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ALGORITHMS FOR UPPER BOUNDS OF LOW DIMENSIONAL GROUP HOMOLOGY

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ABSTRACT OF DISSERTATION

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ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
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Director: Dr. Marian Anton, Professor of Mathematics
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Lexington, Kentucky 2010

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A motivational problem for group homology is a conjecture of Quillen that states, as reformulated by Anton, that the second homology of the general linear group over $R = \mathbb{Z}[1/p, \zeta_p]$, for p an odd prime, is isomorphic to the second homology of the group of units of R, where the homology calculations are over the field of order p. By considering the group extension spectral sequence applied to the short exact sequence $1 \to SL_2 \to GL_2 \to GL_1 \to 1$ we show that the calculation of the homology of SL_2 gives information about this conjecture. We also present a series of algorithms that finds an upper bound on the second homology group of a finitely-presented group. In particular, given a finitely-presented group G, Hopf's formula expresses the second integral homology of G in terms of generators and relators; the algorithms exploit Hopf's formula to estimate $H_2(G; k)$, with coefficients in a finite field k. We conclude with sample calculations using the algorithms.

KEYWORDS: homology, linear groups, Quillen conjecture, finitely-presented groups, GAP

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Date: April 18, 2010

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Chapter 1 Introduction

Experiment has always been at the heart of mathematics. Gauss, widely regarded as the greatest mathematician of all time, has been quoted as saying that his technique of arriving at mathematical truth was "through systematic experimentation"[1]. The importance of mathematical proof cannot be overstated; however, the elegance of a proof may conceal the necessary experimental nature of the mathematics. Computer-based and aided experimentation has long been involved in the areas of applied mathematics and numerical analysis. Relatively recently, more and more attention has been paid to computer/experimental techniques applied to aspects of pure mathematics; moreover, any constructive mathematical proof can be turned into an algorithm. The need for algorithms in pure mathematics arises, for example, when the objects of study are too complicated to construct or compute "by hand". On the other hand, powerful conjectures can be checked or even posed as a result of computations. The results in this dissertation can be considered experimental mathematics in this sense.

This dissertation falls under the research umbrella of algebraic topology, albeit broadly construed. Fundamentally, algebraic topology assigns algebraic invariants to topological spaces. A primary family of invariants for a topological space X is the (integral) homology groups, $H_n(X)$, for $n = 0, 1, 2, \ldots$. In each dimension n, one can think of the homology group of a space as a measure for the connectedness of that space, and the rank of the homology group as counting the number of "holes" of that dimension in the space. For example, the homology of a torus in dimension $n = 0$ is one copy of the integers $\mathbb{Z}$, meaning that the space is path-connected; in dimension $n = 1$ the homology is $\mathbb{Z} \oplus \mathbb{Z}$ meaning that the torus is a closed surface with two "holes", one bounded by a meridian and one by a latitude. With this framework, one can focus on studying topology via algebra or algebra via topology; the results in this work fall into the latter category.

As an example of an algebraic problem treated topologically, consider a group G and its algebraically defined group homology $H_n(G)$. One can construct a topological space BG, called the classifying space of G, by taking the infinite join of G with itself modulo the diagonal G-action. In the case that G has the discrete topology, BG coincides with the Eilenberg-MacLane space $K(G, 1)$. The (singular) homology of

[1] See the Statement of Philosophy of the journal Experimental Mathematics: `http://www.expmath.org/expmath/philosophy.html`

BG is exactly the group homology of G, meaning that there is a natural isomorphism

$$H_n(BG) \cong H_n(G), \tag{1.1}$$

which we review in Section 2.1. Moreover, the field of algebraic K-theory was revolutionized by Quillen when he gave a description, for which we was awarded the Fields Medal, of the K-functors in terms of the homotopy groups of a certain classifying space; we describe this construction in Section 2.2. By virtue of a Hurewicz homomorphism from homotopy groups to homology groups, one can relate the algebraic K-theory groups to group homology.

While the homology of a group is relatively straightforward to define, calculations are another matter. Currently there are several algorithms and computer programs to calculate the homology of finite groups or the homology of certain classes of infinite groups, but there is no general algorithm to calculate the homology of an arbitrary group.

1.0.1 A Conjecture of Quillen

A motivational problem for low dimensional group homology, which is related to algebraic K-theory, is the study of homology for groups of the form $GL_j(A)$, where GL_j is a finite rank general linear group scheme and A is the ring of integers in a number field. An approach to this problem is to consider the diagonal matrices inside GL_j; let D_j denote the subgroup formed by these matrices. Then the canonical inclusions $D_j \subset GL_j$ for $j = 0, 1, ...$ induce homomorphisms on group homology with k-coefficients

$$\rho_{i,j}^{A,p} : H_i(D_j(A); k) \to H_i(GL_j(A); k), \tag{1.2}$$

where k is the field of prime order p, i is called the homological dimension and j the rank. In this context, there is the following celebrated conjecture of Quillen:

Conjecture 1.0.1. [Qui71] *The homomorphism $\rho_{i,j}^{A,p}$, as given above, is an epimorphism for $A = \mathbb{Z}[1/p, \zeta_p]$, p a regular odd prime, ζ_p a primitive pth root of unity and any values of i and j.*

Conjecture 1.0.1 has been proved in a few cases and disproved in infinitely many other cases. For $A = \mathbb{Z}[1/2]$ it was proved by Mitchel in [Mit92] for $j = 2$ and by Henn in [Hen99] for $j = 3$. Anton gave a proof for $A = \mathbb{Z}[1/3, \zeta_3]$ and $j = 2$ in [Ant99].

Dwyer gave a disproof for the conjecture for $A = \mathbb{Z}[1/2]$ and $j = 32$ in [Dwy98] which Henn and Lannes improved to $j = 14$ in [HLS95]; this is an improvement in light of Henn's result in [Hen96] that states that if Conjecture 1.0.1 is false for j_0 then it is false for all $j \geq j_0$. Anton disproved the conjecture for $A = \mathbb{Z}[1/3, \zeta_3]$ and $j \geq 27$ also in [Ant99].

This conjecture was reformulated and, in a sense, corrected by Anton, and he also conjectured the following:

Conjecture 1.0.2. [Ant03] *Given p, k and A as above,*

$$H_2(GL_2(A)) \cong H_2(D_1(A)). \tag{1.3}$$

Anton's conjecture led to a proof of Conjecture 1.0.1 for $\mathbb{Z}[1/5, \zeta_5]$ and $i = j = 2$. For a survey on the current status of conjectures 1.0.1 and 1.0.2 we cite [Ant09].

By a spectral sequence argument applied to the group extension

$$1 \to SL_j(A) \to GL_j(A) \to D_1(A) \to 1 \tag{1.4}$$

given by the determinant map, we can reformulate Quillen's conjecture in terms of $H_i(SL_j(A); k)$. In the particular case $j = 2$, this homology has been studied extensively by using the theory of buildings. However, based on this theory we can calculate this homology only for i sufficiently large [BS76]. The problem of calculating $H_i(SL_2(A); k)$ in low dimensions turns out to be highly nontrivial even when $i = 2$.

1.0.2 Algorithmic Group Homology

Exploiting a classical theorem due to Hopf, we present a series of algorithms in Section 3.1 that give upper bounds on group homology in homological dimensions one and two, provided coefficients are taken in a finite field. In particular, examples in Section 3.1.10 confirm the results in [Ant09], as well as give a new finding:

Theorem. *The dimension of $H_2(SL_2(\mathbb{Z}[1/7, \zeta_7]); \mathbb{F}_7)$ as a vector space over $\mathbb{F}_7$ is at most six.*

Since the algorithms in Section 3.1 depend only upon Hopf's formula for H_2, the usefulness of these algorithms extends to groups beyond the scope of Quillen's Conjecture. Moreover, the algorithms are distinct from existing methods of calculating low dimensional group homology in that they find bounds on the homology of any finitely-presented group.

As a byproduct of the calculations related to Quillen's Conjecture we are involved in a long term project preparing a database for low dimensional group homology of linear groups over number fields and their rings of integers. This work will be extended to other classes of finitely-presented groups of interest to computational group theory and algebraic topology, and the first set of these calculations is found in Section 3.2.

The various algorithms in this dissertation are given in pseudocode and were carried out with the computational algebra system GAP (**G**roups, **A**lgorithms, and **P**rogramming) [GAP07]. The GAP code used to implement the algorithms is given in Appendix 4.1, as well the GAP code to input a selection of linear groups. The calculations were carried out on a refurbished Dell SC1435 server with two Dual-Core Opteron 2220 SE processors running at 2.8GHz with 1MB cache and 16GB of DDR2 RAM, however the memory limit of GAP was set to 2GB.

Finally, we have included Appendix 4.2 which lists some known results on the (co)homology of linear groups, and Appendix 4.3 which explains (1) a technique of finding the abelianization of a finitely-presented group in the context of reductions of integer matrices to Smith Normal Form, and (2) a short description of spectral sequences.

Chapter 2 Background

In this chapter we recall some facts and terminology which will be used throughout this dissertation about group homology and algebraic K-theory. For a more thorough treatment of these topics, we refer the interested reader to the books by Brown [Bro94] and Rosenberg [Ros94]. We also assume familiarity with basic ideas from homological algebra and category theory and give Lang's excellent algebra text [Lan02] as a reference.

2.1 Group Homology and Cohomology

Let G be a group, given multiplicatively, equipped with the discrete topology unless otherwise stated. Our goal in this section is to define two sequences of functors from the category of groups to the category of abelian groups called the homology and cohomology groups of G.

Define $\mathbb{Z}G$ to be the free $\mathbb{Z}$-module on the elements of G. That is, an element of $\mathbb{Z}G$ is a finite sum $\sum_g n_g g$, where $g \in G$ and $n_g \in \mathbb{Z}$. Multiplication in G induces multiplication in $\mathbb{Z}G$ in the obvious way which turns $\mathbb{Z}G$ into a ring, called the **integral group ring** of G. A $\mathbb{Z}G$-module is an abelian group A equipped with a G-action. For right G-module M, the **co-invariants of** M, denoted M_G, is the quotient

$$M_G := \frac{M}{\langle mg - m \rangle}, \tag{2.1}$$

where $\langle mg - m \rangle$ denotes the subgroup generated by all elements of the form $mg - m$ for $m \in M$ and $g \in G$. We say M is a trivial G-module if $mg = g$ for all $m \in M$ and $g \in G$.

Lemma 2.1.1. *Given right G-module M, we have that $M_G \cong M \otimes_G \mathbb{Z}$. Here $\mathbb{Z}$ is a trivial left G-module.*

If M is a left G-module then we can build a free resolution of M by G-modules by letting F_0 be the free module on a set of generators of M and $\epsilon : F_0 \to M$ the canonical surjection. In a similar manner choose a surjection $F_1 \to \ker(\epsilon)$ with F_1 free. Then we have the exact sequence

$$F_1 \to F_0 \to M \to 0. \tag{2.2}$$

Continuing inductively we obtain the infinite exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0 \tag{2.3}$$

and so have the following proposition.

Proposition 2.1.2. *Given a G-module, M such a resolution exists.*

Definition 2.1.3. Let M be a G-module and $P \to M$ a resolution of M by projective G-modules. We define the **homology and cohomology of the group G with coefficients in M** by:

$$\begin{array}{rclcl} H_n(G;M) & := & H_n(P \otimes_G \mathbb{Z}) & = & Tor_n^G(\mathbb{Z}, M) \\ H^n(G;M) & := & H^n(Hom_G(P, M)) & = & Ext_G^n(\mathbb{Z}, M). \end{array} \tag{2.4}$$

In this work, the focus will be upon $H_n(G;\mathbb{Z})$ or $H_n(G;k)$ for a finite field k, where $\mathbb{Z}$ and k are regarded as trivial G-modules. We adopt the convention that $H_n(G)$ denotes $H_n(G;\mathbb{Z})$.

Note 2.1.4. Suppose $\partial : P_n \to P_{n-1}$. Then the induced differential $\delta : Hom_G(P_{n-1}, \mathbb{Z}) \to Hom_G(P_n, \mathbb{Z})$ is defined by $\delta(g(x)) = (g \circ \partial)(x)$ for $g \in Hom_G(P_{n-1}, \mathbb{Z})$ and $x \in P_n$. Also, $\partial \otimes_G 1 : P_n \otimes_G \mathbb{Z} \to P_{n-1} \otimes_G \mathbb{Z}$ is the induced differential on $P \otimes_G \mathbb{Z}$.

The following proposition is standard.

Proposition 2.1.5. *Given a ring R, let P and P' be projective resolutions of M and M' respectively, and let $\alpha : M \to M'$ be an R-linear map.*

(i) There exists an R-linear chain map $P \to P'$ which extends α. Moreover, two such maps are chain homotopic.

(ii) The induced maps on homology and cohomology are canonical isomorphisms.

Proposition 2.1.5 implies that the homology of a group is independent of the resolution chosen.

Example 2.1.6. If $G = \mathbb{Z}_k = \langle t \mid t^k \rangle$ then

$$H_n(G) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & \text{n even} \\ \mathbb{Z}_k, & n > 0 \text{ and n odd.} \end{cases} \tag{2.5}$$

Since $\mathbb{Z}$ is a trivial G-module, a free G-resolution of $\mathbb{Z}$ is:

$$F = \cdots \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{K} \mathbb{Z}G \xrightarrow{1-t} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \tag{2.6}$$

where $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the "augmentation map" $t \mapsto 1$ and "K" denotes multiplication by $1 + t + \cdots + t^{k-1}$. Tensoring with $\mathbb{Z}$ over G gives the complex

$$F \otimes_G \mathbb{Z} = \cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{k} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0\,. \tag{2.7}$$

which implies the result.

2.1.1 The Standard Resolution

The **standard resolution** of a group G consists of free abelian groups F_n, with basis the $(n+1)$-tuples $(g_0, g_1, \ldots, g_n)$, and diagonal G action $g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$ for $g \in G$. The boundary map is given by

$$\partial(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n). \tag{2.8}$$

As a G-module, F_n has as a basis the $(n+1)$-tuples whose first element is 1. We introduce the bar notation

$$[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, \ldots, g_1g_2 \ldots g_n) \tag{2.9}$$

and note that for $n = 0$, there is only one such element which is denoted $[\,]$. For this basis, the boundary map is given by $\partial = \sum_{i=0}^{n}(-1)^i d_i$, where

$$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n], & i = 0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n], & 0 < i < n \\ [g_1|\cdots|g_{i-1}], & i = n. \end{cases} \tag{2.10}$$

Example 2.1.7. For a group G, applying $- \otimes_G \mathbb{Z}$ to the standard resolution and looking at low dimensions gives

$$\bigoplus_{[g|h]} \mathbb{Z} \xrightarrow{\partial} \bigoplus_{[g]} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \tag{2.11}$$

where

$$\partial([g|h]) = g[h] - [gh] + [g] \tag{2.12}$$
$$= [h] - [gh] + [g], \quad \text{since } \mathbb{Z} \text{ is a trivial } G\text{-module.} \tag{2.13}$$

It follows that $H_0(G) = \mathbb{Z}$ and $H_1(G) = G^{\mathrm{ab}}$, where $G^{\mathrm{ab}} := G/[G,G]$ (see Appendix 4.3), the abelianization of G.

The following theorem due to Hopf gives a useful characterization of $H_2(G)$.

Theorem 2.1.8. [Bro94, p. 42] *Let G be a group with presentation $R \to F \to G$. Then there is an isomorphism*

$$H_2(G) \cong \frac{R \cap [F,F]}{[F,R]} \tag{2.14}$$

where, for groups $A, B \subset F$, $[A,B]$ is the subgroup generated by commutator elements $[a,b] = a^{-1}b^{-1}ab$ for $a \in A$ and $b \in B$.

2.1.2 Classifying Spaces

In this section, we explain a topological definition of group homology and show the equivalence of the two approaches. Unless otherwise stated, spaces are connected CW-complexes and G is a group with the discrete topology. Also, since the spaces are connected we suppress the notation of a basepoint when discussing homotopy groups.

Given a group G, a G-bundle is a fiber bundle $P \to X$ with fiber G. A **principle G-bundle** is a locally trivial fibration $p : E \to B$, with fiber G, and a right G-action $E \times G \to B$. Two bundles are isomorphic if there exists a homeomorphism $f : E_1 \to E_2$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{id_X} & B \end{array} \tag{2.15}$$

commutes.

Given a G-bundle $p : E \to B$ and a map $f : A \to B$ the **fiber product**

$$f^*E := E \times_f B = \{(a,e) \in E \times B | f(a) = p(e)\} \tag{2.16}$$

is the pullback

$$\begin{array}{ccc} E \times_f B & \dashrightarrow & E \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array} \tag{2.17}$$

in the category of G-bundles.

Definition 2.1.9. A **classifying space** for a group G is a topological space BG with a principle G-bundle $p : EG \to BG$ where EG is contractible so that $BG = EG/G$. A classifying space is universal in the sense that if $q : E \to B$ is a principle G-bundle then there is a continuous map $f : B \to BG$, unique up to homotopy, such that E is the fiber product f^*EG:

$$\begin{array}{ccc} E & \dashrightarrow & EG \\ \downarrow & & \downarrow \\ B & \dashrightarrow & BG \end{array} \tag{2.18}$$

Note 2.1.10. The universal property of classifying spaces implies that for a space X, there is a 1-1 correspondence between isomorphism classes of G-bundles and homotopy classes of maps $X \to BG$. That is

$$(G\text{-bundles on } X/\cong) \leftrightarrow [X, BG]. \tag{2.19}$$

Construction of Classifying Spaces

The most general construction of a classifying space is the so-called "join-construction" given by Eilenberg and MacLane [EML86, p. 369] and generalized by Milnor [Mil56].

Note 2.1.11. If G is discrete then the homotopy exact sequence of a fibration applied to $G \to EG \to BG$ gives

$$\cdots \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \cdots \tag{2.20}$$

Therefore a classifying space is the familiar Eilenberg-MacLane space $K(G, 1)$.

Theorem 2.1.12. *If G has a presentation $\langle F|R\rangle$ then there is an inductive construction of BG by attaching cells.*

Proof. Take $\{x_\alpha\}$ to be a set of generators for F and $\{r_\beta\}$ to be a set of generators up to conjugacy for R. Let B_0 be a one-point space and B_1 be the space obtained from B_0 by attaching a 1-cell e_α for each x_α. Then the fundamental group of B_1 is isomorphic to F. Now construct B_2 from B_1 by attaching a 2-cell e_β for each r_β such that the attaching map is the word r_β in F. Note that the fundamental group of B_2 is naturally isomorphic to G by the van Kampen theorem.

For $n > 2$, inductively construct B_n from B_{n-1} by attaching n-cells via attaching maps $f_\gamma : S^{n-1} \to B_{n-1}$ where each f_γ is a generator for $\pi_{n-1}B_{n-1}$. Applying the van Kampen theorem again shows that $\pi_1 B_n = G$ and by construction $\pi_k B_n = 0$ for $k > 1$. Now define $B = \bigcup_n B_n$. For a map $f : S^k \to B$, the image of f is contained in some B_n since S^k is compact. So f is nullhomotopic for $k \geq 2$. Therefore B is a $K(G, 1)$ space and, by Note 2.1.11, is a classifying space for G. □

Example 2.1.13. If $G = \mathbb{Z}_2 = \langle t | t^2 \rangle$ then $BG = \mathbb{R}P^\infty$

Let $F = \langle t \rangle$ and R be the normal closure of $\langle t^2 \rangle$ in F. Following Theorem 2.1.12, we construct a sequence of spaces: B_0 is a one-point space and $B_1 = S^1$ since F has only one generator. Next, $B_2 = B_1 \cup e^2$ with the single 2-cell attached according to the word t^2 in B_1, i.e., $\partial(e^2) = S^1 \to B_1 = S^1$ is the double cover. Therefore $B_2 = \mathbb{R}P^2$.

For $B_3 = B_2 \cup \{\text{2-cells}\}$, with one 2-cell for each generator of $\pi_2(B_2) = \pi_2(\mathbb{R}P^2)$ attached along that generator, the long exact homotopy sequence applied to the covering $\mathbb{Z}_2 \to S^2 \to \mathbb{R}P^2$ gives

$$\cdots \to \pi_2(\mathbb{Z}_2) \to \pi_2(S^2) \to \pi_2(\mathbb{R}P^2) \to \pi_1(\mathbb{Z}_2) \to \cdots . \tag{2.21}$$

Thus $\pi_2(\mathbb{R}P^2) \cong \pi_2(S^2) \cong \mathbb{Z}$.

Let $f : S^2 \to \mathbb{R}P^2$ be the generator of $\mathbb{Z}$; that is, $\pi_2(\mathbb{R}P^2) = \langle [f] \rangle$. Since f is also the double cover $S^2 \to \mathbb{R}P^2$ (see [Knu01, p. 154]) we obtain $B_3 = \mathbb{R}P^3$; similarly $B_n = \mathbb{R}P^n$ for $n > 2$. Therefore $B\mathbb{Z}_2 = \bigcup_n B_n = \mathbb{R}P^\infty$ is a classifying space for $\mathbb{Z}_2$. We note that the reduced integral homology of $\mathbb{R}P^\infty$ is zero in even dimensions and $\mathbb{Z}_2$ in odd dimensions; this agrees with the purely algebraic calculation of the homology of $\mathbb{Z}_2$ in Example 2.1.6 for the case $k = 2$.

Theorem 2.1.14. *The group homology (cohomology) of G is isomorphic to the cellular homology (cohomology) of BG. That is,*

$$\begin{aligned} H_n(G) &\cong H_n(BG), \text{ and} \\ H^n(G) &\cong H^n(BG). \end{aligned} \tag{2.22}$$

Proof. Let $C_*(EG)$ be the cellular chain complex of EG; so each $C_i(EG)$ is a free $\mathbb{Z}$-module with a basis element for each i-cell of EG. Note that G acts on EG by deck transformations, i.e., by permuting cells, and therefore there is a G-action that turns each $C_i(EG)$ into a G-module. Note that EG, being contractible, has the homology of a point and so $C_*(EG) \to \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}$ by G-modules.

The map $C_*(EG) \to C_*(BG)$ gives a quotient map $\phi : C_*(EG)_G \to C_*(BG)$. Note that $C_*(EG)_G$ has a $\mathbb{Z}$-basis with one basis element for each G-orbit of cells in X. But $C_*(BG)$ also has a $\mathbb{Z}$-basis with one element for each G-orbit of cells in X. Then since ϕ sends basis elements in $C_*(EG)_G$ to corresponding basis elements in $C_*(BG)$ ϕ is an isomorphism.

Finally, by Lemma 2.1.1 $C_*(BG) = C_*(EG)_G = C_*(EG) \otimes_G \mathbb{Z}$. Then since homology is independent of the resolution chosen by Proposition 2.1.5, $H_*(BG) \cong H_*(G)$. The isomorphism on cohomology follows from duality.

□

2.1.3 Plus Construction

Here we illustrate a method given by Quillen whereby we kill the homotopy of a subgroup of the fundamental group of a space while preserving the homology. The primary application of this construction will be given in Section 2.2 where it will be used to construct the algebraic K-theory functors.

Theorem 2.1.15. [Qui75] *Let X be a connected CW-complex and let π be a perfect normal subgroup of $\pi_1 = \pi_1(X)$. Then there exists a CW-complex X^+, obtained from X by attaching only 2-cells and 3-cells, such that*

(i) $\pi_1(X) \to \pi_1(X^+)$ *is the quotient map* $\pi_1 \to \pi_1/\pi$.

(ii) For a π_1/π-module M, we have $H_*(X^+, X; M) = 0$, where M is viewed as a local coefficient system on X^+.

Moreover, X^+ is unique up to homotopy equivalence.

The proof uses methods similar to those used in the proof of Theorem 2.1.12 to kill the generators of π, and uses the fact that $[\pi, \pi] = 0$ to restore the homology by attaching cells of dimension three. It is a consequence of the van Kampen theorem that attaching these 3-cells does not affect π_1 or π.

Theorem 2.1.16. *The plus construction is functorial in the following sense: Let $f : X \to Y$, π_X be a perfect subgroup of $\pi_1(X)$, and π_Y a perfect subgroup of $\pi_1(Y)$.*

Then f induces a map $f^+ : X^+ \to Y^+$, where X^+ is constructed from π_X and Y^+ is constructed from π_Y, such that

(i) $id_X : X \to X$ induces id_{X^+}, the identity on X^+.

(ii) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(f \circ g)^+ = f^+ \circ g^+$.

Example 2.1.17. Let Σ_5 be the symmetric group on five letters and $A_5 \subset \Sigma_5$ the alternating subgroup. We use that fact that A_5 is a perfect group to construct $B\Sigma_5^+$. Noting that $\{+1, -1\}$ is the group $\mathbb{Z}_2$ written multiplicatively, there is a group homomorphism $sgn : \Sigma_5 \to \{+1, -1\}$ sending even/odd permutations onto $+1/-1$ respectively. Since A_5 is the group of even permutations, it is the kernel of sgn. Therefore $\Sigma_5/A_5 \cong \mathbb{Z}_2$.

This gives that $\pi_1(B\Sigma_5^+) = \mathbb{Z}_2$. Since BA_5^+ is simply connected it is the universal cover of $B\Sigma_5^+$, and we have the $\mathbb{Z}_2$-bundle

$$\mathbb{Z}_2 \to BA_5^+ \to B\Sigma_5^+ \tag{2.23}$$

where $\mathbb{Z}_2$ discrete implies that BA_5^+ is a covering space. And therefore the long exact sequence

$$\cdots \to \pi_n(\mathbb{Z}_2) \to \pi_n(BA_5^+) \to \pi_n(B\Sigma_5^+) \to \pi_{n-1}(\mathbb{Z}_2) \to \cdots \tag{2.24}$$

implies that for $n \geq 2$

$$\pi_n(B\Sigma_5^+) \cong \pi_n(BA_5^+) \tag{2.25}$$

In particular, $\pi_2(B\Sigma_5^+) \cong \pi_2(BA_5^+)$. But since $\pi_1(BA_5^+)$ is trivial, the Hurewicz map $\pi_2(BA_5^+) \to H_2(BA_5^+)$ is an isomorphism; moreover

$$H_2(BA_5^+) = H_2(BA_5) = H_2(A_5). \tag{2.26}$$

So we have that $\pi_2(B\Sigma_5^+) = H_2(A_5)$ and $H_2(A_5) = \mathbb{Z}_2$.

2.2 Algebraic K-Theory

Algebraic K-theory gives a sequence of functors from the category of associative, unitary rings to the category of abelian groups. While the definitions of K_0 and K_1 are straightforward, several definitions of K_2 were proposed and the groups K_n for $n \geq 3$ were not defined until Quillen provided a generalization of the K-groups as homotopy groups of certain topological spaces.

2.2.1 Classical K-Theory

The lower dimensional K-groups are classically defined in the following way. The isomorphism classes of finitely generated projective modules over an associative, unitary ring A form an abelian monoid with addition given by

$$[P] + [Q] := [P \oplus Q]. \tag{2.27}$$

Then $K_0(A)$ is defined as the Grothendieck group of this monoid.

For $K_1(A)$, recall that we can regard the group of $n \times n$ non-singular matrices, $GL_n(A)$, as a subgroup of $GL_{n+1}(A)$: if $M \in GL_n(A)$ then

$$\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(A). \tag{2.28}$$

We define the infinite general linear group as the direct limit $GL(A) := \varinjlim GL_n(A)$. A classic result due to Whitehead [Mil71, p. 25] is that the subgroup of infinite elementary matrices $E(A)$ is the derived group $[GL(A), GL(A)]$ of $GL(A)$. Then

$$K_1(A) := GL(A)/E(A) \cong GL(A)/[GL(A), GL(A)] := GL(A)^{\mathrm{ab}}. \tag{2.29}$$

In [Mil71] Milnor eventually defined $K_2(A)$ following the work of Steinberg on the structure of the group $E(A)$. The so-called Steinberg group over A, denoted $St(A)$, maps canonically onto $E(A)$ and then $K_2(A)$ is defined as the kernel of this canonical map $St(A) \to E(A)$.

2.2.2 Higher K-Theory

Given a ring A as above, let us define a topological space $K(A) := K_0(A) \times BGL(A)^+$, where $K_0(A)$ has the discrete topology. Quillen in [Qui75] defined the K-theory groups as the homotopy groups of this topological space.

$$K_n(A) := \pi_n(K(A)). \tag{2.30}$$

In dimensions 0 and 1 it is clear the Quillen's definition agrees with the classical one. For $n = 2$ we examine the long exact homotopy sequence of the pair of topological spaces $(BGL(A)^+, BGL(A))$. Since $\pi_n(BGL(A))$ is nonzero only for $n = 1$ the sequence then becomes

$$0 \longrightarrow \pi_2(BGL^+) \longrightarrow \pi_2(BGL^+, BGL) \longrightarrow \pi_1(BGL) \longrightarrow \pi_1(BGL^+) \longrightarrow \pi_1(BGL^+, BGL) \longrightarrow 0$$

and it is easy to see that $\pi_1(BGL^+, BGL) = 0$. Therefore we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(BGL^+) & \longrightarrow & \pi_2(BGL^+, BGL) & \longrightarrow & \pi_1(BGL) & \longrightarrow & \pi_1(BGL^+) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow \cong & & \| & & \\
0 & \longrightarrow & K_2 & \longrightarrow & \pi_2(BGL^+, BGL) & \longrightarrow & GL & \longrightarrow & K_1 \cong GL/E & \longrightarrow & 0 \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & E & & & &
\end{array}$$

But by definition

$$0 \longrightarrow K_2(A) \longrightarrow St(A) \longrightarrow GL(A) \longrightarrow K_1(A) \cong GL(A)/E(A) \longrightarrow 0$$

is exact. Thus $\pi_2(BGL(A)^+, BGL(A)) \cong St(A)$ and Quillen's definition for K_2 agrees with the classical one.

Summarizing the above discussion in Figure 2.1 we have the following diagram where the arrows are functors:

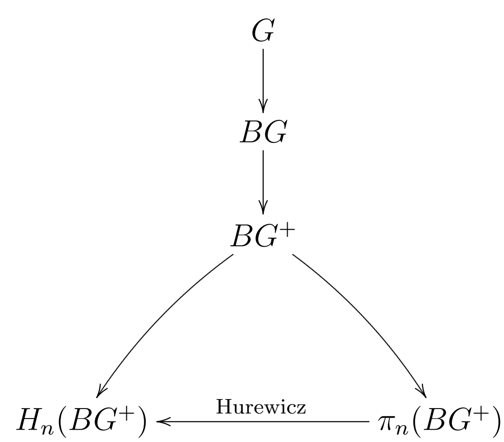

Figure 2.1: The relationship among topology, group homology, and algebraic K-theory

If $G = GL(R)$, for some R, then the last row becomes

$$H_n(GL(R)) \leftarrow K_n(R) \tag{2.31}$$

by the definition of $K_n(R)$ and since $H_n(BGL(R)^+) = H_n(BGL(R)) = H_n(GL(R))$.

Chapter 3 Main Results

3.0.3 A Spectral Sequence Reduction

Given a group extension $1 \to N \to G \to Q \to 1$ there is the Hochschild-Serre Spectral Sequence [McC01, p. 341] with

$$E^2_{p,q} \cong H_p(Q, H_q(N;k)) \Longrightarrow H_{p+q}(G;k), \tag{3.1}$$

where we take coefficients in a field k regarded as a trivial G-module.

By letting R be a Euclidean ring we have that $SL_2(R)$ is a perfect group [Coh66]. Thus, applying the spectral sequence 3.1 to the extension

$$1 \to SL_2(R) \to GL_2(R) \to GL_1(R) \to 1, \tag{3.2}$$

we see that the entries $E^2_{p,1}$ are all 0. Thus for $q < 3$ the E^3 page is equal to the E^2 page. We also note that

$$GL_1(R) = D_1(R) = R^\times, \tag{3.3}$$

where $R^\times$ is the group of units of R.

Figure 3.1 displays the E^2 page of this spectral sequence, and we have included the transgression $\tau : E^3_{3,0} \to E^3_{0,2}$ for reference. Note that since $E^2_{p,q} = E^3_{p,q}$ for all p and for all $q < 3$ then $E^2_{p,1} = E^\infty_{p,1}$. Then since $E^4_{p,q} = E^\infty_{p,q}$ for $p, q+1 < 4$ and $E^4_{0,2} = H_2(SL_2(R))_{GL_1(R)}/Im(\tau)$ we have that

$$\begin{aligned} H_2(GL_2(R)) &= E^4_{2,0} \oplus E^4_{1,1} \oplus E^4_{0,2} && (3.4)\\ &= H_2(SL_2(R))_{GL_1(R)}/Im(\tau) \oplus H_2(GL_1(R)). && (3.5)\end{aligned}$$

Since we have chosen field coefficients, the homology groups displayed above are vector spaces over the field k. Then Equation 3.5 implies that

$$\dim\ H_2(GL_2) \geq \dim\ H_2(GL_1(R)). \tag{3.6}$$

Recall from Section 1.0.1 that the Quillen Conjecture implies that the map induced by inclusion

$$H_2(D_2(R)) \twoheadrightarrow H_2(GL_2(R)) \tag{3.7}$$

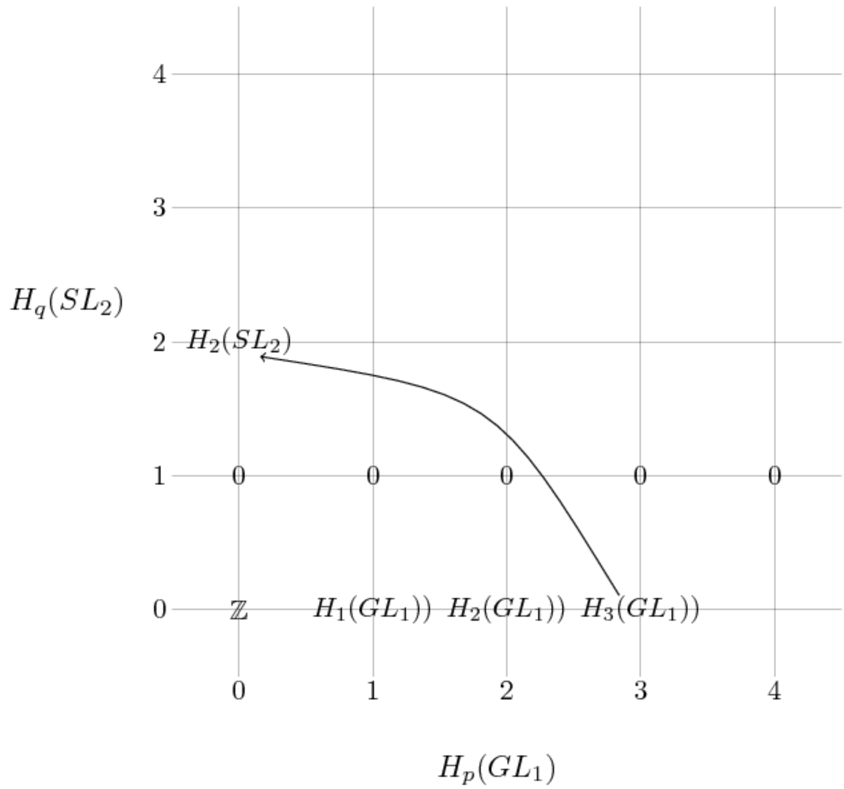

Figure 3.1: E^2 page with $\tau : E^3_{3,0} \to E^3_{0,2}$ displayed

is surjective. Anton's reformulation of Quillen's conjecture in [Ant09] and results in [Ant03] imply that the map 3.7 factorizes thusly:

$$\begin{array}{ccc} H_2(D_2) & \twoheadrightarrow & H_2(GL_2)\ . \\ & \searrow \quad \nearrow\!\!\!\!\nearrow & \\ & H_2(D_1) & \end{array} \tag{3.8}$$

Then $H_2(D_1) \twoheadrightarrow H_2(GL_2)$ is surjective and

$$\dim\, H_2(D_1) \geq \dim\, H_2(GL_2). \tag{3.9}$$

Then equations 3.6, 3.9, and 3.3 imply Conjecture 1.0.2

$$H_2(GL_2) \cong H_2(GL_1), \tag{3.10}$$

which, by Equation 3.5, is equivalent to

$$H_2(SL_2(R))_{GL_1(R)} \cong Im(\tau), \tag{3.11}$$

which is true if and only if τ is surjective. Moreover, for Conjecture 1.0.2 to be true, it is sufficient for the finitely-presented group $SL_2(\mathbb{Z}[1/p, \zeta_p])$ to have trivial second dimensional $\mathbb{F}_p$-homology.

Note 3.0.1. The fact that dim $H_2(GL_2) \geq$ dim $H_2(GL_1(R))$ follows by the spectral sequence argument above and is independent of conjectures 1.0.1 and 1.0.2.

In this context, the purpose of the following section is to give a series of algorithms that allow us to estimate the second homology group of any finitely-presented group. More precisely, given a finitely-presented group G and a finite field k, the second homology group $H_2(G; k)$ with coefficients in k is a finite dimensional vector space over k. Our algorithm gives an upper bound for the dimension of $H_2(G; k)$ and, in particular cases, the algorithm calculates precisely this dimension. This algorithm is an improvement of existing algorithms to compute $H_2(G)$; for example, the algorithms included in the GAP packages "cohomolo" [Hol08] and "HAP" [Ell08] are effective on finite groups and certain classes of infinite groups. The algorithms presented here effectively find a bound for the homology of any finitely-presented group.

3.1 Algorithms for Low Dimensional Group Homology

3.1.1 First Homology Group

We consider a group given by a finite set of generators and a finite set of relators. If we denote this group by G then there is a short exact sequence

$$1 \to R \to F \to G \to 1. \tag{3.12}$$

Here F is a finitely generated free group and R is a normal subgroup of F such that the conjugation action of F on R gives R the structure of a finitely generated F-module. Here if F and R are two groups not necessarily commutative then an F-module structure on R is an assignment $r \mapsto r^f$ for $r \in R$ and $f \in F$ such that

$$r^1 = r \tag{3.13}$$

$$(r_1 r_2)^f = r_1^f r_2^f \tag{3.14}$$

$$r^{f_1 f_2} = (r^{f_1})^{f_2} \tag{3.15}$$

where, if not otherwise stated, all groups are given multiplicatively. As discussed in Example 2.1.7, it is well known that the first homology of a group is just another name for its abelianization. In particular, if we denote by $H_1(G)$ this abelian group then there is a short exact sequence

$$1 \to R[F,F] \to F \to H_1(G) \to 1 \tag{3.16}$$

where $[F,F]$ again denotes the subgroup of F generated by the commutators in F and the juxtaposition $R[F,F]$ denotes the operation of taking the subgroup generated by the product of R and $[F,F]$. Letting F act on $R[F,F]$ by conjugation, we recognize that $R[F,F]$ is a finitely generated F-module. Indeed, the commutator formula

$$[xy,z] = (xy)^{-1}z^{-1}xyz = y^{-1}x^{-1}z^{-1}xzyy^{-1}z^{-1}yz = [x,z]^y[y,z] \tag{3.17}$$

proves that since F is a finitely generated group then $[F,F]$ is a finitely generated F-module under conjugation and the same is assumed about R. This argument leads to a deterministic algorithm that gives the structure of $H_1(G)$. The input is a finite list of generators for F, say S, and a finite list of generators for the F-module R, say T. The output is a list of integers describing the structure of the finitely generated abelian group $H_1(G)$.

3.1.2 The First Homology Algorithm

Algorithm 3.1.1. FirstHomology(F,R)

Input: Free Group F, Relators R

Output: List of abelian invariants of the finitely-presented group F/R

$M :=$ Relation matrix of F/R

$N :=$ Smith normal form of M

return Diagonal entries of N

Note 3.1.2. The GAP command `AbelianInvariants()` carries out (roughly) the above algorithm. A description of the Smith Normal Form of a matrix, and a discussion of some of the applications is given in Appendix 4.3.

Theorem 3.1.3. *For a finitely-presented group G,* FirstHomology *returns the structure of $H_1(G)$ as a finitely generated abelian group.*

Proof. Recall that given a finite presentation for F/R that consists of n generators S and m relators T, there is the associated $n \times m$ relation matrix M whose (i,j) entry is the sum of the exponents of all occurrences of the jth generator in the ith relator. Let G be a group with finite presentation F/R and A the associated relation matrix. By Theorem 4.0.3, A can be brought to Smith Normal Form in a finite number of steps.

The discussion immediately following Definition 4.0.4 and, in particular, Equation 4.11, imply that the diagonal entries of the Smith Normal Form of A, which are a sequence of non-negative integers, are the abelian invariants of $G/[G,G]$. Then the isomorphism

$$G/[G,G] \cong H_1(G), \tag{3.18}$$

as is discussed in Example 2.1.7, gives the invariants of $H_1(G)$. The number of zeros is the rank of $H_1(G)$ and each positive integer n corresponds to a copy of $\mathbb{Z}_n$ in the torsion part of $H_1(G)$. □

This result can be extended to the case when the homology of G is taken with trivial coefficients in a finite field say k. In this case, the first homology group of G is denoted by $H_1(G;k)$ and is a finite dimensional vector space over k. The algorithm takes as input the finite lists S and T from the previous algorithm together with the order p of the finite field k. The output is an integer representing the dimension of the vector space $H_1(G;k)$.

3.1.3 The First Homology with Coefficients Algorithm

Algorithm 3.1.4. FirstHomologyCoefficients(F, R, p)

Input: Free Group F, Relators R, Prime $p = \mathrm{char}(k)$

Output: Dimension of the vector space $k \otimes H_1(G;k)$ over k

A :=FirstHomology(F, R)
$X := [\,]$
for $x \in A$ **do**
  if $x \equiv 0 \bmod p$ **then**
    append x to X
  end if
end for
return Size(X) {number of elements in the list X}

Theorem 3.1.5. *Let G be a finitely-presented group and k a finite field. Then* FirstHomologyCoefficients *returns the dimension of $H_1(G;k)$ as a vector space over k.*

Proof. Recall the universal coefficients [Bro94, p. 36] short exact sequence

$$1 \to k \otimes H_1(G) \to H_1(G;k) \to \mathrm{Tor}(H_0(G),k) \to 1 \tag{3.19}$$

where $H_0(G)$ is the free cyclic group (see Example 2.1.7) and $\mathrm{Tor}(-,k)$ is a functor vanishing on free abelian groups. Then Equation 3.19 reduces to

$$1 \to k \otimes H_1(G) \xrightarrow{\cong} H_1(G;k) \to 1 \tag{3.20}$$

Let A be the list of abelian invariants obtained from FirstHomology(F,R), where F/R is a presentation for G. Since G is finitely-presented, this list A will be finite. For $x \in A$, $x \otimes k = 0$ if and only if $x \equiv 0 \bmod p$, where p is the order of k. If we form a new list X of elements in A that do not vanish modulo p, the cardinality of this list will be the dimension of $k \otimes H_1(G) \cong H_1(G;k)$. □

3.1.4 Second Homology Group

Our investigation can be extended to the second homology group of G which is an abelian group that we denote $H_2(G)$. By Hopf's formula given in Theorem 2.1.8 this group fits into the following exact sequence:

$$1 \to [F,R] \to R \cap [F,F] \to H_2(G) \to 1 \tag{3.21}$$

where $[F,R]$ is the subgroup of F generated by the commutators $[f,r]$ with $f \in F$ and $r \in R$. The commutator formula

$$[x,y^z] = x^{-1}(y^{-1})^z x y^z = x^{-1}z^{-1}y^{-1}zx(yy^{-1})z^{-1}yz = [zx,y][y,z] \tag{3.22}$$

proves that $[F,R]$ is a finitely generated F-module under conjugation. However the intersection $R \cap [F,F]$ is infinitely generated and so is not determined by any algorithm, and we can only estimate the group $H_2(G)$ as a subgroup of the factor group $R/[F,R]$. This factor group is abelian since $[F,R]$ contains $[R,R]$ and if we let F act on it by conjugation then this action is trivial. In particular, since R is a finitely generated F-module it follows that the factor group $R/[F,R]$ is a finitely generated abelian group. Consequently, $H_2(G)$ is a finitely generated abelian group whose structure we would like to determine.

We start with the following exact sequence

$$1 \to H_2(G) \to \frac{R}{[F,R]} \to \frac{F}{[F,F]} \to \frac{F}{R[F,F]} \to 1 \tag{3.23}$$

in which the last two terms are deterministically determined as explained above. Moreover, starting with a finite list of generators T for the F-module R, we can design a deterministic algorithm to find a set of generators for $H_2(G)$.

To simplify the discussion, let k denote the finite field of prime order p and start our investigation with the homology with trivial coefficients in k. By the universal coefficients theorem we have a short exact sequence

$$1 \to k \otimes H_2(G) \to H_2(G;k) \to \mathrm{Tor}(H_1(G),k) \to 1 \tag{3.24}$$

whose last term can be determined as follows. For input we start with the abelian invariants of $H_1(G)$ found by Algorithm 3.1.1 together with the order p of the field k. The output is an integer representing the dimension of the vector space $\mathrm{Tor}(H_1(G),k)$ over k. The algorithm is deterministic.

3.1.5 The Tor Algorithm

Algorithm 3.1.6. $\textsc{Tor}(F,R,p)$

Input: Free Group F, Relators R, Prime $p = \mathrm{char}(k)$

Output: Dimension of $\mathrm{Tor}(H_1(G),k)$ over k

$A :=$ $\textsc{FirstHomology}(F,R)$
$X := [\,]$
for $x \in A$ **do**
    if $x \neq 0$ and $x \equiv 0 \bmod p$ **then**
        append x to X
    end if
end for
return Size(X)

The proof of the following theorem follows immediately from the properties of the $\mathrm{Tor}(-,k)$ functor.

Theorem 3.1.7. *Let G be a finitely-presented group and k a finite field. Then the* $\textsc{Tor}$ *algorithm returns the dimension of Tor$(H_1(G),k)$ as a vector space over k.*

The first term $k \otimes H_2(G)$ of the exact sequence 3.24 is a finite dimensional vector space over k whose dimension we only know how to estimate from above by an algorithm that we will describe next. From the exact sequence (3.23) we extract the short exact sequence

$$1 \to H_2(G) \to \frac{R}{[F,R]} \to \frac{R[F,F]}{[F,F]} \to 1 \tag{3.25}$$

whose last term is a subgroup of the free abelian group $F/[F,F]$. It is a standard fact that any subgroup of a finitely generated free abelian group is free abelian and consequently the above sequence splits. In particular, by tensoring with k we obtain a short exact sequence of vector spaces over k:

$$1 \to k \otimes H_2(G) \to k \otimes \frac{R}{[F,R]} \to k \otimes \frac{R[F,F]}{[F,F]} \to 1 \tag{3.26}$$

where the last term can be rewritten as $R[F,F]/R^p[F,F]$. Here R^p denotes the subgroup of F generated by the p-powers of elements of R. In particular, there is a short exact sequence of finitely generated abelian groups

$$1 \to k \otimes \frac{R[F,F]}{[F,F]} \to \frac{F}{R^p[F,F]} \to \frac{F}{R[F,F]} \to 1 \tag{3.27}$$

whose last two terms are computable by the FirstHomology and FirstHomologyCoefficients algorithms.

Definition 3.1.8. [Fai99, p. 6] For an abelian group A, define the p**-primary subgroup of** A to be

$${}_{p^\infty}(A) = \{a \in A \mid a^{p^i} = 1 \text{ for some } i > 0\}. \tag{3.28}$$

The order of this subgroup is of the form p^e. Call e the p^∞**-rank of** A.

The p^∞ rank of a finitely generated abelian group A can be calculated by taking as input the abelian invariants of A and the prime p.

By passing to p-primary subgroups, sequence 3.27 gives another short exact sequence

$$1 \to k \otimes \frac{R[F,F]}{[F,F]} \to_{p^\infty} \left(\frac{F}{R^p[F,F]}\right) \to_{p^\infty} \left(\frac{F}{R[F,F]}\right) \to 1 \tag{3.29}$$

since the first term is p-torsion. We observe that while $F/R[F,F]$ can be given in terms of S and T, the factor group $F/R^p[F,F]$ can be given in the same way but replacing T by T^p, the finite list of p-powers of elements in T.

3.1.6 The Rank Algorithm

Algorithm 3.1.9. PrimePrimaryRank(F, R, p)

Input: Free Group F, Relators R, Prime p

Output: p^∞-rank of F/R

$A :=$FirstHomology(F, R)
$Y := [\,]$
for $a \in A$ **do**
    if $a \neq 0$ and $a \equiv 0 \bmod p$ **then**
        $y :=$ p-adic valuation of a
        append y to Y
    end if
end for
$s :=$Sum(Y) $\{s$ is the sum of the elements of $Y\}$
return s

Once again, the following theorem is clear.

Theorem 3.1.10. *Let G be a finitely-presented group. The algorithm* PrimePrimaryRank *returns the p^∞-rank of $G/[G, G]$.*

Note 3.1.11. The GAP command `PadicValuation(n,p)` gives the p-adic valuation of an integer n.

We next describe an algorithm that reduces an element of a group via a rewriting system.

3.1.7 Reduce Word Algorithm

Algorithm 3.1.12. ReduceWord(F, R, Z, R', p)

Input: Free Group F, Relators R, Test Word z, Sublist R' of R, Prime p

Output: Reduced word of z in $F/[F, R]R^pR'$

$G := F/[F, R]R^pR'$
$RG :=$Rewriting system for G
$x :=$Reduced word of (z) in the rewriting system RG
return x

We use the rewriting system given by the Knuth-Bendix completion algorithm [KB70] implemented on GAP via the KBMAG package [Hol09].

3.1.8 The Find Basis Algorithm

Algorithm 3.1.13. FINDBASIS(F, R, p, R')

Input: Free Group F, Relators R, Prime p, Sublist R' of R

Output: Size of a generating set for $[F, R]R^pR'/[F, R]R^p$

```
X := R'
for x ∈ X do
    x' := REDUCEWORD(F, R, x, Difference(X, [x]), p) {Difference(A, B) is the complement of B in A}
    if x' = identity then
        X := Difference(X, [x])
    end if
end for
return Size(X)
```

The algorithm attempts to check for linear independence of each element x of R' with respect to $R' - \{x\}$ in $[F, R]R^pR'/[F, R]R^p$. Whenever x is found by the rewriting system to be dependent of $R' - \{x\}$, it is removed from R'. The end result will be a list of potentially linearly independent generators.

We conclude this discussion with the grand scheme algorithm which takes as input a finite list of generators S and a finite list of relators T for a group G together with a prime p and gives as output an integer d representing an upper bound for the dimension of $H_2(G; k)$, where k is a field of order p.

3.1.9 The Second Homology with Coefficients Algorithm

Algorithm 3.1.14. SECONDHOMOLOGYCOEFFICIENTS(F, R, p, R')

Input: Free Group F, Relators R, Prime p, Sublist R' of R generating $R/[F, R]R^p$

Output: An integer d such that dim $(H_2(G; k)) \leq d$

```
a := TOR(F, R, p)
b := PRIMEPRIMARYRANK(F, R[F, F], p)
c := PRIMEPRIMARYRANK(F, R^p[F, F], p)
e := FINDBASIS(F, R, p, R')
d := a + b − c + e
return d
```

Theorem 3.1.15. *Let G be a finitely-presented group. The algorithm* SecondHomologyCoefficients *returns a non-negative integer which is an upper bound for the dimension of $H_2(G; k)$ as a vector space over k.*

Proof. The exact sequence of vector spaces 3.24 implies that

$$\dim (k \otimes H_2(G)) = \dim\, H_2(G; k) - \dim\, \mathrm{Tor}(H_1(G), k). \tag{3.30}$$

Similarly, the exact sequence 3.26 gives

$$\dim (k \otimes H_2(G)) = \dim \left(k \otimes \frac{R}{[F, R]} \right) - \dim \left(k \otimes \frac{R[F, F]}{[F, F]} \right). \tag{3.31}$$

Thus equations 3.30 and 3.31 together show that

$$\dim\, H_2(G; k) - \dim\, \mathrm{Tor}(H_1(G), k) = \dim \left(k \otimes \frac{R}{[F, R]} \right) - \dim \left(k \otimes \frac{R[F, F]}{[F, F]} \right). \tag{3.32}$$

Finally, exact sequence 3.29 implies that

$$\dim \left(k \otimes \frac{R[F, F]}{[F, F]} \right) = \mathrm{rk}_{p^\infty} \left(\frac{F}{R^p[F, F]} \right) - \mathrm{rk}_{p^\infty} \left(\frac{F}{R[F, F]} \right). \tag{3.33}$$

Thus we can express the dimension of $H_2(G; K)$ as

$$\dim\, \mathrm{Tor}(H_1(G), k) + \mathrm{rk}_{p^\infty} \left(\frac{F}{R[F, F]} \right) - \mathrm{rk}_{p^\infty} \left(\frac{F}{R^p[F, F]} \right) + \dim \left(k \otimes \frac{R}{[F, R]} \right) \tag{3.34}$$

To summarize, let

$$\begin{aligned} a &= \text{dimension of } \mathrm{Tor}(H_1(G), k) \\ b &= p^\infty\text{-rank of } \frac{F}{R[F, F]} \\ c &= p^\infty\text{-rank of } \frac{F}{R^p[F, F]} \\ d &= \text{dimension of } H_2(G; k) \\ e &= \text{dimension of } k \otimes \frac{R}{[F, R]} \end{aligned}$$

where a is determined by the Tor Algorithm, b and c by PrimePrimaryRank algorithm, and e is yet to be studied. By the equations above, the following reduction formula holds:

$$d = a + b - c + e. \tag{3.35}$$

The integer e is estimated from above by an integer e' obtained via the algorithm FindBasis(F, R, p, R). Thus

$$d \leq a + b - c + e'. \tag{3.36}$$

□

Note 3.1.16. It is important to note that the reduction of test words in the algorithm ReduceWord is the word problem (for a description of the word problem see [Bri58]). As such, a result of a word not being the identity is an indeterminate result. However, if G is finite, or, more generally, if the rewriting is confluent, then the reduction in the rewriting system is deterministic and a basis is achieved (the confluence for finite groups is guaranteed in theory only; in practice it may take a long time or require more space than is available [KB70]). At any rate, this is not typically the case since the word problem is undecidable in general; thus the result of FindBasis is, in general, the cardinality of a generating set that is not necessarily a basis. Therefore in these cases we do not find the dimension of $H_2(G; k)$, only an upper bound.

3.1.10 Examples

In this section, we apply the grand scheme algorithm above to some select groups. The first example is to illustrate the effect the algorithm has on groups with smallish presentations with confluent rewriting systems. The other three examples are the groups of primary interest since they are relevant to the conjectures discussed in Section 1. In Section 3.1.11 we will discuss these calculations.

Example 3.1.17. The symmetric groups Σ_5 on 5 letters:

$$\begin{aligned} G &= \Sigma_5 \\ S &= \{a, b\} \\ T &= \{a^5, b^2, (a^{-1}b)^4, (a^2ba^{-2}b)^2\} \\ p &= 2 \\ d &= 2 \end{aligned}$$

Next we consider three linear groups over $\mathbb{Z}[1/p, \zeta_p]$, where ζ_p is a primitive pth-root of unity. Presentations for groups of this from can be found in [Ant09, p. 447, 453].

Example 3.1.18.

$$\begin{aligned}
G &= SL_2(\mathbb{Z}[1/3, \zeta_3]) \\
S &= \{z, u_1, a, b, b_0, b_1, b_2, w\} \\
T &= \{b_t^{-1} z^{3t} b z^{3t} a, w^{-1} z^4 u_1 u_2 u_3, z^3, [z, u_1], [u_1, u_1], a^4, [a^2, z], [a^2, u_1], \\
&\quad a^{-1} z a z, a^{-1} u_1 a u_1, [b_s, b_t], b^{-3} a^2, b^{-3} b_0 b_1 b_2, \\
&\quad (b_0 b_1^{-1} a^{-1} u_1)^3, a^{-2} b^{-1} u_1 b z^{-3} b^{-1} b_0^{-1} z^3 b z^{-1} u_1\} \\
p &= 3 \\
d &= 0
\end{aligned}$$

where $s, t \in \{1, 2\}$.

Example 3.1.19.

$$\begin{aligned}
G &= SL_2(\mathbb{Z}[1/5, \zeta_5]) \\
S &= \{z, u_1, u_2, a, b, b_0, b_1, b_2, b_3, b_4, w\} \\
T &= \{b_t^{-1} z^{3t} b z^{3t} a, w^{-1} z^4 u_1 u_2 u_3, z^5, [z, u_i], [u_i, u_j], a^4, [a^2, z], [a^2, u_i], \\
&\quad a^{-1} z a z, a^{-1} u_i a u_i, [b_s, b_t], b^{-3} a^2, b^{-3} b_0 b_1 b_2 b_3 b_4, \\
&\quad (b_0 b_1^{-1} a^{-1} u_1)^3, (b_0 b_2^{-1} a^{-1} u_2)^3, (b_0 b_3^{-1} a^{-1} u_3)^3, \\
&\quad (b_0 b_1^{-1} b_2^{-1} b_3 a^{-1} u_1 u_2)^3, (b_0 b_1^{-1} b_3^{-1} b_4 a^{-1} u_1 u_3)^3, \\
&\quad (b_0 b_2^{-1} b_3^{-1} b_5 a^{-1} u_2 u_3)^3, a^{-2} b^{-1} u_i b z^{-3i} b^{-1} b_0^{-1} z^{3i} b z^{-i} u_i\} \\
p &= 5 \\
d &= 0
\end{aligned}$$

where $i, j \in \{1, 2\}$ and $s, t \in \{1, 2, 3, 4\}$.

Example 3.1.20.

$$\begin{aligned}
G &= SL_2(\mathbb{Z}[1/7,\zeta_7]) \\
S &= \{z, u_1, u_2, u_3, a, b, b_0, b_1, b_2, b_3, b_4, b_5, b_6, w\} \\
T &= \{b_t^{-1}z^{3t}bz^{3t}a, w^{-1}z^4u_1u_2u_3, z^7, [z,u_i], [u_i,u_j], a^4, [a^2,z], [a^2,u_i], \\
&\quad a^{-1}zaz, a^{-1}u_iau_i, [b_s,b_t], b^{-3}a^2, b^{-3}b_0b_1b_2b_3b_4b_5b_6, b_t^{-7}w^{-1}b_t^{-1}w, \\
&\quad (b_0b_1^{-1}a^{-1}u_1)^3, (b_0b_2^{-1}a^{-1}u_2)^3, (b_0b_3^{-1}a^{-1}u_3)^3, \\
&\quad (b_0b_1^{-1}b_2^{-1}b_3a^{-1}u_1u_2)^3, (b_0b_1^{-1}b_3^{-1}b_4a^{-1}u_1u_3)^3, (b_0b_2^{-1}b_3^{-1}b_5a^{-1}u_2u_3)^3, \\
&\quad (b_0b_1^{-1}b_2^{-1}b_3b_4b_5b_6^{-1}a^{-1}u_1u_2u_3)^3, a^{-2}b^{-1}u_ibz^{-3i}b^{-1}b_0^{-1}z^{3i}bz^{-i}u_i\} \\
p &= 7 \\
d &= 6
\end{aligned}$$

where $i,j \in \{1,2,3\}$ and $s,t \in \{1,2,3,4,5,6\}$.

3.1.11 Discussion

Details on the above examples are as follows:

- **Example 3.1.17:** The rewriting system given by the KBMAG package for Σ_5 is confluent; therefore

$$\dim\ H_2(\Sigma_5;\mathbb{F}_2) = 2. \tag{3.37}$$

  The algorithm took about 50 milliseconds to run, reflecting the relatively simple presentation.

- **Example 3.1.18:** The rewriting system given by the KBMAG package for $SL_2(\mathbb{Z}[1/3,\zeta_3])$ is not confluent; the algorithm took about six hours to finish. In this case, the non-confluence of the system did not affect the results as the rewriting system was able to show that all elements of R reduced to identity modulo $[F,R]R^3$, so

$$\dim\ H_2(SL_2(\mathbb{Z}[1/3,\zeta_3];\mathbb{F}_3) = 0. \tag{3.38}$$

- **Example 3.1.19:** The rewriting system given by the KBMAG package for $SL_2(\mathbb{Z}[1/5,\zeta_5])$ is not confluent. As in Example 2 the non-confluence of the system did not affect the results and

$$\dim\ H_2(SL_2(\mathbb{Z}[1/5,\zeta_5]) = 0. \tag{3.39}$$

The algorithm took about two days to finish.

- **Example 3.1.20:** The rewriting system given by the KBMAG package for $SL_2(\mathbb{Z}[1/7, \zeta_7])$ is not confluent. In this case, the algorithm took a total of about five days to finish. Also, in this case the non-confluence actually mattered. Since the algorithms were not able to show that the dimension of $R/[F, R]R^7$ is 0, we only have the upper bound

$$\dim\ H_2(SL_2(\mathbb{Z}[1/7, \zeta_7]); \mathbb{F}_7) \leq 6. \tag{3.40}$$

In implementing these algorithms to find a bound on $H_2(G)$ it is useful to first perform Tietze transforms on the presentations involved to attempt to simplify the presentations. A description of Tietze transformations can be found in [LS01, pp 89-99]. In many cases, the number of generators and relators can be reduced, thus simplifying the calculations. In Example 3.1.20 $SL_2(\mathbb{Z}[1/7, \zeta_7])$ is given via a presentation consisting of 14 generators and 64 relators. A series of Tietze transforms, implemented via GAP, simplifies to a presentation with six generators and 34 relators, and the GAP output of performing this operations is given in Example 4.0.2. This significantly impacts the results of the algorithm.

Finally, we note that for Examples 3.1.19 and 3.1.20, it was necessary to run the algorithm several times to obtain the results above since the parameters of the KBMAG package allow a limited number of equations to be generated in the rewriting system. Each iteration eliminated elements of R from the generating list until the results stabilized. For instance, in Example 3.1.20, the initial iteration gave a result of $e \leq 16$ and $d \leq 10$, the second iteration gave that $e \leq 13$ and $d \leq 7$. The third and fourth iterations each gave a result of $e \leq 12$ and so the upper bound on d is 6. We illustrate this process of iterating Algorithm 3.1.14 in Appendix 4.

3.2 Calculations

In this section we present some additional sample calculations using the algorithms given above. Appendix 4.2 lists the homology of these groups from various sources. We invite the reader to compare the results of our calculations with the calculations found by other means.

3.2.1 Presentations of Groups

The groups below are organized by the references where the presentations can be found, and the GAP code to input the groups may be found in Appendix 4.1.

1. [Alp80]

   a) $SL_2(\mathbb{Z}[\omega]) = \langle a, b | a^3 = b^2 = (ab)^2 = c^3 = (c^{-1}ab)^3 = c^{-3} = (cb^{-1}a)^3 \rangle$, where $\omega = \sqrt[3]{-1}$

2. [Ant09]

   a) $SL_2\left(\mathbb{Z}[1/p, \sqrt[p]{1}]\right)$ for any odd prime p

3. [Bro94]

   a) $SL_2(\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 = \langle a, b | a^4, b^6, a^2 = b^3 \rangle$

   b) $SL_2(\mathbb{F}_2) = \mathbb{Z}_3 \rtimes Z_2 = \langle a, b | a^3, b^2, ab^{-1}a = b \rangle$

   c) $SL_2(\mathbb{F}_3)$ (binary tetrahedral group) $= \langle a, b | (ab)^2 = a^3 = b^3 \rangle$

   d) $SL_2(\mathbb{F}_5)$ (binary icosahedral group) $= \langle a, b | (ab)^2 = a^3 = b \rangle$

4. [Joh90]

   a) $GL_2(\mathbb{Z}) = \langle a, b, c | aba = bab, (aba)^4, c^2, (ca)^2, (cb)^2 \rangle$

   b) $PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b | a^2, b^3 \rangle$

5. [Swa71]

   a) $SL_2(\mathbb{Z}[i]) = \langle a, b, c, d, e, | bc = cb, a^2, a$ central $, d^3 = a, (bd)^2 = a, (cd)^2 = a, (ed)^2 = a, (be)^3 = a, (ced)^3 = a \rangle$

   b) $SL_2\left(\mathbb{Z}[\sqrt{-5}]\right) = \langle a, b, c, d, e, f | a^2, a$ central $, bc = cb, d^2 = a, e^2 = a, (bd)^3 = a, (de)^2 = a, (dcec^{-1}) = a, dfd = abfb^{-1}, cec^{-1}fe = abfb^{-1} \rangle$

3.2.2 Homology Calculations

The next two tables give the results of the algorithms in Section 3.1 applied to the groups listed above. For the second table, a "less than" symbols indicates that the rewriting system involved in the calculation was not confluent, so only an upper bound was found. Otherwise, the rewriting was confluent and the exact dimension was found.

	$H_1(-;\mathbb{F}_2)$	$H_1(-;\mathbb{F}_3)$	$H_1(-;\mathbb{F}_5)$	$H_1(-;\mathbb{F}_7)$
$GL_2(\mathbb{Z})$	2	0	0	0
$SL_2(\mathbb{Z})$	1	1	0	0
$SL_2(\mathbb{Z}_2)$	1	0	0	0
$SL_2(\mathbb{Z}_3)$	0	1	0	0
$SL_2(\mathbb{Z}_5)$	0	0	1	0
$SL_2(\mathbb{Z}[i])$	1	0	0	0
$SL_2(\mathbb{Z}[\omega])$	0	1	0	0
$SL_2(\mathbb{Z}[\sqrt{-5}])$	3	2	1	1
$PSL_2(\mathbb{Z})$	1	1	0	0

Table 3.1: Dimensions of First Homology Groups

	$H_2(-;\mathbb{F}_2)$	$H_2(-;\mathbb{F}_3)$	$H_2(-;\mathbb{F}_5)$	$H_2(-;\mathbb{F}_7)$
$GL_2(\mathbb{Z})$	≤ 4	≤ 2	≤ 2	≤ 2
$SL_2(\mathbb{Z})$	≤ 2	≤ 2	≤ 1	≤ 1
$SL_2(\mathbb{Z}_2)$	1	0	0	0
$SL_2(\mathbb{Z}_3)$	0	1	0	0
$SL_2(\mathbb{Z}_5)$	0	0	1	0
$SL_2(\mathbb{Z}[i])$	1	0	0	0
$SL_2(\mathbb{Z}[\omega])$	≤ 1	≤ 2	≤ 1	≤ 1
$SL_2(\mathbb{Z}[\sqrt{-5}])$	≤ 3	≤ 3	0	0
$PSL_2(\mathbb{Z})$	≤ 1	≤ 1	0	0

Table 3.2: Dimensions of Second Homology Groups

Chapter 4 Conclusion

In the context of Conjecture 1.0.2 the algorithms in Chapter 3 were able to show that the second homology of $SL_2(\mathbb{Z}[1/p, \zeta_p])$, with coefficients in the finite field $\mathbb{F}_p$, is trivial for the cases that $p = 3$ and 5. If we set $R = \mathbb{Z}[1/p, \zeta_p]$ then Equation 3.5, which gives that

$$H_2(GL_2(R); \mathbb{F}_p) = H_2(SL_2(R); \mathbb{F}_p)_{GL_1(R)}/Im(\tau) \oplus H_2(GL_1(R); \mathbb{F}_p),$$

where τ is a transgression map, implies that

$$\dim H_2(GL_2(R); \mathbb{F}_p) = \dim H_2(GL_1(R); \mathbb{F}_p);$$

thus the conjecture is true in the cases that $p = 3$ and 5. This confirms results found (by other methods) in [Ant99] and [Ant09].

For $p = 7$, the algorithms were unable to show that $SL_2(\mathbb{Z}[1/7, \zeta_7])$ has trivial second dimensional $\mathbb{F}_7$-homology. But the following new result was obtained.

Theorem. *The dimension of $H_2(SL_2(\mathbb{Z}[1/7, \zeta_7]); \mathbb{F}_7)$ as a vector space over $\mathbb{F}_7$ is at most six.*

Our future work will involve refining and improving the algorithms above. Initially we were concerned only with writing algorithms that gave results-the efficiency of these algorithms was not a concern. To this end we will analyze, in a rigorous manner, the complexity of the algorithms. For the linear groups above as p increases the number of relators grows exponentially. Going from $p = 3$ to $p = 7$, the time required increased from several hours to several days. For $p = 11$ the algorithm dramatically fails to produce significant results since the number of relators overwhelms the possible bounds allowed by the rewriting systems utilized by the KBMAG package.

We are also developing methods for finding generators of $H_2(G)$ and $H_2(G; k)$ independent from those above. In particular, we attempt to find *lower* bounds on the dimension of $H_2(G; k)$. The strategies for both problems will be based on linear algebra involving rewriting systems and will appear in a future work.

In particular, recall the general form of the decomposition above:

$$H_2(GL_2(R); k) \cong H_2(SL_2(R); k)/Im(\tau) \oplus H_2(GL_1(R); k),$$

where k is a finite field of coefficients and $\tau : H_3(GL_1(R); k) \to H_2(SL_2(R); k)$ is the transgression map. Recall that τ is an epimorphism is equivalent to the conjecture of Anton. In light of these facts, we are developing methods to calculate τ.

Moreover, the algorithms in this dissertation relied heavily on Hopf's formula for a group G given as F/R;

$$H_2(G) \cong \frac{R \cap [F, F]}{[F, R]}.$$

In this context, we plan to investigate generalized Hopf's formulae [Stö89] to attempt to extend the algorithms to higher homology groups. In sufficiently high dimensions the homology of $SL_2(\mathbb{Z}[\sqrt[p]{1}])$ is computable by various spectral sequences, but in low dimensions our calculations are new and highly nontrivial. To this end, we will show how to obtain a finite presentation for $SL_2(\mathbb{Z}[\zeta_p])$ from a finite presentation of $SL_2([\mathbb{Z}[1/p, \zeta_p])$ and extend, using our algorithms, the (short) list of known homology groups.

Appendices

Appendix 1

In this appendix we give the code used in GAP to input and implement the finitely-presented groups and algorithms described in Chapter 3.

Library of Groups

Here we list the code used to input the various finitely-presented groups discussed in this dissertation into GAP. They are given as a list of generators for a free group followed by a list of relators. The group itself is given as a quotient. The first three groups listed are the ones relevant to the conjectures of Quillen and Anton.

```
###############################################
#Case p=3
F3:=FreeGroup(8);
z_3:=F3.1;
u1_3:=F3.2;
a_3:=F3.3;
b_3:=F3.4;
b0_3:=F3.5;
b1_3:=F3.6;
b2_3:=F3.7;
w_3:=F3.8;

k3:=[b0_3^-1*b_3*a_3,
b1_3^-1*z_3*b_3*z_3*a_3,
b2_3^-1*z_3^2*b_3*z_3^2*a_3,
w_3^-1*z_3*u1_3,
z_3^3,
z_3*u1_3*z_3^-1*u1_3^-1,
a_3^4,
a_3^2*z_3*a_3^-2*z_3^-1,
a_3^2*u1_3*a_3^-2*u1_3^-1,
z_3*a_3*z_3*a_3^-1,
```

```
u1_3*a_3*u1_3*a_3^-1,
Comm(b0_3,b1_3),
Comm(b0_3,b2_3),
Comm(b1_3,b2_3),
b_3^3*a_3^-2,
b0_3*b1_3*b2_3*a_3^-2,
b0_3^-3*w_3^-1*b0_3^-1*w_3,
b1_3^-3*w_3^-1*b1_3^-1*w_3,
b2_3^-3*w_3^-1*b2_3^-1*w_3,
(b0_3*b1_3^-1*a_3^-1*u1_3)^3,
a_3^2*b_3^-1*u1_3*b_3*z_3^2*b_3^-1*b0_3^-1*z_3*b_3*z_3^2*u1_3];

G3:=F3/k3;

##############################################
#Case p=5
F5:=FreeGroup(11);
z_5:=F5.1;
u1_5:=F5.2;
u2_5:=F5.3;
a_5:=F5.4;
b_5:=F5.5;
b0_5:=F5.6;
b1_5:=F5.7;
b2_5:=F5.8;
b3_5:=F5.9;
b4_5:=F5.10;
w_5:=F5.11;

k5:=[b0_5^-1*b_5*a_5,
b1_5^-1*z_5^2*b_5*z_5^2*a_5,
b2_5^-1*z_5^4*b_5*z_5^4*a_5,
b3_5^-1*z_5*b_5*z_5*a_5,
b4_5^-1*z_5^3*b_5*z_5^3*a_5,
w_5^-1*z_5*u1_5*u2_5,
z_5^5, z_5*u1_5*z_5^-1*u1_5^-1,
```

```
z_5*u2_5*z_5^-1*u2_5^-1,
u1_5*u2_5*u1_5^-1*u2_5^-1,
a_5^4,
a_5^2*z_5*a_5^-2*z_5^-1,
a_5^2*u1_5*a_5^-2*u1_5^-1,
a_5^2*u2_5*a_5^-2*u2_5^-1,
z_5*a_5*z_5*a_5^-1,
u1_5*a_5*u1_5*a_5^-1,
u2_5*a_5*u2_5*a_5^-1,
b0_5*b1_5*b0_5^-1*b1_5^-1,
b0_5*b2_5*b0_5^-1*b2_5^-1,
b0_5*b3_5*b0_5^-1*b3_5^-1,
b0_5*b4_5*b0_5^-1*b4_5^-1,
b1_5*b2_5*b1_5^-1*b2_5^-1,
b1_5*b3_5*b1_5^-1*b3_5^-1,
b1_5*b4_5*b1_5^-1*b4_5^-1,
b2_5*b3_5*b2_5^-1*b3_5^-1,
b2_5*b4_5*b2_5^-1*b4_5^-1,
b3_5*b4_5*b3_5^-1*b4_5^-1,
b_5^3*a_5^-2,
b0_5*b1_5*b2_5*b3_5*b4_5*a_5^-2,
b0_5^-5*w_5^-1*b0_5*w_5,
b1_5^-5*w_5^-1*b1_5*w_5,
b2_5^-5*w_5^-1*b2_5*w_5,
b3_5^-5*w_5^-1*b3_5*w_5,
b4_5^-5*w_5^-1*b4_5*w_5,
(b0_5*b1_5^-1*a_5^-1*u1_5)^3,
(b0_5*b2_5^-1*a_5^-1*u2_5)^3,
(b0_5*b1_5^-1*b2_5^-1*b3_5*a_5^-1*u1_5*u2_5)^3,
a_5^2*b_5^-1*u1_5*b_5*z_5^3*b_5^-1*b0_5^-1*z_5^2*b_5*z_5^4*u1_5,
a_5^2*b_5^-1*u2_5*b_5*z_5*b_5^-1*b0_5^-1*z_5^4*b_5*z_5^3*u2_5];

G5:=F5/k5;

################################################
#Case p=7
```

```
F7:=FreeGroup(14);
z_7:=F7.1;
u1_7:=F7.2;
u2_7:=F7.3;
u3_7:=F7.4;
a_7:=F7.5;
b_7:=F7.6;
b0_7:=F7.7;
b1_7:=F7.8;
b2_7:=F7.9;
b3_7:=F7.10;
b4_7:=F7.11;
b5_7:=F7.12;
b6_7:=F7.13;
w_7:=F7.14;

k7:=[b0_7^-1*b_7*a_7,
b1_7^-1*z_7^3*b_7*z_7^3*a_7,
b2_7^-1*z_7^6*b_7*z_7^6*a_7,
b3_7^-1*z_7^2*b_7*z_7^2*a_7,
b4_7^-1*z_7^5*b_7*z_7^5*a_7,
b5_7^-1*z_7*b_7*z_7*a_7,
b6_7^-1*z_7^4*b_7*z_7^4*a_7,
w_7^-1*z_7^4*u1_7*u2_7*u3_7,
z_7^7,
z_7*u1_7*z_7^-1*u1_7^-1,
z_7*u2_7*z_7^-1*u2_7^-1,
z_7*u3_7*z_7^-1*u3_7^-1,
u1_7*u2_7*u1_7^-1*u2_7^-1,
u1_7*u3_7*u1_7^-1*u3_7^-1,
u2_7*u3_7*u2_7^-1*u3_7^-1,
a_7^4,
a_7^2*z_7*a_7^-2*z_7^-1,
a_7^2*u1_7*a_7^-2*u1_7^-1,
a_7^2*u2_7*a_7^-2*u2_7^-1,
a_7^2*u3_7*a_7^-2*u3_7^-1,
```

```
z_7*a_7*z_7*a_7^-1,
u1_7*a_7*u1_7*a_7^-1,
u2_7*a_7*u2_7*a_7^-1,
u3_7*a_7*u3_7*a_7^-1,
b0_7*b1_7*b0_7^-1*b1_7^-1,
b0_7*b2_7*b0_7^-1*b2_7^-1,
b0_7*b3_7*b0_7^-1*b3_7^-1,
b0_7*b4_7*b0_7^-1*b4_7^-1,
b0_7*b5_7*b0_7^-1*b5_7^-1,
b0_7*b6_7*b0_7^-1*b6_7^-1,
b1_7*b2_7*b1_7^-1*b2_7^-1,
b1_7*b3_7*b1_7^-1*b3_7^-1,
b1_7*b4_7*b1_7^-1*b4_7^-1,
b1_7*b5_7*b1_7^-1*b5_7^-1,
b1_7*b6_7*b1_7^-1*b6_7^-1,
b2_7*b3_7*b2_7^-1*b3_7^-1,
b2_7*b4_7*b2_7^-1*b4_7^-1,
b2_7*b5_7*b2_7^-1*b5_7^-1,
b2_7*b6_7*b2_7^-1*b6_7^-1,
b3_7*b4_7*b3_7^-1*b4_7^-1,
b3_7*b5_7*b3_7^-1*b5_7^-1,
b3_7*b6_7*b3_7^-1*b6_7^-1,
b4_7*b5_7*b4_7^-1*b5_7^-1,
b4_7*b6_7*b4_7^-1*b6_7^-1,
b5_7*b6_7*b5_7^-1*b6_7^-1,
b_7^3*a_7^-2,
b0_7*b1_7*b2_7*b3_7*b4_7*b5_7*b6_7*a_7^-2,
b0_7^-7*w_7^-1*b0_7^-1*w_7,
b1_7^-7*w_7^-1*b1_7^-1*w_7,
b2_7^-7*w_7^-1*b2_7^-1*w_7,
b3_7^-7*w_7^-1*b3_7^-1*w_7,
b4_7^-7*w_7^-1*b4_7^-1*w_7,
b5_7^-7*w_7^-1*b5_7^-1*w_7,
b6_7^-7*w_7^-1*b6_7^-1*w_7,
(b0_7*b1_7^-1*a_7^-1*u1_7)^3,
(b0_7*b2_7^-1*a_7^-1*u2_7)^3,
```

```
(b0_7*b3_7^-1*a_7^-1*u3_7)^3,
(b0_7*b1_7^-1*b2_7^-1*b3_7*a_7^-1*u1_7*u2_7)^3,
(b0_7*b1_7^-1*b3_7^-1*b4_7*a_7^-1*u1_7*u3_7)^3,
(b0_7*b2_7^-1*b3_7^-1*b5_7*a_7^-1*u2_7*u3_7)^3,
(b0_7*b1_7^-1*b2_7^-1*b4_7*b5_7*b6_7^-1*a_7^-1*u1_7*u2_7*u3_7)^3,
a_7^2*b_7^-1*u1_7*b_7*z_7^-3*b_7^-1*b0_7^-1*z_7^3*b_7*z_7^-1*u1_7,
a_7^2*b_7^-1*u2_7*b_7*z_7*b_7^-1*b0_7^-1*z_7^-1*b_7*z_7^-2*u2_7,
a_7^2*b_7^-1*u3_7*b_7*z_7^-2*b_7^-1*b0_7^-1*z_7^2*b_7*z_7^-3*u3_7];

G7:=F7/k7;

##############################################
#GL_2(Z)
Free6:=FreeGroup(3);
a6:=Free6.1;
b6:=Free6.2;
c6:=Free6.3;

R6:=[a6*b6*a6*b6^-1*a6^-1*b6^-1,
(a6*b6*a6)^4,
c6^2,
(c6*a6)^2,
(c6*b6)^2];

GL_2Z:=Free6/R6;

##############################################
#SL_2(Z)
S:=FreeGroup(2);
a:=S.1;
b:=S.2;

rels:=[a^4,
b^6,
a^2*b^-3];
```

```
SL2:=S/rels;

##############################################
#SL_2(Z_2)
Free0:=FreeGroup(2);
a0:=Free0.1;
b0:=Free0.2;

R0:=[a0^3,
b0^2,
a0*b0^-1*a0*b0^-1];

SL_2Z_2:=Free0/R0;

##############################################
#SL_2(Z_3)
Free3:=FreeGroup(2);
a3:=Free3.1;
b3:=Free3.2;

R3:=[(a3*b3)^2*a3^-3,
a3^3*b3^-3];

SL_2Z_3:=Free3/R3;

##############################################
#SL_2(Z_5)
Free5:=FreeGroup(2);
a5:=Free5.1;
b5:=Free5.2;

R5:=[(a5*b5)^2*a5^-3,
a5^3*b5^-1];

SL_2Z_5:=Free5/R5;
```

```
###############################################
#SL_2Zi
Free8:=FreeGroup(5);
a8:=Free8.1;
b8:=Free8.2;
c8:=Free8.3;
d8:=Free8.4;
e8:=Free8.5;

R8:=[b8*c8*b8^-1*c8^-1,
a8^2,
d8^3*a8^-1,
(b8*d8)^2*a8^-1,
(c8*d8)^2*a8^-1,
(e8*d8)^2*a8^-1,
(b8*e8)*a8^-1,
(c8*e8*d8)^3*a8^-1,
Comm(a8,b8),
Comm(a8,c8),
Comm(a8,d8),
Comm(a8,e8)];

SL_2Zi:=Free8/R8;

###############################################
#SL_2(w), w^3=-1
Free1:=FreeGroup(3);
a1:=Free1.1;
b1:=Free1.2;
c1:=Free1.3;

R1:=[a1^3*b1^-2,
(a1*b1)^2*c1^-3,
(c1^-1*a1*b1)^3*c1^3,
(c1*b1^-1*a1)^3*c1^3];
```

```
SL_2w:=Free1/R1;

##############################################
#SL_2(Z[sqrt(-5)]
Free9:=FreeGroup(6);
a9:=Free9.1;
b9:=Free9.2;
c9:=Free9.3;
d9:=Free9.4;
e9:=Free9.5;
f9:=Free9.6;

R9:=[a9^2,
b9*c9*b9^-1*c9^-1,
d9^2*a9^-1,
e9^2*a9^-1,
(b9*d9)^3*a9^-1,
(d9*e9)^2*a9^-1,
d9*c9*e9*c9^-1*a9^-1,
d9*f9*d9*b9*f9^-1*b9^-1*a9^-1,
c9*e9*c9^-1*e9^-1*f9^-1*b9*f9^-1*b9^-1*a9^-1,
Comm(a9,b9),
Comm(a9,c9),
Comm(a9,d9),
Comm(a9,e9),
Comm(a9,f9)];

SL_2Zneg5:=Free9/R9;

##############################################
#PSL_2(Z)
Free7:=FreeGroup(2);
a7:=Free7.1;
b7:=Free7.2;

R7:=[a7^2,
```

```
b7^3];

PSL_2Z:=Free7/R7;

###############################################
```

The Algorithms

The first set of algorithms outlined below are those discussed in Section 3.1. Following these are two algorithms that have been useful to (i) find presentations for finite groups and (ii) simplify presentations when one is already given.

```
###############################################
#Input: Free group, relators, prime p
#Output: Dimension of H_1(G;F_p)

FirstHomologyCoefficients:=function(Freegroup,Relators,Prime)
local AbelInv, list, x;
AbleInv:=AbelianInvariants(Freegroup/Relators);
list:=[];
for x in AbelInv do
     if x mod Prime = 0 then Add(list,x);
     fi;
     od;
return Size(list);
end;;

###############################################
#Input: Free group, relators, prime p
#Output: Dimension of Tor(H_1(G),F_p)

Tor:=function(Freegroup,Relators,Prime)
local AbelInv, list1, list2, x;
AbelInv:=AbelianInvariants(Freegroup/Relators);
list1:=[];
list2:=[];
for x in AbelInv do
```

```
     if x<>0 then Add(list1,x);
     fi;
     od;
for x in list1 do
     if x mod Prime = 0 then Add(list2,1);
     fi;
     od;
return Sum(list2);
end;;

##############################################
#P-Primary Rank
#Output: P-Primary Rank of Fp Group

PrimePrimaryRank:=function(Freegroup,Relators,Prime)
local AbelInv, list1, list2, x;
AbelInv:=AbelianInvariants(Freegroup/Relators);
list1:=[];
list2:=[];
for x in AbelInv do
     if x <> 0 then Add(list1,x);
     fi;
     od;
for x in list1 do
     if x mod Prime = 0 then Add(list2,PadicValuation(x,Prime));
     fi;
     od;
return Sum(list2);
end;;

##############################################
#The (special) Word Problem
#Output: Reduced word

Reduce_Word:=function(Freegroup,Relators,TestWord,Sublist,Prime)
local Rel_P, GroupGen, comm, G, RG, OR;
```

```
Rel_P:=List(Relators,x->x^Prime);
GroupGen:=GeneratorsOfGroup(Freegroup);
comm:=ListX(GroupGen,Relators,Comm);

G:=Freegroup/Concatenation(comm,Rel_P,Sublist);
RG:=KBMAGRewritingSystem(G);
OR:=OptionsRecordOfKBMAGRewritingSystem(RG);
OR.maxeqns:=500000;
OR.tindyint:=100;
MakeConfluent(RG);

return ReducedWord(RG,TestWord);
end;;

##############################################
#Attempts to reduce a generating set
#Output: List of generators

FindBasis:=function(Freegroup,Relators,Prime,Sublist)
local Gen,TestWord,x,list;
list:=[];
Gen:=Sublist;
for x in Sublist do
     TestWord:=Reduce_Word(Freegroup,Relators,x,
     Difference(Gen,[x]),Prime);
     Add(list,TestWord);
 if IsOne(TestWord)=true then Gen:=Difference(Gen,[x]);
     fi;
     od;
return [Gen, Size(Gen), list];
end;;

##############################################
#Gives the estimate for H_2
#Output: Upper bound on dimension of H_2
```

```
SecondHomologyCoefficients:=function(Freegroup, Relators, Prime,
    Sublist)

local a,b,c,d,e,f,ff,RPrime;

f:=GeneratorsOfGroup(Freegroup);
ff:=ListX(f,f,Comm);
RPrime:=List(Relators,x->x^Prime);

a:=Tor(Freegroup,Relators,Prime);
b:=PrimePrimaryRank(Freegroup,Concatenation(Relators,ff),Prime);
c:=PrimePrimaryRank(Freegroup,Concatenation(RPrime,ff),Prime);
e:=FindBasis(Freegroup,Relators,Prime,Sublist);
d:=a+b-c+e[2];
return [e[1],d,e[3]];
end;;

##############################################
```

The next two algorithms are useful in finding an isomorphic finitely-presented group from a finite group and for reducing the number of generators and relators via Tietze transformations of a given finitely-presented group. The first algorithm, PresentationOfFiniteGroup, receives as input a finite group G and outputs a list $\{G', F, R\}$ where G' is a finitely-presented group isomorphic to G, and

$$R \to F \to G' \tag{4.1}$$

is a presentation of G'.

The second algorithm PresentationOfFpGroup takes as input a finitely-presented group G and outputs a list $\{\phi, G', F, R\}$ where ϕ is an isomorphism $G \to G'$ and

$$R \to F \to G' \tag{4.2}$$

is a presentation of G'.

```
##############################################
#Presentation of a finite group
#Output: [Isomorphic Fp Group, Free Group, Relators]
```

```
PresentationOfFiniteGroup:=function(FiniteGroup)

local Pres, Group, Freegroup, Relators, group;

Pres:=PresentationViaCosetTable(FiniteGroup);
TzGoGo(Pres);
Group:=FpGroupPresentation(Pres);
Freegroup:=FreeGroupOfFpGroup(Group);
Relators:=RelatorsOfFpGroup(Group);

group:=Freegroup/Relators;

return[group,Freegroup,Relators];
end;;

###############################################
#Simplified Presenation of a Fp Group
#Output: [("smaller") Isomorphic Fp group, Free Group, Relators]
PresentationOfFpGroup:=function(FpGroup)

local iso, range, Group, Freegroup, Relators;

iso:=IsomorphismSimplifiedFpGroup(FpGroup);
range:=Range(iso);
Group:=range;
Freegroup:=FreeGroupOfFpGroup(range);
Relators:=RelatorsOfFpGroup(range);

return[iso,Group,Freegroup,Relators];
end;;
###############################################
```

Example 4.0.1. A presentation for the symmetric group on five letters, which we have denoted by Σ_5, can be found using the following procedure on GAP.

```
gap> S5:=SymmetricGroup(5);
Sym( [ 1 .. 5 ] )
```

```
gap> P:=PresentationOfFiniteGroup(S5);
#I  there are 2 generators and 4 relators of total length 27
[ <fp group on the generators [ f1, f2 ]>,
  <free group on the generators [ f1, f2 ]>,
  [ f2^2, f1^5, f1^-1*f2*f1^-1*f2*f1^-1*f2*f1^-1*f2,
      f1^2*f2*f1^-2*f2*f1^2*f2*f1^-2*f2 ] ]
```

Therefore Σ_5 has a presentation $\langle F|R\rangle$ where F is the free group on $\{x, y\}$ and

$$R = \{y^2, x^5, (x^{-1}y)^4, (x^2yx^{-2}y)^2\} \tag{4.3}$$

Example 4.0.2. As noted in Section 3.1.10, a presentation for $SL_2(\mathbb{Z}[1/7, \zeta_7])$, where ζ_7 is primitive 7th root of unity is given in [Ant09]. As given above, this presentation has 14 generators and 64 relators. We can find an isomorphic presentation with fewer generators and relators using the following procedure on GAP.

First, we input a presentation for $SL_2(\mathbb{Z}[1/7, \zeta_7])$ by letting `F7` and `k7` be given as above to obtain:

```
gap> F7/k7;
<fp group on the generators [ f1, f2, f3, f4, f5, f6, f7, f8, f9,
f10, f11, f12, f13, f14 ]>
```

By using the PresentationOfFpGroup algorithm we can find an isomorphic presentation with fewer generators and relators.

```
gap> P:=PresentationOfFpGroup(F7/k7);
[ [ f1, f2, f3, f4, f5, f6, f7, f8, f9, f10, f11, f12, f13, f14 ] ->
    [ f1, f2, f3, f4, f5, f1^-3*f8*f1^3*f5^-1, f1^-3*f8*f1^3, f8,
      f1^3*f8*f1^-3, f1^-1*f8*f1, f1^2*f8*f1^-2, f1^-2*f8*f1^2,
      f1*f8*f1^-1, f1^-2*f2*f1^-1*f3*f4 ],
  <fp group on the generators [ f1, f2, f3, f4, f5, f8 ]>,
  <free group on the generators [ f1, f2, f3, f4, f5, f8 ]>,
  [ f1*f4*f1^-1*f4^-1, f3*f4*f3^-1*f4^-1, f2*f3*f2^-1*f3^-1,
      f4*f5*f4*f5^-1,
      f2*f5*f2*f5^-1, f1*f3*f1^-1*f3^-1, f5^4, f2*f4*f2^-1*f4^-1,
      f1*f5*f1*f5^-1, f1*f2*f1^-1*f2^-1, f3*f5*f3*f5^-1, f1^7,
      f8*f1^-1*f8*f1*f8^-1*f1^-1*f8^-1*f1,
      f8*f1^-2*f8*f1^2*f8^-1*f1^-2*f8^-1*f1^2,
```

```
f1^-3*f8*f1^-1*f5^-1*f8*f1^-1*f5^-1*f8*f1^3*f5,
f8*f1^-3*f8*f1^-2*f8^-2*f1^-1*f5^-1*f4*f5^-1*f1^-1*f8^-1*f4,
f8*f1^-1*f8^-2*f1^-1*f8*f5^-1*f1^3*f3*f5^-1*f1^-2*f8^-1*f3,
f8*f1^-1*f8*f1^-3*f8^-2*f1^-1*f2^-1*f5^-2*f1^-2*f8^-1*f2,
f8*f1^-3*f8*f1^3*f8^-1*f1^-3*f8^-1*f1^3,
f1^-1*f8^-7*f1*f3^-1*f1*f4^-1*f2^-1*f1*f8^-1*f1^-1*f2*f1^-1
      *f3*f4,
f8^-7*f3^-1*f1*f4^-1*f2^-1*f1^2*f8^-1*f1^-3*f2*f3*f4,
f1^-3*f8^-7*f1^-1*f3^-1*f1^-2*f4^-1*f2^-1*f1^-1*f8^-1*f2*
      f3*f4,
f1*f8^-7*f3^-1*f4^-1*f2^-1*f1^3*f8^-1*f1^-3*f2*f1^-1*f3*f4,
f1^3*f8^-7*f3^-1*f1^-2*f4^-1*f2^-1*f1^-2*f8^-1*f1*f2*f3*f4,
f1^2*f8*f1^-3*f8*f1^-3*f8*f1*f8*f1*f8*f1^3*f8*f1^-1*f8*f5^-2,
f1^-3*f8*f1^-1*f8^-1*f3^-1*f5^-1*f8*f1^-1*f8^-1*f1^-3*f3^-1*
      f5^-1*f1^-3*f8*f1^-1*f8^-1*f1^-3*f3^-1*f5^-1,
f1^-1*f8^-1*f1^-2*f8*f1^3*f4^-1*f5^-1*f1^-1*f8^-1*f1^-2*f8*
      f1^3*f4^-1*f5^-1*f1^-1*f8^-1*f1^-2*f8*f1^3*f4^-1*f5^-1,
f8^-1*f1^-3*f8*f1*f5^-1*f1^-2*f2*f8^-1*f1^-3*f8*f1^3*f2^-1*
      f5^-1*f8^-1*f1^-3*f8*f1^3*f2^-1*f5^-1,
f8^-1*f1^-1*f8*f1^-2*f8*f1^-1*f8^-1*f5^-1*f1^3*f2*f3*f1^3*
      f8^-1*f1*f8*f1^2*f8*f1*f8^-1*f2^-1*f5^-1*f3*f1^3*
      f8^-1*f1*f8*f1^2*f8*f1*f8^-1*f2^-1*f5^-1*f3,
f8^-1*f1^-1*f8^-1*f1^-2*f8*f1^-2*f8*f1^-1*f2^-1*f1^-1*f5^-1*
      f4*f1^-3*f8*f1^2*f8^-1*f1*f8^-1*f1^2*f8*f2^-1*f1^-2*
      f5^-1*f4*f1^-1*f8^-1*f1^-2*f8*f1^3*f8^-1*f1^2*f8*f2^-1*
      f1^-2*f5^-1*f4,
f8^-1*f1^3*f8^-1*f1^-1*f8*f1^-1*f8*f1*f5^-1*f1^-2*f4*f3*
      f1^-3*f8*f1^-1*f8^-1*f1^2*f8*f1*f8^-1*f1*f4^-1*f5^-1*
      f3*f1^-3*f8*f1^-1*f8^-1*f1^2*f8*f1*f8^-1*f1*f4^-1*
      f5^-1*f3*f1^3,
f1*f8*f1^-3*f8^-1*f1^-3*f8*f1*f2^-1*f1*f5^-1*f3*f4*f1^-3*f8*
      f1^-1*f8^-1*f1^-1*f8*f1^3*f8*f1^2*f8^-1*f1*f8^-1*f1^-1*
      f2^-1*f5^-1*f3*f4*f1^-3*f8*f1^-1*f8^-1*f1^-1*f8*f1^3*
      f8*f1^2*f8^-1*f1*f8^-1*f1^-1*f2^-1*f5^-1*f3*f4*f8^-1*
      f1^2*f8*f1*f8^-1]]
```

This new presentation has a set of 6 generators with 32 relators,

```
gap> GeneratorsOfGroup(P[2]);
[ f1, f2, f3, f4, f5, f8 ]
gap> Size(P[4]);
32
```

and the isomorphism between the presentations is given by

```
gap> P[1];
[ f1, f2, f3, f4, f5, f6, f7, f8, f9, f10, f11, f12, f13, f14 ] ->
[f1, f2, f3, f4, f5, f1^-3*f8*f1^3*f5^-1, f1^-3*f8*f1^3, f8,
f1^3*f8*f1^-3, f1^-1*f8*f1, f1^2*f8*f1^-2, f1^-2*f8*f1^2,
f1*f8*f1^-1, f1^-2*f2*f1^-1*f3*f4 ]
```

Iterating Algorithm 3.1.14

Here we illustrate the process of iterating Algorithm 3.1.14 and give the GAP output for the group $SL_2(\mathbb{Z}[1/5, \zeta_5])$ at the prime $p = 5$ using the presentation given in Example 3.1.19. We have denoted $SL_2(\mathbb{Z}[1/5, \zeta_5])$ by `F/R`.

```
gap> A:=SecondHomologyCoefficients(F,R,5,R);
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
```

```
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
[ [ f1*f6^-5*f2^-1*f1^-1*f3^-1*f6*f3*f2,
    f6*f1*f6*f1*f6*f1*f6*f1*f6*f1*f4^-2,
    f4^-1*f6^-1*f3*f6*f1^-1*f6^-1*f1^-1*f6*f1*f6^-1*f4^-1*f1^-1*f3,
    f2*f6*f1^2*f6^-2*f1^2*f6*f4^-1*f1^-1*f2*f4^-1*f6^-1,
    f6*f1^-1*f6^-1*f3^-1*f1*f4^-1*f6*f1^-1*f6^-1*f4^-1*f1^-1*f3*f6*
        f1^-1*f6^-1*f4^-1*f1^-1*f3,
    f6*f1^2*f6^-1*f1^-1*f2^-1*f1^-1*f4^-1*f6*f1^2*f6^-1*f2^-1*
        f1^-2*f4^-1*f6*f1^2*f6^-1*f2^-1*f1^-2*f4^-1 ], 1,
  [ <identity ...>, <identity ...>, <identity ...>, <identity ...>,
    <identity ...>, <identity ...>, <identity ...>,
    <identity ...>, <identity ...>, <identity ...>,
    <identity ...>, f1*f2^-1*f4*f1*f2^-1*f4^-1,
    f1*f3*f6^-5*f3^-1*f2^-1*f1^-1*f6*f2, <identity ...>,
    f1*f4^-1*f1^-1*f3*f4^-1*f6^-1*f3*f6*f1^-1*f6^-2*f1^-1*f6,
    f1^2*f6*f4^-1*f1^-1*f2*f4^-1*f6^-1*f2*f6*f1^2*f6^-2,
    f1*f3^-1*f4^-1*f6*f1^-1*f6^-1*f4^-1*f1^-1*f3*f6*f1^-1*
```

```
        f6^-1*f4^-1*f3*f1^-1*f6*f1^-1*f6^-1,
    f1^2*f6*f1^2*f6^-1*f1*f2^-1*f4^-1*f6^-1*f1^-2*f6*f2^-1*f4^-1*
        f6*f1*f3*f1*f3^-1*f6^-1*f2^-1*f4^-1, <identity ...> ] ]
gap> B:=SecondHomologyCoefficients(F,R,5,A[1]);
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
#WARNING: system is not confluent, so reductions may not be to
    normal form.
[ [ f1*f6^-5*f2^-1*f1^-1*f3^-1*f6*f3*f2,
    f6*f1*f6*f1*f6*f1*f6*f1*f6*f1*f4^-2,
    f4^-1*f6^-1*f3*f6*f1^-1*f6^-1*f1^-1*f6*f1*f6^-1*f4^-1*f1^-1*f3,
    f2*f6*f1^2*f6^-2*f1^2*f6*f4^-1*f1^-1*f2*f4^-1*f6^-1,
    f6*f1^-1*f6^-1*f3^-1*f1*f4^-1*f6*f1^-1*f6^-1*f4^-1*f1^-1*f3*
        f6*f1^-1*f6^-1*f4^-1*f1^-1*f3 ], 0,
  [ f1^-1*f2*f4^-1*f1^-1*f2*f4^-3, f1*f4^-2*f6*f1*f6*f1*f6*f1
        *f6*f1*f6, f1^-1*f3*f4*f1^-1*f3*f4^-1,
     f1^-1*f2*f4*f1^-1*f2*f4^-1,
     f1*f3^-1*f4^-1*f6*f1^-1*f6^-1*f4^-1*f1^-1*f3*f6*f1^-1*f6^-1*
     f4^-1*f3*f1^-1*f6*f1^-1*f6^-1, <identity ...> ] ]
```

The first application of SecondHomologyCoefficients found the upper bound

$$\dim H_2(SL_2(\mathbb{Z}[1/5,\zeta_5]);\mathbb{F}_5) \leq 5.$$

Running the algorithm a second time gave the result that $H_2(SL_2(\mathbb{Z}[1/5,\zeta_5]);\mathbb{F}_5)$ is trivial.

Appendix 2

In this appendix we collect from various sources the (co)homology of various linear groups.

1. [Alp80]

   a) $H_i\left(\mathbb{Z}[\omega]\right)$ is annihilated by 24, $i > 0$, and

$$H_n\left(SL_2(\mathbb{Z}[\omega]\right) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}_3 & n = 1(4) \\ \mathbb{Z}_4 & n = 2(4) \\ \mathbb{Z}_{24} \oplus \mathbb{Z}_6 & n = 3(4) \\ 0 & n = 0(4) \end{cases}$$

2. [Ant99] ($k = \mathbb{F}_3, A = \mathbb{Z}[1/3, \sqrt[3]{1}], \Gamma =$ congruence subgroup)

   a) $H^*(GL_2(A)) = P(c_2, c_4) \otimes \Lambda(e_1, e_3) \otimes \Lambda(e'_1, e'_3)$

   b) $H^*(GL_n(A))$ has torsion for $n \geq 27$

   c) $H_1(\Gamma) = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus (0 \text{ or } \mathbb{Z}_2)$

   d) $H_2(\Gamma) = k^{\oplus 3} = \mathbb{F}_3^{\oplus 3}$

   e) $H_1(SL_2(\mathbb{Z}[\sqrt[3]{1}])) = \mathbb{Z}_3$

   f) $H_1(SL_2(A)) = 0$

   g) $H_1(SL_2(A); k) = 0$

   h) $H_2(SL_2(\mathbb{Z}[\sqrt[3]{1}])) = \mathbb{Z}_4$

   i) $H_2(SL_2(A), k) = 0$

   j) $H_2(SL_2(\mathbb{Z}[\sqrt[3]{1}]); k) = k$

   k) $H_3(SL_2(\mathbb{Z}[\sqrt[3]{1}]); k) = H_4(SL_2(\mathbb{Z}[\sqrt[3]{1}]); k) = k^2$

   l) $H_2(\Gamma; k) = 0$

   m) $H_2(SL_2(A); k) = 0$

   n) $H_3(\Gamma; k) = H_4(\Gamma; k) = k^3$

   o) $H_4(SL_2(A); k) = k$

   p) $H_5(SL_2(A); k) = 0$

q) $H_3(SL_2(A);k) = k^2$

3. [Ant09]

   a) $H_2\left(SL_2(\mathbb{Z}[1/5, \sqrt[5]{1}]);\mathbb{F}_5\right) = 0$
   b) $H_2\left(GL_2(\mathbb{Z}[1/5, \sqrt[5]{1}]);\mathbb{F}_5\right) = (\mathbb{F}_5)^{\oplus 4}$

4. [Bro94]

   a) $SL_2(\mathbb{F}_p)$ has periodic cohomology

5. [Knu96] (F a number field with characteristic not zero, r is the number of real embeddings of F, and s is the number of conjugate pairs of complex embeddings)

   a) $H_k(SL_2(F[t,t^{-1}]);\mathbb{Q}) \cong H_{k-1}(F^\times;\mathbb{Q})$ where $k \geq 2r+3s+2$
   b) $H_1(SL_2(F[t,t^{-1}])) = 0$
   c) $SL_2(F) \hookrightarrow SL_2(F[t])$ induces $H_*(SL_2(F)) \cong H_*(SL_2(F[t]))$
   d) $H_*(\Gamma) \cong H_*(F^\times)$
   e) $H_k(SL_2(F[t,t^{-1}]);\mathbb{Q}) = H_{k-1}(\Gamma;\mathbb{Q})$ for $k \geq 2r+3s+2$

6. [Knu01]

   a) $H_i(SL_2(\mathbb{Z})) = H_i(\mathbb{Z}_{12}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_{12} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$
   b) $(p \geq 5)$ $H^1(\mathbb{Z}[1/p]) = 0$

$$H^2(SL_2(\mathbb{Z}[1/p])) = \begin{cases} \mathbb{Z}^{(p-7)/6} \oplus \mathbb{Z}_{12} & p = 1(12) \\ \mathbb{Z}^{(p+1)/6} \oplus \mathbb{Z}_{12} & p = 5(12) \\ \mathbb{Z}^{(p-1)/6} \oplus \mathbb{Z}_{12} & p = 7(12) \\ \mathbb{Z}^{(p+7)/6} \oplus \mathbb{Z}_{12} & p = 11(12) \end{cases}$$

c) $(i \geq 2)$

$$H^{2i}(SL_2(\mathbb{Z}[1/p])) = \begin{cases} \mathbb{Z}_6 & p = 1(12) \\ \mathbb{Z}_2 & p = 5(12) \\ \mathbb{Z}_3 & p = 7(12) \\ 0 & p = 11(12) \end{cases}$$

d) $H^i(SL_2(\mathbb{Z}[1/3])) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} \oplus \mathbb{Z}_4 & i = 2 \\ \mathbb{Z}_{12} \oplus \mathbb{Z}_4 & i = 2j, j > 1 \end{cases}$

e) $H^i(SL_2(\mathbb{Z}[1/2])) = \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} \oplus \mathbb{Z}_3 & i = 1 \\ \mathbb{Z}_{24} \oplus Z_3 & i = 2j, j > 1 \end{cases}$

f) (k an infinite field) $SL_2(k) \hookrightarrow SL_2(k[t])$ induces an isomorphism

$$H_*(SL_2(k)) \cong H_*(SL_2(k[t]))$$

g) $H_i(SL_2(\mathbb{Q}); \mathbb{Q}) = 0, i > 2r + 3s + 1$ where r is the number of real embeddings of k and s is the number conjugate pairs of complex embeddings of k

h) $H_3(SL_2(\mathbb{Q}[t, t^{-1}]); \mathbb{Q}) = H_2(\mathbb{Q}^\times; \mathbb{Q})$

i)

n	$H^n(SL_3(\mathbb{Z}))$
$12m+1$	$(\mathbb{Z}_2)^{6m}$
$12m+2$	$(\mathbb{Z}_2)^{6m}$
$12m+3$	$(\mathbb{Z}_2)^{6m+2}$
$12m+4$	$(\mathbb{Z}_3)^2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^{6m}$
$12m+5$	$(\mathbb{Z}_2)^{6m+1}$
$12m+6$	$(\mathbb{Z}_2)^{6m+4}$
$12m+7$	$(\mathbb{Z}_2)^{6m+3}$
$12m+8$	$(\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^{6m+1}$
$12m+9$	$(\mathbb{Z}_2)^{6m+5}$
$12m+10$	$(\mathbb{Z}_2)^{6m+5}$
$12m+11$	$(\mathbb{Z}_2)^{6m+4}$
$12m+12$	$(\mathbb{Z}_3)^2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^{6m+5}$

j) $(n \geq 3)$ $H_2(SL_n(k[t, t^{-1}]) = H_2(SL_n(k)) \oplus k^\times$

k) $(i = 2, 3)$ $E_2(R) \hookrightarrow E_2(R[t])$ induces $H_*(E_2(R)) \cong H_*(E_2(R[t]))$

l) ($i \geq 1$) $H_i(SL_2(\mathbb{Z}[t]))$ is not finitely generated

7. [Knu08]

a) $H_2(SL_2(\mathbb{Z}[t, t^{-1}]))$ is not finitely-presented

b) $H_2(SL_2(\mathbb{F}_2[t])) = \Lambda^2 t\mathbb{F}_2[t] \oplus \mathbb{F}_2 \oplus t\mathbb{F}_2[t]$

c) **Conjecture:** For all $i \geq 2$, $H_i(SL_2(\mathbb{Z}[t, t^{-1}]))$ is not finitely-presented.

Appendix 3

Abelianization

Given a group G, the commutator subgroup of G, denoted $[G, G]$, is the group generated by all elements of the form $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$. The **abelianization** of G (also described as G **made abelian**) is the quotient group $G^{\mathrm{ab}} := G/[G, G]$ and is the largest quotient of G which is abelian. That is, if N is a normal subgroup of G and G/N is abelian then $[G, G] \subset N$.

In category-theoretic terms, the abelianization satisfies the following universal property. Let $\phi : G \to G^{\mathrm{ab}}$ be the quotient map and $f : G \to H$ be a group homomorphism with H abelian. Then there exists a unique group homomorphism $g : G^{\mathrm{ab}} \to H$ such that

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^{\mathrm{ab}} \\ & {\scriptstyle f}\searrow & \vdots\, {\scriptstyle g} \\ & & H \end{array} \tag{4.4}$$

commutes.

As stated in Example 2.1.7, $H_1(G) \cong G^{\mathrm{ab}}$, which is the main application of the abelianization of G with which we are concerned in this work. Let G have a finite presentation

$$\langle F|R \rangle = \langle x_1, x_2, \ldots x_n \mid r_1, r_2, \ldots, r_k \rangle. \tag{4.5}$$

We describe a technique that uses $\langle F|R \rangle$ to calculate G^{ab}.

Theorem 4.0.3. *Let A be an $m \times n$ matrix with entries in $\mathbb{Z}$. There exists an $\mathbb{Z}$-invertible $m \times m$ matrix S and an $\mathbb{Z}$-invertible $n \times n$ matrix T such that the product SAT is the $m \times n$ matrix*

$$\begin{pmatrix} a_1 & 0 & 0 & & \cdots & & 0 \\ 0 & a_2 & 0 & & \cdots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & a_r & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & 0 \end{pmatrix}, \tag{4.6}$$

where a_i divides a_{i+1} for $1 \leq i < r$.

The matrix SAT is called the **Smith Normal Form** of A and the diagonal entries a_i are the **invariant factors**. The proof utilizes three types of operations similar to Gaussian elimination. Let us denote the rows of A by r_i and the columns by c_j for $1 \leq i \leq n$ and $1 \leq j \leq m$.

- Interchange rows r_k and r_l or interchange columns c_k and c_l
- Multiply r_k or c_k by -1
- Replace r_k by $r_k + qr_l$ or c_k by $c_k + qc_l$ for $q \in \mathbb{Z}$ and $k \neq l$

As is the case for Gaussian elimination over a field, performing one of these operations on A corresponds to multiplying by an appropriate elementary matrix; row operations correspond to multiplying on the left and column operations correspond to multiplying on the right. A proof of the theorem, presented as an algorithm for reducing a matrix to Smith Normal Form, can be found in [HEO05, p 343].

To relate the Smith Normal Form of an integer matrix to the abelianization of a group G, we recall the following definitions and facts.

Definition 4.0.4. Let G have finite presentation $\langle F|R\rangle$, where F is a rank s free group and R is a set of t relators. We define the **abelianized presentation** of G, denoted $\mathrm{Ab}\langle F|R\rangle$, to be the quotient of F by the relators in R made abelian. That is, if $r \in R$ is the word $r = f_1^{n_1} f_2^{n_2} \cdots f_s^{n_s}$ in F, where each $f_i \in F$ and $n_i \in \mathbb{Z}$, then the corresponding relator in $\mathrm{Ab}\langle F|R\rangle$ is

$$\mathrm{Ab}(r) = n_1 f_1 + n_2 f_2 + \cdots + n_s f_s \tag{4.7}$$

It is relatively straightforward to show that $G/[G,G]$ is isomorphic to $\mathrm{Ab}\langle F|R\rangle$. The task now is to deduce the abelian invariants of $G/[G,G]$ as guaranteed by the Fundamental Theorem of Finitely Generated Abelian Groups [DF04, p 158]. Since $G/[G,G] \cong \mathrm{Ab}\langle F|R\rangle$ it follows that $G/[G,G]$ can be expressed as the quotient of $\mathbb{Z}^s$ by the subgroup K generated by the coefficients of the sums in $\mathrm{Ab}(R)$. More explicitly, there is the exact sequence

$$\mathbb{Z}^s \to G/[G,G] \to 0 \tag{4.8}$$

which we extend in the following way. Regard R as a set of generators of $\mathbb{Z}^t$. Define a homomorphism from $\varphi : \mathbb{Z}^t \to \mathbb{Z}^s$ that sends

$$r \mapsto \mathrm{Ab}(r) = n_1 f_1 + n_2 f_2 + \cdots + n_s f_s. \tag{4.9}$$

The matrix corresponding to φ is the **relation matrix** associated to $\langle F|R\rangle$. The column and row operations to put this matrix into Smith Normal Form are equivalent to linear automorphisms of $\mathbb{Z}^s$ and $\mathbb{Z}^t$ respectively.

Thus there is the commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathbb{Z}^s & \xrightarrow{\varphi} & \mathbb{Z}^t & \longrightarrow & G/[G,G] & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ \mathbb{Z}^s & \xrightarrow{\bar{\varphi}} & \mathbb{Z}^t & \longrightarrow & \mathbb{Z}^t/\mathrm{Im}\bar{\varphi} & \longrightarrow & 0, \end{array} \tag{4.10}$$

where $\bar{\varphi}$ is φ composed on the left and right with the appropriate linear automorphism applied to $\mathbb{Z}^s$ and $\mathbb{Z}^t$. We deduce the structure of $G/[G,G]$ from the invariant factors of the matrix associated to $\bar{\varphi}$, which is the Smith Normal Form matrix associated to φ. That is,

$$G/[G,G] \cong \mathbb{Z}^t / \left(\langle a_1\rangle \oplus \cdots \oplus \langle a_s\rangle\right), \tag{4.11}$$

where each a_i is a non-negative integer which is an invariant factor of the matrix associated to $\bar{\varphi}$.

Note 4.0.5. A similar theorem holds for finitely generated modules over any principal ideal domain. This is due to the Structure Theorem for PIDs [DF04, p 462]; the Fundamental Theorem of Finitely Generated Abelian Groups is a special case of this theorem

Example 4.0.6. Suppose a group G has presentation

$$\langle F|R\rangle = \langle x, y, z \mid (xy)^2, (xz)^{-1}x^3, (xy^2z^{-1})^2\rangle. \tag{4.12}$$

Then

$$\mathrm{Ab}\langle F|R\rangle = \langle x, y, z \mid 2x + 2y, 2x - z, 2x + 4x - 2z\rangle \tag{4.13}$$

Therefore $G/[G,G]$ is the quotient of $\mathbb{Z}^3$ by the subgroup generated by

$$\{(2, 2, 0), (2, 0, -1), (2, 4, -2)\}. \tag{4.14}$$

Putting the matrix

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -1 \\ 2 & 4 & -2 \end{pmatrix} \tag{4.15}$$

into Smith Normal Form gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \tag{4.16}$$

and we conclude that $G/[G, G] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

Spectral Sequences

Here we give the basic notions of a spectral sequence as used in this dissertation. We do not explain the various ways that a spectral sequence can arise (filtrations or exact couples) nor give examples, instead we cite [McC01] where proofs, examples, and a large number of spectral sequences are analyzed in detail.

Definition 4.0.7. A **spectral sequence** $\{E^r, d\}$ is a sequence of bimodules $E^r_{p,q}$, for $r = 0, 1, 2, \ldots$ and $p, q \in \mathbb{Z}$, each of which is equipped with a differential d_r of bidegree $(-r, r-1)$ such that $E^{r+1} \cong H(E^r, d_r)$ for each r.

In the definition above, the spectral sequence is of **homological type**. A **cohomological type** spectral sequence is similar, though, as expected, the differential has bidegree $(r, 1-r)$. If we impose a further assumption that $E^r_{p,q} = 0$ for $p, q < 0$ then it is clear that for fixed p and q that $E^r_{p,q}$ becomes fixed for r large. We denote this limiting group by $E^\infty_{p,q}$.

Definition 4.0.8. A spectral sequence is said to **converge** to a graded, filtered object H_n if when

1. $0 \subset F^0_n \subset \cdots \subset F^n_n = H_n$ is the filtration of H_n,
2. the stable terms $E^\infty_{p,n-p}$ are isomorphic to the successive quotients F^p_n / F^{p-1}_n in the filtration above.

Note 4.0.9. If the ground ring is taken to be a field then the modules discussed above are actually vector spaces. Therefore the extension problems associated with finding the convergent object H_n become trivial. In this case,

$$H_n \cong \bigoplus_{p+q=n} E^\infty_{p,q}. \tag{4.17}$$

The following spectral sequence is used in this work.

Theorem 4.0.10. [Bro94, p 171] *Let* $1 \to H \to G \to Q \to 1$ *be an exact sequence groups and let* M *be a* G*-module. Then there is a spectral sequence of the form*

$$E^2_{p,q} \cong H_p\left(Q, H_q(H, M)\right) \to H_{p+q}(G, M). \tag{4.18}$$

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