\chi_y.$$

If m is odd then write,

$$\begin{aligned}\chi_x f(H)\chi_y &= \chi_x(H+M)^{-(m-1)/2}(H+M)^{(m-1)/2}f(H) \\ &\quad \times(H+M)^{(m-1)/2}(H+M)^{-(m-1)/2}\chi_y\end{aligned}$$

Without loss of generality, we consider the m is even case. The m odd case is a straight forward reprise of the even case.

Let $g_M(u) = (u+M)^m f(u)$. Apply the Helffer-Sjöstrand formula, and write the following,

$$\begin{aligned}\chi_x f(H)\chi_y &= \chi_x R_H^{m/2}(-M)g_M(H)R_H^{m/2}(-M)\chi_y \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \chi_x R_H^{m/2}(-M)\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}R_H(z)R_H^{m/2}(-M)\chi_y dvdu.\end{aligned}$$

Just as for Theorem 3.1, we consider the unitary operator U_α with $\alpha \in \mathbb{R}^d$ and from continuing α to $\mathbb{C}^d$ we require $|\alpha| \leq \nu_z$. However, the above equality contains resolvents that are evaluated at two different points in the resolvent set, one is from inserting powers of $(H+M)^{-1}$ and the other comes from the Helffer-Sjöstrand formula, $(H-z)^{-1}$. Let ν_z be defined as before,

$$\nu_z := \frac{3\eta_z\gamma\sqrt{1-a}}{8\sqrt{\eta_z+|z|+b}}$$

Since both $R_{H(\alpha)}(z)$ and $R_{H(\alpha)}(-M)$ will be analytically continued, we choose

$$|\alpha| \leq \hat{\nu} := \min\{\nu_{-M}, \nu_z\}$$

We write,

$$\begin{aligned}\chi_x f(H)\chi_y &= \chi_x U_\alpha R_{H(\alpha)}^{m/2}(-M)g_M(H(\alpha))R_{H(\alpha)}^{m/2}(-M)U_\alpha^{-1}\chi_y \\ &= \frac{1}{\pi}\int_{\mathbb{C}} \chi_x U_\alpha R_{H(\alpha)}^{m/2}(-M)\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}R_{H(\alpha)}(z)R_{H(\alpha)}^{m/2}(-M)U_\alpha^{-1}\chi_y dvdu.\end{aligned}$$

Then, for $|\alpha| \leq \hat{\nu}$, we use Hölder's inequality to obtain the upper bound,

$$\|\chi_x f(H)\chi_y\|_p \;\leq\; \frac{1}{\pi}\int_{\mathbb{C}} \|\chi_x U_\alpha R^{m/2}_{H(\alpha)}(-M)\|_{2p}\left|\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}\right|$$

Then use (6.6) and Lemma 3.9 to get,

$$\begin{aligned} &\cdot\|R_{H(\alpha)}(z)\|\|R^{m/2}_{H(\alpha)}(-M)U_\alpha^{-1}\chi_y\|_{2p}dvdu \\ \leq\; &\int_{\mathbb{C}} \frac{C^{(m,d,2p)}_{\gamma,V,-M}e^{-\hat{\nu}\|p-q\|}}{\eta_z(1-\gamma)}\left|\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}\right| dvdu \end{aligned} \tag{4.6}$$

We must show that the integral of (4.6) is finite. To control the behavior of $\frac{1}{\eta_z}$ we use $e^{-\hat{\nu}\|p-q\|}$ and bound it in the following way. If k is a positive integer then

$$e^{-t} \leq \frac{b_k}{t^k}\text{, where}\quad b_k = e^k k^k.$$

Then choose k to be a positive integer such that $k \geq m$. Also notice that when $z \in \rho(H)$ then $\eta_z \geq |v|$. Write,

$$\begin{aligned} \|\chi_x f(H)\chi_y\|_p \;&\leq\; \int_{\mathbb{C}} \frac{b_k C^{(m,d,2p)}_{\gamma,V,-M}}{\hat{\nu}^k\|x-y\|^k|v|(1-\gamma)}\left|\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}\right| dvdu \\ &\leq\; \frac{b_k C^{(m,d,2p)}_{\gamma,V,-M}}{(1-\gamma)\|x-y\|^k}\int_{\mathbb{C}} \frac{1}{\hat{\nu}^k|v|}\left|\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}\right| dvdu \end{aligned} \tag{4.7}$$

Davies' book [5, p.25] gives us an estimate on $\left|\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}\right|$. Let C be a constant that only depends on τ from equation (4.2).

$$\left|\frac{\partial \tilde{g}_M(z)}{\partial \bar{z}}\right| \leq C\left\{\sum_{r=0}^{n}\frac{1}{r!}|g_M^{(r)}(u)|\frac{|v|^r}{\langle u\rangle}\right\}\chi_A(u,v) + \frac{1}{2n!}|g_M^{(n+1)}(u)||v|^n\chi_B(u,v) \tag{4.8}$$

Where A and B are sets in $\mathbb{C}$ such that

$$A = \{\tfrac{1}{2}\langle u\rangle < |v| < \langle u\rangle\}\quad\text{and}\quad B = \{0 \leq |v| < \langle u\rangle\}.$$

The supports of σ_u and σ_v lie in A and the support of σ lies in B where σ, σ_u and σ_v are defined in (4.2). In the Figure 4.1, A corresponds to the bands above and below the real axis and B corresponds to the entire colored region. χ_A and χ_B are the corresponding characteristic functions.

In the integral of (4.7) notice that $\hat{\nu}$ depends on z. In some regions of $\mathbb{C}$ $\nu_{-M} \geq \nu_z$ and in other regions $\nu_{-M} \leq \nu_z$. Divide the integral in equation (4.7) into two cases:

$\nu_{-M} \leq \nu_z$ and $\nu_{-M} \leq \nu_z$. The final result will be the max of the two cases. First suppose that $\nu_{-M} \leq \nu_z$. Then $\hat{\nu}$ is constant with respect to z.

Combine (4.8), (4.7) and consider the integration only over region A.

$$C\sum_{r=0}^{n}\frac{2b_k C_{\gamma,V,-M}^{(m,d,2p)}}{r!\|x-y\|^k}\int_A \frac{|v|^{r-1}|g_M^{(r)}(u)|}{(1-\gamma)\hat{\nu}^k\langle u\rangle}dvdu \tag{4.9}$$

Combine the constants C, $C_{\gamma,V,-M}^{(m,d,2p)}$, $\frac{1}{\hat{\nu}}$, $1-\gamma$ and the 2 into the constant $\tilde{C}$ and do the following computation with (4.9),

$$\begin{aligned}
& \tilde{C}\sum_{r=0}^{n}\frac{b_k}{r!\|x-y\|^k}\int_A \frac{|v|^{r-1}|g_M^{(r)}(u)|}{\langle u\rangle}dvdu \\
= & \tilde{C}\sum_{r=0}^{n}\frac{b_k}{r!\|x-y\|^k}\int_{-\infty}^{\infty}2\int_{\langle u\rangle}^{2\langle u\rangle}\frac{|v|^{r-1}|g_M^{(r)}(u)|}{\langle u\rangle}dvdu \\
\leq & \tilde{C}\sum_{r=0}^{n}\frac{b_k c_r}{r!\cdot\|x-y\|^k}\int_{-\infty}^{\infty}\langle u\rangle^{r-1}|g_M^{(r)}(u)|du \\
& \text{with } c_0=\ln 2,\text{ and } c_r=\frac{2^{r+1}}{r}\text{ otherwise} \\
\leq & \tilde{C}\frac{b_k c_n}{n!\cdot\|x-y\|^k}\sum_{r=0}^{n}\int_{-\infty}^{\infty}\langle u\rangle^{r-1}|g_M^{(r)}(u)|du
\end{aligned} \tag{4.10}$$

Again, we combine (4.8), (4.7) then consider region B.

$$\begin{aligned}
& & \tilde{C}\frac{b_k}{n!\|x-y\|^k}\int_B |v|^{n-1}|g_M^{(n+1)}(u)|dvdu \\
\text{(recall } n\geq 1) & = & \tilde{C}\frac{b_k}{n!\|x-y\|^k}2\int_{-\infty}^{\infty}\frac{2^n}{n}\langle u\rangle^n g_M^{(n+1)}(u)|dvdu \\
& = & \tilde{C}\frac{b_k c_n}{n!\|x-y\|^k}\int_{-\infty}^{\infty}\langle u\rangle^n g_M^{(n+1)}(u)|dvdu
\end{aligned} \tag{4.11}$$

The estimate (4.10) together with (4.11) show that

$$\|\chi_x f(H)\chi_y\|_p \leq \tilde{C}\frac{b_k c_n}{n!\cdot\|x-y\|^k}|||f(u)(u+M)^k|||_n$$

Next consider the case $\nu_{-M} > \nu_z$. Then $\hat{\nu}$ depends on z as $\hat{\nu} = \frac{3\eta_z\gamma\sqrt{1-a}}{8\sqrt{\eta_z+|z|+b}}$. Since from (4.7) we must integrate over u and v, we will want to manipulate $\hat{\nu}$ so that it is explicitly in terms of u and v. From Theorem 3.1, we may choose γ to be any number between 0 and 1. Following the same technique from [6] we let γ depend on z and choose $\gamma = (2+\sqrt{\eta_z})^{-1}$ which gives,

$$\hat{\nu} = \frac{3\eta_z\sqrt{1-a}}{8(\sqrt{\eta_z}+2)\sqrt{\eta_z+|z|+b}}$$

Notice that $\eta_z \geq |v|$ for every $z \in \rho(H)$. Since $\hat{\nu}$ increases with respect to η_z, the first estimate we do on $\hat{\nu}$ is,

$$\hat{\nu} \geq \frac{3|v|\sqrt{1-a}}{8(\sqrt{|v|}+2)\sqrt{|v|+|z|+b}}$$

Just as before, combine (4.8), (4.7) and consider the integration only over region A just like in (4.9). In this region, we remind ourselves that $\langle u\rangle \leq |v| \leq 2\langle u\rangle$. Now, let's derive a lower estimate on $\hat{\nu}$ that is specific to A. Since $\langle u\rangle \leq |v|$, then $v \geq 1$ and $u^2 \leq v^2 - 1$. Also $|z| = \sqrt{v^2+u^2}$. Therefore $|z| \leq \sqrt{2}|v|$. And we also get that,

$$\begin{aligned}
\hat{\nu} &\geq \frac{3|v|\sqrt{1-a}}{8\sqrt{|v|}(1+2/\sqrt{|v|})\sqrt{|v|(1+\sqrt{2}+b/|v|)}} \\
&\geq \frac{3|v|\sqrt{1-a}}{8\cdot 3|v|\sqrt{1+\sqrt{2}+b}} \\
&\geq \frac{\sqrt{1-a}}{8\sqrt{3+b}}
\end{aligned}$$

Or in other words, $\hat{\nu}$ is bounded below by a constant that is greater than 0. Therefore, one may repeat the calculation for (4.10) and conclude that

$$(4.9) \leq \tilde{C}_2 \frac{b_k c_n}{n! \cdot \|x-y\|^k} \sum_{r=0}^{n} \int_{-\infty}^{\infty} \langle u\rangle^{r-1} |g^{(r)}(u)| du$$

Lastly, we consider combining (4.8), (4.7) then integrating over region B with $\hat{\nu} = \nu$. Just as in the previous calculation we have that

$$\hat{\nu} \geq \frac{3|v|\sqrt{1-a}}{8(\sqrt{|v|}+2)\sqrt{|v|+|z|+b}}$$

Since z is in the region B recall that $|v| \leq 2\langle u\rangle$. We will also need the estimate $|z| \leq 3\langle u\rangle$ which we derive in the following way,

$$\begin{aligned}
|z| &\leq \sqrt{u^2+v^2} \leq \sqrt{u^2+4\langle u\rangle^2} \\
&\leq \sqrt{u^2+4(1+u^2)} \leq \sqrt{5}\langle u\rangle \\
&\leq 3\langle u\rangle
\end{aligned}$$

Another inequality we use is for two positive numbers, say x and y. There is the fundamental inequality $2xy \leq x^2 + y^2$. With the above inequalities we may write,

$$
\begin{aligned}
\hat{\nu} \quad &\geq \quad \frac{3|v|\sqrt{1-a}}{8(\sqrt{2\langle u\rangle}+2)\sqrt{5\langle u\rangle + b}} \geq \frac{3|v|\sqrt{1-a}}{8\frac{1}{2}[(\sqrt{2\langle u\rangle}+2)^2 + 5\langle u\rangle + b]} \\
&\geq \quad \frac{3|v|\sqrt{1-a}}{8\frac{1}{2}[2\langle u\rangle + 2\sqrt{2\langle u\rangle} + 4 + 5\langle u\rangle + b]} \geq \frac{3|v|\sqrt{1-a}}{8\frac{1}{2}[15\langle u\rangle + b]} \\
&\geq \quad \frac{3|v|\sqrt{1-a}}{8\langle u\rangle[8+b]}
\end{aligned}
$$

Use the above estimate while combining (4.8), (4.7). Recall that we chose $\gamma = (2+\sqrt{\eta_z})^{-1}$, therefore $\gamma \leq \frac{1}{2}$. This give the following estimate,

$$
\begin{aligned}
&\frac{b_k C^{(m,d,2p)}_{\gamma,V,-M}}{(1-\gamma)2n!\|x-y\|^k} \int_B \frac{1}{\hat{\nu}^k |v|} |g^{(n+1)}(u)||v|^n dv du \\
\leq \quad &\frac{b_k C^{(m,d,2p)}_{\gamma,V,-M}}{4n!\|x-y\|^k} \int_B \left(\frac{8[8+b]}{3\sqrt{1-a}}\right)^k \frac{|v|^{n-1-k}}{\langle u\rangle^k} |g^{(n+1)}(u)| dv du
\end{aligned}
$$

In the above estimate we are consolidating the constants into $\hat{C}$. In order for the integration in the v variable to be finite, we require that $n-1-k \geq 0$.

$$
\begin{aligned}
&\frac{b_k \hat{C}}{n!\|x-y\|^k} \int_{-\infty}^{\infty} \int_0^{\langle u\rangle} \frac{|v|^{n-1-k}}{\langle u\rangle^k} |g^{(n+1)}(u)| dv du \\
\leq \quad &\frac{b_k \hat{C}}{n!(n-k)\|x-y\|^k} \int_{-\infty}^{\infty} \langle u\rangle^n |g^{(n+1)}(u)| du
\end{aligned}
$$

Then we choose $n = k+1$ and obtain our result.

Chapter 5 Sub-Exponential Decay

The next result explores what happens when f is close to an analytic function. In a paper by J. Bouclet, F. Germinet and A. Klein [3] the *L^1-Gevrey class of order* $a \geq 1$ is used with the Helffer–Sjöstrand formula to show sub-exponential decay in the operator norm. We say a function is *Gevrey class of order* $a \geq 1$ [3] if $f \in C^\infty(\mathbb{R})$ and for each compact subset $K \subset \mathbb{R}$ there is a constant, C, that depend on K such that,

$$|f^{(n)}(x)| \leq C(C(n+1)^a)^n \quad \text{with} \quad x \in K \,\text{and}\, n = 0, 1, 2, ...$$

Recall that a function is real analytic if and only if there exists a positive real constant C that depends on the compact subset $K \subset \mathbb{R}$ such that for each n and every $x \in K$,

$$|f^{(n)}(x)| \leq C^{n+1} n!$$

To see this, Taylor's theorem with remainder says that a real function with $k+1$ derivatives can be written in a series with a remainder as

$$f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1}$$

Then using $|f^{(n)}(x)| \leq C^{n+1} n!$, and apply it to the remainder, we can see that the remainder converges to 0 for a certain radius of convergence. Therefore f has a power series about the point a, and thus f is analytic.

We see, after using Stirling's formula on $n!$, that when $a = 1$ the Gevrey class of functions are analytic. Mixing the idea of slowly decreasing functions with the Gevrey class, L^1-Gevrey is defined as follows.

Definition 5.1. Let I be an open interval and $a \geq 1$, $f \in C^\infty(\mathbb{R})$, and $C_{f,I}$ be a constant greater than one that depends on f and I. The function f is *L^1-Gevrey of class a on I* if for all $k = 0, 1, 2, ...$ we have

$$\left\| f^{(k)} \langle u \rangle^{k-1} \right\|_{L^1(I)} = \int_I |f^{(k)}(u)| \langle u \rangle^{k-1} du \leq C_{f,I}(C_{f,I}(k+1)^a)^k \tag{5.1}$$

Our final result we prove by following the proof from [3]. For a function f that is L^1-Gevery, we prove a sub-exponential result of the form $\|\chi_x f(H)\chi_y\|_p \leq Ce^{-c|x-y|^\alpha}$ for some $\alpha \in (0,1]$ that depends on a and χ_x, χ_y are the same localization functions as in the previous chapters.

Theorem 5.2. *Let $I \subset \mathbb{R}$ be an open interval that contains the spectrum of H. Let $x, y \in \mathbb{R}^d$ such that $|x-y| > 1$. Let M be a positive real number such that $-M \leq H$. And, let m, p be positive real numbers such that $m > d/2p$. Then, given a real number $a \geq 1$, we have:*

- *If $a > 1$, then for each $a' > a$ there exists $C_{a,a',I} > 0$ such that for f such that $f(u)(u+M)^m$ is L^1- Gevrey of class a on I,*

$$\|\chi_x f(H)\chi_y\|_p \leq C_{a,a',I}(eC_{f,I})^{3+|p-q|^{\frac{1}{a'}}} \exp\left(-\frac{a'-a}{2a'}|x-y|^{\frac{1}{a'}} \ln|x-y|\right) \quad (5.2)$$

  Also, there exists a constant $C_{a,a',I,f} > 0$ and $c_{a,a'} > 0$ such that,

$$\|\chi_p f(H)\chi_q\|_p \leq C_{a,a',I,f} \exp\left(-c_{a,a'}|x-y|^{\frac{1}{a'}}\right) \quad (5.3)$$

- *If $a = 1$, then there exists $C_I, c_I > 0$ such that for all L^1- Gevrey functions f of class a on I*

$$\|\chi_x f(H)\chi_y\|_p \leq C_I \frac{C_{f,I}\langle x-y\rangle}{1+\ln C_{f,I}} \exp\left(-\frac{c_I}{C_{f,I}(1+\ln C_{f,I})}|x-y|\right) \quad (5.4)$$

Proof of Theorem 5.2

Notice that if a function is L^1-Gevrey then it is also slowly decreasing. Therefore we may apply the Helffer-Sjöstrand formula the same way we did in proving Theorem 4.3. As before we let $g_M(u) = (u+M)^m f(u)$. Then apply the Helffer-Sjöstrand formula to $g(u)$. Since $\tilde{g}(z)$ may not be analytic when the imaginary part of z is zero, the main idea in the proof is to divide the complex plane into two regions: a region close to the real axis, and everything else.

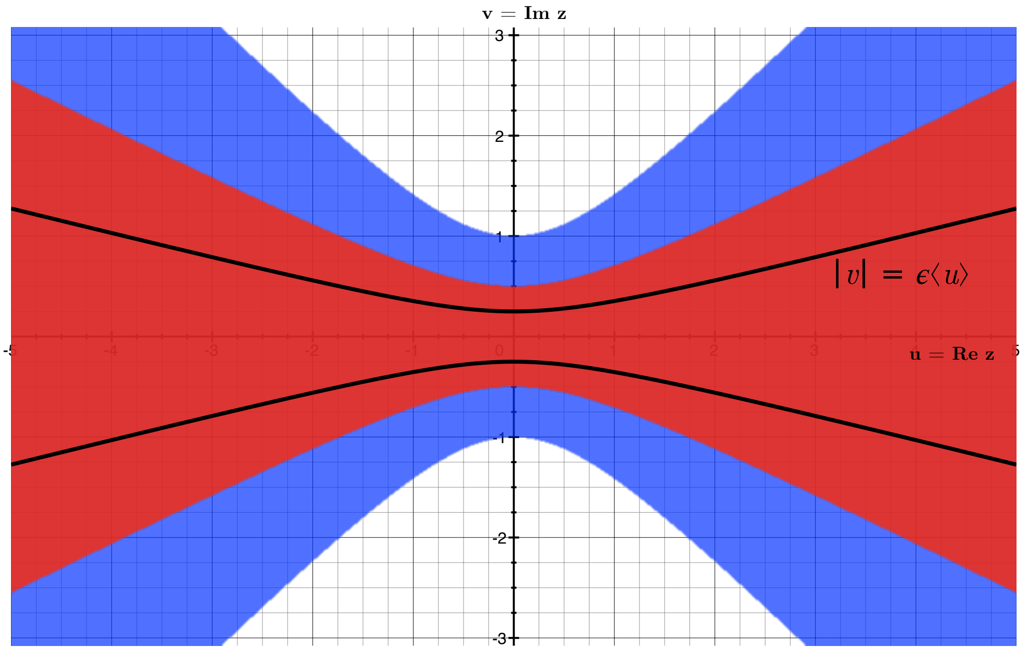

Figure 5.1: ϵ Region

To this end let $\epsilon > 0$ and recall $z = u + iv$. Then,

$$\begin{aligned} g_M(H) &= \frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{g_M}(z)}{\partial \bar{z}} R_H(z) dv du \\ &= \frac{1}{\pi}\int\int_{|v|<\epsilon\langle u\rangle} \frac{\partial \tilde{g_M}(z)}{\partial \bar{z}} R_H(z) dv du \\ &\quad + \frac{1}{\pi}\int\int_{|v|\geq\epsilon\langle u\rangle} \frac{\partial \tilde{g_M}(z)}{\partial \bar{z}} R_H(z) dv du \\ &= \mathrm{I} + \mathrm{II} \end{aligned}$$

In Figure 5.1 region I is the area between the two black lines that surround the real axis and region II is everything outside. To prove our result we tackle I and II by themselves in the next two lemmas.

Lemma 5.3. *There exists $C, c > 0$ that depend on I such that,*

$$\|\chi_x (H+M)^{-m/2} \mathrm{I} (H+M)^{-m/2} \chi_y\|_p \leq \frac{C\epsilon^n}{n!} \left\| g^{(n+1)} \langle u \rangle^n \right\|_{L^1(I)}$$

for all $n \geq 1$, $0 < \epsilon < 1/2$, and $x, y \in \mathbb{R}^d$

Proof. Let $\epsilon < 1/2$ so that we are integrating I in the region that $\sigma(u,v) = 1$ and $\sigma_u(u,v) = \sigma_v(u,v) = 0$. Therefore (4.3) just becomes,

$$\frac{\partial \tilde{g}(z)}{\partial \bar{z}} = \frac{1}{2} g^{n+1}(u)(iv)^n (n!)^{-1} \tag{5.5}$$

Then,

$$\begin{aligned} & \|\chi_x (H+M)^{-m/2} \mathrm{I} (H+M)^{-m/2} \chi_y\|_p \\ = & \left\| \frac{1}{\pi} \int_{\mathbb{R}} \int_{|v| \le \epsilon \langle u \rangle} \chi_x R_H^{m/2}(-M) \frac{1}{2} g^{n+1}(u) \frac{(iv)^n}{n!} R_H(z) R_H^{m/2}(-M) \chi_y dv du \right\|_p \\ \le & \frac{1}{2\pi n!} \int_{\mathbb{R}} \int_{|v| \le \epsilon \langle u \rangle} \left\| \chi_x J^{m/2} R_H^{m/2}(-M) \right\|_{2p} \left\| R_H^{m/2}(-M) \bar{J}^{m/2} \chi_y \right\|_{2p} \\ & \times \left| g^{n+1}(u)(iv)^n \right| \|R_H(z)\| \, dv dy \end{aligned}$$

Using an earlier result from Corollary 3.8 and Lemma 3.4 one can show

$$\left\| \chi_x J^{m/2} R_H^{m/2}(-M) \right\|_{2p} \le \|\chi_x\|_\infty \left\| J^{m/2} R_H^{m/2}(-M) \right\|_{2p} \le C_M$$

Then we can write,

$$\begin{aligned} \|\chi_x (H+M)^{-m/2} \mathrm{I} (H+M)^{-m/2} \chi_y\|_p & \le \frac{C_M}{2\pi n!} \int_I \int_{|v| \le \epsilon \langle u \rangle} |g^{n+1}(u)| |v|^{n-1} dv du \\ & \le C_M \int_I |g^{n+1}(u)| \langle u \rangle^n \epsilon^n du \\ & = \frac{C_M \epsilon^n}{n!} \|g^{(n+1)} \langle u \rangle^n\|_{L^1(I)} \end{aligned}$$

□

Next we address II which deals with integrating $\frac{\partial \tilde{g}}{\partial \bar{z}}$ in the complex plane but away from the real axis. See Figure 5.1. In this region we will apply the Combes-Thomas method and will be able to show exponential decay.

Lemma 5.4. *There exists constants $C, c > 0$ such that*

$$\begin{aligned} & \|\chi_x (H+M)^{-m/2} \mathrm{II} (H+M)^{-m/2} \chi_y\|_p \\ \le & \; C e^{-c|x-y|} \sum_{r=0}^{n} \frac{1}{r!} \left\| f^{(r)} \langle u \rangle^{r-1} \right\|_{L^1(I)} + \frac{C}{n!} e^{-c\epsilon|x-y|} \left\| f^{(n+1)} \langle u \rangle^n \right\|_{L^1(I)} \end{aligned} \tag{5.6}$$

For all $\epsilon > 0$, $n \ge 1$, $x, y \in \mathbb{R}^d$

Proof. We start our calculation,

$$
\begin{aligned}
&\|\chi_x(H+M)^{-m/2}\mathrm{II}(H+M)^{-m/2}\chi_y\|_p \\
\leq\ & \frac{1}{2\pi}\sum_{r=0}^{n}\frac{1}{r!}\int\int_{|v|\geq\epsilon\langle u\rangle}|g^{(r)}(u)||v|^r\left|\frac{\partial\sigma(u,v)}{\partial\bar{z}}\right|\cdot Z\,dvdu \\
&+\frac{1}{2\pi n!}\int\int_{|v|\geq\epsilon\langle u\rangle}\sigma(u,v)|g^{(n+1)}(u)||v|^n\cdot Z\,dvdu \\
&\text{with}\quad Z=\|\chi_x(H+M)^{-m/2}R_H(z)(H+M)^{-m/2}\chi_y\|_p
\end{aligned}
$$

Just as in the proof of Theorem 3.1 we apply the Combes–Thomas method to Z to obtain,

$$Z\leq\frac{C}{\eta_z}e^{-\hat{\nu}|x-y|}$$

where C does not depend on z or x,y. Therefore,

$$
\begin{aligned}
&\|\chi_x(H+M)^{-m/2}\mathrm{II}(H+M)^{-m/2}\chi_y\|_p \\
\leq\ & \frac{C}{2\pi}\sum_{r=0}^{n}\frac{1}{r!}\int\int_{|v|\geq\epsilon\langle u\rangle}|g^{(r)}(u)|\frac{|v|^r}{\eta_z}\left|\frac{\partial\sigma(u,v)}{\partial\bar{z}}\right|e^{-\hat{\nu}|x-y|}\,dvdu \\
&+\frac{C}{2\pi n!}\int\int_{|v|\geq\epsilon\langle u\rangle}\sigma(u,v)|g^{(n+1)}(u)|\frac{|v|^n}{\eta_z}e^{-\hat{\nu}|x-y|}\,dvdu
\end{aligned}
$$

Next notice that from the definition of $\sigma(u,v)$ we can estimate $\left|\frac{\partial\sigma(u,v)}{\partial\bar{z}}\right|\leq\frac{C}{\langle u\rangle}$ where C only depends on $\tau(t)$. Also notice that since $\epsilon<\frac{1}{2}$ the support of $\left|\frac{\partial\sigma(u,v)}{\partial\bar{z}}\right|$ and $\sigma(u,v)$ allow us to write,

$$
\begin{aligned}
&\|\chi_x(H+M)^{-m/2}\mathrm{II}(H+M)^{-m/2}\chi_y\|_p \\
\leq\ & \frac{C}{2\pi}\sum_{r=0}^{n}\frac{1}{r!}\int_{\mathbb{R}}\frac{2}{\langle u\rangle}\int_{\frac{1}{2}\langle u\rangle}^{\langle u\rangle}|g^{(r)}(u)|\frac{|v|^r}{\eta_z}e^{-\hat{\nu}|x-y|}\,dvdu \\
&+\frac{C}{2\pi n!}\int_{\mathbb{R}}2\int_{\epsilon\langle u\rangle}^{\langle u\rangle}|g^{(n+1)}(u)|\frac{|v|^n}{\eta_z}e^{-\hat{\nu}|x-y|}\,dvdu
\end{aligned}
$$

On the domain of integration from II, we can compute that $\frac{|v|}{\langle u\rangle}\leq\hat{\nu}$. Furthermore, since $|v| \geq \epsilon\langle u\rangle$ we have $\epsilon \leq \frac{|v|}{\langle u\rangle}$. Also notice that $\langle u\rangle \leq |v| \leq \eta_z$. Then,

$$\begin{aligned}
&\|\chi_x(H+M)^{-m/2}\mathrm{II}(H+M)^{-m/2}\chi_y\|_p \\
\leq\ & \frac{Ce^{-\frac{1}{2}|x-y|}}{\pi}\sum_{r=0}^{n}\frac{1}{r!}\int_I \frac{1}{\langle u\rangle^2}\int_{\frac{1}{2}\langle u\rangle}^{\langle u\rangle}|g^{(r)}(u)||v|^r\,dvdu \\
&+\frac{Ce^{-\epsilon|x-y|}}{\pi n!}\int_I\int_{\epsilon\langle u\rangle}^{\langle u\rangle}|g^{(n+1)}(u)||v|^{n-1}\,dvdu \\
\leq\ & \frac{Ce^{-\frac{1}{2}|x-y|}}{\pi}\sum_{r=0}^{n}\frac{1}{r!}\int_I |g^{(r)}(u)|\langle u\rangle^{r-1}\,du+\frac{Ce^{-\epsilon|x-y|}}{\pi n!}\int_I|g^{(n+1)}(u)|\langle u\rangle^n\,du
\end{aligned}$$

□

To lemma 5.3 and 5.4 we apply the $L^1(I)$-Gevery of class a estimates to write,

$$\begin{aligned}
&\|\chi_x f(H)\chi_y\|_p \\
\leq\ & Ce^{-c|x-y|}\sum_{r=0}^{n}\frac{1}{r!}\left\|g^{(r)}\langle u\rangle^{r-1}\right\|_{L^1(I)}+\frac{C}{n!}\left[\epsilon^n+e^{-c\epsilon|x-y|}\right]\left\|g^{(n+1)}\langle u\rangle^n\right\|_{L^1(I)} \\
\leq\ & Ce^{-c|x-y|}\sum_{r=0}^{n}\frac{C_{g,I}(C_{g,I}(r+1)^a)^r}{r!}+\frac{C}{n!}\left[\epsilon^n+e^{-c\epsilon|x-y|}\right]C_{g,I}(C_{g,I}(n+2)^a)^{n+1}
\end{aligned}$$

Next we want to estimate $\frac{(r+1)^{ar}}{r!}$. Use Stirling's approximation we know $r! \geq (r/e)^r$ for $r \geq 1$. Therefore, $\frac{(r+1)^{ar}}{r!} \leq \frac{e^r(r+1)^{ar}}{r^r}$. To estimate $\frac{(r+1)^{ar}}{r^r}$ we start by looking at the series expansion of the exponential function to see $1+\frac{1}{r} \leq e^{\frac{1}{r}}$. Then

$$\ln\left(\frac{r+1}{r}\right) \leq \frac{1}{r} \quad \text{and further} \quad \left(\frac{r+1}{r}\right)^r \leq e.$$

But then $(r+1)^{ra} \leq e^a r^{ra}$, so $\frac{(r+1)^{ra}}{r^r} \leq e^a r^{r(a-1)}$.

To estimate $(n+2)^{a(n+1)}/n^n$ and show $(n+2)^{a(n+1)}/n^n \leq e^{4a}n^{a(n+1)-n}$ notice that it is equivalent to $(n+2)^{a(n+1)} \leq e^{4a}n^{a(n+1)}$ then start with the inequalities $1+\frac{4}{n+1} \leq e^{\frac{4}{n+1}}$ and $n+1 \leq 2n$. Therefore $1+2/n \leq e^{\frac{4}{n+1}}$ or $\frac{n+1}{n} \leq e^{\frac{n}{n+1}}$. With a little more algebra it is clear that $(n+2)^{a(n+1)}/n^n \leq e^{4a}n^{a(n+1)-n}$. Now we can estimate $\|\chi_x f(H)\chi_y\|_p$ in the following way,

$$\begin{aligned}
&\|\chi_x f(H)\chi_y\|_p \\
\leq\ & Ce^{-c|x-y|}\sum_{r=0}^{n}C_gC_{g,I}^r e^a e^r r^{r(a-1)}+C\left[\epsilon^n+e^{-c\epsilon|x-y|}\right]C_{g,I}C_{g,I}^{n+1}e^{4a}e^n n^{a(n+1)-n} \\
\leq\ & Ce^{-c|x-y|}n^aC_{g,I}C_{g,I}^n e^a e^n n^{n(a-1)}+C\left[\epsilon^n+e^{-c\epsilon|x-y|}\right]C_{g,I}C_{g,I}^{n+1}e^{4a}e^n n^a n^{n(a-1)} \\
\leq\ & Ce^{4a}\left[e^{n\ln\epsilon}+e^{-c\epsilon|x-y|}+e^{-c|x-y|}\right](eC_{g,I})^{n+2}e^{a\ln n+(a-1)n\ln n} \qquad (5.7)
\end{aligned}$$

Since the Gevrey class of functions is supposed to be analytic functions when $a = 1$, we expect to recover exponential decay in the above estimate and when $a > 1$ we do not. To this end, we consider two cases.

In the first case take $a = 1$. Fix ϵ. We will let n depend on x and y and let it grow as $\delta|x-y|$. Formally we write $n = \delta|x-y|$ so n is the smallest integer that is greater than $\delta|x-y|$. So,

$$\|\chi_x f(H)\chi_y\|_p \;\leq\; Cn(eC_{g,I})^{n+2}\left[e^{-c|x-y|} + e^{n\ln\epsilon} + e^{-c\epsilon|x-y|}\right] \tag{5.8}$$

The term $(eC_{g,I})^{n+2}$, by itself, does not exponentially decay. So it must be compared agains each of the terms $e^{-c|p-q|} + e^{n\ln\epsilon} + e^{-c\epsilon|x-y|}$. Notice $(eC_{g,I})^{n+2} = (eC_{g,I})^2 e^{n\ln(eC_{g,I})}$. Comparing to the first term means δ must be such that $\delta\ln(eC_{g,I}) < c$. Comparing to the second term means we must choose ϵ such that $\ln(eC_{g,I}) < -\ln\epsilon$. So let $\epsilon = \frac{1}{e^2C_{g,I}}$. Comparing the third term, δ and ϵ must be chosen so that $\delta\ln(eC_{g,I}) < c\epsilon$. We can combine these estimates and see that we can choose $\delta = \frac{c}{2\ln(eC_{g,I})e^2C_{g,I}}$. With our choice of δ, ϵ and $n = \delta|x-y|$ the equation (5.8) can be written

$$\|\chi_x f(H)\chi_y\|_1 \leq Cn(eC_{g,I})^{n+2}e^{n\ln\epsilon}$$

Then feeding δ, ϵ and $n = \delta|x-y|$ into the above will yield our result.

Next let $a > 1$. The sub-exponential decay of Theorem 5.2 centers around using our control over n and ϵ to simultaneously cause the following expressions from the exponents of (5.7), $(a-1)n\ln n - c|p-q|$, $(a-1)n\ln n + n\ln\epsilon$, and $(a-1)n\ln n - c\epsilon|p-q|$, to be negative.

For $s, \delta > 0$ pick $n \sim |x-y|^\delta$ then we formally choose $n = |x-y|^\delta$ and let $\epsilon = |x-y|^{-s}$. Or, formally, we can write $n^{1/\delta} = |x-y|$. So furthermore, $\epsilon = n^{-s/\delta}$. Then we have

$$\begin{aligned}
&\|\chi_x f(H)\chi_y\|_p \\
\leq\;& Cn^a(eC_{g,I})^{n+2}\left[e^{(a-1)n\ln n - cn^{1/\delta}} + e^{(a-1)n\ln n + \frac{-s}{\delta}n\ln n} + e^{(a-1)n\ln n - cn^{(1-s)/\delta}}\right] \\
\leq\;& C(eC_{g,I})^{n+2}\left[e^{a\ln n + (a-1)n\ln n - cn^{1/\delta}} + e^{a\ln n + (a-1)n\ln n + \frac{-s}{\delta}n\ln n} + e^{a\ln n + (a-1)n\ln n - cn^{(1-s)/\delta}}\right]
\end{aligned}$$

In this case we look at each term of

$$e^{a\ln n+(a-1)n\ln n-cn^{1/\delta}}+e^{a\ln n+(a-1)n\ln n+\frac{-s}{\delta}n\ln n}+e^{a\ln n+(a-1)n\ln n-cn^{(1-s)/\delta}}$$

and examine how the exponent causes it to decay. But in the first term, $cn^{1/\delta}$ dominates over $n\ln n$ provided $1/\delta>1$. Similarly for the last term $(1-s)/\delta>1$. The middle term is what gives us our slowest decay. Observe that $(a-1)-s/\delta<0$ comes from the middle term. Define ν and a' as $\nu=a'-a>0$. Use ν to consider the following equalities,

$$\frac{1-s}{\delta}=1+\nu \quad \text{and} \quad (a-1)-\frac{s}{\delta}=-\frac{1}{2}\nu$$

and you will discover that $\delta=1/a'$ then $1-s=\frac{1/2\nu+1}{a+\nu}$. Since the middle term above had the slowest decay, we substitute back in δ, $1-s$, and $n=|x-y|^{1/a'}$ to get

$$\|\chi_p f(H)\chi_q\|_p\leq C_{a,a'}C_{g,I}^{|x-y|^{1/a'}+2}e^{\left(\frac{a-a'}{2}\right)|x-y|^{1/a'}\ln|x-y|^{1/a'}}$$

Chapter 6 Appendix

Proof of Lemma 3.2 and related results.

Lemma 3.2 *Let V be Δ bounded with relative bound less than 1 and let $z \in \rho(H)$. Then*

$$\|\nabla R_H(z)\| \leq \frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}} \tag{6.1}$$

Proof. First, because $V$is Δ bounded we can write

$$\|Vu\| \leq a\| - \Delta u\| + b\|u\|$$

where $u \in H^2(\mathbb{R}^d)$ and $a, b \in \mathbb{R}$. Let $u = R_H v$ with $v \in L^2(\mathbb{R}^d)$. Then

$$\|VR_H(z)\| \leq a\|\Delta R_H(z)\| + b\|R_H(z)\| \tag{6.2}$$

Add and subtract $V - z$ between Δ and $R_H(z)$ to compute

$$\Delta R_H(z) = 1 - VR_H(z) + zR_H(z)$$

$$\|\Delta R_H(z)\| \leq 1 + \|VR_H(z)\| + |z|\|R_H(z)\| \tag{6.3}$$

Recall that $\|R_H(z)\| \leq \frac{1}{\eta_z}$ and combine (6.2) and (6.3) to write

$$\|VR_H(z)\| \leq a + a\|VR_H(z)\| + \frac{a|z|}{\eta_z} + \frac{b}{\eta_z}$$

and so,

$$\|VR_H(z)\| \leq \frac{a(\eta_z + |z|) + b}{\eta_z(1-a)}$$

Next is to show $\|\nabla R_H(z)\| \leq \frac{1}{\eta_z}\sqrt{\frac{\eta_z+|z|+b}{1-a}}$. We start with $\|\nabla R_H(z)\|^2 = \|(\nabla R_H(z))^* \nabla R_H(z)\|$,

$$\begin{aligned}
\|\nabla R_H(z)\|^2 &= \|R_H(x)WR_H(x)\| \\
&\leq \frac{1}{\eta_z}\|WR_H(z)\| \\
&\leq \frac{1}{\eta_z}\left(1 + \|VR_H(z)\| + \frac{|z|}{\eta_z}\right) \\
&\leq \frac{1}{\eta_z}\left(1 + \frac{a(\eta_z + |z|) + b}{\eta_z(1-a)} + \frac{|z|}{\eta_z}\right) \\
&= \frac{\eta_z + |z| + b}{\eta_z^2(1-a)}
\end{aligned}$$

And so

$$\|\nabla R_H(z)\| \leq \frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}$$

□

Recall that,

$$\nu_z := \frac{3\eta_z\gamma\sqrt{1-a}}{8\sqrt{\eta_z + |z| + b}}$$

Lemma 3.3 *Let $\alpha \in \mathbb{C}^d$, $z \in \rho(H)$, and V be a potential that is relatively Δ bounded with relative $a < 1$. Then $R_{H(\alpha)}(z)$ is a bounded operator on $L^2(\mathbb{R}^d)$ when $|\alpha| \leq \nu_z$*

Proof. Recall $B = 2i\alpha \cdot \nabla R_H(z) + |\alpha| R_H(z)$. We wish to bound $\|B\| \leq \gamma < 1$.

$$\|B\| \leq 2\|\alpha \cdot \nabla R_H(z)\| + |\alpha|\|R_H(z)\| \tag{6.4}$$

Let $C := \sqrt{\frac{\eta_z+|z|+b}{1-a}}$. Then $\|B\|$ is bounded by,

$$\|B\| \leq |\alpha|^2 \frac{1}{\eta_z} + |\alpha| \frac{2}{\eta_z} C$$

We use the parameter $0 < \gamma < 1$ and stipulate that,

$$|\alpha|^2 \frac{1}{\eta_z} + |\alpha| \frac{2}{\eta_z} C \leq \gamma$$

Fix the potential V, Δ, and z. Let γ be a free parameter, then α will be constrained by solving,

$$|\alpha|^2 \frac{1}{\eta_z} + |\alpha| \frac{2C}{\eta_z} - \gamma \leq 0$$

To which we get,

$$|\alpha| \leq C\left[\sqrt{1 + \frac{\eta_z\gamma}{C^2}} - 1\right]$$

Let $\tilde{\nu} := C\left[\sqrt{1 + \frac{\eta_z\gamma}{C^2}} - 1\right]$ and we will use $\sqrt{1+x} \geq 1 + \frac{1}{2}x - \frac{1}{8}x^2$, $0 < x$, then,

$$\begin{aligned} \tilde{\nu} &\geq C\left[\frac{1}{2}\frac{\eta_z\gamma}{C^2} - \frac{1}{8}\frac{\eta_z^2\gamma^2}{C^4}\right] \\ &= \frac{1}{2}\frac{\eta_z\gamma}{C}\left[1 - \frac{1}{4}\frac{\eta_z\gamma}{C^2}\right] \end{aligned} \tag{6.5}$$

We want to estimate $\frac{1}{4}\frac{\eta_z\gamma}{C^2}$ in the above expression. Write

$$\begin{aligned} \frac{\eta_z}{1-a} &\leq \frac{\eta_z + |z| + b}{1-a} \\ \frac{\eta_z}{1-a} &\leq C^2 \\ \frac{\eta_z\gamma}{4C^2} &\leq \frac{(1-a)\gamma}{4} \qquad \left(< \frac{1}{4}\right) \end{aligned}$$

Feeding this estimate back into (6.5) we can write,

$$\begin{aligned}\tilde{\nu} &\geq \frac{1}{2}\frac{\eta_z \gamma}{C}\left[1 - \frac{(1-a)\gamma}{4}\right] \\ &\geq \frac{3}{8}\frac{\eta_z \gamma}{C}\end{aligned}$$

$\square$

Lemma 3.5 *Let* γ *be as defined above in* (3.3)*, then*

$$\|R_{H(\alpha)}(z)\| \leq \frac{1}{\eta_z(1-\gamma)} \tag{6.6}$$

and

$$\|\nabla R_{H(\alpha)}(z)\| \leq \frac{1}{\eta_z(1-\gamma)}\sqrt{\frac{\eta_z + |z| + b}{1-a}} \tag{6.7}$$

Let ∇V *be relatively bounded with respect to* W *with bounding constants* $\tilde{a}, \tilde{b}$*. Then,*

$$\|\nabla R_H(z) A\| \leq \frac{\eta_z + |z| + b}{(1-a)\eta_z} + \frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\left[\frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\tilde{a} + \frac{\tilde{b}}{\eta_z}\right]$$

and

$$\|\nabla R_{H(\alpha)} \nabla\| \leq \frac{9}{16}\frac{\gamma^2}{1-\gamma} + \left(1 + \frac{3}{4}\frac{\eta_z \gamma}{\sqrt{\eta_z + |z| + b}}\right)\|\nabla R_H \nabla\|.$$

Proof. The first result,

$$\|R_{H(\alpha)}(z)\| \leq \frac{1}{\eta_z(1-\gamma)}$$

follows almost immediately from (3.1) and the Neumann series.

For the bound on $\|A R_{H(\alpha)}(z)\|$ we start with the second resolvent formula,

$$R_{H(\alpha)}(z) = R_H(z) + R_{H(\alpha)}\tilde{B} R_H. \tag{6.8}$$

Where,

$$\tilde{B} := 2i\alpha \cdot \nabla + |\alpha|^2$$

Then,

$$\begin{aligned}\nabla R_{H(\alpha)}(z) &= \nabla R_H(z) + \nabla R_{H(\alpha)}\tilde{B} R_H \\ \nabla R_{H(\alpha)}(z)(1 - \tilde{B} R_H) &= \nabla R_H(z).\end{aligned}$$

But, we have already chosen α so that $\tilde{B}R_H \leq \gamma < 1$. So we can invert, and take an operator norm to get,

$$\|\nabla R_{H(\alpha)}(z)\| \leq \frac{1}{1-\gamma}\|\nabla R_H(z)\|$$

or

$$\|\nabla R_{H(\alpha)}(z)\| \leq \frac{1}{\eta_z(1-\gamma)}\sqrt{\frac{\eta_z+|z|+b}{1-a}}$$

Next, we prove the bound on $\|\nabla R_H(z)\nabla\|$. Let $R_H = R_H(z)$.

$$\begin{aligned}
\nabla R_H \nabla &= \Delta R_H + \nabla[R_H, \nabla] \\
&= \Delta R_H + \nabla R_H [V, \nabla] R_H \\
&= \Delta R_H - \nabla R_H \nabla V R_H
\end{aligned}$$

$\tilde{a}$ and $\tilde{b}$ are the relative and operator bounds for ∇V. Also we know how to bound ΔR_H form (6.3). Therefore,

$$\begin{aligned}
\|\nabla R_H \nabla\| &\leq \|\Delta R_H\| + \|\nabla R_H\|\|\nabla V R_H\| \\
&\leq \|\Delta R_H\| + \|\nabla R_H\| \left[\tilde{a}\|\Delta R_H\| + \tilde{b}\|R_H\|\right]
\end{aligned}$$

Next, revisit (6.3) to get a bound on $\|\Delta R_H\|$.

$$\begin{aligned}
\|\Delta R_H\| &\leq 1 + \|V R_H\| + |z|\|R_H\| \\
&\leq \frac{\eta_z + |z|}{\eta_z} + a\|\Delta R_H\| + b\|R_H\| \\
&\leq \frac{\eta_z + |z| + b}{(1-a)\eta_z} \qquad (6.9)
\end{aligned}$$

So then,

$$\|\nabla R_H \nabla\| \leq \frac{\eta_z + |z| + b}{(1-a)\eta_z} + \frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\left[\frac{1}{\eta_z}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\tilde{a} + \frac{\tilde{b}}{\eta_z}\right].$$

To show the bound on $\|\nabla R_{H(\alpha)}\nabla\|$, recall the 2nd resolvent formula (6.8). Then

$$\begin{aligned}
\nabla R_{H(\alpha)}\nabla &= \nabla R_{H(\alpha)}(2i\alpha\cdot\nabla + |\alpha|^2)R_H\nabla + \nabla R_H \nabla \\
&= |\alpha|^2 \nabla R_{H(\alpha)} R_H \nabla + (\vec{1} + 2i\alpha)\nabla R_H \nabla \\
\|\nabla R_{H(\alpha)}\nabla\| &\leq \nu^2 \|\nabla R_{H(\alpha)}\|\|\nabla R_H\| + (1+2\nu)\|\nabla R_H \nabla\| \\
&\leq \frac{9}{16}\frac{\gamma^2}{1-\gamma} + \left(1 + \frac{3}{4}\frac{\eta_z\gamma}{\sqrt{\eta_z + |z| + b}}\right)\|\nabla R_H \nabla\|
\end{aligned}$$

□

Proof of Lemma 3.6

Let

$$g_z(x) := \frac{1}{x^2 - z}$$

Lemma 3.6 *Suppose $x \in \mathbb{R}^d$, $z \in \mathbb{C} - [0, \infty)$, and $p > d/2$. Then, $g_z(x) \in L^p(\mathbb{R}^d)$. Furthermore, if $Re(z) \geq 0$ there exists a constant, $\tilde{C}_{d,p}$, that depends on d, p, such that,*

$$\|g_z\|_p^p \leq \frac{\tilde{C}_{d,p}|z|^{d/2}}{|Im(z)|^p} \tag{6.10}$$

If $Re(z) < 0$ there exists a constant, $C_{d,p}$, that depends on d, p, such that,

$$\|g_z\|_p^p \leq \frac{C_{d,p}}{|z|^{p-d/2}} \tag{6.11}$$

Proof. Compute $\|g_z\|_p^p$. Let c_d be the volume of a unit $d-1$ dimensional sphere.

$$\begin{aligned} \|g_z\|_p^p &= c_d \int_{\mathbb{R}^d} \frac{1}{|x^2 - z|^p} dx \\ &= \int_0^\infty \frac{r^{d-1}}{|r^2 - z|^p} dr \end{aligned}$$

Let $z = v + wi$. Then $|r^2 - z|^2 = (r^2 - v)^2 + w^2$.

$$\begin{aligned} \|g_z\|_p^p &= c_d \int_0^\infty \frac{r^{d-1}}{(r^4 - 2r^2 v + v^2 + w^2)^{\frac{p}{2}}} dr \\ &= \frac{c_d}{|z|^p} \int_0^\infty \frac{r^{d-1}}{(\frac{r^4}{|z|^2} - \frac{2r^2 v}{|z|^2} + 1)^{\frac{p}{2}}} dr \end{aligned}$$

Now suppose that Re$(z) < 0$. Then $-\frac{2r^2 v}{|z|^2} > 0$. So

$$\|g_z\|_p^p \leq \frac{c_d}{|z|^p} \int_0^\infty \frac{r^{d-1}}{(\frac{r^4}{|z|^2} + 1)^{\frac{p}{2}}} dr$$

Make the substitution $u = \frac{r}{\sqrt{|z|}}$ and get,

$$\begin{aligned} \|g_z\|_p^p &\leq \frac{c_d}{|z|^p} \int_0^\infty \frac{u^{d-1}|z|^{\frac{d-1}{2}}|z|^{\frac{1}{2}}}{(u^4 + 1)^{\frac{p}{2}}} dr \\ &\leq \frac{C_{d,p}}{|z|^{p-\frac{d}{2}}} \end{aligned} \tag{6.12}$$

For the above integral to converge, we must stipulate that $2p - d + 1 > 1$ or $p > d/2$

Next suppose that $\mathrm{Re}(z) \geq 0$. Recall $z = v + iw$. Then $|r^2 - z| \geq |v|$. Also, since $|r^2 - z| > |r^2 - w|$, so $|r^2 - z|^{-1} \leq |w|^{-1}$ and $|r^2 - z|^{-1} < |r^2 - v|^{-1}$. So write the integral,

$$\begin{aligned} \|g_z\|_p^p &= \int_0^{2\sqrt{v}+\sqrt{|w|}} \frac{r^{d-1}}{|r^2 - z|^p} dr + \int_{2\sqrt{v}+\sqrt{|w|}}^{\infty} \frac{r^{d-1}}{|r^2 - z|^p} dr \\ &= \mathrm{I} + \mathrm{II} \end{aligned}$$

For $0 < r < 2\sqrt{v} + \sqrt{|w|}$ we use $|r^2 - z|^{-1} < |w|^{-1}$ so the first integral is bounded by,

$$\begin{aligned} \mathrm{I} &\leq \int_0^{2\sqrt{v}+\sqrt{|w|}} \frac{r^{d-1}}{|w|^p} dr \\ &\leq \frac{(2\sqrt{v} + \sqrt{|w|})^d}{d|w|^p} \end{aligned}$$

As for the region $2\sqrt{v} + \sqrt{|w|} < r \leq \infty$ we use $|r^2 - z|^{-1} < |r^2 - v|^{-1}$ so the second integral II is bounded as,

$$\begin{aligned} \mathrm{II} &\leq \int_{2\sqrt{v}+\sqrt{|w|}}^{\infty} \frac{\tilde{C} r^{d-1}}{|r^2 - v|^p} dr \\ &\leq \frac{\tilde{C} v^{p-d/2}}{v^{p-\frac{d}{2}}} |w|^{d/2-p} \end{aligned}$$

Therefore, combining I and II we get,

$$\begin{aligned} \|g_{z,m}\|_p^p &\leq \frac{\tilde{C}_{d,p,m}(2\sqrt{v} + \sqrt{|w|})^d}{|w|^p} \\ &\leq \frac{\tilde{C}_{d,p,m}|z|^{d/2}}{|w|^p} \end{aligned}$$

□

Proof of Corollary 3.8

Corollary 3.8 *Assume the hypothesis from lemma 3.3 and corollary 3.7. Let* $p > d/2$. *Then* $\chi R_H(z)$ *and,* $\chi R_{H(\alpha)}(z)$ *are all in the p-th trace ideal. And they have the following estimates.*

$$\|\chi R_H(z)\|_p \leq \frac{C_{d,p}}{|z|^{1-d/2p}}\left[1 + \frac{a(\eta_z + |z|) + b}{\eta_z(1-a)}\right] \qquad Re(z) < 0 \tag{6.13}$$

$$\|\chi R_H(z)\|_p \leq \frac{\tilde{C}_{d,p}|z|^{d/2p}}{|Im(z)|}\left[1 + \frac{a(\eta_z + |z|) + b}{\eta_z(1-a)}\right] \qquad Re(z) \geq 0 \tag{6.14}$$

$$\|\chi R_{H(\alpha)}(z)\|_p \leq \frac{C_{d,p}(1+\gamma)}{|z|^{1-d/2p}} D_{z,\gamma,V} \qquad Re(z) < 0 \tag{6.15}$$

$$\|\chi R_{H(\alpha)}(z)\|_p \leq \frac{\tilde{C}_{d,p}|z|^{d/2p}}{|Im(z)|} D_{z,\gamma,V} \qquad Re(z) \geq 0 \tag{6.16}$$

With,

$$D_{z,\gamma,V} = \left(1 + \frac{a(\eta_z + |z|) + b)}{\eta_z(1-a)}\right)\left(1 + \frac{2\gamma}{\eta_z(1-\gamma)}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\right) \tag{6.17}$$

Proof. To prove this, combine 3 things: Corollary 3.7, Lemma 3.2, and the second resolvent formula, $R_0 - R_H = R_0 V R_H$ and $R_H - R_{H(\alpha)} = 2i\alpha \cdot R_H \nabla R_{H(\alpha)}$.

First, use $R_0 - R_H = R_0 V R_H$ to do the computation,

$$\begin{aligned}
\chi R_H &= \chi R_0(1 - V R_H) \\
\|\chi R_H\|_p &\leq \|\chi R_0\|_p \|(1 - V R_H)\| \\
\|\chi R_H\|_p &\leq \|\chi R_0\|_p \left(1 + \frac{a(\eta_z + |z|) + b)}{\eta_z(1-a)}\right)
\end{aligned}$$

Then apply Corollary 3.7 with $m = 1$ to get the result.

Next, use $R_H - R_{H(\alpha)} = 2i\alpha \cdot R_H \nabla R_{H(\alpha)}$ to do the following computation,

$$\begin{aligned}
\chi R_{H(\alpha)} &= \chi R_H(1 - 2i\alpha \cdot \nabla R_{H(\alpha)}) \\
\|\chi R_{H(\alpha)}\|_p &\leq \|\chi R_H\|_p \left(1 + 2\gamma \|\nabla R_{H(\alpha)}\|\right) \\
\|\chi R_{H(\alpha)}\|_p &\leq \|\chi R_0\|_p \left(1 + \frac{a(\eta_z + |z|) + b)}{\eta_z(1-a)}\right)\left(1 + \frac{2\gamma}{\eta_z(1-\gamma)}\sqrt{\frac{\eta_z + |z| + b}{1-a}}\right)
\end{aligned}$$

Thus proves our result. $\square$

Proof of Lemma 3.4

Lemma 3.4 *For an integer $s \geq 2$ there exists $J_{\gamma\delta} \in \mathcal{J}$ and $Q_{\gamma\delta} \in \mathcal{Q}$ such that,*

$$J^s R^s_{H(\alpha)} = \sum_{\delta=1}^{N} \prod_{\beta=1}^{s} J_{\beta\delta} R_{H(\alpha)} Q_{\beta\delta} \tag{6.18}$$

with $B = 2i\alpha \cdot \nabla + |\alpha|^2$ and $H(\alpha) = H + B$. Formally that is, one may exchange powers on $J^s R^s$ for a product of s JRQ terms.

Proof. The proof goes by induction. The statement trivially holds for the $m = 1$ case. We assume that the following is true.

$$J^m R^m_{H+B} = \sum_{\delta=1}^{N} \prod_{\gamma=1}^{m} J_{\gamma\delta} R_{H+B} Q_{\gamma\delta}$$

and we want to calculate $JJ^m R^m_{H+B} R_{H+B}$, or,

$$JJ^m R^m_{H+B} R_{H+B} = \sum_{\delta=1}^{N} J \left[\prod_{\delta=1}^{m} J_{\gamma\delta} R_{H+B} Q_{\gamma\delta} \right] R_{H+B} \tag{6.19}$$

Considering one additive term from above,

$$JJ_{\gamma 1} R_{H+B} Q_{\gamma 1} \left[\prod_{\delta=2}^{m} J_{\gamma\delta} R_{H+B} Q_{\gamma\delta} \right] R_{H+B} \tag{6.20}$$

Further consider $JJ_{1\delta} R_{H+B} Q_{1\delta}$, and commute J past R_{H+B} to get,

$$\begin{aligned} = \quad & J_{1,\delta}(RJ + RQ_2 J_1) Q_{1,\delta} + \\ & J_{1,\delta}(RJ_2 R + RQ_2 J_3 R + RQ_4 J_4 R + RQ_2 J_{5,\alpha} R + RJ_{6,\alpha} R) Q_{1,\delta} \end{aligned} \tag{6.21}$$

The second additive term above is exactly what we want. Feed it back into (6.20) then into (6.19) to see that you get $m + 1$ more mutiplicative terms, and $N + 4$ more additive terms. Let's focus on $J_{1,\delta}(RJ + RQ_2 J_1) Q_{1,\delta}$, and commute J_* past $Q_{1,\delta}$.

$$\begin{aligned} & J_{1,\delta}(RJ + RQ_2 J_1) Q_{1,\delta} \\ = \quad & J_{1,\delta}(RQ_{1,\delta} J + RQ_2 Q_{1,\delta} J_1) + J_{1,\delta}(R[J, Q_{1,\delta}] + RQ_2 [J_1, Q_{1,\delta}]) \end{aligned} \tag{6.22}$$

Computing $[J, Q_{1,\delta}]$ and $[J_1, Q_{1,\delta}]$ amounts to computing $[J, Q_i]$, $i = 1, 2, 3, 4$. You can appeal to appendix A in [11] lemma 14 to see $[J_*, Q_*]$, but we'll write it here for convenience. Assume $R = R_{H(\alpha)}(z)$

$$\begin{aligned}
[J, Q_1] &= RQ_2J_3 + RJ_1R + RQ_2J_4R + RQ_4 \cdot J_5R \\
&\quad + RQ_2\alpha \cdot J_6R + R\alpha \cdot J_2R && (6.23) \\
[J, Q_2] &= J_*R + Q_2\{Q_2J_* + J_*R + Q_2J_*R + Q_4J_*R \\
&\quad + \alpha \cdot [Q_2J_*R + J_*R]\} && (6.24) \\
[J, Q_3] &= RJ_* + Q_3\{Q_3J_* + J_*Q_3 + RJ_*Q_3 + RJ_*Q_4 + RJ_*Q_3 \\
&\quad + \alpha \cdot [RJ_*R\nabla + J_*R\nabla]\} && (6.25)
\end{aligned}$$

The calculation of $[J, Q_4]$ is similar to the above comutators.

We appeal to appendix A in [11] lemma 14 to calculate $[J_*, Q_*]$. The lemma says that

$$[J_*, Q_*] = \sum \tilde{Q}_i \tilde{S}_i + \sum \bar{Q}_i \tilde{J}_i \tag{6.26}$$

where S_* is as we defined it before, $S_* = J_*RB_*$. Feed (6.22) and (6.26) back into (6.21) and (6.20), reindex Q_*, J_* and expand another multiplicative term from (6.20) to get,

$$(6.20) = \sum_{\gamma=\tilde{N}+1}^{\hat{N}} S_\gamma J_{\gamma 1} J_{\gamma 2} R Q_{\gamma 2} \prod_{\delta=3}^{m} J_{\gamma\delta} R Q_{\gamma\delta} + \sum_{\delta=1}^{\tilde{N}} \prod_{\gamma=1}^{m+1} J_{\gamma\delta} R Q_{\gamma\delta} \tag{6.27}$$

To the first term of (6.27), we play the same game that happened after (6.20). Repeat this process, and observe from (6.22) and (6.26) that at the end of this process,

$$(6.20) = \left[\prod_{i}^{m} S_i\right] J_{\hat{N}1} + \sum_{\delta=1}^{\tilde{N}} \prod_{\gamma=1}^{m+1} J_{\gamma\delta} R Q_{\gamma\delta}$$

Recall form (6.19), R_{H+B} multiplies on the right side, and this gives us our result. $\square$

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