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Solutions to the L^p Mixed Boundary Value Problem in $C^{1,1}$ Domains

Laura D. Croyle
University of Kentucky, lauradcroyle@gmail.com
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Laura D. Croyle, Student

Dr. Russell Brown, Major Professor

Dr. Peter Hislop, Director of Graduate Studies

Solutions to the L^p Mixed Boundary Value Problem in $C^{1,1}$ Domains

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Laura Croyle
Lexington, Kentucky

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ABSTRACT OF DISSERTATION

Solutions to the L^p Mixed Boundary Value Problem in $C^{1,1}$ Domains

We look at the mixed boundary value problem for elliptic operators in a bounded $C^{1,1}(\mathbb{R}^n)$ domain. The boundary is decomposed into disjoint parts, D and N, with Dirichlet and Neumann data respectively. Expanding on work done by Ott and Brown, we find a larger range of values of p, $1 < p < \frac{n}{n-1}$, for which the L^p mixed problem has a unique solution with the non-tangential maximal function of the gradient in $L^p(\partial\Omega)$.

KEYWORDS: Boundary value problem, mixed problem, Besov spaces

Author's signature: Laura Croyle

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Solutions to the L^p Mixed Boundary Value Problem in $C^{1,1}$ Domains

By
Laura Croyle

Director of Dissertation: Russell Brown

Director of Graduate Studies: Peter Hislop

Date: July 21, 2016

Dedicated to Ralph Oliver Croyle
for everything

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CONTENTS

Chapter 1 Introduction

1.1 Introduction and History

In this dissertation, we will prove a result for a L^p mixed boundary value problem. This and related problems have been studied for decades. Proving existence of solutions to the Dirichlet Boundary Value Problem on Lipschitz domains was done by Dahlberg in 1976 [5]. Jerison and Kenig studied the non-tangential maximal function in L^2 for the Neumann Problem for Lipschitz domains in 1981 [10]. Optimal results have been given for the L^p Neumann Problem on Lipschitz domains by Dahlberg and Kenig in 1987 [6].
In this paper we look at the L^p mixed problem on a $C^{1,1}$ domain. The mixed problem has both Neumann and Dirichlet data, given as follows

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f_D & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ \nabla u^* \in L^p(\partial\Omega) \end{cases}$$

where D and N given a Lipschitz dissection of $\partial\Omega$, defined later. Also, ∇u^* is the non-tangential maximal function of ∇u and is defined in detail in the Section 1.2. Brown and Ott proved existence of solutions to this problem on Lipschitz domains, but did not get a specific range of p-values [11]. Brown, Ott, and Taylor, also proved existence to this problem on Lipschitz domains with general decompositions of the boundary, but again did not get a specific range of p-values [16]. By putting stricter conditions on the domain, we are able to prove in this paper existence of solutions for $1 < p < \frac{n}{n-1}$. Moreover, when $n = 2$ this range of p-values is optimal. For our paper, we will prove a Reverse Hölder Inequality for solutions to the weak mixed problem, which will in turn allow us to get these better results for the L^p mixed problem.

The work of Cafarelli and Peral showed that we could use Reverse Hölder Inequalities to get better L^p results [3]. This method was adapted to boundary value problems by Shen, whose work we will use to prove our result [14]. We will prove our Reverse Hölder Inequality by utilizing the work by Savaré on $\mathbb{R}^n_+$ [13].

1.2 Definitions and Preliminaries

We start by introducing notation and problems of interest. Given a domain, Ω, we define for $k \in \mathbb{N}$ and $1 \le p < \infty$, $\mathbf{W^{k,p}(\Omega)}$ to be the set of functions, $u \in L^p(\Omega)$, such that for each multiindex, α, with $|\alpha| \le k$, the weak derivative, $\frac{\partial^\alpha u}{\partial x^\alpha}$, exists and is in $L^p(\Omega)$. We define the norm on this space as follows

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \left\| \frac{\partial^\alpha u}{\partial x^\alpha} \right\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Note, that for our L^p spaces, we use the Lebesgue measure on Ω and on the boundary we use the surface measure, σ. For $D \subseteq \partial\Omega$, we define $C_D^\infty(\bar{\Omega})$ to be the set of C^∞ functions on $\bar{\Omega}$ that are zero in a neighborhood of $\bar{D}$ and are compactly supported. Moreover, we let $\mathbf{W_D^{k,p}(\Omega)}$ be the closure of $C_D^\infty(\bar{\Omega})$ in the $W^{k,p}(\Omega)$ norm. We say that $u \in \mathbf{W^{k,p}_{\mathrm{loc}}(\Omega)}$, if for each V compactly contained in Ω, $u \in W^{k,p}(V)$. We define $\mathbf{C^{1,1}}$ to be the set of Lipschitz functions that have Lipschitz first partial derivatives. We say that Ω is a **$\mathbf{C^{1,1}}$ domain** if locally the boundary of Ω is the graph of a $C^{1,1}$ function. We say that a matrix a is **elliptic** if there is a constant $\theta > 0$, such that

$$\xi^T a \xi \ge \theta |\xi|^2$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. For all of the following we assume that our matrix a is elliptic and symmetric with coefficients that are Lipschitz. Now we introduce the problems of interest in this paper. Given $g \in (W_0^{1,2}(\Omega))^*$ and $f_D \in \mathrm{tr}((W^{1,2}(\Omega))^*)$, the **Weak Dirichlet Boundary Value Problem** is the problem of finding a function, u in $W^{1,2}(\Omega)$, that is a weak solution to

$$\begin{cases} -\mathrm{div} a\nabla u = g & \text{in } \Omega \\ u = f_D & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

A function, u, is a weak solution if $u - f_D \in W_0^{1,2}(\Omega)$ and it satisfies

$$\int_\Omega a\nabla u \nabla \Psi \, dx = \langle g, \Psi \rangle_\Omega$$

for all $\Psi \in W_0^{1,2}(\Omega)$.

Next, given $g \in (W^{1,2}(\Omega))^*$ and $f_N \in \mathrm{tr}((W^{1,2}(\Omega))^*)$, the **Weak Neumann Boundary Value Problem** is the problem of finding a function, $u \in W^{1,2}(\Omega)$, that is a weak solution to

$$\begin{cases} -\mathrm{div} a\nabla u = g & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

A function, u, is a weak solution if it satisfies

$$\int_\Omega a\nabla u \nabla \Psi \, dx = \langle g, \Psi \rangle_\Omega + \langle f_N, \Psi \rangle_{\partial\Omega}$$

for all $\Psi \in W^{1,2}(\Omega)$.

One of the boundary value problems we will look at in this paper is the **Weak Mixed Boundary Value Problem**. First we let D closed and N open give a disjoint partition of $\partial\Omega$. Given $g \in (W_D^{1,2}(\Omega))^*$ and $f_N \in \mathrm{tr}((W_D^{1,2}(\Omega))^*)$, we are searching for $u \in W_D^{1,2}(\Omega)$, a weak solution, to

$$\begin{cases} -\mathrm{div} a\nabla u = g & \text{in } \Omega \\ u = 0 & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N. \end{cases} \tag{1.3}$$

A function, u, is a weak solution if it satisfies

$$\int_\Omega a\nabla u \nabla \Psi \, dx = \langle g, \Psi \rangle_\Omega + \langle f_N, \Psi \rangle_{\partial\Omega}$$

for all $\Psi \in W_D^{1,2}\Omega$.
Now to introduce a problem closely related to each of these, we need to define the **non-tangential maximal function**. For a function u defined on Ω, we define the non-tangential maximal function on $\partial\Omega$ by

$$u^*(x) = \sup_{y \in \Gamma_\alpha(x)} |u(y)|$$

where $\Gamma_\alpha(x)$ is the non-tangential approach region to x given by

$$\Gamma_\alpha(x) = \{y \in \Omega : |x - y| < (1+\alpha)\mathrm{dist}(y, \partial\Omega)\}$$

for some $\alpha > 0$.
For this paper we use ∇u^* to stand for $(\nabla u)^*$ to simplify notation. Now we turn to the main problem of interest for this paper. We say $u \in C^2(\Omega)$ is a solution to the **Lp-Mixed Problem**, if

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \\ \nabla u^* \in L^p(\partial\Omega) \end{cases} \tag{1.4}$$

where $\frac{\partial u}{\partial \nu} = \nu \cdot \nabla u$ is the outward normal derivative of u and ∇u is defined in the non-tangential sense, meaning that for $x \in \partial\Omega$

$$\nabla u(x) = \lim_{\substack{y \to x \\ y \in \Gamma_\alpha(x)}} \nabla u(y). \tag{1.5}$$

In the case where $N = \varnothing$ and we only specify Dirichlet data, we call (1.4) the **Lp-Regularity Problem**. When $D = \varnothing$ and we only specify Neumann data, we call (1.4) the **Lp-Neumann Problem**.
For our problem we will consider D and N that give a **Lipschitz dissection of the boundary**. To define what this means, we begin by assuming that our domain, Ω, is $\mathbb{R}^n_+$. In this case, we say that D and N give a Lipschitz dissection of the boundary if there exists a Lipschitz function $\Psi : \mathbb{R}^{n-2} \to \mathbb{R}$ such that

$$D = \{(x_1, x'', 0) : \Psi(x'') \le x_1\}$$

and

$$N = \{(x_1, x'', 0) : \Psi(x'') > x_1\}$$

where $x'' \in \mathbb{R}^{n-2}$.
Moving on to a more general case, suppose that Ω is a $C^{1,1}$ graph domain, meaning there exists a $C^{1,1}$ function, $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, such that

$$\Omega = \{(x', x_n) : x_n > \varphi(x')\}$$

where $x' \in \mathbb{R}^{n-1}$. To have a Lipschitz dissection in this case, we require

$$D = \partial\Omega \cap \{\Psi(x'') \leq x_1\}$$

and

$$N = \partial\Omega \cap \{\Psi(x'') > X_1\}.$$

For our general case where Ω is any $C^{1,1}$ domain, we say that we have a Lipschitz dissection of the boundary if there exists $r > 0$ small enough, such that for each $x \in \partial\Omega$, there is a $C^{1,1}$ graph domain, Ω_x, such that

$$\Omega \cap B_r(x) = \Omega_x \cap B_r(x),$$

$$N \cap B_r(x) = N_x \cap B_r(x),$$

and

$$D \cap B_r(x) = D_x \cap B_r(x),$$

where D_x and N_x give a Lipschitz dissection of $\partial\Omega_x$. For this paper we refer to bounded $C^{1,1}$ domains, Ω, that have a Lipschitz dissection of the boundary, given by D and N, as a **standard domain**.
We may fix $r_0 > 0$ small enough so that for each $x \in \partial\Omega$, we have a $C^{1,1}$ function that agrees with the boundary of Ω on $B_{100r_0}(x)$. Similarly, we choose r_0 small enough so that our Lipschitz function, φ, gives a Lipschitz dissection of the boundary of a graph domain Ω_x, as in the definition, that agrees with $\partial\Omega$, D, and N on $B_{100r_0}(x)$. Since Ω is bounded, we have that $\partial\Omega$ is compact and we can guarantee existence of such a value for r_0 that works for all $x \in \partial\Omega$.
For simplification of notation, we will use the following notations for our local sets

$$\Upsilon_r(x) = B_r(x) \cap \Omega$$

and we define surface balls by

$$\zeta_r(x) = B_r(x) \cap \partial\Omega.$$

1.3 Main Result

The goal of this dissertation is to prove the following theorem.

Theorem 1.1 (Main Theorem)**.** *Let Ω, D, and N be a standard domain, then there exists a unique solution u to the L^p Mixed Problem (1.4) with $f_N \in L^p(N)$, such that*

$$\|\nabla u^*\|_{L^p(\partial\Omega)} \leq c\|f_N\|_{L^p(N)}$$

for $1 < p < \frac{n}{n-1}$.

In the case where $n = 2$, we have that our range of p-values is optimal. Consider the following example. Let $0 < \alpha < 2\pi$. Define

$$\Omega_\alpha = B_1(0) \cap \{(r,\theta) : 0 < \theta < \alpha\}.$$

We let $D = \{(x,0) : 0 < x < 1\}$ and $N = \partial\Omega_\alpha \backslash D$. Consider

$$u(r,\theta) = r^{\frac{\pi}{2\alpha}} \sin\left(\theta\frac{\pi}{2\alpha}\right).$$

The main interest of this paper is that we are able to find a larger range of p-values. Straight forward calculation gives that $\Delta u = 0$ on Ω_α, $u = 0$ on D and $\frac{\partial u}{\partial \nu} \in L^\infty$ on N. When $\alpha = \pi$, it is clear that $\nabla u* \in L^p(\partial\Omega)$, only if $p < 2$. This example proves that our result is maximal in 2-dimensions.

We prove this theorem in multiple steps. We will show that solutions to the weak mixed problem, which satisfy extra conditions will, in fact, be solutions to the L^p mixed problem. In Chapter 2, we will work through the proof of a Reverse Hölder Inequality for solutions to the weak mixed problem. This will utilize the work of Savaré [13]. In Chapter 3, we will follow the work of Ott and Brown, [11], by using our results from Chapter 2, to prove a related Reverse Hölder Inequality on the boundary for the non-tangential maximal function. Finally, in Chapter 4, we will use the results from Chapter 3 to prove conditions necessary for a theorem by Shen [14] and prove our Main Theorem.

Chapter 2 Reverse Hölder Inequality

2.1 Reverse Hölder Inequality

In order to prove the main result we first need to prove a weak Reverse Hölder's Inequality for solutions to weak mixed boundary problems. Recall that Hölder's Inequality implies that for S with positive measure, that

$$\left(\fint_S |f|^p \, dx\right)^{\frac{1}{p}} \leq \left(\fint_S |f|^q \, dx\right)^{\frac{1}{q}}$$

for $q > p$. A Reverse Hölder Inequality is the same result, but in the case where $p > q$. To prove our main theorem, we will prove a weak reverse Hölder inequality. A weak result refers to a local result which bounds integrals on a neighborhood by integrals on larger neighborhoods. For instance,

$$\left(\fint_{B_r(x)} |f|^p \, dx\right)^{\frac{1}{p}} \leq \left(\fint_{B_{2r}(x)} |f|^q \, dx\right)^{\frac{1}{q}}$$

for $p > q$. For the duration of this paper, we will be referring to the Reverse Hölder Inequality stated in the following theorem.

Theorem 2.1 (Reverse Hölder Inequality)**.** *Let $x \in \partial\Omega$ and $0 < r < r_0$. Suppose u is a weak solution to our mixed problem (1.3) with $g = 0$ and $f_N = \kappa$, a constant, on $B_r(x) \cap N$, then we have*

$$\left(\fint_{\Upsilon_{r/2}(x)} |\nabla u|^s \, dy\right)^{\frac{1}{s}} \leq c\left(\fint_{\Upsilon_r(x)} |\nabla u|^2 \, dy\right)^{\frac{1}{2}} + c|\kappa|$$

for $2 < s < \frac{2n}{n-1}$.

2.2 Flattening the Boundary

Before we begin our proof of the Reverse Hölder Inequality, Theorem 2.1, we need to flatten the boundary, so we can utilize work already done for $\mathbb{R}^n_+$ by Savaré [13]. Although we eventually want to consider the case where $g = 0$ and a is the identity matrix in (1.3) for the proof of Theorem 1.1, we are able to prove many of these results in a more general setting. We first need some preliminary results.

Lemma 2.2. *Given a graph domain, Ω, given by a Lipschitz function, $\varphi : \mathbb{R}^{n-1} :\to \mathbb{R}$, we define $\Phi : \mathbb{R}^n_+ \to \Omega$ by*

$$\Phi(x', x_n) = (x', \varphi(x') + x_n)$$

where $x' \in \mathbb{R}^{n-1}$. We can show that Φ is a bi-Lipschitz function differentiable almost everywhere with

$$\det D\Phi = 1.$$

Proof. (of Lemma 2.2) We begin by showing that Φ is Lipschitz with a constant of $1+M$, since

$$\begin{aligned}
|\Phi(x)-\Phi(y)| &= |(x',\varphi(x')+x_n)-(y',\varphi(y')+y_n)| \\
&= |(x-y)+\mathbf{e}_n(\varphi(x')-\varphi(y'))| \\
&\le |x-y|+|\varphi(x')-\varphi(y')| \\
&\le |x-y|+M|x'-y'| \\
&\le (1+M)|x-y|
\end{aligned}$$

where we use that φ is Lipschitz, with constant M, in the second to last step. It is clear that $\Phi^{-1}:\Omega\to\mathbb{R}^n_+$ defined by

$$\Phi^{-1}(y) = (y', y_n - \varphi(y'))$$

is in fact an inverse for Φ. It follows similarly that Φ^{-1} is also Lipschitz with constant $1+M$, giving that Φ is a bi-Lipschitz function. Since φ is Lipschitz, we know that φ is differentiable almost everywhere by Rademacher's Theorem. Now, we see that

$$\frac{\partial\Phi}{\partial x_i} = \begin{cases} (0,\dots,0,1,0,\dots,0,\frac{\partial\varphi}{\partial x_i}) & \text{if } i \ne n \\ (0,\dots,0,1) & \text{if } i = n. \end{cases}$$

Hence,

$$D\Phi = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial\varphi}{\partial x_1} & \frac{\partial\varphi}{\partial x_2} & \cdots & \frac{\partial\varphi}{\partial x_{n-1}} & 1 \end{bmatrix}.$$

Therefore, we have that $\det D\Phi = 1$. □

Once we flatten the boundary and have a new mixed problem, the following lemma will allow us to prove that the matrix in our new problem is still elliptic.

Lemma 2.3. *If a matrix a is elliptic, then for an invertible matrix s, $s^T as$ is also elliptic.*

Proof. (of Lemma 2.3) Let $\xi\in\mathbb{R}^n$. Note that

$$\begin{aligned}
s^T as\xi\cdot\xi &= \xi^T s^T as\xi \\
&= (s\xi)^T a(s\xi) \\
&\ge c|s\xi|^2
\end{aligned}$$

using that a is elliptic. Next note

$$|\xi| = |s^{-1}s\xi| \leq \|s^{-1}\||s\xi|.$$

Together these bounds give the result. Note that the ellipticity constant depends on the norm of the matrix s^{-1}. □

The following two lemmas will allow us to show that our functions in the flattened domain lie in the correct spaces. Although we will eventually use Lemma 2.2 and have a function with $\det D\Phi = 1$, we are able to prove the following results with weaker conditions.

Lemma 2.4. *Given domains Ω and Ω', $u \in W^{1,1}_{\text{loc}}(\Omega)$, and $\Phi : \Omega' \to \Omega$, a bi-Lipschitz function with*

$$m \leq |\det D\Phi| \leq M,$$

we have the following chain rule

$$\frac{\partial}{\partial x_i}(u \circ \Phi) = \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \circ \Phi \frac{\partial \Phi_j}{\partial x_i} \tag{2.1}$$

and $u \circ \Phi \in W^{1,1}_{\text{loc}}(\Omega')$.

Proof. (of Lemma 2.4) We will prove this result in steps.
Step 1: If $u \in C_c^\infty(\Omega)$ and $\Phi \in C^\infty(\Omega')$, then (2.1) holds by the chain rule for smooth functions and the result is obvious.

Step 2: If $u \in C_c^\infty(\Omega)$ and Φ is as in the statement of the lemma, we can choose a mollifier, ξ, and define

$$\Phi_\varepsilon(x) = \xi_\varepsilon * \Phi(x).$$

Since φ is Lipschitz,

$$|\Phi_\varepsilon(x) - \Phi(x)| \leq c\varepsilon$$

for all $x \in \Omega$ and $\frac{\partial \Phi_{\varepsilon,j}}{\partial x_i}$ converges to $\frac{\partial \Phi_j}{\partial x_i}$ pointwise almost everywhere. Given Ψ be a test function in $C_c^\infty(\Omega')$, we have that

$$\int_{\Omega'} \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \circ \Phi \frac{\partial \Phi_j}{\partial x_i} \Psi \, dx = \lim_{\varepsilon \to 0^+} \int_{\Omega'} \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \circ \Phi_\varepsilon \frac{\partial \Phi_{\varepsilon,j}}{\partial x_i} \Psi \, dx$$

by the Lebesgue Dominated Convergence Theorem (LDCT). By the first case and integration by parts we have that,

$$\int_{\Omega'} \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \circ \Phi_\varepsilon \frac{\partial \Phi_{\varepsilon,j}}{\partial x_i} \Psi \, dx = -\int_{\Omega'} u \circ \Phi_\varepsilon \frac{\partial \Psi}{\partial x_i} \, dx.$$

Again using LDCT, we have

$$-\lim_{\epsilon\to 0^+}\int_{\Omega'} u\circ\Phi_\epsilon \frac{\partial\Psi}{\partial x_i}\,dx = -\int_{\Omega'} u\circ\Phi\frac{\partial\Psi}{\partial x_i}\,dx.$$

This result gives (2.1). Now, we know that $\frac{\partial\Phi}{\partial x_i}$ exists and is in $L^\infty(\Omega')$. We also have that

$$\frac{\partial(\Phi_{j,\varepsilon})}{\partial x_i} = \left(\frac{\partial\Phi_j}{\partial x_i}\right)_\varepsilon.$$

By the Lebesgue Differentiation Theorem, we have that

$$\left(\frac{\partial\Phi_j}{\partial x_i}\right)_\varepsilon \to \frac{\partial\Phi_j}{\partial x_i}$$

almost everywhere in Ω. Lastly, since $\frac{\partial\Phi_j}{\partial x_i} \in L^\infty(\Omega')$ and

$$\left\|\left(\frac{\partial\Phi_j}{\partial x_i}\right)_\varepsilon\right\|_{L^\infty(\Omega')} \le \left\|\frac{\partial\Phi_j}{\partial x_i}\right\|_{L^\infty(\Omega')}\int_{\Omega'}|\Phi_\varepsilon|\,dx = \left\|\frac{\partial\Phi_j}{\partial x_i}\right\|_{L^\infty(\Omega')},$$

Since $u\circ\Phi$ is bounded as is $\frac{\partial u}{\partial x_i}\circ\Phi\frac{\partial\Phi}{\partial x_i}$, we have $u\circ\Phi\in W^{1,1}_{\text{loc}}(\Omega')$.

Step 3: For this case let u and Φ be as in the statement of the lemma. Let $V'\subset\subset\Omega$ and set $V=\Phi(V')$. Easily, this gives that $V\subset\subset\Omega$. Choose $u_k\in C_c^\infty(\Omega)$, where u_k is supported on V and $u_k\to u$ on V in the $W^{1,1}$ norm. Now we have

$$\int_{V'}|u_k\circ\Phi - u\circ\Phi|\,dx = \int_V |u_k(y)-u(y)|\frac{1}{|\det D\Phi|}\,dy.$$

Since $u_k\to u$ in $L^1(V)$, the above gives that $u_k\circ\Phi\to u\circ\Phi$ in $L^1(V')$. Similarly, since $\frac{\partial u_k}{\partial y_j}\to\frac{\partial u}{\partial y_j}$ in L^1 and

$$\begin{aligned}\int_{V'}\left|\sum_{j=1}^n\frac{\partial u_k}{\partial y_j}\circ\Phi\frac{\partial\Phi_j}{\partial x_i} - \sum_{j=1}^n\frac{\partial u}{\partial y_j}\circ\Phi\frac{\partial\Phi_j}{\partial x_i}\right|dx \\ = \int_{V)}\sum_{j=1}^n\left|\frac{\partial u_k}{\partial y_j}-\frac{\partial u}{\partial y_j}\right|\left|\frac{\partial\Phi_j}{\partial x_i}(\Phi^{-1}(y))\right|\frac{1}{|\det D\Phi|}\,dy\end{aligned}$$

we have that $\sum_{j=1}^n\frac{\partial u_k}{\partial y_j}(\Phi)\frac{\partial\Phi_j}{\partial x_i}\to\sum_{j=1}^n\frac{\partial u}{\partial y_j}(\Phi)\frac{\partial\Phi_j}{\partial x_i}$ in $L^1(V')$. Lastly, we get the result as follows by letting Ψ be a test function and getting

$$\begin{aligned}\int_{V'}\sum_{j=1}^n\frac{\partial u}{\partial y_j}\circ\Phi\frac{\Phi_j}{\partial x_i}\Psi\,dx &= \lim_{k\to\infty}\int_{V'}\sum_{j=1}^n\frac{\partial u_k}{\partial y_j}\circ\Phi\frac{\partial\Phi_j}{\partial x_i}\Psi\,dx \\ &= -\lim_{k\to\infty}\int_{V'}u_k\circ\Phi\frac{\partial\Psi}{\partial x_i}\,dx \\ &= -\int_{V'}u\circ\Phi\frac{\partial\Psi}{\partial x_i}\,dx\end{aligned}$$

where we used Step 2 to get the second equality. □

Next, we observe that if $u \in W^{1,2}_D(\Omega)$, then $u \circ \Phi \in W^{1,2}_{D'}(\Omega')$.

Lemma 2.5. *Let Ω and Ω' be $C^{1,1}$ graph domains and $u \in W^{1,2}_D(\Omega)$. Given $\Phi : \Omega' \to \Omega$ a bi-Lipschitz function with*

$$m \leq |\det D\Phi| \leq M$$

we have that $u \circ \Phi \in W^{1,2}_{D'}(\Omega')$ where $D' = \Phi(D)$.

Proof. (of Lemma 2.5) First, since $W^{1,2}_D(\Omega) \subset W^{1,1}_{\mathrm{loc}}(\Omega)$, we have by the chain rule in Lemma 2.4, that $u \circ \Phi$ is in $W^{1,1}_{\mathrm{loc}}(\Omega')$. Moreover, we have that

$$\frac{\partial (u \circ \Phi)}{\partial x_i} = \sum_{j=1}^n \frac{\partial u}{\partial y_j} \circ \Phi \frac{\partial \Phi_j}{\partial x_i}. \tag{2.2}$$

Hence,

$$\begin{aligned} \left\| \frac{\partial}{\partial x_i}(u \circ \Phi) \right\|_{L^2(\Omega')} &= \left\| \sum_{j=1}^n \frac{\partial u}{\partial y_j} \circ \Phi \frac{\partial \Phi_j}{\partial x_i} \right\|_{L^2(\Omega')} \\ &\leq M \sum_{j=1}^n \left\| \frac{\partial u}{\partial y_j} \circ \Phi \right\|_{L^2(\Omega')} \end{aligned}$$

using that the derivatives of Φ are bounded by the Lipschitz coefficient M and the triangle inequality. Changing variables gives that

$$\begin{aligned} \left\| \frac{\partial}{\partial x_i}(u \circ \Phi) \right\|_{L^2(\Omega')} &\leq M \sum_{j=1}^n \left\| \frac{\partial u}{\partial y_j} \frac{1}{|\det D\Phi|^{1/2}} \right\|_{L^2(\Omega)} \\ &\leq \frac{M}{m^{1/2}} \sum_{j=1}^n \left\| \frac{\partial u}{\partial y_j} \right\|_{L^2(\Omega)} \\ &\leq c \|\nabla u\|_{L^2(\Omega)}. \end{aligned}$$

Since this is finite, we have that $\frac{\partial}{\partial x_i}(u \circ \Phi) \in L^2(\Omega')$. Using the same reasoning we can show that $u \circ \Phi \in L^2(\Omega')$. Therefore we have that $u \circ \Phi \in W^{1,2}(\Omega')$. To show that $u \circ \Phi$ has the correct boundary condition, pick $\{u_k\}_{k=1}^\infty \subset C^\infty_D(\Omega)$ such that $u_k \to u$ in $W^{1,2}(\Omega)$. Now we consider $u_k \circ \Phi$. We know that u_k is zero in a neighborhood of D, so $u_k \circ \Phi$ is zero in a neighborhood of D'. Since u_k is bounded and Φ is Lipschitz, we have that $u_k \circ \Phi$ is also Lipschitz. Since the boundary of Ω' is $C^{1,1}$, we can extend $u_k \circ \Phi$ to a Lipschitz function in a neighborhood of $\bar{\Omega}'$ using a reflection. We call this extended domain $E(\Omega')$. It is clear that the extension of our function is still zero in a neighborhood of D'. Next, choose $\varepsilon > 0$ small enough so that shrinking the extended domain by ε doesn't cut into the original domain. Now we can mollify to give $(u_k \circ \Phi)_{\varepsilon_k}$. Choosing $\varepsilon_k = \frac{1}{k}$ will suffice, and starting at $k \geq K$ large enough so that $\varepsilon > \frac{1}{K}$. Keep in mind that we are mollifying the extension of our function. Now we know that $(u_k \circ \Phi)_{\varepsilon_k} \in C^\infty_{D'}(E(\Omega'))$ and

$$(u_k \circ \Phi)_{\varepsilon_k} \to u \circ \Phi$$

in $W^{1,2}(\Omega')$. This gives that $u \circ \Phi \in W^{1,2}_{D'}(\Omega')$. □

We are now ready to flatten the boundary, so we will define our new functions on $\mathbb{R}^n_+$. Given a function $u \in W^{1,2}_D(\Omega)$, $0 < r < r_0$, and $x = (x', x_n) \in \partial\Omega$, where $x' \in \mathbb{R}^{n-1}$, we define v by

$$v(y) = \begin{cases} \eta(y)(u \circ \Phi)(y) & \text{on } \mathbb{R}^n_+ \cap B_{r/M}(x', 0) \\ 0 & \text{else} \end{cases} \tag{2.3}$$

where $\Phi^{-1}(y) = (y', y_n - \varphi(y'))$ is the function that flattens the boundary of Ω and $\eta \in C^\infty(\mathbb{R}^n)$ satisfies

$$\begin{cases} \eta = 1 & \text{on } B_{\frac{r}{2M}}(x', 0) \\ \eta = 0 & \text{on } \mathbb{R}^n \backslash B_{\frac{2r}{3M}}(x', 0) \\ |\nabla \eta| \le \frac{c}{r} & \text{everywhere.} \end{cases}$$

When we flatten the boundary, we will want to ensure we still have a Lipschitz dissection of the boundary. As a result, we must choose carefully how we define the dissection of $\partial\mathbb{R}^n_+$. Given our $x \in \partial\Omega$, we know there exists a Lipschitz function, Ψ, a graph domain, Ω_x, and a Lipschitz dissection of its boundary given by D_x and N_x that agree with D and N, respectively, on $B_{100r_0}(x)$. We know $D_x = \{(x_1, x'', x_n) : x_1 \ge \Psi(x'')\}$. We define

$$\tilde{D} = \{(x_1, x'', 0) : x_1 \ge \Psi(x'')\}$$

and

$$\tilde{N} = \{(x_1, x'', 0) : x_1 < \Psi(x'')\}.$$

If we assume that our original function u is a solution to (1.3), we will show that v is a weak solution to the following

$$\begin{cases} -\mathrm{div}(b\nabla v) + v &= \tilde{g} + v \quad \text{in } \mathbb{R}^n_+ \\ v &= 0 \quad \text{on } \tilde{D} \\ \frac{\partial v}{\partial \nu} &= \tilde{f}_N \quad \text{on } \tilde{N} \end{cases} \tag{2.4}$$

where $\frac{\partial v}{\partial \nu} = \sum b_j \frac{\partial v}{\partial x_j} \nu_j$,

$$
\begin{aligned}
b_{\ell m} &= \sum_{i,j=1}^{n} |\det D\Phi| (a_{ij}\circ\Phi)\left(\frac{\partial(\Phi^{-1})_\ell}{\partial x_i}\circ\Phi\right)\left(\frac{\partial(\Phi^{-1})_m}{\partial x_j}\circ\Phi\right),\\
\tilde{g} &= \begin{cases} -(u\circ\Phi)\mathrm{div}(b\nabla\eta) - 2b\nabla(u\circ\Phi)\cdot\nabla\eta & \text{on } \Upsilon_{\frac{r}{M}}(x',0)\\ \qquad +|\det D\Phi|(g\circ\Phi^{-1}) & \\ 0 & \text{else,}\end{cases}
\end{aligned}
$$

and

$$
\tilde{f}_N = \begin{cases} (u\circ\Phi)b\nabla\eta\cdot\nu + \sqrt{1+|\nabla\varphi|^2}(f_N\circ\Phi^{-1}) & \text{on } \Upsilon_{\frac{r}{M}}(x',0)\\ 0 & \text{else.}\end{cases}
$$

Next, we give a careful definition of the operator $(-\mathrm{div} b\nabla + I)$ in (2.4). For $v \in W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+)$, we define a map given by (2.4), denoted $B : W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+) \to (W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+))^*$, as follows

$$
Bv(\Psi)) = \int_{\mathbb{R}^n_+} (b\nabla v\cdot\nabla\Psi + v\Psi)\,dy, \qquad \Psi\in W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+).
$$

In order to prove our Reverse Hölder Inequality, we want to be able to ensure that B has a right inverse. We define $\tilde{B} : W^{1,2}(\mathbb{R}^n_+) \to (W^{1,2}_0(\mathbb{R}^n_+))^*$ as

$$
\tilde{B}v(\Psi)) = \int_{\mathbb{R}^n_+} (b\nabla v\cdot\nabla\Psi + v\Psi)\,dy \tag{2.5}
$$

for all $\Psi \in W^{1,2}_0(\mathbb{R}^n_+)$. Also note that the bilinear form $\tilde{B}v(\Psi)$ is coercive on $W^{1,2}_0(\mathbb{R}^n_+)\times W^{1,2}_0(\mathbb{R}^n_+)$.

Lemma 2.6. *Given $u \in W^{1,2}_D(\Omega)$, a weak solution to (1.3), and a fixed $x = (x', x_n) \in \partial\Omega$, we have that v defined by (2.3) is in $W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+)$. Moreover, v is a weak solution to (2.4) and the matrix b is symmetric and elliptic with Lipschitz coefficients.*

Proof. (of Lemma 2.6) From Lemma 2.3, we have that b is elliptic. That b is symmetric and has Lipschitz coefficients follows from its definition and our assumption that Ω is $C^{1,1}$. Now, recall that we flatten a piece of the boundary of Ω using $\Phi : \mathbb{R}^n_+ \to \Omega_x$ where Ω_x is the region above the graph of a $C^{1,1}$ function and $\Omega\cap\Phi(\mathbb{R}^n_+\cap B_{\frac{1}{M}}(x',0)) = \Omega_x\cap\Phi(B_{\frac{1}{M}}(x',0)$. Define $S = \Phi(\mathbb{R}^n_+\cap B_{\frac{r}{M}}(x',0))$. We know that $u \in W^{1,2}_{D\cap\bar{S}}(S)$. By Lemma 2.2, we have that Φ is bi-Lipschitz and $\det D\Phi = 1$. Using $\Omega = \Omega_x$ and $\Omega' = \mathbb{R}^n_+$ in Lemma 2.5, we have that $v \in W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+)$. Now let $\Psi \in W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+)$ be a test function. Define

$$
\tilde{\Psi}(x) = \begin{cases} \eta(\Phi^{-1}(x))\Psi(\Phi^{-1}(x)) & \text{on } S\\ 0 & \text{else}\end{cases}
$$

Again, by Lemma 2.5, we have that $\tilde{\Psi} \in W^{1,2}_D(\Omega)$. Hence using the weak formulation of our original mixed problem we know that

$$\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx = \int_\Omega g\tilde{\Psi}\, dx + \int_N f_N \tilde{\Psi}\, d\sigma. \tag{2.6}$$

We start by changing variables on the left-hand side of equation (2.6).

$$\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx = \int_S \sum_{i,j}^n a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial \tilde{\Psi}}{\partial x_j}\, dx.$$

Letting $u = u \circ \Phi \circ \Phi^{-1}$ and using the chain rule from Lemma 2.4, we have

$$\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx = \int_S \sum_{i,j,\ell}^n a_{ij} \frac{\partial\,(u\circ\Phi)}{\partial y_\ell} \circ \Phi^{-1} \frac{\partial (\Phi^{-1})_\ell}{\partial x_i}\frac{\partial \tilde{\Psi}}{\partial x_j}\, dx.$$

Changing variables by letting $x = \Phi(y)$, we get

$$\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx = \int_{\Phi^{-1}(S)} \sum_{i,j,\ell}^n (a_{ij}\circ\Phi) \frac{\partial\,(u\circ\Phi)}{\partial y_\ell} \left(\frac{\partial (\Phi^{-1})_\ell}{\partial x_i}\circ\Phi\right)\left(\frac{\partial \tilde{\Psi}}{\partial x_j}\circ\Phi\right) dy.$$

From the definition of $\tilde{\Psi}$, straight forward calculation gives that

$$\frac{\partial \tilde{\Psi}}{\partial x_j}\circ\Phi = \sum_{m=1}^n \left(\Psi \frac{\partial \eta}{\partial y_m}\frac{\partial(\Phi^{-1})_m}{\partial x_j}\circ\Phi + \eta \frac{\partial \Psi}{\partial y_m}\frac{\partial(\Phi^{-1})_m}{\partial x_j}\circ\Phi\right).$$

Using this and the definition we gave for $b_{\ell m}$, we have

$$\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx = \int_{\Phi^{-1}(S)} (\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta + \eta b\nabla(u\circ\Phi)\cdot\nabla\Psi)\, dy.$$

Next, subtracting and adding $(u\circ\Phi)b\nabla\Psi\cdot\nabla\eta$, we have

$$\begin{aligned}\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx &= \int_{\Phi^{-1}(S)} (\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta - (u\circ\Phi)b\nabla\Psi\cdot\nabla\eta \\ &\quad +(u\circ\Phi)b\nabla\Psi\cdot\nabla\eta + \eta b\nabla(u\circ\Phi)\cdot\nabla\Psi)\, dy.\end{aligned}$$

Combining the last two terms in this equation gives us the following

$$\begin{aligned}\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx &= \int_{\Phi^{-1}(S)} (\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta - (u\circ\Phi)b\nabla\Psi\cdot\nabla\eta \\ &\quad +b\nabla(\eta(u\circ\Psi))\cdot\nabla\Psi)\, dy.\end{aligned}$$

Recalling our definition of v, we have

$$\begin{aligned}\textstyle\int_\Omega a\nabla u \cdot \nabla\tilde{\Psi}\, dx &= \textstyle\int_{\Phi^{-1}(S)} (\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta - (u\circ\Phi)b\nabla\Psi\cdot\nabla\eta \\ &\quad +b\nabla v\cdot\nabla\Psi)\, dy.\end{aligned} \tag{2.7}$$

Now, we need to work through the second term of this equation. Adding and subtracting $\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta$ gives

$$\int_{\Phi^{-1}(S)} (u\circ\Phi) b\nabla\Psi\cdot\nabla\eta\, dy = \int_{\Phi^{-1}(S)} ((u\circ\Phi) b\nabla\Psi\cdot\nabla\eta + \Psi b\nabla(u\circ\Phi)\cdot\nabla\eta - \Psi b\nabla(u\circ\Phi)\cdot\nabla\eta)\, dy.$$

Combining the first two terms of the right-hand side, we have

$$\int_{\Phi^{-1}(S)} (u\circ\Phi) b\nabla\Psi\cdot\nabla\eta\, dy = \int_{\Phi^{-1}(S)} (b\nabla((u\circ\Phi)\Psi)\cdot\nabla\eta - \Psi b\nabla(u\circ\Phi)\cdot\nabla\eta)\, dy.$$

Now using integration by parts on the first term of this equation gives

$$\begin{aligned}\int_{\Phi^{-1}(S)} (u\circ\Phi) b\nabla\Psi\cdot\nabla\eta\, dy &= -\int_{\Phi^{-1}(S)} (u\circ\Phi)\Psi \operatorname{div}(b\nabla\eta)\, dy \\ &\quad + \int_{\partial(\Phi^{-1}(S))} (u\circ\Phi)\Psi b\nabla\eta\cdot\nu\, d\sigma \\ &\quad - \int_{\Phi^{-1}(S)} \Psi b\nabla(u\circ\Phi)\cdot\nabla\eta)\, dy.\end{aligned}$$

Substituting this into (2.7), gives

$$\begin{aligned}\int_\Omega a\nabla u\cdot\nabla\tilde{\Psi}\, dx &= \int_{\Phi^{-1}(S)} (2\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta + (u\circ\Phi)\Psi\operatorname{div}(b\nabla\eta))\, dy \\ &\quad + \int_{\Phi^{-1}(S)} b\nabla v\cdot\nabla\Psi\, dy - \int_{\partial(\Phi^{-1}(S))} (u\circ\Phi)\Psi b\nabla\eta\cdot\nu\, d\sigma.\end{aligned}$$

By definition of η, $\nabla\eta$ is zero on $\partial\Upsilon_{\frac{r}{M}}(x',0)\cap\mathbb{R}^n_+$ and $\Psi = 0$ on $\Phi^{-1}(D\cap S)$. This gives

$$\begin{aligned}\textstyle\int_\Omega a\nabla u\cdot\nabla\tilde{\Psi}\, dx &= \textstyle\int_{\Phi^{-1}(S)} (2\Psi b\nabla(u\circ\Phi)\cdot\nabla\eta + (u\circ\Phi)\Psi\operatorname{div}(b\nabla\eta))\, dy \\ &\quad \textstyle+ \int_{\Phi^{-1}(S)} b\nabla v\cdot\nabla\Psi\, dy - \int_{\tilde{N}\cap B_{\frac{r}{M}}(x',0)} (u\circ\Phi)\Psi b\nabla\eta\cdot\nu\, d\sigma.\end{aligned} \tag{2.8}$$

Using the same change of variables as earlier, but on the left-hand side of (2.6), we get

$$\begin{aligned}\textstyle\int_\Omega g\tilde{\Psi}\, dx + \int_N f_N\tilde{\Psi}\, d\sigma &= \textstyle\int_{\Phi^{-1}(S)} |\det D\Phi|(g\circ\Phi^{-1})\Psi\, dy \\ &\quad \textstyle+ \int_{\Phi^{-1}(N)\cap B_{\frac{r}{M}}(x',0)} \sqrt{1+|\nabla\varphi|^2}(f_N\circ\Phi^{-1})\Psi\, d\sigma.\end{aligned} \tag{2.9}$$

Now, combining (2.8), (2.9), and (2.6) gives

$$\begin{aligned}\int_{\Phi^{-1}(S)} b\nabla v\cdot\nabla\Psi\, dy &= \int_{\tilde{N}\cap B_{\frac{r}{M}}(x',0)} \left(\sqrt{1+|\nabla\varphi|^2}(f_N\circ\Phi^{-1}) + (u\circ\Phi)b\nabla v\cdot\nu\right)\Psi\, d\sigma \\ &\quad + \int_{\Phi^{-1}(S)} (|\det D\Phi|(g\circ\Phi^{-1}) - 2b\nabla(u\circ\Phi)\cdot\nabla\eta \\ &\quad -(u\circ\Phi)\operatorname{div}(b\nabla\eta))\Psi\, dy.\end{aligned}$$

Finally, recalling the definition of $\tilde{g}$ and $\tilde{f}_N$, along with recalling the support of v, we have

$$\int_{\mathbb{R}^n_+} b\nabla v \cdot \nabla \Psi \, dy \quad = \quad \int_{\tilde{N}} \tilde{f}_N \Psi \, d\sigma + \int_{\mathbb{R}^n_+} \tilde{g} \Psi \, dy$$

for all $\Psi \in W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+)$ as desired. □

It is obvious from the definition that $\tilde{D}$ and $\tilde{N}$ in (2.4) still give a Lipschitz dissection of the boundary. The following Lemma will allow us to ensure that the difference quotient of our test functions still lie in the same test space.

Lemma 2.7. *Let $\Omega = \mathbb{R}^n_+$ with $\tilde{N}$ and $\tilde{D}$, a Lipschitz dissection of the boundary, given by one Lipschitz function, Ψ. For $x \in \tilde{D}$ and $h > 0$, we have that*

$$x + h\alpha \in \tilde{D}$$

for all $\alpha \in \{(\alpha_1, \alpha'', 0) : \alpha_1 > M|\alpha''|\}$, where M is the Lipschitz coefficient for Ψ.

Proof. (of Lemma 2.7) Let $x \in \tilde{D}$, $h > 0$, and α be as in the Lemma. First note that

$$\begin{aligned} x + h\alpha &= (x_1, x'', 0) + h(\alpha_1, \alpha'', 0) \\ &= (x_1 + h\alpha_1, x'' + h\alpha'', 0) \end{aligned}$$

This gives that $x + h\alpha \in \partial\mathbb{R}^n_+$. Next note,

$$\begin{aligned} \Psi(x'' + h\alpha'') &= \Psi(x'' + h\alpha'') - \Psi(x'') + \Psi(x'') \\ &\leq Mh|\alpha''| + \Psi(x'') \end{aligned}$$

since Ψ is Lipschitz. Now, since $x \in \tilde{D}$

$$\begin{aligned} \Psi(x'' + h\alpha'') &\leq Mh|\alpha''| + x_1 \\ &\leq h\alpha_1 + x_1 \end{aligned}$$

by choice of α. Hence we have that $x + h\alpha \in \tilde{D}$. □

Remark: It is clear that we can choose a basis for $\partial\mathbb{R}^n_+$ in $\{(\alpha_1, \alpha'', 0) : \alpha_1 > M|\alpha''\}$. We will choose such a basis later in this chapter.

2.3 Besov Spaces

We want to show that our function, v, is in a **Besov space**. We begin by defining the Besov spaces we are interested in, $\mathbf{B^s_{p,q}}(\mathbb{R}^\mathbf{n}_+)$, as the real interpolation space,

$$B^s_{p,q}(\mathbb{R}^n_+) = (L^p(\mathbb{R}^n_+), W^{1,p}(\mathbb{R}^n_+))_{s,q}.$$

For a definition of interpolation spaces see Bergh and Löfström [2]. We also define some related spaces found in Savaré [13]. For $\theta > 0$, define

$$D'_{1+\theta}(\mathbb{R}^n_+) = \left\{ w \in W^{1,2}(\mathbb{R}^n_+) : \sup_{\substack{h>0 \\ k=1,\dots,n-1}} \left(\frac{\|\nabla w(\cdot + he_k) - \nabla w\|_{L^2(\mathbb{R}^n_+)}}{h^\theta} \right) < \infty \right\}.$$

For this paper, we are interested in $B^{1/2}_{2,\infty}$ and $B^{3/2}_{2,\infty}$. In order to prove our function is in one of these Besov spaces, we need an alternate definition for $B^{3/2}_{2,\infty}(\mathbb{R}^n_+)$. We start by proving some necessary results.

Lemma 2.8. *Let $\{\beta_k\}_{k=1}^{n-1}$ give a basis for $\partial\mathbb{R}^n_+$, then*

$$\|u\|_{D'^{\beta}_{1+\theta}} = \|u\|_{W^{1,2}(\mathbb{R}^n_+)} + \sup_{\substack{h>0 \\ k=1,\dots,n-1}} \left\{ \frac{\|\nabla u(\cdot + h\beta_k) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)}}{h^\theta} \right\}$$

is an equivalent norm on $D'_{1+\theta}$.

Proof. (of Lemma 2.8) First, since we know that $\{e_1, \dots, e_{n-1}\}$ is a basis for $\partial\mathbb{R}^n_+$, we have that $\beta_i = \sum_{j=1}^{n-1} c_{ij} e_j$, so

$$\begin{aligned} \|\nabla(\cdot + h\beta_i) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)} &= \|\nabla u(\cdot + h\sum_{j=1}^{n-1} c_{ij}e_j) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)} \\ &\leq \|\nabla u(\cdot + h\sum_{j=1}^{n-1} c_{ij}e_j) - \nabla u(\cdot + hc_{i,n-1}e_{n-1})\|_{L^2(\mathbb{R}^n_+)} \\ &\quad + \|\nabla(\cdot + hc_{i,n-1}e_{n-1}) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)} \end{aligned}$$

where we subtracted and added $\nabla u(\cdot + hc_{i,n-1}e_{n-1})$ and used the triangle inequality. Note that we can change variables on the first term using $y = x + hc_{i,n-1}e_{n-1}$. If $c_{i,n-1} < 0$, we can translate the last term as well to get the following result

$$\begin{aligned} \|\nabla u(\cdot + h\beta_i) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)} &\leq \|\nabla u(\cdot + h\sum_{j=1}^{n-2} c_{ij}e_j) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)} \\ &\quad + h^\theta \sup_{k>0} \frac{\|\nabla u(\cdot + ke_{n-1}) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)}}{k^\theta}. \end{aligned}$$

Continuing in this matter for each e_j, we get

$$\|\nabla(\cdot + h\beta_i) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)} \leq h^\theta \sum_{j=1}^{n-1} \sup_{k>0} \frac{\|\nabla u(\cdot + ke_j) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)}}{k^\theta}.$$

From this, it follows easily that

$$\|u\|_{D'^{\beta}_{1+\theta}} \leq c\|u\|_{D'_{1+\theta}}.$$

The other direction follows analogously. □

In order to prove several of the following results, we need to utilize a basic result for difference quotients found in Evans [7, p292-293]. First, we define for $\xi \in \mathbb{R}^{n-1} \times \{0\} \backslash \{\mathbf{0}\}$,

$$\tau_\xi u = u(\cdot + \xi) \tag{2.10}$$

and

$$\Delta_\xi u = \frac{u(\cdot + \xi) - u(\cdot)}{|\xi|}. \tag{2.11}$$

Note that this notation is a little confusing, Δ_ξ is not the Laplacian.

Lemma 2.9. *Let $v \in W^{1,p}(\mathbb{R}^n_+)$ for $1 \leq p < \infty$ be supported on a unit ball, then for $\xi \in \mathbb{R}^n$ with $\xi_n = 0$, we have*

$$\|\Delta_\xi v\|_{L^p(\mathbb{R}^n_+)} \leq c\|\nabla v\|_{L^p(\mathbb{R}^n_+)}.$$

Lemma 2.10. *For D'_2 defined earlier ($\theta = 1$), we have that*

$$W^{2,2}(\mathbb{R}^n_+) = \{u \in D'_2(\mathbb{R}^n_+) : \tilde{B}u \in L^2(\mathbb{R}^n_+)\}$$

where $\tilde{B}$ defined in (2.5).

Proof. (of Lemma 2.10) If $u \in W^{2,2}(\mathbb{R}^n_+)$, easily we have that $u \in D'_2(\mathbb{R}^n_+)$ and integration by parts gives that

$$(\tilde{B}u)(\Psi) = \int_{\mathbb{R}^n_+} \ell(z)\Psi(z)dz$$

for some $\ell \in L^2(\mathbb{R}^n_+)$. This is what we mean for $\tilde{B}u \in L^2(\mathbb{R}^n_+)$, which gives the first direction.

For the other direction, suppose that $u \in D'_2(\mathbb{R}^n_+)$ and $\tilde{B}u \in L^2(\mathbb{R}^n_+)$. We will begin by proving that $\frac{\partial^2 u}{\partial x_n^2} \in L^2(\mathbb{R}^n_+)$ will follow from showing that $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(R^n_+)$ for $(i,j) \neq (n,n)$. By Theorem 1 on interior H^2-regularity on page 327 of Evans [7], we have that $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^n_+)$, since $\tilde{B}u \in L^2(\mathbb{R}^n_+)$. Moreover, if $\ell \in L^2(\mathbb{R}^n_+)$ is the function satisfying

$$(\tilde{B}u)(\Psi) = \int_{\mathbb{R}^n_+} \ell(z)\Psi(z)dz,$$

then by the remarks following Theorem 1 on page 328 of Evans [7], we have that

$$-\mathrm{div}(b\nabla u) + u = \ell$$

almost everywhere. Writing this in non-divergence form gives

$$-\sum_{i,j=1}^{n} b_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} c_k \frac{\partial u}{\partial x_k} + u = \ell$$

almost everywhere. Rearranging, we have

$$b_{nn}\frac{\partial^2 u}{\partial x_n^2} = -\sum_{\substack{i,j=1\\ i+j\neq 2n}}^{n} b_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} c_k \frac{\partial u}{\partial x_k} + u - \ell.$$

Since b is elliptic, we know that

$$0 < \theta \leq \theta |e_n|^2 \leq \sum_{i,j=1}^{n} b_{ij}(e_n)_i(e_n)_j = b_{nn}.$$

This gives that $b_{nn} > 0$ and therefore we can divide to obtain,

$$\frac{\partial^2 u}{\partial x_n^2} = -\sum_{\substack{i,j=1\\ i+j\neq 2n}}^{n} \frac{b_{ij}}{b_{nn}}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} \frac{c_k}{b_{nn}} \frac{\partial u}{\partial x_k} + \frac{u}{b_{nn}} - \frac{\ell}{b_{nn}}$$

almost everywhere.
Now, c_k and b_{ij} are in $L^\infty(\mathbb{R}^n_+)$, $\frac{1}{b_{nn}} \leq \frac{1}{\varepsilon}$, and we know that u, $\frac{\partial u}{\partial x_k}$, and ℓ are in $L^2(\mathbb{R}^n_+)$. If we know that $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\mathbb{R}^n_+)$ for $(i,j) \neq (n,n)$, then we would have that $\frac{\partial^2 u}{\partial x_n^2} \in L^2(\mathbb{R}^n_+)$ and thus $u \in W^{2,2}(\mathbb{R}^n_+)$ as desired. We only need to prove that $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\mathbb{R}^n_+)$ for $(i,j) \neq (n,n)$. Since $u \in D_2'(\mathbb{R}^n_+)$, we have that for $i \neq n$

$$\frac{\|\nabla u(\cdot + he_i) - \nabla u(\cdot)\|_{L^2(\mathbb{R}^n_+)}}{h} \leq c$$

for all $h > 0$. Thus we have a sequence of $\{h_k\}_{k=1}^{\infty}$ for which

$$\frac{\frac{\partial u}{\partial x_j}(\cdot + h_k e_i) - \frac{\partial u}{\partial x_j}(\cdot)}{h_k}$$

converges to $v_{ij} \in L^2(\mathbb{R}^n_+)$ weakly. Given $\Phi \in C_c^\infty(\mathbb{R}^n_+)$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}\,dy &= \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_j} \lim_{k\to\infty} \frac{\Phi(y - h_k e_i) - \Phi(y)}{-h_k}\,dy \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_j}\left(\frac{\Phi(y - h_k e_i) - \Phi(y)}{-h_k}\right) dy
\end{aligned}$$

by the Dominated Convergence Theorem. Now changing variables gives

$$\begin{aligned}\int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_j}\frac{\partial \Phi}{\partial x_i}\,dy &= \lim_{k\to\infty}\int_{\mathbb{R}^n_+}\left(\frac{\frac{\partial u}{\partial x_j}(y+h_k e_i) - \frac{\partial u}{\partial x_j}(y)}{-h_k}\right)\Phi(y)\,dy \\ &= -\int_{\mathbb{R}^n_+} v_{ij}(y)\Phi(y)\,dy\end{aligned}$$

by weak convergence of the difference quotient. Hence the weak derivative of $\frac{\partial u}{\partial x_j}$, namely $\frac{\partial^2 u}{\partial x_j \partial x_i}$, is equal to v_{ij}. Hence we have for $(i,j) \neq (n,n)$ that $\frac{\partial^2 u}{\partial x_j \partial x_i} \in L^2(\mathbb{R}^n_+)$ and so $\frac{\partial^2 u}{\partial x_n^2} \in L^2(\mathbb{R}^n_+)$. This gives us the result. □

Remark: It is obvious that

$$W^{1,2}(\mathbb{R}^n_+) = \{u \in W^{1,2}(\mathbb{R}^n_+) : \tilde{B}u \in (W_0^{1,2}(\mathbb{R}^n_+))^*\}$$

since for $u \in W^{1,2}(\mathbb{R}^n_+)$, we know $\tilde{B}u \in (W_0^{1,2}(\mathbb{R}^n_+))^*$.
Now, we need a basic interpolation result before we can continue, but first we take the following definition of compatible couples of Banach Spaces from Bergh and Löfström [2]. We say that X_0 and X_1 is a **compatible couple** if there is a normed Hausdorff topological vector space $\mathcal{X}$ such that X_0 and X_1 are subspaces of $\mathcal{X}$. Then, $X_0 + X_1$ and $X_0 \cap X_1$ are also subspaces of $\mathcal{X}$. We denote compatible couples by (X_0, X_1). The following result can be found in Theorem 3.1.2 in Bergh and Löfström [2].

Lemma 2.11. *Suppose there exists an operator T, such that for compatible couples (X_0, X_1) and (Y_0, Y_1) we have*

$$\begin{aligned} T &: X_0 \to Y_0 \\ T &: Y_1 \to Y_1 \end{aligned}$$

then for $0 < \theta < 1$ and $1 \le p \le \infty$, we have that

$$T : (X_0, X_1)_{\theta,p} \to (Y_0, Y_1)_{\theta,p}.$$

We also need the following Lemma from Baiocchi [1].

Lemma 2.12. *Let (X_0, X_1) and (Z_0, Z_1) be compatible couples of Banach Spaces and $A : X_0 + X_1 \to Z_0 + Z_1$. Suppose a right inverse, $\hat{A}$, to A exists such that*

$$\begin{aligned} &\hat{A} : Z_0 \to X_0 \\ &\hat{A} : Z_1 \to X_1 \\ &\hat{A} \circ A : X_0 \to X_0 \\ &\hat{A} \circ A : X_1 \to X_1 \\ &A \circ \hat{A} = id. \end{aligned}$$

Let

$$Y_0 = \{u \in X_0 : Au \in Z_0\}$$
$$Y_1 = \{u \in X_1 : Au \in Z_1\},$$

then we have for $0 < \theta < 1$ *and* $1 \leq q \leq \infty$ *that*

$$(Y_0, Y_1)_{\theta,q} = \{u \in (X_0, X_1)_{\theta,q} : Au \in (Z_0, Z_1)_{\theta,q}\}.$$

For completeness, we include the proof of this Lemma.

Proof. (of Lemma 2.12) First, to prove the forward direction of the containment, note that we have

$$i : Y_k \to X_k$$

where i is the natural inclusion and

$$A : Y_k \to Z_k$$

for $k = 0, 1$ easily. Lemma 2.11 gives for $0 < \theta < 1$ and $1 \leq q \leq \infty$, that

$$i : (Y_0, Y_1)_{\theta,q} \to (X_0, X_1)_{\theta,q}$$

and

$$A : (Y_0, Y_1)_{\theta,q} \to (Z_0, Z_1)_{\theta,q}.$$

This means that if $u \in (Y_0, Y_1)_{\theta,q}$, then $u \in (X_0, X_1)_{\theta,q}$ and $Au \in (Z_0, Z_1)_{\theta,q}$. Therefore,

$$(Y_0, Y_1)_{\theta,q} \subseteq \{u \in (X_0, X_1)_{\theta,q} : Au \in (Z_0, Z_1)_{\theta,q}\}.$$

Now to get the other direction of the containment, start by defining

$$P(u, f) = u - \hat{A}(Au - f).$$

If $u \in X_0$ and $f \in Z_0$, then $Au - f \in Z_0$. We know by assumption that $\hat{A}(Au - f) \in X_0$, hence

$$P(u, f) : X_0 \times Z_0 \to X_0.$$

Similarly, we have that

$$P(u, f) : X_1 \times Z_1 \to X_1.$$

Next note,

$$\begin{aligned} A(P(u,f)) &= A(u - \hat{A}(Au - f)) \\ &= Au - A\hat{A}(Au - f) \\ &= Au - (Au - f) \end{aligned}$$

since $A\hat{A} = id$. Hence $A(P(u,f)) = f$. In either case we have that,

$$P : X_k \times Z_k \to Y_k$$

for $k = 0, 1$. Again by Lemma 2.11, we know that

$$P : (X_0 \times Z_0, X_1 \times Z_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$$

for $0 < \theta < 1$ and $1 \le q \le \infty$. Let $u \in \{u \in (X_0, X_1)_{\theta,q} : Au \in (Z_0, Z_1)_{\theta,q}\}$, then $(u, Au) \in (X_0 \times Z_0, X_1 \times Z_1)_{\theta,q}$. This means that $P(u, Au) \in (Y_0, Y_1)_{\theta,q}$.
Now note that $P(u, Au) = u$. Therefore we have that $u \in (Y_0, Y_1)_{\theta,q}$ as desired. □

Lemma 2.13. *Our operator $\tilde{B}$, defined in (2.5) satisfies the conditions from Lemma 2.12. Meaning there exists a one-sided inverse, $\hat{B}$, such that*

1. $\hat{B} : (W_0^{1,2}(\mathbb{R}_+^n))^* \to W_0^{1,2}(\mathbb{R}_+^n) \subseteq W^{1,2}(\mathbb{R}_+^n)$
2. $\hat{B} : L^2(\mathbb{R}^F n_+) \to D_2'(\mathbb{R}_+^n)$
3. $\tilde{B} \circ \hat{B} = id$
4. $\hat{B} \circ \tilde{B} : W^{1,2}(\mathbb{R}_+^n) \to W^{1,2}(\mathbb{R}_+^n)$
5. $\hat{B} \circ \tilde{B} : D_2'(\mathbb{R}_+^n) \to D_2'(\mathbb{R}_+^n)$

Proof. (of Lemma 2.13) First, given $f \in (W_0^{1,2}(\mathbb{R}_+^n))^*$, Lax-Milgram gives that there exists $w \in W_0^{1,2}(\mathbb{R}_+^n)$ such that

$$(\tilde{B}w)(\Psi) = f(\Psi)$$

for all $\Psi \in W_0^{1,2}(\mathbb{R}_+^n)$, since $\tilde{B}$ is coercive. Define $\hat{B}(f) = w$. Note that **1** is clear directly from Lax-Milgram. Result **4** follows directly from **1** and the fact that

$$\tilde{B} : W^{1,2}(\mathbb{R}_+^n) \to W^{-1,2}(\mathbb{R}_+^n).$$

Next we have that,

$$\tilde{B}(\hat{B}(f))(\Psi) = (\tilde{B}w)(\Psi) = f(\Psi)$$

which gives **3**. Now to show **5**, let $v = \hat{B} \circ \tilde{B}u$ with $u \in D_2'(\mathbb{R}_+^n)$. We want to show that $v \in D_2'(\mathbb{R}_+^n)$. By **3**, we know that $\tilde{B}v = \tilde{B}u$. Hence

$$\tilde{B}u(\Psi) = \tilde{B}v(\Psi) = \int_{\mathbb{R}_+^n} (b\nabla v \cdot \nabla\Psi + v\Psi)\, dy.$$

Also, we know that $v \in W_0^{1,2}(\mathbb{R}^n_+)$ by definition of $\hat{B}$. Easily, we have $\Delta_\xi v \in W_0^{1,2}(\mathbb{R}^n_+)$, so we can use it as a test function. Hence

$$(\tilde{B}u)(\Delta_\xi v) = \int_{\mathbb{R}^n_+} (b\nabla v \cdot \nabla(\Delta_\xi v) + v\Delta_\xi v)\, dy. \tag{2.12}$$

Now define,

$$\tau_\xi(\tilde{B}v)(\Psi) = \int_{\mathbb{R}^n_+} (\tau_\xi(b\nabla v) \cdot \nabla \Psi + \tau_\xi v \Psi)\, dy$$

and

$$\Delta_\xi(\tilde{B}v)(\Psi) = \frac{\tau_\xi(\tilde{B}v)(\Psi) - \tilde{B}v)(\Psi)}{|\xi|}.$$

Changing variables gives,

$$\begin{aligned}
\tau_\xi(\tilde{B}v)(\Psi) &= \int_{\mathbb{R}^n_+} (b\nabla v \cdot \nabla(\tau_{-\xi}\Psi) + v\tau_{-\xi}\Psi)\, dy \\
&= (\tilde{B}v)(\tau_{-\xi}\Psi) \\
&= (\tilde{B}u)(\tau_{-\xi}\Psi) \\
&= \tau_\xi(\tilde{B}u)(\Psi)
\end{aligned}$$

where we use the same reasoning for the last step as on the first two, but in the reverse direction. Hence letting $\Psi = \Delta_\xi v$ we have that

$$(\tau_\xi \tilde{B}u)(\Delta_\xi v) = \int_{\mathbb{R}^n_+} (\tau_\xi(b\nabla v) \cdot \nabla(\Delta_\xi v) + \tau_\xi v \Delta_\xi v)\, dy. \tag{2.13}$$

Now subtracting (2.12) from (2.13) and dividing by $|\xi|$ gives that

$$(\Delta_\xi \tilde{B}u)(\Delta_\xi v) = \frac{1}{|\xi|} \int_{\mathbb{R}^n_+} ((\tau_\xi(b\nabla v) - b\nabla v) \cdot \nabla(\Delta_\xi v) + (\tau_\xi v - v)\Delta_\xi v)\, dy.$$

Adding and subtracting $(\tau_\xi b)\nabla v$ to the integrand gives

$$\begin{aligned}
\tau_\xi(b\nabla v) - b\nabla v &= (\tau_\xi b)\tau_\xi(\nabla v) - (\tau_\xi b)\nabla v + (\tau_\xi b)\nabla v - b\nabla v \\
&= (\tau_\xi b)(\tau_\xi \nabla v - \nabla v) + (\tau_\xi b - b)\nabla v.
\end{aligned}$$

Hence,

$$(\Delta_\xi \tilde{B}u)(\Delta_\xi v) = \int_{\mathbb{R}^n_+} ((\tau_\xi b)\nabla(\Delta_\xi v) \cdot \nabla(\Delta_\xi v) + (\Delta_\xi b)\nabla v \cdot \nabla(\Delta_\xi v) + (\Delta_\xi v)(\Delta_\xi v))\, dy.$$

Since $\tau_\xi b$ is elliptic, we have

$$\begin{aligned}\min(\lambda,1)\int_{\mathbb{R}^n_+}(|\nabla(\Delta_\xi v)|^2+|\Delta_\xi v|^2)\,dy &\le (\Delta_\xi\tilde{B}u)(\Delta_\xi v)-\int_{\mathbb{R}^n_+}\Delta_\xi b\nabla\cdot\nabla(\Delta_\xi v)\,dy\\ &\le |(\Delta_\xi\tilde{B}u)(\Delta_\xi v)|\\ &\quad +M_b\|\nabla v\|_{L^2(\mathbb{R}^n_+)}\|\nabla(\Delta_\xi v)\|_{L^2(\mathbb{R}^n_+)}\end{aligned}$$

where we use that b is Lipschitz with constant M_b and Cauchy-Schwarz's Inequality. Next use Cauchy's Inequality with an ε on the last term to get

$$\begin{aligned}\min(\lambda,1)\int_{\mathbb{R}^n_+}(|\nabla(\Delta_\xi v)|^2+|\Delta_\xi v|^2)\,dy &\le |(\Delta_\xi\tilde{B}u)(\Delta_\xi v)|+c\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}\\ &\quad +\frac{\min(\lambda,1)}{2}\|\nabla(\Delta_\xi v)\|^2_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

Combining the similar terms, we have

$$\int_{\mathbb{R}^n_+}(|\nabla(\Delta_\xi v)|^2+|\Delta_\xi v|^2)\,dy\le c|(\Delta_\xi\tilde{B}u)(\Delta_\xi v)|+c\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}. \tag{2.14}$$

Now,

$$\begin{aligned}|(\Delta_\xi\hat{B}u)(\Delta_\xi v)| &= \left|\int_{\mathbb{R}^n_+}(\Delta_\xi(b\nabla u)\cdot\nabla(\Delta_\xi v)+\Delta_\xi u\Delta_\xi v)\,dy\right|\\ &= \left|\int_{\mathbb{R}^n_+}((\tau_\xi b\Delta_\xi\nabla u+\Delta_\xi b\nabla u)\nabla(\Delta_\xi v)+\Delta_\xi u\Delta_\xi v)\,dy\right|\end{aligned}$$

using our work from earlier. Next,

$$\begin{aligned}|(\Delta_\xi\tilde{B}u)(\Delta_\xi v)| &\le (\|b\|_{L^\infty(\mathbb{R}^n_+)}\|\Delta_\xi\nabla u\|_{L^2(\mathbb{R}^n_+)}+M_b\|\nabla u\|_{L^2(\mathbb{R}^n_+)})\|\nabla(\Delta_\xi v)\|_{L^2(\mathbb{R}^n_+)}\\ &\quad +\|\Delta_\xi u\|_{L^2(\mathbb{R}^n_+)}\|\Delta_\xi v\|_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

Again Cauchy's Inequality with an ε, used twice, gives

$$\begin{aligned}|(\Delta_\xi\tilde{B}u)(\Delta_\xi v)| &\le c(\|\Delta_\xi u\|^2_{L^2(\mathbb{R}^n_+)}+\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)})+(1/2)\|\nabla(\Delta_\xi v)\|^2_{L^2(\mathbb{R}^n_+)}\\ &\quad +\|\Delta_\xi u\|_{L^2(\mathbb{R}^n_+)}\|\Delta_\xi v\|_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

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Hence, substituting this into (2.14) gives

$$\begin{aligned}\int_{\mathbb{R}^n_+}(|\nabla\Delta_\xi v|^2+|\Delta_\xi v|^2)\,dy &\le c(\|\Delta_\xi u\|^2_{L^2(\mathbb{R}^n_+)}+\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)}\\ &\quad +\|\Delta_\xi u\|_{L^2(\mathbb{R}^n_+)}\|\Delta_\xi v\|_{L^2(\mathbb{R}^n_+)}+\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}).\end{aligned}$$

Since $u \in D_2'(\mathbb{R}^n_+)$ and $v \in W^{1,2}(\mathbb{R}^n_+)$, setting $\xi = h\mathbf{e}_i$ for $i = 1, \dots, n-1$, we easily get $v \in D_2'(\mathbb{R}^n_+)$. Hence we have **5**. To show **2**, let $f \in L^2(\mathbb{R}^n_+)$. We know that for $u = \hat{B}f$, that $u \in W_0^{1,2}(\mathbb{R}^n_+)$ and

$$\int_{\mathbb{R}^n_+} (b\nabla u \cdot \nabla \Psi + u\Psi)\, dy = \int_{\mathbb{R}^n_+} f\Psi\, dy$$

for all $\Psi \in W^{1,2}(\mathbb{R}^n_+)$. Using $\Delta_\xi u$ for a test function gives

$$\int_{\mathbb{R}^n_+} (b\nabla u \cdot \nabla(\Delta_\xi u) + u\Delta_\xi u)\, dy = \int_{\mathbb{R}^n_+} f\Delta_\xi u\, dy. \tag{2.15}$$

Letting $\tau_{-\xi}\Delta_\xi u$ be a test function and translating gives that

$$\int_{\mathbb{R}^n_+} (\tau_\xi(b\nabla u) \cdot \nabla(\Delta_\xi u)\tau_\xi u\Delta_\xi u)\, dy = \int_{\mathbb{R}^n_+} f\tau_{-\xi}(\Delta_\xi u)\, dy. \tag{2.16}$$

Subtracting (2.15) and (2.16) gives

$$\int_{\mathbb{R}^n_+} ((\tau_\xi(b\nabla u) - b\nabla u) \cdot \nabla\Delta_\xi u + (\tau_\xi u - u)\Delta_\xi u)\, dy = \int_{\mathbb{R}^n_+} f(\tau_{-\xi}(\Delta_\xi u) - \Delta_\xi u)\, dy.$$

Now the same work as before gives that

$$\int_{\mathbb{R}^n_+} (|\nabla\Delta_\xi u|^2 + |\Delta_\xi u|^2)\, dy \le \int_{\mathbb{R}^n_+} f\nabla_{-\xi}(\Delta_\xi u)\, dy + c\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)}.$$

Fix $w = \Delta_\xi u$ and use Cauchy-Schwarz Inequality to get

$$\int_{\mathbb{R}^n_+} (|\nabla\Delta_\xi u|^2 + |\Delta_\xi u|^2)\, dy \le \|f\|_{L^2(\mathbb{R}^n_+)}\|\Delta_{-\xi}(w)\|_{L^2(\mathbb{R}^n_+)} + c\|\nabla u\|^2_{L^2(\mathbb{R}^n_+)}.$$

Next, using Lemma 2.9 we have that

$$\int_{\mathbb{R}^n_+} (|\nabla\Delta_\xi u|^2 + |\Delta_\xi u|^2)\, dy \le \|f\|_{L^2(\mathbb{R}^n_+)}\|\nabla\Delta_\xi u\|_{L^2(\mathbb{R}^n_+)} + c\|\nabla u\|_{L^2(\mathbb{R}^n_+)}.$$

Using Cauchy's Inequality with an ε on the first term on the right-hand side and rearranging gives

$$\int_{\mathbb{R}^n_+} (|\nabla\Delta_\xi u|^2 + |\Delta_\xi u|^2)\, dy \le c\|f\|_{L^2(\mathbb{R}^n_+)} + c\|\nabla u\|_{L^2(\mathbb{R}^n_+)}.$$

Since $f \in L^2(\mathbb{R}^n_+)$ and $\nabla u \in L^2(\mathbb{R}^n_+)$, we have that $u \in D_2'$ by using $\xi = h\mathbf{e}_i$ for $i = 1, \dots, n-1$. Hence we have the lemma. □

Lemma 2.14.

$$D_{3/2}'(\mathbb{R}^n_+) = (W^{1,2}(\mathbb{R}^n_+), D_2'(\mathbb{R}^n_+))_{1/2,\infty}$$

Proof. (of Lemma 2.14) As noted in Savaré, the proof can be carried out using semigroup theory (See Bergh and Löfstrom) [2, p. 156]. □

Lemma 2.15. *We have the following alternate characterization of the Besov Space, $B^{3/2}_{2,\infty}(\mathbb{R}^n_+)$,*

$$B^{3/2}_{2,\infty}(\mathbb{R}^n_+) = \left\{ w \in D'_{3/2}(\mathbb{R}^n_+) : \tilde{B}w \in (W^{-1,2}(\mathbb{R}^n_+), L^2(\mathbb{R}^n_+))_{1/2,\infty} \right\}$$

where $\tilde{B}$ is defined in (2.5).

Proof. (of Lemma 2.15) This alternate definition for our Besov Space follows from Lemma 2.12, Lemma 2.13, and Lemma 2.14. □

From this definition, the following gives a norm for $B^{3/2}_{2,\infty}(\mathbb{R}^n_+)$

$$\begin{aligned} \|v\|_{B^{3/2}_{2,\infty}(\mathbb{R}^n_+)} &= \|v\|_{W^{1,2}(\mathbb{R}^n_+)} + \sup_{\substack{h>0 \\ k=1,2,\dots,n-1}} \left\{ \frac{\|\nabla v(\cdot + h\beta_k - \nabla v(\cdot))\|_{L^2(\mathbb{R}^n_+)}}{\sqrt{h}} \right\} \\ &\quad + \|\tilde{B}v\|_{(W^{-1,2}(\mathbb{R}^n_+), L^2(\mathbb{R}^n_+))_{1/2,\infty}} \end{aligned} \tag{2.17}$$

where $\{\beta_k\}$ is a basis of $\mathbb{R}^{n-1} \times \{0\}$. Next, we need a couple of basic results about Besov spaces.

Lemma 2.16. *If $v \in B^{3/2}(\mathbb{R}^n_+)$, then $\nabla v \in B^{1/2}_{2,\infty}(\mathbb{R}^n_+)$. Moreover,*

$$\|\nabla v\|_{B^{1/2}_{2,\infty}(\mathbb{R}^n_+)} \leq c\|v\|_{B^{3/2}_{2,\infty}(\mathbb{R}^n_+)}.$$

The Lemma above is obvious and follows directly from the definition of interpolation spaces.

Lemma 2.17. *If $v \in B^{1/2}_{2,\infty}(\Omega)$ for a C^1 domain, Ω, then for $2 < q < \frac{2n}{n-1}$, we have*

$$\|v\|_{L^q(\Omega)} \leq c\|v\|_{B^{1/2}_{2,\infty}(\Omega)}$$

if v is supported on a ball. The constant, c, depends on this support.

Proof. (of Lemma 2.17) Theorem 2 from pg 279 of Evans [7] gives that $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, meaning for $u \in W^{1,2}(\Omega)$ we have

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq c\|u\|_{W^{1,2}(\Omega)}.$$

We also have that $L^2(\Omega) \hookrightarrow L^2(\Omega)$. Hence, by Lemma 2.11 we have that

$$(L^2(\Omega), W^{1,2}(\Omega))_{\theta,q} \hookrightarrow (L^2(\Omega), L^{\frac{2n}{n-2}}(\Omega))_{\theta,q}.$$

Recall that $B^{\theta}_{2,q} = (L^2(\Omega), W^{1,2}(\Omega))_{\theta,q}$. We define the Lorentz space by

$$L^{p,q}(\Omega) = (L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q}$$

where $0 < p_0 < p_1 \leq \infty$, $1 \leq\leq \infty$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

with $0 < \theta < 1$. Note that this is not the standard definition of a Lorentz space, but rather an equivalent definition given by Theorem 5.2.1 on page 109 of Interpolation Spaces by Bergh and Löfström [2]. In our particular case, $p_0 = 2$, $p_1 = \frac{2n}{n-2}$, $\theta = \frac{1}{2}$, and $q = \infty$. This gives us that $p = \frac{2n}{n-1}$. Hence we have that

$$B^{1/2}_{2,\infty} \hookrightarrow L^{\frac{2n}{n-1},\infty}(\Omega).$$

This means for $u \in B^{1/2}_{2,\infty}(\Omega)$, we have that

$$\|u\|_{L^{\frac{2n}{n-1},\infty}(\Omega)} \leq c\|u\|_{B^{1/2}_{2,\infty}(\Omega)}. \tag{2.18}$$

Next we have another definition of the Lorentz Space which holds only when $q = \infty$ given by

$$L^{p,\infty}(\Omega) = \{f : \sup_{t>0}\left(t^p m(\{|f| > t\})\right) < \infty\}.$$

We define

$$W_p(f) = \left\{\sup_{t>0}\left(t^p m(\{|f| > t\})\right)\right\}^{\frac{1}{p}}$$

and we have that $W_p(f) \leq c\|f\|_{L^{p,\infty}(\Omega)}$.
Fix $f \in L^{p,\infty}(\Omega)$. Let $E \subset \Omega$ be a set with finite measure that contains the support of f. Let $2 < s < p$, then

$$\int_E |f|^s dy = \int_E s\int_0^{|f|} t^{s-1}\,dt\,dy.$$

Next using Fubini to change the order of integration, we have

$$\begin{aligned}
\int_E |f|^s dy &= \int_0^\infty st^{s-1}\int_{\{y\in E:|f(y)|>t\}} dy\,dt \\
&= \int_0^\infty st^{s-1}m(\{|f| > t\})\,dt \\
&= s\int_0^A t^{s-1}m(\{|f| > t\})dt + s\int_A^\infty t^{s-1}m(\{|f| > t\})\,dt \\
&\leq cA^s m(E) + cW_p^p(f)\int_A^\infty t^{s-p-1}\,dt
\end{aligned}$$

where we use that $t^p m(\{|f| > t\})$ is bounded by the pth-power of of $W_p(f)$. Hence, we have

$$\int_E |f|^s dy \leq cA^s m(E) + cW_p^p(f)A^{s-p}.$$

Setting $A = W_p(f)$ gives

$$\|f\|_{L^s(\Omega)} \le cW_p(f).$$

Combining this with (2.18) gives the result. □

2.4 Proof of Reverse Hölder Inequality

In this section, we will use a difference quotient method to prove our Reverse Hölder Inequality. To prove Theorem 2.1, we begin by assuming that $r = 1$.

Lemma 2.18. *Assuming that our Neumann data f_N has a constant value of κ on $B_1(x)$, then for our function v, defined in (2.3), we have*

$$v \in B^{3/2}_{2,\infty}(\mathbb{R}^n_+).$$

Moreover, recalling u in the definition of v, we have

$$\|v\|_{B^{3/2}_{2,\infty}(\mathbb{R}^n_+)} \le c\|\nabla u\|_{L^2(\Upsilon_1(x))} + c\|u\|_{L^2(\Upsilon_1(x))} + c\|g\|_{L^2(\Upsilon_1(x))} + c|\kappa|.$$

Proof. (of Lemma 2.18) Recalling Lemma 2.15, we will begin by getting a bound on the $D'_{3/2}(\mathbb{R}^n_+)$ norm of v and also the $(W^{-1,2}(\mathbb{R}^n_+), L^2(\mathbb{R}^n_+))_{1/2,\infty}$ norm of $\tilde{B}v$. Let $\{\beta_i\}_{i=1}^{n-1}$ be a basis for $\partial\mathbb{R}^n_+$ chosen so that each $\beta \in \{(\alpha_1, \alpha'', 0) : \alpha_1 > M|\alpha''|\}$ from Lemma 2.7. Let $h > 0$. From this Lemma we easily see that $v(\cdot + h\beta_i) \in W^{1,2}_{\tilde{D}}(\mathbb{R}^n_+)$. So we can let $v - \tau_{h\beta_i}v$ be a test function, which gives that

$$\begin{aligned}\int_{\mathbb{R}^n_+} b\nabla v \cdot \nabla(v - \tau_{h\beta_i}v)\,dy + \int_{\mathbb{R}^n_+} v(v - \tau_{h\beta_i}v)\,dy \;&=\; \int_{\mathbb{R}^n_+} (\tilde{g} + v)(v - \tau_{h\beta_i}v)\,dy \\ &\quad + \int_{\tilde{N}} \tilde{f}_N(v - \tau_{h\beta_i}v)\,dy.\end{aligned}$$

Let's look at just the first term on the left-hand side to see,

$$\begin{aligned}\int_{\mathbb{R}^n_+} b\nabla v \cdot \nabla(v - \tau_{h\beta_i}v)\,dy \;&=\; \frac{1}{2}\Big\{\int_{\mathbb{R}^n_+} b\nabla(v - \tau_{h\beta_i}v) \cdot \nabla(v - \tau_{h\beta_i}v)\,dy \\ &\quad + \int_{\mathbb{R}^n_+} b\nabla\tau_{h\beta_i}v \cdot \nabla(v - \tau_{h\beta_i}v)\,dy \\ &\quad + \int_{\mathbb{R}^n_+} b\nabla v \cdot \nabla(v - \tau_{h\beta_i}v)\,dy\Big\}\end{aligned}$$

by straight forward calculation and utilizing that b is symmetric. Now using ellipticity on the first term of the right-hand side gives

$$\begin{aligned}\int_{\mathbb{R}^n_+} b\nabla v\cdot\nabla(v-\tau_{h\beta_i}v)\,dy &\geq \frac{1}{2}\Big\{c\int_{\mathbb{R}^n_+}|\nabla(v-\tau_{h\beta_i}v)|^2\,dy+\int_{\mathbb{R}^n_+} b\nabla\tau_{h\beta_i}v\cdot\nabla v\,dy\\ &\qquad -\int_{\mathbb{R}^n_+} b\nabla\tau_{h\beta_i}v\cdot\nabla\tau_{h\beta_i}v\,dy+\int_{\mathbb{R}^n_+} b\nabla v\cdot\nabla v\,dy\\ &\qquad -\int_{\mathbb{R}^n_+} b\nabla v\cdot\nabla\tau_{h\beta_i}v\,dy\Big\}\\ &= \frac{1}{2}\Big\{c\int_{\mathbb{R}^n_+}|\nabla(v-\tau_{h\beta_i}v)|^2\,dy-\int_{\mathbb{R}^n_+} b\nabla\tau_{h\beta_i}v\cdot\nabla\tau_{h\beta_i}v\,dy\\ &\qquad +\int_{\mathbb{R}^n_+} b\nabla v\cdot\nabla v\,dy\Big\}.\end{aligned}$$

Now we will focus on the last two terms on the right-hand side. By changing variables on the last term we have that

$$\int_{\mathbb{R}^n_+} b\nabla v\cdot\nabla v\,dy-\int_{\mathbb{R}^n_+} b\nabla\tau_{h\beta_i}v\cdot\nabla\tau_{h\beta_i}v\,dy=\int_{\mathbb{R}^n_+}(b-\tau_{-h\beta_i}b)\nabla v\cdot\nabla v\,dy.$$

Altogether, we now have that

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla(v-\tau_{h\beta_i}v)|^2\,dy &\leq c\Big\{\int_{\mathbb{R}^n_+}(\tau_{-h\beta_i}b-b)\nabla v\cdot\nabla v\,dy+2\int_{\mathbb{R}^n_+}(\tilde{g}+v)(v-\tau_{h\beta_i}v)\,dy\\ &\qquad +2\int_{\tilde{N}}\tilde{f}_N(v-\tau_{h\beta_i}v)\,dy-2\int_{\mathbb{R}^n_+}v(v-\tau_{h\beta_i}v)\,dy\Big\}.\end{aligned}$$

Now, notice that

$$\int_{\mathbb{R}^n_+}(\tau_{-h\beta_i}b-b)\nabla v\cdot\nabla v\,dy\leq ch\int_{\mathbb{R}^n_+}|\nabla v|^2\,dy$$

since b is Lipschitz. Hence,

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla(v-\tau_{h\beta_i}v)|^2\,dy &\leq c\Big\{ch\int_{\mathbb{R}^n_+}|\nabla v|^2\,dy+2\int_{\mathbb{R}^n_+}(\tilde{g}+v)(v-\tau_{h\beta_i}v)\,dy\\ &\qquad +2\int_{\tilde{N}}\tilde{f}_N(v-\tau_{h\beta_i}v)\,dy-2\int_{\mathbb{R}^n_+}v(v-\tau_{h\beta_i}v)\,dy\Big\}\\ &\leq ch\int_{\mathbb{R}^n_+}|\nabla v|^2\,dy+c\int_{\mathbb{R}^n_+}|\tilde{g}||v-\tau_{h\beta_i}v|\,dy\\ &\qquad +c\int_{\mathbb{R}^n_+}\tilde{f}_N(v-\tau_{h\beta_i}v)\,dy\end{aligned}$$

where we notice some terms cancel. For the second term, note that

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\tilde{g}||v-\tau_{h\beta_i}v|\,dy &\leq \|\tilde{g}\|_{L^2(\mathbb{R}^n_+)}\|v-\tau_{h\beta_i}v\|_{L^2(\mathbb{R}^n_+)}\\ &\leq \|\tilde{g}\|_{L^2(\mathbb{R}^n_+)}(ch\|\nabla v\|_{L^2(\mathbb{R}^n_+)})\end{aligned}$$

where we used Lemma 2.9 and the fact that v is supported in a unit ball. Next, using that $ab \le 1/2(a^2+b^2)$, we have

$$\int_{\mathbb{R}^n_+} |\tilde{g}||v-\tau_{h\beta_i}v|\,dy \le ch\|\tilde{g}\|^2_{L^2(\mathbb{R}^n_+)} + ch\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}.$$

Hence, putting it all back together we now have

$$\int_{\mathbb{R}^n_+} |\nabla(v-\tau_{h\beta_i}v)|^2\,dy \le ch\int_{\mathbb{R}^n_+}|\nabla v|^2\,dy + ch\int_{\mathbb{R}^n_+}|\tilde{g}|^2\,dy + c\int_{\partial\mathbb{R}^n_+}\tilde{f}_N(v-\tau_{h\beta_i}v)\,dy. \quad (2.19)$$

Now we consider the last term,

$$\int_{\partial\mathbb{R}^n_+}\tilde{f}_N(v-\tau_{h\beta_i}v)\,d\sigma(y) = \int_{\partial\mathbb{R}^n_+}(u\circ\Phi)b\nabla\eta\cdot\nu(v-\tau_{h\beta_i})\,d\sigma + \int_{\partial\mathbb{R}^n_+}\kappa\sqrt{1+|\nabla\varphi|^2}(v-\tau_{h\beta_i}v)\,d\sigma$$

using the definition of $\tilde{f}_N$. For the second term, changing variables to push the difference onto $\sqrt{1+|\nabla\varphi|^2}$, using Cauchy-Schwarz, and recalling that $\nabla\varphi$ is Lipschitz, gives

$$\begin{aligned}\int_{\partial\mathbb{R}^N_+}\kappa\sqrt{1+|\nabla\varphi|^2}(v-\tau_{h\beta_i}v)\,d\sigma &\le ch|\kappa|\|v\|_{L^2(\mathbb{R}^n_+)}\\ &\le ch|\kappa|^2 + ch\|v\|^2_{L^2(\mathbb{R}^n_+)}\end{aligned}$$

where we again use that $ab \le \frac{1}{2}(a^2+b^2)$. By the divergence theorem we have

$$\int_{\partial\mathbb{R}^n_+}(u\circ\Phi)b\nabla\eta\cdot\nu(v-\tau_{h\beta_i}v)\,d\sigma = \int_{\mathbb{R}^n_+}\operatorname{div}((u\circ\Phi)(v-\tau_{h\beta_i}v)b\nabla\eta)\,dy.$$

Straight forward calculation gives

$$\begin{aligned}\int_{\partial\mathbb{R}^n_+}(u\circ\Phi)b\nabla\eta\cdot\nu(v-\tau_{h\beta_i}v)\,d\sigma &= \int_{\mathbb{R}^n_+}b\nabla\eta\cdot\nabla((u\circ\Phi)(v-\tau_{h\beta_i}v))\,dy\\ &\quad+\int_{\mathbb{R}^n_+}(u\circ\Phi)(v-\tau_{h\beta_i}v)\operatorname{div}(b\nabla\eta)\,dy\\ &\le \int_{\mathbb{R}^n_+}((u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)\\ &\quad+(v-\tau_{h\beta_i}v)\nabla(u\circ\Phi))b\nabla\eta\,dy\\ &\quad+c\|u\circ\Phi\|_{L^2(\Upsilon_{1/M}(x',0))}\|v-\tau_{h\beta_i}v\|_{L^2(\mathbb{R}^n_+)}\end{aligned}$$

where we use the product rule in the first integral and for the second term we use Cauchy-Shwarz and the fact that η is bounded. Again, using that $ab \le \frac{1}{2}(a^2+b^2)$ on the last term as well as Lemma 2.9, gives

$$\int_{\partial\mathbb{R}^n_+}(u\circ\Phi)b\nabla\eta\cdot\nu(v-\tau_{h\beta_i}v)\,d\sigma$$
$$\leq \int_{\mathbb{R}^n_+}((u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)+(v-\tau_{h\beta_i}v)\nabla(u\circ\Phi))b\nabla\eta\,dy + ch\|u\circ\Phi\|^2_{L^2(\Upsilon_{1/M}(x',0))}+ch\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}.$$

We look at the first term on the right-hand side, to see

$$\begin{aligned}\int_{\mathbb{R}^n_+}&((u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)+(v-\tau_{h\beta_i}v)\nabla(u\circ\Phi))b\nabla\eta\,dy\\ &=\int_{\mathbb{R}^n_+}(u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)\cdot b\nabla\eta\,dy+\int_{\mathbb{R}^n_+}(v-\tau_{h\beta_i}v)\nabla(u\circ\Phi)\cdot b\nabla\eta\,dy\\ &\leq\int_{\mathbb{R}^n_+}(u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)\cdot b\nabla\eta\,dy+ch\|\nabla(u\circ\Phi)\|^2_{L^2(\Upsilon_{1/M}(x',0))}\\ &\quad+ch\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

Combining these bounds with (2.19), we have

$$\begin{aligned}\int_{\mathbb{R}^n_+}|\nabla(v-\tau_{h\beta_i}v)|^2\,dy \quad\leq\quad & ch\int_{\mathbb{R}^n_+}|\nabla v|^2\,dy+ch\int_{\mathbb{R}^n_+}|\tilde g|^2\,dy+ch\|u\circ\Phi\|^2_{W^{1,2}(\Upsilon_{1/M}(x',0))}\\ &+\int_{\mathbb{R}^n_+}(u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)\cdot b\nabla\eta\,dy+ch|\kappa|^2.\end{aligned}$$

Now we turn our attention to the second to last term. Changing variables after splitting up terms gives

$$\begin{aligned}\int_{\mathbb{R}^n_+}(u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)\cdot b\nabla\eta\,dy \quad&=\quad \int_{\mathbb{R}^n_+}((u\circ\Phi)b\nabla\eta-\tau_{h\beta_i}((u\circ\Phi)b\nabla\eta))\cdot\nabla v\,dy\\ &\leq\quad ch\|(u\circ\Phi)b\nabla\eta\|_{L^2(\Upsilon_{1/M}(x',0))}\|\nabla v\|_{L^2(\mathbb{R}^n_+)}.\end{aligned}$$

Using the same ideas as before, we have

$$\int_{\mathbb{R}^n_+}(u\circ\Phi)\nabla(v-\tau_{h\beta_i}v)\cdot b\nabla\eta\,dy\leq ch\|u\circ\Phi\|^2_{L^2(\Upsilon_{1/M}(x',0))}+ch\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)}.$$

Together we now have

$$\int_{\mathbb{R}^n_+}|\nabla(v-\tau_{h\beta_i}v)|^2\,dy\leq ch\left(\int_{\mathbb{R}^n_+}|\nabla v|^2\,dy+\int_{\mathbb{R}^n_+}|\tilde g|^2\,dy+\|u\circ\Phi\|^2_{W^{1,2}(\Upsilon_{1/M}(x',0))}+|\kappa|^2\right).$$

Now, we need to deal with $\tilde g$ term. Recalling the definition of $\tilde g$, we have

$$\begin{aligned}\|\tilde{g}\|_{L^2(\mathbb{R}^n_+)} &\le \|(u\circ\Phi)\mathrm{div}(b\nabla\eta)\|_{L^2(\Upsilon_{1/M}(x',0))} + \|2b\nabla(u\circ\Phi)\cdot\nabla\eta\|_{L^2(\Upsilon_{1/M}(x',0))} \\ &\quad + \||\det\nabla\Phi| g\circ\phi^{-1}\|_{L^2(\Upsilon_{1/M}(x',0))} \\ &\le c\|u\circ\Phi\|_{L^2(\Upsilon_{1/M}(x',0))} + c\|\nabla(u\circ\Phi)\|_{L^2(\Upsilon_{1/M}(x',0))} \\ &\quad + c\|g\circ\Phi^{-1}\|_{L^2(\Upsilon_{1/M}(x',0))}.\end{aligned}$$

Altogether now we have,

$$\begin{aligned}\int_{\mathbb{R}^n_+} |\nabla(v-\tau_{h\beta_i}v)|^2\,dy &\le ch\Big(\|\nabla v\|^2_{L^2(\mathbb{R}^n_+)} + \|g\circ\Phi^{-1}\|^2_{L^2(\Upsilon_{1/M}(x',0))} \\ &\qquad + \|u\circ\Phi\|^2_{W^{1,2}(\Upsilon_{1/M}(x',0))} + |\kappa|^2\Big).\end{aligned}$$

Recalling the definition of v and changing variables back to Ω gives

$$\int_{\mathbb{R}^n_+} |\nabla(v-\tau_{h\beta_i}v)|^2\,dy \le ch\left\{\int_{\Upsilon_1(x)} |u|^2\,dy + \int_{\Upsilon_1(x)} |\nabla u|^2\,dy + \|g\|^2_{L^2(\Upsilon_1(x))} + |\kappa|^2\right\},$$

which gives us a bound for the $D'_{3/2}$ norm. Using the definition of the norm of the Besov space given in (2.17), we have the result by bounding the $\tilde{B}v$ term its L^2 norm and then by $\|v\|_{W^{1,2}(\mathbb{R}^n_+)}$ and changing variables back to Ω. □

To prove our Reverse Hölder Inequality, we need to prove a Poincaré Inequality. The following domains are useful in that they allow us to find a family of domains for which the Poincaré inequality holds.
We say that Ω is **star-shaped Lipschitz with constant M and scale r** if there is function, $\rho : \partial(B_1(x)) \to [1, M+1]$, that is Lipschitz with constant M such that

$$\Omega = \left\{ y \ne x : |x-y| < r\rho\left(\frac{y-x}{|y-x|}\right)\right\} \cup \{x\}$$

for some $x \in \Omega$, which is the domain's center. For the following lemmas, we will assume that our domains are centered at $x = 0$ to simplify the proofs.

Lemma 2.19. *If Ω is star-shaped Lipschitz with constant M and scale r, then there is a bi-Lipschitz function f that maps $B_r(0)$ onto Ω.*

Proof. (of Lemma 2.19) Define $f : B_r(0) \to \mathbb{R}^n$ by

$$f(y) = \begin{cases} y\rho\left(\frac{y}{|y|}\right) & \text{if } y \ne 0 \\ 0 & \text{if } y = 0. \end{cases}$$

If $y = 0$, easily $f(y) \in \Omega$. If $y \ne 0$, then

$$|f(y)| \le |y|\rho\left(\frac{y}{|y|}\right) \le r\rho\left(\frac{y}{|y|}\right).$$

Hence $f(y) \in \Omega$. Let $z \in \Omega$. If $z = 0$, then easily z is in the range of f. If $z \ne 0$, define

$$y = \frac{z}{\rho\left(\frac{z}{|z|}\right)}.$$

Since $|y| < r$ and $f(y) = z$ easily, we have that f is onto Ω. Since it is clear that $y = 0$ is the only point that maps to 0, let $y, w \ne 0$ by two points in $B_r(0)$ with $f(w) = f(y)$. Now we know that

$$y\rho\left(\frac{y}{|y|}\right) = w\rho\left(\frac{w}{|w|}\right). \tag{2.20}$$

Now, since ρ maps into $[1, M+1]$, it must be that y and w are in the same direction, meaning

$$\frac{y}{|y|} = \frac{w}{|w|}.$$

This gives that $\rho\left(\frac{y}{|y|}\right) = \rho\left(\frac{w}{|w|}\right)$. Dividing these out of (2.20), gives that $y = w$. Thus we know that f has an inverse and it is given by

$$f^{-1}(z) = \begin{cases} \frac{z}{\rho\left(\frac{z}{|z|}\right)} & \text{if } z \ne 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Now we want to show that f and f^{-1} are Lipschitz. Let x and y be in $B_r(0)$. Note,

$$\begin{aligned}
|f(x) - f(y)| &= \left|x\rho\left(\frac{x}{|x|}\right) - y\rho\left(\frac{y}{|y|}\right)\right| \\
&= \left|x\rho\left(\frac{x}{|x|}\right) - x\rho\left(\frac{y}{|y|}\right) + x\rho\left(\frac{y}{|y|}\right) - y\rho\left(\frac{y}{|y|}\right)\right| \\
&\le |x|\left|\rho\left(\frac{x}{|x|}\right) - \rho\left(\frac{y}{|y|}\right)\right| + |x-y|\left|\rho\left(\frac{y}{|y|}\right)\right| \\
&\le M|x|\left|\frac{x}{|x|} - \frac{y}{|y|}\right| + (M+1)|x-y|
\end{aligned}$$

where we use that ρ is Lipschitz and bounded by $M+1$. Now,

$$\begin{aligned}
|f(x)-f(y)| &\le M|x|\left|\frac{x|y|-y|x|}{|x||y|}\right| + (M+1)|x-y| \\
&= M\left|\frac{x|y|-y|y|+y|y|-y|x|}{|y|}\right| + (M+1)|x-y| \\
&\le M\frac{|y||x-y|+|y|\ ||x|-|y||}{|y|} + (M+1)|x-y| \\
&\le 2M|x-y| + (M+1)|x-y| \\
&= (3M+1)|x-y|.
\end{aligned}$$

We have assumed that x and y are not zero, but the result is even easier if one or both of them equal zero. Hence we now have that f is Lipscitz. Now to show that f^{-1} is Lipschitz, note that ρ maps into $[1, M+1]$, so $\frac{1}{\rho}$ maps into $[\frac{1}{1+M}, 1]$. Now if $y = 0$ and $x \neq 0$, then

$$|f^{-1}(x) - f^{-1}(y)| = |f^{-1}(x)| = \frac{|x|}{\rho\left(\frac{x}{|x|}\right)} \le |x - 0|$$

so the Lipschitz condition is satisfied. If both $x, y \neq 0$, then we have

$$\begin{aligned}
|f^{-1}(x) - f^{-1}(y)| &= \left|\frac{x}{\rho\left(\frac{x}{|x|}\right)} - \frac{y}{\rho\left(\frac{y}{|y|}\right)}\right| \\
&= \left|\frac{x\rho\left(\frac{y}{|y|}\right) - y\rho\left(\frac{x}{|x|}\right)}{\rho\left(\frac{x}{|x|}\right)\rho\left(\frac{y}{|y|}\right)}\right| \\
&\le \left|x\rho\left(\frac{y}{|y|}\right) - y\rho\left(\frac{x}{|x|}\right)\right|
\end{aligned}$$

using that $\frac{1}{\rho}$ maps into $[\frac{1}{M+1}, 1]$. Next

$$\begin{aligned}
|f^{-1}(x) - f^{-1}(y)| &\le \left|x\rho\left(\frac{y}{|y|}\right) - x\rho\left(\frac{x}{|x|}\right) + x\rho\left(\frac{x}{|x|}\right) - y\rho\left(\frac{x}{|x|}\right)\right| \\
&\le |x|\left|\rho\left(\frac{y}{|y|}\right) - \rho\left(\frac{x}{|x|}\right)\right| + \rho\left(\frac{x}{|x|}\right)|x-y| \\
&\le (3M+1)|x-y|
\end{aligned}$$

by the work we already did. Hence f^{-1} is Lipschitz. This gives the result. □

Lemma 2.20. *Suppose that $u \in C^\infty(\bar{\Omega})$, where Ω is star-shaped Lipschitz with constant M and scale r and $u = 0$ on $S \subseteq \partial\Omega$, where S has positive measure, then*

$$\left(\int_\Omega |u|^{\frac{np}{n-p}}\,dy\right)^{\frac{1}{p}-\frac{1}{n}} \le c\left(\int_\Omega |\nabla u|^p\,dy\right)^{\frac{1}{p}}$$

where c depends on $\sigma(S)$, if and only if $u \circ f$ where f is from Lemma 2.19, satisfies the same inequality on $B_r(0)$.

This result follows from a change of variables. Note that the previous two lemmas enable us to get the desired inequality on our star-shaped domains by proving that we have the result on a ball.
Given a set S of positive measure, we using the following notation to denote the average value of a function over S

$$\bar{u}_S = \fint_S u\,dy.$$

Now, we introduce the following well-known result.

Lemma 2.21. *Let $u \in C^\infty(\bar{\Omega})$, where Ω is star-shaped Lipschitz with constant M and scalar r, then for $1 < p < n$, we have*

$$\|u - \bar{u}_S\|_{L^q(\Omega)} \le \frac{cd^n}{n\,m(S)}\|\nabla u\|_{L^p(\Omega)}$$

where $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, $d = \operatorname{diam}(\Omega)$, and $S = B_{\epsilon r}(x)$.

Proof. (of Lemma 2.21) Let $x \in \Omega$ and $y \in S$. By the Fundamental Theorem of Calculus, we have

$$u(x)m(S) - \int_S u(y)\,dy = \int_S \int_0^{|y-x|} \nabla u\left(x + \frac{y-x}{|y-x|}t\right) \cdot \frac{y-x}{|y-x|}\,dt\,dy.$$

Now, letting $\xi = |y-x|$ and $\hat{\omega} = \frac{y-x}{|y-x|}$, we can switch to polar coordinates by using that $dy = \xi^{n-1}\,d\xi\,d\hat{\omega}$. Dividing by $m(S)$ gives

$$u(x) - \fint_S u(y)\,dy = \frac{1}{m(S)}\int_0^r \int_{\partial B_1(0)} \int_0^\xi \nabla u(x + t\hat{\omega}) \cdot \hat{\omega}\chi_\Omega(x + \hat{\omega}t)\,dt\,\xi^{n-1}\,d\hat{\omega}\,d\xi.$$

Taking the absolute value gives

$$\begin{aligned} |u(x) - \bar{u}_S| &\le \frac{1}{m(S)}\int_0^r \int_{\partial B_1(0)} \int_0^\xi |\nabla u(x + t\hat{\omega}) \cdot \hat{\omega}\chi_\Omega(x + \hat{\omega}t)|\,dt\,\xi^{n-1}\,d\hat{\omega}\,d\xi \\ &\le \frac{1}{m(S)}\int_0^d \int_{\partial B_1(0)} \int_0^\xi |v(x + \hat{\omega}t)|\,dt\,\xi^{n-1}\,d\hat{\omega}\,d\xi \end{aligned}$$

where we use that $|\hat{\omega}| = 1$, set $v(z) = \chi_\Omega(z)\nabla u(z)$, and use that $d = \operatorname{diam}(\Omega)$, $r < d$. Switching the order of integration gives,

$$
\begin{aligned}
|u(x) - \bar{u}_S| &\leq \frac{1}{m(S)} \int_{\partial B_1(0)} \int_0^d |v(x + t\hat{\omega})| \int_t^d \xi^{n-1}\, d\xi\, dt\, d\hat{\omega} \\
&\leq \frac{1}{m(S)} \int_{\partial B_1(0)} \int_0^d |v(x + t\hat{\omega})| \int_0^d \xi^{n-1}\, d\xi\, dt\, d\hat{\omega} \\
&= \frac{d^n}{n\, m(S)} \int_{\partial B_1(0)} \int_0^d |v(x + t\hat{\omega})|\, dt\, d\hat{\omega} \\
&\leq \frac{d^n}{n\, m(S)} \int_{\partial B_1(0)} \int_0^\infty |v(x + t\hat{\omega})|\, dt\, d\hat{\omega} \\
&= \frac{d^n}{n\, m(S)} \int_{\partial B_1(0)} \int_0^\infty \frac{|v(x + t\hat{\omega})|}{t^{n-1}} t^{n-1}\, dt\, d\hat{\omega}.
\end{aligned}
$$

Letting $y = x + t\hat{\omega}$ and $dy = t^{n-1}\, dt\, d\hat{\omega}$, we have

$$
|u(x) - \bar{u}_S| \leq \frac{d^n}{n\, m(S)} \int_{\mathbb{R}^n_+} \frac{|v(y)|}{|x - y|^{n-1}}\, dy
$$

Define $I_1|v|(x) = \int_\Omega \frac{v(y)}{|x-y|^{n-1}}\, dy$ and letting $|v| = |\chi_\Omega \nabla u|$, we have

$$
|u(x) - \bar{u}_S| \leq \frac{d^n}{n\, m(S)} (I_1|v|)(x).
$$

Use the Hardy Littlewood Sobolev Inequality from [15] to show

$$
\|I_1 g\|_{L^q(\mathbb{R}^n_+)} \leq c \|\chi_\Omega \nabla u\|_{L^p(\mathbb{R}^n_+)}.
$$

Thus,

$$
\begin{aligned}
\|u(x) - \bar{u}_S\|_{L^q(\Omega)} &\leq \frac{cd^n}{n\, m(S)} \|I_1 g\|_{L^p(\mathbb{R}^n_+)} \\
&= \frac{cd^n}{n\, m(S)} \|\chi_\Omega \nabla u\|_{L^p(\mathbb{R}^n_+)} \\
&= \frac{cd^n}{n\, m(S)} \|\nabla u\|_{L^p(\Omega)}
\end{aligned}
$$

□

Lemma 2.22. *Let Ω be a star-shaped Lipschitz domains with scalar M, constant r, and centered at x. Given $u \in C^\infty(\Omega)$, then if $T \subseteq \Omega$ is a set of positive measure, we have that*

$$
\|u - \bar{u}_T\|_{L^q(\Omega)} \leq \frac{cd^n}{n\, m(B_r(x))} \left(1 + \frac{|\Omega|^{1/q}}{|T|^{1/q}}\right) \|\nabla u\|_{L^p(\Omega)}
$$

where $1 < p < n$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$.

Proof. (of Lemma 2.22) Recall S from Lemma 2.21. By the triangle inequality,

$$\begin{aligned}
\|u-\bar{u}_T\|_{L^q(\Omega)} &\le \|u-\bar{u}_S\|_{L^q(\Omega)} + \|\bar{u}_S-\bar{u}_T\|_{L^q(\Omega)} \\
&\le \frac{cd^n}{n\,m(S)}\|\nabla u\|_{L^p(\Omega)} + |\Omega|^{1/q}|\bar{u}_S-\bar{u}_T| \\
&= \frac{cd^n}{n\,m(S)}\|\nabla u\|_{L^p(\Omega)} + |\Omega|^{1/q}\left|\fint_T (u-\bar{u}_S)\,dy\right| \\
&\le \frac{cd^n}{n\,m(S)}\|\nabla u\|_{L^p(\Omega)} + \frac{|\Omega|^{1/q}}{|T|}\int_T |u-\bar{u}_S|\,dy
\end{aligned}$$

where we used Lemma 2.21. On the first term, using Hölder's Inequality on the second term (with $u(y)-\bar{u}_S$ and 1) to gives

$$\begin{aligned}
\|u-\bar{u}_T\|_{L^q(\Omega)} &\le \frac{cd^n}{n\,m(S)}\|\nabla u\|_{L^p(\Omega)} + \frac{|\Omega|^{1/q}}{|T|}\left(\int_T |u-\bar{u}_S|^q\,dy\right)^{1/q}|T|^{1-1/q} \\
&= \frac{cd^n}{n\,m(S)}\|\nabla u\|_{L^p(\Omega)} + \frac{|\Omega|^{1/q}}{|T|^{1/q}}\|u-\bar{u}_S\|_{L^q(\Omega)}.
\end{aligned}$$

Using Lemma 2.21 on the second term gives

$$\|u-\bar{u}_T\|_{L^q(\Omega)} \le \frac{cd^n}{n\,m(S)}\left(1+\frac{|\Omega|^{1/q}}{|T|^{1/q}}\right)\|\nabla u\|_{L^p(\Omega)}.$$

□

Lemma 2.23. *If $u \in C^\infty(\overline{B_r(0)})$ and $u = 0$ on $T \subseteq \partial B_r(0)$, where T has positive measure, then*

$$\left(\int_{B_r(0)} |u|^{\frac{np}{n-p}}\,dy\right)^{\frac{n-p}{np}} \le c_1\left(\int_{B_r(0)} |\nabla u|^p\,dy\right)^{\frac{1}{p}}$$

where we assume that $\sigma(T) \ge c_2 r^{n-1}$.

Proof. (of Lemma 2.23) First define

$$\hat{T} = \left\{y \in B_r(0) : \frac{y}{|y|}r \in T\right\}$$

and for $y \in \hat{T}$, let

$$\hat{y} = \frac{y}{|y|}.$$

We first want to get a bound on $\bar{u}_{\hat{T}}$. By the Fundamental Theorem of Calculus we have that

$$u(r\hat{y}) - u(s\hat{y}) = \int_s^r \frac{\partial}{\partial t}(u(t\hat{y}))\,dt.$$

Since $r\hat{y} \in T$ we have that $u(r\hat{y}) = 0$, and so

$$u(s\hat{y}) = -\int_s^r \hat{y} \cdot \nabla u(t\hat{y})\,dt.$$

Let $\frac{1}{r}T = \{\frac{1}{r}y : y \in T\}$, then we can integrate both sides over all $y = s\hat{y}$ to get

$$\int_{\frac{1}{r}T} \int_0^r u(s\hat{y}) s^{n-1}\,ds\,d\sigma(\hat{y}) = -\int_{\frac{1}{r}T} \int_0^r \int_s^r \hat{y} \cdot \nabla u(t\hat{y})\,dt\,s^{n-1}\,ds\,d\sigma(\hat{y}).$$

Switching out of polar coordinates for the left-hand side and switching the order of integration for the right-hand side gives

$$\begin{aligned}
\int_{\hat{T}} u(y)\,dy &= -\int_{\frac{1}{r}T} \int_0^r \hat{y} \cdot \nabla u(t\hat{y})\,dt \int_0^t s^{n-1}\,ds\,d\sigma(\hat{y}) \\
&= -\int_{\frac{1}{r}T} \int_0^r \hat{y} \cdot \nabla u(t\hat{y}) \left(\frac{1}{n}t^n\right) dt\,d\sigma(\hat{y}).
\end{aligned}$$

Switching out of polar coordinates for the right-hand side gives

$$\int_{\hat{T}} u(y)\,dy = -\frac{1}{n} \int_{\hat{T}} y \cdot \nabla u(y)\,dy.$$

Dividing by $m(\hat{T})$ and taking the absolute values gives

$$\begin{aligned}
|u_{\hat{T}}| &= \frac{1}{n\,m(\hat{T})} \left| \int_{\hat{T}} y \cdot \nabla u(y)\,dy \right| \\
&\leq \frac{1}{n\,m(\hat{T})} \int_{\hat{T}} |y||\nabla u(y)|\,dy \\
&\leq \frac{r}{n\,m(\hat{T})} \int_{\hat{T}} |\nabla u|\,dy.
\end{aligned}$$

A straight forward geometric argument gives that $m(\hat{T}) = \frac{r}{n}\sigma(T)$. Hence we have that

$$|u_{\hat{T}}| \leq \frac{1}{\sigma(T)} \int_{\hat{T}} |\nabla u|\,dy. \tag{2.21}$$

Now let $q = \frac{np}{n-p}$ and we have

$$\|u\|_{L^q(B_r(0))} \leq \|u - u_{\hat{T}}\|_{L^q(B_r(0))} + \|u_{\hat{T}}\|_{L^q(B_r(0))}.$$

For the first term, we apply Lemma 2.21 with $\Omega = S = B_r(0)$ and $\hat{T}$ to get

$$
\begin{aligned}
\|u\|_{L^q(B_r(0))} &\le \frac{c(2r)^n}{cr^n}\left(1+\left(\frac{cr^n}{m(\hat{T})}\right)^{1/q}\right)\|\nabla u\|_{L^p(B_r(0))} + \|u_{\hat{T}}\|_{L^q(B_r(0))} \\
&= c\left(1+\left(\frac{cr^n}{m(\hat{T})}\right)^{1/q}\right)\|\nabla u\|_{L^p(B_r(0))} + |u_{\hat{T}}|(m(B_r(0)))^{1/q}.
\end{aligned}
$$

Next, from the relationship between $m(\hat{T})$ and $\sigma(T)$, as well as the fact that $\sigma(T) \ge cr^{n-1}$, we have that

$$
\frac{r^n}{m(\hat{T})} \le c.
$$

Now using (2.21), we have

$$
\|u\|_{L^q(B_r(0))} \le c\|\nabla u\|_{L^p(B_r(0))} + \frac{(cr^n)^{1/q}}{\sigma(T)}\|\nabla u\|_{L^1(B_r(0))}.
$$

Using Hölder's Inequality on the second term gives

$$
\begin{aligned}
\|u\|_{L^q(B_r(0))} &\le c\|\nabla u\|_{L^p(B_r(0))} \\
&\quad + \frac{(cr^n)^{1/q}}{\sigma(T)}(m(B_r(0)))^{1-\frac{1}{p}}\|\nabla u\|_{L^p(B_r(0))} \\
&= \left(c + \frac{(cr^n)^{1+\frac{1}{q}-\frac{1}{p}}}{\sigma(T)}\right)\|\nabla u\|_{L^p(B_r(0))}.
\end{aligned}
$$

Note $1+\frac{1}{q}-\frac{1}{p} = \frac{n-1}{n}$. Hence we have

$$
\begin{aligned}
\|u\|_{L^q(B_r(0))} &\le \left(c + \frac{cr^{n-1}}{\sigma(T)}\right)\|\nabla u\|_{L^p(B_r(0))} \\
&\le c\|\nabla u\|_{L^p(B_r(0))}
\end{aligned}
$$

where we once again use that $\sigma(T) \ge c_2 r^{n-1}$. □

Now, we have the tools necessary to prove the following Poincaré inequality.

Lemma 2.24. *Let* Ω, D, *and* N *be a standard domain,* $x \in \partial\Omega$, *and* $0 < r < r_0$, *then for* $u \in W^{1,2}_D(\Omega)$

$$
\left(\int_{\Upsilon_{r/2}(x)} |u - \bar{u}_{x,r}|^2 dy\right)^{\frac{1}{2}} \le c\left(\int_{\Upsilon_r(x)} |\nabla u|^2 dy\right)^{\frac{1}{2}}
$$

where

$$\bar{u}_{x,r} = \begin{cases} \fint_{\Upsilon_r(x)} u(y)dy & B_r(x) \cap D = \varnothing \\ 0 & B_r(x) \cap D \neq \varnothing \end{cases}$$

Proof. (of Lemma 2.24) Since we can approximate functions in $W^{1,2}$ by C^∞ functions, we may assume that $u \in C^\infty$. We will prove this result in two cases.
Case 1: Suppose that $B_r(x) \cap D \neq \varnothing$ and assume that $D = \partial\Omega \cap \{z_1 > \Psi(z'')\}$, where Ψ is the Lipschitz function with constant M that gives the Lipschitz dissection of the boundary. This gives that

$$\{(z_1, z'', \varphi(z_1, z'')) : z_1 > M|z''|\} \cap B_{2r}(y) \subset D.$$

It is clear that this set has measure comparable to cr^{n-1}. Hence

$$m(B_{2r}(x) \cap D) \geq cr^{n-1}.$$

Now we can use Lemma 2.19, Lemma 2.20, and Lemma 2.23, to get

$$\left(\int_{\Upsilon_{2r}(x)} |u|^{\frac{np}{n-p}} dy \right)^{\frac{1}{p}-\frac{1}{n}} = c \left(\int_{\Upsilon_{2r}(x)} |\nabla u|^p dy \right)^{\frac{1}{p}}$$

Setting $\frac{np}{n-p} = 2$ gives $p = \frac{2n}{n+1}$ and so we have

$$\left(\int_{\Upsilon_{2r}(x)} |u|^2 dy \right)^{\frac{1}{2}} \leq c \left(\int_{\Upsilon_{2r}(x)} |\nabla u|^{\frac{n+2}{2n}} dy \right)^{\frac{n+2}{2n}}$$

For use in Hölder's Inequality, let the first function equal $|\nabla u|^{\frac{2n}{n+2}}$ and the second equal to 1. For the powers, let the first by $\frac{n+2}{n}$ and the second $\frac{n+2}{2}$ and then Holder's Inequality gives

$$\left(\int_{\Upsilon_{2r}(x)} |u|^2 dy \right)^{\frac{1}{2}} \leq c \left(\int_{\Upsilon_{2r}(x)} |\nabla u|^2 dy \right)^{1/2}$$

as desired.
Case 2: Suppose that $B_r(x) \cap D = \varnothing$. Note that Lemma 2.22 gives

$$\left(\int_{\Upsilon_r(x)} |u - \bar{u}_{x,r}|^2 dy \right)^{1/2} \leq \frac{2r^n}{ncr^n} \left(\int_{\Upsilon_r(x)} |\nabla u|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}}$$

using Hölder's inequality as before gives the result. □

Now we are ready to prove our Reverse Hölder Inequality.

Proof. (of Theorem 2.1) First, if we assume that $r = 1$, we have by Lemma 2.16 and Lemma 2.17 along with Lemma 2.18, for $2 < p < \frac{2n}{n-1}$, and our functions u and v, that

$$\|\nabla v\|_{L^p(\mathbb{R}^n_+)} \leq c \left(\|\nabla u\|_{L^2(\Upsilon_1(x))} + \|u\|_{L^2(\Upsilon_1(x))} + |\kappa| \right).$$

We want to be able to apply Lemma 2.24 to estimate the L^2-norm of u. If we are in the case where the constant is not zero, we apply Lemma 2.18 with $u - c$ to get the necessary term. In either case, we get

$$\|\nabla v\|_{L^p(\mathbb{R}^n_+)} \le c\left(\|\nabla u\|_{L^2(\Upsilon_1(x))} + |\kappa|\right).$$

Changing variables back to u on the left-hand side gives

$$\|\nabla u\|_{L^p(\Upsilon_{1/2}(x))} \le c\left(\|\nabla u\|_{L^2(\Upsilon_1(x))} + |\kappa|\right).$$

Rescaling gives the desired result. □

Chapter 3 Main Estimate

The goal of this chapter is to utilize the Reverse Hölder Inequality proved in the previous chapter to get a related result on the boundary, which will allow us to obtain our L^p estimates in Chapter 4. This idea, of using a Reverse Hölder Inequality to get better L^p results, is due to Caffarelli and Peral [3]. Shen, as well as Ott and Brown used this idea for the L^p Mixed Problem on Lipschitz domains [11, 14]. We will work through these arguments.

3.1 Reverse Hölder Inequality on the Boundary

The following lemma is our goal for this section.

Lemma 3.1. *Let Ω, D, and N be a standard domain for the mixed problem. Let u be a weak solution of the mixed problem with Neumann data f_N in $L^\infty(N)$ and zero Dirichlet data. Fix q satisfying $1 < q < \frac{n}{n-1}$. For $x \in \partial\Omega$ and $0 < r < r_0$ we have*

$$\left(⨍_{\zeta_r(x)} (\nabla u^*_{cr})^q \, d\sigma\right)^{\frac{1}{q}} \leq C\left[⨍_{\Upsilon_{2r}(x)} |\nabla u| \, dy + \|f_N\|_{L^\infty(N\cap B_{2r}(x))}\right].$$

The constant $c = \frac{1}{16}$ and C depends on M, n, and q.

In order to prove this lemma we must first work through some preliminary results. First, we define

$$\delta(x) = \text{dist}(x, \Lambda) \tag{3.1}$$

where $\Lambda = \partial D$ is taken in the relative topology in $\partial\Omega$.

Lemma 3.2. *(4.9 from Ott and Brown) Let Ω, D and N be a standard domain. Let u be a weak solution to*

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } D \\ \frac{\partial u}{\partial \nu} = f_N & \text{on } N \end{cases}$$

for $f_N \in L^2(N)$. Given $\varepsilon > 0$, $x \in \partial\Omega$, and $0 < r < r_0$, such that for some $A > 0$, we have that $\delta(x) \leq Ar$, then

$$\int_{\zeta_r(x)} (\nabla u^*_{c\delta})^2 \delta^{1-\varepsilon} \, d\sigma \leq c\left(\int_{\zeta_{2r}(x)} |f_N|^2 \delta^{1-\varepsilon} \, d\sigma + \int_{\Upsilon_{2r}(x)} |\nabla u|^2 \delta^{-\varepsilon} \, dy\right)$$

where $c = c(M, n, \varepsilon, A)$.

Define the **tangential gradient** of u on $\partial\Omega$ by

$$\nabla_{\tan} u = \nabla u - \frac{\partial u}{\partial \nu} \cdot \nu.$$

Next we introduce two lemmas needed to prove Lemma 3.2

Lemma 3.3. *(4.4 From Ott and Brown) Let Ω be a Lipschitz domain. Let $x \in \partial\Omega$ and $0 < r < r_0$. Let u be a harmonic function on $\Upsilon_{4r}(x)$ with $\nabla u \in L^2(\Upsilon_{4r}(x))$ and $\frac{\partial u}{\partial \nu} \in L^2(\zeta_{4r}(x))$, then*

$$\nabla u \in L^2(\zeta_r(x))$$

and moreover

$$\int_{\zeta_r(x)} (\nabla u_r^*)^2 \, d\sigma \leq c \bigg(\int_{\zeta_{4r}(x)} |\frac{\partial u}{\partial \nu}|^2 \, d\sigma + \frac{1}{r} \int_{\Upsilon_{4r}(x)} |\nabla u|^2 \, dy \bigg)$$

where c depends on M and n.

Lemma 3.4. *(4.8 from Ott and Brown) Let Ω be a Lipschitz domain. Let $x \in \partial\Omega$ and $0 < r < r_0$. Let u be a harmonic function on $\Upsilon_{4r}(x)$. If $\nabla u \in L^2(\Upsilon_{4r}(x))$ and $\nabla_{\tan} u \in L^2(\zeta_{4r}(x))$, then $\nabla u \in L^2(\zeta_r(x))$ and*

$$\int_{\zeta_r(x)} (\nabla u_r^*)^2 \, d\sigma(y) \leq c \bigg(\int_{\zeta_{4r}(x)} |\nabla_{\tan} u|^2 \, d\sigma(y) + \frac{1}{r} \int_{\Upsilon_{4r}(x)} |\nabla u|^2 \, dy \bigg).$$

where the constant c depends only on M.

The proof of Lemma 3.3 and 3.4 can be found in the work of Ott and Brown and depends on the work of Jerison and Kenig as well as Rellich [11, 12].

Lemma 3.5. *If $u = 0$ on $B_{100r_0}(x) \cap \partial\Omega$, we have that $\nabla_{\tan} u = 0$ on $B_{100r_0}(x) \cap \partial\Omega$.*

Proof. (of Lemma 3.5) First we define for $i = 1, \ldots, n-1$,

$$T_i(x', \varphi(x')) = e_i + \frac{\partial \varphi}{\partial x_i} e_n$$

on $B_{100r_0}(x) \cap \partial\Omega$. First, we will show that $T_i \cdot \nu = 0$. We know that

$$\nu = \frac{1}{\sqrt{1 + |\nabla \varphi|^2}} \begin{bmatrix} \nabla \varphi \\ -1 \end{bmatrix}.$$

Now,

$$
\begin{aligned}
T_i(x',\varphi(x'))\cdot\nu &= (e_i + \frac{\partial\varphi}{\partial x_i}(x')e_n)\cdot\frac{1}{\sqrt{1+|\nabla\varphi|^2}}\begin{bmatrix}\nabla\varphi\\ -1\end{bmatrix}\\
&= \begin{bmatrix}0\\ \cdots\\ 1\\ \cdots\\ 0\\ \frac{\partial\varphi}{\partial x_i}(x')\end{bmatrix}\cdot\frac{1}{\sqrt{1+|\nabla\varphi|^2}}\begin{bmatrix}\frac{\partial\varphi}{\partial x_1}(x')\\ \cdots\\ \frac{\partial\varphi}{\partial x_i}(x')\\ \cdots\\ \frac{\partial\varphi}{\partial x_{n-1}}(x')\\ -1\end{bmatrix}\\
&= \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\frac{\partial\varphi}{\partial x_i}(x') - \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\frac{\partial\varphi}{\partial x_i}(x')\\
&= 0
\end{aligned}
$$

as desired. We also easily have that the vectors $\{T_i(P)\}_{i=1}^{n-1}$ are linearly independent and therefore form a basis of

$$\{y : y\cdot\nu(P) = 0\}$$

for $P\in B_{100r_0}(x)\cap\partial\Omega$. Next note that

$$
\begin{aligned}
T_i(x',\varphi(x'))\cdot(\nabla u(x',\varphi(x'))) &= \begin{bmatrix}0\\ \cdots\\ 1\\ \cdots\\ 0\\ \frac{\partial\varphi}{\partial x_i}(x')\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial u}{\partial x_1}(x',\varphi(x'))\\ \cdots\\ \frac{\partial u}{\partial x_i}(x',\varphi(x'))\\ \cdots\\ \frac{\partial u}{\partial x_n}(x',\varphi(x'))\end{bmatrix}\\
&= \frac{\partial u}{\partial x_i}(x',\varphi(x')) + \frac{\partial\varphi}{\partial x_i}(x')\frac{\partial u}{\partial x_n}(x',\varphi(x')).
\end{aligned}
$$

Using the chain rule gives

$$T_i(x',\varphi(x'))\cdot(\nabla u(x',\varphi(x'))) = \frac{\partial}{\partial x_i}(u(x',\varphi(x'))).$$

Since $f(x',\varphi(x'))$ is always zero,

$$T_i(x',\varphi(x'))\cdot(\nabla f(x',\varphi(x'))) = 0.$$

Next, $\nabla_{\tan}u = 0$, if and only if $\nabla u - (\nabla u\cdot\nu)\nu = 0$. Note

$$
\begin{aligned}
T_i(x',\varphi(x'))\cdot\nabla_{\tan}u(x',\varphi(x')) &= T_i(x',\varphi(x'))\cdot\nabla u(x',\varphi(x'))\\
&\quad -(\nabla u(x',\varphi(x'))\cdot\nu)T_i(x',\varphi(x'))\cdot\nu\\
&= 0.
\end{aligned}
$$

Since $\nabla_{\tan}u\cdot\nu = 0$ as well and $\{T_1,\cdots,T_{n-1},\nu\}$ forms a basis of $\mathbb{R}^n$, we must have that $\nabla_{\tan}u = 0$ on the boundary. This finishes the proof of the lemma. □

Now, we want to recall the idea of a Whitney Decomposition given in Stein [15]. Let O be an open set. Define $\mathcal{D}$ to be the set of dyadic cubes, Q, satisfying the following condition,

$$\frac{\text{dist}(Q, O^c)}{\ell(Q)} \geq C_1$$

where $\ell(Q)$ is the side length of Q. Note that we can cover O by cubes in $\mathcal{D}$. The larger the cube, the further away it must be from the boundary. Now let $\mathcal{W} \subseteq \mathcal{D}$ be the set of cubes that are maximal with respect to inclusion. We know that as long as $O^c \neq \emptyset$, we can't have an infinite increasing chain of cubes in $\mathcal{D}$ of the form $Q_1 \subseteq Q_2 \subseteq Q_3 \ldots$

If $Q_j, Q_k \in \mathcal{W}$, then $Q_j^\circ \cap Q_k^\circ = \emptyset$. Let $\tilde{Q}_j$ be the parent of Q_j. We know that $\tilde{Q}_j \notin \mathcal{D}$, otherwise Q_j is not maximal. Therefore, we have that

$$\frac{\text{dist}(\tilde{Q}_j, O^c)}{\ell(\tilde{Q}_j)} < C_1.$$

Now note $\ell(\tilde{Q}_j) = 2\ell(Q_j)$, thus we have that

$$\frac{\text{dist}(\tilde{Q}_j, O^c)}{\ell(Q_j)} < 2C_1.$$

Now, we have

$$\begin{aligned} \frac{\text{dist}(Q_j, O^c)}{\ell(Q_j)} &\leq \frac{\text{dist}(\tilde{Q}_j, O^c) + \sqrt{n}\,\ell(Q_j)}{\ell(Q_j)} \\ &\leq 2C_1 + \sqrt{n}. \end{aligned}$$

using basic geometry. We say these $Q_j \in \mathcal{W}$ give a **Whitney Decomposition** of O. Note that we can choose C_1 as large as we like.

Proof. (of Lemma 3.2) Note that we will be using cubes instead of balls for our local neighborhoods. Since, every cube can be contained in slightly larger ball with their side length and diameter ratio not depending on the original cube (and vice versa for fitting balls into cubes), a simple covering argument concludes we can use either. Recall from earlier, that

$$\delta(y) = \text{dist}(y, \Lambda).$$

Now since $0 < r < r_0$, we know that $\Upsilon_{2r}(x)$ is contained in some $B_{2r_0}(z)$. Now we first consider two simpler cases, where $\zeta_{100r}(z) \subseteq N$ or $\zeta_{100r}(z) \subseteq D$. In either of these cases for $y \in \Upsilon_{2r}(x)$,

$$\delta(y) \geq 50r.$$

We also have by assumption that $\delta(x) \leq Ar$ for some $A > 0$. Hence

$$\begin{aligned}\delta(y) &\le |x-y|+\delta(x)\\ &= 2r+Ar\\ &= (2+A)r.\end{aligned}$$

Hence,

$$c_1 r \le \delta(y) \le c_2 r.$$

In other words, $\delta(y)$ is equivalent to r. Therefore we have that

$$\int_{\zeta_r(x)} (\nabla u^*_{c\delta})^2 \delta(y)^{1-\varepsilon}\, d\sigma(y) \le c \int_{\zeta_r(x)} (\nabla u^*_r)^2 r^{1-\varepsilon}\, d\sigma(y).$$

Step 1: Suppose $\zeta_{100r}(z) \subset N$. In this step, we can use Lemma 3.3 since we know $\frac{\partial u}{\partial \nu} \in L^2(\zeta_r(x))$ and $\nabla u \in L^2$ giving

$$\begin{aligned}\int_{\zeta_r(x)} (\nabla u^*_{c\delta})^2 \delta(y)^{1-\varepsilon}\, d\sigma(y) &\le c\Big(\int_{\zeta_{4r}(x)} |\frac{\partial u}{\partial \nu}|^2 r^{1-\varepsilon}\, d\sigma(y)\\ &\qquad + \frac{1}{r}\int_{\Upsilon_{4r}(x)} |\nabla u|^2 r^{1-\varepsilon}\, dy\Big)\\ &\le c\Big(\int_{\zeta_{4r}(x)} |f_N|^2 \delta(y)^{1-\varepsilon}\, d\sigma(y)\\ &\qquad + \int_{\Upsilon_{4r}(x)} |\nabla u|^2 \delta(y)^{-\varepsilon}\, dy\Big)\end{aligned}$$

where we use that $\frac{\partial u}{\partial \nu} = f_N$ on N and again that $\delta(y)$ is equivalent to r. Now we can scale the domain and use a covering argument to get the result.
Step 2: Suppose $\zeta_{100r}(z) \subseteq D$. In this step we know that $u = 0$ on $\zeta_{100r}(z) \subseteq D$. Again, we know that

$$\int_{\zeta_r(x)} (\nabla u^*_{c\delta})^2 \delta(y)^{1-\varepsilon}\, d\sigma(y) \le c \int_{\zeta_r(x)} (\nabla u^*_r)^2 r^{1-\varepsilon}\, d\sigma(y).$$

Now we use Lemma 3.4 and the fact that $\nabla_{\tan} u = 0$ from Lemma 3.5, to give

$$\begin{aligned}\int_{\zeta_r(x)} (\nabla u^*_{c\delta})^2 \delta(y)^{1-\varepsilon}\, d\sigma(y) &\le \frac{c}{r}\int_{\Upsilon_{4r}(x)} r^{1-\varepsilon}|\nabla u|^2 dy\\ &= c\int_{\Upsilon_{4r}(x)} r^{-\varepsilon}|\nabla u|^2\, dy\\ &\le c\int_{\Upsilon_{4r}(x)} |\nabla u|^2 \delta(y)^{-\varepsilon}\, dy\end{aligned}$$

where we use that $\delta(y)$ is equivalent to r. Again this will give the result.
Step 3: Now we consider the third and final case, where $\zeta_{100r_0}(z)$ intersects both D and N. Recall that Λ is the set where D and N meet and that we have Ψ which gives the Lipschitz dissection of D and N. Consider

$$F = \mathbb{R}^{n-1} \backslash \{(\Psi(x''), x'') : x'' \in \mathbb{R}^{n-2}\}.$$

Let $\{G_i\}$ give a Whitney Decomposition of F . Now we will map these onto the boundary by defining

$$Q_i = \{(x', \varphi(x')) : x' \in G_i\}.$$

Since the cubes, G_i, are connected and our mapping is continuous, each Q_i is connected too. This means $\delta(y)$ never vanishes on Q_i and $Q_i \subseteq N$ or $Q_i \subseteq D$. Since the G_i's give a Whitney Decomposition we know,

$$c_1 \text{diam}(G_i) \le \text{dist}(G_i, F^c) \le c_2 \text{diam}(G_i)$$

where we can choose c_1 later. Recall that c_2 dependeds on c_1. Next note, for $\alpha \in \Lambda$ and $y \in Q_i$,

$$\begin{aligned} |\alpha - y| &= |\alpha' - y'| + |\varphi(\alpha') - \varphi(y')| \\ &\le |\alpha' - y'| + M|\alpha' - y'| \\ &= (1 + M)|\alpha' - y'|. \end{aligned}$$

Since this is true for all α,

$$\begin{aligned} \delta(y) &= \text{dist}(y, \Lambda) \\ &\le (1 + M)\text{dist}(y', F^c) \\ &\le (1 + M)(\text{diamG}_\text{i} + \text{dist}(G_i, F^c)) \\ &\le (1 + M)(1 + c_2)\text{diam}(G_i) \\ &\le (1 + M)(1 + c_2)\text{diam}(Q_i). \end{aligned}$$

Hence,

$$\text{diam}(Q_i) \ge C_2 \delta(y)$$

where $C_2 = \frac{1}{(1+M)(1+c_2)}$. Next,

$$\begin{aligned} c_1 \text{diam}(Q_i) &\le c_1(1 + M)\text{diam}(G_i) \\ &\le (1 + M)\text{dist}(G_i, F^c) \\ &\le (1 + M)\text{dist}(Q_i, \Lambda) \\ &\le (1 + M)\text{dist}(y, \Lambda) \\ &\le (1 + M)\delta(y) \end{aligned}$$

for $y \in Q_i$. Hence

$$\text{diam}(Q_i) \le C_1 \delta(y)$$

where

$$C_1 = \frac{1+M}{c_1}.$$

Now choose c_1, so that C_1 is small enough to guarantee that if $Q_i \cap B_r(x) \neq \varnothing$, then $T(2Q_i) \subseteq \Upsilon_{2r}(x)$. Now define

$$T(Q) = \{y \in \Omega : \operatorname{dist}(y,Q) < \operatorname{diam}(Q)\}.$$

Since we know that for $y \in Q_i$, $\delta(y)$ is equivalent to $\operatorname{diam}(Q_i)$ we may use the same reasoning as in either Step 1 or Step 2 to obtain

$$\begin{aligned}\int_{Q_i} (\nabla u^*_{c\delta})^2 \delta(y)^{1-\varepsilon}\, d\sigma(y) \quad \leq \quad & c\Big(\int_{2Q_i \cap N} |f_N|^2 \delta(y)^{1-\varepsilon}\, d\sigma(y) \\ & + \int_{T(2Q_i)} |\nabla u|^2 \delta(y)^{-\varepsilon}\, dy\Big).\end{aligned}$$

Now sum over all Q_i that intersect $B_r(x)$ to get

$$\int_{\zeta_r(x)} |\nabla u|^2 \delta^{1-\varepsilon}\, d\sigma \leq c\Big(\int_{B_{2r}(x)\cap N} |f_N|^2 \delta^{1-\varepsilon}\, d\sigma + \int_{\Upsilon_{2r}(x)} |\nabla u|^2 \delta^{-\varepsilon}\, dy\Big)$$

where we use that $\{T(2Q_i)\}$ have bounded overlap. □

Lemma 3.6. *Given Ω, D, and N, a standard domain, then*

$$\int_{\zeta_{8r}(x)} \delta(y)^{\ell}\, d\sigma \leq c r^{\ell+n-1} \tag{3.2}$$

for $-1 < \ell < 0$. Moreover, if $-2 < \ell < 0$, we can show that

$$\int_{\Upsilon_{8r}(x)} \delta(y)^{\ell}\, dy \leq c r^{\ell+n} \tag{3.3}$$

Proof. (of Lemma 3.6)
First to prove (3.2), define for $k \in \mathbf{Z}$

$$S_k = \{y \in \zeta_{4r}(x) : r2^{-(k+1)} \leq \delta(y) < r2^{-k}\}.$$

Easily we have

$$\bigcup_{k\in\mathbf{Z}} S_k = \{y \in \zeta_{4r}(x) : \delta(y) > 0\}.$$

Now,

$$\begin{aligned}\int_{\zeta_{4r}(x)} \delta(y)^{\ell}\, d\sigma \quad &= \quad \int_{\{y\in\zeta_{4r}(x):\delta(y)=0\}} \delta(y)^{\ell}\, d\sigma + \sum_{k\in\mathbf{Z}} \int_{S_k} \delta(y)^{\ell}\, d\sigma \\ &= \quad \sum_{k\in\mathbf{Z}} \int_{S_k} \delta(y)^{\ell}\, d\sigma\end{aligned}$$

Now we need to compute $\int_{S_k} \delta(y)^\ell \, d\sigma$ where $-1 < \ell < 0$. On S_k

$$r2^{-(k+1)} \le \delta(y) < r2^{-k}.$$

Thus

$$(r2^{-k})^\ell < \delta(y)^\ell \le (r2^{-(k+1)})^\ell.$$

Hence

$$\begin{aligned}\int_{S_k} \delta(y)^\ell \, d\sigma &\le \int_{S_k} r^\ell 2^{-\ell(k+1)} \, d\sigma \\ &= r^\ell 2^{-\ell(k+1)} \sigma(S_k).\end{aligned}$$

Now we need to get a bound on $\sigma(S_k)$. First note

$$\zeta_{4r}(x) = \{(y', \varphi(y')) : y' \in \mathbb{R}^{n-1}\} \cap B_{4r}(x)$$

and

$$\Lambda \cap B_{4r}(x) = \{(\Psi(y''), y'', \varphi(\Psi(y''), y'')) : y'' \in \mathbb{R}^{n-2}\} \cap B_{4r}(x).$$

Easily,

$$\begin{aligned}\delta(y) &\le |(y_1, y'', \varphi(y_1, y'')) - (\Psi(y''), y'', \varphi(\Psi(y''), y''))| \\ &= \left(|y_1 - \Psi(y'')|^2 + |\varphi(y_1, y'') - \varphi(\Psi(y''), y'')|^2\right)^{1/2}.\end{aligned}$$

Since φ is Lipschitz,

$$\begin{aligned}\delta(y) &\le \left(|y_1 + \Psi(y'')|^2 + M^2 |y_1 - \Psi(y'')|^2\right)^{1/2} \\ &= (1 + M^2)^{1/2} |y_1 - \Psi(y'')|.\end{aligned}$$

Define for fixed y

$$\Gamma_y = \{z \in \zeta_{r_0}(x) : |y'' - z''| < \frac{1}{M} |z_1 - \Psi(y'')|\}.$$

If $z \in \Gamma_y$, then

$$\begin{aligned}|y'' - z''| &< \frac{1}{M} |z_1 - \Psi(y'')| \\ &\le \frac{1}{M} |z_1 - \Psi(z'')| + \frac{1}{M} |\Psi(z'') - \Psi(y'')| \\ &\le \frac{1}{M} |z_1 - \Psi(z'')| + |z'' - y''|\end{aligned}$$

using the Ψ is Lipschitz. Hence

$$|z_1 - \Psi(z'')| > 0.$$

This means that $z \notin \Lambda$ and so $\Lambda \cap \Gamma_y = \varnothing$. Now, let

$$z \in E = B_{\frac{|y_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}}}(y) \cap (\zeta_{r_0}(x)).$$

We want to show that $z \in \Gamma_y$. First

$$\begin{aligned} |z-y| &< \frac{|y_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}} \\ &\leq \frac{|y_1 - z_1|}{\sqrt{2(1+M^2)}} + \frac{|z_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}} \end{aligned}$$

and so

$$|z-y| < \frac{|y-z|}{\sqrt{2(1+M^2)}} + \frac{|z_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}}.$$

Combining the $|z-y|$ terms gives that

$$|z-y|\left(1 - \frac{1}{\sqrt{2(1+M^2)}}\right) < \frac{|z_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}}.$$

Next note that $M^2 > 0$, which gives that $1 - \frac{1}{\sqrt{2(1+M^2)}} > 0$. Thus, dividing by this constant gives

$$\begin{aligned} |z-y| &\leq \frac{|z_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}\left(1 - \frac{1}{\sqrt{2(1+M^2)}}\right)} \\ &= \frac{|z_1 - \Psi(y'')|}{\sqrt{2(1+M^2)} - 1}. \end{aligned}$$

Now to bound the constant in the above term, note

$$\begin{aligned} (1+M)^2 &\leq 2(1+M^2) \\ (1+M) &\leq \sqrt{2(1+M^2)} \\ M &\leq \sqrt{2(1+M^2)} - 1 \end{aligned}$$

and so we get

$$\frac{1}{\sqrt{2(1+M^2)}-1} \le \frac{1}{M}.$$

From this we have that

$$|y'' - z''| \le |y - z| < \frac{|z_1 - \Psi(y'')|}{M}$$

which gives that $z \in \Gamma_y$. Since $\Gamma_y \cap \Lambda = \varnothing$, and $E \subset \Gamma_y$, we have that

$$\delta(y) \ge \frac{|y_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}}.$$

Combining the upper and lower bounds for $\delta(y)$, we have shown that

$$\frac{|y_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}} \le \delta(y) \le \sqrt{(1+M^2)}|y_1 - \Psi(y'')|.$$

Choose N large enough so that

$$\sqrt{2(1+M^2)} \le N$$

and so

$$\frac{1}{N} \le \frac{1}{\sqrt{2(1+M^2)}}.$$

Easily, we can now rewrite our upper and lower bounds for $\delta(y)$ as follows

$$\frac{|y_1 - \Psi(y'')|}{N} \le \delta(y) \le N|y_1 - \Psi(y'')|.$$

Now, we will define a set whose measure will be easier to determine. Define for $k \in \mathbf{Z}$

$$E_k = \{y \in \zeta_{4r}(x) : \frac{r2^{-(k+1)}}{N} \le |y_1 - \Psi(y'')| \le Nr^{-k}\}.$$

If $y \in S_k$,

$$2^{-(k+1)}r \le \delta(y) \le 2^{-k}r.$$

Therefore,

$$|y_1 - \Psi(y'')| \le N\delta(y) \le N2^{-k}r$$

and

$$|y_1 - \Psi(y'')| \ge \frac{\delta(y)}{N} \ge \frac{2^{-(k+1)}r}{N}.$$

Hence $y \in E_k$. Thus,

$$\sigma(S_k) \le \sigma(E_k).$$

Note,

$$E_k \subseteq \{y \in \partial\Omega : |x'' - y''| \leq 4r. \frac{r2^{-(k+1)}}{N} \leq |y_1 - \Psi(y'')| \leq r2^{-k}N\}.$$

Thus we have our bound on $\sigma(S_k)$ as follows,

$$\sigma(S_k) \leq \sigma(E_k) \leq cr^{n-2}\left(r2^{-k}N - \frac{r2^{-(k+1)}}{N}\right) = cr^{n-1}2^{-k}$$

where our constant, c, depends only on N and as a result depends only on the Lipschitz coefficient, M.

Case 1: Suppose $\operatorname{dist}(\zeta_{4r}(x), \Lambda) < r$. This means for $y \in \zeta_{4r}(x)$,

$$\begin{aligned} \delta(y) &\leq \operatorname{diam}(B_{4r}(x)) + r \\ &= 9r < 2^4 r. \end{aligned}$$

Hence $S_k = \varnothing$ for $k \leq -5$, so

$$\begin{aligned} \int_{\zeta_{4r}(x)} \delta(y)^\ell \, d\sigma &\leq \sum_{k=-5}^{\infty} 2^{-(k+1)\ell} r^\ell \sigma(S_k) \\ &\leq \sum_{k=-5}^{\infty} 2^{-(k+1)\ell} r^\ell c 2^{-k} r^{n-1} \\ &= cr^\ell r^{n-1} \sum_{k=-5}^{\infty} 2^{-k\ell} 2^{-k} \end{aligned}$$

Now,

$$\frac{2^{-(k+1)\ell}2^{-(k+1)}}{2^{-k\ell}2^{-k}} = 2^{-\ell}2^{-1},$$

since $-\ell < 1$ we have

$$2^{-\ell}2^{-1} < 1$$

and the ratio test tells us that the series converges. Hence

$$\int_{\zeta_{4r}(x)} \delta(y)^\ell \, d\sigma \leq cr^{\ell+n-1}$$

as desired.

Case 2: Suppose $\operatorname{dist}(\zeta_{4r}(x), \Lambda) \geq r$. In this case for $y \in \zeta_{4r}(x)$, $\delta(y) \geq r$. Also, since

$$\delta(y) \leq \delta(x) + |x - y| \leq cr,$$

we are only summing over finitely many terms and we have the same result.

Now to prove (3.3), define for $k \in \mathbb{Z}$,

$$S_k = \{y \in \Upsilon_{8r}(x) : r2^{-(k+1)} \le \delta(y) \le r2^{-k}\}.$$

As before we have that

$$\begin{aligned}\int_{\Upsilon_{8r}(x)} \delta(y)^\ell \, dy &= \sum_{k\in\mathbb{Z}} \int_{S_k} \delta(y)^\ell \, dy \\ &\le \sum_{k\in\mathbb{Z}} c2^{-(k+1)\ell} m(S_k).\end{aligned}$$

To compute $m(S_k)$, we notice that

$$\Lambda \cap B_{8r}(x) = \{(\Psi(y''), y'', \varphi(\Psi(y''), y'')) : y'' \in \mathbb{R}^{n-2}\} \cap B_{8r}(x).$$

Therefore, we have that

$$\begin{aligned}\delta(y) &\le |(y_1, y'', y_n) - (\Psi(y''), y'', \varphi(\Psi(y''), y''))| \\ &= (|y_1 - \Psi(y'')|^2 + |y_n - \varphi(y_1, y'') + \varphi(\Psi(y''), y'') - \varphi(\Psi(y''), y'')|^2)^{\frac{1}{2}} \\ &\le (|y_1 - \Psi(y'')|^2 + 2|y_n - \varphi(y_1, y'')|^2 + 2|\varphi(y_1, y'') - \varphi(\Psi(y''), y'')|^2)^{\frac{1}{2}} \\ &\le (|y_1 - \Psi(y'')|^2 + 2\delta(y)^2 + 2M^2|y_1 - \Psi(y'')|^2)^{\frac{1}{2}} \\ &\le \sqrt{1 + 2M^2}|y_1 - \Psi(y'')| + \sqrt{2}\delta(y)\end{aligned}$$

On S_k,

$$\delta(y) \le \sqrt{1 + 2M^2}|y_1 - \Psi(y'')| + \sqrt{2}r2^{-k}.$$

As we did for the previous result, we define

$$\Gamma_y = \{z \in \Omega : |y'' - z''| < \frac{1}{M}|z_1 - \Psi(y'')|\},$$

but notice that this set is a subset of Ω now, not $\partial\Omega$. Now let

$$z \in B_{\frac{|y_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}}}(y).$$

Since the boundary of Ω is Lipschitz with constant M it is straight forward that $z \in \Omega$. Following the same idea as the first proved result gives

$$|z'' - y''| < \frac{1}{M}|z_1 - \Psi(y'')|$$

and $z \in \Gamma_y$. This gives that

$$\delta(y) \ge \frac{|y_1 - \Psi(y'')|}{\sqrt{2(1 + M^2)}}.$$

Altogether we now have that

$$\frac{|y_1 - \Psi(y'')|}{\sqrt{2(1+M^2)}} \le \delta(y) \le \sqrt{2(1+M^2)}|y_1 - \Psi(y'')| + \sqrt{2}r2^{-k}.$$

Again choosing $N \ge \sqrt{2(1+M^2)}$ gives

$$\frac{|y_1 - \Psi(y'')|}{N} \le \delta(y) \le N|y_1 - \Psi(y'')| + \sqrt{2}r2^{-k}.$$

Next, if $y \in S_k$, then

$$\begin{aligned} |y_1 - \Psi(y'')| &\le N\delta(y) \\ &= Nr2^{-k}. \end{aligned}$$

Hence using the same ideas as before,

$$S_k \subseteq \{y \in \partial\Omega : |y_1 - \Psi(y'')| \le Nr2^{-k}, |x'' - y''| \le 8r, |y_n - \varphi(y_1, y'')| \le r2^{-k}\}.$$

Therefore, we can get an easy upper bound for $m(S_k)$, as follows,

$$\begin{aligned} m(S_k) &\le cr2^{-k}r^{n-2}r2^{-k} \\ &= cr^n 2^{-2k}. \end{aligned}$$

Case 1: If $\operatorname{dist}(B_{8r}(x), \Lambda) < r$, then

$$\begin{aligned} \delta(y) &\le \operatorname{diam}(B_{8r}(x)) + r \\ &= 16r + r \\ &\le 2^5 r. \end{aligned}$$

This means $S_k = \varnothing$ for $k \le -6$. Hence

$$\begin{aligned} \int_{\Upsilon_{8r}(x)} \delta(y)^\ell \, d\sigma &\le \sum_{k=-5}^{\infty} (cr2^{-(k+1)})^\ell (cr^n 2^{-2k}) \\ &= cr^{\ell+n} \sum_{k=-5}^{\infty} 2^{-k\ell-2k}. \end{aligned}$$

Next note

$$\frac{2^{-(k+1)\ell} - 2(k+1)}{2^{-k\ell-2k}} = 2^{-\ell}2^{-2}$$

since $\ell > -2$ gives $1 > 2^{-\ell-2}$. Hence our sum converges and we have

$$\int_{\Upsilon_{8r}(x)} \delta(y)^\ell \, d\sigma \le cr^{\ell+n}.$$

Case 2: As before, we have that only finitely many of the S_k are nonempty and we get the same result. □

Before we can prove Lemma 3.1, we must first a couple more results.

Lemma 3.7. *Let Ω, D, and N be a standard domain. Let u be a weak solution of the mixed problem with Neumann data f_N in $L^\infty(N)$ and zero Dirichlet data. Fix q satisfying $1 < q < \frac{n}{n-1}$. For $x \in \partial\Omega$ and $0 < r < r_0$, then we have*

$$\left(\fint_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma \right)^{\frac{1}{q}} \le C \left(\fint_{\Upsilon_{16r}(x)} |\nabla u| \, dy + \|f_N\|_{L^\infty(N \cap B(x,16r))} \right)$$

where C depends only on M, n, and q.

Proof. (of Lemma 3.7) We will work through this proof in two cases: *Case 1:* $\delta(x) \le 8r\sqrt{1+M^2}$ and *Case 2:* $\delta(x) > 8r\sqrt{1+M^2}$.

Case 1: Choose $2 < p < \frac{2n}{n-1}$ such that $q < \frac{p}{2}$. From this we have that

$$2 - \frac{2}{q} < 2 - \frac{4}{p}.$$

This means that we can choose ε such that

$$2 - \frac{2}{q} < \varepsilon < 2 - \frac{4}{p}. \tag{3.4}$$

Now since $1 < q$, we have that $0 < 2 - \frac{2}{q}$ and therefore, $\varepsilon > 0$. Easily, we see that

$$\left(\int_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma \right)^{1/q} = \left(\int_{\zeta_{4r}(x)} \left(|\nabla u|^q \delta(y)^{\frac{(1-\varepsilon)q}{2}} \right) \left(\delta(y)^{\frac{(\varepsilon-1)q}{2}} \right) d\sigma \right)^{1/q}.$$

Since $q < \frac{n}{n-1}$ gives that $\frac{2(n-1)}{n} < \frac{2}{q}$ and $n > 2$ gives $\frac{2(n-1)}{n} > 1$ we can apply Hölder's Inequality with $\frac{2}{q} > 1$ and $p_0 = \frac{2}{2-q}$ to give that

$$\begin{aligned} \left(\int_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma \right)^{1/q} \quad &\le \quad \left(\int_{\zeta_{4r}(x)} |\nabla u|^2 \delta(y)^{1-\varepsilon} \, d\sigma \right)^{1/2} \\ &\qquad \times \left(\int_{\zeta_{4r}(x)} \delta(y)^{\frac{(\varepsilon-1)q}{2-q}} \, d\sigma \right)^{\frac{2-q}{2q}}. \end{aligned}$$

To compute the second integral, note by (3.4), we have that

$$1 - \frac{2}{q} < \varepsilon - 1 < 1 - \frac{4}{p}.$$

Multiplying by q gives

$$q - 2 < q(\varepsilon - 1) < q - \frac{4q}{p}.$$

Since $2 - q > 0$, we can multiply by $\frac{1}{2-q}$ to give

$$\frac{q-2}{2-q} < \frac{q(\varepsilon-1)}{2-q} < \frac{1}{2-q}\left(q - \frac{4q}{p} \right)$$

and so with simplification, we have

$$-1 < \frac{q(\varepsilon-1)}{2-q} < \frac{q}{2-q}\left(1-\frac{4}{p}\right).$$

Now $\frac{q}{2-q} > 0$ and $p < \frac{2n}{n-1}$ gives $1-\frac{4}{p} < \frac{2}{n}-1$, so we have that

$$-1 < \frac{q(\varepsilon-1)}{2-q} < \frac{q}{2-q}(\frac{1}{n}-1) \leq 0.$$

Hence by (3.2),

$$\left(\int_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma\right)^{1/q} \leq c\left(\int_{\zeta_{4r}(x)} |\nabla u|^2 \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2} r^{\frac{\varepsilon-1}{2}+(n-1)\frac{2-q}{2q}}.$$

Now since $\lim_{\substack{z\in\Gamma(y)\\ z\to y}} \nabla u(z) = \nabla u(y)$,

$$\left(\int_{\zeta_{4r}(x)} |\nabla u|^2 \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2} \leq \left(\int_{\zeta_{4r}(x)} (\nabla u^*_{c\delta})^2 \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2}.$$

Since $\delta(x) \leq Ar$, we can use Lemma 3.2 to get

$$\begin{aligned}
\left(\int_{\zeta_{4r}(x)} |\nabla u|^2 \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2} &\leq c\left(\int_{\zeta_{8r}(x)} |f_N|^2 \delta(y)^{1-\varepsilon} \, d\sigma\right. \\
&\qquad \left. + \int_{\Upsilon_{8r}(x)} |\nabla u|^2 \delta(y)^{-\varepsilon} \, dy\right)^{1/2} \\
&= c\left(\int_{\zeta_{8r}(x)} |f_N|^2 \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2} \\
&\qquad + \left(\int_{\Upsilon_{8r}(x)} |\nabla u|^2 \delta(y)^{-\varepsilon} \, dy\right)^{1/2}.
\end{aligned}$$

We can pull out the L^∞ norm of f_N on the first term to get

$$\left(\int_{\zeta_{8r}(x)} |f_N|^2 \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2} \leq \|f_N\|_{L^\infty(\zeta_{8r}(x))}\left(\int_{\zeta_{8r}(x)} \delta(y)^{1-\varepsilon} \, d\sigma\right)^{1/2}.$$

Once again using (3.4), we can show that

$$\frac{2}{q} - 1 > 1 - \varepsilon > \frac{4}{p} - 1.$$

Now since $p < \frac{2n}{n-1}$, we have that

$$1 - \frac{2}{n} < \frac{4}{p} - 1.$$

Since $n \geq 2$, we have that $\frac{4}{p} - 1 > 0$. This means that $0 < 1-\varepsilon$. Since $\delta(y) \leq 8r\sqrt{1+M^2}$, we have

$$\begin{aligned}\delta(y) &\le \delta(x) + |x-y| \\ &\le 8r\sqrt{1+M} + 8r \\ &= cr.\end{aligned}$$

Since $1-\varepsilon > 0$, we now have

$$\delta(y)^{1-\varepsilon} \le cr^{1-\varepsilon}.$$

Hence, we can compute this integral as follows

$$\begin{aligned}\left(\int_{\zeta_{8r}(x)} \delta(y)^{1-\varepsilon}\, d\sigma\right)^{1/2} &= cr^{\frac{1-\varepsilon}{2}} \sigma(\zeta_{8r}(x))^{1/2} \\ &\le cr^{\frac{1-\varepsilon}{2}} r^{\frac{n-1}{2}} \\ &= cr^{\frac{n-\varepsilon}{2}}.\end{aligned}$$

Now for the second term, notice that since $p > 2$, we can apply Hölder's Inequality with $\frac{p}{2}$ and $\frac{p}{p-2}$ to get

$$\left(\int_{\Upsilon_{8r}(x)} |\nabla u|^2 \delta(y)^{-\varepsilon}\right)^{1/2} = \left(\int_{\Upsilon_{8r}(x)} |\nabla u|^p\, dy\right)^{\frac{1}{p}} \left(\int_{\Upsilon_{8r}(x)} \delta(y)^{\frac{-\varepsilon p}{p-2}}\, dy\right)^{\frac{p-2}{2p}}.$$

Since our function f_N is in L^∞, we can still use Theorem 2.1 to obtain

$$\left(⨍_{\Upsilon_{8r}(x)} |\nabla u|^p\, dy\right)^{\frac{1}{p}} \le c\left(⨍_{\Upsilon_{32r}(x)} |\nabla u|^2\, dy\right)^{\frac{1}{2}} + c\|f_N\|_{L^\infty(\Upsilon_{8r}(x))}.$$

Hence

$$\begin{aligned}\left(\int_{\Upsilon_{8r}(x)} |\nabla u|^2 \delta(y)^{-\varepsilon}\, dy\right)^{\frac{1}{2}} &\le c\left(r^{\frac{n}{p}-\frac{n}{2}}\left(\int_{\Upsilon_{32r}(x)} |\nabla u|^2\, dy\right)^{\frac{1}{2}} + r^{\frac{n}{p}} \|f_N\|_{L^\infty(B_{8r}(x))}\right) \\ &\quad \times \left(\int_{\Upsilon_{8r}(x)} \delta(y)^{\frac{-\varepsilon p}{p-2}}\right)^{\frac{p-2}{2p}}.\end{aligned}$$

Now, to compute the second integral note from (3.4) and the fact that $\frac{p}{p-2} > 0$, we have

$$\left(2-\frac{2}{q}\right)\frac{p}{p-2} < \frac{\varepsilon p}{p-2} < \frac{p}{p-2}\left(2-\frac{4}{p}\right).$$

Simplifying this expression gives that

$$\left(\frac{2}{q}-2\right)\frac{p}{p-2} > \frac{-\varepsilon p}{p-2} > \frac{p}{p-2}\left(\frac{4}{p}-2\right) = -2.$$

Therefore, we have that

$$0 > \frac{-\varepsilon p}{p-2} > -2.$$

Hence by (3.3), we have

$$\begin{aligned}\left(\int_{\Upsilon_{8r}(x)} \delta(y)^{\frac{-\varepsilon p}{p-2}}\right)^{\frac{p-2}{2p}} &\le \left(cr^{\frac{-\varepsilon p}{p-2}+n}\right)^{\frac{p-2}{2p}} \\ &= cr^{\frac{n}{2}-\frac{\varepsilon}{2}-\frac{n}{p}}.\end{aligned}$$

Now, putting these results together we have that

$$\begin{aligned}&\left(\int_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma\right)^{\frac{1}{q}} \\ &\le \Bigg(\|f_N\|_{L^\infty(\zeta_{8r}(x))} r^{\frac{n-\varepsilon}{2}} + c\Bigg(r^{\frac{n}{p}-\frac{n}{2}}\left(\int_{\Upsilon_{32r}(x)} |\nabla u|^2 \, dy\right)^{\frac{1}{2}} \\ &\qquad + r^{\frac{n}{p}}\|f_N\|_{L^\infty(\zeta_{8r}(x))}\Bigg)\left(cr^{\frac{-\varepsilon}{2}+\frac{n}{2}-\frac{n}{p}}\right)\Bigg) r^{\frac{\varepsilon-1}{2}+(n-1)\frac{2-q}{2q}} \\ &= c\|f_N\|_{L^\infty(\zeta_{8r}(x))} r^{\frac{n-\varepsilon}{2}} r^{\frac{\varepsilon-1}{2}+(n-1)\frac{2-q}{2q}} \\ &\qquad + cr^{\frac{n}{p}-\frac{n}{2}} r^{\frac{-\varepsilon}{2}+\frac{n}{2}-\frac{n}{p}} r^{\frac{\varepsilon-1}{2}+(n-1)\frac{2-q}{2q}}\left(\int_{\Upsilon_{32r}(x)} |\nabla u|^2 \, dy\right)^{\frac{1}{2}} \\ &\qquad + cr^{\frac{n}{p}} r^{\frac{-\varepsilon}{2}+\frac{n}{2}-\frac{n}{p}} r^{\frac{\varepsilon-1}{2}+(n-1)\frac{2-q}{2q}} \|f_N\|_{L^\infty(\zeta_{8r}(x))}\end{aligned}$$

Now to compute our powers of r we have the following simplifications

$$\frac{n}{2} - \frac{\varepsilon}{2} + \frac{\varepsilon}{2} - \frac{1}{2} + (n-1)\left(\frac{1}{q} - \frac{1}{2}\right) = \frac{n-1}{q},$$

$$\frac{n}{p} - \frac{n}{2} - \frac{\varepsilon}{2} + \frac{n}{2} - \frac{n}{p} + \frac{\varepsilon}{2} - \frac{1}{2} + (n-1)\left(\frac{1}{q} - \frac{1}{2}\right) = \frac{n-1}{q} - \frac{n}{2},$$

and

$$\frac{n}{p} - \frac{\varepsilon}{2} + \frac{n}{2} - \frac{n}{p} + \frac{\varepsilon}{2} - \frac{1}{2} + (n-1)\left(\frac{1}{q} - \frac{1}{2}\right) = \frac{n-1}{q}.$$

Therefore, we have

$$\begin{aligned}\left(\int_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma\right)^{\frac{1}{q}} &\le cr^{\frac{n-1}{q}} \|f_N\|_{L^\infty(\zeta_{8r}(x))} \\ &\qquad + cr^{\frac{n-1}{q}} r^{\frac{-n}{2}}\left(\int_{\Upsilon_{32r}(x)} |\nabla u|^2 \, dy\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, dividing by our powers of r gives the desired result

$$\left(\fint_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma\right)^{\frac{1}{q}} \le c\|f_N\|_{L^\infty(\zeta_{8r}(x))} + c\left(\fint_{\Upsilon_{32r}(x)} |\nabla u|^2 \, dy\right)^{\frac{1}{2}}.$$

Case 2: In this case, it can be show that $\zeta_{8r}(x) \subset D$ or $\zeta_{8r}(x) \subset N$ and the proof is simpler. For more details see Ott and Brown [11].
Lastly, using the techniques of [9, p. 80-82] or [8, p. 1004-5] we can obtain the same inequality with an L^1-norm on the right-hand side and obtain the result. □

We need one more result and we will be ready to prove the main result of this Chapter. The proof of the following can be found in Ott and Brown [11] and follows from Lemma 3.2 and a theorem of Coifman, McIntosh and Meyer [4].

Lemma 3.8. *For our function u and $1 < q < \frac{n}{n-1}$, we have that*

$$\left(\fint_{\zeta_r(x)} (\nabla u_r^*)^q \, d\sigma\right)^{1/q} \le c\left\{\fint_{\Upsilon_{4r}(x)} |\nabla u| \, dy + \left(\fint_{\zeta_{4r}(x)} |\nabla u|^q \, d\sigma\right)^{1/q}\right\}.$$

Proof. (of Lemma 3.1) First combining Lemma 3.8 and Lemma 3.7, we have

$$\left(\fint_{\zeta_r(x)} (\nabla u_r^*)^q \, d\sigma\right)^{1/q} \le c\fint_{\Upsilon_{16r}(x)} |\nabla u| \, dy + c\|f_N\|_{L^\infty}(N \cap B_{16r}(x). \tag{3.5}$$

Choose $\{x_k\}_{k=1}^n$ such that $B_r(x) \subset \bigcup_{k=1}^n B_{\frac{r}{16}}(x_k)$ such that $B_r(x_k) \subset B_{2r}(x)$. Now applying (3.5) on $B_{\frac{1}{16}}(x_k)$ and using a covering argument gives the result. □

Chapter 4 Final Details

The goal of this Chapter is to prove the conditions necessary for a theorem by Shen and then utilize his theorem to get our main result. Given Ω, the graph domain of a Lipschitz function, φ. We have that $\partial\Omega = \{(x', \varphi(x') : x' \in \mathbb{R}^{n-1})\}$. Define $\Psi : \partial\Omega \to \mathbb{R}^{n-1}$ by

$$\Psi(x', x_n) = x'.$$

We say that $Q \subset \partial\Omega$ is **surface cube** if $\Psi(Q)$ is a cube in $\mathbb{R}^{n-1}$. Note that we define a dilation of Q by

$$\alpha Q = \Psi^{-1}(\alpha\Phi(Q))$$

for $\alpha > 0$.
The following is the theorem we will need from [14, p. 224] of Shen, modified for the boundary.

Theorem 4.1 (Shen)**.** *Let Q_0 be a surface cube in $\partial\Omega$ and assume that q and s satisfy $1 < q < s$. Suppose that $F \in L^1(2Q_0)$ and $f \in L^q(2Q_0)$. Next, if for each dyadic surface subcube Q of Q_0 with $|Q| \le \beta|Q_0|$, there exist integrable functions F_Q and R_Q defined on $2Q$ such that*

$$|F| \le c\left(|F_Q| + |R_Q|\right), \tag{4.1}$$

$$\left(\fint_{2Q} |R_Q|^s \, d\sigma\right)^{\frac{1}{s}} \le c_1 \left(\fint_{2Q} |F| \, d\sigma + \sup_{Q_1 \supset Q} \fint_{Q_1} |f| \, d\sigma\right), \tag{4.2}$$

and

$$\fint_{2Q} |F_Q| \, d\sigma \le c_2 \sup_{Q_1 \supset Q} \fint_{Q_1} |f| \, d\sigma \tag{4.3}$$

where $c_1,\ c_2 > 0$ and $0 < \beta < 1 < \alpha$, then we have

$$\left(\fint_{Q_0} |F|^q \, d\sigma\right)^{\frac{1}{q}} \le c\left(\fint_{2Q_0} |F| \, d\sigma + \left(\fint_{2Q_0} |f|^q \, d\sigma\right)^{\frac{1}{q}}\right) \tag{4.4}$$

for some constant c depending on c_1, c_1, p, q, α, β, n, and $\|\nabla\varphi\|_{L^\infty}$.

4.1 Shen's Result

The goal of this section is to prove a local result by utilizing the theorem from Shen, which will enable us to prove our main theorem. Before stating this result, we need to introduce a related non-tangential maximal function. Let $\beta > 0$. Define

$$\tilde{\Gamma}_\beta(0) = \{(y', y_n) : |y'| < \beta y_n\}.$$

Now we can cover $\partial\Omega$ by a collection of boundary cubes $\{Q_i\}_{i=1}^n$. We can choose these boundary cubes small enough so that for each Q_i, we have a coordinate system and a Lipschitz function, φ so that $\partial\Omega$ is given by φ on $B_{100r_0}(0)$. Note that we have translated our coordinate system so that is is centered at $x = 0$. We can also rotate our coordinate system, so that $\nabla\varphi(0) = 0$. Moreover, we have that

$$100Q_i \subset B_{r_0}(0).$$

We define for $x \in 100Q_i$,

$$\tilde{\Gamma}_\beta(x) = \{x + y : y \in \tilde{\Gamma}_\beta(0)\} \bigcap \{z \in \mathbb{R}^n : z_n < c_1 r_0\}.$$

First we need to show that we can choose c_1 so that points in this set lie in $B_{100r_0}(0)$. Let $z \in \tilde{\Gamma}_\beta(x)$ for $x \in 100Q_i$. First note

$$\begin{aligned} |z| &\leq |z'| + |z_n| \\ &\leq |z' - x'| + |x'| + |z_n| \\ &< \beta(z_n - x_n) + |x'| + |z_n| \end{aligned}$$

since $z \in \tilde{\Gamma}_\beta(x)$. Now since $z_n < c_1 r_0$,

$$\begin{aligned} |z| &< \beta c_1 r_0 + (\beta + 1)|x| + |z_n| \\ &\leq \beta c_1 r_0 + (\beta + 1) r_0 + |z_n - x_n| + |x_n| \end{aligned}$$

where we used that $x \in 100Q_i \subset B(0, r_0)$ in the second step. Next, we can use that $z - x \in \tilde{\Gamma}_\beta(0)$ which implies that $z_n - x_n > 0$. This gives that

$$\begin{aligned} |z| &\leq \beta c_1 r_0 + (\beta + 1) r_0 + z_n - x_n + |x_n| \\ &\leq \beta c_1 r_0 + (\beta + 1) r_0 + c_1 r_0 + 2r_0 \\ &= (\beta c_1 + \beta + c_1 + 3) r_0. \end{aligned}$$

Now we want $|z| < 100 r_0$, so we can choose $\beta > 0$ and c_1 such that

$$\beta c_1 + \beta + c_1 + 3 < 100.$$

This means that

$$\beta < \frac{97 - c_1}{c_1 + 1}$$

is needed to ensure that we can talk about $\varphi(x')$. Now we want to show that $z \in \Omega$. First, we know that $z - x \in \tilde{\Gamma}_\beta(0)$ and therefore

$$|z' - x'| < \beta(z_n - x_n).$$

Since $x_n = \varphi(x')$, we have that

$$z_n > \varphi(x') + \frac{1}{\beta}|z' - x'|. \tag{4.5}$$

We will prove our result by contradiction by assuming that $z \notin \Omega$. This means that

$$\begin{aligned} z_n &\leq \varphi(z') \\ &= \varphi(z') - \varphi(x') + \varphi(x') \\ &\leq |\nabla\varphi(x' + \theta(x' - z'))||x' - z'| + \varphi(x') \end{aligned}$$

for some $\theta \in (0,1)$, since φ is in C^1. Now, since $\nabla\varphi(0) = 0$,

$$\begin{aligned} z_n &\leq |\nabla\varphi(x' + \theta(x' - z')) - \nabla\varphi(0)||x' - z'| + \varphi(x') \\ &\leq M|x' + \theta(x' - z')||x' - z'| + \varphi(x') \end{aligned}$$

where we use that $\nabla\varphi$ is Lipschitz with constant M. It is not difficult to see that $x' + \theta(x' - z') \in B_{100r_0}(0)$, so

$$z_n \leq 100Mr_0|x' - z'| + \varphi(x'). \tag{4.6}$$

Comparing (4.5) and (4.6) shows that choosing $\beta < \frac{1}{100Mr_0}$ guarantees that we have a contradiction and so $z \in \Omega$. Thus, we require that

$$0 < \beta < \min\left(\frac{97 - c_1}{c_1 + 1}, \frac{1}{100Mr_0}\right). \tag{4.7}$$

Now we define our related non-tangential maximal function as follows

$$\nabla u^\dagger(x) = \sup_{y \in \tilde{\Gamma}_\beta(x)} |\nabla u(y)|$$

and

$$F_i(x) = \begin{cases} \sup\limits_{y \in \tilde{\Gamma}_\beta(x)} |\nabla u(y)| & x \in 2Q_i \\ 0 & \text{else.} \end{cases}$$

Theorem 4.2. *Let Ω, D, and N be a standard domain. Let u be a solution to the weak mixed problem (1.3) with $g = 0$ and a the identity matrix. For a surface cube, Q, small enough, $1 < t < 1 + \epsilon$, and $1 < q < \frac{n}{t(n-1)}$ we have*

$$\left(\fint_Q |\left(\nabla u^\dagger\right)^{tq}\, d\sigma\right)^{\frac{1}{q}} \leq c\left\{\fint_{2Q} \left(\nabla u^\dagger\right)^t\, d\sigma + \left(\fint_{2Q} |f_N|^{tq}\, d\sigma\right)^{\frac{1}{q}}\right\}. \tag{4.8}$$

The following Lemma gives us a way to relate what will be our new non-tangential maximal function and the standard one.

Lemma 4.3. *For $1 < p < \infty$, we have that*

$$\|\nabla u^*\|_{L^p(\partial\Omega)} \le c \sum_{i=1}^{N} \|F_i\|_{L^p(\partial\Omega)}$$

where c depends on $\partial\Omega$, α, and β.

Before we can prove this result we need to introduce a few preliminary results.

Lemma 4.4. *Let*

$$\widehat{\partial\Omega} = \{y \in \Omega : \mathrm{dist}(y, \partial\Omega) = \frac{r_0}{2}\}$$

then

$$\sup_{x \in \widehat{\partial\Omega}} |\nabla u(x)| \le c \sum_{i=1}^{N} \|F_i\|_{L^p(\partial\Omega)}.$$

Proof. (of Lemma 4.4) Let $y \in \Gamma_\alpha(x)$. For $x \in Q_i$. We assume that $\mathrm{dist}(y, \partial\Omega) \le \frac{r_0}{2}$. We assume that r_0 is small enough such that

$$y = (y', \varphi(y') + t)$$

where $(y', \varphi(y')) \in 2Q_i$. Define $\mathrm{dist}(y, \partial\Omega)$.
First we claim the following

$$1 \le \frac{|y_n - \varphi(y')}{d} \le \sqrt{1 + M^2}. \tag{4.9}$$

We easily have that $d \le |y_n - \varphi(y')|$. The other inequality follows from a geometric argument. Now, if β is small enough, we have that

$$B_{c|y_n - \varphi(y;)|}(\hat{y}) \cap \partial\Omega \subset \{z : y \in \tilde{\Gamma}_\beta(z)\}$$

where we let $\hat{y} = (y', \varphi(y'))$.
By (4.9) d is comparable to $|y_n - \varphi(y')|$, so

$$B_{cd}(\hat{y}) \cap \partial\Omega \subset \{z : y \in \tilde{\Gamma}_\beta(z)\}.$$

Next note that

$$\begin{aligned} |\hat{y} - x| &\le |y' - x'| + |\varphi(y') + |\varphi(x')| \\ &\le (1 + M)|y' - x'| \\ &\le (1 + \alpha)(1 + M)d \end{aligned}$$

since $y \in \Gamma_\alpha(x)$. Thus,

$$B_{c_1 d}(\hat{y}) \subset B_{c_2 d}(x).$$

From this we have that

$$\sigma(B_{cd}(x) \cap \{z : y \in \tilde{\Gamma}_\beta(z)\}) \geq c d^{n-1}.$$

It follows that

$$|\nabla u(y)| \leq c \fint_{B_{cd}(x)\cap\partial\Omega} \nabla u^\dagger \, d\sigma \tag{4.10}$$

Now,

$$\sup_{x\in\widehat{\partial\Omega}} |\nabla u(x)| \leq c \sup_{x\in\widehat{\partial\Omega}} \fint_{B_{cd}(x)} \nabla u^\dagger \, d\sigma.$$

Now since $d = frac r_0 2$ and we have that

$$\sup_{x\in\widehat{\partial\Omega}} |\nabla u(x)| \leq c \sum_{i=1}^{N} \fint_{2Q_i} \nabla u^\dagger \, d\sigma.$$

The result follows from this. □

Lemma 4.5. *If* $y \in \Gamma_\alpha(x) \cap \{\delta(y) < \frac{r_0}{2}\}$ *and* $q > 0$*, then*

$$|\nabla u(y)| \leq c \sum_{i=1}^{N} M(F_i^q)^{1/q}(x)$$

where M *is the Hardy Littlewood Maximal function.*

Proof. (of Lemma 4.5) First, we can use (4.10) from the proof of Lemma 4.4. The result follows directly from this. □

Proof. (of Lemma 4.3) First, by the Maximum Principle

$$\sup_{\delta(y)\geq r_0/2} |\nabla u(y)| = \sup_{\delta(y)=r_0/2} |\nabla u(y)|.$$

Now using Lemma 4.4, we have that

$$\sup_{\delta(y)\geq r_0/2} |\nabla u(y)| \leq c \sum_{i=1}^{n} \|F_i\|_{L^p(\partial\Omega)}.$$

Combining this result with Lemma 4.5, we have

$$\sup_{y\in\Gamma_\alpha(x)} |\nabla u(y)| \leq c \sum_{i=1}^{n} \|F_i\|_{L^p(\partial\Omega)} + c \sum_{i=1}^{N} M(F_i^q)^{1/q}(x).$$

Note that the left-hand side of this equation is $\nabla u^*(x)$. Taking the $L^p(\partial\Omega)$ norm of both sides and using the Hardy Littlewood Maximal Inequality gives the result. □

Lemma 4.6. *If $y \in \tilde{\Gamma}_\beta(x)$, $|x-y| > r_1$, and $x \in Q_i$, then*

$$|\nabla u(y)| \le \frac{c}{r_1^{n-1}} \left(\int_{B_r(x)\cap 2Q_i} F_i^q \, d\sigma \right)^{1/q}.$$

Proof. (of Lemma 4.6) Start by defining for $y \in \tilde{\Gamma}_\beta(x)$,

$$E_y = \{z \in 2Q_i : y \in \tilde{\Gamma}_\beta(z)\}.$$

Basic geometry gives that

$$\sigma(E_y \cap B(x,r)) \ge r_1^{n-1}$$

since $|x-y| > r$. Hence, we have

$$\begin{aligned} |\nabla u(y)| &\le \inf_{z\in E_y \cap B(x,r)} F_i(z) \\ &\le \left(⨍_{E_y\cap B(x,r)} F_i^q \, d\sigma \right)^{1/q} \\ &\le \left(⨍_{2Q_i \cap B(x,r)} F_i^q \, d\sigma \right)^{1/q}. \end{aligned}$$

Since $\sigma(B(x,r)\cap 2Q_i)$ is comparable to $\sigma(B(x,r)\cap E_y)$. □

Lemma 4.7. *Given our value for β from (4.7), then there exists $\alpha > 0$ such that*

$$\tilde{\Gamma}_\beta(x) \subset \Gamma_\alpha(x).$$

Therefore,

$$\nabla u^\dagger(x) \le \nabla u^*(x).$$

Proof. (of Lemma 4.7) We have on $B_{100r_0}(x)$ that the boundary of $\partial\Omega$ is given by a function φ. We know locally that the graph of $\partial\Omega$ is contained in

$$\{y : |y_n - \varphi(x')| \le M|x'-y'|\}.$$

We also know for $y \in \tilde{\Gamma}_\beta(x)$ that

$$|y'-x'| < \beta |y_n - \varphi(x')|.$$

From this we see that we can choose $\alpha > 0$ that give the result from this. □

Lemma 4.8. *If $\alpha_1 < \alpha_2$, then for any $q > 0$*

$$\nabla u^*_{\alpha_2} \le cM(\nabla u^{*q}_{\alpha_1})^{1/q}(x).$$

Proof. (of Lemma 4.8) If $y \in \Gamma_{\alpha_2}(x)$, let $\hat{y}$ be a point on $\partial\Omega$, so that

$$|y - \hat{y}| = \operatorname{dist}(y, \partial\Omega) = d.$$

By the triangle inequality

$$\begin{aligned} |x - \hat{y}| &\leq |x - y| + |y - \hat{y}| \\ &\leq (1 + \alpha_2)d + d \\ &= (2 + \alpha_2)d. \end{aligned}$$

If $z \in B(\hat{y}, \alpha_1 d)$, then

$$\begin{aligned} |y - z| &\leq |y - \hat{y}| + |\hat{y} - z| \\ &< d + \alpha_1 d \\ &= (1 + \alpha_1)d. \end{aligned}$$

Thus for such z, $y \in \Gamma_{\alpha_1}(z)$ and so we have

$$\begin{aligned} |\nabla u(y)| &\leq \inf_{z \in B(\hat{y}, \alpha_1 d)} \nabla u^*_{\alpha_1}(z) \\ &\leq \left(\fint_{B(\hat{y}, \alpha_1 d) \cap \partial\Omega} (\nabla u^*_{\alpha_1}(z))^q \, d\sigma \right)^{1/q} \\ &\leq \left(\fint_{B(x, (\alpha_1 + \alpha_2 + 2)d) \cap \partial\Omega} (\nabla u^*_{\alpha_1}(z))^q \, d\sigma \right)^{1/q} \end{aligned}$$

The result follows from this. □

Now, we need the following theorem from the work of Ott and Brown.

Theorem 4.9. *(Theorem 7.7 from Ott and Brown) Let Ω, D, and N be a standard domain, then there exists an $\varepsilon > 0$, such that for $1 < t < 1 + \varepsilon$ and $f_N \in L^p(N)$ given, there exists a solution u to (1.4), and the solution satisfies the estimate*

$$\|\nabla u^*\|_{L^t(\partial\Omega)} \leq c \|f_N\|_{L^t(N)}.$$

The constant c depends on the Lipschitz constant for the domain and the index t.

Finally, we are ready to prove that the conditions from Theorem 4.1 are satisfied. We start by fixing a surface cube on $\partial\Omega$, called Q. Recall that we are interested in solutions to (1.4). Define

$$g = \chi_{4Q} \left(f_N - \fint_{4Q} f_N \, d\sigma \right)$$

and

$$h = f_N - g.$$

Now by Theorem 4.9 we have for $1 < t < 1 + \varepsilon$, that there exists $C^2(\Omega)$ functions, v and w that solve

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \\ v = 0 & \text{on } D \\ \frac{\partial v}{\partial \nu} = g & \text{on } N \\ \nabla v^* \in L^t(\partial\Omega) & \end{cases} \tag{4.11}$$

and

$$\begin{cases} -\Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } D \\ \frac{\partial u}{\partial \nu} = h & \text{on } N \\ \nabla w^* \in L^t(\partial\Omega) & \end{cases}, \tag{4.12}$$

respectively. Now define for use in Lemma 4.1,

$$\begin{aligned} f &= |f_N|^t \\ F &= \left(\nabla u^\dagger\right)^t \\ F_Q &= \left(\nabla v^\dagger\right)^t \\ R_Q &= \left(\nabla w^\dagger\right)^t. \end{aligned}$$

Let $1 < q < s < \frac{n}{t(n-1)}$ for t from Theorem 4.9. We start by proving (4.1) by noting that

$$|F(y)| = \left(\sup_{z \in \tilde{\Gamma}_\beta(y)} |\nabla u| \right)^t = \sup_{z \in \tilde{\Gamma}_\beta(y)} |\nabla u|^t$$

since $t > 1$. Now noticing that $u = v + w$, we have that

$$|F(y)| \le \sup_{z \in \tilde{\Gamma}_\beta(y)} |\nabla v + \nabla w|^t.$$

We can show easily that $(a+b)^t \le c(a^t + b^t)$ for a and b positive. Hence,

$$\begin{aligned} |F(y)| &\le \sup_{z \in \tilde{\Gamma}_\beta(y)} (|\nabla v| + |\nabla w|)^t \\ &\le c \sup_{z \in \tilde{\Gamma}_\beta(y)} (|\nabla v|^t + |\nabla w|^t). \end{aligned}$$

Again, since $t > 1$,

$$
\begin{aligned}
|F(y)| &\leq c\left((\sup_{z \in \tilde{\Gamma}_\beta(y)} |\nabla v|)^t + (\sup_{z \in \tilde{\Gamma}_\beta(y)} |\nabla w|)^t \right) \\
&= c(\nabla v^\dagger)^t(y) + c(\nabla w^\dagger)^t(y) \\
&= c(F_Q(y) + R_Q(y))
\end{aligned}
$$

This gives us (4.1). Next to show (4.3), note

$$
\begin{aligned}
⨍_{2Q} |F_Q| \, d\sigma &= \frac{1}{m(2Q)} \int_{2Q} (\nabla v^\dagger)^t \, d\sigma \\
&\leq \frac{1}{m(2Q)} \int_{2Q} (\nabla v^*)^t \, d\sigma
\end{aligned}
$$

where we recall from Lemma 4.7, we can choose α to guarantee that $\nabla v^\dagger \leq \nabla v^*$. Thus, by Lemma 3.3,

$$
\begin{aligned}
⨍_{2Q} |F_Q| \, d\sigma &\leq \frac{c}{m(2Q)} \int_N g^t \, d\sigma \\
&= \frac{c}{m(2Q)} \int_{4Q} |f_N|^t \, d\sigma
\end{aligned}
$$

where we use the definition of g. The desired result, (4.3), follows from this. Now we move on to show (4.2), the most difficult of the three conditions. Fix $r_1 = c\,\ell(Q)$, where we allow for c to be chosen later and $\ell(Q)$ is the side length of Q. We define two related non-tangential maximal functions by

$$
u_r^\dagger(x) = \sup_{y \in \tilde{\Gamma}_\beta(x) \cap B_r(x)} |u(y)| \tag{4.13}
$$

and

$$
u_{r^+}^\dagger(x) = \sup_{y \in \tilde{\Gamma}_\beta(x) \setminus B_r(x)} |u(y)|. \tag{4.14}
$$

Now,

$$
\begin{aligned}
R_Q &= (\nabla w^\dagger)^t \\
&\leq (\nabla w_{r_1}^\dagger + \nabla w_{r_1^+}^\dagger)^t \\
&\leq c(\nabla w_{r_1}^\dagger)^t + (\nabla w_{r_1^+}^\dagger)^t
\end{aligned}
$$

using the same ideas as before. By Minkowski's Inequality

$$
\left(⨍_{2Q} |R_Q|^s \, d\sigma \right)^{\frac{1}{s}} \leq c \left(\left(⨍_{2Q} \left(\nabla w_{r_1}^\dagger \right)^{st} \, d\sigma \right)^{\frac{1}{s}} + \left(⨍_{2Q} \left(\nabla w_{r_1^+}^\dagger \, d\sigma \right) \right)^{\frac{1}{s}} \right). \tag{4.15}
$$

Now, to bound these two terms, we introduce the following lemmas.

Lemma 4.10. *Given our function w from (4.12) and $1 < p < \frac{n}{n-1}$, we have*

$$\left(\fint_{2Q} \left(\nabla w^{\dagger}_{r_0}\right)^p \, d\sigma \right)^{\frac{1}{p}} \le c \fint_{4Q} \nabla w^{\dagger} \, d\sigma + c\|h\|_{L^\infty(4Q)}. \tag{4.16}$$

Proof. (of Lemma 4.10) Let $Q_0 \subset \partial\Omega$ be a surface cube with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. We can assume that $10Q \subset Q_0$. Define

$$\hat{Q}_0 = \{x = y + te_n : x \in \Omega, y \in Q_0, t > 0, \text{ and } x_n < r_0\}.$$

Let $\hat{E} \subset \hat{Q}_0$. Define,

$$E = \{y \in Q_0 : y + te_n \in \hat{E} \text{ for some } t > 0\}.$$

Next, let $z \in \hat{E}$, then we know there exists $y \in E$ such that $z = y + te_n$. It is clear that $z \in \tilde{\Gamma}_\beta(y)$. Thus

$$|v(z)| \le v^\dagger(y).$$

Integrating over all $z \in \hat{E}$ gives,

$$\int_{\hat{E}} |v(z)| \, dz \quad \le \quad \int_E v^\dagger(y) \sup_{y+te_n \in \hat{E}} (t) \, d\sigma(y).$$

Letting $\hat{E} = \Upsilon_r(x)$. It is clear that $E = \zeta_r(x)$. Hence

$$\int_{\Upsilon_r(x)} |v(y)| \, dy \le 2r \int_{\zeta_r(x)} |v^\dagger(y)| \, d\sigma(y).$$

Dividing each side by a multiple of r^n gives

$$\fint_{\Upsilon_r(x)} |\nabla w| \, dy \le c \fint_{\zeta_r(x)} \nabla w^\dagger_r \, d\sigma \tag{4.17}$$

We also know that

$$\left(\fint_{\zeta_r(x)} (\nabla w^\dagger_{cr})^{tq} \, d\sigma \right)^{\frac{1}{tq}} \le \left(\fint_{\zeta_r(x)} (\nabla w^*_{cr})^{tq} \, d\sigma \right)^{\frac{1}{tq}}.$$

If we let $c = \frac{1}{16}$, then by Lemma 3.1, we have

$$\left(\fint_{\zeta_r(x)} (\nabla w^\dagger_{cr})^{tq} \, d\sigma \right)^{\frac{1}{tq}} \le c \left(\fint_{\Upsilon_{2r}(x)} |\nabla w| \, dy + \|h\|_{L^\infty(\zeta_{2r}(x))} \right).$$

Now assuming $0 < 2r$ is small enough, we have from the above equation and (4.17)

$$\left(\fint_{\zeta_r(x)} (\nabla w^\dagger_{cr})^{tq} \, d\sigma \right)^{\frac{1}{tq}} \le c \left(\fint_{\zeta_{2r}(x)} \nabla w^\dagger_r \, d\sigma + \|h\|_{L^\infty(\zeta_{2r}(x))} \right). \tag{4.18}$$

Now, let $r = \ell(Q)$ and $r_1 = \frac{\ell(Q)}{16}$, we can cover $2Q$ by a finite number of balls of radius r with centers called y_k in $2Q$. In this case, $\zeta_{2r}(y_k) \subseteq 4Q$. Now, using the triangle inequality

$$\begin{aligned}\left(\fint_{2Q} (\nabla w^{\dagger}_{r_1})^{tq}\, d\sigma\right)^{\frac{1}{tq}} &\leq \frac{c}{(r^{n-1})^{\frac{1}{tq}}} \sum_{k-1}^{N} \left(\int_{\zeta_r(y_k)} (\nabla w^{\dagger}_{r_1})^{tq}\, d\sigma\right)^{\frac{1}{tq}} \\ &= c\sum_{k-1}^{N} \left(\fint_{\zeta_r(y_k)} (\nabla w^{\dagger}_{cr})^{tq}\, d\sigma\right)^{\frac{1}{tq}}.\end{aligned}$$

Hence by (4.18),

$$\begin{aligned}\left(\fint_{2Q} (\nabla w^{\dagger}_{r_1})^{tq}\, d\sigma\right)^{\frac{1}{tq}} &\leq c\sum_{k=1}^{N} \left(\fint_{\zeta_{2r}(y_k)} \nabla w^{\dagger}_{r}\, d\sigma + \|h\|_{L^{\infty}(\zeta_{2r}(y_k))}\right) \\ &\leq c\left(\fint_{4Q} \nabla w^{\dagger}_{r}\, d\sigma + \|h\|_{L^{\infty}(4Q)}\right) \\ &\leq c\left(\fint_{4Q} \nabla w^{\dagger}\, d\sigma + \|h\|_{L^{\infty}(4Q)}\right)\end{aligned}$$

as desired. □

Now, we turn our attention to the far-away result.

Lemma 4.11. *For our function w from (4.12) and the s and t values from above, we have*

$$\left(\fint_{2Q} \left(\nabla w^{\dagger}_{r_1^+}\right)^{st} d\sigma\right)^{\frac{1}{st}} \leq c \fint_{4Q} \nabla w^{\dagger}\, d\sigma. \tag{4.19}$$

Proof. (of Lemma 4.11) Fix $y \in 2Q$. Let $z \in \tilde{\Gamma}_{\beta}(y)$ and $|z-y| \geq r_1$. Define

$$E_z = \{\xi \in 4Q : z \in \tilde{\Gamma}_{\beta}(\xi)\}.$$

Using basic geometry, it is clear that

$$\sigma(E_z \cap B(y, r_1)) \geq c r_1^{n-1}.$$

Now, we know that

$$|\nabla w(z)| \leq \nabla w^{\dagger}(\xi)$$

for all $\xi \in E_z \cap B(y, r_1)$. Hence, we have that

$$\begin{aligned}|\nabla w(z)| &\leq \inf_{\xi \in E_z \cap B(y,r_1)} \nabla w^{\dagger}(\xi) \\ &\leq c\left(\fint_{E_z \cap B(y,r_1)} (\nabla w^{\dagger})^{q}\, d\sigma\right)^{1/q}.\end{aligned}$$

Since $r_1 = c\ell(Q)$ and $\sigma(E_z \cap B(y, r_1)) \geq r_1^{n-1}$, we have

$$|\nabla w(z)| \leq c\left(\fint_{4Q} (\nabla w^\dagger)^q \, d\sigma\right)^{1/q}.$$

Now since this is true for all such z, we now have

$$\nabla w_{r_1}^\dagger(y) \leq c\left(\fint_{4Q} (\nabla w^\dagger)^q \, d\sigma\right)^{1/q}$$

for all $y \in 2Q$. Hence integrating over $2Q$, we have

$$\left(\fint_{2Q} (\nabla w_{r_1^\dagger})^{st} \, d\sigma\right)^{\frac{1}{st}} \leq c\left(\fint_{4Q} (\nabla w^\dagger)^q \, d\sigma\right)^{1/q}.$$

We can choose $q = 1$, which gives us the result. □

Together, (4.16) and (4.19), together with (4.15), we have

$$\fint_{2Q} |R_Q|^s \, d\sigma \leq c\left(\fint_{4Q} \nabla w^\dagger \, d\sigma + \|h\|_{L^\infty(4Q)}\right)^t.$$

Recalling the definition of h, we have

$$\left(\fint_{2Q} |R_Q|^s \, d\sigma\right)^{\frac{1}{s}} \leq c\left(\fint_{4Q} \nabla w^* \, d\sigma + \fint_{4Q} |f_N| \, d\sigma\right)^t.$$

Now since $t > 1$,

$$\left(\fint_{2Q} |R_Q|^s \, d\sigma\right)^{\frac{1}{s}} \leq c\left\{\left(\fint_{4Q} \nabla w^* \, d\sigma\right)^t + \left(\fint_{4Q} |f_N| \, d\sigma\right)^t\right\}.$$

Using Hölder's inequality on both terms gives

$$\left(\fint_{2Q} |R_Q|^s \, d\sigma\right)^{\frac{1}{s}} \leq c\left\{\fint_{4Q} (\nabla w^*)^t \, d\sigma + \fint_{4Q} |f_N|^t \, d\sigma\right\}.$$

Recalling that $u = v + w$, we have

$$\begin{aligned}\fint_{4Q} \left(\nabla w^\dagger\right)^t \, d\sigma &\leq \fint_{4Q} \left(\nabla u^\dagger\right)^t \, d\sigma + \fint_{4Q} \left(\nabla v^\dagger\right)^t \, d\sigma \\ &\leq c\left\{\fint_{4Q} |F| \, d\sigma + \fint_{8Q} |f| \, d\sigma\right\},\end{aligned}$$

where we use the proof of the (4.3). Recalling the definitions of F, we have

$$\fint_{4Q} \left(\nabla w^\dagger\right)^t \, d\sigma \leq c\left\{\fint_{4Q} F \, d\sigma + \fint_{8Q} f \, d\sigma\right\}$$

which gives (4.2). Now we are ready to prove Theorem 4.2.

Proof. (of Theorem 4.2) First, we have met the conditions of Theorem 4.1, so we have

$$\left(\fint_Q |F|^q \, d\sigma\right)^{\frac{1}{q}} \le c\left(\fint_{2Q} |F| \, d\sigma + \left(\fint_{2Q} |f|^q \, d\sigma\right)^{\frac{1}{q}}\right).$$

Now recalling the definition of F and f, we have that

$$\left(\fint_Q | \left(\nabla u^\dagger\right)^{tq} \, d\sigma\right)^{\frac{1}{q}} \le c\left\{\fint_{2Q} \left(\nabla u^\dagger\right)^t \, d\sigma + \left(\fint_{2Q} |f_N|^{tq} \, d\sigma\right)^{\frac{1}{q}}\right\}. \tag{4.20}$$

□

4.2 Proof of Main Theorem

First using (4.20), we have by raising each side to the power of $\frac{1}{t}$,

$$\left(\fint_Q | \left(\nabla u^\dagger\right)^{tq} \, d\sigma\right)^{\frac{1}{tq}} \le c\left\{\left(\fint_{2Q} \left(\nabla u^\dagger\right)^t \, d\sigma\right)^{\frac{1}{t}} + \left(\fint_{2Q} |f_N|^{tq} \, d\sigma\right)^{\frac{1}{tq}}\right\}.$$

Letting $p = tq$, we have for $1 < p < \frac{n}{n-1}$

$$\left(\fint_Q | \left(\nabla u^\dagger\right)^{p} \, d\sigma\right)^{\frac{1}{p}} \le c\left\{\left(\fint_{2Q} \left(\nabla u^\dagger\right)^t \, d\sigma\right)^{\frac{1}{t}} + \left(\fint_{2Q} |f_N|^{p} \, d\sigma\right)^{\frac{1}{p}}\right\}. \tag{4.21}$$

Next, let $\{Q_i\}_{i=1}^N$ be a set of cubes that cover $\partial\Omega$ each with size equal to $c\sigma(\partial\Omega)$. We have by Lemma 4.3

$$\left(\fint_{\partial\Omega} (\nabla u^*)^p \, d\sigma\right)^{\frac{1}{p}} \quad \le \quad c\sum_{i=1}^{N}\left(\fint_{2Q_i} (\nabla u^\dagger)^p \, d\sigma\right)^{\frac{1}{p}}.$$

Combining this with (4.21), we have

$$\left(\fint_{\partial\Omega} (\nabla u^*)^p \, d\sigma\right)^{\frac{1}{p}} \quad \le \quad c\sum_{i=1}^{N}\left\{\left(\fint_{2Q} \left(\nabla u^\dagger\right)^t \, d\sigma\right)^{\frac{1}{t}} + \left(\fint_{2Q} |f_N|^{p} \, d\sigma\right)^{\frac{1}{p}}\right\}.$$

Hence,

$$\left(\fint_{\partial\Omega} (\nabla u^*)^p \, d\sigma\right)^{\frac{1}{p}} \quad \le \quad c\left\{\left(\fint_{\partial\Omega} \left(\nabla u^\dagger\right)^t \, d\sigma\right)^{\frac{1}{t}} + \left(\fint_{\partial\Omega} |f_N|^{p} \, d\sigma\right)^{\frac{1}{p}}\right\}.$$

Since we can bound † by $*$, we have

$$\left(\fint_{\partial\Omega} (\nabla u^*)^p \, d\sigma \right)^{\frac{1}{p}} \;\; \leq \;\; c \left\{ \left(\fint_{\partial\Omega} (\nabla u^*)^t \, d\sigma \right)^{\frac{1}{t}} + \left(\fint_{\partial\Omega} |f_N|^p \, d\sigma \right)^{\frac{1}{p}} \right\}.$$

Since $1 < t < 1 + \epsilon$ as in Theorem 4.9, we can bound the first term on the right-hand side as follows

$$\left(\fint_{\partial\Omega} (\nabla u^*)^p \, d\sigma \right)^{\frac{1}{p}} \;\; \leq \;\; c \left\{ \left(\fint_{\partial\Omega} |f_N|^t \, d\sigma \right)^{\frac{1}{t}} + \left(\fint_{\partial\Omega} |f_N|^p \, d\sigma \right)^{\frac{1}{p}} \right\}.$$

Using Hölder's Inequality on the first term gives us the desired result

$$\left(\fint_{\partial\Omega} (\nabla u^*)^p \, d\sigma \right)^{\frac{1}{p}} \;\; \leq \;\; c \left(\fint_{\partial\Omega} |f_N|^p \, d\sigma \right)^{\frac{1}{p}}.$$

Chapter 5 Future Problems

From here there are many problems of interest. For instance, our example from earlier proved that our range of exponents is maximal in 2-dimensions. Determining the maximal range in other dimensions is a question of interest. In addition, in order to get these concrete results, we had to make our domain $C^{1,1}$. Can we find the maximal ranges in the case where we only require our domain to be Lipschitz?

Some of the results that we proved in this paper do not require our problem be for the Laplacian. We can also ask what about the case where we have a general second order linear operator. It would also be interesting to look at other types of Mixed Boundary Value Problems. Another problem of interest is the **Robin Boundary Value Problem**. Here we are searching for a function $u \in W^{1,2}(\Omega)$ that is a weak solution to

$$\begin{cases} -\mathrm{div} a\nabla u = f & \text{in } \Omega \\ c_1 u + \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

for some given functions $f \in L^2(\Omega)$ and constant c_1. We could consider a Mixed problem, where on part of the boundary we put Robin boundary conditions and the other part Dirichlet.

We could also consider simplifying the domains further, for example, Polygonal domains and see what happens to results. In general, we can continue studying for what domains and operators we can find solutions to these problems on.

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Vita

Academic Appointments

2010–Present **Graduate Teaching Assistant**, *University of Kentucky*, *Department of Mathematics*

Summer 2012 Summer 2013 **Graduate Research Assistant**, *University of Kentucky*, *Department of Mathematics*

Education

2010–2016 **Masters in Mathematics**, *University of Kentucky* , 6

2010–2015 **Certificate in College Teaching and Learning**, *University of Kentucky*

2006–2010 **B.S. in Mathematics**, *Miami University* , *cum laude*

2006–2010 **B.A. in Chemistry**, *Miami University* , *cum laude*

Awards

2014 **Arts and Sciences Certificate of Outstanding Teaching**, *University of Kentucky*

2011 **Mathematics Fellowship**, *University of Kentucky*

2008 **Koehler Prize**, *Miami University*, for outstanding work in Mathematics

2008 **Joseph A. Culler Prize**, *Miami University*, for outstanding work in Physics

Classroom Experience

Spring 2016 **Elementary Calculus 1**, *Teaching Assistant*, 4 sections

Fall 2015 **Differential Equations**, *Grader*, 2 sections

Spring 2015 **Problem Solving for Middle School Teachers**, *Primary Instructor*, 1 section

Fall 2014 **Mathematics for Elementary Teachers**, *Primary Instructor*, 2 section

2013-2014 **Math Excel**, *Calculus 1*, Workshop Leader, 2 sections

Summer 2013 **FastTrack to Calculus**, *Teaching Assistant*

Spring 2013 **Calculus 2 for the Life Sciences** , Teaching Assistant, 2 sections

Fall 2012 **Calculus 1 for the Life Sciences** , Teaching Assistant, 2 sections

Summer 2012 **Online Calculus 1** , Teaching Assistant

Spring 2012 **College Algebra**, *Primary Instructor*, 1 section

Fall 2011 **Elementary Calculus 1**, *Teaching Assistant*, 3 sections

Summer 2011 **Finite Mathematics**, *Primary Instructor*, 1 section

Spring 2011 **Finite Mathematics**, *Teaching Assistant*, 4 sections

Fall 2010 **Algebra and Trigonometry**, *Teaching Assistant*, 2 sections

.Outreach

2014 **Math Circle**, *Teaching Assistant*, 4 meetings

2012 **STEM Camp**, *Teaching Assistant*, 1 meeting, July 17th, 2012

2011 **Math Circle**, *Teaching Assistant*, 4 meetings

Research Experience

2009 **Research Experience for Undergraduates (REU)**, *Brigham Young University*, Topic: Convex Combinations of Minimal Graphs, Advisor: Michael Dorff

Conference Talks

2016 **Joint Mathematics Meeting**, *Seattle, Washington*, January 9th, 2016, *"Solutions to the L^p mixed boundary value problem in $C^{1,1}$ domains."*

2015 **Southeastern-Atlantic Regional Conference on Differential Equations**, *University of North Carolina at Greensboro*, *"L^p solutions to the mixed boundary value problem in C^2 domains."*

2009 **MAA Mathfest**, *Portland, Oregon*, *"Convex Combinations of Minimal Graphs"*, Award for Outstanding Presentation

2009 **Pi Mu Epsilon Student Conference**, *Miami University*, *"Convex Combinations of Minimal Graphs"*

Attended Conferences

2014 **Ohio River Analysis Meeting**, *University of Cincinnati*, February 28th-March 1st,2015

2014 **Spring Opportunities Workshop for Women in the Mathematical Sciences**, *University of Tennessee*, NIMBioS, April 9th-11th, 2014

2012 **Finite Element Circus**, *University of Pittsburgh*, October 9th-20th, 2012

2011 **MAA Mathfest**, *Lexington, Kentucky*, August 3rd-6th, 2011

Service

Summer 2014 **Math Excel Recruiting**, *University of Kentucky Chapter*, 4
Summer 2015 weeks each

2014-2015 **Noetherian Ring**, *University of Kentucky Chapter of AWM*, Vice-President

2011-2012 **Graduate Student Council**, *Department of Mathematics*, University of Kentucky, Secretary