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ABSTRACT OF DISSERTATION

Matthew J. Wells

The Graduate School
University of Kentucky
2009

ASPECTS OF THE GEOMETRY OF METRICAL CONNECTIONS

ABSTRACT OF DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Matthew J. Wells
Lexington, Kentucky

Director: Dr. Richard Millman, Professor of Mathematics
Lexington, Kentucky 2009

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ABSTRACT OF DISSERTATION

ASPECTS OF THE GEOMETRY OF METRICAL CONNECTIONS

Differential geometry is about space (a manifold) and a geometric structure on that space. In Riemann's lecture (see [17]), he stated that "Thus arises the problem, to discover the matters of fact from which the measure-relations of space may be determined...". It is key then to understand how manifolds differ from one another geometrically. The results of this dissertation concern how the geometry of a manifold changes when we alter metrical connections. We investigate how diverse geodesics are in different metrical connections. From this, we investigate a new class of metrical connections which are dependent on the class of smooth functions. Specifically, we fix a Riemannian metric and investigate the geometry of the manifold when we change the metrical connections associated with the fixed Riemannian metric. We measure the change in the Riemannian curvatures associated with this new class of metrical connections, and then give uniqueness and existence criterion for curvature of compact 2-manifolds. These results depend on the use of Hodge Theory and ultimately on the function f we choose to define a metrical connection.

KEYWORDS: Riemannian Geometry, metrical connections, Riemannian curvature, compact 2-manifold, Gauss-Bonnet Theorem

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Date: June 26, 2009

ASPECTS OF THE GEOMETRY OF METRICAL CONNECTIONS

By
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Throughout my educational career in mathematics, there have been many people who have influenced my philosophy in mathematics. I would first like to thank Dr. Richard Millman for his years of guidance and patience with me. I learned much from him about classical geometry, mathematics education, and how to appreciate and adore the intricate details in all areas of mathematics. His love of teaching and adoration of mathematics made the time here at the University of Kentucky seem short.

I would also like to thank the rest of my doctoral committee, which consisted of Dr. Serge Ochanine, Dr. David Leep, Dr. Tim Gorringe, and Dr. Samson Cheung for the time they had taken from their schedule to look over this body of work.

Finally, I would also like to thank any others who aided in the process of getting me to this point in my academic career. The office assistants in the Mathematics Department have been ever helpful, and graduate students such as Erik Stokes, Eric Kahn, and Tricia Muldoon have aided in answering questions that came about during this lengthy process. To those that I may have omitted, I am grateful for your help as well.

Here, I take the opportunity to dedicate the research that is presented in this paper. If only you glance upon the work that has been done in this paper, know that years and years of preparation and education were needed to get to this point. Therefore, these dedications are for those who have been there for me through this journey, even though they (in most cases) did not understand the math I was doing! Ultimately, I am thankful that God has seen me to this point. What ensues is a list of those who, through God's will, have played an important role.

First and of upmost importance, I dedicate this work to my parents, David and Patricia Wells, and to my brother, David, and my sister, Jamie. Through the many changes I have undergone in my life thus far, my family has never wavered in their love for me. Indeed, when times got tough, I knew that I could always count on them for support and guidance, and occasionally a laugh or two. To my family: as you look through the following collection of incomprehensible symbols and mathematics, know that you played an important role in my accomplishment. To my parents: you raised me to believe that I can accomplish anything that I put my mind to, and instilled in me values that I am thankful for every day of my life. You taught me to work hard for myself and more importantly for others, and so this work is yours as much as it is mine. You played an important role in my life, and my love for you grows ever stronger.

Furthermore, I dedicate this work to family and friends that have been there throughout the years. There are far to many to list out and properly acknowledge, but I remember each of you in my heart. Some of you have moved on while others have not, but I will never forget the support that you gave or continue to give as I journey through life.

Lastly, I give a special dedication to an old friend who has passed on, Ivan Sheets. As I think back to people who influenced me in mathematics and life, Ivan comes to mind. He taught me to appreciate math, to never let others deter me from what I am passionate about, and was a genuine person who cared about others. Ivan was a teacher of basketball to me, showing me how to play the right way. I continue to utilize his teachings and play the best defense of anyone that I play against. He gave me confidence at a young age when I could not speak. This played a larger role than he knew in influencing me to keep working hard. Ivan was also a lover of horses. It seemed fitting, then, that I am ending my graduate student career in Kentucky, where I studied math amid an area that thrives on horse racing and basketball. I am certain that Ivan looked down upon me through these years, and I am thankful.

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Overview and Background of Research

To study and quantify the changes that happen to a mathematical object as one of its properties vary has been a philosophy in many areas of mathematics, whether it be algebra, geometry, topology, or analysis. In Riemannian geometry as well as this thesis, we follow this philosophy with respect to the geometry of a fixed space (manifold). We are interested in how the geometric properties vary as we change objects associated with the space, such as Riemannian metrics, linear connections, etc. Of particular emphasis here, we are interested in how the curvature and geodesics of a space are influenced by metrical connections defined on a manifold.

In Chapters 1 and 2, the author outlines the essential definitions and well known results we will need for the results in later chapters. The author introduces the idea of a differentiable manifold and ideas associated with the differentiable structure in Chapter 1. One topic essential to understanding later work is that of tensor fields, which is given in Section 1.3. The idea of a metrical connection is defined in Section 2.2, and some common analysis ideas are presented through the machinery of tensor fields and linear connections. In Chapter 2, the Riemannian curvature tensors of type $(1, 3)$ and $(0, 4)$ are defined. The author then concludes the chapter by discussing the independent nature of curvature by looking at Gauss's Theorema Egregium.

Chapter 3 starts with a discussion of conformal Riemannian metrics given by a function $f \in \mathfrak{F}(M)$. As we move forward, the author will be comparing geometrical objects with respect to manifolds which are conformally related. Therefore, in Section 3.2 the author establishes notation that will aid in later work. The main idea of Chapter 3 is to compare geodesics which are given by two metrical connections (which may be metrical with respect to conformal metrics). Millman's article ([11]) is motivation for the main ideas. Millman showed that the Q-tensors need to be equal if two metrical connections are to have the same geodesics (in his work, the metrical connections are metrical with respect to the same Riemannian metric). In Section 3.5, the author generalizes Millman's ideas via Theorem 3.5.4. A corollary to this theorem is that the conformal function is constant if we wish for two metrical connections to produce the same geodesics.

In Chapter 4, metrical connections are examined with respect to curvature and new classes of metrical connections are introduced. Motivated by the relationship of the Levi-Civita connections between conformally related Riemannian manifolds, the author defines three types of metrical connections in Section 4.1, each dependent on a smooth function f. The author then describes the Riemannian curvature of two of these classes of metrical connections in the last two sections of Chapter 4. Specifically, the author looks at the metrical connection ${}^{f(L)}\nabla_X Y$ which is defined using the Levi-

Civita connection and a smooth function f. Comparing the Riemannian curvature of ${}^{f(L)}\nabla$ to the Riemannian curvature of ${}^{L}\nabla$, the author establishes the key equation

$$ {}^{f(L)}K(X,Y) \;=\; {}^{L}K(X,Y) + (Xf)^2 + (Yf)^2 - \|\mathrm{grad} f\|^2 - H_f(X,X) - H_f(Y,Y) $$

in Theorem 4.3.1, which is found with respect to an orthonormal basis of (local) vector fields. Here, H_f is the Hessian of the function f.

In Chapter 5, the author presents results concerning the Riemannian curvature of ${}^{f(L)}\nabla$ on manifolds that are oriented, compact, connected, and of dimension 2. With the condition that a manifold is dimension 2, the equation established in Chapter 4 (see above) can be reduced to

$$ {}^{f(L)}K(X,Y) \;=\; {}^{L}K(X,Y) + \Delta f. $$

To attain the results given in Sections 5.5 and 5.6, the author reviews basic Hodge theory as it pertains to compact manifolds and specializes the results to dimension 2 in Section 5.3. In Section 5.4, the Gauss-Bonnet Theorem is presented as context for later ideas. Utilizing the tools established in Chapter 4 and the first sections of Chapter 5, the author shows that the Riemannian Curvature ${}^{f(L)}K$ associated with ${}^{f(L)}\nabla$ necessarily must satisfy the integral given in Theorem 5.5.2,

$$ \int_M {}^{f(L)}K = 2\pi\chi(M). $$

In Section 5.6, the author then gives existence and uniqueness criterion (in terms of the function f) for the Riemannian curvature of ${}^{f(L)}\nabla$ in Theorems 5.6.1 and 5.6.2. The results show that any function $\mathfrak{h}$ which integrates to $2\pi\chi(M)$ is necessarily and sufficiently a Riemannian curvature for some metrical connection ${}^{f(L)}\nabla$. Also, the author shows that only when functions f_1 and f_2 differ by a constant can the Riemannian curvatures of ${}^{f_1(L)}\nabla$ and ${}^{f_2(L)}\nabla$ to be the same. The work is concluded by outlining further research and indicating how the ideas presented in this paper cannot be immediately extended to higher dimensions. Generalizing to higher dimensions will be the next major research focus of the author.

Chapter 1 Preliminaries of Differential Geometry

1.1 Introduction

Differential geometry is the study of space (a manifold) and a geometric structure of the space using an analytical viewpoint. Early work (see Spivak [16]) in differential geometry focused on the geometry of spaces that were embedded in some Euclidean space. The basic ideas of a manifold were limited to what the structure of the ambient space was and the properties the manifold inherited. For instance, if one wanted to discuss the geometrical properties of the 2-sphere $\mathbb{S}^2$, we would first need to find maps which described $\mathbb{S}^2$ in terms of lying in Euclidean 3-space $\mathbb{R}^3$. Even though there are many ways to place a sphere into $\mathbb{R}^3$, Riemann discussed how we could envision manifolds as intrinsic objects, not spaces that were embedded in Euclidean space. A well written, intuitive discussion of the modern history of differential geometry can be found in M. Berger, A Panoramic View of Riemannian Geometry, especially pages 101-104 (see [1]).

In Riemann's lecture of 1854 (see [17]), he stated that "Thus arises the problem, to discover the matters of fact from which the measure-relations of space may be determined...". Riemann thus considered looking at manifolds from an intrinsic point of view; i.e. looking at the geometry of a manifold independent of any ambient spaces. Initially, this viewpoint was not practiced by classical differential geometers, even though Carl Friedrich Gauss embraced Riemann's abstract approach. In the beginning of the twentieth century, much work was being done in the field of differential geometry using this intrinsic point of view. Of significant importance, it was shown that certain properties of manifolds were independent of how the manifold was embedded in Euclidean (or other) space. The power of results like this allows us to more completely describe the structure of a given manifold. In Chapter 5, we will visit some of these results and discuss how we can use these to describe different classes of manifolds.

Given the evolution of differential geometry, we naturally start by describing the basic concepts of differentiable manifolds. We will only give a basic sketch here, and encourage the reader to read Lee (see [10]) for a modern interpretation of the ideas we assume as basic knowledge. Other texts, such as do Carmo (see [4]), Kobayashi-Nomizu (see [7]), or Hicks (see [6]), have greatly influenced my interpretation of ideas in differential geometry, and are also suggested to the reader (though Lee will suffice for the knowledge we will need). After the introduction, we will examine other ideas we will need in more depth than we do here.

Throughout the paper, M will be an *n-dimensional smooth (differentiable) manifold*. Topologically, M is Hausdorff, connected, and second countable. With these properties, M is also necessarily normal, metrizable, paracompact, and path connected (see Munkres [13] for definitions and the reasons these four properties follow from the other). In particular, there exist smooth partitions of unity on M which allow us to construct functions on M, even though M need not be Euclidean space. Warner presents these ideas in the early chapter of [19], but since these are details that get away from the main ideas of this paper, we do not present them here. The other main topological properties are that M is locally like n-dimensional Euclidean space, meaning that the manifold has the structure of Euclidean space in neighborhoods on M, and the overlaps are smooth.

Analytically, M has a C^∞ differentiable structure. Throughout the paper, it is understood that the structure on all of our manifolds is C^∞, and we refer to M as a smooth manifold. The *differentiable structure* is simply a maximal collection of open sets of $\mathbb{R}^n$ mapping into M which overlap smoothly (see Figure 1.1 below; $\psi \circ \phi^{-1}$ is smooth) and cover all of M (see chapter 1 of Warner [19]). Let $U \subset M$ be an open set in the differentiable structure. Thus U is homeomorphic to an open subset in $\mathbb{R}^n$. As we proceed, $(x_1, x_2, \ldots, x_n)$ will represent the local coordinates of U in terms of this mapping into $\mathbb{R}^n$ (see Figure 1.1 for U). Note that geometrical concepts are dependent on the choice of coordinate chart we are in.

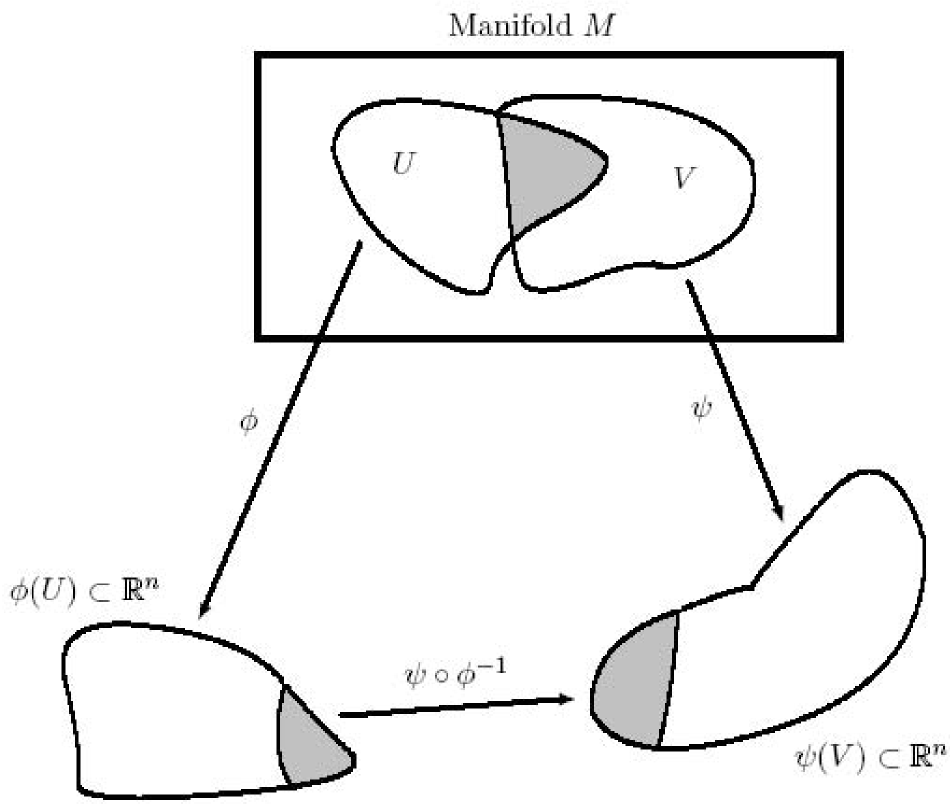

Figure 1.1: Smooth Overlap of Coordinate Charts

1.2 Brief Overview of Definitions in Differential Geometry

Let N be an m-dimensional smooth manifold. Hence, N also has a smooth structure which depends on open sets in $\mathbb{R}^m$. A function $F : M \to N$ is *smooth* if the function is infinitely differentiable through the smooth structures defined on M and N. Said another way, for $p \in M$ mapping to $q \in N$, we can find charts projecting into $\mathbb{R}^m$ and $\mathbb{R}^n$ such that the coordinate transformations (the map $\psi \circ f \circ \phi^{-1}$) in the picture below are smooth.

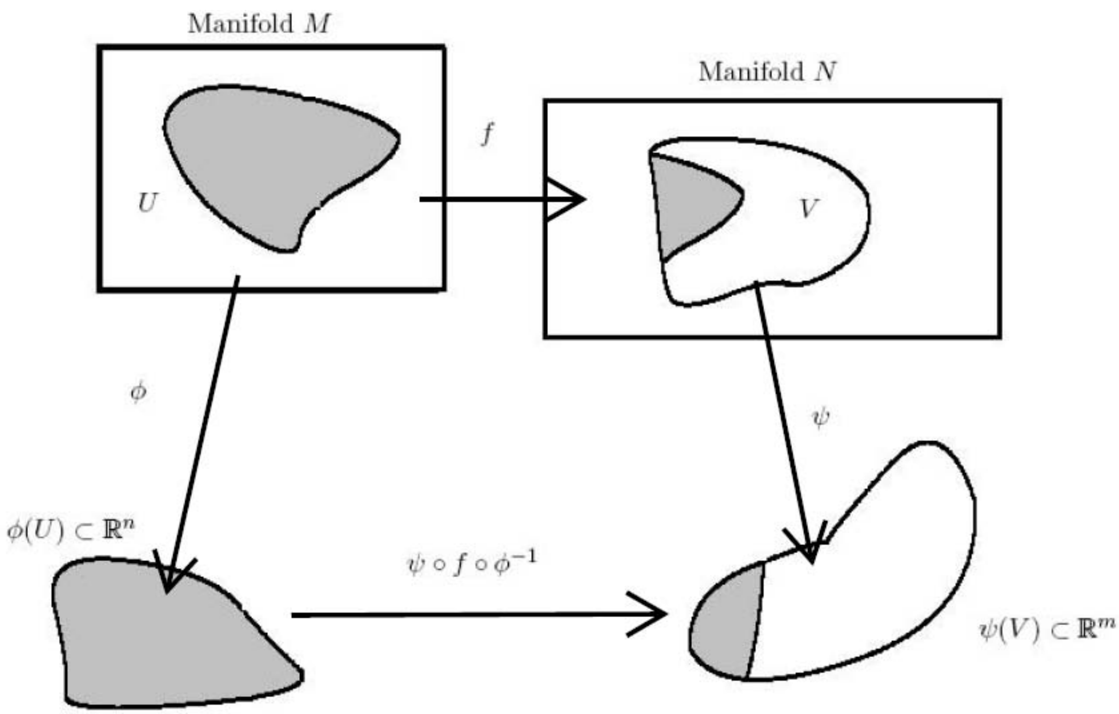

Figure 1.2: Smoothness of a Function Through Coordinate Charts

Generally, we will not concern ourselves with maps from M to N, but rather maps involving M and $\mathbb{R}$, where $\mathbb{R}$ is the real line with the usual structure. We denote the *space of smooth functions* from M to $\mathbb{R}$ as

$$\mathfrak{F}(M) \quad = \quad \{f\,\text{smooth}\,|\, f : M \to \mathbb{R}\}$$

Having established the idea of smooth functions, we define the concept of a tangent vector X_m at a point $m \in M$ as a derivation on $\mathfrak{F}(M)$ (See Lee [10] chapter 3). Using coordinate charts, we can define tangent vectors in terms of the coordinates back in $\mathbb{R}^n$. Given $f \in \mathfrak{F}(M)$, we look locally through a coordinate chart about $m \in M$. Thus, for the coordinate x_i in a coordinate chart about m, we define a *tangent vector* $(\frac{\partial}{\partial x_i})_m$ as the derivation induced by looking through $(x_1, \ldots, x_n)$ (see Lee [10] p.70). A consequence of this definition is that the tangent space at a point m, usually denoted $T_m(M)$, has the same dimension as the manifold (n in this case) and the set

$$\{(\frac{\partial}{\partial x_i})_m \text{ for } 1 \leq i \leq n\}$$

forms a basis for the tangent space at m. Using these facts, we can conclude that $T_p(M)$ is a real vector space of dimension n. For the coordinate chart used above to find coordinate tangent vectors, we can in fact find a basis for each m in the chart. As we proceed, the subscript m will be omitted from our notation. If a result is dependent on $m \in M$, we will include the subscript.

Lastly, we define a *smooth vector field* X as a smooth choice of a tangent vector at each point $m \in M$. For X to be smooth, it is necessary that Xf is smooth for every smooth function f. Observe that Xf makes sense, as at a point $m \in M$, $X_m f$ gives a real value, so Xf is indeed a function. We define the *set of smooth vector fields* on M as $\mathfrak{X}(M)$. Using the fact that $T_p(M)$ is a real vector space, we can deduce with little work that $\mathfrak{X}(M)$ is also a real vector space (of infinite dimension). In a coordinate chart $(x_1, \ldots, x_n)$, the set $\{\frac{\partial}{\partial x_i}\}$ forms a basis for each point in the coordinate chart. We view this basis as a collection of local smooth vector fields. For a smooth vector field X, we can represent X locally with this basis as

$$X = \sum_i X^i \frac{\partial}{\partial x_i},$$

where X^i are smooth functions in the coordinate chart. As is typical convention, we do not provide limits on our sums, as it is assumed we are summing to the dimension of our manifold (which is n). Keep this in mind as we proceed. Also, observe that fX is a smooth vector field for any $f \in \mathfrak{F}(M)$.

For $X, Y \in \mathfrak{X}(M)$, we define the *Lie Bracket* as the smooth vector field $[X, Y]$, where

$$[X, Y]_m \quad = \quad X_m(Yf) - Y_m(Xf) \tag{1.1}$$

for $f \in \mathfrak{F}(M)$ and $m \in M$. For the purposes of this paper, we limit the presentation of the basics to these brief paragraphs.

1.3 Tensors of Type (r, s)

We provide the mathematics of tensors from the viewpoint of differential geometry in this section. The presentation of tensors in differential geometry varies from text to text, as some authors present a rigorous algebraic definition (see Lee [9]), while others utilize tangent bundles over smooth manifolds (see Lang [8]). Since we only consider tensors over the tangent space $T_m(M)$ (which is a real vector space), the definitions which ensue are with respect to this and later to $\mathfrak{X}(M)$, which is also a vector space over the reals. The issue of point dependence in M will also be discussed. We then briefly discuss the concept of a covector, which is used in Chapter 5.

Definition 1.3.1 *Let $r, s \geq 0$ be integers. We define a* (smooth) tensor of type (r, s) *at $m \in M$ as a map of a product of s copies of $T_m(M)$ into r copies of $T_m(M)$*

$$\mathcal{T}_m : \underbrace{T_m(M) \times \cdots \times T_m(M)}_{\text{s times}} \longrightarrow \underbrace{T_m(M) \times \cdots \times T_m(M))}_{\text{r times}}$$

which is multilinear over $\mathbb{R}$. Said another way, given $a, b \in \mathbb{R}$,

$$\mathcal{T}_m(\ldots, aX_m + bY_m, \ldots) \quad = \quad a\mathcal{T}_m(\ldots, X_m, \ldots) + b\mathcal{T}(\ldots, Y_m, \ldots)$$

When r (or s) is 0, the product becomes $\mathbb{R}$. Thus, a tensor of type $(0, s)$ is a map from s copies of the tangent space into $\mathbb{R}$. Notice that Definition 1.3.1 is defined for a point in M. Of particular importance to us will be tensor fields. We present the definition of a tensor field, and discuss what smooth means after our definition.

Definition 1.3.2 *Let $r, s \geq 0$ be integers. We define a (*smooth) tensor field of type *(r, s) as a choice of a tensor of type (r, s) for each $m \in M$ which is compatible with $\mathfrak{X}(M)$.*

Let $\mathcal{T}$ be a tensor field of type (r, s). Thus $\mathcal{T}$ is a choice of a tensor of type (r, s) at each $m \in M$. We think of $\mathcal{T}$ as a map

$$\mathcal{T} : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{s\ times} \longrightarrow \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r\ times}. \tag{1.2}$$

In this map, a product of s smooth vector fields maps to a product of r smooth vector fields, while maintaining point dependence. For smooth vector fields $X_1, \ldots, X_s$ and $Y_1, \ldots, Y_r$, we write

$$\mathcal{T}(X_1, \ldots, X_s) = (Y_1, \ldots, Y_r),$$

with the understanding that at $m \in M$,

$$\mathcal{T}_m((X_1)_m, \ldots, (X_s)_m) = ((Y_1)_m, \ldots, (Y_r)_m).$$

We can approach tensor fields by evaluating them with smooth vector fields, or by evaluating the individual tensor field at the point $m \in M$. The smoothness criterion is linked to the compatibility with smooth vector fields. That means that the Y_j above must be in $\mathfrak{X}(M)$. Also note that since smooth tensors are linear over $\mathbb{R}$, tensor fields are function linear over $\mathfrak{F}(M)$. In fact, since tensor fields are linear, we need only be concerned with basis elements when showing that a candidate $\mathcal{T}$ is indeed a tensor field. Therefore, an equivalent condition for $\mathcal{T}$ to be a tensor field is to check that $\mathcal{T}$ is linear over $\mathfrak{F}(M)$, which is to say that $\mathcal{T}$ is addition linear in each slot and

$$\mathcal{T}(\ldots, fX, \ldots) = f\mathcal{T}(\ldots, X, \ldots)$$

As before, unless the dependence of $m \in M$ is vital to a result, we will drop the subscript. We will reference Definition 1.3.1 and encourage the reader to think of $\mathcal{T}$ as the map described in Equation (1.2).

Now, we take note of some specific types of tensor fields. First, let us consider a tensor field $\mathcal{T}$ of type $(0, 1)$. Notice that $\mathcal{T}_m$ is simply a map from $T_m(M)$ into $\mathbb{R}$. Thus, $\mathcal{T}_m$ is simply a vector in the dual of $T_m(M)$. We call $\mathcal{T}_m$ a 1-covector, and $\mathcal{T}$ a differential (or smooth) 1-form. We will revisit differential forms again in Chapter 5.

Lastly, we make a note about tensor fields when we change the arguments within them. Continue to let $\mathcal{T}$ be a tensor field of type (r, s). For $X, Y \in \mathfrak{X}(M)$, we say that $\mathcal{T}$ is a *symmetric tensor field* if, regardless of where X and Y are placed,

$$\mathcal{T}(\ldots, X, \ldots, Y, \ldots) = \mathcal{T}(\ldots, Y, \ldots, X, \ldots). \tag{1.3}$$

Additionally, we say $\mathcal{T}$ is a *skew-symmetric tensor field* if, regardless of where X and Y are placed,

$$\mathcal{T}(\ldots, X, \ldots, Y, \ldots) = -\mathcal{T}(\ldots, Y, \ldots, X, \ldots). \tag{1.4}$$

Within Equations (1.3) and (1.4), we have only swapped the X and Y in the arguments of the tensor field $\mathcal{T}$. Not all tensor fields are symmetric or skew-symmetric, as we shall see, and in general we cannot hope to describe a tensor field in terms of symmetric and skew-symmetric tensor fields. As Hicks (see [6]) observed, there is a nice decomposition when dealing with tensor fields of type $(1, 2)$.

Lemma 1.3.3 *Let $\mathcal{T}$ be a tensor field of type $(1, 2)$ for M a smooth manifold and $X, Y \in \mathfrak{X}(M)$. Then $\mathcal{T}$ can be decomposed as*

$$\mathcal{T}(X, Y) = \mathcal{S}(X, Y) + \mathcal{A}(X, Y),$$

where $\mathcal{S}$ is a symmetric tensor field of type $(1, 2)$ and $\mathcal{A}$ is a skew-symmetric tensor of type $(1, 2)$.

Proof Observe that $\mathcal{T}(X, Y) = \frac{1}{2}(\mathcal{T}(X, Y) + \mathcal{T}(Y, X)) + \frac{1}{2}(\mathcal{T}(X, Y) - \mathcal{T}(Y, X))$ is the decomposition. **Q.E.D.**

There is one last easy result about tensor fields of type $(1, 2)$ that needs to be discussed. Suppose $\mathcal{T}$ is a tensor field of type $(1, 2)$. For $X, Y \in \mathfrak{X}(M)$, we have

$$\mathcal{T}(X + Y, X + Y) = \mathcal{T}(X, X) + \mathcal{T}(Y, Y) + \mathcal{T}(X, Y) + \mathcal{T}(Y, X). \tag{1.5}$$

Equation (1.5) has a useful consequence. If $\mathcal{T}$ is a symmetric tensor field, then $\mathcal{T}(X,Y)$ is determined by $\mathcal{T}(Q,Q)$ for all $Q \in \mathfrak{X}(M)$. In fact,

$$\mathcal{T} \text{ symmetric} \Rightarrow \mathcal{T}(X,Y) = \frac{\mathcal{T}(X+Y,X+Y) - \mathcal{T}(X,X) - \mathcal{T}(Y,Y)}{2}.$$

Thus for symmetric tensor fields $\mathcal{T}$ of type $(1,2)$, knowing $\mathcal{T}$ on a diagonal suffices to know $\mathcal{T}$. Also, we see that if we evaluate a skew-symmetric tensor field $\mathcal{A}$ of type $(1,2)$ on a diagonal, we get zero; i.e. for all $X \in \mathfrak{X}(M)$,

$$\mathcal{A}(X,X) = 0.$$

1.4 The Differential of a Map

In this section, we define what the differential of a smooth map f is. Any smooth function induces a linear map from the tangent space of a manifold to the tangent space of another tangent space at the image point. We present the definition for arbitrary smooth manifolds M, N of dimension n, m respectively.

Definition 1.4.1 *Let $F : M \to N$ be smooth for smooth manifolds M and N. Define the* differential *dF of F at $p \in M$ as*

$$(dF)_p(X_p)(h) = X_p(h \circ F)$$

for $X_p \in \mathfrak{F}(M)$ and $h \in \mathfrak{F}(N)$. Thus, dF is the map

$$dF_p : T_p(M) \to T_{F(p)}(N).$$

Observe that the differential map transforms tangent vectors in M to tangent vectors in N, defined in terms of the original map F. It is easy to check that this definition does indeed define a tangent vector at $F(p)$ by showing that $(dF)_p(X_p)$ is a derivation on $\mathfrak{F}(N)$. Suppose $N = \mathbb{R}$. Since the tangent space to a point on $\mathbb{R}$ is naturally isomorphic to $\mathbb{R}$, dF would be the map, for each $p \in M$,

$$T_p(M) \longrightarrow \mathbb{R},$$

which can be interpreted as a covector at p. The map dF is therefore a differential 1-form for $f \in \mathfrak{F}(M)$. In chapter 5, we re-visit this concept and use the differential to define a map from p-forms to $(p+1)$-forms.

1.5 Concepts of Linear Connections

We start this section by introducing the idea of linear connections (also called covariant derivatives) of smooth vector fields (by other smooth vector fields), then present some common analysis concepts as they relate to linear connections. The ideas discussed in this section are standard in the literature, and can be found in most introductory differential geometry texts. Assume M to be a smooth manifold of dimension n.

Definition 1.5.1 *The map* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *is a* linear connection *if, for* $X, Y, Z \in \mathfrak{X}(M)$ *and* $f \in \mathfrak{F}(M)$*,*

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ *and* $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$ *,*

(2) $\nabla_{fX} Y = f \nabla_X Y$*,*

(3) $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$*.*

We look at a linear connection $\nabla_X Y$ as taking a derivative of Y in the direction of X. As such, ∇ does not behave like a tensor field. Indeed, $\nabla_X Y$ is not function linear in Y. To define a linear connection in local coordinates $(x_1, \ldots, x_n)$, we define the *Christoffel symbols* ${\Gamma_{ij}}^k$, locally by

$$\nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial}{\partial x_j} \right) = \sum_k {\Gamma_{ij}}^k \frac{\partial}{\partial x_k}. \tag{1.6}$$

Once we discuss Riemannian metrics in the next chapter, we can establish how to generate linear connections on M and obtain a formula for the Christoffel symbols in terms of a Riemannian metric.

Even though linear connections are not tensor fields, we can use them to define tensor fields. What follows are some ideas which naturally extend familiar concepts from analysis to differential manifolds. Although these topics are typically discussed in conjunction with the Levi-Civita connection, which will be presented in Chapter 2, we present them here, as they will be needed in the last two chapters. The purpose of this is to emphasize that these definitions are needed for any arbitrary linear connection ∇.

Definition 1.5.2 *Let* M *be a smooth manifold and let* $X, Y \in \mathfrak{X}(M)$*. Suppose* ∇ *is a linear connection. We define the* torsion tensor Tor *of* ∇*, or* Tor^∇*,as*

$$\mathrm{Tor}^\nabla(X, Y) = \mathrm{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

where $[X, Y]$ *is the Lie Bracket.*

Definition 1.5.3 *Let M be a smooth manifold, $X, Y \in \mathfrak{X}(M)$, and $f \in \mathfrak{F}(M)$. Suppose ∇ is a linear connection. We define the* Hessian *of f, denoted H_f, as*

$$H_f(X, Y) = X(Yf) - (\nabla_X Y)\, f.$$

Proposition 1.5.4 *For ∇ a linear connection on a smooth manifold M and $f \in \mathfrak{F}(M)$,* Tor *is a tensor field of type $(1, 2)$ and H_f is a tensor field of type $(0, 2)$.*

Proof We need to show that Tor and H_f are linear and function-linear over $\mathfrak{F}(M)$. Showing this for the torsion tensor is straightforward. As for the Hessian of f, we show that H_f is function-linear in the second argument (the other conditions are obvious). Let $h \in \mathfrak{F}(M)$ and $X, Y \in \mathfrak{X}(M)$. Using the definition of a linear connection,

$$\begin{aligned} H_f(X, hY) &= X(hY)f - (\nabla_X(hY))\, f \\ &= ((Xh)Yf + h(XY)f) - (h\,(\nabla_X Y)\, f + (Xh)Yf) \\ &= h \cdot H_f(X, Y). \end{aligned}$$

Thus the Hessian of f is a tensor field of type $(0, 2)$. **Q.E.D.**

A few words need to be said about the torsion and Hessian tensors. First, each depends on the choice of linear connections. The torsion tensor specifically is easily seen to be skew-symmetric. Notice that the torsion tensor measures, in a sense, the difference between the Lie bracket and $\nabla_X Y - \nabla_Y X$; indeed, some write $\mathrm{Tor}(X, Y) = [\nabla_X, \nabla_Y] - [X, Y]$. In Chapter 2, we will relate the torsion to certain types of linear connections and give existence and uniqueness criterion for certain linear connections. This is how we interpret torsion in general.

The Hessian, on the other hand, will not necessarily be symmetric or skew-symmetric. As we see in Chapter 2, specific classes of connections (those for which $\mathrm{Tor}^{\nabla}(X, Y) \equiv 0$) cause H_f to be symmetric. In the case of Euclidean space (where the Chistoffel symbols are zero), the Hessian is exactly the matrix of second order partial derivatives. That is, for flat $\mathbb{R}^n$, H_f has the representation

$$(H_f)_{ij} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

Lastly, given a coordinate chart $(x_1, \ldots, x_n)$, we can express the torsion tensor in terms of the basis $\{\frac{\partial}{\partial x_i}\}$ as

$$\mathrm{Tor}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \sum_k \mathrm{Tor}_{ij}{}^k \left(\frac{\partial}{\partial x_k} \right). \tag{1.7}$$

We use this notation as we proceed; since we will not use the coefficients of the Hessian, we do not introduce a similar notation.

The last topic we discuss is the notion of a "line" on a manifold. First, we introduce a smooth curve. The concept of a curve is simply a smooth map

$$\alpha : (a, b) \subset \mathbb{R} \longrightarrow M.$$

The definition of a curve is natural, yet curves can help us understand the geometry involved with a smooth manifold M of dimension n. Consider the differential of α. Observe that $d\alpha$ sends a tangent vector in $\mathbb{R}$ to a tangent vector in M. Thus, if we have a smooth curve α going through a point $m \in M$ at $t = 0$, we get $d\alpha(\frac{d}{dt}) = X_m$ for some $X_m \in T_m(M)$. This suggests that we can think of tangent vectors at m as the differential of $\frac{d}{dt}$ for some curve α, which is true as we shall see.

For Euclidean real space, the idea of a line is what we normally envision, where we can represent a line with a linear equation. With the abstraction to smooth manifolds, the concept is not as obvious as it once was. One way to visualize a line on a manifold is a curve which has zero "wobble" as you travel along it. We have the tool of the linear connection to help measure the deviation that a curve has. There is a rather important issue of how to evaluate a curve using a linear connection, since linear connections work on smooth vector fields. But, as J.H.C. Whitehead demonstrated (see [18]), given a smooth curve α, we can construct a coordinate chart $(x_1, \ldots, x_n)$ so that the curve given by x_1 is α. Thus $\frac{\partial}{\partial x_1}$ represents the tangent vectors along α. More generally, if α is a curve, then $\frac{d\alpha}{dt}$ can be extended locally to a vector field $\widetilde{X}$ on M so that $\nabla_{\widetilde{X}}\widetilde{X}$ is independent of which extension $\widetilde{X}$ of $\frac{d\alpha}{dt}$ is used in a neighborhood. Thus we can define a geodesic as our interpretation of a line as:

Definition 1.5.5 *Let $\alpha : (a, b) \to M$ be a smooth curve on a smooth manifold M. Then α is a* geodesic *with respect to a linear connection ∇ if, for all $a < t < b$,*

$$\nabla_{\frac{d\alpha}{dt}}\left(\frac{d\alpha}{dt}\right) = 0.$$

The existence of geodesics is not obvious when the spaces we deal with do not behave as Euclidean space. There is much detail left to the reader, but computation shows that α is a geodesic if and only if α satisfies the system of differential equations given by

$$\frac{d^2\alpha_k}{dt^2} + \sum_{i,j} \Gamma_{ij}{}^k \frac{d\alpha_i}{dt}\frac{d\alpha_j}{dt} = 0 \tag{1.8}$$

for $k = 1, \ldots, n$. Here, α_i represents the i^{th} component of α when we look at α locally through a coordinate chart; that is, $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ locally. The solutions of this system yield the following useful result.

Theorem 1.5.6 *Let M be a smooth n-manifold and let $m \in M$. For $X_m \in T_m(M)$, there exists a geodesic $\alpha : (-\epsilon, \epsilon) \to M$ for $\epsilon > 0$ such that $\alpha(0) = m$ and*

$$d\alpha \left(\frac{d}{dt} \right)_{(t=0)} = X_m.$$

This theorem says we can produce coordinate charts to "fit" a geodesic just as we would fit a coordinate chart to any curve. We will utilize a coordinate chart constructed this way to make conclusions about geodesics associated with metrical connections in Chapter 3. Observe that in Euclidean space, geodesics, which we can visualize as linear equations between a pair of points, represents the shortest distance between these two points. The main reason why this is true depends on the nature of the space we are working in. When we abstract to smooth manifolds, this is not necessarily true globally. In fact, how do we speak of distance on a manifold? From calculus, we defined the length of a curve as the integral of the length of the tangent vectors as we integrated over the curve. Notice that we do not have any notion of length of a tangent vector, which is ultimately depends on an inner product on the tangent space. It is key then to understanding how manifolds differ from one another geometrically after we introduce the idea of an inner product on a tangent space and this will be discussed in the next chapter.

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Chapter 2 Preliminaries of Riemannian Geometry

2.1 Introduction of a Riemannian Metric on a Manifold

To this point, we have described the abstract idea of a smooth n-manifold M, and developed machinery for a smooth manifold which relates to what we see in any Calculus course (the concepts of tangent vectors, vector calculus, etc...). Early differential geometers worked with these ideas under the condition that the smooth manifolds were embedded in some Euclidean space. Using this classical approach, we could then assign an inner product to $T_m(M)$ using the ambient Euclidean space.

In the nineteenth century, Bernhard Riemann was asked by Gauss to generalize the foundations of geometry. Riemann described this abstraction at the University of Göttingen (see a translation in [17]), which involved varying the inner product from point to point, and doing so independent of any embedding into Euclidean space. This choice of inner product is called the Riemannian metric for a smooth manifold M, and adds geometry to the topological (or space) concept of manifold as in Riemann's *Habilitationsschrift.* Another way of "adding geometry" is to start with a smooth manifold M and put a linear connection ∇ on it. In section 2.2, we'll see the way a Riemannian metric naturally gives a special linear connection, the Levi-Civita connection.

The purpose of this chapter is to introduce the definition of Riemannian metric, and re-visit some of the ideas of Chapter 1 with respect to a given Riemannian metric. Later, we discuss a variety of tools to aid in distinguishing the geometry of one manifold from another.

Definition 2.1.1 *Let M be a smooth manifold. Define a* Riemannian metric *g on M to be a $(0,2)$ symmetric tensor field that has the following property: for $m \in M$ and $X_m \in T_m(M)$,*

$$g_m(X_m, X_m) \geq 0 \text{ with } g_m(X_m, X_m) = 0 \text{ only when } X_m = 0.$$

Fundamentally, this definition make sense, as a Riemannian metric takes pairs of tangent vectors and returns a real number. The condition stated in Definition 2.1.1 is equivalent to requiring that non-zero tangent vectors have positive length. For a smooth manifold M of dimension n, there are many different ways to choose a Riemannian metric. The choice of a Riemannian metric determines the geometry of a space. Therefore, we call a smooth manifold M with a choice of Riemannian metric g a *Riemannian manifold*, and denote this with (M, g). From here on, our work will be on Riemannian manifolds.

Suppose we are given a coordinate chart $(x_1, \ldots, x_n)$ on a Riemannian manifold (M, g). Since $\{\frac{\partial}{\partial x_j}\}$ form a basis for the tangent space at each point of M, we can describe the Riemannian metric g (locally) in terms of this basis. Define the *Riemannian metric coeffients* associated with the coordinate chart $(x_1, \ldots, x_n)$ as

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right). \tag{2.1}$$

Be warned that this description is only valid when discussing the given coordinate chart. As soon as we change coordinate charts, the Riemannian metric coefficients will change. We can represent the Riemannian metric coefficients locally as an $n \times n$ matrix, which is positive definite by definition of Riemannian metric.

Another tool that a Riemannian metric g allows us is the use of an orthonormal basis. Let $X_1, \ldots, X_n \in \mathfrak{X}(M)$ be a basis of vector fields on M. Then $\{X_i\}$ is *orthonormal* if

$$g(X_i, X_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker-delta. In general, we are not guaranteed an orthonormal basis of smooth vector fields that cover the entire manifold. Locally, though, we can take the basis $\{\frac{\partial}{\partial x_i}\}$ and apply the Gram-Schmidt algorithm to attain an orthonormal basis at a point $m \in M$. We can find an orthonormal basis about a point $m \in M$ (which need not come from a coordinate chart).

Euclidean Space Consider real n-dimensional space $\mathbb{R}^n$. To think of $\mathbb{R}^n$ as an abstract smooth manifold, we need to construct coordinate charts to cover $\mathbb{R}^n$. The natural way to do this is to consider the identity map

$$id : \mathbb{R}^n \to \mathbb{R}^n.$$

Notice that this one coordinate chart covers $\mathbb{R}^n$. Because of this, we can describe any Riemannian metric on $\mathbb{R}^n$ in terms of this single chart. The natural way to define a Riemannian metric on $\mathbb{R}^n$ is with the identity matrix; i.e. for δ_{ij} the Kronecker-delta,

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{ij}.$$

This definition indeed gives us the usual dot product that we are familiar with from Calculus.

In general, describing a Riemannian metric on a smooth manifold M is not easy. When dealing with the 2-sphere, for example, we would need to describe the Riemannian metric of the upper half as well as the lower half, the right and left hemispheres, and front and back hemispheres. Then, we would need to check that the Riemannian metric was indeed smooth where any coordinate charts overlap. We reference Kobayashi and Nomizu (see [7]) to see how this is done. On the other hand, given a Riemannian metric g, we can easily define a new Riemannian metric $\widetilde{g}$ in terms of g by multiplying g by some $f \in \mathfrak{F}(M)$ such that $f > 0$, or for $X, Y \in \mathfrak{X}(M)$,

$$\widetilde{g}(X, Y) = f \cdot g(X, Y). \tag{2.2}$$

It is easily seen that $\widetilde{g}$ is a symmetric tensor field of type $(0, 2)$. Since $f > 0$, $\widetilde{g}$ satisfies the condition in Definition 2.1.1, so that $\widetilde{g}$ is a Riemannian metric. We call $\widetilde{g}$ *conformal to the Riemannian metric* g; this idea will be discussed in detail at the beginning of Chapter 3. For now, we present a well-known space which is conformal to Euclidean space.

Hyperbolic Upper-Half Plane Consider the upper-half Euclidean 2-Plane, $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 | y > 0\}$, where the Riemannian metric is the same as $\mathbb{R}^2$ (and so all coefficients described in the above example are still 0). We define a Riemannian metric conformal to g using $f \in \mathfrak{F}(\mathbb{R}^2_+)$, where $f(x, y) = \frac{1}{y^2}$, as

$$\widetilde{g}(X, Y) = \left(\frac{1}{y^2}\right) g(X, Y).$$

Note here that the Riemannian metric is dependent on the point (x, y), which we have suppressed in the notation. Thus, where before the Riemannian metric (on $\mathbb{R}^2_+$) was represented by the identity matrix, now the matrix has non-constant entries dependent on the function $\frac{1}{y^2}$. Since we have altered the Riemannian metric, we will refer to the upper-half plane with $\widetilde{g}$ as *Hyperbolic Space* $\mathcal{H}$. In terms of the same coordinate chart used in the previous example, we easily see that

$$\widetilde{g}_{ij} = \left(\frac{1}{y^2}\right) \delta_{ij}.$$

Another concept that can be discussed since the introduction of a Riemannian metric is the idea of distance on a manifold. Recall that we can define the length of a curve in calculus as the integral of the length of the tangent vector as we run over the curve. Distance is quite similar for an abstract smooth Riemannian manifold M. Given a curve α, we can calculate the distance between two points on the curve by computing the integral along the curve of the length of the tangent vectors. Thus, define distance between points on a manifold as:

Definition 2.1.2 *Let $p, q \in M$ for (M, g) a Riemannian manifold. Define the* distance d from p to q *as*

$$d(p, q) = \inf_{\alpha} |\alpha|_p^q,$$

where the infimum is taken over all piece-wise smooth curves which are smooth except at finitely many points and go from p to q.

The distance function d is commonly seen as a metric in any topology course. Be careful of the notation; when we say Riemannian metric, we are referring to a choice of inner product for the tangent spaces on a manifold. We present this concept of distance to give the reader a deeper understanding of the implications of Riemannian metrics; that is, each Riemannian metric gives rise to a measure of distance $d(p, q)$ as given in Definition 2.1.2. Earlier, we discussed the notion of geodesics as lines on a smooth manifold. In fact, there exist neighborhoods (see Whitehead [18]) where any two points can be connected by a geodesic, such that the geodesic realizes the infimum and represents the shortest distance between the two points. Note that a geodesic need not represent the shortest distance between two points (think of the 2-sphere and the equator). Even though we do not discuss the distance function again, the importance as it relates to geodesics should be kept in mind as we proceed.

2.2 Differential Geometry Concepts Revisited: Linear Connections and Riemannian Manifolds

Suppose (M, g) is a Riemannian manifold of dimension n. Equipped with a Riemannian metric g, we now turn to describing some of our earlier tools through the Riemannian metric. Specifically, we describe linear connections which are naturally associated to the Riemannian metric, and present the concept of a metrical connection. We then show that torsion tensor fields determine metrical connections. The main focus of this section will be on linear connections; we develop other tools that help describe the geometry of M as it pertains to concepts seen in analysis in the next section.

Let ∇ be a linear connection (Definition 1.5.1) on M. Recall that we think of $\nabla_X Y$ as a covariant derivative of Y in the direction of X. To this point, there are many different ways to define such a covariant derivative. Of particular interest to us are linear connections which satisfy the following definition.

Definition 2.2.1 *Let (M, g) be a Riemannian manifold and let $X, Y, Z \in \mathfrak{F}(M)$. We say that a linear connection ∇ is* metrical with respect to g *if*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{2.3}$$

One interpretation of Equation (2.3) is that we are taking a derivative of $g(Y, Z)$ in the direction of X. In other words, this is the derivative of the dot product. Not every linear connection will be metrical with respect to g. As the next proposition shows, though, we can describe metrical connections by specifying the torsion tensor. Recall the definition of the torsion tensor associated with ∇ from Definition 1.5.2. This next proposition is usually proven in the case where $\mathcal{T}$ is zero (see Millman-Parker [12] p.236). We include it for its increased generality, although the proof of the case where $\mathcal{T}$ is zero has the same proof.

Proposition 2.2.2 *Let (M, g) be a Riemannian manifold. Then, given a $(1, 2)$ skew-symmetric tensor field $\mathcal{T}$, there exists a unique metrical connection ∇ on (M, g) whose torsion is given by $\mathcal{T}$.*

Proof We prove uniqueness first. Suppose ∇ is a metrical connection whose torsion tensor is $\mathcal{T} = \mathrm{Tor}$. Writing the metrical condition with respect to the definition of the torsion tensor for $X, Y, Z \in \mathfrak{X}(M)$, we get

$$\begin{aligned} g(\nabla_X Y, Z) &= Xg(Y, Z) - g(Y, \nabla_X Z) \\ &= Xg(Y, Z) - g(Y, \nabla_Z X) - g(Y, [X, Z]) - g(Y, \mathcal{T}(X, Z)). \end{aligned} \tag{2.4}$$

Notice that we can continue to substitute for the terms on the right of Equation (2.4) that involve ∇. Eventually, $-g(\nabla_X Y, Z)$ will show up on the right side of Equation (2.4). Solving for $g(\nabla_X Y, Z)$, we get

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) - Zg(X, Y) + Yg(X, Z) \\ &\quad -g(Y, [X, Z]) + g(X, [Z, Y]) - g(Z[Y, X]) \\ &\quad -g(Y, \mathcal{T}(X, Z)) + g(X, \mathcal{T}(Z, Y)) - g(Z, \mathcal{T}(Y, X)) \end{aligned} \tag{2.5}$$

Notice that Equation (2.5) depends only (on the right hand side) on the metric g, the Lie bracket, and the torsion tensor $\mathcal{T}$. If ${}^1\nabla$ is another metrical connection with torsion $\mathcal{T}$, then Equation (2.5) can be used with a basis to show ∇ and ${}^1\nabla$ are the same. Furthermore, given a skew-symmetric $(1, 2)$ tensor field $\mathcal{T}$, we can define a metrical connection whose torsion is $\mathcal{T}$ by means of Equation (2.5), which proves existence because Equation (2.5) gives the g_{ij}'s. **Q.E.D.**

Definition 2.2.3 *For a Riemannian manifold (M, g), denote by ${}^L\nabla$ as the (unique) metrical connection which is torsion-free, called the* Levi-Civita connection*; that is, for $X, Y \in \mathfrak{X}(M)$, ${}^L\nabla$ is metrical with respect to g and*

$${}^L\nabla_X Y - {}^L\nabla_Y X - [X, Y] = {}^L\mathrm{Tor}(X, Y) = 0$$

Proposition 2.2.2 is typically proven in the case when torsion is 0. In fact, we generally approach Riemannian geometry with the focus of looking at the Levi-Civita connection given in Definition (2.2.3) and how the geometry of M can be described with respect to ${}^L\nabla$ (we use the superscript L to denote the unique Levi-Civita connection; in Chapter 3, we support our choice of notation). For example, using a coordinate chart $(x_1, \ldots, x_n)$, we can describe the *Christoffel symbols of* ${}^L\nabla$ by using Equation (2.5). We note that the torsion is zero and each Lie bracket will vanish, leaving

$$ {}^L\Gamma_{ij}{}^k = \frac{1}{2} \sum_l g^{lk} \left(\frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} + \frac{\partial g_{il}}{\partial x_j} \right) \tag{2.6} $$

where g^{lk} is the inverse of the matrix of (g_{ij}). Further work in Riemannian geometry is centered about studying the Levi-Civita connection as it pertains to the curvature of a manifold (see Lee [9] or Sakai [15]). We want to maintain a general approach to metrical connections whose torsion need not necessarily be zero. Thus, as we proceed, the reader should be aware that any linear connection discussed will not necessarily be the Levi-Civita connection. Of particular interest to us will be the behavior of geodesics and describing curvature properties for arbitrary metrical connections. Later, we use the Levi-Civita connection (Definition 2.2.3) to aid in describing other metrical connections.

2.3 The Gradient, Laplacian, and Hessian of a Function

Let $f \in \mathfrak{F}(M)$ for a Riemannian manifold (M, g). The purpose of this section is to describe two ways to operate on a function: one is the gradient of f, which is a vector field, and the other is the Laplacian of f, which is a function. Both depend on the Riemannian metric g. Recall from Definition 1.5.3 that we described the Hessian of a function. As we will see now, the Hessian can be described nicely in terms of the gradient of a function. Thus, we naturally start with defining the gradient of a function.

Definition 2.3.1 *For (M, g) a Riemannian manifold and $f \in \mathfrak{F}(M)$, define the* gradient of f *to be the vector field* $\mathrm{grad} f$ *which satisfies*

$$ g(\mathrm{grad} f, X) = Xf $$

for $X \in \mathfrak{X}(M)$.

It can easily be seen that $\mathrm{grad} f$ is a unique vector field from the definition. Supposing that M is dimension n, we can describe the gradient of f in terms of an (local) orthonormal basis $\{X_j\}$ as

$$ \mathrm{grad} f = \sum_j (X_j f) X_j. $$

Suppose ∇ is a metrical connection on (M, g) and $X, Y \in \mathfrak{X}(M)$. With Definition 2.3.1 as our definition of gradient, we can use the metrical condition of ∇ to see the following Lemma.

Lemma 2.3.2 *Let (M, g) be a Riemannian manifold and $f \in \mathfrak{F}(M)$. For $X, Y \in \mathfrak{X}(M)$ and ∇ a metrical connection of (M, g),*

$$g(\nabla_X \mathrm{grad} f, Y) \;=\; H_f(X, Y), \tag{2.7}$$

where H_f and grad$f$ *are the Hessian and gradient of f respectively with respect to the linear connection ∇.*

Proof Use the metrical condition of ∇ to see

$$\begin{aligned} g(\nabla_X \mathrm{grad} f, Y) &= Xg(\mathrm{grad} f, Y) - g(\mathrm{grad} f, \nabla_X Y) \\ &= X(Yf) - (\nabla_X Y)f \\ &= H_f(X, Y). \end{aligned}$$

Q.E.D.

Equation (2.7) shows how the gradient is related to the Hessian. Note that this equation is only true for metrical connections, which is why we discussed the previous section first. In the case where $\nabla = {}^L\nabla$ is the Levi-Civita connection, H_f is in fact a symmetric tensor field. Indeed, we use the fact that ${}^L\nabla$ is torsion-free to see this fact. Since we will be dealing with metrical connections, either interpretation of the Hessian (Equation (2.7) or Definition 1.5.3) will suffice.

The last topic involving smooth functions we discuss is the concept of the Laplacian of a function. From analysis, we know the Laplacian in $\mathbb{R}^n$ as opposite the trace of the Hessian. Since the Hessian is a tensor field of type $(0, 2)$, we can think of the Hessian (with respect to a basis) as a square matrix of dimension n (for M of dimension n). Thus, the trace of the Hessian is nothing more than the sum of the diagonal elements of this matrix. Suppose we are given an orthonormal basis $\{X_j\}$ in a neighborhood of $m \in M$. Because H_f is a tensor field, we can construct vector fields from this basis which give $\nabla_{X_j} X_j$ as zero at the given point $m \in M$. This is done as follows:

Choose Vectors Using X_j, choose $(X_j)_m$

Fit Geodesics Find geodesic α_X which fit X_j at m (see Millman-Parker [12] p.231)

Define New Vector Fields Use result of (see Hicks [6]) to find a new (local) vector field $\widetilde{X}$ which fits geodesic from last step using normal coordinates.

For convenience, we will use $\{X_j\}$ to represent the vector fields constructed. As we proceed, remember that the X_j are orthonormal at m, and need not be orthonormal in a neighborhood. Notice that $\nabla_{X_j} X_j = 0$ at $m \in M$ by construction. Therefore, the Hessian of f at the point m reduces to

$$H_f(X_j, X_j)_m = (X_j)_m((X_j)f) = ((X_j)_m)^2 f = X_j^2 f \tag{2.8}$$

Hence, we can define the *Laplacian of* f, denoted Δf, as, for any orthonormal smooth vector fields in a neighborhood,

$$\Delta f = -\text{trace}\,(H_f(X_i, X_j)) = -\sum_j X_j^2 f. \tag{2.9}$$

The definition of the Laplacian resembles how we typically think of the Laplacian in analysis. As we will see in Chapter 5, there is another interpretation of the Laplacian of f that is equivalent, yet describes the Laplacian in terms of operators on differential forms.

Functions on Hyperbolic Space Recall Hyperbolic Space $\mathcal{H}$ from our example earlier. We looked at $\mathcal{H}$ using the usual coordinate chart, and were able to describe the $\widetilde{g}_{ij}$ coefficients. We use the same formula from Equation (2.6) to compute the Christoffel symbols of the Levi-Civita connection ${}^L\widetilde{\nabla}$ for $\mathcal{H}$ as

$$\begin{aligned}
\widetilde{\Gamma}_{11}{}^1 = \widetilde{\Gamma}_{12}{}^1 = \widetilde{\Gamma}_{21}{}^2 = \widetilde{\Gamma}_{22}{}^1 &= 0 \\
\widetilde{\Gamma}_{12}{}^1 = \widetilde{\Gamma}_{21}{}^1 = \widetilde{\Gamma}_{22}{}^2 &= -\frac{1}{y} \\
\widetilde{\Gamma}_{11}{}^2 &= \frac{1}{y}.
\end{aligned}$$

With the obvious change in Christoffel symbols, it comes as no surprise that the geodesics of $\mathcal{H}$ are different than those of $\mathbb{R}^2_+$. We will re-visit geodesics and this example later in following chapters. Notice that the Christoffel symbols are with respect to the Levi-Civita connection of $\mathcal{H}$. Let $f \in \mathfrak{F}(\mathcal{H})$. We investigate arbitrary f under the machinery of the gradient, Laplacian, and Hessian. From Equation (2.3.1), we see for $X \in \mathfrak{X}(\mathcal{H})$,

$$\widetilde{g}(\widetilde{\text{grad} f}, X) = Xf.$$

Since $\widetilde{g} = \frac{1}{y^2} g$, we can see that the gradient of f as it relates to Euclidean space is

$$g(\widetilde{\text{grad} f}, X) = (y^2)Xf.$$

We use Equation (2.7) to compute the Hessian of f. Note that, using $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ to be the usual orthonormal basis of Euclidean $\mathbb{R}^2$, that

$$\{\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$$

is an orthonormal basis for hyperbolic space $\mathcal{H}$. Thus, we use this basis to write $\Delta_{\mathcal{H}}$ for the Laplacian on $\mathcal{H}$:

$$\begin{aligned}
\Delta_{\mathcal{H}} f &= -\left(\frac{\partial^2 f}{\partial x^2} + y\frac{\partial}{\partial y}\left(y\frac{\partial f}{\partial y}\right)\right) \\
&= -\left(\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + y\frac{\partial f}{\partial y}\right).
\end{aligned}$$

2.4 Two Riemannian Curvature Tensor Fields on a Manifold with Linear Connection

The goal of this section is to discuss well-known tensor fields with respect to metrical connections. As we look through the standard presentation of curvature (see Lee [9], Sakai [15], etc...), we notice that curvature is almost always discussed with respect to the Levi-Civita connection. This approach has merits, as we are allowed the useful fact that the Levi-Civita connection has zero torsion, which provides many additional properties. Within our presentation, we seek to describe Riemannian manifolds whose linear connections are more general. Therefore, as we proceed, we should question what we have seen about curvature, and essentially work with the basic definitions.

Let (M, g) be a Riemannian manifold and let ∇ be a linear connection of M. Using the Lie bracket (Equation (1.1)), we define the following curvatures:

Definition 2.4.1 *For ∇ a linear connection on a smooth manifold M, define the* Riemannian curvature tensor R of type $(1,3)$ *as the map*

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

where for $X, Y, Z \in \mathfrak{X}(M)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Definition 2.4.2 *For ∇ a linear connection on a Riemannian manifold (M, g), define the* Riemannian curvature tensor Rm of type $(0,4)$ *as the map*

$$Rm : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R},$$

where for $X, Y, Z, W \in \mathfrak{X}(M)$ and R the Riemannian curvature tensor of type $(1,3)$ associated with ∇,

$$Rm(X, Y, Z, W) = g(R(X, Y)Z, W)$$

Proposition 2.4.3 *Let (M, g) be a Riemannian manifold and ∇ be a linear connection on M. The Riemannian curvature tensors R and Rm (dependent on ∇) are tensor fields of type $(1,3)$ and $(0,4)$ respectively.*

Proof We need to check that each is function multi-linear. As this is straight forward, we omit the proof.

A few words are in line about the Riemannian curvature tensors. First, we follow the convention of Lee (see [9] p.117-118), where we use R for Riemannian curvature of type $(1,3)$ and Rm for the Riemannian curvature of type $(0,4)$. As such, we need to be careful not to regard the m in Rm as a point on the manifold. There will never arise a situation in the later chapters where we confuse these ideas. Moreover, we classically look at Rm as "lowering an index" from the curvature tensor R (see Kobayashi-Nomizu [7]). Thus we refer to each as Riemannian curvature, and distinguish them by R and Rm. Within a coordinate chart $(x_1, \ldots, x_n)$ with local basis $\{\frac{\partial}{\partial x_i}\}$, we represent Rm and R locally as

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \sum_l R_{ijk}{}^l\left(\frac{\partial}{\partial x_l}\right) \tag{2.10}$$

$$Rm\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) = R_{ijkl}. \tag{2.11}$$

For the coefficients, we also follow the convention of Lee (see [9] p. 118). Furthermore, we can use the expression of ∇ in terms of Christoffel symbols (see Equation (1.6)) to express R in terms of Christoffel symbols locally. Since the computations are tedious and straightforward, we omit the computations and describe $R_{ijk}{}^l$ in terms of Christoffel symbols.

$$R_{ijk}{}^l = \frac{\partial}{\partial x_i}\Gamma_{jk}{}^l - \frac{\partial}{\partial x_j}\Gamma_{ik}{}^l + \sum_m \left(\Gamma_{im}{}^l\Gamma_{jk}{}^m - \Gamma_{jm}{}^l\Gamma_{ik}{}^m\right). \tag{2.12}$$

Within the Riemannian curvature tensors, we see obvious symmetries (and skew-symmetries). Some are quite obvious, such as

$$\begin{aligned} R(X, Y)Z &= -R(Y, X)Z \\ Rm(X, Y, Z, W) &= -Rm(Y, X, Z, W), \end{aligned}$$

while others are not as obvious (such as the Bianchi identities (see Lee [9] p.122 or Boothby [3]), which assumes that ∇ is torsion-free). The following proposition outlines one other property that is true for metrical connections, and is needed when we discuss sectional curvature. The proof is the same for the Levi-Civita connection (see Lee [9] p.122), but we give it here for added emphasis.

Proposition 2.4.4 *Let (M, g) be a Riemannian manifold and ∇ be metrical with respect to g on M. For $X, Y, Z, W \in \mathfrak{X}(M)$,*

$$Rm(X, Y, Z, W) \;=\; -Rm(X, Y, W, Z),$$

where Rm is defined with respect to ∇.

Proof To show skew-symmetry, it is enough to show $Rm(X, Y, Z, Z) = 0$, for the result follows from expanding $Rm(X, Y, W + Z, W + Z)$. Since ∇ is metrical with respect to g, we use Equation (2.2.1) to write

$$\begin{aligned} XYg(Z, Z) = X(2g(\nabla_Y Z, Z)) &= 2g(\nabla_X \nabla_Y Z, Z) + 2g(\nabla_Y Z, \nabla_X Z) && (2.13)\\ YXg(Z, Z) = Y(2g(\nabla_X Z, Z)) &= 2g(\nabla_Y \nabla_X Z, Z) + 2g(\nabla_X Z, \nabla_Y Z) && (2.14)\\ [X, Y]g(Z, Z) &= 2g(\nabla_{[X,Y]} Z, Z). && (2.15) \end{aligned}$$

If we subtract Equations (2.14) and (2.15) from Equation (2.13), we get the desired result. **Q.E.D.**

Note that since we cannot assume the connection is metrical AND torsion-free, the common identities cannot be assumed true. We keep this in mind as we proceed.

Lastly, we interpret the Riemannian curvature tensor of type $(1, 3)$. From Definition 2.4.1, we can think of R as measuring the difference between $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$ and $\nabla_{[X,Y]} Z$. This is occasionally written as $R(X, Y)Z = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. In an abstract sense, R and Rm measures how much the manifold curves with respect to the linear connection ∇. Notice that if we know either R or Rm, the other curvature can be found. In the next section, we present other ways to measure the curvature of a Riemannian manifold and conclude with Gauss's Theorema Egregium

2.5 Sectional Curvature and Gauss's Theorema Egregium

In classical differential geometry, manifolds were studied as objects embedded in Euclidean 3-space. Geometers studied the properties of $M \subset \mathbb{R}^3$, where M inherits a Riemannian metric from $\mathbb{R}^3$. The linear connection involved was assumed to be the Levi-Civita connection. Of particular interest, we could study the Riemannian curvature of M. Surprisingly, Gauss saw that the Riemannian curvature was an isometry invariant; i.e. the Riemannian curvature does not depend on the Riemannian metric determined by the imbedding. We state the original result here, which is due to Gauss (and can be found in Lee [9] p.143). The proof can also be seen in Lee. Again, take note that the following results are true for Rm the Riemannian curvature associated with the Levi-Civita connection.

Theorem 2.5.1 (Gauss's Theorema Egregium) *Let $M \subset \mathbb{R}^3$ be a submanifold of dimension 2 and let g be the induced metric on M. For any $p \in M$ and any basis $\{X, Y\}$ for $T_p(M)$, the Gaussian curvature of M at p is given by*

$$K = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - (g(X,Y))^2}.$$

Notice that for a surface M of dimension 2, $\{X, Y\}$ spans the tangent space at p. Also, we think of Gaussian curvature as the product of the principal curvatures of the surface M as it sets in $\mathbb{R}^3$, or alternatively K is the determinant of the scalar second fundamental form. With this in mind, we see that the formula given in Gauss's Theorema Egregium is true for any basis that is chosen in $T_p(M)$. The number K_p describes the Riemannian curvature of M independent of the embedding of M, which can be seen by the following formula (for $X, Y, Z, W \in \mathfrak{X}(M)$):

$$Rm(X,Y,Z,W) = K \cdot (g(X,W)g(Y,Z) - g(X,Z)g(Y,W)). \tag{2.16}$$

Keep in mind that Equation (2.16) describes Rm in terms of K for M a submanifold of dimension 2. We are able to generalize these concepts to higher dimensions, which is what we look at next.

Let M be a Riemannian manifold of dimension n. Here, M is not assumed to be a submanifold or embedded in another space. Let $p \in M$ and consider $T_p(M)$. Denote $\Pi_p \subset T_p(M)$ as a 2-dimensional plane contained in the tangent space of p. Gauss noticed that we could build a 2-dimensional submanifold of M using Π_p as follows: choose a tangent vector $X_p \in \Pi_p$, fit a geodesic to X_p, and extend away from the point. If we do this for all tangent vectors in Π_p, we can build a 2-dimensional submanifold (to build properly, we would use the exponential map). Therefore, we could apply the machinery of Gauss's Theorema Egregium to find a number K_Π, which describes the 2-dimensional submanifold.

Definition 2.5.2 *Let (M, g) be a Riemannian manifold and let ${}^L Rm$ be the Riemannian curvature tensor associated with ${}^L\nabla$. For $X, Y \in \mathfrak{X}(M)$ a basis, denote the* sectional curvature *${}^L K(X,Y)$ as*

$${}^L K(X,Y) = \frac{{}^L Rm(X,Y,Y,X))}{|X|^2|Y|^2 - (g(X,Y))^2}.$$

Here, we can think of X and Y as spanning a plane in the tangent space at a point (locally). The results of Gauss's Theorema Egregium are then naturally extended to planes of the tangent space in the next theorem.

Theorem 2.5.3 *Let (M, g) be a Riemannian manifold and let ${}^{L}Rm$ be the Riemannian curvature tensor associated with ${}^{L}\nabla$. Let $p \in M$ and $\Pi_p \subset T_p(M)$ be a 2-plane in the tangent space. Then the sectional curvature of Π_p is independent of basis; i.e for any $X, Y \in \mathfrak{X}(M)$ which span Π_p at p,*

$$
{}^{L}K(\Pi) = {}^{L}K(X,Y) = \frac{{}^{L}Rm(X,Y,Y,X)}{|X|^2|Y|^2 - (g(X,Y))^2}.
$$

We follow conventional notation and suppress the point p in this last theorem. What we notice is that we can describe the Riemannian curvature with respect to planes in the tangent space. The proof centers about looking at orthonormal bases and extending to any basis (see Lee [9] p.143). There is a subtle detail in the proof, where the fact that

$$
Rm(X,Y,Z,W) = -Rm(Y,X,Z,W) = -Rm(X,Y,W,Z)
$$

is used. From Proposition 2.4.4, we know these properties to be true for ANY metrical connection. Thus, we can think of sectional curvature with regards to any metrical connection.

Theorem 2.5.4 *Let (M, g) be a Riemannian manifold and let Rm be the Riemannian curvature tensor associated with a* metrical *connection ∇. Let $p \in M$ and $\Pi_p \subset T_p(M)$ be a 2-plane in the tangent space. Then the sectional curvature of Π_p is independent of basis; i.e for any $X, Y \in \mathfrak{X}(M)$ which span Π_p at p,*

$$
K(\Pi) = K(X,Y) = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - (g(X,Y))^2}.
$$

Notice, though, that there are many planes in the tangent space when the dimension of M is large. We will only need the facts given in Theorem 2.5.4 for manifolds of dimension 2. By specifying to this class of manifolds, we inherit the property that sectional curvature determines Riemannian curvature. In higher dimensions, we cannot assume sectional curvature determines Riemannian curvature, as we would need the Bianchi identities to prove this fact (see Boothby [3] p.385). When dealing with manifolds of dimension 2, there is only one sectional curvature we need to find, as the tangent space is a plane. We will visit curvature again in chapters 4 and 5.

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Chapter 3 The Geometry of Manifolds with Conformal Metrics

3.1 Introduction

In the last chapter, we encountered different ways to describe the geometry of a Riemannian manifold (M, g). We discussed interpretations of the different curvatures, the idea of "distance" on a manifold, as well as the idea of shortest path between two points on a manifold. Underlying in these geometrical ideas is the dependence on the Riemannian metric g of the manifold M. Recall (Definition 2.1.1) that a Riemannian metric g is a positive definite $(0, 2)$ tensor field and can be represented locally by smooth functions $\{g_{ij}\}$. If we think about this locally as a matrix of functions (g_{ij}), then we see there are many ways in which we can define a Riemannian metric for M. One way to generate many Riemannian metrics on a manifold is to start with one (which exists since the manifold is assumed to be paracompact) and to look at the Riemannian metrics which are conformal to the original one.

Definition 3.1.1 *Let g be a Riemannian metric on M,let $X, Y \in \mathfrak{X}(M)$, and let $f \in \mathfrak{F}(M)$. Define $\widetilde{g}(X, Y) = e^{2f} \cdot g(X, Y)$ as the* Riemannian metric conformal to g *determined by f.*

In the previous chapter (see Equation (2.2)), we saw that as long as a function $h \in \mathfrak{F}(M)$ is positive, then $h \cdot g(X, Y)$ would make sense (for $X, Y \in \mathfrak{X}(M)$). Note that for any $h > 0$, we can find an f (not necessarily positive) such that $h \cdot g = e^{2f} \cdot g$ (by using natural logarithms). We will assume Definition 3.1.1 as our definition of conformal for the rest of this paper.

The importance of conformal Riemannian metrics is seen in many different areas of Riemannian Geometry. In the *Proceedings of the American Mathematical Society* (see [14]), for example, Nomizu and Ozeki showed how we could construct complete or bounded metrics on any manifold M. Given any Riemannian metric g, we could build Riemannian metrics conformal to g by using the fact that a manifold is locally like $\mathbb{R}^n$. Nomizu and Ozeki did this, and found functions locally that they "glued" together and produced Riemannian metrics which were complete or bounded (and were conformal to g). After Nomizu and Ozeki showed the existence of complete (or bounded) Riemannian metrics, they combined these to prove the following impressive theorem:

Theorem 3.1.2 *If every Riemannian metric g on a smooth manifold M is complete, then M must be compact.*

The results are quite powerful, as we know we can always find a complete Riemannian metric for ANY manifold, and compact goes hand in hand with every Riemannian metric being complete!

Other work related to conformal Riemannian metrics deal with Riemannian metrics which are changed independently of one another (the ideas are motivated by conformal Riemannian metrics). Fegan and Millman (see [5]) showed that we could take a complete Riemannian metric g_1 and an arbitrary Riemannian metric g_2, and look at a linear combination (using positive coefficients) of these after constant conformal changes to the metrics. The result is a Riemannian metric that is complete. Bishop and O'neill (see [2]) utilize warped products to produce a Riemannian metric that has negative curvature (under the right conditions). The ideas involved in warping a Riemannian metric are closely related to the idea of building a conformal Riemannian metric, except that we change the Riemannian metric on part of the metric associated with part of a product of manifolds. The use of conformal Riemannian metrics extends into other fields, and is one focus for this paper. Conformal Riemannian metrics are also used in Physics.

3.2 Notation and Examples

From Definition 3.1.1, we defined a conformal Riemannian metric $\widetilde{g}$ by multiplying a Riemannian metric g by a function dependent on $f \in \mathfrak{F}(M)$. Of course, changing the Riemannian metric on a manifold alters the geometry of the space. To help us compare the changes in the geometrical properties, we will use the tilde notation for properties associated with the conformal Riemannian metric $\widetilde{g}$ (and no tilde for those associated with g). Within the context of ideas presented later, we will not be dealing with more than these two specific Riemannian metrics, so this notation should suffice. Here are some examples to demonstrate our notation, where δ_{ij} is the Kronecker delta and ${\Gamma_{ij}}^k$ are the Christoffel symbols, as usual. Some of the coefficients found earlier will be restated here to aid in demonstrating notation.

Euclidean Plane Consider $\mathbb{R}^2$. Recall that we defined the (usual) Riemannian metric in terms of the identity coordinate chart, for coordinates $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$, as

$$\begin{aligned} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= g_{ij} \\ &= \delta_{ij} \end{aligned}$$

Using Equation (2.6) to compute the Christoffel symbols for the Levi-Civita connection ${}^L\nabla$, we can easily see that

$$\begin{aligned} \left({}^L\nabla_{\frac{\partial}{\partial x_i}}\left(\frac{\partial}{\partial x_j}\right)\right)^k &= {\Gamma_{ij}}^k \frac{\partial}{\partial x_k}, \\ &= 0. \end{aligned}$$

Since the Christoffel symbols are 0 for all $1 \leq i, j, k \leq 2$, we can conclude that the Riemannian curvature coefficients of the (usual) 2-dimensional Euclidean plane must be 0; i.e.

$$R_{ijkl} = R_{ijk}{}^{l} = 0.$$

Hyperbolic Upper-Half Plane Recall Hyperbolic space from Chapter 2. We define a conformal Riemannian metric to g using $f \in \mathfrak{F}(\mathbb{R}^2_+)$, where $f(x, y) = -\ln(y)$, as

$$\widetilde{g}(X, Y) = e^{2f} \cdot g(X, Y) = \left(\frac{1}{y^2}\right) g(X, Y).$$

Note here that each Riemannian metric is dependent on the point (x, y), which we have suppressed. We still ge the same Riemannian metric as before, but the difference is in how we look at the function f. In terms of the same coordinate chart used in the previous example, we easily see that

$$\widetilde{g}_{ij} = \left(\frac{1}{y^2}\right) \delta_{ij}.$$

and the Christoffel symbols were

$$\begin{aligned} \widetilde{\Gamma}_{11}{}^{1} = \widetilde{\Gamma}_{12}{}^{1} = \widetilde{\Gamma}_{21}{}^{2} = \widetilde{\Gamma}_{22}{}^{1} &= 0 \\ \widetilde{\Gamma}_{12}{}^{1} = \widetilde{\Gamma}_{21}{}^{1} = \widetilde{\Gamma}_{22}{}^{2} &= -\frac{1}{y} \\ \widetilde{\Gamma}_{11}{}^{2} &= \frac{1}{y}. \end{aligned}$$

We compute the Riemannian curvature coefficients using Equation (2.12) to find the coefficients and list them below. Since most of the coefficients will be zero, only the non-zero coefficients are listed. The computations are omitted, as these are rather straightforward.

$$\begin{aligned} \widetilde{R}_{1,2,2}{}^{1} = \widetilde{R}_{2,1,1}{}^{2} &= -\frac{1}{y^2} \\ \widetilde{R}_{1,2,1}{}^{2} = \widetilde{R}_{2,1,2}{}^{1} &= \frac{1}{y^2}. \end{aligned}$$

As shown in the example, we will associate any invariants with a tilde with the conformal Riemannian metric $\widetilde{g}$, and those without a tilde with g. If we extended our computations to find the scalar curvatures (see Lee [9] p.124-126) for both $\mathcal{H}$ and $\mathbb{R}^2_+$, we would find that the scalar curvatures are both 0 (which is not surprising, since scalar curvature is found using only some of the geometrical information of a space). This does not set well as there is an obvious difference in the geometry of these two spaces. Can we quantify how different two Riemannian manifolds are from one another in terms of invariants like curvature and other geometrical objects? In a general sense (for conformally related Riemannian manifolds), much work has been done relating the geometrical invariants. We present some of these ideas to finish out this chapter while giving new ways to aid in the comparison of conformally related Riemannian manifolds.

3.3 Geometrical Objects of Conformal Metrics

Assume that $\widetilde{g} = e^{2f} \cdot g$ on a Riemannian manifold (M, g). We are particularly interested in how the geometry of (M, g) is related to the geometry of $(M, \widetilde{g})$. Recall from Definition 2.2.3 that we can find a *unique* torsion free metrical linear connection ${}^L\nabla$ for each Riemannian metric g on a manifold, where

$$\begin{aligned} Xg(Y,Z) &= g({}^L\nabla_X Y, Z) + g(Y, {}^L\nabla_X Z) \\ [X,Y] &= {}^L\nabla_X Y - {}^L\nabla_Y X. \end{aligned}$$

From solving the systems given in Equation (1.8), we are able to get an idea of what geodesics look like given a linear connection. We give graphical representations of geodesics in $\mathcal{H}$ in figure 3.1; note that geodesics in $\mathbb{R}^2_+$ are simply lines.

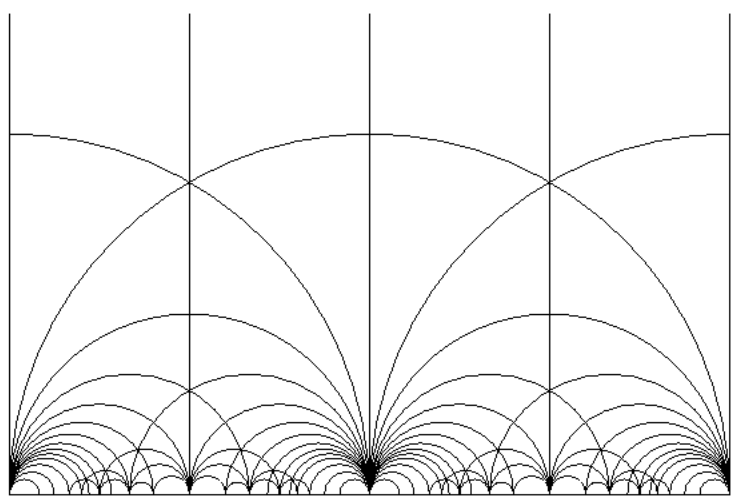

Figure 3.1: Geodesics in Hyperbolic Space

It should be emphasized that in order to solve the systems involved to find a geodesic is quite exhaustive. In any event, we can see that the geodesics of $\mathbb{R}^2_+$ are simply lines in classical Euclidean geometry. The geodesics of $\mathcal{H}$ are vastly different, consisting of arcs of circles (whose center lies on the x-axis) and vertical lines. Clearly, the geodesics of these two spaces are different. How "different" are they? To answer this question, we need to understand first how the Levi-Civita connections of $\mathcal{H}$ and $\mathbb{R}^2_+$ are related. A well-known result in Riemannian geometry gives us insight into the relationship between conformally related manifolds.

Proposition 3.3.1 *Let (M, g) and $(M, \widetilde{g})$ be conformally related Riemannian manifolds; let* $\mathrm{grad}f$ *be the gradient of $f \in \mathfrak{F}(M)$ with respect to g. Then the Levi-Civita connections ∇ and ${}^L\nabla$ with respect to $g, \widetilde{g}$ respectively are related by the following equation:*

$$ {}^L\widetilde{\nabla}_X Y \;=\; {}^L\nabla_X Y + (Xf)Y + (Yf)X - g(X,Y)\mathrm{grad}f. \tag{3.1} $$

Proof To prove that this equation is true, we would only need to show that ${}^L\widetilde{\nabla}$ is indeed the Levi-Civita connection associated with $\widetilde{g}$; i.e. that ${}^L\widetilde{\nabla}$ is torsion-free and metrical with respect to $\widetilde{g}$. Since these facts are not too hard to show, the proof will be left to the reader.

We note here that Equation (3.1) depends on the gradient of f, and that the gradient is given with respect to the Riemannian metric g (not $\widetilde{g}$). Observe how the Levi-Civita connections are related between $\mathcal{H}$ and $\mathbb{R}^2_+$ in the following example.

> **$\mathcal{H}$ versus $\mathbb{R}^2_+$** Recall that g was the usual Riemannian metric on $\mathbb{R}^2_+$. For $\mathcal{H}$, we defined the metric
>
> $$ \widetilde{g}(X,Y) = e^{2f} \cdot g(X,Y), $$
>
> where $f(x,y) = -\ln(y)$. Consider the usual chart on the upper half plane, where $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ span the tangent space at each point. Set $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$. The Levi-Civita connections of g and $\widetilde{g}$ are related by Equation (3.1) in the next equation:
>
> $$ {}^L\widetilde{\nabla}_X Y \;=\; {}^L\nabla_X Y + (-\frac{1}{y})X. \tag{3.2} $$
>
> Notice that the basis $\{X, Y\}$ is orthogonal, which is why the term with the gradient vector field vanishes. If this had not been the case, the gradient vector field (which is associated with g) is the same object we see in an introductory analysis course. We can use Equation (3.2) to understand the geodesic behavior in $\mathcal{H}$. Observe that ${}^L\nabla$ has Christoffel symbols zero. Therefore the geodesics in $\mathcal{H}$ are heavily influenced by $(-\frac{1}{y})X$. As we see, the "weight" of the Riemannian metric on $\mathcal{H}$ is much heavier near the x-axis since $\frac{1}{y^2} \to \infty$ as $y \to 0$. This has the effect of requiring geodesics which are not vertical to resemble arcs of circles by bending back towards the y-axis.

From the comparison of $\mathcal{H}$ and $\mathbb{R}^2_+$, we can conclude that the geodesic behavior is different between the spaces. It is not too far-fetched to believe that manifolds that are related conformally would have different geodesics with respect to the Levi-Civita connections. As we shall see, looking to the class of metrical connections potentially gives many different ways to place geodesics on manifolds. Recall that a linear connection is metrical with respect to a Riemannian metric g if the connection satisfies the "product rule":

$$Xg(Y,Z) \;=\; g({}^L\nabla_X Y, Z) + g(Y, {}^L\nabla_X Z).$$

A metrical connection need not be torsion-free. For a Riemannian metric g on M, there happen to be many ways to define metrical connections associated with g. Each metrical connection will produce geodesics on M. We shall see some classes of metrical connections later in the paper. For now, we are interested in how metrical connections are related in terms of their geodesics.

3.4 Comparing Metrical Connections using the Q-Tensor

We see from the past section how the Levi-Civita connections are related between conformal metrics. What Levi-Civita noticed (demonstrated in Theorem 2.2.2) was that metrical connections are uniquely determined by their torsion. One way to look at this is that two metrical connections that have the same torsion have the same geodesics (and are in fact the same connection). Millman (see [11]) showed that the torsion, as a geometrical invariant, was too strong to determine geodesics. In fact, he found a weaker invariant (the Q-tensor) which we could use to compare the geodesics of metrical connections, which is based on the following definition.

Definition 3.4.1 *Let (M,g) be a Riemannian manifold and $\{X_k\}$ be an orthonormal basis (in a neighborhood of x) for $k = 1, 2, \ldots n$. Let T be a $(1,2)$ skew symmetric tensor field. For vector fields X, Y, define the Q*-tensor of T *as*

$$Q^T(X,Y) \;=\; -\frac{1}{2}\sum_k \{g(T(X,X_k),Y) + g(T(Y,X_k),X)\}X_k \tag{3.3}$$

$$\;=\; -\frac{1}{2}\sum_k (Q^T)^k(X,Y)X_k. \tag{3.4}$$

The fact that Q is a tensor field should be verified by the reader. We note here that the Q-tensor is defined for a local orthonormal basis, but through a change of basis argument is easy to show to be defined globally. Of interest to us with the Q-tensor is the torsion tensor that is associated with a metrical connection. From here on, we will drop the T, as the Q-tensor will only be associated with (metrical) connections and their associated torsion tensor. Also, Q^k will represent the k-th term of the Q-tensor in terms of the orthornormal basis $\{X_k\}$. Millman (see [11]) focused on looking at metrical connections which were metrical to the same metric g. We present the main theorem of his paper while maintaining the notation we have been using here.

Theorem 3.4.2 *Let (M, g) be a Riemannian manifold with metrical connections ${}^1\nabla$, ${}^2\nabla$. Let $Q^{({}^1\mathrm{Tor})}, Q^{({}^2\mathrm{Tor})}$ be the Q-tensors associated with the torsions of ${}^1\nabla$ and ${}^2\nabla$ respectively. Then ${}^1\nabla, {}^2\nabla$ have the same geodesics if and only if $Q^{({}^1\mathrm{Tor})} = Q^{({}^2\mathrm{Tor})}$.*

For notational convenience, we will let ${}^1Q, {}^2Q$ and ${}^1\mathrm{Tor}, {}^2\mathrm{Tor}$ be the Q-tensor and torsion tensor associated with the metrical connections ${}^1\nabla$ and ${}^2\nabla$ respectively. Hicks (ref) noticed that the difference of two connections is a tensor of type $(1, 2)$. This fact is indeed not hard to show; we provide the main ideas of the proof after the definition. We use the conventional notation that Hicks (see [6] p.64) used to denote the difference of two connections,

Definition 3.4.3 *For ${}^2\nabla, {}^1\nabla$ linear connections on a smooth manifold M, let $B^{2,1}$ represent the difference of ${}^2\nabla$ and ${}^1\nabla$, or for $X, Y \in \mathfrak{X}(M)$,*

$$B^{2,1}(X, Y) = B(X, Y) = {}^2\nabla_X Y - {}^1\nabla_X Y.$$

Lemma 3.4.4 *For ${}^2\nabla, {}^1\nabla$ linear connections on a smooth manifold M, $B(X, Y)$ is a tensor field of type $(1, 2)$.*

Proof We need to show that B is bi-linear and function linear. Since ${}^1\nabla_X Y$ and ${}^2\nabla_X Y$ are linear in X and Y, B is bi-linear. Similarly, since each of the connections are function linear in X, B is function linear in X. Since Y is not as easily seen to be function linear, we show this fact. Let $h \in \mathfrak{F}(M)$. Then

$$\begin{aligned} B(X, hY) = {}^2\nabla_X hY - {}^1\nabla_X hY &= ((Xh)Y + h^2\nabla_X Y) - ((Xh)Y + h^1\nabla_X Y) \\ &= h^2\nabla_X Y - h^1\nabla_X Y \\ &= hB(X, Y) \end{aligned}$$

Thus B is indeed function linear in Y, and so B is a $(1, 2)$ tensor field. **Q.E.D.**

Hicks (Lemma 1.3.3 or [6] p.64) noticed that any $(1, 2)$ tensor field can be decomposed as a sum of a symmetric and skew-symmetric tensor field. Looking at the B-tensor using an orthonormal basis, Millman was able to decompose the difference (of metrical connections). Millman (see [11]) described these tensors for metrical connections in the following theorem. We take this approach for ideas presented later in this chapter.

Theorem 3.4.5 *Let (M, g) be a Riemannian manifold and let ${}^2\nabla$ be a metrical connection. Let ${}^1\nabla$ be a connection and let $B(X, Y)$ represent the difference of ${}^2\nabla$ and ${}^1\nabla$. Then ${}^1\nabla$ is metrical if and only if*

$$B(X, Y) = {}^2Q(X, Y) - {}^1Q(X, Y) + \frac{{}^2\mathrm{Tor}(X, Y) - {}^1\mathrm{Tor}(X, Y)}{2}, \tag{3.5}$$

where ${}^2Q$ and ${}^1Q$ are defined using the torsion ${}^2\mathrm{Tor}, {}^1\mathrm{Tor}$ of ${}^2\nabla$ and ${}^1\nabla$ respectively.

Here, the torsion tensor fields are the skew-symmetric tensor fields of the decomposition, while the Q-tensor (from Equation (3.3)) is the symmetric piece of the decomposition. From Equation (3.5), we could use the skew-symmetry of the torsion to prove Theorem 3.4.2. We note also that these ideas came about when we specified our work to metrical connections, which are metrical to the same metric g. Naturally, we are interested in the geometry of metrical connections, whether they be metrical with respect to the same Riemannian metric or related through a conformal change in Riemannian metric. The next section discusses the latter idea.

3.5 Generalizing the Q-Tensor

In this section, we turn our attention to the metrical connections that are associated with conformal Riemannian metrics and prove new theorems which generalize those of Section 3.4. Throughout the following, g and $\widetilde{g}$ are conformal Riemannian metrics, which are conformal using the smooth function $f : M \to \mathbb{R}$. We continue to use the notation established for metrical connections. Motivated by Millman (see [11]), we generalize his results to conformal metrics. We are interested in the difference of metrical connections, which in our case are metrical with respect to different Riemannian metrics. Recalling Definition 3.4.3, we will continue to use the B-tensor. For our purposes, the B-tensor will represent the difference between ∇ and $\widetilde{\nabla}$ (Note: later, we shall see easily that this definition of the B-tensor coincides with the previous definition). Recall that the Levi-Civita connection has zero-torsion, so that ${}^{L}Q = {}^{L}\widetilde{Q} = 0$. Therefore, we know

$$\begin{aligned} \widetilde{\nabla}_X Y - {}^{L}\widetilde{\nabla}_X Y &= \widetilde{Q}(X,Y) + \frac{\widetilde{\mathrm{Tor}}(X,Y)}{2} && (3.6)\\ \nabla_X Y - {}^{L}\nabla_X Y &= Q(X,Y) + \frac{\mathrm{Tor}(X,Y)}{2}. && (3.7) \end{aligned}$$

If we combine Equations (3.6) and (3.7) to form the B-tensor, which is defined as $B(X,Y) = B^{\widetilde{\nabla},\nabla}(X,Y)$, we get

$$\begin{aligned} B(X,Y) &= \widetilde{\nabla}_X Y - \nabla_X Y \\ &= \widetilde{Q}(X,Y) - Q(X,Y) + \frac{\widetilde{\mathrm{Tor}}(X,Y) - \mathrm{Tor}(X,Y)}{2} \\ &\quad + ({}^{L}\widetilde{\nabla}_X Y - {}^{L}\nabla_X Y). && (3.8) \end{aligned}$$

We denote B_L *as the B-tensor of the Levi-Civita connections* of conformal Riemannian metrics, which is the last term of Equation (3.8) above. Thus, $B(X,Y)$ is written as

$$B(X,Y) = \widetilde{Q}(X,Y) - Q(X,Y) + \frac{\widetilde{\mathrm{Tor}}(X,Y) - \mathrm{Tor}(X,Y)}{2} + B_L(X,Y). \quad (3.9)$$

Of particular interest, we would like to understand B_L (which is a tensor of type $(1,2)$). Recall Equation (3.1), which shows how the Levi-Civita connections are related across conformal Riemannian metrics. We can easily see the following corollary:

Corollary 3.5.1 *The difference between ${}^L\nabla$ and ${}^L\widetilde{\nabla}$, denoted $B_L(X,Y)$ for vector fields X and Y is*

$$B_L(X,Y) = (Xf)Y + (Yf)X - g(X,Y)\text{grad}f. \tag{3.10}$$

Proof This is immediate from Theorem 3.3.1 **Q.E.D**.

Millman (see [11]) saw the functionality of looking at the Q-tensor in terms of a local orthonormal basis. Particularly, we can simplify ideas involving the Riemannian metric while still describing the geometry of these metrical connections. Therefore, we consider Equation (3.10) in terms of an orthonormal basis.

Definition 3.5.2 *Let $\{X_k\}$ be an orthonormal basis with respect to the n-dimensional Riemannian manifold (M,g); let $f \in \mathfrak{F}(M)$ and $X,Y \in \mathfrak{X}(M)$. Define the C-*tensor and its components ${}^fC^k$ *as*

$$\begin{aligned} {}^fC(X,Y) &= \sum_k \Big\{(Xf)g(Y,X_k) + (Yf)g(X,X_k) - (X_kf)g(X,Y)\Big\}X_k \\ &= \sum_k \Big\{{}^fC^k(X,Y)\Big\}X_k. \end{aligned}$$

It is straight forward to see that Definition 3.5.2 is the B_L-tensor when we look locally in an orthonormal basis. The notation B_L follows from Definition 3.4.3 as a difference of two connections, which in this case are the Levi-Civita connections. We introduce ${}^fC$ to represent the "conformal difference" between conformal metrics. The results we will show will emphasize the importance of the conformal difference when comparing metrical connections across conformal Riemannian metrics. We will write ${}^fC^k$ to represent the k-th term of ${}^fC$ in terms of the orthornormal basis $\{X_k\}$. We re-write Equation(3.9) as

$$B(X,Y) = \widetilde{Q}(X,Y) - Q(X,Y) + \frac{\widetilde{\text{Tor}}(X,Y) - \text{Tor}(X,Y)}{2} + {}^fC(X,Y). \tag{3.11}$$

For now, we note that because of the symmetry of the terms in Definition 3.5.2,the ${}^fC$-tensor is easily seen to be symmetric.

Theorem 3.5.3 *Let $\nabla, \widetilde{\nabla}$ be metrical connections with respect to $(M,g),(M,\widetilde{g})$, where g and $\widetilde{g}$ are related conformally using $f \in \mathfrak{F}(M)$. Let Q and $\widetilde{Q}$ be the Q-tensors associated with the torsion of ∇ and $\widetilde{\nabla}$ respectively. Then $\nabla, \widetilde{\nabla}$ have the same geodesics if and only if*

$$\widetilde{Q}(X,Y) - Q(X,Y) + {}^fC(X,Y) = 0.$$

Proof The result hinges on Equation (3.11). Assume that ∇ and $\widetilde{\nabla}$ have the same geodesics. We want to conclude information about $\widetilde{Q} - Q + {}^fC$, which is a symmetric tensor field of type $(1,2)$. Recall from Equation (1.5) that we need only evaluate this tensor field with diagonal elements. From our discussion of tensor fields in Chapter 1, it suffices to evaluate these tensor fields at $m \in M$. For any $X_m \in T_m(M)$, we can fit a geodesic to X_m and ultimately a smooth vector field X to fit this geodesic (see section 1.5 of Chapter 1). Using this construction, $B(X,X) = 0$ at m, and since the torsion is skew-symmetric, we get (again at the point m)

$$\widetilde{Q}(X,X) - Q(X,X) + {}^fC(X,X) \;\; = \;\; 0.$$

The result follows from the discussion right after Lemma 1.3.3.

Conversely, suppose $\widetilde{Q} - Q + {}^fC = 0$. Therefore,

$$B(X,Y) \;\; = \;\; \frac{\widetilde{\mathrm{Tor}}(X,Y) - \mathrm{Tor}(X,Y)}{2}.$$

Since torsion is skew symmetric, $B(X,X) = 0$, and it follows that ∇ and $\widetilde{\nabla}$ have the same geodesics, where we use the technique of constructing a smooth vector field about any geodesic as described in Chapter 1. **Q.E.D**.

From Proposition 3.5.3, we gain some insight into what must happen with the ${}^fC$ and Q tensors in order for metrical connections to have the same geodesics (for connections that are metrical to different Riemannian metrics). The next theorem generalizes Millman's (from [11]) results as the function f of the conformal change also plays a role in deciding the behavior of geodesics in metrical connections.

Theorem 3.5.4 *Let $\nabla, \widetilde{\nabla}$ be metrical with respect to connected Riemannian manifolds $(M,g), (M,\widetilde{g})$, where g and $\widetilde{g}$ are related conformally using $f \in \mathfrak{F}(M)$. Let Q and $\widetilde{Q}$ be the Q-tensors associated with the torsion of ∇ and $\widetilde{\nabla}$ respectively. Then $\nabla, \widetilde{\nabla}$ have the same geodesics if and only if $\widetilde{Q} = Q$ and f is constant on M.*

Proof Recall that we expressed Q and ${}^fC$ in terms of an orthonormal basis $\{X_k\}$. Using Theorem 3.5.3 and the independence of $\{X_k\}$, we have

$$\nabla, \widetilde{\nabla} \text{have same geodesics} \;\; \Leftrightarrow \;\; \widetilde{Q}(X,Y) - Q(X,Y) + {}^fC(X,Y) = 0 \quad (3.12)$$

Equation (3.12) is true for any vector fields X and Y. Of note, Equation (3.12) is true for X_i for all i. Therefore statement (3.12) is true

$$\begin{aligned} &\Leftrightarrow \;\; \widetilde{Q}(X_i,X_i) - Q(X_i,X_i) + {}^fC(X_i,X_i) = 0 \\ &\Leftrightarrow \;\; \widetilde{Q}^k(X_i,X_i) - Q^k(X_i,X_i) + {}^fC^k(X_i,X_i) = 0 \end{aligned} \qquad (3.13)$$

Because $\{X_k\}$ is an independent basis, Equation (3.13) is true for any $0 \leq k \leq n$. Notice that if $k = i$, then the torsion tensors inside of the Q-tensors are identically 0. Thus Equation (3.13) can be reduced (keeping in mind that $\{X_k\}$ is orthonormal) using the definition of the ${}^fC$ tensor to

$$\begin{aligned} 0 &= {}^fC^k(X_i, X_i) \\ &= 2X_if + 2X_if - 2X_if \\ &= 2X_if. \end{aligned}$$

Since this is true for any i, we must have that f is indeed constant. From Theorem 3.5.3, f is constant if and only if ${}^fC \equiv 0$, which happens if and only if $Q = \widetilde{Q}$, and the theorem is proven. **Q.E.D**.

The results from Theorem 3.5.3 and Theorem 3.5.4 generalize the ideas that Millman had shown from Theorem 3.4.2. But Theorem 3.5.4 says more about the class of metrical connections than what might have been thought. Observe that metrical connections from conformal metrics need the conformal function f to be constant to have any chance that the metrical connections have the same geodesics. Therefore, metrical connections look “geodesically different” in general. Up to this point, we have not given many examples of metrical connections to consider. We have not done this deliberately, but rather we cannot give examples due to a lack of concrete metrical connections. We know of metrical connections that are torsion-free (Levi-Civita connections), but there is very little literature discussing other classes of metrical connections that may have torsion. We will discuss how we can construct a few classes of metrical connections, and the role these classes play in terms of the geometry of a manifold in the coming chapters.

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Chapter 4 Classes of Metrical Connections

4.1 Defining Certain Classes of Metrical Connections

Clearly, looking into conformal metrics on a Riemannian manifold M gives us insight into how we can relate the Levi-Civita connections between two conformal metrics. Hidden in this connection (no pun intended!) of the Levi-Civita connections is the motivation for creating other metrical connections (metrical with respect to either g or $\widetilde{g}$) by using $f \in \mathfrak{F}(M)$ (see Theorem 3.1). We maintain the same notation of letting g be a Riemannian metric, and letting $\widetilde{g}$ be a conformal metric to g. Using Theorem 3.1 as motivation, we define classes of metrical connections based off of smooth functions $f \in \mathfrak{F}(M)$. While ${}^{C}\widetilde{\nabla}$ seems well known (see Sakai [15] p.50), the definition of ${}^{f}\nabla$ does not appear in the literature. Once we show that ${}^{f}\nabla$ is metrical (with respect to g), we will have a whole new family of metrical connections which are parameterized by C^{∞} functions on M.

Definition 4.1.1 *Let $f \in \mathfrak{F}(M)$ for the Riemannian manifold (M, g), and let ∇ be a metrical connection with respect to g. Let X, Y be smooth vector fields on M. Define the following:*

$$
\begin{aligned}
{}^{C(f)}\widetilde{\nabla}_X Y &= \nabla_X Y + (Xf)Y + (Yf)X - g(X,Y)\mathrm{grad} f && (4.1)\\
{}^{f}\nabla_X Y &= \nabla_X Y + (Yf)X - g(X,Y)\mathrm{grad} f && (4.2)
\end{aligned}
$$

Proposition 4.1.2 *For smooth vector fields X and Y and $f \in \mathfrak{F}(M)$, ${}^{C(f)}\nabla_X Y$, ${}^{f}\nabla_X Y$ are connections which are metrical to $\widetilde{g}$ and g respectively.*

Proof We prove that ${}^{f}\nabla$ is a metrical connection with respect to g, as the proof that ${}^{C(f)}\nabla$ is a metrical connection is similar to this proof and Theorem 3.1. Since the space of smooth vector fields is linear and g is linear, it follows that ${}^{f}\nabla_X Y$ is (real) linear in the X and Y position.

We now show that ${}^{f}\nabla$ behaves like a connection with regards to functions. Let $h \in \mathfrak{F}(M)$. Since $\nabla_X Y$ is function linear in the X position,

$$
\begin{aligned}
{}^{f}\nabla_{hX} Y &= \nabla_{hX} Y + (Yf)(hX) - g(hX,Y)\mathrm{grad} f\\
&= h({}^{f}\nabla_X Y).
\end{aligned}
$$

In a similar fashion, we use the fact that ∇ is a connection to show

$$
\begin{aligned}
{}^{f}\nabla_X hY &= \nabla_X hY + (hY)(f)X - g(X,hY)\mathrm{grad} f\\
&= h\nabla_X Y + (Xh)Y + h(Yf)X - hg(X,Y)\mathrm{grad} f\\
&= h^{f}\nabla_X Y + (Xh)Y.
\end{aligned}
$$

Therefore, ${}^f\nabla$ is indeed a connection. Let Z be a smooth vector field of M. Using that ∇ is metrical with respect to g, we get

$$
\begin{aligned}
g({}^f\nabla_X Y, Z) + g(Y, {}^f\nabla_X Z) &= g(\nabla_X Y + (Yf)X - g(X,Y)\mathrm{grad}f, Z) \\
&\quad + g(Y, \nabla_X Z + (Zf)X - g(X,Z)\mathrm{grad}f) \\
&= [g(\nabla_X Y, Z) + g(Y, \nabla_X Z)] \\
&\quad + [(Yf)g(X,Z) - g(X,Y)(Zf) \\
&\quad + (Zf)g(Y,Z) - g(X,Z)(Yf)] \\
&= Xg(Y,Z).
\end{aligned}
$$

Thus ${}^f\nabla$ is metrical with respect to g. **Q.E.D**.

Before we move on to looking at the geometry of some classes of metrical connections, we discuss the notation we are using and introduce other notation we will need. From Chapter 3, recall we use a tilde ($\sim$) with geometrical objects which are associated with the conformal metric $\widetilde{g}$, and omit the tilde for those associated with g. Since ${}^C\widetilde{\nabla}$ is metrical with respect to $\widetilde{g}$, we justify including a tilde for the definition (${}^f\nabla$ has no tilde as it is metrical with respect to g) from Proposition 4.1.2.

When we discuss classes of metrical connections in the rest of this paper, we want to be reminded that the metrical connections as defined above are quite dependent on the function f. Therefore, we will use a superscript before our geometrical invariants to help organize. Here is a sample comparing the different notations:

Connection	Torsion	R. Curvature	Sectional Curvature
$\nabla_X Y$	$\mathrm{Tor}(X,Y)$	$Rm(X,Y,Z,W)$	$K(X,Y)$
${}^f\nabla_X Y$	${}^f\mathrm{Tor}(X,Y)$	${}^f Rm(X,Y,Z,W)$	${}^f K(X,Y)$
${}^{C(f)}\widetilde{\nabla}_X Y$	${}^{C(f)}\widetilde{\mathrm{Tor}}(X,Y)$	${}^{C(f)}\widetilde{Rm}(X,Y,Z,W)$	${}^{C(f)}\widetilde{K}(X,Y)$

Table 4.1: Notation for Metrical Connections

Millman (see [11]) showed that for two metrical connections that were metrical with respect to the same metric g, it was necessary and sufficient for the $Q-$ tensors to be equal in order that the geodesics were the same. From Chapter 3, we also learned that metrical connections which were metrical with respect to conformal Riemannian metrics would not have the same geodesics unless the conformal function was constant on connected components. We examine these ideas using our newly defined metrical connection ${}^f\nabla$.

Suppose ∇ is metrical with respect to (M, g) and define ${}^f\nabla$ for $f \in \mathfrak{F}(M)$ as in Equation (4.2). When could we expect ∇ and ${}^f\nabla$ to have the same geodesics? We look at this through the machinery of the $Q-$tensor.

Proposition 4.1.3 *Suppose ∇ is metrical on a connected Riemannian manifold (M, g). Let $f \in \mathfrak{F}(M)$. Then ∇ and ${}^f\nabla$ have the same geodesics if and only if f is constant.*

Proof Clearly, if f is constant, ∇ and ${}^f\nabla$ are the same as the terms of $(Yf)X - g(X, Y)\mathrm{grad}f$ vanish in the definition of ${}^f\nabla$.

Suppose ∇ and ${}^f\nabla$ have the same geodesics. From Theorem 3.4.2, it follows that $Q(X, Y) = {}^fQ(X, Y)$. Thus, given an orthonormal basis $\{X_k\}$ on M, the coefficients from Equation (3.3) must be equal for Tor and ^{f}Tor; if $X = Y$, this looks like

$$g(\mathrm{Tor}(X, X_k), X) = g({}^f\mathrm{Tor}(X, X_k), X). \tag{4.3}$$

Since g and torsion are tensor fields, the linearity gives us

$$g(\mathrm{Tor}(X, Z), X) = g({}^f\mathrm{Tor}(X, Z), X), \tag{4.4}$$

which is true for any $X, Z \in \mathfrak{X}(M)$. Using a coordinate chart on M with coordinates $(x_1, \ldots, x_n)$, set $X = \frac{\partial}{\partial x_j}$ and $Z = \frac{\partial}{\partial x_k}$ for $j \neq k$. With this, Equation (4.4) tells us that ${}^f\mathrm{Tor}_{jk}{}^j = \mathrm{Tor}_{jk}{}^j$. By the definition of torsion (see Definition 1.5.2), we get

$$\begin{aligned} {}^f\mathrm{Tor}(X, Z) &= {}^f\nabla_X Z - {}^f\nabla_Z X - [X, Z] \\ &= \big\{\nabla_X Z + (Zf)X - g(X, Z)\mathrm{grad}f\big\} \\ &\quad -\big\{\nabla_Z X + (Xf)Z - g(Z, X)\mathrm{grad}f\big\} - [X, Z] \\ &= \mathrm{Tor}(X, Z) + (Zf)X - (Xf)Z. \end{aligned}$$

If we write each side of this last equation in terms of our basis $\{\frac{\partial}{\partial x_i}\}$, we can equate the j^{th} coefficients to get

$${}^f\mathrm{Tor}_{jk}{}^j = \mathrm{Tor}_{jk}{}^j + \frac{\partial f}{\partial x_k},$$

and it follows that $\frac{\partial f}{\partial x_k} = 0$. Since this is true for any k, f follows to be constant on connected components. **Q.E.D**.

This result gives some insight into ${}^f\nabla$ for arbitrary f. Up until now, we have maintained a focus on (metrical) connections and how they are related to one another geodesically. As we move forward, we are interested in how other geometrical invariants are related. Naturally, we move onto curvature tensors, and observe relationships between the classes of metrical connections defined in this section and "other" connections.

4.2 Riemannian Curvature of a Class of Metrical Connections

We continue to let ∇ be a metrical connection with respect to (M, g) and $f \in \mathfrak{F}(M)$. Define $\widetilde{\nabla}$ as a metrical connection (with respect to $\widetilde{g}$) as

$$\begin{aligned} \widetilde{\nabla}_X Y &= {}^{C(f)}\left(\widetilde{({}^{-f}\nabla)}\right)_X Y \\ &= \nabla_X Y + (Xf)Y, \end{aligned} \tag{4.5}$$

for $X, Y \in \mathfrak{F}(M)$ (as usual, $\widetilde{g} = e^{2f} \cdot g$). Notice that $\widetilde{\nabla}$ is metrical with respect to $\widetilde{g}$, as what's given in Equation (4.5) is a combination of Equations (4.1) and (4.2) given in the previous section. Since ∇ and $\widetilde{\nabla}$ are related by definition, it is natural to see how the Riemannian curvatures are related. Recall the Riemannian curvature tensor of type $(1, 3)$ from Chapter 2, given in Definition 2.4.1.

Proposition 4.2.1 *Let ∇ be metrical with respect to a Riemannian manifold (M, g) and define $\widetilde{\nabla}_X Y = \nabla_X Y + (Xf)Y$ for $f \in \mathfrak{F}(M)$. Let $R(X, Y)Z, \widetilde{R}(X, Y)Z$ be the Riemannian curvature tensors of type $(1, 3)$ for $\nabla, \widetilde{\nabla}$ respectively and $X, Y, Z \in \mathfrak{X}(M)$. Then*

$$R(X, Y)Z = \widetilde{R}(X, Y)Z.$$

Proof The proof is a straightforward application of the definitions. Applying Definition 2.4.1 for our connections,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{4.6}$$

$$\widetilde{R}(X, Y)Z = \widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z + \widetilde{\nabla}_{[X,Y]} Z \tag{4.7}$$

Using the definition of $\widetilde{\nabla}$ and using Equation (4.6), we write the terms of Equation (4.7) as follows:

$$\begin{aligned} \widetilde{\nabla}_X \widetilde{\nabla}_Y Z &= \widetilde{\nabla}_X(\nabla_Y Z + (Yf)Z) \\ &= \nabla_X \nabla_Y Z + (Xf)\nabla_Y Z + \nabla_X(Yf)Z + (Xf)(Yf)Z \\ &= \nabla_X \nabla_Y Z + (Xf)\nabla_Y Z + X(Yf)Z + (Yf)\nabla_X Z + (Xf)(Yf)Z \end{aligned} \tag{4.8}$$

$$-\widetilde{\nabla}_Y \widetilde{\nabla}_X Z = -\nabla_Y \nabla_X Z - (Yf)\nabla_X Z - Y(Xf)Z - (Xf)\nabla_Y Z - (Yf)(Xf)Z \tag{4.9}$$

$$\begin{aligned} -\widetilde{\nabla}_{[X,Y]} Z &= -\nabla_{[X,Y]} Z - ([X, Y]f)\, Z \\ &= -\nabla_{[X,Y]} Z - X(Yf)Z + Y(Xf)Z. \end{aligned} \tag{4.10}$$

Combining Equations (4.8), (4.9), and (4.10), we get the desired result. **Q.E.D.**

Corollary 4.2.2 *The Riemannian curvature tensors of type* $(0,4)$ *for* ∇ *and* $\widetilde{\nabla}$ *are related as*

$$e^{2f} \cdot Rm(X,Y,Z,W) = \widetilde{Rm}(X,Y,Z,W).$$

One immediate observation is that we have compared Riemannian curvature tensors across Riemannian metrics. Said another way, given two metrical connections which were metrical to different Riemannian metrics, we proved that the Riemannian curvature tensors of type $(1,3)$ were equal. We shall re-visit Proposition 4.2.1 and Corollary 4.2.2 again in Chapter 5.

4.3 The Sectional Curvature of ${}^f\nabla_X Y$

So far, we have defined some classes of metrical connections parameterized by $f \in \mathfrak{F}(M)$, which are metrical with respect to either g or $\widetilde{g}$, and seen one result concerning the Riemannian curvature tensor. We now turn our interest to the class of connections defined by ${}^f\nabla_X Y$. In particular, we are interested in how the sectional curvature of ${}^f\nabla_X Y$ relates to the sectional curvature of the Levi-Civita connection of a Riemannian manifold (M,g). Define the metrical connection of Equation (4.2) which uses the Levi-Civita connection as

$$^{f(L)}\nabla_X Y = {}^L\nabla_X Y + (Yf)X - g(X,Y)\mathrm{grad} f. \tag{4.11}$$

Note that Equation (4.11) is Equation (4.2) in the special case that ∇ is the Levi-Civita connection of the Riemannian manifold (M,g). We will continue to use the superscript notation $f(L)$ for geometrical objects which are dependent on the connection ${}^{f(L)}\nabla$. Keep in mind that each of the objects in Equation (4.11) is dependent on the metric g and not $\widetilde{g}$. This fact will aid later in demonstrating how the sectional curvature of ${}^{f(L)}\nabla$ relates to ${}^L\nabla$. This theorem will be used significantly in sections 5.4 and 5.5.

Theorem 4.3.1 *Let* Π_m *be a plane in the tangent space to* M *at* $m \in M$*. Let* X *and* Y *be local vector fields which span* Π_m *at the point* m *and also form an orthonormal basis locally with respect to a Riemannian manifold* (M,g) *and let* $f \in \mathfrak{F}(M)$*. Denote the sectional curvature of the plane spanned by this basis as* ${}^{f(L)}K(X,Y), {}^L K(X,Y)$ *for the respective connections* ${}^{f(L)}\nabla, {}^L\nabla$*. Then we have*

$$^{f(L)}K(X,Y) \;=\; {}^L K(X,Y) + (Xf)^2 + (Yf)^2 - \|\mathrm{grad} f\|^2 - H_f(X,X) - H_f(Y,Y). \tag{4.12}$$

Proof The proof uses the definition of sectional curvature and is rather computational. Recall that $K(\Pi_m) = K(X,Y)_m = Rm(X,Y,Y,X)_m$ on orthonormal bases, so for arbitrary connections ∇ (suppressing the dependence on $m \in M$),

$$\begin{aligned}
K(X,Y) &= Rm(X,Y,Y,X) \\
&= g(R(X,Y)Y,Y) \\
&= g(\nabla_X\nabla_Y Y - \nabla_Y\nabla_X Y - \nabla_{[X,Y]}Y, X). \qquad (4.13)
\end{aligned}$$

Using the definition of ${}^{f(L)}\nabla$, we will write out a term from the Riemannian curvature. Through the following, keep in mind that $\{X,Y\}$ is orthonormal.

$$\begin{aligned}
{}^{f(L)}\nabla_X({}^{f(L)}\nabla_Y Y) &= {}^{f(L)}\nabla_X({}^{L}\nabla_Y Y + (Yf)Y - g(Y,Y)\mathrm{grad}f) \\
&= {}^{L}\nabla_X{}^{L}\nabla_Y Y + {}^{L}\nabla_X[(Yf)Y] - {}^{L}\nabla_X\mathrm{grad}f \\
&\quad +[({}^{L}\nabla_Y Y)f]X + (Yf)(Yf)X - [(\mathrm{grad}f)f]X \\
&\quad -g(X,{}^{L}\nabla_Y Y)\mathrm{grad}f - g(X,(Yf)Y)\mathrm{grad}f + g(X,\mathrm{grad}f)\mathrm{grad}f \\
&= {}^{L}\nabla_X{}^{L}\nabla_Y Y + X(Yf)Y + (Yf){}^{L}\nabla_X Y - {}^{L}\nabla_X\mathrm{grad}f \\
&\quad +[({}^{L}\nabla_Y Y)f]X + (Yf)^2X - \|\mathrm{grad}f\|^2X \\
&\quad -g(X,{}^{L}\nabla_Y Y)\mathrm{grad}f + (Xf)\mathrm{grad}f. \qquad (4.14)
\end{aligned}$$

Since we want to eventually get an equation similar to Equation (4.13), we take the inner product of Equation (4.14) with X. Again, noting that X and Y are orthonormal, we get

$$\begin{aligned}
g({}^{f(L)}\nabla_X{}^{f(L)}\nabla_Y Y, X) &= g({}^{L}\nabla_X{}^{L}\nabla_Y Y, X) + (Yf)g({}^{L}\nabla_X Y, X) - g({}^{L}\nabla_X\mathrm{grad}f, X) \\
&\quad +{}^{L}\nabla_Y Y + (Yf)^2 - \|\mathrm{grad}f\|^2 \\
&\quad -(Xf)g(X,{}^{L}\nabla_Y Y) + (Xf)^2. \qquad (4.15)
\end{aligned}$$

Our goal is to reduce Equation (4.15) and compare the sectional curvatures. We recall here that ${}^{L}\nabla$ is metrical (with respect to g) and also torsion free. Using the metrical condition with the orthonormal basis, we can deduce the following:

$$\begin{aligned}
2g({}^{L}\nabla_X Y, Y) &= Xg(Y,Y) = 0 \qquad (4.16) \\
g({}^{L}\nabla_Y X, Y) + g(X,{}^{L}\nabla_Y Y) &= Yg(X,Y) = 0 \qquad (4.17) \\
H_f(X,X) &= X(Xf) - ({}^{L}\nabla_X X)f \\
&= Xg(\mathrm{grad}f, X) - ({}^{L}\nabla_X X)f \\
&= g({}^{L}\nabla_X\mathrm{grad}f). \qquad (4.18)
\end{aligned}$$

Using Equations (4.16),(4.17),and (4.18), we can reduce Equation (4.15) to get

$$\begin{aligned}
g({}^{f(L)}\nabla_X{}^{f(L)}\nabla_Y Y, X) &= g({}^{L}\nabla_X{}^{L}\nabla_Y Y, X) + (Yf)g({}^{L}\nabla_X Y, X) \\
&\quad -H_f(X,X) + {}^{L}\nabla_Y Y + (Yf)^2 + (Xf)^2 \\
&\quad -\|\mathrm{grad}f\|^2 + (Xf)g({}^{L}\nabla_Y X, Y). \qquad (4.19)
\end{aligned}$$

In a similar fashion, we can find equations for the other terms of the Riemannian curvature in Equation (4.13) as

$$\begin{aligned} g({}^{f(L)}\nabla_Y {}^{f(L)}\nabla_X Y, X) &= g({}^L\nabla_Y {}^L\nabla_X Y, X) + Y(Yf) \\ &\quad +(Yf)g({}^L\nabla_Y X, X) \qquad (4.20) \\ g({}^{f(L)}\nabla_{[X,Y]} Y, X) &= g({}^L\nabla_{[X,Y]} Y, X) + (Yf)g([X,Y],X) \\ &\quad -(Xf)g([X,Y],Y). \qquad (4.21) \end{aligned}$$

We are interested in subtracting Equations (4.20) and (4.21) from (4.19), or [(4.19) − (4.20) − (4.21)]. Utilizing Equation (4.16), the definition of the Hessian, and the fact that ${}^L\nabla$ is torsion-free, we get

$$\begin{aligned} (Yf)g({}^L\nabla_X Y, X) - (Yf)g({}^L\nabla_Y X, X) - (Yf)g([X,Y],X) &= 0 \\ (Xf)g({}^L\nabla_Y X, Y) + (Xf)g([X,Y],Y) &= (Xf)g({}^L\nabla_X Y, Y) \\ &= 0 \\ ({}^L\nabla_Y Y)f - Y(Yf) &= -H_f(Y,Y). \end{aligned}$$

Using these equations, we can thus reduce [(4.19) − (4.20) − (4.21)] to the result that we wished to show. **Q.E.D.**

Curvatures of Different Connections

Recall our previous example with $\mathbb{R}^2_+$ and $\mathcal{H}$, where $f(x,y) = -\ln(y)$ was the conformal function between g (usual Riemannian metric) and $\widetilde{g}$. Using the usual coordinate chart with coordinate vector fields $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$, we give the Riemannian curvature coefficients for the metrical connections previously described. Two of the metrical connections will be metrical on $\mathbb{R}^2_+$: the Levi-Civita connection ${}^L\nabla$ and ${}^f\nabla$ defined earlier. The other two metrical connections will be metrical on $\mathcal{H}$: the Levi-Civita connection ${}^L\widetilde{\nabla}$ and $\widetilde{\nabla}$ defined in Equation (4.5) (using $f(x,y) = -\ln(y)$). In the ensuing table, we will use R to signify the Riemannian curvature coefficient of type $(0,4)$ and write R_{ijkl} to denote the curvature coefficients for both $\mathbb{R}^2_+$ and $\mathcal{H}$ (the purpose here is to see all the coefficients next to one another). Also, we use symmetries of the coefficients so that we need not list all of them (see Chapter 2 or Lee [9]).

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R. Coefficients	${}^{L}\nabla$	${}^{f}\nabla$	${}^{L}\widetilde{\nabla}$	$\widetilde{\nabla}$
$R_{1,1,1,1}$	0	0	0	0
$R_{1,1,1,2}$	0	0	0	0
$R_{1,1,2,2}$	0	0	0	0
$R_{1,2,2,1}$	0	$-\dfrac{1}{y^2}$	$-\dfrac{1}{y^4}$	0
$R_{1,2,1,2}$	0	$\dfrac{1}{y^2}$	$\dfrac{1}{y^4}$	0
$R_{1,2,2,2}$	0	0	0	0
$R_{2,2,2,2}$	0	0	0	0

Table 4.2: Comparison of Coefficients of Riemannian Curvature Tensors, $\mathbb{R}^2_+$ and $\mathcal{H}$

Chapter 5 Curvature of ${}^{f(L)}\nabla$ on Compact Riemannian 2-Manifolds as f Varies

5.1 Overview

In the previous section, we introduced ${}^{f(L)}\nabla$ and saw that this was a viable way to produce a vast array of connections which are all metrical with respect to the same Riemannian metric. We continue to look at metrical connections given by ${}^{f(L)}\nabla$ for some smooth function f on a manifold M. Specifically, we are interested in looking at properties of curvatures which are associated with this class of metrical connections, and ask which curvatures can be generated.

The central problem of Chapter 5 is to focus on a problem of global analysis in the realm of differential geometry. In general, and to put it in an oversimplified way, if one has a manifold with a variety of geometric structures, a global analysis approach means looking at how various well-known properties of the geometric structure vary as the geometric structure changes. For example, our work in previous chapters gives a method in which each smooth function gives rise to a certain type of metrical linear connection and describes the curvature of these connections with respect to the Levi-Civita connection (Theorem 4.3.1) .

We outline in this first section of Chapter 5 the specific global analysis problem we are considering. On a smooth manifold M of dimension n, each linear connection gives rise to a tensor of type $(0,4)$, specifically its Riemannian curvature (See Definition 2.4.2). Since each smooth function f on M gives a linear connection which is metrical with respect to a given Riemannian metric, we now have a way of generating a family of Riemannian curvature tensors by varying the function f. Said differently, we have a mapping from $\mathfrak{F}(M)$ into the set of possible curvature tensors on M. The general question of this section then concerns itself with seeing which tensors can be written as a curvature tensor defined by ${}^{f(L)}\nabla$ for some $f \in \mathfrak{F}(M)$, and whether the curvature tensor (uniquely?) determines such an f. In more informal words, these ideas lead to asking existence and uniqueness questions for curvatures coming from linear connections which are metrical with respect to a given Riemannian metric.

The major result for manifolds of dimension 2 is that the possible curvature tensors (here, $n = 2$ necessarily means the curvature tensors are smooth functions of M) obtained as described earlier must average to 2π times the Euler characteristic of M. Of course, the Euler characteristic is independent of the choice of Riemannian metric. Furthermore, two functions give metrical linear connections which produce the same curvature exactly when they differ by a constant. Our primary theorems (Theorems 5.6.1 and 5.6.2), which give criterion for the existence and uniqueness in the case $n = 2$, are proven using the theory of the Hodge Decomposition Theorem (see Warner [19] p.223).

5.2 Riemannian 2-Manifolds and ${}^{f(L)}\nabla$

We now describe the global analysis problem, which can be pictured as

$$\mathfrak{F}(M) \to \left\{ \begin{array}{c} \text{Metrical Linear Connections} \\ \text{with respect to } g \end{array} \right\} \to \{\text{Curvatures on } M\} \subset \left\{ \begin{array}{c} \text{Tensor Fields} \\ \text{of type } (0,4) \end{array} \right\}$$

From Theorem 4.3.1, we see how the sectional curvature of ${}^{f(L)}\nabla$ relates to the sectional curvature of the Levi-Civita connection ${}^{L}\nabla$. Let M be an $n-$dimensional manifold and let $\{X_1, X_2, \ldots, X_n\}$ be an orthonormal basis at a point $m \in M$. It is well known that the sectional curvatures of all planes spanned by vectors in this basis will determine the Riemannian curvature tensor of the Levi-Civita connection (see Boothby [3]); i.e. the collection

$$\{{}^{L}K(X_i, X_j) \text{ for } 0 \leq i, j \leq n \text{ and } i \neq j\}$$

determines the Riemannian curvature tensor. Therefore, in knowing information about the Levi-Civita connection, we can use Equation (4.12) to potentially give information about the curvatures associated with ${}^{f(L)}\nabla$. In terms of Equation (4.12) and a pair of orthonormal vectors X_i, X_j with $i \neq j$, the curvatures are related as

$$\begin{aligned} {}^{f(L)}K(X_i, X_j) &= {}^{L}K(X_i, X_j) + (X_i f)^2 + (X_j f)^2 \\ &\quad -\|\mathrm{grad} f\|^2 - H_f(X_i, X_i) - H_f(X_j, X_j). \end{aligned}$$

For arbitrary manifolds of dimension n, this last equation is really a system of differential equations. This system gets complicated to solve in dimensions that are three or larger. Thus, we focus our attention first to manifolds that are dimension 2 and compact, for which we have explicit results. The assumption of compactness gives us the tools to use to solve the $n = 2$ case.

For the next few sections, consider (M, g) to be a compact Riemannian 2-manifold. Recall that we defined the sectional curvature of a plane Π_m in the tangent space at $m \in M$ (see Definition 2.5.2) as, for $\{X, Y\}$ a basis for Π_m,

$$K(\Pi_m) = \frac{Rm(X, Y, Y, X)}{|X|^2|Y|^2 - (g(X, Y))^2}$$

Since we are working with 2-dimensional manifolds, the sectional curvature and Riemannian curvature are the same. In the case where the 2-manifold is actually a surface in $\mathbb{R}^3$, as in Gauss' Theorema Egregium, the sectional curvature is in fact the Gaussian curvature of M, which is the product of the eigenvalues of the Weingarten map at the point. This is described later in Theorem 5.4.1. Therefore, Equation

(4.12) relates the curvature of ${}^{f(L)}\nabla$ to that of the Gaussian curvature of ${}^{L}\nabla$ in the case when we look at surfaces in $\mathbb{R}^3$.

Now we turn our attention to some facts about 2-manifolds that will aid in the main result of this chapter. Consider the Hessian from Equation (4.12) on a 2-manifold. Recall that the Hessian of f with respect to a linear connection ∇ is given as

$$H_f(X,Y) = X(Yf) - (\nabla_X Y)\, f$$

for vector fields $X, Y \in \mathfrak{X}(M)$. Notice that Equation (4.12) involves the Hessian of vector fields with themselves, i.e. $H_f(X,X)$. We have seen the Hessian described in terms of the Laplacian in Equation (2.9). For an orthonormal basis $\{X,Y\}$,

$$\Delta f = -X^2 f - Y^2 f. \tag{5.1}$$

Therefore, the Hessian terms in Equation (4.12) reduce to the Laplacian:

$$-H_f(X,X) - H_f(Y,Y) = -X^2 f - Y^2 f = \Delta f. \tag{5.2}$$

Lastly, we consider the gradient term in Equation (4.12), which was $||\text{grad}f||^2$. Recall Definition 2.3.1. Using our basis $\{X,Y\}$ which is orthonormal at m, the gradient vector field can be written locally as

$$\text{grad}f = (Xf)X + (Yf)Y.$$

Therefore, the gradient term reduces to

$$\begin{aligned} ||\text{grad}f||^2 &= g(\text{grad}f, \text{grad}f) \qquad (5.3) \\ &= (Xf)^2 + (Yf)^2 \qquad (5.4) \end{aligned}$$

Using the facts established for 2-manifolds in Equations (5.2) and (5.3), we see that Equation (4.12) for 2-manifolds reduces to

$$\begin{aligned} {}^{f(L)}K(X,Y) &= {}^{L}K(X,Y) + (Xf)^2 + (Yf)^2 - \|\text{grad}f\|^2 - H_f(X,X) - H_f(Y,Y) \\ &= {}^{L}K(X,Y) + \Delta f. \qquad (5.5) \end{aligned}$$

Again, we emphasize that Equation (5.5) is true at each point m. As we have seen, looking to manifolds of dimension 2 has enabled us to simplify the equation relating the curvature of ${}^{f(L)}\nabla$ to that of ${}^{L}\nabla$. Knowing this relationship, what can we learn about the curvature of ${}^{f(L)}\nabla$? To answer this, we look to Hodge Theory and use the Gauss-Bonnet Theorem. Note that in dimension n, where $n \geq 3$, there is not the simplification that Equation (5.5) affords. That is why the higher dimensional cases, which have non-linearity in them, are much more difficult and will be a natural topic to consider as a research agenda for a future research program.

5.3 Basic Hodge Theory

The purpose of this section is to present some of the tools of Hodge Theory for our result later in the chapter. As such, we do not give a detailed view of Hodge Theory, but rather discuss the ideas that we find particularly useful for this chapter, and present the material for manifolds of any dimension (not just $n = 2$) as we did earlier. We encourage the reader to reference Frank Warner (see [19]) or John Lee (see Chapter 12 [10]) for the details of basic ideas of Hodge Theory.

For the initial exposition of Hodge Theory, we suppose (M, g) is an $n-$manifold and $f \in \mathfrak{F}(M)$. Recall Definition 1.4 of a skew-symmetric (alternating) tensor field (defined on the tangent space of a point $m \in M$). We define a differential p-form as follows.

Definition 5.3.1 *Let $X_1, X_2, \ldots, X_p \in \mathfrak{X}(M)$ for $p \leq n$ on an n-dimensional smooth manifold M. Define a* differential $p-$form ω on M *as a skew-symmetric tensor field of type* $(0, p)$

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow \mathbb{R},$$

where $\omega(X_1, X_2, \ldots, X_p) \in \mathfrak{F}(M)$.

Observe that this definition is similar to the definition of a Riemannian metric, with the difference being that we are choosing skew-symmetric tensor fields instead of symmetric positive definite tensor fields. We will denote the space of differential $p-$forms by $\Omega^p(M)$. Of particular interest, since Definition 5.3.1 is in terms of skew-symmetric tensors, we can conclude that "p-forms" for $p > n$ will be 0 (see Lee Chapter 12 [10]), since when two or more of the arguments in the $p-$form are the same, the form is 0. Also, $\Omega^0(M)$ is $\mathfrak{F}(M)$, which will make sense later when we discuss the differential operator. It follows that $\Omega^1(M)$ is simply the dual of the space of vector fields, or the set of co-vectors in M. In particular, within a coordinate chart$(x_1, \ldots, x_n)$, dx^i is the 1-form associated with the coordinate vector field $\frac{\partial}{\partial x_i}$ such that

$$dx^i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

To get a better idea of what a $p-$form for $p \neq 1$ looks like locally, we introduce the wedge product (or exterior product) on forms.

Definition 5.3.2 *Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^p(M)$ for M a smooth n-manifold. For $X_1, X_2, \ldots, X_{k+p} \in \mathfrak{X}(M)$, define the* wedge product *of ω and η as the (alternating) $(k + l)$-form*

$$(\omega \wedge \eta)(X_1, \ldots, X_{k+p}) = \frac{1}{k!p!} \sum_{\sigma \in S_k} \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) \cdot \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+p)})$$

The proof that $\omega \wedge \eta$ is indeed a $(k+p)$-form can be found in the literature and is quite technical. We omit the proof here, and maintain a focus on the use of forms for our later results. Using the wedge product and a basis argument for $\Omega^k(M)$, we conclude that in a coordinate chart with n coordinates $(x_1, x_2, \ldots, x_n)$, ω can be expressed as

$$\omega = \sum_I \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Here, the ω_I are smooth functions in the coordinate chart. Also, I represents k of the n indices in increasing order. We are therefore summing over possible combinations of k indices included in the $k-$form ω. Here are some examples of forms in $\mathbb{R}^2$.

Forms in $\mathbb{R}^n$ Let $x_1, x_2, \ldots x_n$ be coordinates for the usual chart on $\mathbb{R}^n$.

$dx^1, \ldots, dx^n$

2-Forms: $dx^3 \wedge dx^1 + dx^2 \wedge dx^1 + dx^2 \wedge dx^n$, $\sin(x_1 x_3) dx^1 \wedge dx^n$

$\vdots$

n-**Forms:** $(e^{x_2}) dx^1 \wedge \cdots \wedge dx^n$.

Now that we have established what differential forms are, we present operators (exterior derivative, Hodge star operator, and Laplace-Beltrami operator) on $\Omega^p(M)$, which send $p-$forms to forms in $\Omega^k(M)$ for some $k \leq n$.

Exterior Derivative

Suppose $f \in \mathfrak{F}(M)$. We defined the differential *df* in Chapter 1 (Definition 1.4.1) as a map which sends tangent vectors of M into $\mathbb{R}$. Notice that this is exactly that *df* is a co-vector or 1-form. We extend the operator d to $p-$forms naturally. In a coordinate chart, let

$$\omega = \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

be a basis $p-$form in $\Omega^p(M)$. Here, ω_I is a smooth function in the coordinate chart. We define $d\omega$ as

$$d\omega = d\omega_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

There is a rather important issue of whether the operator d is well-defined with respect to coordinate charts. The fact is well-known and technical, and we refer the reader to Lee (see [10] P.306). As such, we have thus defined an operator which increases forms by 1 dimension; i.e.

Definition 5.3.3 *Let $p < n$ and let M be a smooth n-manifold. Define d on forms as discussed above. We call d the* exterior derivative*, and d is the map*

$$d : \Omega^p(M) \longrightarrow \Omega^{(p+1)}(M)$$

Looking back, defining differential 0-forms to be smooth functions fits with how we defined the differential of a function in Chapter 1, as dF is a differential 1-form. That $d^2 = 0$ is an important conclusion, which is again quite technical.

Hodge Star Operator

To properly present the Hodge star operator, we need to discuss orientation of manifolds. Observe that for M an n-dimensional smooth manifold, each differential p-forms (where $p = n$) in a coordinate chart can be described by

$$dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n. \tag{5.6}$$

Thus the space of differential n-forms is one dimensional. Thinking of the wedge in Equation (5.6) as a skew-symmetric tensor field $\mathcal{T}$, an orientation on M is a choice of either $\mathcal{T}^{-1}(\mathbb{R}_+)$ or $\mathcal{T}^{-1}(\mathbb{R}_-)$. Thus we define the Hodge star operator (see Warner [19] p.79):

Definition 5.3.4 *Let M be an oriented smooth n-manifold and let $\varepsilon^1, \ldots, \varepsilon^n$ be differential 1-forms in a coordinate chart on M which are orthonormal. For $p \leq n$, we define the* Hodge star operator $*$ *on basis p-forms as*

$$*(e^1 \wedge \cdots \wedge e^p) = \pm e^{p+1} \wedge \cdots \wedge e^n,$$

where the $(n-p)$-form on the right is "+" if $e^1 \wedge \cdots \wedge e^n$ lies in the positive component of the orientation and $-$ otherwise.

The action of the Hodge star operator takes p-forms to $(n-p)$-forms and is dependent on the orientation of the manifold M.

Laplace-Beltrami Operator

This last operator we define connects the previous 2 operators to the well-known Laplacian operator (which we have seen in Equation (5.5)). First, define δ as an operator from $p-$forms to $(p-1)$-forms as

$$\delta = (-1)^{n(p+1)+1} * d * .$$

One can easily follow the actions of the operators $*$ and d to see that δ does indeed take p-forms to $(p-1)$-forms.

Definition 5.3.5 *Let M be an oriented smooth n-manifold. The* Laplace-Beltrami operator Δ *is an operator taking $p-$forms to $p-$forms defined as*

$$\Delta = \delta d + d\delta.$$

This definition of Δ looks different than our previous definition of the Laplacian Δ seen in Equation (2.9). Lang establishes (see [8] p.410) that these two definitions of Δ are indeed the same. Therefore, we will refer to the Laplace-Beltrami operator as the Laplacian from here on, with the understanding that Δ has multiple descriptions.

Having established these operators, we now present some ideas that will aid in establishing the result we use from Hodge Theory. At this point, we now suppose that M is a connected compact oriented Riemannian manifold (not necessarily a 2-manifold). Recall (see Warner Chapter 2 [19]) that an arbitrary differential n-form ω can be expressed (uniquely) in a coordinate chart as

$$\omega = \omega_I \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

for some $\omega_I \in \mathfrak{F}(M)$. We define integration of differential n-forms as

$$\int_M \omega = \int_M \omega_I \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \int_M (\omega_I) dx^1 dx^2 \cdots dx^n.$$

In simple terms, we think of the abstract idea of integration on M of a differential p-form by using the local Euclidean structure of M and integrating as usual in $\mathbb{R}^n$. Since M is compact, our definition will make sense for all differential forms (for non-compact manifolds, more work needs to be done; this exposition will suffice for our needs).

Another advantage of compactness is that we can define an inner product on differential $p-$forms, since integration on compact smooth manifolds is defined. Define the inner product of $\alpha, \beta \in \Omega^p(M)$ as

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta.$$

Proposition 5.3.6 *Let $\alpha, \beta \in \Omega^p(M)$, where (M, g) is a compact Riemannian manifold. The operator δ is the adjoint of d in the inner product $\langle , \rangle$; that is*

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

Proof See Warner ([19] p.221)

Proposition 5.3.6 has an especially useful consequence in terms of compact Riemannian manifolds and differential 0-forms. We turn to understanding information about Δf found in Equation (5.5) of the previous section. Recall that differential 0-forms are nothing more than smooth functions on the manifold M. Therefore, the ideas established in Proposition 5.3.6 and the inner product defined on differential $p-$forms applies to differential 0-forms as well.

Theorem 5.3.7 *Let h be a differential 0-form on a connected, compact, oriented Riemannian manifold (M, g). Then h is harmonic, i.e. $\Delta h = 0$, if and only if h is a constant function*

Proof We show that it is necessary and sufficient for $dh = 0$. If this happens, h must be constant. Observe that if h is constant, then $\Delta h = 0$ by the definition of Δ given in Equation (2.9).

Conversely, suppose $\Delta h = 0$. Using Theorem 5.3.6 and the definition of Δ, we see

$$\begin{aligned} 0 &= \langle \Delta h, h \rangle \\ &= \langle (d\delta + \delta d)h, h \rangle \\ &= \langle \delta h, \delta h \rangle + \langle dh, dh \rangle. \end{aligned}$$

Since δh is a (-1)-form, as h is a 0-form, $\delta h = 0$. Therefore, dh must be 0, and h is constant. **Q.E.D.**

Notice that without M compact, we are unable to nicely define an inner product on forms, and could not obtain the result shown in Theorem 5.3.7 (although one could work with compactly supported differential forms on manifolds). The result that is of particular interest is that only constant functions are harmonic on connected components of compact manifolds. We shall revisit these ideas later in the proof of our main result.

5.4 Gauss-Bonnet Theorem as Context for Section 5.6

In classical differential geometry, mathematicians considered 2-dimensional surfaces which are embedded in $\mathbb{R}^3$. Using this approach, the structure of the embedded surface was described in terms of the embedding, and eventually some results were shown to be independent of the actual embedding into $\mathbb{R}^3$. We present some of the classical ideas here, and connect these ideas to the curvature equation given in Equation (5.5).

Let K be the Gaussian curvature of a surface in $\mathbb{R}^3$. That is, K is the Riemannian curvature of the Levi-Civita connection of the Riemannian metric which is induced from (flat) $\mathbb{R}^3$. Recall from Gauss' Theorema Egregium that K, which is a function on M, is independent of the basis chosen to define it (because K is a tensor).

The next important result is that the Gaussian curvature of a surface is independent of how the surface is embedded in $\mathbb{R}^3$. The Gaussian curvature can be written (and is usually defined as such in introductory differential geometry courses) in terms of the determinant of the Weingarten map. Since the Weingarten map L depends on the normal to M, it depends on the embedding. However, we find by Gauss' Theorema Egregium (stated differently than in Theorem 2.5.1) that

Theorem 5.4.1 *For M a surface embedded in $\mathbb{R}^3$, where $g = \det(g_{ij})$, we locally get*

$$K = \det({L_j}^k) = \sum_i {R_{121}}^i (g_{i2}/g). \tag{5.7}$$

Proof See Millman-Parker ([12] p.143)

Within Equation (5.7), observe that ${R_{121}}^i$ corresponds to the Riemannian curvature tensor in a local chart, and the coefficients ${L_j}^k$ come from the Weingarten map in the same local chart.

The last aspect of the independent nature of K on a Riemannian manifold (not necessarily a surface in $\mathbb{R}^3$) is that different Riemannian metrics on M usually give different curvatures. Yet the average of curvature on a 2-manifold is a multiple of the Euler characteristic, $\chi(M)$, of M. This result is stated here, which is given as seen in Hicks (see [6]) or Lee (see [9]).

Theorem 5.4.2 (Gauss-Bonnet Theorem) *Let M be a compact, connected, oriented Riemannian 2-manifold with Riemannian (Gaussian) curvature function K. Then*

$$\int_M K = 2\pi\chi(M) \tag{5.8}$$

Of particular importance in the midst of these results is the independence of the embedding of the 2-manifold. We interpret the results of the Gauss-Bonnet Theorem as stating the average of the Riemannian Curvature over M is a multiple of its Euler characteristic. The astounding fact (hence the word "egregium") is that the integral of a purely geometric property (curvature) is independent of the geometry and just depends on the topology of M. Here, the Riemannian curvature is associated with the Levi-Civita connection. As we will see, this result is not just for the special case of the connection being metrical and having zero-torsion.

5.5 The Integral of the Curvature of ${}^f\nabla_X Y$ on Compact 2-Manifolds

Assume that (M, g) is a compact, oriented, connected Riemannian 2-manifold. We have seen that ${}^f\nabla$ is a metrical connection with respect to g and have established

Equation (5.5) to relate the sectional curvature of ${}^f\nabla$ to that of ${}^L\nabla$ (which depends on M being compact, connected, and oriented). We re-state that relationship here:

$$^{f(L)}K(X,Y) = {}^LK(X,Y) + \Delta f.$$

We view this partial differential equation on a manifold as a relationship which relates a smooth function f, which defines a metrical connection ${}^{f(L)}\nabla$, to the sectional curvature of ${}^{f(L)}\nabla$ and the sectional curvature of ${}^L\nabla$, the Levi-Civita connection of the manifold. We turn to the tools of the Hodge Decomposition Theorem to exploit this relationship and prove the major theorems in this chapter.

Moving forward, we will need to know information about the role Δf plays in this equation. Let ω be a differential p-form of M. We say that ω is harmonic if

$$\Delta\omega = 0.$$

We define some important subsets of $\Omega^p(M)$ as

$$\begin{aligned}\mathcal{H}^p &= \text{the space of harmonic } p\text{-forms}\\ \Delta(\Omega^p(M)) &= \{\beta \in \Omega^p(M) | \Delta\alpha = \beta\} = \text{image of the Laplacian.}\end{aligned}$$

To understand Δf, we consider the following powerful decomposition.

Theorem 5.5.1 (Hodge Decomposition Theorem) *Let $0 \leq p \leq n$ for M an $n-$dimensional Riemannian manifold. The space of $p-$forms decomposes as an orthogonal direct sum as follows:*

$$\Omega^p(M) = \Delta(\Omega^p(M)) \oplus \mathcal{H}^p.$$

Furthermore, $\Delta\omega = \alpha$ has a solution $\omega \in \Omega^p(M)$ if and only if α is orthogonal to $\mathcal{H}^p$.

Proof See Warner ([19]).

The Hodge Decomposition Theorem is a much stronger result than we will need, but it beautifully describes $p-$forms in terms of the Laplacian and harmonic forms. As we move forward, we will only need this result to help describe smooth functions (which are 0-forms).

We now state one of the main theorems of this chapter for ${}^{f(L)}\nabla$. For emphasis, we re-write the definition of ${}^{f(L)}\nabla$:

$$^{f(L)}\nabla_X Y = {}^L\nabla_X Y + (Yf)X - g(X,Y)\text{grad}f.$$

Theorem 5.5.2 *If M is a connected, compact, oriented, Riemannian 2-manifold, then*

$$\int_M ({}^{f(L)}K) \quad = \quad 2\pi\chi(M),$$

where ${}^{f(L)}K$ is the Gaussian curvature associated with the metrical connection ${}^{f(L)}\nabla$.

Proof Recall that on a 2-dimensional manifold, sectional curvature is a smooth function $K \in \mathfrak{F}(M)$. Let $\{X, Y\}$ be a basis in a neighborhood of $m \in M$ orthonormal at m. From Equation (5.5) we see the curvatures of ${}^{L}\nabla$ and ${}^{f(L)}\nabla$ are related as

$$ {}^{f(L)}K(X,Y) \quad = \quad {}^{L}K(X,Y) + \Delta f. \tag{5.9}$$

Since the sectional curvatures and the Laplacian are independent of the basis we choose (about a point), we are left with the differential equation defining $h \in \mathfrak{F}(M)$

$$\begin{aligned} \Delta f &= {}^{f(L)}K - {}^{L}K \\ &= h. \end{aligned} \tag{5.10}$$

What Equation (5.10) tells us is that ${}^{f(L)}K$ is the curvature of ${}^{f(L)}\nabla$ as long as f satisfies the equation given. Viewing each of the terms in Equation (5.10) as functions in $\mathfrak{F}(M)$, and furthermore as 0-forms, we can apply the Hodge Decomposition Theorem. Thus to solve Equation (5.10) it is necessary and sufficient for h to be the image of $\Delta(\Omega^p(M))$ with $p = 0$. That is the same as h being orthogonal to the space of harmonic 0-forms. From Theorem 5.3.7, harmonic 0-forms are simply constant functions on M. Let $C > 0$ be a constant function on M. Using properties of integration, the star operator, and the inner product of forms, we see

$$\begin{aligned} 0 &= \langle h, C\rangle \\ &= \int_M h \wedge *C \\ &= \int_M C(h \wedge *1) \\ &= C\int_M h. \end{aligned}$$

Hence ${}^{f(L)}K$ is the sectional curvature provided that $\int_M h = 0$. Using this fact and also that ${}^{L}K$ is the Gaussian curvature for the Levi-Civita connection, the Gauss-Bonnet Theorem provides us the result:

$$\begin{aligned} \int_M h = 0 \quad &\Leftrightarrow \quad \int_M ({}^{f(L)}K) - \int_M ({}^{L}K) = 0 \\ &\Leftrightarrow \quad \int_M ({}^{f(L)}K)\, dA = 2\pi\chi(M). \end{aligned}$$

Q.E.D.

Putting this result into the similar words used earlier, the sectional curvature of ${}^{f(L)}\nabla$ has average $2\pi\chi(M)$ when we integrate over the entire manifold M. We have successfully extended the result from connections with zero-torsion to a class of connections which are defined using smooth functions. This holds true for Riemannian manifolds which are compact, oriented, and dimension 2. Yet we can learn much more about curvature on compact 2-manifolds and discuss extensions to Riemannian manifolds of dimension 3 in future research.

5.6 A Global Analysis Result: Characterization of the Metrical Connection ${}^{f}\nabla$ with a given Curvature

In the last section, we saw that the sectional curvature associated with ${}^{f(L)}\nabla$ must have an "average" of $2\pi\chi(M)$. Recall that the sectional curvature ${}^{f(L)}K$ can be thought of as a smooth function when dealing with 2-manifolds. There are many functions whose "average" over M is $2\pi\chi(M)$. Naturally, we wonder for which functions can we find an f so that ${}^{f(L)}\nabla$ gives sectional curvature which is a given function. In other words, can we find a change of metrical connection to give a desired curvature whose integral is $2\pi\chi(M)$?

Suppose we start with $\mathfrak{h} \in \mathfrak{F}(M)$, where (M, g) continues to be a compact, connected, oriented, Riemannian 2-manifold. We deal with the question of whether there is a metrical connection with respect to g whose curvature is given by the function $\mathfrak{h}$. Using the ideas of Hodge Theory, we will show that $\mathfrak{h}$ can be expressed as a curvature of the form ${}^{f(L)}K$ for some $f \in \mathfrak{F}(M)$ if and only if the results of Theorem 5.5.2 hold. We state and prove here.

Theorem 5.6.1 *Let $\mathfrak{h} \in \mathfrak{F}(M)$, where (M, g) is a compact, oriented, connected Riemannian 2-manifold. Then there exists $f \in \mathfrak{F}(M)$ such that the sectional curvature of ${}^{f(L)}\nabla$ is $\mathfrak{h}$ if and only if*

$$\int_M \mathfrak{h} = 2\pi\chi(M).$$

Proof If such an f exists, we have shown in Theorem 5.5.2 that $\mathfrak{h}$ follows to integrate to $2\pi\chi(M)$. Suppose $\mathfrak{h}$ integrates to $2\pi\chi(M)$. By the Hodge Decomposition Theorem, $\Delta f = h$ has a solution if and only if h integrates to zero (shown in Theorem 5.5.2); that is, h is orthogonal to harmonic functions (constants). If h is the difference between $\mathfrak{h}$ and ${}^{L}K$, i.e.

$$h = \mathfrak{h} - {}^{L}K,$$

then such an f exists (as $\int_M h = 0$). We choose this f, which induces ${}^{f(L)}\nabla$ to have curvature $\mathfrak{h}$ by construction. **Q.E.D.**

From this result, we can see that the possible sectional curvatures that exist for fixed (M, g) are quite diverse. Even more, as we show in the next theorem, the only way two functions f_1 and f_2 could hope to produce the same sectional curvature (via ${}^{f_1(L)}\nabla$ and ${}^{f_2(L)}\nabla$) is when the functions are the same up to an additive constant.

Theorem 5.6.2 *Let (M, g) be a compact, connected, Riemannian 2-manifold. Let $f_1, f_2 \in \mathfrak{F}(M)$ and let ${}^{f_1(L)}K, {}^{f_2(L)}K$ represent the sectional curvature of ${}^{f_1(L)}\nabla, {}^{f_2(L)}\nabla$ respectively. Then ${}^{f_1(L)}K = {}^{f_2(L)}K$ if and only if the difference of f_1 and f_2 is constant.*

Proof Using Equation (5.5), we find

$$\begin{aligned} \Delta f_1 &= {}^{f_1(L)}K - {}^{L}K && (5.11) \\ \Delta f_2 &= {}^{f_2(L)}K - {}^{L}K. && (5.12) \end{aligned}$$

Looking at the difference of Equation (5.11) and Equation (5.12), we get

$$\Delta(f_1 - f_2) = {}^{f_1(L)}K - {}^{f_2(L)}K. \tag{5.13}$$

If the right side of Equation (5.13) is 0, then $(f_1 - f_2)$ is harmonic on M (compact). So by Theorem 5.3.7, $(f_1 - f_2)$ is constant. If the left hand side of Equation (5.13) is such that $(f_1 - f_2)$ is constant, then clearly the sectional curvatures ${}^{f_1(L)}K$ and ${}^{f_2(L)}K$ are equal. **Q.E.D**.

What we have shown for compact Riemannian 2-manifolds is very intriguing. Once we defined a new class of metrical connections ${}^{f(L)}\nabla$ which depend on smooth functions, we saw that the sectional curvature induced from this could be any function whose "average" was a multiple of the characteristic function of M. Then, we proved that the functions used to define our metrical connections would only produce the same curvature if the functions were the same up to an additive constant. In a sense, f_1 must be uniquely different from f_2 (up to a constant) if we ever wish the curvatures coming from them to be the same.

5.7 The Sectional Curvature of ${}^{f(L)}\nabla$ on Higher Dimensional Manifolds

For compact manifolds of dimension 2, we were able to describe the sectional curvature associated with ${}^{f(L)}\nabla$ and $f \in \mathfrak{F}(M)$. Looking back to how we accomplished this, the key was to simplify

$$\begin{aligned} {}^{f(L)}K(X, Y) &= {}^{L}K(X, Y) + (Xf)^2 + (Yf)^2 \\ &\quad - \|\mathrm{grad} f\|^2 - H_f(X, X) - H_f(Y, Y) && (5.14) \end{aligned}$$

using the Laplacian to get the following equation:

$$ {}^{f(L)}K(X,Y) = {}^{L}K(X,Y) + \Delta f. \tag{5.15} $$

Simplifying Equation (5.14) to Equation (5.15) was accomplished precisely because the compact manifold was assumed to have dimension 2. In turn, because the analysis of the Laplacian is so well understood, we were able to use Equation (5.15) to prove the key necessary and sufficient conditions of Section 5.6.

Naturally, the next step is to learn about the sectional curvature for ${}^{f(L)}\nabla$ for manifolds of dimension $n \geq 3$. To gain insight, we consider what happens when $n = 3$. Suppose (M, g) is a Riemannian manifold of dimension 3 and $f \in \mathfrak{F}(M)$. Let $X, Y, Z \in \mathfrak{X}(M)$ be an orthonormal basis locally (about a point $m \in M$) which is constructed as in Equation (2.8). We investigate Equation (5.14) with respect to the plane spanned by X and Y. Observe that $\text{grad} f$ and the Hessian terms in Equation (2.8) with respect to our basis are

$$ \begin{aligned} -\|\text{grad} f\|^2 &= -(Xf)^2 - (Yf)^2 - (Zf)^2 && (5.16) \\ -H_f(X,X) - H_f(Y,Y) &= -X^2 f - Y^2 f && \\ &= \Delta f + Z^2 f. && (5.17) \end{aligned} $$

Here, we used Equation (2.8) and Equation (2.9). The addition of a third vector in the basis does not give us the nice cancelations and substitutions that we had before. Equation (5.14) thus reduces to

$$ {}^{f(L)}K(X,Y) = {}^{L}K(X,Y) + \Delta f - (Zf)^2 + Z^2 f. \tag{5.18} $$

Here, we need to emphasize that Equation (5.18) relates the sectional curvatures of ${}^{f(L)}\nabla$ to that of ${}^{L}\nabla$ with respect to the plane spanned by X and Y. With regards to our problem of working on a manifold of dimension 3, we need to determine facts about sectional curvature of planes spanned by X, Y and Y, Z and X, Z (for a basis $\{X, Y, Z\}$ spanning $T_p(M)$ at $p \in M$). Therefore, we are working with the system

$$ \begin{aligned} {}^{f(L)}K(X,Y) &= {}^{L}K(X,Y) + \Delta f - (Zf)^2 + Z^2 f \\ {}^{f(L)}K(Y,Z) &= {}^{L}K(Y,Z) + \Delta f - (Xf)^2 + X^2 f \\ {}^{f(L)}K(Z,X) &= {}^{L}K(Z,X) + \Delta f - (Yf)^2 + Y^2 f \end{aligned} $$

This system of equations shows the difficulty if the dimension of M is greater than 2. The dimension 3 case and the dimension 2 case both involve the Laplacian on a manifold. However, the dimension 2 case gives linear partial differential equations to solve. Because of the $(Zf)^2$ term (for example), the dimension 3 (and higher) case is non-linear. The non-linearity means that additional analytical tools will need to be brought to the task when the dimension is greater than 2.

There is a technical point here. There may be a difference between knowing the Riemann curvature and knowing the sectional curvature. In fact, either curvature determines the other. The following well-known result (given in Boothby p. 385 [3]) shows the relationship between the sectional curvatures of planes in the tangent space and the Riemannian curvature tensors.

Theorem 5.7.1 *If* dim $M \geq 3$ *and the sectional curvature is known on all sections of* $T_p(M)$*, then the Riemann curvature tensor is uniquely determined at* $p \in M$*.*

In other words, using the linearity of sectional curvature and Riemannian curvature, it is enough to find the sectional curvature of all possible planes spanned by pairs of tangent vectors included in a basis which spans the tangent space.

Notice that the result given in Theorem 5.7.1 is given with respect to the Levi-Civita connection. Within the proof of Theorem 5.7.1, we use properties of the Levi-Civita connection which rely on the fact that the connection is torsion-free. Recall that ${}^{f(L)}\nabla$ is not necessarily torsion-free. Thus, we cannot easily hope to directly use this result coupled with the system of equations given above to help relate the Riemannian curvature of ${}^{f(L)}\nabla$ to that of ${}^{L}\nabla$. Furthermore, we are not allowed the luxury of the results we used for compact manifolds of dimension 2. Hence, even if we hoped to compare the Riemannian curvatures, the results we established here in Chapter 5 would not hold true without further work. Questions like these are left unanswered. Yet, as shown with compact 2-manifolds, there is potential in looking through the machinery we set up throughout this paper on other classes of manifolds, and is an avenue for future research.

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