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EIGENVALUE MULTIPLICITES OF THE HODGE LAPLACIAN ON COEXACT 2-FORMS FOR GENERIC METRICS ON 5-MANIFOLDS

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EIGENVALUE MULTIPLICITES OF THE HODGE LAPLACIAN ON COEXACT 2-FORMS FOR GENERIC METRICS ON 5-MANIFOLDS

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
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Lexington, Kentucky

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ABSTRACT OF DISSERTATION

EIGENVALUE MULTIPLICITES OF THE HODGE LAPLACIAN ON COEXACT 2-FORMS FOR GENERIC METRICS ON 5-MANIFOLDS

In 1976, Uhlenbeck used transversality theory to show that for certain families of elliptic operators, the property of having only simple eigenvalues is generic. As one application, she proved that on a closed Riemannian manifold, the eigenvalues of the Laplace-Beltrami operator Δ_g are all simple for a residual set of C^r metrics. In 2012, Enciso and Peralta-Salas established an analogue of Uhlenbeck's theorem for differential forms, showing that on a closed 3-manifold, there exists a residual set of C^r metrics such that the nonzero eigenvalues of the Hodge Laplacian $\Delta_g^{(k)}$ on k-forms are all simple for $0 \leq k \leq 3$. In this dissertation, we continue to address the question of whether Uhlenbeck's theorem can be extended to differential forms. In particular, we prove that for a residual set of C^r metrics, the nonzero eigenvalues of the Hodge Laplacian $\Delta_g^{(2)}$ acting on coexact 2-forms on a closed 5-manifold have multiplicity 2. To prove our main result, we structure our argument around a study of the Beltrami operator $*_g d$, which is related to the Hodge Laplacian by $\Delta_g^{(2)} = -(*_g d)^2$ when the operators are restricted to coexact 2-forms on a 5-manifold. We use techniques from perturbation theory to show that the Beltrami operator has only simple eigenvalues for a residual set of metrics. We further establish even eigenvalue multiplicities for the Hodge Laplacian acting on coexact k-forms in the more general setting $n = 4\ell + 1$ and $k = 2\ell$ for $\ell \in \mathbb{N}$.

KEYWORDS: Hodge Laplacian, Beltrami operator, perturbation theory, eigenvalue multiplicities, geometric analysis

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EIGENVALUE MULTIPLICITES OF THE HODGE LAPLACIAN ON COEXACT 2-FORMS FOR GENERIC METRICS ON 5-MANIFOLDS

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To my grandparents, Paul and Rae Grimmig.

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Praise be to the name of God for ever and ever;
wisdom and power are his.
He gives wisdom to the wise
and knowledge to the discerning.
He reveals deep and hidden things;
he knows what lies in darkness,
and light dwells with him.
Daniel 3:20-22

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Chapter 1 Introduction

Determining the spectrum of linear operators is a prominent topic in functional analysis. Let A be a linear operator on a Banach space X with domain $D(A)$. The *spectrum* $\sigma(A)$ of A consists of all values $\lambda \in \mathbb{C}$ such that the operator $A - \lambda$ is not boundedly invertible. A point $\lambda \in \sigma(A)$ is called an *eigenvalue* of A if $\ker(A - \lambda) \neq \{0\}$. The *eigenspace* of A at λ is defined to be $E(A, \lambda) = \ker(A - \lambda)$, and the dimension of $E(A, \lambda)$ is the *multiplicity* of λ.

The spectrum of a self-adjoint elliptic operator on a compact manifold consists of isolated eigenvalues, each of which has finite multiplicity. One such operator is the Laplacian Δ on $L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$. Its eigenvalues are given by $\lambda = \ell(\ell + 1)$ for $\ell = 0, 1, 2, \ldots$ and have multiplicity $2\ell + 1$. While each eigenspace is finite-dimensional, the dimension of $E(\Delta, \lambda)$ grows unboundedly large as λ approaches infinity.

The example of the Laplacian on $L^2(S^2)$ might cause one to suppose that multi-dimensional eigenspaces are typical of self-adjoint elliptic operators. However, Uhlenbeck [28] showed in 1976 that for certain families of elliptic operators, the property of having only simple eigenvalues – that is, eigenvalues of multiplicity 1 – is generic.

Theorem 1.0.1. (Uhlenbeck, [28]) *Let M be a connected compact manifold and L_b be a family of self-adjoint second order elliptic operators on M, where the parameter b lies in an open subset of a Banach space B. Let*

$$S_k^p = \{u \in H_k^p(M) \cap H_{1,0}(M) \mid \|u\|_{L^2(M)} = 1\},$$

and define $\varphi : S_k^p \times \mathbb{R} \times B \to H_{k-2}^p(M)$ by

$$\varphi(u, \lambda, b) = (L_b + \lambda)u.$$

If 0 *is a regular value of* φ*, then the set*

$$\{b \in B \mid L_b \textit{ has one-dimensional eigenspaces}\}$$

is residual in B*.*

Theorem 1.0.1 indicates that self-adjoint second order elliptic operators on a compact manifold will in general have simple eigenvalues. This result may seem counterintuitive given that many operators with computable eigenvalues, such as the Laplacian on $L^2(S^2)$, have multidimensional eigenspaces. However, the repeated eigenvalues often spring from the high degree of symmetry present in these examples, and as symmetry is an exceptional property, it is reasonable that the operators will more typically have simple eigenvalues. Uhlenbeck's proof of Theorem 1.0.1 employs techniques from infinite-dimensional transversality theory based on works by Abraham [1], Smale [27], and Quinn [23].

As an application of Theorem 1.0.1, Uhlenbeck gives the example of the family of operators $L_b = L + b$, where L is any self-adjoint second order elliptic operator with smooth coefficients and b comes from the space $B = C_0^k(U)$ for some open subset $U \subset M$. This family of operators had been studied in 1975 by Albert [3], who likewise proved the genericty of simple eigenvalues. While Albert's approach from perturbation theory is more direct than Uhlenbeck's proof based on transversality, it requires him to take the space of functions to be $B = C^\infty(M)$, making Uhlenbeck's the stronger of the two results.

Uhlenbeck's second example applies Theorem 1.0.1 to the family of Laplace-Beltrami operators Δ_g. In this setting, the parameter space is

$$\mathfrak{M}_r = \{g \in \mathcal{G}^r(M) \mid (g - g_0)|_{M-U} = 0\},$$

the set of all C^r metrics on a manifold M which differ from a fixed metric g_0 only on some open subset $U \subset M$. In particular, she establishes the following:

Theorem 1.0.2. (Uhlenbeck, [28]) *Let Δ_g be the Laplace-Beltrami operator for a metric $g \in \mathfrak{M}_r$ for $r > n + 3$. Then the set*

$$\{g \in \mathfrak{M}_r \mid \Delta_g \textit{ has one-dimensional eigenspaces}\}$$

is residual in $\mathfrak{M}_r$.

In 1980, Bleecker and Wilson [9] proved a similar result using eigenvalue perturbation theory. They show that under conformal perturbations $g_f = e^f g_0$ of a fixed metric g_0, the Laplace-Beltrami operator Δ_{g_f} has only simple eigenvalues for a residual set of functions $f \in C^\infty(M)$. Their method is more constructive than Uhlenbeck's, but they lose the ability to restrict the metrics' deviance from g_0 to an open set $U \subset M$.

In light of Uhlenbeck's result for the Laplace-Beltrami operator on functions, one might wonder if the eigenvalues of the Hodge Laplacian $\Delta_g^{(k)}$ acting on k-forms might likewise be simple for a residual set of metrics. However, Theorem 1.0.2 does not have an automatic analogue to the Hodge Laplacian. In 1980, Millman [20] observed that on a manifold of dimension $n = 2k$, the McKean-Singer Télescopage Theorem [7] implies that all eigenvalues of the Hodge Laplacian acting on k-forms have even multiplicity. Consequently, when M is a manifold of even dimension, the eigenvalues of the Hodge Laplacian acting on forms of middle rank cannot be simple.

While Millman's observation precludes a general extension of Uhlenbeck's theorem to the Hodge Laplacian, the possibility remains that an analogue might hold under appropriate hypotheses. In 2012, Enciso and Peralta-Salas [12] established a result similar to Theorem 1.0.2 for the Hodge Laplacian on a closed manifold of dimension 3.

Theorem 1.0.3. (Enciso and Peralta-Salas, [12]) *Let M be a closed 3-manifold and $r \geq 2$ be an integer. There exists a residual subset Γ of the space of C^r metrics on*

M such that, for all $g \in \Gamma$, the nonzero eigenvalues of the Hodge Laplacian Δ_g on k-forms have multiplicity 1 *for all* $0 \leq k \leq 3$.

Enciso and Peralta-Salas structure their proof around the study of the Beltrami operator $*_g d$ on coexact 1-forms, which they show to have simple spectrum using transversality theory. This fact, when combined with Uhlenbeck's Theorem 1.0.2, allows them to conclude that on a 3-manifold, the nonzero eigenvalues of the Hodge Laplacian are generically simple.

In light of Millman's comment regarding manifolds of even dimension, the next natural case to which we might hope to extend Uhlenbeck's Theorem 1.0.2 is the Hodge Laplacian on a five-dimensional manifold. In this dissertation, we use perturbation theory to study the generic eigenvalue multiplicities of the Hodge Laplacian on a closed 5-manifold. In particular, we prove the following result:

Theorem 4.2.4. *Let M be a closed* 5*-manifold, and let r be an integer, $r \geq 2$. There exists a residual subset Γ of the space of all C^r metrics on M such that, for all $g \in \Gamma$, the eigenvalues of the restriction of the Hodge Laplacian $\Delta_g^{(2)}$ to coexact* 2*-forms have multiplicity* 2.

Note that Theorem 4.2.4 is not a direct extension of Theorem 1.0.2, for the generic behavior of the nonzero eigenvalues of $\Delta_g^{(2)}$ on coexact 2-forms is to have multiplicity 2, not 1.

To provide context for Theorem 4.2.4 and justify its validity, we structure this dissertation as follows. In Chapter 2, we review definitions from Riemannian geometry and introduce several operators on differential forms which will be used extensively throughout the dissertation. Chapter 3 provides a study of the Beltrami operator $*_g d$, which is related to the Hodge Laplacian by

$$\Delta_g^{(2)} = -(*_g d)^2$$

when the operators are restricted to coexact 2-forms on a 5-manifold. We use perturbation theory to show that the Beltrami operator has only simple eigenvalues for a residual set of metrics (Theorem 3.4.3). In Chapter 4, we make observations on the eigenspaces of the Hodge Laplacian and apply Theorem 3.4.3 to show that for a residual set of metrics, the eigenvalues of the restriction of the Hodge Laplacian to coexact 2-forms have multiplicity 2, thereby proving Theorem 4.2.4. Chapter 5 offers concluding comments about related problems and establishes even eigenvalue multiplicities for the Hodge Laplacian acting on coexact k-forms in the more general setting $n = 4\ell + 1$ and $k = 2\ell$ for $\ell \in \mathbb{N}$ (Theorem 5.2.1).

Chapter 2 Background

In this chapter, we introduce the definitions, notation, and concepts which provide context for our subsequent discussion. We begin with an overview of tangent spaces and differential forms. We then define several key operators before considering how to decompose the spaces $\Lambda^k(M)$ into harmonic, exact, and coexact forms using the Hodge Decomposition Theorem 2.4.1.

2.1 Tangent Vectors and Vector Fields

We begin with a brief review of differential forms on Riemannian manifolds, as discussed in [11, 21, 26]. Let M be a closed Riemannian manifold of dimension n with atlas $\{U_\alpha, \phi_\alpha\}$. A *tangent vector* to M at a point $p \in M$ is a linear map $v : C^\infty(M) \to \mathbb{R}$ that satisfies the property $v(fg) = v(f)g(p) + f(p)v(g)$ for all $f, g \in C^\infty(M)$. The set of all tangent vectors at $p \in M$ is called the *tangent space* at the point p of M, which we denote by T_pM.

If (U, ϕ) is a local coordinate system about p with coordinate functions $x_1, \ldots, x_n$, of special interest are the tangent vectors $\left(\frac{\partial}{\partial x_i}\right)_p \in T_pM$, $1 \leq i \leq n$, defined by

$$\left(\frac{\partial}{\partial x_i}\right)_p (f) = \frac{\partial(f \circ \phi^{-1})}{\partial x_i}(\phi(p))$$

for $f \in C^\infty(M)$. When the point p is made clear from context, we sometimes write $\frac{\partial}{\partial x_i}$ for these vectors. The tangent space T_pM is an n-dimensional vector space with basis elements

$$\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p.$$

Therefore, we may uniquely represent each tangent vector $X_p \in T_pM$ as

$$X_p = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}.$$

Here, we use the alternate notation X_p for tangent vectors, which will be convenient in our subsequent discussion.

The *tangent bundle* of M is the union

$$TM = \bigcup_{p \in M} T_pM$$

consisting of points $(p, X_p) \in M \times T_pM$. The tangent bundle is structured so that if $\pi : TM \to M$ is the projection from the tangent bundle onto M, then $\pi^{-1}(p) = T_pM$.

A *vector field* X on M is a mapping $X : M \to TM$ that takes each $p \in M$ to a tangent vector $X(p) = X_p \in T_pM$ so that X_p is of class C^∞ with respect to p. Let $\mathfrak{X}(M)$ denote the space of all smooth vector fields on M. In local coordinates, we may write

$$X_p = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

for $p \in U$ and smooth functions $a_i : U \to \mathbb{R}$. Observe, then, that a vector field $X \in \mathfrak{X}(M)$ acts on a function $f \in C^\infty(M)$ to produce a new function $Xf \in C^\infty(M)$ given by

$$(Xf)(p) = X_p(f) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p).$$

2.2 Differential Forms

Consider now the *cotangent space* T_p^*M, which is the dual space of T_pM at a point $p \in M$ and hence contains all bounded linear functionals mapping T_pM to $\mathbb{R}$. We define dx_i to be the dual of $\dfrac{\partial}{\partial x_i}$ for $1 \leq i \leq n$; that is,

$$dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta function. Thus, $\{dx_1, \ldots, dx_n\}$ forms the dual basis of T_p^*M. We call

$$T^*M = \bigcup_{p \in M} T_p^*M$$

the *cotangent bundle* of M.

A *differential form of degree* k, or k*-form*, is a smooth section of the kth exterior power of the cotangent bundle of M. As such, a k-form ω assigns to each point $p \in M$ an alternating multilinear map

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ factors}} \to \mathbb{R}.$$

We use $\Lambda^k(M)$ to denote the space of all C^∞ k-forms on M.

As an alternative approach to k-forms, let $x_1, \ldots, x_n$ be coordinate functions on a coordinate neighborhood U in M. Consider the algebra generated by

$$dx_1, \ldots, dx_n$$

over $\mathbb{R}$ with unity 1 and product $\wedge$ defined so that $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for $1 \leq i, j \leq n$. The antisymmetry of the wedge product implies $dx_i \wedge dx_i = 0$ for all $i = 1, \ldots, n$, and hence $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$ for $k \geq n+1$. In local coordinates, a k-form ω can be written uniquely as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for functions $\omega_{i_1 \cdots i_k} \in C^\infty(U)$. The action of the elements

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n\} \tag{2.1}$$

upon tangent vectors $X_1, \ldots, X_k \in T_pM$ is defined by

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(X_1, \ldots, X_k) = \frac{1}{k!} \det(dx_{i_\ell}(X_j)).$$

By linearity, we may extend this definition to a general k-form $\omega \in \Lambda^k(M)$.

Though antisymmetry of the wedge product allows us to define a k-form locally in terms of $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where the indices are strictly increasing, we will hereafter use the Einstein convention that summation from 1 to n takes place over repeated indices. To accommodate this convention, we define coefficients of ω for nonincreasing indices by $\omega_{\sigma(i_1)\ldots\sigma(i_k)} = \epsilon_{\sigma(i_1)\cdots\sigma(i_k)}\omega_{i_1\ldots i_k}$, where $i_1 < \cdots < i_k$, σ is a permutation on $\{i_1, \ldots, i_k\}$, and

$$\epsilon_{\sigma(i_1)\cdots\sigma(i_k)} = \begin{cases} 1, & \text{if } \sigma \text{ is even;} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

In this way, we may write

$$\omega = \frac{1}{k!}\omega_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where the sum now ranges over all indices.

The wedge product of two differential forms $\omega \in \Lambda^k(M)$ and $\eta \in \Lambda^\ell(M)$ is a $(k+\ell)$-form $\omega \wedge \eta$. If $\omega = \dfrac{1}{k!}\omega_{i_1\ldots i_k} dx_{i_1} \wedge dx_{i_k}$ and $\eta = \dfrac{1}{\ell!}\omega_{j_1\ldots j_\ell} dx_{j_1} \wedge dx_{j_\ell}$ in local coordinates, then

$$\omega \wedge \eta = \frac{1}{k!\ell!}\omega_{i_1\ldots i_k}\eta_{j_1\ldots j_\ell} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}.$$

Observe that if $k + \ell > n$, then $\omega \wedge \eta = 0$. Moreover,

$$\eta \wedge \omega = (-1)^{k\ell}\omega \wedge \eta. \tag{2.2}$$

2.3 Operators on Differential Forms

In this section, we will introduce several operators on differential forms which will feature prominently in the following chapters: the exterior differential operator, the Hodge star operator, the codifferential operator, and the Hodge Laplacian. Our primary references are [10] and [21].

Exterior Differential Operator

One of our main operators of interest is the *exterior differential operator*

$$d : \Lambda^k(M) \to \Lambda^{k+1}(M),$$

defined by

$$
\begin{aligned}
d\omega(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i X_i\left(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)\right)\\
&\quad+\sum_{i<j}(-1)^{i+j}\omega\left([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k\right)
\end{aligned}
$$

for $\omega \in \Lambda^k(M)$ and $X_0,\dots,X_k \in \mathfrak{X}(M)$. The notation $\hat{X}_i$ indicates that the vector field X_i is omitted. Given a k-form $\omega = \frac{1}{k!}\omega_{i_1\dots i_k}dx_{i_1}\wedge dx_{i_k}$, the local expression of the components of $d\omega \in \Lambda^{k+1}(M)$ is

$$
(d\omega)_{i_1\cdots i_{k+1}} = \sum_{\ell=1}^{k+1}(-1)^{\ell-1}\frac{\partial\omega_{i_1\dots i_{\ell-1}i_{\ell+1}\cdots i_{k+1}}}{\partial x_{i_\ell}}. \tag{2.3}
$$

The exterior differential is a first order linear operator and satisfies the following properties:

$$
d\circ d = 0; \tag{2.4}
$$

$$
d(\omega_1\wedge\omega_2) = d\omega_1\wedge\omega_2+(-1)^k\omega_1\wedge d\omega_2 \tag{2.5}
$$

for all $\omega_1 \in \Lambda^k(M)$ and $\omega_2 \in \Lambda^\ell(M)$.

The Hodge Star Operator

When M is an n-dimensional manifold, it is evident from (2.1) that as a vector space, $\Lambda^k(M)$ has dimension $\binom{n}{k}$. Since $\Lambda^k(M)$ and $\Lambda^{n-k}(M)$ can be viewed as vector spaces of the same dimension, there is an isomorphism $*_g : \Lambda^k(M) \to \Lambda^{n-k}(M)$ between the two spaces. We call $*_g$ the *Hodge star operator* and include the subscript g to highlight the operator's dependence on the metric.

Definition 2.3.1. *The Hodge star operator*

$$
*_g : \Lambda^k(M) \to \Lambda^{n-k}(M)
$$

is the zeroth order differential operator defined as follows:

(i) $*_g$ *is a* $C^\infty(M)$*-linear mapping, that is,*

$$*_g(f_1\omega_1 + f_2\omega_2) = f_1 *_g \omega_1 + f_2 *_g \omega_2$$

for all $f_1, f_2 \in C^\infty(M)$ *and* $\omega_1, \omega_2 \in \Lambda^k(M)$;

(ii) *In local coordinates,*

$$*_g(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \frac{1}{(n-k)!}|g|^{1/2}\epsilon_{j_1\cdots j_k, j_{k+1}\cdots j_n} g^{i_1 j_1} \cdots g^{i_k j_k} dx_{j_{k+1}} \wedge \ldots \wedge dx_{j_n},$$

where $|g| = \det g$ *and* $\epsilon_{j_1\cdots j_n}$ *is the Levi-Civita symbol defined by*

$$\epsilon_{j_1\cdots j_n} = \begin{cases} 1, & \textit{if } (j_1, \ldots, j_n) \textit{ is an even permutation of } (1, \ldots, n); \\ -1, & \textit{if } (j_1, \ldots, j_n) \textit{ is an odd permutation of } (1, \ldots, n); \\ 0, & \textit{otherwise.} \end{cases}$$

In subsequent computations, it will be useful to note that Definition 2.3.1 implies that for $\omega \in \Lambda^k(M)$, the local coordinate expression of $*_g\omega \in \Lambda^{n-k}(M)$ is

$$(*_g\omega)_{i_1\cdots i_{n-k}} = \frac{1}{k!}|g|^{1/2}\epsilon_{j_1\cdots j_k i_1\cdots i_{n-k}} g^{j_1\ell_1} \cdots g^{j_k\ell_k}\omega_{\ell_1\cdots\ell_k}. \tag{2.6}$$

We also observe from Definition 2.3.1 that

$$*_g(1) = d\mu_g \quad \text{and} \quad *_g(d\mu_g) = 1,$$

where $d\mu_g = |g|^{1/2}dx_1 \wedge \cdots \wedge dx_n$ is the volume element of (M, g). Moreover, the Hodge star operator has the property that

$$*_g *_g \omega = (-1)^{k(n-k)}\omega \tag{2.7}$$

for any $\omega \in \Lambda^k(M)$.

The Codifferential Operator

The *de Rham complex* for (M, g) consists of the spaces $\Lambda^k(M)$ of k-forms on M and the exterior differential operators $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ for $k = 0, \ldots, n$. Each space

$\Lambda^k(M)$ is equipped with inner product given by

$$(\alpha, \beta)_g \;=\; \int_M \alpha \wedge *_g \beta \;\text{ for } \alpha, \beta \in \Lambda^k(M). \tag{2.8}$$

As a consequence of (2.2) and (2.7), the inner product satisfies

$$(*_g\, \omega, *_g\, \eta)_g \;=\; (\omega, \eta)_g \tag{2.9}$$

for all $\omega, \eta \in \Lambda^k(M)$.

The adjoint of d with respect to the inner product (2.8) is the *codifferential operator*

$$\delta_g : \Lambda^{k+1}(M) \to \Lambda^k(M),$$

defined by

$$\delta_g = (-1)^{n(k+1)+1} *_g d *_g . \tag{2.10}$$

As with the Hodge star operator, we include the subscript g in our notation δ_g to indicate that the codifferential operator is dependent on the choice of metric. From (2.7) and (2.10), we obtain the following useful identities on $\Lambda^k(M)$:

$$*_g\delta_g \;=\; (-1)^k d*_g; \tag{2.11}$$

$$\delta_g *_g \;=\; (-1)^{k+1} *_g d; \tag{2.12}$$

$$\delta_g \circ \delta_g \;=\; 0. \tag{2.13}$$

To prove our claim that δ_g is the adjoint of d under the inner product (2.8), we take M to be a closed manifold and follow the argument found in [21]. Let $\omega \in \Lambda^k(M)$ and $\eta \in \Lambda^{k+1}(M)$. By equations (2.5) and (2.11), we find

$$\begin{aligned} d\omega \wedge *_g\eta &= d(\omega \wedge *_g\eta) - (-1)^k \omega \wedge d *_g \eta \\ &= d(\omega \wedge *_g\eta) + \omega \wedge *_g\delta_g\eta. \end{aligned}$$

We integrate each side over M and apply Stokes' Theorem to obtain

$$(d\omega, \eta)_g = \int_M d(\omega \wedge *_g\eta) + (\omega, \delta_g\eta)_g = (\omega, \delta_g\eta)_g$$

as desired. □

The Hodge Laplacian

Our primary operator of interest is the *Hodge Laplacian*, the second order differential operator

$$\Delta_g^{(k)} : \Lambda^k(M) \to \Lambda^k(M)$$

given by

$$\Delta_g^{(k)} = d\delta_g + \delta_g d. \tag{2.14}$$

Note that on 0-forms, the Hodge Laplacian $\Delta_g^{(0)}$ is simply the Laplace-Beltrami operator, defined locally by

$$\Delta f = -|g|^{-1/2} \frac{\partial}{\partial x_i} \left(|g|^{1/2} g^{ij} \frac{\partial f}{\partial x_j} \right)$$

for $f \in C^\infty(M)$. The next proposition outlines a few convenient properties of the Hodge Laplacian.

Proposition 2.3.2. ([10]) *The Hodge Laplacian $\Delta_g^{(k)}$, for $0 \leq k \leq n$, has the following properties:*

(i) $\Delta_g^{(k)}$ *is formally self-adjoint;*

(ii) $\Delta_g^{(k)}$ *is formally non-negative;*

(iii) $\Delta_g^{(k)} \omega = 0$ *if and only if* $d\omega = 0$ *and* $\delta_g \omega = 0$.

Proof. In the following computations, context will determine the rank of the forms upon which d and δ_g act.

(i) Let $\omega, \eta \in \Lambda^k(M)$. Expressing $\Delta_g^{(k)}$ using (2.14) and noting that δ_g is the adjoint of d, we obtain

$$\begin{aligned} (\Delta_g^{(k)}\omega, \eta)_g &= (d\delta_g\omega, \eta)_g + (\delta_g d\omega, \eta)_g \\ &= (\delta_g\omega, \delta_g\eta)_g + (d\omega, d\eta)_g \\ &= (\omega, d\delta_g\eta)_g + (\omega, \delta_g d\eta)_g \\ &= (\omega, \Delta_g^{(k)}\eta)_g. \end{aligned}$$

Thus, $\Delta_g^{(k)}$ is formally self-adjoint.

(ii) For each $\omega \in \Lambda^k(M)$, we have

$$\begin{aligned} (\Delta_g^{(k)}\omega, \omega)_g &= (d\delta_g\omega, \omega)_g + (\delta_g d\omega, \omega)_g \\ &= (\delta_g\omega, \delta_g\omega)_g + (d\omega, d\omega)_g \\ &= \|\delta_g\omega\|_g^2 + \|d\omega\|_g^2 \\ &\geq 0, \end{aligned}$$

and so $\Delta_g^{(k)}$ is formally a non-negative operator.

(iii) Let $\omega \in \Lambda^k(M)$. First, suppose $\Delta_g^{(k)}\omega = 0$. By our computation in (ii), we find that

$$\|\delta_g\omega\|_g^2 + \|d\omega\|_g^2 = (\Delta_g^{(k)}\omega, \omega)_g = 0,$$

thereby implying $\delta_g\omega = 0$ and $d\omega = 0$.

Conversely, suppose that $\delta_g\omega = 0$ and $d\omega = 0$. Then

$$\Delta_g^{(k)}\omega = d\delta_g\omega + \delta_g d\omega = 0.$$

□

In addition to satisfying the properties of Proposition 2.3.2, the Hodge Laplacian commutes with the operators $*_g$, d, and δ_g.

Proposition 2.3.3. ([10]) *Let $0 \leq k \leq n$. For all $\omega \in \Lambda^k(M)$, the following equalities hold:*

(i) $*_g(\Delta_g^{(k)}\omega) = \Delta_g^{(n-k)}(*_g\omega)$;

(ii) $d(\Delta_g^{(k)}\omega) = \Delta_g^{(k+1)}(d\omega)$;

(iii) $\delta_g(\Delta_g^{(k)}\omega) = \Delta_g^{(k-1)}(\delta_g\omega)$.

Proof. Let $\omega \in \Lambda^k(M)$. In the computations which follow, the rank of the forms upon which the operators $*_g$, d, and δ_g act will be clear from context.

(i) By the identities given in (2.11) and (2.12), we obtain

$$\begin{aligned} *_g(\Delta_g^{(k)}\omega) &= (*_g d)\delta_g\omega + (*_g\delta_g)d\omega \\ &= (-1)^k\delta_g(*_g\delta_g)\omega + (-1)^{k+1}d(*_g d)\omega \\ &= (-1)^{2k}\delta_g d *_g \omega + (-1)^{2(k+1)}d\delta_g *_g \omega \\ &= \Delta_g^{(n-k)}(*_g\omega). \end{aligned}$$

Thus, the Hodge Laplacian and Hodge star operators commute.

(ii) We apply (2.4) to find

$$\begin{aligned} d(\Delta_g^{(k)}\omega) &= d(d\delta_g + \delta_g d)\omega \\ &= d\delta_g d\omega \\ &= (d\delta_g + \delta_g d)d\omega \\ &= \Delta_g^{(k+1)}(d\omega), \end{aligned}$$

thereby establishing the commutativity of the Hodge Laplacian and exterior differential operator.

(iii) Using (2.13), we compute

$$\begin{aligned}\delta_g(\Delta_g^{(k)}\omega) &= \delta_g(d\delta_g + \delta_g d)\omega \\ &= \delta_g d\delta_g\omega \\ &= (\delta_g d + d\delta_g)\delta_g\omega \\ &= \Delta_g^{(k-1)}(\delta_g\omega)\end{aligned}$$

to verify that the Hodge Laplacian and codifferential operator commute. □

2.4 Hodge Decomposition

One reason why the operators $\Delta_g^{(k)}$, d, and δ_g are so fundamental is that they decompose $\Lambda^k(M)$ into orthogonal subspaces. We define the space of harmonic k-forms on (M, g) by

$$\mathcal{H}^k(M) = \{\omega \in \Lambda^k(M) | \, \Delta_g^{(k)}\omega = 0\},$$

the space of exact k-forms by

$$d\Lambda^{k-1}(M) = \{\omega \in \Lambda^k(M) | \, \omega = d\eta \text{ for some } \eta \in \Lambda^{k-1}(M)\}, \tag{2.15}$$

and the space of coexact k-forms by

$$\delta_g\Lambda^{k+1}(M) = \{\omega \in \Lambda^k(M) | \, \omega = \delta_g\zeta \text{ for some } \zeta \in \Lambda^{k+1}(M)\}.$$

The Hodge Decomposition Theorem guarantees that any k-form can be uniquely written as the sum of a harmonic form, an exact form, and a coexact form.

Theorem 2.4.1. (Hodge Decomposition Theorem, [21]) *On an oriented compact Riemannian manifold (M, g), the space $\Lambda^k(M)$ can be decomposed as*

$$\Lambda^k(M) = \mathcal{H}^k(M) \oplus_g d\Lambda^{k-1}(M) \oplus_g \delta_g\Lambda^{k+1}(M).$$

Many of our computations in subsequent chapters will make use of the decomposition given in Theorem 2.4.1. For now, we will consider the behavior of the operators defined in the previous section when restricted to the subspaces $\mathcal{H}^k(M), d\Lambda^{k-1}(M)$, and $\delta_g\Lambda^{k+1}(M)$.

Proposition 2.4.2. *The Hodge star operator* $*_g$ *has the following properties:*

(i) $*_g : d\Lambda^{k-1}(M) \to \delta_g\Lambda^{n-k+1}(M)$ *is an isomorphism between exact* k*-forms and coexact* $(n-k)$*-forms;*

(ii) $*_g : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$ *is an isomorphism between harmonic* k*-forms and harmonic* $(n-k)$*-forms.*

Proof. We first observe that identity (2.7) implies that the linear operator

$$(-1)^{k(n-k)} *_g : \Lambda^{n-k}(M) \to \Lambda^k(M)$$

is the inverse of $*_g : \Lambda^k(M) \to \Lambda^{n-k}(M)$.

(i) Suppose $\omega \in d\Lambda^{k-1}(M)$ so that $\omega = d\eta$ for some $\eta \in \Lambda^{k-1}(M)$. Then

$$*_g\omega = *_g(d\eta) = (-1)^k\delta_g(*_g\eta)$$

is in $\delta_g\Lambda^{n-k+1}(M)$. To show that $*_g$ is injective, note that $*_g\omega = 0$ implies that

$$\omega = (-1)^{k(n-k)} *_g (*_g\omega) = (-1)^{k(n-k)} *_g (0) = 0.$$

For surjectivity, let $\tau \in \delta_g\Lambda^{n-k+1}(M)$ so that $\tau = \delta_g\zeta$ for some $\zeta \in \Lambda^{n-k+1}(M)$. The k-form $\omega = (-1)^{k(n-k)} *_g \tau$ satisfies

$$*_g\omega = *_g[(-1)^{k(n-k)} *_g \tau] = \tau$$

and belongs to $d\Lambda^{k-1}(M)$ since

$$\omega = (-1)^{k(n-k)} *_g (\delta_g\zeta) = (-1)^{n(k+1)+1} d *_g \zeta.$$

Thus, $*_g$ provides an isomorphism between exact k-forms and coexact $(n-k)$-forms.

(ii) Let $\omega \in \mathcal{H}^k(M)$. By the commutativity of the Hodge Laplacian and Hodge star operator established in Proposition 2.3.3, we have

$$\Delta_g^{(n-k)}(*_g\omega) = *_g(\Delta_g^{(k)}\omega) = 0$$

so that $*_g\omega \in \mathcal{H}^{n-k}(M)$. As in part (i), we find that $*_g\omega = 0$ implies $\omega = 0$, thereby proving the injectivity of $*_g$. Moreover, for each $\zeta \in \mathcal{H}^{n-k}(M)$, the k-form $\omega = (-1)^{k(n-k)} *_g \zeta$ belongs to $\mathcal{H}^k(M)$ and satisfies

$$*_g\omega = (-1)^{k(n-k)} *_g *_g\zeta = \zeta,$$

so $*_g : \mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$ is surjective. Therefore, $*_g$ is an isomorphism between harmonic k-forms and harmonic $(n-k)$-forms. □

By identity (2.4) and Proposition 2.3.2, we have $d\omega = 0$ whenever ω is in the subspace $\mathcal{H}^k(M) \oplus d\Lambda^{k-1}(M)$. However, when restricted to coexact k-forms, the exterior differential operator provides an isomorphism between $\delta_g\Lambda^{k+1}(M)$ and $d\Lambda^k(M)$.

Proposition 2.4.3. *The exterior differential operator $d : \delta_g\Lambda^{k+1}(M) \to d\Lambda^k(M)$ is an isomorphism between the spaces of coexact k-forms and exact $(k+1)$-forms.*

Proof. By (2.15), we readily see that the range of d is contained in $d\Lambda^k(M)$. To show that $d : \delta_g\Lambda^{k+1}(M) \to d\Lambda^k(M)$ is injective, suppose that $\omega = \delta_g\eta$ is an element of $\delta_g\Lambda^{k+1}(M)$ such that $d\omega = 0$. Since ω also satisfies

$$\delta_g\omega = \delta_g(\delta_g\eta) = 0,$$

Proposition 2.3.2 implies that ω is in $\mathcal{H}^k(M)$. However, the spaces $\mathcal{H}^k(M)$ and $\delta_g\Lambda^{k+1}(M)$ are orthogonal by the Hodge Decomposition Theorem 2.4.1, thereby implying that $\omega = 0$.

For surjectivity, let $\tau \in d\Lambda^k(M)$. Then $\tau = d\omega$ for some $\omega \in \Lambda^k(M)$, which can be written $\omega = \alpha + d\zeta + \delta_g\eta$ for some $\alpha \in \mathcal{H}^k(M)$, $\zeta \in \Lambda^{k-1}(M)$, and $\eta \in \Lambda^{k+1}(M)$

by the Hodge Decomposition Theorem 2.4.1. Consequently,

$$\tau = d\omega = d(\alpha + d\zeta + \delta_g \eta) = d(\delta_g \eta)$$

for $\delta_g \eta \in \delta_g \Lambda^{k+1}(M)$, and so $d : \delta_g \Lambda^{k+1}(M) \to d\Lambda^k(M)$ is surjective. Hence, d provides a linear isomorphism between the spaces of coexact k-forms and exact $(k+1)$-forms. ☐

Parallel to our observations regarding the exterior differential operator, equation (2.13) and Proposition 2.3.2 imply that the range of the restriction of the codifferential operator to $\mathcal{H}^k(M) \oplus \delta_g \Lambda^{k+1}(M)$ is $\{0\}$. When the domain of δ_g is restricted to exact forms, we obtain a result analogous to Proposition 2.4.3.

Proposition 2.4.4. *The codifferential operator $\delta_g : d\Lambda^{k-1}(M) \to \delta_g \Lambda^k(M)$ is an isomorphism between the spaces of exact k-forms and coexact $(k-1)$-forms.*

The proof of Proposition 2.4.4 is similar to that of Proposition 2.4.3 and will be left to the reader. As a final observation, we consider the restriction of the Hodge Laplacian to the spaces of exact and of coexact k-forms.

Proposition 2.4.5. *The Hodge Laplacian $\Delta_g^{(k)}$ is an automorphism when its domain is restricted to either $d\Lambda^{k-1}(M)$ or $\delta_g \Lambda^{k+1}(M)$.*

Proof. Consider the Hodge Laplacian $\Delta_g^{(k)} : d\Lambda^{k-1}(M) \to d\Lambda^{k-1}(M)$ restricted to exact k-forms. In this case, the Hodge Laplacian reduces to

$$\Delta_g^{(k)} = d\delta_g.$$

Since the operators $d : \delta_g \Lambda^k(M) \to d\Lambda^{k-1}(M)$ and $\delta_g : d\Lambda^{k-1}(M) \to \delta_g \Lambda^k(M)$ are both isomorphisms by Propositions 2.4.3 and 2.4.4, we conclude that

$$\Delta_g^{(k)} : d\Lambda^{k-1}(M) \to d\Lambda^{k-1}(M)$$

is an automorphism. The proof for the restriction of $\Delta_g^{(k)}$ to $\delta_g \Lambda^{k+1}(M)$ is similar. ☐

2.5 Spaces of Differential Forms

Up to this point, we have considered the spaces $\Lambda^k(M)$ of k-forms on M with smooth, real-valued coefficient functions. Define the pointwise inner product $\langle\omega,\eta\rangle_g$ for k-forms $\omega,\eta\in\Lambda^k(M)$ to be the function on M satisfying

$$\langle\omega,\eta\rangle_g d\mu_g \quad = \quad \omega\wedge *_g\eta, \tag{2.16}$$

where $d\mu_g = |g|^{1/2}dx_1\wedge\ldots\wedge dx_n$ is the volume element of (M,g). Following [6], we define the space $L^p(M,\Lambda^k)$ of L^p k-forms on M to be the completion of $\Lambda^k(M)$ with respect to the norm

$$\|\omega\|_{L^p} \quad = \quad \left(\int_M |\omega|_g^p\, d\mu_g\right)^{1/p}, \tag{2.17}$$

where $|\omega|_g = \langle\omega,\omega\rangle_g^{1/2}$ is the pointwise norm of the k-form ω. Likewise, the Sobolev space $H^s(M,\Lambda^k)$ is the completion of $\Lambda^k(M)$ in the norm

$$\|\omega\|_{H^s} \quad = \quad \left(\sum_{\ell=0}^{s}\int_M |\nabla^\ell\omega|_g^2\, d\mu_g\right)^{1/2}, \tag{2.18}$$

where $\nabla^\ell\omega$ is the ℓ-th covariant derivative of ω. Note that while the pointwise inner product (2.16) depends on the C^r metric g, the choice of metric does not affect $L^p(M,\Lambda^k)$ or $H^s(M,\Lambda^k)$ when viewed as topological vector spaces.

While we are primarily concerned with real differential forms, the skew-symmetry of the Beltrami operator will require us to consider differential forms with complex coefficients throughout much of Chapter 3. By extending the underlying field of scalars to include complex numbers, we obtain the complexification of $\Lambda^k(M)$, defined as

$$\Lambda^k_{\mathbb{C}}(M) = \{\alpha + i\beta \,|\, \alpha,\beta\in\Lambda^k(M)\}.$$

To account for complex-valued coefficients, we define the pointwise inner product

$$\langle\omega,\eta\rangle_g d\mu_g \quad = \quad \omega\wedge\overline{(*_g\eta)} \tag{2.19}$$

and corresponding global inner product

$$(\omega, \eta)_g = \int_M \omega \wedge \overline{(*_g \eta)} \tag{2.20}$$

for $\omega, \eta \in \Lambda^k_{\mathbb{C}}(M)$. By substituting the complex pointwise inner product (2.19) in the norms (2.17) and (2.18) and considering complex forms, we obtain the spaces $L^2(M, \Lambda^k_{\mathbb{C}})$ and $H^s(M, \Lambda^k_{\mathbb{C}})$ of L^2 and H^s complex k-forms, respectively. The Hodge Decomposition Theorem 2.4.1 extends to complex forms to give

$$\Lambda^k_{\mathbb{C}}(M) = \mathcal{H}^k_{\mathbb{C}}(M) \oplus d\Lambda^{k-1}_{\mathbb{C}}(M) \oplus \delta_g \Lambda^{k+1}_{\mathbb{C}}(M),$$

where $\mathcal{H}^k_{\mathbb{C}}(M), d\Lambda^{k-1}_{\mathbb{C}}(M)$ and $\delta_g \Lambda^{k+1}_{\mathbb{C}}(M)$ respectively denote the spaces of harmonic, exact, and coexact complex k-forms.

Chapter 3 Generic Simplicity of the Eigenvalues of the Beltrami Operator

Given the background provided in Chapter 2, the next two chapters will build towards a proof of our main result, Theorem 4.2.4. Let M be a closed 5-manifold, and let $\mathcal{G}^r(M)$ denote the space of metrics on M of class C^r for some integer $r \geq 2$. Within this setting, we wish to formulate a theorem in the spirit of Uhlenbeck, which would predict the eigenvalue multiplicities of the Hodge Laplacian $\Delta_g^{(k)}$ for a residual set of metrics in $\mathcal{G}^r(M)$.

3.1 Explanation of Approach

To provide justification for why we restrict our attention to coexact 2-forms and study the Beltrami operator rather than work directly with $\Delta_g^{(2)}$, we make a few preliminary comments regarding the eigenvalues of the Hodge Laplacian on a 5-manifold. First, observe that since 0-forms are simply functions, Uhlenbeck's Theorem 1.0.2 guarantees the nonzero eigenvalues of $\Delta_g^{(0)}$ will all be simple for a residual set of metrics in $\mathcal{G}^r(M)$. The commutativity of the Hodge Laplacian and $*_g$ established in Proposition 2.3.3 implies that the nonzero eigenvalues of $\Delta_g^{(5)}$ are also generically simple. The same conclusion holds for $\Delta_g^{(1)}$ restricted to exact 1-forms and to $\Delta_g^{(4)}$ restricted to coexact 4-forms by Corollary 4.1.3.

Our observations above reveal that Uhlenbeck's Theorem 1.0.2 is sufficient to ensure generic simplicity of the nonzero eigenvalues of the Hodge Laplacian when its domain is restricted to the space of 0-forms, exact 1-forms, coexact 4-forms, or 5-forms. However, Theorem 1.0.2 has no direct implications for the Hodge Laplacian restricted to coexact 1-forms, 2-forms, 3-forms, or exact 4-forms. Fortunately, we do not have to consider each of these remaining types of forms individually. The results of Section 4.1 imply that the following operators have isomorphic eigenspaces:

1. $\Delta_g^{(1)}$ restricted to coexact 1-forms
2. $\Delta_g^{(2)}$ restricted to exact 2-forms
3. $\Delta_g^{(3)}$ restricted to coexact 3-forms
4. $\Delta_g^{(4)}$ restricted to exact 4-forms

Likewise, $\Delta_g^{(2)}$ and $\Delta_g^{(3)}$ acting on coexact 2-forms and exact 3-forms, respectively, have isomorphic eigenspaces. Therefore, if we could determine the generic eigenvalue multiplicities of the Hodge Laplacian restricted to coexact 1-forms and to coexact 2-forms, we would obtain a full characterization of the generic eigenvalue multiplicities of the Hodge Laplacian on a 5-manifold.

Following the approach of Enciso and Peralta-Salas [12], we determine the generic eigenvalue multiplicities of the Hodge-Laplacian on coexact 2-forms by first studying the eigenvalues of the related Beltrami operator. Lemma 3.2.1 and equation (3.1) indicate that the Beltrami operator will only give insight into the eigenvalues of the Hodge Laplacian when $\Delta_g^{(2)}$ is restricted to coexact 2-forms, and thus our discussion will focus on forms of this type. We build our argument around the Beltrami operator because it has simpler structure than the Hodge Laplacian, which makes computations in local coordinates more manageable.

As a final note, we focus on the nonzero eigenvalues of the Hodge-Laplacian since the set of all eigenforms of $\Delta_g^{(k)}$ with eigenvalue 0 is precisely the space of harmonic forms $\mathcal{H}^k(M)$. Since M is a closed manifold, the Hodge Theorem implies $\mathcal{H}^k(M)$ is isomorphic to the k-th dimensional de Rham cohomology group of M and therefore has dimension equal to $b_k(M)$, the k-th Betti number of M. Now, $b_k(M)$ is a topological invariant, and thus the dimension of $\mathcal{H}^k(M)$ is independent of the Riemannian metric $g \in \mathcal{G}^r(M)$.

One might observe that Theorems 3.4.3 and 4.2.4 do not specify that the eigenvalues of $*_g d$ and $\Delta_g^{(2)}$ restricted to coexact 2-forms must be nonzero. As we just observed, all eigenforms of the Hodge Laplacian with eigenvalue 0 are harmonic, so 0 will not be an eigenvalue of $\Delta_g^{(2)}$ when restricted to coexact 2-forms. In the case of the Beltrami operator, $*_g du = 0$ for $u \in H^1(M, \Lambda^2_{\mathbb{C}})$ implies that the coexact component of u is 0. Thus, 0 is not an eigenvalue of $*_g d$ or $\Delta_g^{(2)}$ when the operators act on spaces of coexact forms.

3.2 The Beltrami Operator

Our objective is to determine the eigenvalue multiplicities of the Hodge Laplacian acting on coexact 2-forms for a residual set of metrics. We would like to use perturbation theory to obtain this residual set of metrics, but local coordinate computations are difficult if we work directly with $\Delta_g^{(2)}$. We observe, however, that if ω is a coexact 2-form then

$$\Delta_g^{(2)}\omega = \delta_g d\omega = -(*_g d)^2\omega.$$

Thus, in order to gain insight into the eigenvalue multiplicities of the Hodge Laplacian, we first direct our attention to the unbounded first order operator $*_g d$, which is more conducive to perturbation theory and thus has more easily determined eigenvalue multiplicities. We call $*_g d$ the *Beltrami operator*.

Before narrowing our focus to coexact 2-forms on a 5-manifold, we consider the more general properties of the Beltrami operator acting on k-forms on an n-manifold. First, observe that since the Beltrami operator is the composition of $*_g$ and d, Propositions 2.4.2 and 2.4.3 imply it is an isomorphism between $\delta_g\Lambda^{k+1}(M)$ and $\delta_g\Lambda^{n-k}(M)$, that is, the spaces of real coexact k-forms and coexact $(n-k-1)$-forms. By extension to complex forms, $*_g d : \delta_g\Lambda^{k+1}_{\mathbb{C}}(M) \to \delta_g\Lambda^{n-k}_{\mathbb{C}}(M)$ is also an isomorphism. We have already noted that $\Delta_g^{(2)} = -(*_g d)^2$ on coexact 2-forms when $n = 5$, which is a special case of a more general relationship.

Lemma 3.2.1. *Let M be an n-manifold. Then*

$$\Delta_g^{(k)} = (-1)^{nk+1}(*_g d)^2$$

when restricted to coexact k-forms.

Proof. If $\omega \in \delta_g \Lambda_{\mathbb{C}}^{k+1}(M)$, then

$$\begin{aligned}
\Delta_g^{(k)}\omega &= \delta_g d\omega \\
&= (-1)^{n(k+2)+1}(*_g d *_g) d\omega \\
&= (-1)^{nk+1}(*_g d)^2\omega.
\end{aligned}$$

□

Lemma 3.2.1 implies that when restricted to coexact forms, the Hodge Laplacian is given by $\Delta_g^{(k)} = (*_g d)^2$ if n and k are both odd; otherwise $\Delta_g^{(k)} = -(*_g d)^2$. The parity of n and k also determine whether the Beltrami operator is self-adjoint or skew-adjoint.

Lemma 3.2.2. *Let M be an n-manifold, $\omega \in H^1(M, \Lambda_{\mathbb{C}}^k)$, and $\eta \in H^1(M, \Lambda_{\mathbb{C}}^{n-k-1})$. Then*

$$(*_g d\omega, \eta)_g = (-1)^{nk+1}(\omega, *_g d\eta)_g.$$

Proof: Let $\omega \in H^1(M, \Lambda_{\mathbb{C}}^k)$ and $\eta \in H^1(M, \Lambda_{\mathbb{C}}^{n-k-1})$. By applying properties (2.9),(2.7), and (2.12), we obtain

$$\begin{aligned}
(*_g d\omega, \eta)_g &= (*_g(*_g d\omega), *_g\eta)_g \\
&= (-1)^{(k+1)(n-k-1)}(d\omega, *_g\eta)_g \\
&= (-1)^{(k+1)(n-k-1)}(\omega, \delta_g *_g \eta)_g \\
&= (-1)^{(k+1)(n-k-1)+(n-k)}(\omega, *_g d\eta)_g \\
&= (-1)^{nk+1}(\omega, *_g d\eta)_g.
\end{aligned}$$

□

In general, the Beltrami operator maps k-forms to $(n-k-1)$-forms so that the ranks of the forms in its domain and range coincide precisely when

$$k = \frac{n-1}{2}. \tag{3.1}$$

In this case, Lemma 3.2.2 reveals that $*_g d$ is self-adjoint if n and k are both odd and is skew-adjoint otherwise. In particular, the Beltrami operator will be skew-adjoint when $n = 5$ and $k = 2$, our case of interest.

In order for the Beltrami operator to have eigenvalues, k and n must satisfy equation (3.1). In particular, (3.1) can only hold if M is an odd-dimensional manifold, and so the Beltrami operator will not give insight into the eigenvalue multiplicities of the Hodge-Laplacian when M has even dimension. Since $n = 5$ and $k = 2$ satisfy (3.1), it is reasonable to discuss the eigenvalues of $*_g d$ acting on 2-forms on a 5-manifold. As a consequence of the skew-adjointness of the Beltrami operator in this case, all of its eigenvalues must be purely imaginary.

Lemma 3.2.3. *Let M be a manifold of odd dimension n, and consider $*_g d$ acting on $H^1(M, \Lambda^k_{\mathbb{C}})$, where $k = (n-1)/2$. If k is odd, then $*_g d$ has only real eigenvalues. If k is even, then all eigenvalues of $*_g d$ are purely imaginary.*

Proof: Let $\omega \in H^1(M, \Lambda^k_{\mathbb{C}})$ be an eigenform of $*_g d$ with eigenvalue $\lambda \in \mathbb{C}$. By Lemma 3.2.2,

$$\lambda \|\omega\|_g^2 = (*_g d\omega, \omega)_g = (-1)^{nk+1}(\omega, *_g d\omega)_g = (-1)^{nk+1}\bar{\lambda}\|\omega\|_g^2.$$

If k is even, the relationship $\lambda = \bar{\lambda}$ implies λ is real. If k is odd, then $\lambda = -\bar{\lambda}$ implies λ is purely imaginary. $\square$

To accommodate for the fact that $*_g d$ has purely imaginary eigenvalues when $n = 5$ and $k = 2$, we will primarily be working with complex differential forms for the remainder of this chapter.

3.3 Variation of the Beltrami Operator in Local Coordinates

Now that we have established general properties of the Beltrami operator $*_g d$ on a manifold of dimension n, we hereafter assume that M is a closed 5-manifold and the metrics g belong to $\mathcal{G}^r(M)$ for $r \geq 2$. Define

$$\mathcal{K} = \{u \in L^2(M, \Lambda^2) \,|\, du = 0\},$$

which is the set of all L^2 exact and harmonic 2-forms on M. Because we will be switching between metrics in the arguments that follow, we will use $\perp_g$ to specify orthogonality with respect to the inner product $(\cdot,\cdot)_g$. By Hodge decomposition, $\mathcal{K}^{\perp_g}$ is the set of all L^2 coexact 2-forms on (M, g). The spaces $\mathcal{K}$ and $\mathcal{K}^{\perp_g}$ consist of real 2-forms and will be pertinent to our discussion in Chapter 4. In this chapter, however, we will focus on the analogous spaces of complex 2-forms, $\mathcal{K}_{\mathbb{C}}$ and $\mathcal{K}_{\mathbb{C}}^{\perp_g}$.

To prove the generic simplicity of the nonzero eigenvalues of the Beltrami operator, we will be using methods from perturbation theory. In particular, we must determine how $*_g d$ behaves under variation of the metric, which we will do using local coordinates. A few comments regarding notation in local coordinates are in order. We use g_{ij} and g^{ij} to signify the components of the metric tensor $g \in \mathcal{G}^r(M)$ and its inverse matrix g^{-1}, respectively. The inverse g^{-1} can be used to raise the indices of a covariant $(0, k)$-tensor field $T_{j_1 \dots j_k}$ to produce a contravariant $(k, 0)$-tensor field

$$T^{i_1 \dots i_k} = g^{i_1 j_1} \cdots g^{i_k j_k} T_{j_1 \dots j_k}.$$

The trace of a $(0, 2)$-tensor h is given by $\mathrm{tr}_g h = g^{ij} h_{ij}$.

The space $\mathcal{S}^r(M)$ consists of all symmetric tensor fields of class C^r and type $(0, 2)$ and can be identified with the tangent space $T_g\mathcal{G}^r(M)$ at any $g \in \mathcal{G}^r(M)$. Thus, $D(*d)_g(h)$ represents the variation of the Beltrami operator at the metric $g \in \mathcal{G}^r(M)$ in the direction of a C^r symmetric $(0, 2)$-tensor h. The following lemma gives the local coordinate representation of $D(*d)_g(h)$ acting on an eigenform of the Beltrami operator.

Lemma 3.3.1. *Let $u \in H^1(M, \Lambda^2_{\mathbb{C}})$ be an eigenform of $*_g d$ with eigenvalue $i\lambda$. Then for any $h \in \mathcal{S}^r(M)$,*

$$(D(*d)_g(h)u)_{ij} = i\lambda \left[-\frac{1}{2}(tr_g h)u_{ij} + g^{mt}h_{ti}u_{mj} + g^{mt}h_{tj}u_{im} \right].$$

The full proof of Lemma 3.3.1 is given in Appendix A, but we will here provide an overview of the computations involved. First, we express the Beltrami operator in local coordinates by

$$(*_g du)_{ij} = \frac{1}{6}\varepsilon_{klmij}|g|^{1/2}g^{kn}g^{lp}g^{mq}\left(\frac{\partial u_{np}}{\partial x_q} - \frac{\partial u_{nq}}{\partial x_p} + \frac{\partial u_{pq}}{\partial x_n}\right).$$

Using the formulas

$$D(g^{ij})(h) = -h^{ij} \quad \text{and} \quad D(|g|^s)(h) = s|g|^s(\mathrm{tr}_g h) \ \text{ for } s > 0,$$

we compute

$$\begin{aligned}(D(*d)_g(h)u)_{ij} &= \frac{1}{6}\varepsilon_{klmij}|g|^{1/2}\left(\frac{\partial u_{np}}{\partial x_q} - \frac{\partial u_{nq}}{\partial x_p} + \frac{\partial u_{pq}}{\partial x_n}\right) \\ &\quad \times \left[\frac{1}{2}(\mathrm{tr}_g h)g^{kn}g^{lp}g^{mq} - g^{kn}g^{lp}h^{mq} - g^{kn}g^{mq}h^{lp} - g^{lp}g^{mq}h^{kn}\right].\end{aligned}$$

Finally, we utilize the relationship $*_g du = i\lambda u$ and simplify the expression for $(D(*d)_g(h)u)_{ij}$ to arrive at the desired formula.

In our proof that the eigenvalues of the Beltrami operator are generically simple, we will need the following density result, which allows any compactly-supported 2-form to be locally expressed in terms of a given non-vanishing form and a symmetric $(0,2)$-tensor.

Lemma 3.3.2. *Let $w \in C^r(M, \Lambda^2_{\mathbb{C}})$, $r \geq 1$, and consider a compact subset $K \subset M\backslash w^{-1}(0)$. Then for any $v \in C^r(M, \Lambda^2_{\mathbb{C}})$ with supp $v \subset K$, there exists a symmetric complex $(0,2)$-tensor $t \in \mathcal{S}^r_{\mathbb{C}}(M)$ such that $v_{ij} = t_{ik}g^{kl}w_{lj} + w_{ik}g^{kl}t_{lj}$.*

Proof. Let $w \in C^r(M, \Lambda^2_{\mathbb{C}})$, let K be a compact subset of $M\backslash w^{-1}(0)$, and let v be any 2-form in $C^r(M, \Lambda^2_{\mathbb{C}})$ with supp $v \subset K$. To make our computations clearer, we will use matrix representations of the various forms and tensors. The 2-forms w and v correspond to the antisymmetric 5×5 matrices

$$W = \begin{bmatrix} 0 & w_{12} & w_{13} & w_{14} & w_{15} \\ -w_{12} & 0 & w_{23} & w_{24} & w_{25} \\ -w_{13} & -w_{23} & 0 & w_{34} & w_{35} \\ -w_{14} & -w_{24} & -w_{34} & 0 & w_{45} \\ -w_{15} & -w_{25} & -w_{35} & -w_{45} & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & v_{12} & v_{13} & v_{14} & v_{15} \\ -v_{12} & 0 & v_{23} & v_{24} & v_{25} \\ -v_{13} & -v_{23} & 0 & v_{34} & v_{35} \\ -v_{14} & -v_{24} & -v_{34} & 0 & v_{45} \\ -v_{15} & -v_{25} & -v_{35} & -v_{45} & 0 \end{bmatrix},$$

while g^{-1} and t naturally correspond to the symmetric matrices

$$G^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} & g^{14} & g^{15} \\ g^{12} & g^{22} & g^{23} & g^{24} & g^{25} \\ g^{13} & g^{23} & g^{33} & g^{34} & g^{35} \\ g^{14} & g^{24} & g^{34} & g^{44} & g^{45} \\ g^{15} & g^{25} & g^{35} & g^{45} & g^{55} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\ t_{12} & t_{22} & t_{23} & t_{24} & t_{25} \\ t_{13} & t_{23} & t_{33} & t_{34} & t_{35} \\ t_{14} & t_{24} & t_{34} & t_{44} & t_{45} \\ t_{15} & t_{25} & t_{35} & t_{45} & t_{55} \end{bmatrix}.$$

Note that the entries of W, V, G^{-1}, and T are functions of $p \in M$, so these matrices are in fact matrix-valued functions. For ease of notation, we suppress the point of evaluation p.

The condition $v_{ij} = t_{ik}g^{kl}w_{lj} + w_{ik}g^{kl}t_{lj}$ for $1 \leq i, j \leq 5$ translates into the matrix equation

$$V = TG^{-1}W + WG^{-1}T.$$

Since G^{-1} is a symmetric positive-definite matrix, it has a symmetric positive-definite square root $G^{-1/2}$. We thus obtain the equivalent equation

$$\tilde{V} = \tilde{T}\tilde{W} + \tilde{W}\tilde{T}, \tag{3.2}$$

where the matrices $\tilde{V} = G^{-1/2}VG^{-1/2}$ and $\tilde{W} = G^{-1/2}WG^{-1/2}$ are antisymmetric and $\tilde{T} = G^{-1/2}TG^{-1/2}$ is symmetric.

Let $\mathcal{M}$ denote the set of all C^r 5×5 matrix-valued functions on M, and define a linear operator $L : \mathcal{M} \to \mathcal{M}$ by

$$L(X) \;=\; X\tilde{W} + \tilde{W}X. \tag{3.3}$$

Satisfying condition (3.2) amounts to finding a symmetric $\tilde{T} \in \mathcal{M}$ such that $L(\tilde{T}) = \tilde{V}$. The Sylvester equation

$$L(X) = X\tilde{W} + \tilde{W}X = \tilde{V}$$

has a solution if $\tilde{V}$ is orthogonal to $\ker L$. We show in Appendix C that each $E \in \ker L$ is symmetric. By the antisymmetry of $\tilde{V}$, the matrix inner product of $\tilde{V}$ with each $E \in \ker L$ is

$$\begin{aligned}
E \cdot \tilde{V} &= \sum_{i,j=1}^{5} \overline{e_{ij}}\tilde{v}_{ij} \\
&= \sum_{i<j} \overline{e_{ij}}\tilde{v}_{ij} + \sum_{i>j} \overline{e_{ij}}\tilde{v}_{ij} \\
&= \sum_{i<j} \overline{e_{ij}}\tilde{v}_{ij} + \sum_{i>j} \overline{e_{ji}}(-\tilde{v}_{ji}) \\
&= \sum_{i<j} \overline{e_{ij}}\tilde{v}_{ij} - \sum_{i<j} \overline{e_{ij}}\tilde{v}_{ij} \quad \text{(reindexing)} \\
&= 0.
\end{aligned}$$

Since $\tilde{V}$ is orthogonal to $\ker L$, there exists an $X \in \mathcal{M}$ such that $\tilde{V} = X\tilde{W} + \tilde{W}X$ on K. The antisymmetry of $\tilde{V}$ and $\tilde{W}$ gives

$$\begin{aligned}
(X\tilde{W} + \tilde{W}X)^T &= \tilde{V}^T \\
\tilde{W}^T X^T + X^T \tilde{W}^T &= \tilde{V}^T \\
-\tilde{W}X^T - X^T\tilde{W} &= -\tilde{V} \\
X^T\tilde{W} + \tilde{W}X^T &= \tilde{V}
\end{aligned}$$

so that X^T solves the same equation as X. Thus, we define $\tilde{T}$ to be the symmetrization

$$\tilde{T} = \frac{1}{2}[X + X^T].$$

Hence, $T = G^{1/2}\tilde{T}G^{1/2}$ is a symmetric C^r matrix-valued function such that

$$V = TG^{-1}W + WG^{-1}T,$$

and thus we obtain from T the desired symmetric complex $(0,2)$-tensor $t \in \mathcal{S}^r_{\mathbb{C}}(M)$.

$\square$

In (2.20), we defined the global inner product of $\alpha, \beta \in \Lambda^k_{\mathbb{C}}(M)$ to be

$$(\alpha, \beta)_g \quad = \quad \int_M \alpha \wedge \overline{(*\beta)}.$$

In our case $n = 5$ and $k = 2$, we wish to express this inner product using local coordinate representations of the differential forms. We claim that the local coordinate representation of the pointwise inner product (2.19) is given by

$$\langle \alpha, \beta \rangle_g \quad = \quad \frac{1}{2} g^{ik} g^{j\ell} \alpha_{ij} \overline{\beta_{k\ell}} \tag{3.4}$$

for complex 2-forms $\alpha = \frac{1}{2}\alpha_{st}\, dx_s \wedge dx_t$ and $\beta = \frac{1}{2}\beta_{k\ell}\, dx_k \wedge dx_\ell$. Indeed, observe that

$$\begin{aligned}
\alpha \wedge \overline{(*_g\beta)} \quad &= \quad \left(\frac{1}{2}\alpha_{st}\, dx_s \wedge dx_t\right) \wedge \left(\frac{1}{3!} \cdot \frac{1}{2} |g|^{1/2} \epsilon_{ijpqr} g^{ik} g^{j\ell} \overline{\beta_{k\ell}}\, dx_p \wedge dx_q \wedge dx_r\right) \\
&= \quad \frac{1}{24} |g|^{1/2} \epsilon_{stpqr} \epsilon_{ijpqr} g^{ik} g^{j\ell} \alpha_{st} \overline{\beta_{k\ell}}\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \\
&= \quad \frac{1}{2} |g|^{1/2} g^{ik} g^{j\ell} \alpha_{ij} \overline{\beta_{k\ell}}\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \\
&= \quad \frac{1}{2} g^{ik} g^{j\ell} \alpha_{ij} \overline{\beta_{k\ell}}\, d\mu_g \\
&= \quad \langle \alpha, \beta \rangle_g\, d\mu_g,
\end{aligned}$$

where $d\mu_g = |g|^{1/2} dx_1 \wedge \cdots \wedge dx_5$ is the volume element. We may thereby use the local inner product (3.4) to express the global inner product in terms of local coordinates as

$$(\alpha, \beta)_g \quad = \quad \int_M \alpha \wedge \overline{(*\beta)} \quad = \quad \int_M \langle \alpha, \beta \rangle_g\, d\mu_g \quad = \quad \frac{1}{2} \int_M g^{ik} g^{j\ell} \alpha_{ij} \overline{\beta_{k\ell}}\, d\mu_g \tag{3.5}$$

for $\alpha, \beta \in \Lambda^2_{\mathbb{C}}(M)$.

3.4 Eigenvalue Perturbation for the Beltrami Operator

To establish the generic simplicity of the eigenvalues of the Beltrami operator, we use standard results from perturbation theory as discussed in Rellich [24] and Kato [19]. In particular, observe that the skew-adjointness of the Beltrami operator $*_g d$ when $n = 5$ and $k = 2$ implies that the operator $i *_g d : H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g} \to \mathcal{K}_{\mathbb{C}}^{\perp_g}$ is self-adjoint with respect to the metric g and has real, isolated eigenvalues of finite multiplicity. We consequently have the following perturbation theorem for linear perturbations of the metric:

Theorem 3.4.1. *Let λ be an eigenvalue of $i *_g d : H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g} \to \mathcal{K}_{\mathbb{C}}^{\perp_g}$ of multiplicity m, and let $g(\epsilon) = g + \epsilon h$ for some $h \in S^r(M)$. Then there are m functions $\ell_1^h(\epsilon), \ldots, \ell_m^h(\epsilon)$ real-analytic at $\epsilon = 0$, and m functions $U_1^h(\epsilon), \ldots, U_m^h(\epsilon)$ analytic in $H^1(M, \Lambda^2_{\mathbb{C}})$ at $\epsilon = 0$ such that the following conditions hold:*

(1) *$\ell_j^h(0) = \lambda$ for $j = 1, \ldots, m$;*

(2) *$i *_{g(\epsilon)} dU_j^h(\epsilon) = \ell_j^h(\epsilon) U_j^h(\epsilon)$ for $j = 1, \ldots, m$;*

(3) *For ϵ in a small enough neighborhood of 0, $\{U_1^h(\epsilon), \ldots, U_m^h(\epsilon)\}$ is an orthonormal set in $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_{g(\epsilon)}}$;*

(4) *For every open interval $(a, b) \subset \mathbb{R}$ such that λ is the only eigenvalue of $i *_g d$ in $[a, b]$, there are exactly m eigenvalues (counting multiplicity) $\ell_1^h(\epsilon), \ldots, \ell_m^h(\epsilon)$ of $i *_{g(\epsilon)} d$ in (a, b), for ϵ sufficiently small.*

As a technical point, we may apply perturbation theory when the domains of the perturbed operators $i *_{g(\epsilon)} d$ are taken to be $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_{g(\epsilon)}}$ since there exists a bijection between the spaces $\mathcal{K}_{\mathbb{C}}^{\perp_g}$ and $\mathcal{K}_{\mathbb{C}}^{\perp_{g(\epsilon)}}$. To see this, let $P^{\bar{g}} : L^2(M, \Lambda^2_{\mathbb{C}}) \to \mathcal{K}_{\mathbb{C}}^{\perp_{\bar{g}}}$ denote the $\bar{g}$-orthogonal projection onto $\mathcal{K}_{\mathbb{C}}^{\perp_{\bar{g}}}$ for a metric $\bar{g} \in \mathcal{G}^r(M)$, and consider the restriction $P_g^{\bar{g}} : \mathcal{K}_{\mathbb{C}}^{\perp_g} \to \mathcal{K}_{\mathbb{C}}^{\perp_{\bar{g}}}$ defined by

$$P_g^{\bar{g}} = P^{\bar{g}}|_{\mathcal{K}_{\mathbb{C}}^{\perp_g}}.$$

We have the following lemma:

Lemma 3.4.2. *For any $g, \bar{g} \in \mathcal{G}^r(M)$, the bounded operator $P_g^{\bar{g}} : \mathcal{K}_{\mathbb{C}}^{\perp_g} \to \mathcal{K}_{\mathbb{C}}^{\perp_{\bar{g}}}$ is a bijection.*

Proof. Let $g, \bar{g} \in \mathcal{G}^r(M)$. If $\omega \in \mathcal{K}_{\mathbb{C}}^{\perp_g}$ satisfies $P_g^{\bar{g}}\omega = 0$, then ω is also contained in $\mathcal{K}_{\mathbb{C}}$. Since $\mathcal{K}_{\mathbb{C}}^{\perp_g} \cap \mathcal{K}_{\mathbb{C}} = \{0\}$, we conclude that $P_g^{\bar{g}}$ is injective. Moreover, since

$$P^{\bar{g}} : L^2(M, \Lambda^2_{\mathbb{C}}) = \mathcal{K}_{\mathbb{C}} \oplus_g \mathcal{K}_{\mathbb{C}}^{\perp_g} \to \mathcal{K}_{\mathbb{C}}^{\perp_{\bar{g}}}$$

is surjective and $P^{\bar{g}}(\mathcal{K}_{\mathbb{C}}) = \{0\}$, the restriction $P^{\bar{g}}|_{\mathcal{K}_{\mathbb{C}}^{\perp_g}} = P_g^{\bar{g}}$ is surjective. Hence, $P_g^{\bar{g}}$ is a bijection. $\square$

Given the perturbation theorem 3.4.1, we may now proceed to show that the eigenvalues of $*_g d$ on coexact 2-forms are generically simple.

Theorem 3.4.3. *The eigenvalues of the Beltrami operator $*_g d$ acting on the space $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g}$ are all simple for a residual set of C^r metrics.*

Proof. For a metric $g \in \mathcal{G}^r(M)$, we label the eigenvalues $i\lambda_n$ of the Beltrami operator $*_g d$ so that

$$\lambda^2_{n+1}(g) \geq \lambda^2_n(g).$$

Define the subsets

$$\Gamma = \{g \in \mathcal{G}^r(M) \,|\, \text{ all eigenvalues of } *_g d|_{H^1(M,\Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g}} \text{ are simple}\}$$

and

$$\Gamma_n = \{g \in \mathcal{G}^r(M) \,|\, \text{ the first } n \text{ eigenvalues of } *_g d|_{H^1(M,\Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g}} \text{ are simple}\}$$

so that

$$\Gamma \subset \cdots \subset \Gamma_n \subset \Gamma_{n+1} \subset \cdots \subset \Gamma_1 \subset \Gamma_0 = \mathcal{G}^r(M)$$

and

$$\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n.$$

By the stability of simple eigenvalues under small perturbations of the metric, each set Γ_n is open in $\mathcal{G}^r(M)$. Thus, to prove that Γ is residual in $\mathcal{G}^r(M)$, it is sufficient to show that Γ_{n+1} is dense in Γ_n for all $n = 0, 1, 2, \ldots$.

Let $g \in \Gamma_n$ so that the first n eigenvalues of

$$*_g d : H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g} \to \mathcal{K}_{\mathbb{C}}^{\perp_g}$$

are simple. Suppose that the $(n+1)$-st eigenvalue $i\lambda \neq 0$ of $*_g d$ has multiplicity m, and define $g(\epsilon) = g + \epsilon h$ for some $h \in S^r(M)$. Theorem 3.4.1 implies there are m functions $\ell_1^h(\epsilon), \ldots, \ell_m^h(\epsilon)$ real-analytic at $\epsilon = 0$, and m functions $U_1^h(\epsilon), \ldots, U_m^h(\epsilon)$ analytic in $H^1(M, \Lambda^2_{\mathbb{C}})$ at $\epsilon = 0$ such that the following conditions hold:

(1) $\ell_j^h(0) = \lambda$ for $j = 1, \ldots, m$;

(2) $*_{g(\epsilon)} dU_j^h(\epsilon) = i\ell_j^h(\epsilon) U_j^h(\epsilon)$ for $j = 1, \ldots, m$;

(3) For ϵ in a small enough neighborhood of 0, $\{U_1^h(\epsilon), \ldots, U_m^h(\epsilon)\}$ is an orthonormal set in $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_{g(\epsilon)}}$;

(4) If $\lambda \in (a, b) \subset \mathbb{R}$ and no other eigenvalue $i\mu$ of $*_g d$ satisfies $\mu \in [a, b]$, then for ϵ sufficiently small, $i\ell_1^h(\epsilon), \ldots, i\ell_m^h(\epsilon)$ are the only eigenvalues of $*_{g(\epsilon)} d$ of the form $i\mu$ for $\mu \in (a, b)$.

When $\epsilon = 0$, each set $\{U_1^h(0), \ldots, U_m^h(0)\}$ forms an orthonormal basis of $E(*_g d, i\lambda)$. However, the basis may depend on our choice of $h \in \mathcal{S}^r(M)$ in the linear perturbation $g(\epsilon) = g + \epsilon h$, which is why we include the superscript in our notation.

If we differentiate

$$*_{g(\epsilon)}dU_j^h(\epsilon) \quad = \quad i\ell_j^h(\epsilon)U_j^h(\epsilon)$$

with respect to ϵ and evaluate at $\epsilon = 0$, we obtain

$$\begin{aligned} D(*d)_g(h)U_j^h(0) + *_g d(U_j^h)'(0) &= i(\ell_j^h)'(0)U_j^h(0) + i\ell_j^h(0)(U_j^h)'(0) \\ D(*d)_g(h)U_j^h(0) + (*_g d - i\lambda)(U_j^h)'(0) &= i(\ell_j^h)'(0)U_j^h(0) \\ D(*d)_g(h)u_j^h + (*_g d - i\lambda)(U_j^h)'(0) &= i(\ell_j^h)'(0)u_j^h, \end{aligned} \tag{3.6}$$

where we have introduced the notation $u_j^h = U_j^h(0)$. Observing that $\{u_1^h, \ldots, u_m^h\}$ is an orthonormal basis of $E(*_g d, i\lambda)$ and taking the inner product of (3.6) with another eigenform u_k^h, we find

$$\begin{aligned} i(\ell_j^h)'(0)(u_j^h, u_k^h)_g &= (D(*d)_g(h)u_j^h, u_k^h)_g + ((*_g d - i\lambda)(U_j^h)'(0), u_k^h)_g \\ i(\ell_j^h)'(0)\delta_{jk} &= (D(*d)_g(h)u_j^h, u_k^h)_g. \end{aligned}$$

We may express the inner product $(D(*d)_g(h)u_j^h, u_k^h)_g$ in local coordinates using (3.5) and Lemma 3.3.1 to obtain

$$(\ell_j^h)'(0)\delta_{jk} \quad = \quad \frac{\lambda}{2}\int g^{pr}g^{qs}\left[-\frac{1}{2}(\mathrm{tr}_g h)(u_j^h)_{pq} + g^{lt}h_{tp}(u_j^h)_{lq} + g^{lt}h_{tq}(u_j^h)_{pl}\right]\overline{(u_k^h)_{rs}}\, d\mu_g,$$

which we may express more concisely as

$$(\ell_j^h)'(0)\delta_{jk} \quad = \quad \lambda(S(h, u_j^h), u_k^h)_g \tag{3.7}$$

by defining $S : \mathcal{S}^r_{\mathbb{C}}(M) \times L^2(M, \Lambda^2_{\mathbb{C}}) \to L^2(M, \Lambda^2_{\mathbb{C}})$ by

$$[S(h, w)]_{pq} = -\frac{1}{2}(\mathrm{tr}_g h)w_{pq} + g^{lt}h_{tp}w_{lq} + g^{lt}h_{tq}w_{pl}$$

for $h \in \mathcal{S}^r_{\mathbb{C}}(M)$ and $w \in L^2(M, \Lambda^2_{\mathbb{C}})$. Observe that S is linear in both h and w.

Our goal is to show that there exists an $h \in \mathcal{S}^r(M)$ such that

$$(\ell_j^h)'(0) \neq (\ell_k^h)'(0)$$

for some pair $j, k \in \{1, \ldots, m\}$. This fact implies that under the metric perturbation $g(\epsilon) = g + \epsilon h$ for ϵ sufficiently small, the perturbed eigenvalues $i\ell_j^h(\epsilon)$ and $i\ell_k^h(\epsilon)$ of $*_{g(\epsilon)}d$ are distinct. While $i\ell_n^h(\epsilon)$ and $i\ell_k^h(\epsilon)$ are not guaranteed to be simple, they each have multiplicity less than m, so we may repeat the argument finitely many times to obtain $g(\epsilon) \in \Gamma_{n+1}$.

To this end, assume to the contrary that $(\ell_j^h)'(0) = (\ell_k^h)'(0)$ for all $h \in \mathcal{S}^r(M)$ and all $j, k \in \{1, \ldots, m\}$. By (3.7), this assumption implies

$$\begin{aligned} (S(h, u_j^h), u_j^h)_g &= (S(h, u_k^h), u_k^h), \quad 1 \leq j, k \leq m && (3.8) \\ (S(h, u_j^h), u_k^h)_g &= 0, \quad j \neq k && (3.9) \end{aligned}$$

for all $h \in \mathcal{S}^r(M)$. As previously noted, each set $\{u_1^h, \ldots, u_m^h\}$ forms an orthonormal basis of $E(*_g d, i\lambda)$, but we cannot assume that $u_j^{h_1} = u_j^{h_2}$ when $h_1 \neq h_2$. Let us therefore fix an orthonormal basis $\{u_1, \ldots, u_m\}$ of $E(*_g d, i\lambda)$. For a given $h \in \mathcal{S}^r(M)$, we may write each u_j in terms of the basis elements $\{u_1^h, \ldots, u_m^h\}$ as

$$u_j = c_{j,1}u_1^h + \cdots + c_{j,m}u_m^h$$

for constants $c_{j,1}, \ldots, c_{j,m} \in \mathbb{C}$. The fact that $\{u_1, \ldots, u_m\}$ and $\{u_1^h, \ldots, u_m^h\}$ are both orthonormal bases of $E(*_g d, i\lambda)$ implies

$$\begin{aligned} \delta_{jk} &= (u_j, u_k)_g \\ &= (c_{j,1}u_1^h + \cdots + c_{j,m}u_m^h, c_{k,1}u_1^h + \cdots + c_{k,m}u_m^h)_g \\ &= c_{j,1}\overline{c_{k,1}} + \cdots + c_{j,m}\overline{c_{k,m}}. && (3.10) \end{aligned}$$

Combining (3.10) with (3.8) and (3.9) yields

$$\begin{aligned} (S(h, u_j), u_k)_g &= c_{j,1}(S(h, u_1^h), u_k)_g + \cdots + c_{j,m}(S(h, u_m^h), u_k)_g \\ &= c_{j,1}\overline{c_{k,1}}(S(h, u_1^h), u_1^h)_g + \cdots + c_{j,m}\overline{c_{k,m}}(S(h, u_m^h), u_m^h)_g \\ &= (c_{j,1}\overline{c_{k,1}} + \cdots + c_{j,m}\overline{c_{k,m}})(S(h, u_j^h), u_j^h)_g \\ &= \delta_{jk}(S(h, u_j^h), u_j^h)_g. \end{aligned}$$

Thus, for all $h \in \mathcal{S}^r(M)$, the elements in the orthonormal basis $\{u_1, \ldots, u_m\}$ satisfy

$$
\begin{aligned}
(S(h, u_j), u_j)_g &= (S(h, u_k), u_k)_g, \quad 1 \leq j, k \leq m \\
(S(h, u_j), u_k)_g &= 0, \quad j \neq k.
\end{aligned}
$$

Observe that setting

$$
h_T = T - (\mathrm{tr}_g T) g
$$

for $T \in \mathcal{S}^r(M)$ yields

$$
\begin{aligned}
[S(h_T, u_j)]_{pq} &= -\frac{1}{2}[(\mathrm{tr}_g T) - 5(\mathrm{tr}_g T)](u_j)_{pq} + g^{lt}[T_{tp} - (\mathrm{tr}_g T) g_{tp}](u_j)_{lq} \\
&\quad + g^{lt}[T_{tq} - (\mathrm{tr}_g T) g_{tq}](u_j)_{pl} \\
&= 2(\mathrm{tr}_g T)(u_j)_{pq} + g^{lt} T_{tp}(u_j)_{lq} - (\mathrm{tr}_g T)(u_j)_{pq} + g^{lt} T_{tq}(u_j)_{pl} \\
&\quad - (\mathrm{tr}_g T)(u_j)_{pq} \\
&= T_{pt} g^{tl}(u_j)_{lq} + (u_j)_{pl} g^{lt} T_{tq}.
\end{aligned}
$$

By decomposing a complex symmetric $(0,2)$-tensor $T \in \mathcal{S}^r_{\mathbb{C}}(M)$ into $T = T_1 + iT_2$ for $T_1, T_2 \in \mathcal{S}^r(M)$, we find that

$$
[S(h_{T_1 + iT_2}, u_j)]_{pq} = [S(h_{T_1}, u_j)]_{pq} + i[S(h_{T_2}, u_j)]_{pq},
$$

thereby implying

$$
\begin{aligned}
(S(h_T, u_j), u_j)_g &= (S(h_{T_1 + iT_2}, u_j), u_j)_g \\
&= (S(h_{T_1}, u_j), u_j)_g + i(S(h_{T_2}, u_j), u_j)_g \\
&= (S(h_{T_1}, u_k), u_k)_g + i(S(h_{T_2}, u_k), u_k)_g \\
&= (S(h_{T_1 + iT_2}, u_k), u_k)_g \\
&= (S(h_T, u_k), u_k)_g
\end{aligned}
$$

for all $T \in \mathcal{S}^r_{\mathbb{C}}(M)$. Likewise, we obtain

$$
(S(h_T, u_j), u_k)_g = 0, \qquad j \neq k \tag{3.11}
$$

for all complex tensors $T \in S^r_{\mathbb{C}}(M)$.

Without loss of generality, fix $j = 1$ and $k = 2$. Equation (3.11) implies

$$(S(h_T, u_1), u_2)_g \quad = \quad 0 \tag{3.12}$$

for all $T \in \mathcal{S}^r_{\mathbb{C}}(M)$. Now, it follows from $(*_g d - i\lambda)u_1 = 0$ that

$$\Delta_g^{(2)} u_1 = -(*_g d)^2 u_1 = \lambda^2 u_1,$$

and so u_1 is an eigenform of the Hodge Laplacian $\Delta_g^{(2)}$ with eigenvalue λ^2. Unique continuation then indicates that u_1 cannot vanish in any open subset of M [4, 5], and consequently, the set

$$\mathscr{S} = \{S(h_T, u_1) \,|\, T \in \mathcal{S}^r_{\mathbb{C}}(M)\}$$

is dense in $L^2(M, \Lambda^2_{\mathbb{C}})$ by Lemma 3.3.2. Since (3.12) implies u_2 is orthogonal to the dense set $\mathscr{S}$, we obtain $u_2 = 0$ on M, contradicting that u_2 is an eigenform.

We hence conclude that there exists an $h \in \mathcal{S}^r(M)$ such that $(\ell^h_j)'(0) \neq (\ell^h_k)'(0)$ for some $j, k \in \{1, \ldots, m\}$. Repeating the above argument as necessary, we obtain a metric $g(\epsilon) = g + \epsilon h$ in Γ^{n+1} for ϵ sufficiently small. Since $g(\epsilon)$ can be taken arbitrarily close to g in the C^r topology, we conclude that Γ^{n+1} is dense in Γ^n. Additionally, each Γ^n is open in $\mathscr{G}^r(M)$, so we infer that

$$\Gamma = \bigcap_{n=1}^{\infty} \Gamma_n$$

is residual $\mathscr{G}^r(M)$. Thus, for a residual set of metrics $\Gamma \subset \mathscr{G}^r(M)$, the Beltrami operator acting on $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}^{\perp_g}_{\mathbb{C}}$ has only simple eigenvalues. $\square$

Chapter 4 The Hodge Laplacian on Coexact 2-Forms

In the previous chapter, we established that when M is a 5-manifold, the Beltrami operator $*_g d$ acting on $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g}$ has only simple eigenvalues for a residual set of metrics in $\mathcal{G}^r(M)$. We will utilize this fact in proving our main result, Theorem 4.2.4, which establishes the generic eigenvalue multiplicities of the Hodge Laplacian acting on coexact 2-forms.

4.1 The Spectrum of the Hodge Laplacian

Before continuing our discussion of coexact 2-forms on a 5-manifold, we pause to make a few general observations about the spectrum of the Hodge Laplacian $\Delta_g^{(k)}$ on a manifold (M, g) of dimension n. Since we will not be varying the metric in this section, we temporarily suppress the subscript in our notation for the Hodge Laplacian and codifferential operators. The following spectral theorem holds for the Hodge Laplacian.

Theorem 4.1.1. ([10]) *Let (M, g) be a closed, connected, oriented, n-dimensional Riemannian manifold. The eigenvalue problem*

$$\Delta^{(k)}\omega = \lambda\omega$$

has a complete orthonormal system $\omega_1, \omega_2, \ldots$ of smooth eigenforms in $L^2(M, \Lambda^k)$ with corresponding eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, where $\lambda_j \to \infty$ as $j \to \infty$.

Let $E(\Delta^{(k)}, \lambda)$ denote the space of eigenforms of $\Delta^{(k)}$ in $L^2(M, \Lambda^k)$ with eigenvalue λ. In light of the decomposition

$$\Lambda^k(M) = \mathcal{H}^k(M) \oplus d\Lambda^{k-1}(M) \oplus \delta\Lambda^{k+1}(M)$$

given in Theorem 2.4.1, the following theorem allows us to decompose the eigenspace $E(\Delta^{(k)}, \lambda)$ into exact and coexact subspaces. Let $E_d^k(\lambda)$ and $E_\delta^k(\lambda)$ denote the spaces of exact and coexact eigenforms of $\Delta^{(k)}$ with eigenvalue $\lambda \neq 0$. We adopt the convention of adding a superscript to the operators d and δ to indicate the rank of the forms upon which they are acting.

Theorem 4.1.2. ([10]) *The following equalities are valid:*

(i) $E(\Delta^{(k)}, \lambda) = E_d^k(\lambda) \oplus E_\delta^k(\lambda)$,

(ii) $E_d^k(\lambda) = d^{k-1} E_\delta^{k-1}(\lambda)$,

(iii) $E_\delta^k(\lambda) = \delta^{k+1} E_d^{k+1}(\lambda)$.

Proof. (i) The linearity of $\Delta^{(k)}$ implies $E_d^k(\lambda) \oplus E_\delta^k(\lambda) \subset E(\Delta^{(k)}, \lambda)$. For the reverse inclusion, suppose $\omega \in E(\Delta^{(k)}, \lambda)$ so that $\Delta^{(k)}\omega = \lambda\omega$. By the Hodge Decomposition Theorem 2.4.1, there exists a unique decomposition

$$\omega = \omega_{\mathcal{H}} + \omega_d + \omega_\delta$$

such that $\omega_{\mathcal{H}} \in \mathcal{H}^k(M)$, $\omega_d \in d\Lambda^{k-1}(M)$, and $\omega_\delta \in \delta\Lambda^{k+1}(M)$. We then obtain

$$\begin{aligned} \Delta^{(k)}\omega &= \lambda\omega \\ \Delta^{(k)}(\omega_{\mathcal{H}} + \omega_d + \omega_\delta) &= \lambda(\omega_{\mathcal{H}} + \omega_d + \omega_\delta) \\ \Delta^{(k)}\omega_d + \Delta^{(k)}\omega_\delta &= \lambda\omega_{\mathcal{H}} + \lambda\omega_d + \lambda\omega_\delta. \end{aligned}$$

Proposition 2.4.5 implies that $\Delta^{(k)}\omega_d \in d\Lambda^{k-1}(M)$ and $\Delta^{(k)}\omega_\delta \in \delta\Lambda^{k+1}(M)$, so uniqueness of the Hodge decomposition gives

$$\omega_{\mathcal{H}} = 0, \quad \Delta^{(k)}\omega_d = \lambda\omega_d, \ \text{ and } \ \Delta^{(k)}\omega_\delta = \lambda\omega_\delta.$$

Thus, $\omega = \omega_d + \omega_\delta$ is contained in $E_d^k(\lambda) \oplus E_\delta^k(\lambda)$.

(ii) Let $\omega \in E_d^k(\lambda)$. Since $d^{k-1} : \delta^k \Lambda^k(M) \to d^{k-1}\Lambda^{k-1}(M)$ is an isomorphism by Proposition 2.4.3, there exists a unique $\eta \in \delta^k \Lambda^k(M)$ such that $\omega = d^{k-1}\eta$. The commutativity of the Hodge Laplacian and d established in Proposition 2.3.3 gives

$$d^{k-1}(\Delta^{(k-1)}\eta) = \Delta^k \omega = \lambda\omega = d^{k-1}(\lambda\eta).$$

By the injectivity of d^{k-1} restricted to coexact forms, we conclude $\Delta^{(k-1)}\eta = \lambda\eta$ so that $\eta \in E_\delta^{k-1}(\lambda)$. The reverse inclusion $d^{k-1}E_\delta^{k-1}(\lambda) \subset E_d^k(\lambda)$ easily follows from the commutativity of $\Delta^{(k)}$ and d.

(iii) Using that $\delta^{k+1} : d^k\Lambda^k(M) \to \delta^{k+1}\Lambda^{k+1}(M)$ is an isomorphism (Proposition 2.4.4), the proof of the equality

$$E_\delta^k(\lambda) = \delta^{k+1}E_d^{k+1}(\lambda)$$

mirrors the argument given in (ii). $\square$

As a consequence of Theorem 4.1.2 (i), all eigenforms in $E(\Delta^{(k)}, \lambda)$ can be decomposed into exact and coexact components which are individually eigenforms of $\Delta^{(k)}$ with eigenvalue λ. Statement (ii) can be improved upon in the following corollary:

Corollary 4.1.3. ([10]) *For every $k = 1, \ldots, n$, the linear mapping*

$$d^{k-1} : E_\delta^{k-1}(\lambda) \to E_d^k(\lambda)$$

induced by the exterior derivative d^{k-1} is an isomorphism of norm $\|d^{k-1}\| = \sqrt{\lambda}$.

Corollary 4.1.3 implies that λ is a simple eigenvalue of $\Delta_g^{(k)}$ acting on exact k-forms if and only if it is a simple eigenvalue of $\Delta_g^{(k-1)}$ acting on coexact $(k-1)$-forms. In addition to $E_d^k(\lambda)$ being isomorphic to $E_\delta^{k-1}(\lambda)$, the commutativity of $*_g$ and $\Delta^{(k)}$ established in Proposition 2.3.3 implies that $E_d^k(\lambda)$ is also isomorphic to $E_\delta^{n-k}(\lambda)$. As a consequence of these isomorphisms, determining the eigenvalues of the operators $\Delta^{(k)}$, $0 \le k \le n$, reduces to identifying the eigenvalues of the restriction of $\Delta^{(k)}$ to exact forms for select values of k.

Corollary 4.1.4. ([10]) *The eigenvalues of the operators* $\Delta^{(k)}$, $0 \le k \le n$ *are completely determined by the eigenvalues of the restriction of* $\Delta^{(k)}$ *to exact* k*-forms for* $0 < k \le \lfloor (n+1)/2 \rfloor$.

Note that we could just as easily determine the eigenvalues of $\Delta^{(k)}$, $0 \le k \le n$ by studying the restriction of $\Delta^{(k)}$ to coexact forms.

4.2 Generic Eigenvalue Multiplicities of the Hodge Laplacian on Coexact 2-Forms

In order to determine the generic eigenvalue multiplicities of the Hodge Laplacian on coexact 2-forms on a 5 manifold, we must determine the relationship between the eigenvalues and eigenforms of the Hodge Laplacian and the Beltrami operator. Our next two lemmas hold in the more general setting of $n = 4\ell + 1$ and $k = 2\ell$ for some $\ell \in \mathbb{N}$ and in particular apply when $n = 5$ and $k = 2$.

Lemma 4.2.1. *Let* M *be a manifold of dimension* $n = 4\ell + 1$ *for some* $\ell \in \mathbb{N}$, *and let* $k = 2\ell$. *Let* $\omega = \alpha + i\beta$ *be a nonzero complex* k*-form with* $\alpha, \beta \in H^1(M, \Lambda^k)$. *Then* $*_g d\omega = i\lambda\omega$ *if and only if*

$$*_g d\alpha \;=\; -\lambda\beta \quad \textit{and} \quad *_g\, d\beta = \lambda\alpha. \tag{4.1}$$

Proof. First, suppose that $\omega = \alpha + i\beta \in H^1(M, \Lambda^k_{\mathbb{C}})$ solves $*_g d\omega = i\lambda\omega$. Then

$$\begin{aligned} *_g d\omega &= i\lambda\omega \\ *_g d(\alpha + i\beta) &= i\lambda(\alpha + i\beta) \\ *_g d\alpha + i *_g d\beta &= -\lambda\beta + i\lambda\alpha, \end{aligned}$$

so equating real and imaginary parts yields (4.1).

Conversely, suppose that $\alpha, \beta \in H^1(M, \Lambda^k)$ satisfy (4.1), and let $\omega = \alpha + i\beta$. Then

$$*_g d\omega = *_g d\alpha + i *_g d\beta = -\lambda\beta + i\lambda\alpha = i\lambda(\alpha + i\beta) = i\lambda\omega$$

so that ω is an eigenfunction of $*_g d$ with eigenvalue $i\lambda$. $\square$

Note 4.2.2. It is important to recognize that condition (4.1) implies that α and β are nonzero, linearly independent forms over $\mathbb{R}$. To see this, observe that $\beta = c\alpha$ implies

$$\beta = c\alpha = \frac{c}{\lambda} *_g d\beta = \frac{c^2}{\lambda} *_g d\alpha = -c^2\beta,$$

which gives $c = \pm i$ in contradiction to $c \in \mathbb{R}$. Even more notably,

$$(\alpha, \beta)_g = \frac{1}{\lambda}(*_g d\beta, \beta)_g = -\frac{1}{\lambda}(\beta, *_g d\beta)_g = -(\beta, \alpha)_g$$

reveals that

$$(\alpha, \beta)_g \quad = \quad 0.$$

The next lemma follows from our observations in Lemma 4.2.1.

Lemma 4.2.3. *Let M be a manifold of dimension $n = 4\ell + 1$ for some $\ell \in \mathbb{N}$, and let $k = 2\ell$. Let $\alpha, \beta \in H^2(M, \Lambda^k) \cap \mathcal{K}^{\perp_g}$. If $\omega = \alpha + i\beta$ is an eigenform of the Beltrami operator $*_g d$ with eigenvalue $i\lambda$, then both α and β are eigenforms of the Hodge Laplacian $\Delta_g^{(k)}$ with eigenvalue λ^2.*

Proof. Let $\alpha, \beta \in H^2(M, \Lambda^k) \cap \mathcal{K}^{\perp_g}$, and suppose $\omega = \alpha + i\beta$ is an eigenform of $*_g d$ with eigenvalue $i\lambda$. By Lemma 4.2.1, α and β satisfy

$$*_g d\alpha \quad = \quad -\lambda\beta \quad \text{and} \quad *_g d\beta = \lambda\alpha.$$

Since α is a coexact form, $n = 4\ell + 1$ is odd, and $k = 2\ell$ is even, Lemma 3.2.1 implies

$$\Delta_g^{(k)}\alpha = -(*_g d)^2\alpha = \lambda *_g d\beta = \lambda^2\alpha.$$

Similarly,

$$\Delta_g^{(k)}\beta = -(*_g d)^2\beta = -\lambda *_g d\alpha = \lambda^2\beta$$

so that α and β are both eigenforms of $\Delta_g^{(k)}$ with eigenvalue λ^2. $\square$

Given that Lemmas 4.2.1 and 4.2.3 apply in the case $n = 5$ and $k = 2$, we are now ready to prove our main theorem.

Theorem 4.2.4. *Let M be a closed 5-manifold, and let r be an integer, $r \geq 2$. There exists a residual subset Γ of the space of all C^r metrics on M such that, for all $g \in \Gamma$, the eigenvalues of the restriction of the Hodge Laplacian $\Delta_g^{(2)}$ to $H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$ have multiplicity 2.*

Proof. By Theorem 3.4.3, there exists a residual set Γ of C^r metrics on M such that the eigenvalues of the Beltrami operator $*_g d$ acting on $H^1(M, \Lambda^2_{\mathbb{C}}) \cap \mathcal{K}_{\mathbb{C}}^{\perp_g}$ are all simple. Take $g \in \Gamma$, and consider an eigenvalue $\lambda^2 > 0$ of the restriction of $\Delta_g^{(2)}$ to coexact 2-forms. Let $\eta \in H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$ be an eigenform of $\Delta_g^{(2)}$ with eigenvalue λ^2. Since η is coexact, Lemma 3.2.1 implies that

$$-(*_g d)^2 \eta = \lambda^2 \eta. \tag{4.2}$$

Now, since $*_g d$ maps $H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$ to $H^1(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$, we have $*_g d\eta = \lambda\zeta$ for some $\zeta \in H^1(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$. Equation (4.2) then yields

$$\begin{aligned} - *_g d(\lambda\zeta) &= \lambda^2 \eta \\ *_g d\zeta &= -\lambda\eta \end{aligned}$$

so that ζ is in fact contained in $H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$. Since η and ζ together satisfy condition (4.1), Lemma 4.2.1 implies that $\zeta + i\eta$ is an eigenform of $*_g d$ with eigenvalue $i\lambda$. It follows from Lemma 4.2.3 that ζ is also an eigenform of $\Delta_g^{(2)}$ with eigenvalue λ^2. As mentioned in 4.2.2, the eigenforms η and ζ are linearly independent, indicating that the eigenvalue λ^2 of $\Delta_g^{(2)}$ has multiplicity of at least 2.

To prove that λ^2 has a multiplicity of precisely 2, suppose that $\Delta_g^{(2)}\tau = \lambda^2\tau$ for some $\tau \in H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$. By our previous argument, there must exist a coexact 2-form $\xi \in H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$ such that $\xi + i\tau$ is an eigenform of $*_g d$ with eigenvalue

$i\lambda$. Since g is contained in the residual set Γ, the eigenvalue $i\lambda$ is simple. Thus, $\xi + i\tau$ must be a complex multiple of the eigenform $\zeta + i\eta$; that is,

$$\xi + i\tau \quad = \quad (a + ib)(\zeta + i\eta) \quad = \quad (a\zeta - b\eta) + i(b\zeta + a\eta) \tag{4.3}$$

for some $a + ib \in \mathbb{C}$. Equating the imaginary parts of equation (4.3) gives

$$\tau = b\zeta + a\eta$$

so that τ is a linear combination of the eigenforms η and ζ of $\Delta_g^{(2)}$. Thus, λ^2 has multiplicity 2. We therefore conclude that for a residual set of metrics $\Gamma \subset \mathcal{G}^r(M)$, all eigenvalues of the restriction of the Hodge Laplacian $\Delta_g^{(2)}$ to $H^2(M, \Lambda^2) \cap \mathcal{K}^{\perp_g}$ have multiplicity 2. $\square$

As an immediate consequence of Corollary 4.1.3, we obtain an analogous result for exact 3-forms.

Corollary 4.2.5. *Let M be a closed 5-manifold, and let r be an integer, $r \geq 2$. There exists a residual subset Γ of the space of all C^r metrics on M such that, for all $g \in \Gamma$, the eigenvalues of the restriction of the Hodge Laplacian $\Delta_g^{(3)}$ to the space of exact forms in $H^2(M, \Lambda^3)$ have multiplicity 2.*

Chapter 5 Concluding Comments

Uhlenbeck's Theorem 1.0.1 is a powerful result with many applications. While Theorem 4.2.4 provides an analogue to Uhlenbeck's Theorem 1.0.2 in the case of the Hodge Laplacian acting on coexact 2-forms on a closed 5-manifold, there are still many questions to be explored.

5.1 The Hodge Laplacian on a Closed 5-Manifold

Our goal for this dissertation was to establish the nonzero eigenvalue multiplicities of the Hodge Laplacian $\Delta_g^{(k)}$ on a closed 5-manifold for a residual set of C^r metrics. As outlined in Section 3.1, Uhlenbeck's Theorem 1.0.2 ensures the generic simplicity of the nonzero eigenvalues of following operators:

(i) $\Delta_g^{(0)}$;

(ii) $\Delta_g^{(1)}$ restricted to exact 1-forms;

(iii) $\Delta_g^{(4)}$ restricted to coexact 4-forms;

(iv) $\Delta_g^{(5)}$.

Moreover, there exists a residual set of C^r metrics such that the operators

(v) $\Delta_g^{(2)}$ restricted to coexact 2-forms,

(vi) $\Delta_g^{(3)}$ restricted to exact 3-forms

have eigenvalues of multiplicity 2 (Theorem 4.2.4 and Corollary 4.2.5). In order to completely characterize the generic nonzero eigenvalue multiplicities of the Hodge Laplacian on a closed 5-manifold, we still need information about the eigenspaces of the operators

(vii) $\Delta_g^{(1)}$ restricted to coexact 1-forms,

(viii) $\Delta_g^{(2)}$ restricted to exact 2-forms,

(ix) $\Delta_g^{(3)}$ restricted to coexact 3-forms,

(x) $\Delta_g^{(4)}$ restricted to exact 4-forms.

Since operators (vii)-(x) have isomorphic eigenspaces, it suffices to determine the eigenvalue multiplicities of the Hodge Laplacian restricted to coexact 1-forms. However, it is unclear how to best approach this problem. On a 5-manifold, (3.1) implies that the Beltrami operator only has eigenvalues when acting on 2-forms. Thus, the eigenvalue multiplicities of the Beltrami operator will not give insight into the eigenvalues of $\Delta_g^{(1)}$ on coexact 1-forms, and so we cannot rely on the methods of Chapters 3 and 4. It may be possible, however, to apply perturbation theory or transversality theory directly to the Hodge Laplacian in this case, though the local coordinate computations might prove difficult.

5.2 The Hodge Laplacian on a Closed n-Manifold

Thus far, the results regarding the eigenvalue multiplicities of the Hodge Laplacian on a closed manifold M have been highly dependent on the dimension n of the manifold. In the case $n = 3$, Theorem 1.0.3 establishes that all nonzero eigenvalues of $\Delta_g^{(k)}$, $0 \leq k \leq 3$, are simple for a residual set of C^r metrics. When $n = 5$, generic simplicity of nonzero eigenvalues holds for the operators (i)-(iv), while the eigenspaces of the operators (v) and (vi) have dimension 2 (Theorem 4.2.4).

We hope to employ our methods to study the generic eigenvalue multiplicities of the Hodge Laplacian in the more general context of a closed n-manifold, where n is odd. When $k = (n-1)/2$, the Hodge Laplacian on coexact k-forms can be written $\Delta_g^{(k)} = (-1)^{nk+1}(*_g d)^2$ (Lemma 3.2.1), implying that the Beltrami operator is self-adjoint when k is odd and skew-adjoint when k is even. This leads us to conjecture that on a closed n-manifold, n odd, the nonzero eigenvalues of $\Delta_g^{(k)}$ on coexact k-forms generically have multiplicity 1 when $k = (n-1)/2$ is odd and have multiplicity 2 when k is even. At the very least, the relationship between the eigenvalues of the Beltrami operator and Hodge Laplacian on coexact forms established in Chapter 4 indicate that all eigenvalues of $\Delta_g^{(k)}$ restricted to coexact forms have even multiplicity when $n = 4\ell + 1$ and $k = 2\ell$. Redefining

$$\mathcal{K} = \{u \in L^2(M, \Lambda^k) \,|\, du = 0\}$$

to be the space of L^2 coexact k-forms for $k = 2\ell$, we have the following theorem.

Theorem 5.2.1. *Let (M, g) be a closed Riemannian manifold of dimension $n = 4\ell+1$ for some $\ell \in \mathbb{N}$, and let $k = 2\ell$. Then all eigenvalues of the restriction of $\Delta_g^{(k)}$ to $H^2(M, \Lambda^k) \cap \mathcal{K}^{\perp_g}$ have even multiplicity.*

Proof. Let (M, g) be a closed Riemannian manifold of dimension $n = 4\ell + 1$, and let $k = 2\ell$. Suppose $\lambda^2 > 0$ is an eigenvalue of the restriction of $\Delta_g^{(k)}$ to coexact 2-forms. We will show that the eigenspace $E(\Delta_g^{(k)}, \lambda^2)$ has a basis

$$\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}$$

of $2m$ linearly independent eigenforms of $\Delta_g^{(k)}$ with eigenvalue λ^2. Let η be an eigenform of $\Delta_g^{(k)}$ in $H^2(M, \Lambda^k) \cap \mathcal{K}^{\perp_g}$ with eigenvalue λ^2. Since $n = 4\ell + 1$ is odd and $k = 2\ell$ is even, Lemma 3.2.1 implies that the action of $\Delta_g^{(k)}$ on the coexact form η is given by

$$\Delta_g^{(k)}\eta \;=\; -(*_g d)^2\eta \;=\; \lambda^2\eta. \tag{5.1}$$

Now,

$$\frac{n-1}{2} = \frac{(4\ell+1)-1}{2} = 2\ell = k,$$

so k and n satisfy relationship (3.1). Then

$$*_g d : H^2(M, \Lambda^k) \cap \mathcal{K}^{\perp_g} \to H^1(M, \Lambda^k) \cap \mathcal{K}^{\perp_g}$$

maps coexact k-forms in $H^2(M,\Lambda^k)$ to coexact k-forms in $H^1(M,\Lambda^k)$, yielding

$$*_g d\eta = \lambda\zeta$$

for some $\zeta \in H^1(M,\Lambda^k)\cap\mathcal{K}^{\perp_g}$. Equation (5.1) then implies

$$\begin{aligned} -*_g d(\lambda\zeta) &= \lambda^2\eta \\ *_g d\zeta &= -\lambda\eta \end{aligned}$$

so that ζ is in fact contained in $H^2(M,\Lambda^k)\cap\mathcal{K}^{\perp_g}$. Since η and ζ together satisfy condition (4.1), Lemma 4.2.1 indicates that $\zeta + i\eta$ is an eigenform of $*_g d$ with eigenvalue $i\lambda$. It follows from Lemma 4.2.3 that ζ is also an eigenform of $\Delta_g^{(k)}$ with eigenvalue λ^2.

Suppose that the eigenvalue $i\lambda$ of $*_g d$ has multiplicity $m \in \mathbb{N}$. Then $E(*_g d, i\lambda)$ is spanned by m eigenforms

$$\alpha_1 + i\beta_1, \ldots, \alpha_m + i\beta_m$$

that are linearly independent over $\mathbb{C}$. We may therefore write

$$\begin{aligned} \zeta + i\eta &= \sum_{j=1}^{m}(p_j + iq_j)(\alpha_j + i\beta_j) \\ &= \sum_{j=1}^{m}[(p_j\alpha_j - q_j\beta_j) + i(q_j\alpha_j + p_j\beta_j)] \end{aligned} \tag{5.2}$$

for some $p_j, q_j \in \mathbb{R}$. By equating the imaginary parts of (5.2), we find

$$\eta = \sum_{j=1}^{m}(q_j\alpha_j + p_j\beta_j). \tag{5.3}$$

Lemma 4.2.3 implies that all α_j and β_j are eigenforms of $\Delta_g^{(k)}$ with eigenvalue λ^2, so it only remains to establish that

$$\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}$$

is a linearly independent set over $\mathbb{R}$. Without loss of generality, suppose, to the contrary, that there exist $p_j, q_j \in \mathbb{R}$ such that

$$\alpha_m = \sum_{j=1}^{m-1}(p_j\alpha_j + q_j\beta_j). \tag{5.4}$$

Note that we may exclude β_m from the sum since α_m and β_m are linearly independent over $\mathbb{R}$ by 4.2.2. Lemma 4.2.1 gives the equalities

$$*_g d\alpha_j = -\lambda\beta_j \quad \text{and} \quad *_g\, d\beta_j = \lambda\alpha_j$$

for $1 \leq j \leq m$, so applying the Beltrami operator to (5.4) yields

$$\begin{aligned}
-\lambda\beta_m &= *_g d\alpha_m \\
&= \sum_{j=1}^{m-1}(p_j *_g d\alpha_j + q_j *_g d\beta_j) \\
&= \sum_{j=1}^{m-1}(-\lambda p_j\beta_j + \lambda q_j\alpha_j) \\
&= -\lambda\sum_{j=1}^{m-1}(p_j\beta_j - q_j\alpha_j).
\end{aligned}$$

Thus, $\beta_m = \sum_{j=1}^{m-1}(p_j\beta_j - q_j\alpha_j)$. We therefore obtain

$$\begin{aligned}
\alpha_m + i\beta_m &= \sum_{j=1}^{m-1}[(p_j\alpha_j + q_j\beta_j) + i(p_j\beta - q_j\alpha)] \\
&= \sum_{j=1}^{m-1}(p_j - iq_j)(\alpha_j + i\beta_j),
\end{aligned}$$

which contradicts the independence of the forms $\alpha_1+i\beta_1, \ldots, \alpha_m+i\beta_m$ over $\mathbb{C}$. Hence, the set

$$S = \{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m\}$$

is linearly independent over $\mathbb{R}$. Since (5.3) implies any eigenform $\eta \in E(\Delta_g^{(k)}, \lambda^2)$ can be written as a linear combination of forms in S, we conclude that $E(\Delta_g^{(k)}, \lambda^2)$ has dimension $2m$. Therefore, all nonzero eigenvalues of $\Delta_g^{(k)}$ have even multiplicity. $\square$

By Corollary 4.1.3, the eigenvalues of the restriction of the Hodge Laplacian to exact $(2\ell+1)$-forms will likewise have even multiplicities.

Corollary 5.2.2. *Let (M, g) be a closed Riemannian manifold of dimension $n = 4\ell+1$ for some $\ell \in \mathbb{N}$, and let $k = 2\ell+1$. Then all eigenvalues of the restriction of $\Delta_g^{(k)}$ to exact forms in $H^2(M, \Lambda^k)$ have even multiplicity.*

In the case of closed manifolds of even dimension $n = 2k$, little is known regarding the eigenvalue multiplicities of the Hodge Laplacian beyond Millman's observation that $\Delta_g^{(k)}$ has eigenvalues of even multiplicity [20]. Enciso and Peralta-Salas [12] note that when $n = 2$, Uhlenbeck's Theorem 1.0.2 implies that the nonzero eigenvalues of the Hodge Laplacian on k-forms are generically simple when $k = 0$ and $k = 2$ and have multiplicity 2 when $k = 1$. To justify this last claim, observe that if λ is a simple eigenvalue of $\Delta_g^{(0)}$ then it is also a simple eigenvalue of $\Delta_g^{(1)}$ restricted to exact forms by Theorem 4.1.2. Since λ is likewise a simple eigenvalue of $\Delta_g^{(2)}$, and hence of $\Delta_g^{(1)}$ restricted to coexact forms, we find that $E(\Delta_g^{(1)}, \lambda) = E_d^1(\lambda) \oplus E_\delta^1(\lambda)$ has dimension 2. We will need to develop new techniques to study the eigenvalue multiplicities of the Hodge Laplacian on manifolds of even dimension $n > 2$, for (3.1) indicates the Beltrami operator will not be of use.

5.3 Perturbation of Boundary

The applications of Uhlenbeck's Theorem 1.0.1 extend beyond metric perturbations. As another example, she establishes that if N is a compact n-manifold with boundary, then the Laplacian on $\mathrm{Im}(F)$ will have simple eigenvalues for generic C^r embeddings $F : N \to \mathbb{R}^n$.

Theorem 5.3.1. (Uhlenbeck, [28]) Let $r > n-2$, and let $\Delta_{\mathrm{Im}(F)}$ denote the Laplace-Beltrami operator on the image of F with Dirichlet boundary conditions. Then the set

$$\{F \in \mathrm{Emb}_r(N, \mathbb{R}^n) \,|\, \Delta_{\mathrm{Im}(F)} \text{ has one-dimensional eigenspaces}\}$$

is residual in $\mathrm{Emb}_r(N, \mathbb{R}^n)$.

Henry [15] likewise considers the generic simplicity of eigenvalues of partial differential operators under perturbation of the boundary.

We ask whether an analogue of Theorem 5.3.1 might hold for the Hodge Laplacian with relative boundary conditions. Ho [17] raises this question in his consideration of the Hodge Laplacian acting on a family of symmetric regions in $\mathbb{R}^n$ consisting of two cavities connected by a thin tube. He assumes simple first relative eigenvalues of the Hodge Laplacian with relative boundary conditions, noting that there is no general classification of domains which satisfy this assumption.

Appendix A Computation of Derivatives

We here provide the proof of Lemma 3.3.1 by calculating the variation of the Beltrami operator at the metric $g \in \mathcal{G}^r(M)$ in the direction of a symmetric $(0,2)$-tensor h in $S^r(M)$. A preliminary step in this computation is determining how variation of the metric effects the local coordinate expressions of the inverse metric g^{-1} and the determinant $|g|$.

Lemma A1. *Let $g \in \mathcal{G}^r(M)$. Then $D(g^{ij}) = -h^{ij}$ for all $h \in S^r(M)$.*

Proof. We wish to compute

$$D(g^{ij})(h) = \lim_{t\to 0} \frac{1}{t}[(g+th)^{ij} - g^{ij}].$$

Utilizing the geometric series expansion, we write

$$(g+th)^{-1} = (1+tg^{-1}h)^{-1}g^{-1} = \left(\sum_{m=0}^{\infty}(-1)^m t^m (g^{-1}h)^m\right) g^{-1},$$

which allows us to compute

$$\begin{aligned}
D(g^{-1})(h) &= \lim_{t\to 0} \frac{1}{t}[(g+th)^{-1} - g^{-1}] \\
&= \lim_{t\to 0} \frac{1}{t}\left[\left(g^{-1} + \sum_{m=1}^{\infty}(-1)^m t^m (g^{-1}h)^m g^{-1}\right) - g^{-1}\right] \\
&= \lim_{t\to 0} \frac{1}{t}[-tg^{-1}hg^{-1} + O(t^2)] \\
&= -g^{-1}hg^{-1}.
\end{aligned}$$

Since the ij-th component of the matrix $(-g^{-1}hg^{-1})$ is $-g^{il}g^{kj}h_{lk} = -h^{ij}$, we see that $D(g^{ij})(h) = -h^{ij}$ as claimed. □

Lemma A2. *Let $g \in \mathcal{G}^r(M)$. Then $D(|g|^s)(h) = s|g|^s(tr_g h)$ for all $h \in S^r(M)$.*

Proof. For any $s > 0$, we compute

$$\begin{aligned} D(|g|^s)(h) &= \lim_{t\to 0}\frac{1}{t}(|g+th|^s - |g|^s) \\ &= \lim_{t\to 0}\frac{1}{t}(|g|^s|1+tg^{-1}h|^s - |g|^s) \\ &= \lim_{t\to 0}\frac{1}{t}[|g|^s(1+tg^{ij}h_{ij}+O(t^2))^s - |g|^s] \\ &= \lim_{t\to 0}\frac{1}{t}[|g|^s(1+st(\mathrm{tr}_g h)+O(t^2)) - |g|^s] \\ &= s|g|^s(\mathrm{tr}_g h). \end{aligned}$$

□

Lemmas A1 and A2 allow us to calculate the derivative of $*_g d$ under variation of the metric.

Lemma 3.3.1. *Let $u \in H^1(M, \Lambda^2_{\mathbb{C}})$ be an eigenform of $*_g d$ with eigenvalue $i\lambda$. Then for any $h \in \mathcal{S}^r(M)$,*

$$(D(*d)_g(h)u)_{ij} = i\lambda\left[-\frac{1}{2}(tr_g h)u_{ij} + g^{mt}h_{ti}u_{mj} + g^{mt}h_{tj}u_{im}\right].$$

Proof. Let $u \in H^1(M, \Lambda^2_{\mathbb{C}})$ be an eigenform of $*_g d$ with eigenvalue $i\lambda$. Since u is a 2-form, (2.3) allows us to express the differential du in local coordinates as

$$(du)_{npq} = \frac{\partial u_{np}}{\partial x_q} - \frac{\partial u_{nq}}{\partial x_p} + \frac{\partial u_{pq}}{\partial x_n},$$

and thus by (2.6), the Beltrami operator acting on u has local coordinate expression

$$(*_g du)_{ij} = \frac{1}{6}\varepsilon_{klmij}|g|^{1/2}g^{kn}g^{lp}g^{mq}\left(\frac{\partial u_{np}}{\partial x_q} - \frac{\partial u_{nq}}{\partial x_p} + \frac{\partial u_{pq}}{\partial x_n}\right).$$

Utilizing the calculations $D(g^{ij})(h) = -h^{ij}$ and $D(|g|^s)(h) = s|g|^s(\mathrm{tr}_g h)$ of Lemmas A1 and A2, we find that the ij-th component of the differential of $*_g d$ with respect to g in the direction of $h \in \mathcal{S}^r(M)$ acting on u is

$$\begin{aligned} (D(*d)_g(h)u)_{ij} &= \frac{1}{6}\varepsilon_{klmij}|g|^{1/2}\left(\frac{\partial u_{np}}{\partial x_q} - \frac{\partial u_{nq}}{\partial x_p} + \frac{\partial u_{pq}}{\partial x_n}\right) \\ &\quad\times\left[\frac{1}{2}(\mathrm{tr}_g h)g^{kn}g^{lp}g^{mq} - g^{kn}g^{lp}h^{mq} - g^{kn}g^{mq}h^{lp} - g^{lp}g^{mq}h^{kn}\right]. \end{aligned} \tag{5}$$

The formula $A^{-1} = (\det A)^{-1}\mathrm{adj}(A)$ implies that

$$\begin{aligned}
|g|\varepsilon_{klmij}\,\varepsilon_{npqrs}g^{kn} \;=\;& g_{lp}g_{mq}g_{ir}g_{js} - g_{lp}g_{mq}g_{is}g_{jr} - g_{lp}g_{mr}g_{iq}g_{js} + g_{lp}g_{mr}g_{is}g_{jq} \\
& + g_{lp}g_{ms}g_{iq}g_{jr} - g_{lp}g_{ms}g_{ir}g_{jq} - g_{lq}g_{mp}g_{ir}g_{js} + g_{lq}g_{mp}g_{is}g_{jr} \\
& + g_{lq}g_{mr}g_{ip}g_{js} - g_{lq}g_{mr}g_{is}g_{jp} - g_{lq}g_{ms}g_{ip}g_{jr} + g_{lq}g_{ms}g_{ir}g_{jp} \\
& + g_{lr}g_{mp}g_{iq}g_{js} - g_{lr}g_{mp}g_{is}g_{jq} - g_{lr}g_{mq}g_{ip}g_{js} + g_{lr}g_{mq}g_{is}g_{jp} \\
& + g_{lr}g_{ms}g_{ip}g_{jq} - g_{lr}g_{ms}g_{iq}g_{jp} - g_{ls}g_{mp}g_{iq}g_{jr} + g_{ls}g_{mp}g_{ir}g_{jq} \\
& + g_{ls}g_{mq}g_{ip}g_{jr} - g_{ls}g_{mq}g_{ir}g_{jp} - g_{ls}g_{mr}g_{ip}g_{jq} + g_{ls}g_{mr}g_{iq}g_{jp}.
\end{aligned}$$

By this formula and the fact that u is an eigenform of $*_g d$ with eigenvalue $i\lambda$, we obtain

$$
\begin{aligned}
i\lambda\varepsilon_{npqrs}|g|^{1/2}u^{rs} &= \varepsilon_{npqrs}|g|^{1/2}g^{ri}g^{sj}(i\lambda u_{ij}) \\
&= \varepsilon_{npqrs}|g|^{1/2}g^{ri}g^{sj}(*_g du)_{ij} \\
&= \frac{1}{6}(|g|\varepsilon_{npqrs}\varepsilon_{klmij}g^{sj})g^{ka}g^{lb}g^{mc}g^{ri}\left(\frac{\partial u_{ab}}{\partial x_c}-\frac{\partial u_{ac}}{\partial x_b}+\frac{\partial u_{bc}}{\partial x_a}\right) \\
&= \frac{1}{6}\big(g_{nk}g_{pl}g_{qm}g_{ri}-g_{nk}g_{pl}g_{qi}g_{rm}-g_{nk}g_{pm}g_{ql}g_{ri}+g_{nk}g_{pm}g_{qi}g_{rl} \\
&\quad +g_{nk}g_{pi}g_{ql}g_{rm}-g_{nk}g_{pi}g_{qm}g_{rl}-g_{nl}g_{pk}g_{qm}g_{ri}+g_{nl}g_{pk}g_{qi}g_{rm} \\
&\quad +g_{nl}g_{pm}g_{qk}g_{ri}-g_{nl}g_{pm}g_{qi}g_{rk}-g_{nl}g_{pi}g_{qk}g_{rm}+g_{nl}g_{pi}g_{qm}g_{rk} \\
&\quad +g_{nm}g_{pk}g_{ql}g_{ri}-g_{nm}g_{pk}g_{qi}g_{rl}-g_{nm}g_{pl}g_{qk}g_{ri}+g_{nm}g_{pl}g_{qi}g_{rk} \\
&\quad +g_{nm}g_{pi}g_{qk}g_{rl}-g_{nm}g_{pi}g_{ql}g_{rk}-g_{ni}g_{pk}g_{ql}g_{rm}+g_{ni}g_{pk}g_{qm}g_{rl} \\
&\quad +g_{ni}g_{pl}g_{qk}g_{rm}-g_{ni}g_{pl}g_{qm}g_{rk}-g_{ni}g_{pm}g_{qk}g_{rl}+g_{ni}g_{pm}g_{ql}g_{rk}\big) \\
&\quad \times g^{ka}g^{lb}g^{mc}g^{ri}\left(\frac{\partial u_{ab}}{\partial x_c}-\frac{\partial u_{ac}}{\partial x_b}+\frac{\partial u_{bc}}{\partial x_a}\right) \\
&= \frac{1}{6}\big(5\delta_n^a\delta_p^b\delta_q^c-\delta_n^a\delta_p^b\delta_q^r\delta_r^c-5\delta_n^a\delta_p^c\delta_q^b+\delta_n^a\delta_p^c\delta_q^r\delta_r^b+\delta_n^a\delta_p^r\delta_a^b\delta_r^c \\
&\quad -\delta_n^a\delta_p^r\delta_q^c\delta_r^b-5\delta_n^b\delta_p^a\delta_q^c+\delta_n^b\delta_p^c\delta_q^r\delta_r^a+5\delta_n^b\delta_p^c\delta_q^a-\delta_n^b\delta_p^c\delta_q^r\delta_r^a \\
&\quad -\delta_n^b\delta_p^r\delta_q^a\delta_r^c+\delta_n^b\delta_p^r\delta_q^c\delta_r^a+5\delta_n^c\delta_p^a\delta_q^b-\delta_n^c\delta_p^a\delta_q^r\delta_r^b-5\delta_n^c\delta_p^b\delta_q^a \\
&\quad +\delta_n^c\delta_p^b\delta_q^r\delta_r^a+\delta_n^c\delta_p^r\delta_q^a\delta_r^b-\delta_n^c\delta_p^r\delta_q^b\delta_r^a-\delta_n^r\delta_p^a\delta_q^b\delta_r^c+\delta_n^r\delta_p^a\delta_q^c\delta_r^b \\
&\quad +\delta_n^r\delta_p^b\delta_q^a\delta_r^c-\delta_n^r\delta_p^b\delta_q^c\delta_r^a-\delta_n^r\delta_p^c\delta_q^a\delta_r^b+\delta_n^r\delta_p^c\delta_q^b\delta_r^a\big) \\
&\quad \times\left(\frac{\partial u_{ab}}{\partial x_c}-\frac{\partial u_{ac}}{\partial x_b}+\frac{\partial u_{bc}}{\partial x_a}\right) \\
&= \frac{1}{6}\left[2\left(\frac{\partial u_{np}}{\partial x_q}-\frac{\partial u_{nq}}{\partial x_p}+\frac{\partial u_{pq}}{\partial x_n}\right)-2\left(\frac{\partial u_{nq}}{\partial x_p}-\frac{\partial u_{np}}{\partial x_q}+\frac{\partial u_{qp}}{\partial x_n}\right)\right. \\
&\quad -2\left(\frac{\partial u_{pn}}{\partial x_q}-\frac{\partial u_{pq}}{\partial x_n}+\frac{\partial u_{nq}}{\partial x_p}\right)+2\left(\frac{\partial u_{qn}}{\partial x_p}-\frac{\partial u_{qp}}{\partial x_n}+\frac{\partial u_{np}}{\partial x_q}\right) \\
&\quad \left.+2\left(\frac{\partial u_{pq}}{\partial x_n}-\frac{\partial u_{pn}}{\partial x_q}+\frac{\partial u_{qn}}{\partial x_p}\right)-2\left(\frac{\partial u_{qp}}{\partial x_n}-\frac{\partial u_{qn}}{\partial x_p}+\frac{\partial u_{pn}}{\partial x_q}\right)\right] \\
&= 2\left(\frac{\partial u_{np}}{\partial x_q}-\frac{\partial u_{nq}}{\partial x_p}+\frac{\partial u_{pq}}{\partial x_n}\right)
\end{aligned}
$$

Substituting this expression for $\frac{\partial u_{np}}{\partial x_q} - \frac{\partial u_{nq}}{\partial x_p} + \frac{\partial u_{pq}}{\partial x_n}$ into (5) yields

$$\begin{aligned}(D(*d)_g(h)u)_{ij} &= \frac{i\lambda}{12}\,\varepsilon_{klmij}\varepsilon_{npqrs}|g|u^{rs} \times \\ &\quad \left[\frac{1}{2}(\mathrm{tr}_g h)g^{kn}g^{lp}g^{mq} - g^{kn}g^{lp}h^{mq} - g^{kn}g^{mq}h^{lp} - g^{lp}g^{mq}h^{kn}\right] \\ &= A + B + C + D\end{aligned}$$

where

$$A = \frac{i\lambda}{24}|g|(\mathrm{tr}_g h)\,\varepsilon_{klmij}\,\varepsilon_{npqrs}\,g^{kn}g^{lp}g^{mq}g^{ra}g^{sb}u_{ab}$$

$$B = -\frac{i\lambda}{12}|g|\,\varepsilon_{klmij}\,\varepsilon_{npqrs}\,g^{kn}g^{lp}g^{ra}g^{sb}h^{mq}u_{ab}$$

$$C = -\frac{i\lambda}{12}|g|\,\varepsilon_{klmij}\,\varepsilon_{npqrs}\,g^{kn}g^{mq}g^{ra}g^{sb}h^{lp}u_{ab}$$

$$D = -\frac{i\lambda}{12}|g|\,\varepsilon_{klmij}\,\varepsilon_{npqrs}\,g^{lp}g^{mq}g^{ra}g^{sb}h^{kn}u_{ab}.$$

Note that by symmetry, $B = C = D$. We compute

$$
\begin{aligned}
A &= \frac{i\lambda}{24}(\mathrm{tr}_g h)(|g|\,\varepsilon_{klmij}\,\varepsilon_{npqrs}\,g^{kn})g^{lp}g^{mq}g^{ra}g^{sb}u_{ab} \\
&= \frac{i\lambda}{24}(\mathrm{tr}_g h)\,(g_{lp}g_{mq}g_{ir}g_{js} - g_{lp}g_{mq}g_{is}g_{jr} - g_{lp}g_{mr}g_{iq}g_{js} + g_{lp}g_{mr}g_{is}g_{jq} \\
&\quad + g_{lp}g_{ms}g_{iq}g_{jr} - g_{lp}g_{ms}g_{ir}g_{jq} - g_{lq}g_{mp}g_{ir}g_{js} + g_{lq}g_{mp}g_{is}g_{jr} \\
&\quad + g_{lq}g_{mr}g_{ip}g_{js} - g_{lq}g_{mr}g_{is}g_{jp} - g_{lq}g_{ms}g_{ip}g_{jr} + g_{lq}g_{ms}g_{ir}g_{jp} \\
&\quad + g_{lr}g_{mp}g_{iq}g_{js} - g_{lr}g_{mp}g_{is}g_{jq} - g_{lr}g_{mq}g_{ip}g_{js} + g_{lr}g_{mq}g_{is}g_{jp} \\
&\quad + g_{lr}g_{ms}g_{ip}g_{jq} - g_{lr}g_{ms}g_{iq}g_{jp} - g_{ls}g_{mp}g_{iq}g_{jr} + g_{ls}g_{mp}g_{ir}g_{jq} \\
&\quad + g_{ls}g_{mq}g_{ip}g_{jr} - g_{ls}g_{mq}g_{ir}g_{jp} - g_{ls}g_{mr}g_{ip}g_{jq} + g_{ls}g_{mr}g_{iq}g_{jp}) \\
&\quad \times g^{lp}g^{mq}g^{ra}g^{sb}u_{ab} \\
&= \frac{i\lambda}{24}(\mathrm{tr}_g h)\left(25\delta_i^a\delta_j^b - 25\delta_i^b\delta_j^a - 5\delta_m^a\delta_i^m\delta_j^b + 5\delta_m^a\delta_i^b\delta_j^m + 5\delta_m^b\delta_i^m\delta_j^a - 5\delta_m^b\delta_i^a\delta_j^m \right.\\
&\quad -\delta_l^m\delta_m^l\delta_i^a\delta_j^b + \delta_l^m\delta_m^l\delta_i^b\delta_j^a + \delta_l^m\delta_m^a\delta_i^l\delta_j^b - \delta_l^m\delta_m^a\delta_i^b\delta_j^l - \delta_l^m\delta_m^b\delta_i^l\delta_j^a + \delta_l^m\delta_m^b\delta_i^a\delta_j^l \\
&\quad +\delta_l^a\delta_m^l\delta_i^m\delta_j^b - \delta_l^a\delta_m^l\delta_i^b\delta_j^m - 5\delta_l^a\delta_i^l\delta_j^b + 5\delta_l^a\delta_i^b\delta_j^l + \delta_l^a\delta_m^b\delta_i^l\delta_j^m - \delta_l^a\delta_m^b\delta_i^m\delta_j^l \\
&\quad \left. -\delta_l^b\delta_m^l\delta_i^m\delta_j^a + \delta_l^b\delta_m^l\delta_i^a\delta_j^m + 5\delta_l^b\delta_i^l\delta_j^a - 5\delta_l^b\delta_i^a\delta_j^l - \delta_l^b\delta_m^a\delta_i^l\delta_j^m + \delta_l^b\delta_m^a\delta_i^m\delta_j^l\right) u_{ab} \\
&= \frac{i\lambda}{24}(\mathrm{tr}_g h)(6u_{ij} - 6u_{ji}) \\
&= \frac{i\lambda}{2}(\mathrm{tr}_g h)u_{ij}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
B &= -\frac{i\lambda}{12}(|g|\,\varepsilon_{klmij}\,\varepsilon_{npqrs}\,g^{kn})g^{lp}g^{qv}g^{ra}g^{sb}g^{mt}h_{tv}u_{ab} \\
&= -\frac{i\lambda}{12}\big(g_{lp}g_{mq}g_{ir}g_{js} - g_{lp}g_{mq}g_{is}g_{jr} - g_{lp}g_{mr}g_{iq}g_{js} + g_{lp}g_{mr}g_{is}g_{jq} \\
&\quad + g_{lp}g_{ms}g_{iq}g_{jr} - g_{lp}g_{ms}g_{ir}g_{jq} - g_{lq}g_{mp}g_{ir}g_{js} + g_{lq}g_{mp}g_{is}g_{jr} \\
&\quad + g_{lq}g_{mr}g_{ip}g_{js} - g_{lq}g_{mr}g_{is}g_{jp} - g_{lq}g_{ms}g_{ip}g_{jr} + g_{lq}g_{ms}g_{ir}g_{jp} \\
&\quad + g_{lr}g_{mp}g_{iq}g_{js} - g_{lr}g_{mp}g_{is}g_{jq} - g_{lr}g_{mq}g_{ip}g_{js} + g_{lr}g_{mq}g_{is}g_{jp} \\
&\quad + g_{lr}g_{ms}g_{ip}g_{jq} - g_{lr}g_{ms}g_{iq}g_{jp} - g_{ls}g_{mp}g_{iq}g_{jr} + g_{ls}g_{mp}g_{ir}g_{jq} \\
&\quad + g_{ls}g_{mq}g_{ip}g_{jr} - g_{ls}g_{mq}g_{ir}g_{jp} - g_{ls}g_{mr}g_{ip}g_{jq} + g_{ls}g_{mr}g_{iq}g_{jp}\big) \\
&\quad \times g^{lp}g^{qv}g^{ra}g^{sb}g^{mt}h_{tv}u_{ab} \\
&= -\frac{i\lambda}{12}\big(5\delta^v_m\delta^a_i\delta^b_j - 5\delta^v_m\delta^b_i\delta^a_j - 5\delta^a_m\delta^v_i\delta^b_j + 5\delta^a_m\delta^b_i\delta^v_j + 5\delta^b_m\delta^v_i\delta^a_j - 5\delta^b_m\delta^a_i\delta^v_j \\
&\quad - \delta^v_l\delta^l_m\delta^a_i\delta^b_j + \delta^v_l\delta^l_m\delta^b_i\delta^a_j + \delta^v_l\delta^a_m\delta^l_i\delta^b_j - \delta^v_l\delta^a_m\delta^b_i\delta^l_j - \delta^v_l\delta^b_m\delta^l_i\delta^a_j + \delta^v_l\delta^b_m\delta^a_i\delta^l_j \\
&\quad + \delta^a_l\delta^l_m\delta^v_i\delta^b_j - \delta^a_l\delta^l_m\delta^b_i\delta^v_j - \delta^a_l\delta^v_m\delta^l_i\delta^b_j + \delta^a_l\delta^v_m\delta^b_i\delta^l_j + \delta^a_l\delta^b_m\delta^l_i\delta^v_j - \delta^a_l\delta^b_m\delta^v_i\delta^l_j \\
&\quad - \delta^b_l\delta^l_m\delta^v_i\delta^a_j + \delta^b_l\delta^l_m\delta^a_i\delta^v_j + \delta^b_l\delta^v_m\delta^l_i\delta^a_j - \delta^b_l\delta^v_m\delta^a_i\delta^l_j - \delta^b_l\delta^a_m\delta^l_i\delta^v_j + \delta^b_l\delta^a_m\delta^v_i\delta^l_j\big) \\
&\quad \times g^{mt}h_{tv}u_{ab} \\
&= -\frac{i\lambda}{12}\big(2g^{mt}h_{tm}u_{ij} - 2g^{mt}h_{tm}u_{ji} - 2g^{mt}h_{ti}u_{mj} + 2g^{mt}h_{tj}u_{mi} + 2g^{mt}h_{ti}u_{jm} \\
&\quad - 2g^{mt}h_{tj}u_{im}\big) \\
&= -\frac{i\lambda}{3}[(\mathrm{tr}_g h)u_{ij} - g^{mt}h_{ti}u_{mj} - g^{mt}h_{tj}u_{im}].
\end{aligned}
$$

We combine the expressions for A and B to obtain

$$
\begin{aligned}
(D(*d)_g(h)u)_{ij} &= A + 3B \\
&= \frac{i\lambda}{2}(\mathrm{tr}_g h)u_{ij} - i\lambda[(\mathrm{tr}_g h)u_{ij} - g^{mt}h_{ti}u_{mj} - g^{mt}h_{tj}u_{im}] \\
&= i\lambda\left[-\frac{1}{2}(\mathrm{tr}_g h)u_{ij} + g^{mt}h_{ti}u_{mj} + g^{mt}h_{tj}u_{im}\right].
\end{aligned}
$$

□

Appendix B Kernel of Sylvester Equation

In the proof of Lemma 3.3.2, we defined $\mathcal{M}$ to be the set of C^r 5×5 matrix-valued functions on M and introduced the linear operator $L : \mathcal{M} \to \mathcal{M}$ given by

$$L(X) = X\tilde{W} + \tilde{W}X, \tag{3.3}$$

where $\tilde{W}$ was a given antisymmetric matrix corresponding to a 2-form. We stated that all matrices $E \in \ker L$ are symmetric, a claim which we now prove.

Lemma B1. *Let $U \in \mathbb{C}^{5\times5}$ be an antisymmetric matrix with zeros along the diagonal, and define the linear operator $L : \mathbb{C}^{5\times5} \to \mathbb{C}^{5\times5}$ by*

$$L(X) = UX + XU.$$

Then each matrix in the kernel of L is symmetric.

Proof. Let $U \in \mathbb{C}^{5\times5}$ be an antisymmetric matrix of the form

$$U = \begin{bmatrix} 0 & u_{12} & u_{13} & u_{14} & u_{15} \\ -u_{12} & 0 & u_{23} & u_{24} & u_{25} \\ -u_{13} & -u_{23} & 0 & u_{34} & u_{35} \\ -u_{14} & -u_{24} & -u_{34} & 0 & u_{45} \\ -u_{15} & -u_{25} & -u_{35} & -u_{45} & 0 \end{bmatrix}.$$

A matrix

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} \\ e_{51} & e_{52} & e_{53} & e_{54} & e_{55} \end{bmatrix}$$

is contained in $\ker L$ if

$$UE + EU = 0. \tag{6}$$

We wish to show that all such matrices $E \in \ker L$ are symmetric.

Since $UE+EU$ is a 5×5 matrix, the matrix equation (6) corresponds to a system of 25 linear equations in which the e_{ij} are unknown. From these 25 equations, we obtain the equivalent matrix equation

$$Kv = 0, \tag{7}$$

where

$$v = [e_{11}, \ldots, e_{15}, e_{21}, \ldots, e_{25}, e_{31}, \ldots, e_{35}, e_{41}, \ldots, e_{45}, e_{51}, \ldots, e_{55}]^T$$

is the vectorization of E and K is the 25×25 matrix

$$K = \left[\begin{array}{c|c|c|c|c} U^T & J_{12} & J_{13} & J_{14} & J_{15} \\ \hline -J_{12} & U^T & J_{23} & J_{24} & J_{25} \\ \hline -J_{13} & -J_{23} & U^T & J_{34} & J_{35} \\ \hline -J_{14} & -J_{24} & -J_{34} & U^T & J_{45} \\ \hline -J_{15} & -J_{25} & -J_{35} & -J_{45} & U^T \end{array}\right],$$

where $J_{ij} = u_{ij} I_{5\times5}$ for $1 \leq i < j \leq 5$.

Using Maple, we find that the reduced row echelon form of K is the matrix

$$A = \left[\begin{array}{c|c} I_{20\times20} & B_{20\times5} \\ \hline \multicolumn{2}{c}{0_{5\times25}} \end{array}\right].$$

Since the last 5 rows of A are zero, the system of equations given by

$$Av = 0 \tag{8}$$

is underdetermined, indicating there exist nonzero solutions v. We claim that all such v satisfy the symmetry condition $e_{ij} = e_{ji}$. To see this, first consider the condition

$e_{12} = e_{21}$. Letting b_{ij} denote the (i, j)-th entry of B, the second row of (8) gives the equation

$$e_{12} + b_{21}e_{51} + b_{22}e_{52} + b_{23}e_{53} + b_{24}e_{54} + b_{25}e_{55} \;=\; 0, \tag{9}$$

while the sixth row yields

$$e_{21} + b_{61}e_{51} + b_{62}e_{52} + b_{63}e_{53} + b_{64}e_{54} + b_{65}e_{55} \;=\; 0. \tag{10}$$

By Maple, we find that $b_{2i} = b_{6i}$ for $1 \leq i \leq 5$, and hence equations (9) and (10) combine to give $e_{12} = e_{21}$. A similar comparison shows

$$e_{13} = e_{31}, \quad e_{14} = e_{41}, \quad e_{23} = e_{32}, \quad e_{24} = e_{42}, \text{ and } e_{34} = e_{43}.$$

Furthermore, Maple indicates that the fifth row of B is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and so we obtain the equation

$$e_{15} - e_{51} = 0$$

from the fifth row of (8). We likewise find

$$e_{25} = e_{52}, \quad e_{35} = e_{53}, \text{ and } e_{45} = e_{54}$$

by examining rows 10, 15, and 20 of (8), respectively. Therefore, we conclude that all $E \in \ker L$ are symmetric. $\square$

$> with(LinearAlgebra):$

$> interface(rtablesize{=}25):$

$$> U := \begin{bmatrix} 0 & u_{12} & u_{13} & u_{14} & u_{15} \\ -u_{12} & 0 & u_{23} & u_{24} & u_{25} \\ -u_{13} & -u_{23} & 0 & u_{34} & u_{35} \\ -u_{14} & -u_{24} & -u_{34} & 0 & u_{45} \\ -u_{15} & -u_{25} & -u_{35} & -u_{45} & 0 \end{bmatrix}:$$

$$> E := \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} \\ e_{41} & e_{42} & e_{43} & e_{44} & e_{45} \\ e_{51} & e_{52} & e_{53} & e_{54} & e_{55} \end{bmatrix}:$$

$> U.E + E.U$

$$[\,[u_{12}e_{21} + u_{13}e_{31} + u_{14}e_{41} + u_{15}e_{51} - u_{12}e_{12} - u_{13}e_{13} - u_{14}e_{14} - u_{15}e_{15},$$

$$u_{12}e_{22} + u_{13}e_{32} + u_{14}e_{42} + u_{15}e_{52} + u_{12}e_{11} - e_{13}u_{23} - e_{14}u_{24} - e_{15}u_{25},$$

$$u_{12}e_{23} + u_{13}e_{33} + u_{14}e_{43} + u_{15}e_{53} + u_{13}e_{11} + e_{12}u_{23} - e_{14}u_{34} - e_{15}u_{35},$$

$$u_{12}e_{24} + u_{13}e_{34} + u_{14}e_{44} + u_{15}e_{54} + u_{14}e_{11} + e_{12}u_{24} + e_{13}u_{34} - e_{15}u_{45},$$

$$u_{12}e_{25} + u_{13}e_{35} + u_{14}e_{45} + u_{15}e_{55} + u_{15}e_{11} + e_{12}u_{25} + e_{13}u_{35} + e_{14}u_{45}]\,,$$

$$[-u_{12}e_{11} + u_{23}e_{31} + u_{24}e_{41} + u_{25}e_{51} - u_{12}e_{22} - e_{23}u_{13} - e_{24}u_{14} - e_{25}u_{15},$$

$$-u_{12}e_{12} + u_{23}e_{32} + u_{24}e_{42} + u_{25}e_{52} + u_{12}e_{21} - u_{23}e_{23} - u_{24}e_{24} - u_{25}e_{25},$$

$$-u_{12}e_{13} + u_{23}e_{33} + u_{24}e_{43} + u_{25}e_{53} + e_{21}u_{13} + u_{23}e_{22} - e_{24}u_{34} - e_{25}u_{35},$$

$$-u_{12}e_{14} + u_{23}e_{34} + u_{24}e_{44} + u_{25}e_{54} + e_{21}u_{14} + u_{24}e_{22} + e_{23}u_{34} - e_{25}u_{45},$$

$$-u_{12}e_{15} + u_{23}e_{35} + u_{24}e_{45} + u_{25}e_{55} + e_{21}u_{15} + u_{25}e_{22} + e_{23}u_{35} + e_{24}u_{45}]\,,$$

$$[-u_{13}e_{11} - u_{23}e_{21} + u_{34}e_{41} + u_{35}e_{51} - e_{32}u_{12} - u_{13}e_{33} - e_{34}u_{14} - e_{35}u_{15},$$

$$-u_{13}e_{12} - u_{23}e_{22} + u_{34}e_{42} + u_{35}e_{52} + e_{31}u_{12} - u_{23}e_{33} - e_{34}u_{24} - e_{35}u_{25},$$

$$-u_{13}e_{13} - u_{23}e_{23} + u_{34}e_{43} + u_{35}e_{53} + u_{13}e_{31} + u_{23}e_{32} - u_{34}e_{34} - u_{35}e_{35},$$

$$-u_{13}e_{14} - u_{23}e_{24} + u_{34}e_{44} + u_{35}e_{54} + e_{31}u_{14} + e_{32}u_{24} + u_{34}e_{33} - e_{35}u_{45},$$

$-u_{13}e_{15} - u_{23}e_{25} + u_{34}e_{45} + u_{35}e_{55} + e_{31}u_{15} + e_{32}u_{25} + u_{35}e_{33} + e_{34}u_{45}]$,

$[-u_{14}e_{11} - u_{24}e_{21} - u_{34}e_{31} + u_{45}e_{51} - e_{42}u_{12} - e_{43}u_{13} - u_{14}e_{44} - e_{45}u_{15},$

$-u_{14}e_{12} - u_{24}e_{22} - u_{34}e_{32} + u_{45}e_{52} + e_{41}u_{12} - e_{43}u_{23} - u_{24}e_{44} - e_{45}u_{25},$

$-u_{14}e_{13} - u_{24}e_{23} - u_{34}e_{33} + u_{45}e_{53} + e_{41}u_{13} + e_{42}u_{23} - u_{34}e_{44} - e_{45}u_{35},$

$-u_{14}e_{14} - u_{24}e_{24} - u_{34}e_{34} + u_{45}e_{54} + u_{14}e_{41} + u_{24}e_{42} + u_{34}e_{43} - u_{45}e_{45},$

$-u_{14}e_{15} - u_{24}e_{25} - u_{34}e_{35} + u_{45}e_{55} + e_{41}u_{15} + e_{42}u_{25} + e_{43}u_{35} + u_{45}e_{44}]$,

$[-u_{15}e_{11} - u_{25}e_{21} - u_{35}e_{31} - u_{45}e_{41} - e_{52}u_{12} - e_{53}u_{13} - e_{54}u_{14} - u_{15}e_{55},$

$-u_{15}e_{12} - u_{25}e_{22} - u_{35}e_{32} - u_{45}e_{42} + e_{51}u_{12} - e_{53}u_{23} - e_{54}u_{24} - u_{25}e_{55},$

$-u_{15}e_{13} - u_{25}e_{23} - u_{35}e_{33} - u_{45}e_{43} + e_{51}u_{13} + e_{52}u_{23} - e_{54}u_{34} - u_{35}e_{55},$

$-u_{15}e_{14} - u_{25}e_{24} - u_{35}e_{34} - u_{45}e_{44} + e_{51}u_{14} + e_{52}u_{24} + e_{53}u_{34} - u_{45}e_{55},$

$-u_{15}e_{15} - u_{25}e_{25} - u_{35}e_{35} - u_{45}e_{45} + u_{15}e_{51} + u_{25}e_{52} + u_{35}e_{53} + u_{45}e_{54}]]$

$>$ $K := Matrix([[0, -u_{12}, -u_{13}, -u_{14}, -u_{15}, u_{12}, 0, 0, 0, 0, u_{13}, 0, 0, 0, 0, u_{14}, 0, 0, 0,$
$0, u_{15}, 0, 0, 0, 0],$

$[u_{12}, 0, -u_{23}, -u_{24}, -u_{25}, 0, u_{12}, 0, 0, 0, 0, u_{13}, 0, 0, 0, 0, u_{14}, 0, 0, 0, 0, u_{15}, 0, 0, 0],$

$[u_{13}, u_{23}, 0, -u_{34}, -u_{35}, 0, 0, u_{12}, 0, 0, 0, 0, u_{13}, 0, 0, 0, 0, u_{14}, 0, 0, 0, 0, u_{15}, 0, 0],$

$[u_{14}, u_{24}, u_{34}, 0, -u_{45}, 0, 0, 0, u_{12}, 0, 0, 0, 0, u_{13}, 0, 0, 0, 0, u_{14}, 0, 0, 0, 0, u_{15}, 0],$

$[u_{15}, u_{25}, u_{35}, u_{45}, 0, 0, 0, 0, 0, u_{12}, 0, 0, 0, 0, u_{13}, 0, 0, 0, 0, u_{14}, 0, 0, 0, 0, u_{15}],$

$[-u_{12}, 0, 0, 0, 0, 0, -u_{12}, -u_{13}, -u_{14}, -u_{15}, u_{23}, 0, 0, 0, 0, u_{24}, 0, 0, 0, 0, u_{25}, 0, 0, 0, 0],$

$[0, -u_{12}, 0, 0, 0, u_{12}, 0, -u_{23}, -u_{24}, -u_{25}, 0, u_{23}, 0, 0, 0, 0, u_{24}, 0, 0, 0, 0, u_{25}, 0, 0, 0],$

$[0, 0, -u_{12}, 0, 0, u_{13}, u_{23}, 0, -u_{34}, -u_{35}, 0, 0, u_{23}, 0, 0, 0, 0, u_{24}, 0, 0, 0, 0, u_{25}, 0, 0],$

$[0, 0, 0, -u_{12}, 0, u_{14}, u_{24}, u_{34}, 0, -u_{45}, 0, 0, 0, u_{23}, 0, 0, 0, 0, u_{24}, 0, 0, 0, 0, u_{25}, 0],$

$[0, 0, 0, 0, -u_{12}, u_{15}, u_{25}, u_{35}, u_{45}, 0, 0, 0, 0, 0, u_{23}, 0, 0, 0, 0, u_{24}, 0, 0, 0, 0, u_{25}],$

$[-u_{13}, 0, 0, 0, 0, -u_{23}, 0, 0, 0, 0, 0, -u_{12}, -u_{13}, -u_{14}, -u_{15}, u_{34}, 0, 0, 0, 0, u_{35}, 0, 0, 0, 0],$

$[0, -u_{13}, 0, 0, 0, 0, -u_{23}, 0, 0, 0, u_{12}, 0, -u_{23}, -u_{24}, -u_{25}, 0, u_{34}, 0, 0, 0, 0, u_{35}, 0, 0, 0],$

$[0, 0, -u_{13}, 0, 0, 0, 0, -u_{23}, 0, 0, u_{13}, u_{23}, 0, -u_{34}, -u_{35}, 0, 0, u_{34}, 0, 0, 0, 0, u_{35}, 0, 0],$

$[0, 0, 0, -u_{13}, 0, 0, 0, 0, -u_{23}, 0, u_{14}, u_{24}, u_{34}, 0, -u_{45}, 0, 0, 0, u_{34}, 0, 0, 0, 0, u_{35}, 0],$

$[0, 0, 0, 0, -u_{13}, 0, 0, 0, 0, -u_{23}, u_{15}, u_{25}, u_{35}, u_{45}, 0, 0, 0, 0, 0, u_{34}, 0, 0, 0, 0, u_{35}],$

$[-u_{14}, 0, 0, 0, 0, -u_{24}, 0, 0, 0, 0, -u_{34}, 0, 0, 0, 0, 0, -u_{12}, -u_{13}, -u_{14}, -u_{15}, u_{45}, 0, 0,$

$0, 0]$,

$[0, -u_{14}, 0, 0, 0, 0, -u_{24}, 0, 0, 0, 0, -u_{34}, 0, 0, 0, u_{12}, 0, -u_{23}, -u_{24}, -u_{25}, 0, u_{45}, 0, 0, 0],$

$[0, 0, -u_{14}, 0, 0, 0, 0, -u_{24}, 0, 0, 0, 0, -u_{34}, 0, 0, u_{13}, u_{23}, 0, -u_{34}, -u_{35}, 0, 0, u_{45}, 0, 0],$

$[0, 0, 0, -u_{14}, 0, 0, 0, 0, -u_{24}, 0, 0, 0, 0, -u_{34}, 0, u_{14}, u_{24}, u_{34}, 0, -u_{45}, 0, 0, 0, u_{45}, 0],$

$[0, 0, 0, 0, -u_{14}, 0, 0, 0, 0, -u_{24}, 0, 0, 0, 0, -u_{34}, u_{15}, u_{25}, u_{35}, u_{45}, 0, 0, 0, 0, 0, u_{45}],$

$[-u_{15}, 0, 0, 0, 0, -u_{25}, 0, 0, 0, 0, -u_{35}, 0, 0, 0, 0, -u_{45}, 0, 0, 0, 0, 0, -u_{12}, -u_{13}, -u_{14},$

$-u_{15}]$,

$[0, -u_{15}, 0, 0, 0, 0, -u_{25}, 0, 0, 0, 0, -u_{35}, 0, 0, 0, 0, -u_{45}, 0, 0, 0, u_{12}, 0, -u_{23}, -u_{24},$

$-u_{25}]$,

$[0, 0, -u_{15}, 0, 0, 0, 0, -u_{25}, 0, 0, 0, 0, -u_{35}, 0, 0, 0, 0, -u_{45}, 0, 0, u_{13}, u_{23}, 0, -u_{34}, -u_{35}],$

$[0, 0, 0, -u_{15}, 0, 0, 0, 0, -u_{25}, 0, 0, 0, 0, -u_{35}, 0, 0, 0, 0, -u_{45}, 0, u_{14}, u_{24}, u_{34}, 0, -u_{45}],$

$[0, 0, 0, 0, -u_{15}, 0, 0, 0, 0, -u_{25}, 0, 0, 0, 0, -u_{35}, 0, 0, 0, 0, -u_{45}, u_{15}, u_{25}, u_{35}, u_{45}, 0]])$:

$> A := ReducedRowEchelonForm_Q(K)$

[Length of output exceeds limit of 1000000]

$> A[1..20,\ 1..20]$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$> A[21..25\,1..25]$

$$\left[\begin{array}{ccccccccccccccccccccccccc} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{array}\right]$$

$> A[2,\ 21..25] - A[6,\ 21..25]$

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$> A[3,\ 21..25] - A[11,\ 21..25]$

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$> A[4,\ 21..25] - A[16,\ 21..25]$

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$> A[8,\ 21..25] - A[12,\ 21..25]$

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$> A[9,\ 21..25] - A[17,\ 21..25]$

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$> A[14,\ 21..25] - A[18,\ 21..25]$

$$\left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$> A[5,\ 21..25]$

$$\left[\begin{array}{ccccc} -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

> $A[10, 21..25]$

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$

> $A[15, 21..25]$

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

> $A[20, 21..25]$

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

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