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The Bourgain Spaces and Recovery of Magnetic and Electric Potentials of Schrödinger Operators

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The Bourgain spaces and recovery of magnetic and electric potentials of Schrödinger operators

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

The Bourgain spaces and recovery of magnetic and electric potentials of Schrödinger operators

We consider the inverse problem for the magnetic Schrödinger operator with the assumption that the magnetic potential is in C^λ and the electric potential is of the form $p_1 + \operatorname{div} p_2$ with $p_1,\, p_2 \in C^{\tilde{\lambda}}$. We use semiclassical pseudodifferential operators on semiclassical Sobolev spaces and Bourgain type spaces. The Bourgain type spaces are defined using the symbol of the operator $h^2\Delta + h\mu \cdot D$. Our main result gives a procedure for recovering the curl of the magnetic field and the electric potential from the Dirichlet to Neumann map. Our results are in dimension three and higher.

KEYWORDS: Schrödinger operator, Pseudodifferential operators,
Bourgain type spaces

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Chapter 1 Introduction

Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$ be a bounded domain with $C^{1,1}$ boundary. The magnetic Schrödinger operator is

$$H_{W,p} = \sum_{j=1}^{n} (D_j + W_j)^2 + p. \tag{1.1}$$

where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, W is the magnetic potential, and p is the electric potential. We assume that $W \in C_c^{\lambda}(\mathbf{R}^n; \mathbf{C}^n)$, $p = p_1 + \operatorname{div} p_2$ with p_1, $p_2 \in C_c^{\tilde{\lambda}}(\mathbf{R}^n; \mathbf{C})$. We are using $C_c^{\alpha}(\mathbf{R}^n)$ to denote the space of functions which are Hölder continuous of exponent α and compactly supported in $\mathbf{R}^n$.

If we assume 0 is not a Dirichlet eigenvalue of $H_{W,p}$, then the boundary value problem

$$\begin{cases} H_{W,p} u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u = u_f \in H^1(\Omega)$ for any $f \in H^{1/2}(\partial\Omega)$.

We will use the Dirichlet to Neumann map (DN map) to describe our boundary measurements. The DN map is defined by

$$\Lambda_{W,p} : f \to \frac{\partial u_f}{\partial \nu}|_{\partial\Omega} + i(W \cdot \nu) f$$

where ν is the outer unit normal to $\partial\Omega$. Furthermore, for f, $g \in H^{1/2}(\partial\Omega)$, if we assume $H_{W,p} u_f = 0$ in Ω, $u_f = f$ on $\partial\Omega$, $H_{-W,p} v_g = 0$ in Ω, $v_g = g$ on $\partial\Omega$, φ_f, φ_g are any functions in $H^1(\Omega)$ with $\varphi_f = f$, $\varphi_g = g$ on $\partial\Omega$, then the weak formulation for $\Lambda_{W,p}$ is

$$\langle \Lambda_{w,p} f, g \rangle = \int_{\Omega} (\nabla u_f \cdot \nabla \varphi_g + iW \cdot (u_f \nabla \varphi_g - \varphi_g \nabla u_f) + (W \cdot W) u_f \varphi_g) \, dx + \langle p u_f, \varphi_g \rangle. \tag{1.2}$$

Using the adjoint of $\Lambda_{W,p}$, we also have

$$\langle \Lambda_{w,p} f, g \rangle = \int_{\Omega} (\nabla \varphi_f \cdot \nabla v_g + iW \cdot (\varphi_f \nabla v_g - v_g \nabla \varphi_f) + (W \cdot W) \varphi_f, v_g) \, dx + \langle p \varphi_f, v_g \rangle. \tag{1.3}$$

From this definition it follows that $\Lambda_{W,p}$ is bounded on $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Since p is not a function, following Brown [4], we define the "multiplication by p" by $\langle pu, v \rangle = \int p_1 uv - \int p_2 \cdot \nabla(uv)$ when u and v are sufficently smooth.

The inverse problem for the magnetic Schrödinger operator is the problem of recovering $\operatorname{curl} W$ and p from $\Lambda_{W,p}$. This problem related to the inverse conductivity problem of Calderón [5]. Previous results for this problem concern different assumptions of W, p (starting with Sylvester and Uhlmann[22] and later work in

[26],[27]). The inverse problem of the closely related magnetic Schrödinger equation, was first worked by Sun [21]. Nakamura, Sun and Uhlmann [15] considered W, $p \in C^\infty$(Tolmasky [24] improved to C^1 and Salo [17] to Dini continuous). Salo [18] solved this in the case when W is continuous with $D \cdot W \in L^\infty$, $p \in L^\infty$. Krupchyk and Uhlmann [12] worked on this problem when $W \in L^\infty$ and $q \in L^\infty$. Pohjola [16] studied the case when $W \in C^{2/3+\varepsilon}$, $p = p_1 + \operatorname{div} p_2$ with p_1, $p_2 \in C^{2/3+\varepsilon}$ for any small ε. Haberman[9] proves the case that on a ball $B \subset \mathbf{R}^3$ when W is small in $W^{s,3}$ for some $s > 0$, and $p \in W^{-1,3}$.

We recover p by using the complex geometrical optics (CGO) solutions of $H_{W,p}u = 0$ in Bourgain type spaces. CGO solutions are solutions of the form

$$u = e^{ix\cdot\zeta}v$$

where $\zeta \in \mathbf{C}^n$, and satisfies $\zeta \cdot \zeta = 0$. The use of CGO solutions for inverse problems first appeared in Sylvester-Uhlmann [22]. The Bourgain type spaces we use were introduced to the study of inverse problems by Haberman and Tataru [10]. The original Bourgain spaces were defined by Bourgain in [2]. The definition is as follows. In this definition and throughout this paper, we use $a \cdot b = \sum_i a_i b_i$ for vectors a and b in $\mathbf{C}^n$.

Definition 1.1. *Let $q(\xi) = \xi \cdot \xi + 2\mu \cdot \xi$, where $\mu \in \mathbf{C}^n$ with $|\mu| = \sqrt{2}$ and $\mu \cdot \mu = 0$, we define spaces $\dot{X}^b_{\mu,h}$, $X^b_{\mu,h}$ and $X^b_{\mu,h,\sigma}$ with the following norms*

$$\|f\|_{\dot{X}^b_{\mu,h}} = \||q(h\cdot)|^b \hat{f}(\cdot)\|_2$$
$$\|f\|_{X^b_{\mu,h}} = \|(h + |q(h\cdot)|)^b \hat{f}(\cdot)\|_2$$
$$\|f\|_{X^b_{\mu,h,\sigma}} = \|(h^{2(1-\sigma)} + |q(h\cdot)|^2)^{b/2} \hat{f}(\cdot)\|_2$$

for $f \in \mathcal{S}'$, $\hat{f} \in L^1_{loc}$ is a function, b, h, $\sigma \in \mathbf{R}$ with $|b| < 1$, $\sigma \in [0, 1)$, $h > 0$.

In this definition and below, we use $\mathcal{S}'(\mathbf{R}^n)$ to denote the space of tempered distributions and $\mathcal{S}(\mathbf{R}^n)$ will denote the space of Schwartz functions.

Lemma 1.2. *For $b < 1$, the the spaces $\dot{X}^b_{\mu,h}$, $X^b_{\mu,h}$ and $X^b_{\mu,h,\sigma}$ are Banach spaces.*

Proof. It is clear that the spaces $X^b_{\mu,h}$ and $X^b_{\mu,h,\sigma}$ are Banach spaces. The interesting point is to show that $\dot{X}^b_{\mu,h}$ is complete.

Suppose we have a Cauchy sequence $\{f_n\}$ in the space $\dot{X}^b_{\mu,h}$. Then by the definition of the space $\dot{X}^b_{\mu,h}$, $\hat{f}_n$ is an element of a weighted L^2-space where the norm of a function g is $\|q(h\cdot)g\|_{L^2}$. Since the weighted L^2-space is complete, we can define

$$g := \lim_{n\to\infty} \hat{f}_n.$$

We need to show that g is a tempered distribution and thus that $g = \hat{f}$. Then we will have that $f = \lim_{n\to\infty} f_n$ where the limit occurs in the space $\dot{X}^b_{\mu,h}$. To show that g is a tempered distribution, observe that we have

$$\int_{B(0,R)} |g(\xi)|\, d\xi \leq \left(\int_{B(0,R)} |g(\xi)|^2 |q(h\xi)|^{2b}\, d\xi\right)^{1/2} \left(\int_{B(0,R)} |q(h\xi)|^{-2b}\, d\xi\right)^{1/2}. \quad (1.4)$$

From estimate (2.1), since $b < 1$

$$\int_{B(0,R)} |q(h\xi)|^{-2b}\, d\xi \leq CR^{n-2b}/h^{2b}.$$

Using this in (1.4) implies that g is a tempered distribution. This proves the Lemma. $\square$

To recover p, we need to prove the existence of CGO solutions in the space $\dot{X}^{1/2}_{\mu,h}$ space. We will need to define semiclassical pseudodifferential operators and define their action on Bourgain spaces in order to establish the existence of the CGO solutions. We need a variant of the pseudodifferential cutoff technique used by Takeuchi [23] and Kenig-Ponce-Vega [11]. Zworski [28] provides a good introduction to semiclassical pseudodifferential operators. To recover W, we follow the method of Salo [18, Section 6]. Even though our W is less regular than in Salo, we are still able to adapt his proof. Once the CGO solutions are constructed, we can use a non-physical scattering transform to recover p. The main result is

Theorem 1.3. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$ be a bounded $C^{1,1}$ domain. If $W \in C^{\lambda}(\bar{\Omega}; \mathbf{C}^n)$, $p = p_1 + div p_2$ with p_1, $p_2 \in C^{\tilde{\lambda}}(\bar{\Omega}; \mathbf{C}) \cap H^{\lambda}(\mathbf{C}^n)$ and 0 is not a Dirichlet eigenvalue of $H_{W,p}$. Then when $\lambda \in (1/2, 1)$, $\tilde{\lambda} \in (0, 1)$, $curl W$ is determined by $\Lambda_{W,p}$. Further, when $(\lambda + 1)\tilde{\lambda} > \frac{3}{2}$, $0 < \tilde{\lambda}, \lambda < 1$, $\Lambda_{W,p}$ determines p uniquely.*

The main interest of this result is the low regularity on W and p. If we let $\lambda = \tilde{\lambda}$, then our theorem applies for $\lambda > (-1 + \sqrt{7})/2 \approx 0.82$. Thus our theorem allows potentials which are not functions and represents an improvement over the work of Krupchyk and Uhlmann. The work of Pohjola assumes that $\lambda = \tilde{\lambda}$ and thus, our work is slightly more general. Unlike Haberman's recent work, there is not a restriction that W is small. However, the result above is far from sharp.

Here is the structure of this dissertation. Chapter 2 talks about the properties of Bourgain type spaces and semiclassical pseudodifferential operators. In chapter 3 and chapter 4, we obtain the existence of CGO solutions in the space $\dot{X}^{1/2}_{\mu,h}$ and in semiclassical Sobolev spaces. We show the boundary value of CGO solutions can be determined by $\Lambda_{W,p}$ in chapter 5. chapter 6 contains the recovery of $\operatorname{curl} W$. We recover p in chapter 7. Chapter 8 talks about the future work.

Chapter 2 Bourgain type spaces and semiclassical pseudodifferential operators

Bourgain type spaces

We first present some properties of Bourgain type spaces, which are critical when we prove the existence of CGO solutions in the space $\dot{X}^{1/2}_{\mu,h}$.

Let $\langle x \rangle = (1+|x|^2)^{1/2}$, before we come to the proposition, we need this lemma.

Lemma 2.1. *let $B_r(\eta)$ be the ball of radius r with center η, $k \in (0,2)$, N is a large positive number, then*

$$\int_{B_r(\eta)} \left|\frac{1}{q(h\xi)}\right|^k \leq Cr^{n-k}/h^k \tag{2.1}$$

and

$$\int \langle \eta - \xi \rangle^{-N} \left|\frac{h}{q(h\xi)}\right| \leq C \tag{2.2}$$

for some constant C depending only on the dimension n.

Proof. The proof of the first estimate (2.1) follows from R. Brown [3, page 82]. The second estimate is from the first estimate and is also in Haberman and Tataru [10, Lemma 2.2]. I will show how the first estimate works.

By a rotation in the variable ξ, it suffices to consider the case where $\mu = e_1 + ie_2$. We define the zero set of $q(h\xi)$ by $\Sigma_\mu = \{\xi : q(h\xi) = h^2|\xi|^2 + 2\mu\cdot(h\xi) = 0\}$. We consider 3 cases of the ball $B_r(\eta)$.

Case 1: When $r < \frac{1}{100}h^{-1}$ and $dist(\eta, \Sigma_\mu) < 2r$. We can rotate the variables $(\xi_2, \xi_3, \cdots, \xi_n)$ about the center of Σ_μ, so that $B_r(\eta) \subset B_{3r}(0)$. Since $q(h\xi) = |h\xi|^2 + 2(Re\mu + iIm\mu)\cdot(h\xi) = |h\xi + Re\mu|^2 - |Re\mu|^2 + 2iIm\mu\cdot(h\xi)$, we can define new variables $x_1 = |h\xi + Re\mu|^2 - |Re\mu|^2$, $x_2 = Im\mu\cdot(h\xi)$, $x_j = h\xi_j$, $j = 3, 4, \cdots, n$. Then we have $\frac{dx_1}{d\xi_j} = 2h^2\xi_j + 2h(Re\mu)_j$, $\frac{dx_2}{d\xi_j} = 2h(Im\mu)_j$. So we obtain that

$$\begin{aligned}\int_{B_r(\eta)} \left|\frac{1}{q(h\xi)}\right|^k &\leq Ch^{-n}\int_{B_{Chr}(0)} \left|\frac{1}{x_1 + ix_2}\right|^k dx_1\, dx_2 \cdots dx_n \\ &\leq Ch^{-n}(hr)^{n-k} \\ &\leq Cr^{n-k}/h^k.\end{aligned} \tag{2.3}$$

Case 2: When $\mathrm{dist}(\eta, \Sigma_\mu) > 2r$. Since $|q(h\xi)|$ is comparable to $h\,\mathrm{dist}(\xi, \Sigma_\mu)$ when $|\xi| < 8h^{-1}$ and comparable to $|h\xi|^2$ when $|\xi| > 8h^{-1}$. Thus

$$\frac{1}{|q(h\xi)|} \leq C\frac{1}{hr}. \tag{2.4}$$

This gives

$$\int_{B_r(\eta)} \left|\frac{1}{q(h\xi)}\right|^k \leq C\left(\frac{1}{hr}\right)^k \int_{B_r(\eta)} 1 \leq Cr^{n-k}/h^k. \tag{2.5}$$

Case 3: When $r > \frac{1}{100}h^{-1}$ and $dist(\eta, \Sigma_\mu) < 2r$. We write $B_r(\eta) = B_0 \cup B_\infty$, where $B_0 = B_r(\eta) \cap B_{4h^{-1}}(0)$ and $B_\infty = B_r(\eta) \setminus B_{4h^{-1}}(0)$. By the argument in Case 1 and Case 2, we know

$$\int_{B_0} |\frac{1}{q(h\xi)}|^k \leq C/h^n \tag{2.6}$$

We know that $\Sigma_\mu \subset B_{4h^{-1}}(0)$, so

$$\frac{1}{|q(h\xi)|} \leq C\frac{1}{|h\xi|^2}. \tag{2.7}$$

on B_∞. Thus

$$\int_{B_\infty} |\frac{1}{q(h\xi)}|^k \leq C\frac{1}{h^{2k}}r^{n-2k} \tag{2.8}$$

which implies 2.1 since $r > \frac{1}{100}h^{-1}$. □

Proposition 2.2. *For any $\delta > 0$, we have a constant $C = C(n, \delta)$ so that*

$$\|\langle x\rangle^{-1/2-\delta} f\|_{X^{1/2}_{\mu,h}} \leq C\|f\|_{\dot{X}^{1/2}_{\mu,h}} \tag{2.9}$$

$$\|\langle x\rangle^{-1/2-\delta} f\|_{\dot{X}^{-1/2}_{\mu,h}} \leq C\|f\|_{X^{-1/2}_{\mu,h}} \leq Ch^{-1/2}\|f\|_{L^2} \tag{2.10}$$

$$\|\langle x\rangle^{-1/2-\delta} f\|_{L^2} \leq Ch^{-1/2}\|f\|_{\dot{X}^{1/2}_{\mu,h}}. \tag{2.11}$$

$$\|f\|_{\dot{X}^{1/2}_{\mu,h}} \leq \|f\|_{X^{1/2}_{\mu,h}} \leq \|f\|_{X^{1/2}_{\mu,h,\sigma}} \tag{2.12}$$

$$\|f\|_{X^{-1/2}_{\mu,h,\sigma}} \leq \|f\|_{X^{-1/2}_{\mu,h}} \leq \|f\|_{\dot{X}^{-1/2}_{\mu,h}} \tag{2.13}$$

$$\|f\|_{X^{1/2}_{\mu,h,\sigma}} \leq h^{-\sigma/2}\|f\|_{X^{1/2}_{\mu,h}} \tag{2.14}$$

$$\|f\|_{X^{-1/2}_{\mu,h}} \leq h^{-\sigma/2}\|f\|_{X^{-1/2}_{\mu,h,\sigma}} \tag{2.15}$$

The following proof uses ideas from Haberman and Tataru [10].

Proof. Let $\varphi(x) = \langle x\rangle^{-1/2-\delta}$, then

$$|\hat{\varphi}(\xi)| \leq \frac{C_1}{|\xi|^{n-1/2-\delta}} e^{-C_2|\xi|} \tag{2.16}$$

for some $C_1 \in \mathbf{R}$, $C_2 \in \mathbf{R}$ from Stein [19, p. 132].

To prove (2.9), let $v(\xi) = h + |q(h\xi)|$ and $w(\xi) = |q(h\xi)|$. It suffices to prove that the operator S with $Sf = v^{1/2}(\hat{\varphi} * \frac{f}{w^{1/2}})$ is bounded on L^2. Suppose that S^* is the adjoint of S, since $\|S^*\|_{L^2\to L^2} = \|S\|_{L^2\to L^2}$, it suffices to show that S^* is bounded on L^2. We have

$$\begin{aligned}\langle S^* f, g\rangle &= \langle f, Sg\rangle \\ &= \int f(\xi)v(\xi)^{1/2}\overline{\left(\int \hat{\varphi}(\xi-\eta)\frac{g(\eta)}{w(\eta)^{1/2}}\,d\eta\right)}\,d\xi \\ &= \int w(\eta)^{-1/2}\left(\int \overline{\hat{\varphi}(\xi-\eta)}v(\xi)^{1/2}f(\xi)\,d\xi\right)\overline{g(\eta)}\,d\eta\end{aligned} \tag{2.17}$$

Thus

$$S^* f(\xi) = w(\xi)^{-1/2}\int \overline{\hat{\varphi}(\eta-\xi)}v(\eta)^{1/2}f(\eta)\,d\eta$$

Since

$$\begin{aligned}&\|S^* f\|_2^2 = \\ &= \int \left(\int \overline{\hat{\varphi}(\eta-\xi)}v(\eta)^{1/2}f(\eta)\,d\eta\right)\cdot\overline{\left(\int \overline{\hat{\varphi}(\tilde{\eta}-\xi)}v(\tilde{\eta})^{1/2}f(\tilde{\eta})\,d\tilde{\eta}\right)}w(\xi)^{-1}\,d\xi \\ &= \iiint \overline{\hat{\varphi}(\eta-\xi)}v(\eta)^{1/2}f(\eta)\hat{\varphi}(\tilde{\eta}-\xi)v(\tilde{\eta})^{1/2}\overline{f(\tilde{\eta})}w(\xi)^{-1}\,d\eta\,d\tilde{\eta}\,d\xi\end{aligned}$$

and

$$\begin{aligned}&\overline{\hat{\varphi}(\eta-\xi)}v(\eta)^{1/2}f(\eta)\hat{\varphi}(\tilde{\eta}-\xi)v(\tilde{\eta})^{1/2}\overline{f(\tilde{\eta})} \\ &\qquad = \left(\overline{\hat{\varphi}(\eta-\xi)}\hat{\varphi}(\tilde{\eta}-\xi)\left|\frac{\eta-\xi}{\tilde{\eta}-\xi}\right|^{1/2}\right)^{1/2} v(\eta)^{1/2}f(\eta) \\ &\qquad\quad \cdot\left(\overline{\hat{\varphi}(\eta-\xi)}\hat{\varphi}(\tilde{\eta}-\xi)\left|\frac{\tilde{\eta}-\xi}{\eta-\xi}\right|^{1/2}\right)^{1/2} v(\tilde{\eta})^{1/2}\overline{f(\tilde{\eta})}.\end{aligned}$$

Using $a\cdot b \le \frac{1}{2}(a^2+b^2)$, we have

$$\|S^* f\|_2^2 \le \iiint |\overline{\hat{\varphi}(\eta-\xi)}||\hat{\varphi}(\tilde{\eta}-\xi)|\left|\frac{\eta-\xi}{\tilde{\eta}-\xi}\right|^{1/2} v(\eta)\,|f(\eta)|^2 w(\xi)^{-1}\,d\tilde{\eta}\,d\eta\,d\xi$$

Now we integrate in the variable $\tilde{\eta}$, and use that

$$\int |\hat{\varphi}(\tilde{\eta}-\xi)|\left|\frac{1}{\xi-\tilde{\eta}}\right|^{1/2}\,d\tilde{\eta} \le C.$$

This implies that

$$\begin{aligned}\|S^* f\|_2^2 \le& C\iint |\overline{\hat{\varphi}(\eta-\xi)}||\eta-\xi|^{1/2}v(\eta)|f(\eta)|^2 w(\xi)^{-1}\,d\eta\,d\xi \\ \le& C\int\left(\int |\overline{\hat{\varphi}(\eta-\xi)}||\eta-\xi|^{1/2}v(\eta)w(\xi)^{-1}\,d\xi\right)|f(\eta)|^2\,d\eta\end{aligned}$$

It follows that

$$\|S^*\|_{L^2\to L^2} \le C\max_{\eta}\left(\int |\overline{\hat{\varphi}(\eta-\xi)}||\eta-\xi|^{1/2}v(\eta)w(\xi)^{-1}\,d\xi\right)^{1/2}. \tag{2.18}$$

We need to show the right side of (2.18) is bounded by some finite number.

Since $|q(h\xi)| \approx |h\xi|^2$ when $|h\xi| > 4$ and $|q(h\eta)| \leq |q(h\xi)| + |q(h(\eta-\xi))| + h^2|\xi||\eta-\xi|$, we have

$$
\begin{aligned}
|\frac{v(\eta)}{w(\xi)}| &= |\frac{h + |q(h\eta)|}{|q(h\xi)|}| \\
&\leq C\frac{h + |q(h\xi)| + |q(h(\eta-\xi))| + h^2|\xi||\eta-\xi|}{|q(h\xi)}| \\
&\leq C(1 + \frac{h + h^2|\eta-\xi|^2 + h|\eta-\xi| + h^2|\xi||\eta-\xi|}{|q(h\xi)|}) \\
&\leq C(1 + \frac{h + h^2|\eta-\xi|^2 + h|\eta-\xi|}{|q(h\xi)|})
\end{aligned}
\tag{2.19}
$$

Now we consider the right side of (2.18),

$$
\begin{aligned}
&|\int |\overline{\hat{\varphi}(\eta-\xi)}||\eta-\xi|^{1/2} v(\eta) w(\xi)^{-1}\, d\xi| \\
&\leq C\int e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1/2-\delta}}|\eta-\xi|^{1/2}(1 + \frac{h + h^2|\eta-\xi|^2 + h|\eta-\xi|}{|q(h\xi)|})\, d\xi \\
&\leq C\int e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1-\delta}}(1 + \frac{h + h^2|\eta-\xi|^2 + h|\eta-\xi|}{|q(h\xi)|})\, d\xi \\
&\leq C\int e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1-\delta}}(1 + \frac{h}{|q(h\xi)|})\, d\xi \\
&\leq C + C\int e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1-\delta}}\frac{h}{|q(h\xi)|}\, d\xi.
\end{aligned}
\tag{2.20}
$$

For the integral

$$
\begin{aligned}
&\int e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1-\delta}}\frac{h}{|q(h\xi)|}\, d\xi \\
&\leq \int_{|\eta-\xi|>1} e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1-\delta}}\frac{h}{|q(h\xi)|}\, d\xi \\
&\qquad + \int_{B_1(\eta)} e^{-C_2|\eta-\xi|}\frac{1}{|\eta-\xi|^{n-1-\delta}}\frac{h}{|q(h\xi)|}\, d\xi \\
&\leq \int_{|\eta-\xi|>1} e^{-C_2|\eta-\xi|}\frac{h}{|q(h\xi)|}\, d\xi + \int_{B_1(\eta)}\frac{1}{|\eta-\xi|^{n-1-\delta}}\frac{h}{|q(h\xi)|}\, d\xi \\
&= A + B
\end{aligned}
\tag{2.21}
$$

By estimate (2.2) in Lemma 2.1, A is bounded by finite number. For B, by estimate (2.1) in Lemma 2.1, we have

$$\begin{aligned} B &\le \sum_{k=0}^{\infty} \int_{B_{2^{-k}}(\eta)\setminus B_{2^{-k-1}}(\eta)} \frac{1}{|\eta-\xi|^{n-1-\delta}} \frac{h}{|q(h\xi)|}\, d\xi \\ &\le C \sum_{k=0}^{\infty} 2^{k(n-1-\delta)} 2^{-k(n-1)} \\ &\le C \sum_{k=0}^{\infty} 2^{-k\delta} \\ &\le C. \end{aligned} \tag{2.22}$$

Thus we have proven the estimate (2.9). The estimate (2.10) follows from (2.9) by duality. For the estimate (2.11), since $1 < \frac{h+|q(h\xi)|}{h}$, it follows that $\|\langle x\rangle^{-1/2-\delta} f\|_{L^2} \le h^{-1/2}\|\langle x\rangle^{-1/2-\delta} f\|_{X^{1/2}_{\mu,h}}$, which is bounded by $Ch^{-1/2}\|f\|_{\dot{X}^{1/2}_{\mu,h}}$ from estimate (2.9). The estimates (2.12) and (2.13) follow easily from the definition of the Bourgain type spaces. □

The semiclassical pseudodifferential operators

In our proof of existence of solutions, we need to use semiclassical pseudodifferential operators and we give the definition of these operators.

Definition 2.3. *Let $0 \le \sigma_1,\ \sigma_2 \le 1$ with $\sigma_1+\sigma_2 \le 1$. We define S_{σ_1,σ_2} to be the space of all functions $a(x,\xi;h)$ where $x,\xi \in \mathbf{R}^n$ and $0 < h < h_0$, $h_0 < 1$, such that for each h, we have $a(x,\xi;h) \in C^\infty(\mathbf{R}^{2n})$ and*

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) \le C_{\alpha\beta} h^{-\sigma_1|\alpha|-\sigma_2|\beta|}$$

for any multi-index α, β.

Next we give more general symbol classes, which are defined by using an order function. This is similar to the symbol classes in R. Beals [1, p.3] but we give a semiclassical version of his definition.

Definition 2.4. *Let $\varphi(x,\xi;h) : \mathbf{R}^n \times \mathbf{R}^n \times (0,h_0] \to [0,\infty)$, we say φ is an order function if there exist $\sigma_1 \ge 0, \sigma_2 \ge 0, N_1 \ge 0$ and $N_2 \ge 0$ so that φ satisfies*

$$|\varphi(x,\xi;h)| \le C\langle h^{-\sigma_1}(x-y)\rangle^{N_1} \langle h^{-\sigma_2}(\xi-\eta)\rangle^{N_2} |\varphi(y,\eta;h)|$$

for all $x,y,\xi,\eta \in \mathbf{R}^n$.

Definition 2.5. *Let $\sigma_1,\ \sigma_2 \in [0,1]$, and $\sigma_1+\sigma_2 \le 1$, we define the symbol class $S^\varphi_{\sigma_1,\sigma_2}(\mathbf{R}^n)$ as the space of all functions $a : \mathbf{R}^n \times \mathbf{R}^n \times (0,h_0] \to \mathbf{C}$ such that $a(\,\cdot\,;h) \in C^\infty(\mathbf{R}^{2n})$ and*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha,\beta} h^{-|\alpha|\sigma_1-|\beta|\sigma_2} \varphi(x,\xi;h).$$

When the order function $\varphi = 1$, we use S_{σ_1,σ_2} for symbols in place of $S^1_{\sigma_1,\sigma_2}$ in order to avoid confusion with other common notations. Given a symbol a in one of the classes defined above, there is a natural way to define an operator $A = \mathrm{Op}_h(a) : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$. This is the subject of the next definition.

Definition 2.6. *For $a \in S^{\varphi}_{\sigma_1,\sigma_2}$, we define the operator $A = Op_h(a) = a(x, hD)$ by*

$$Af(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} a(x, h\xi; h) \hat{f}(\xi)\, d\xi$$

where $\hat{f}$ is the Fourier transform of f. The function $a(x, \xi; h)$ is called the symbol of the operator A.

Lemma 2.7. *Let $b \in \mathbf{R}$, then the function $\phi(\xi) = (h^{2(1-\sigma)} + |q(\xi)|^2)^{b/2}$ is an order function and a symbol in $S^{\phi}_{0,1-\sigma}$.*

Proof. We know $|q(\xi)| = |q(\xi - \eta + \eta)| \leq |q(\xi - \eta)| + |q(\eta)| + 2|\xi - \eta||\eta|$. Thus

$$\begin{aligned}
\frac{(h^{2(1-\sigma)} + |q(\xi)|^2)^{b/2}}{(h^{2(1-\sigma)} + |q(\eta)|^2)^{b/2}} \leq & (1 + \frac{(|q(\xi-\eta)| + 2|\xi - \eta||\eta|)^2}{h^{2(1-\sigma)} + |q(\eta)|^2})^{b/2} \\
\leq & (1 + \frac{(|\xi - \eta|^2 + 2|\mu \cdot (\xi - \eta)| + 2|\xi - \eta||\eta|)^2}{h^{2(1-\sigma)} + |q(\eta)|^2})^{b/2} \\
\leq & \Big(1 + (|h^{-2(1-\sigma)}|\xi - \eta|^4 + 2h^{-2(1-\sigma)}|\xi - \eta|^2 \\
& \quad + \frac{(2|\xi - \eta||\eta|)^2}{h^{2(1-\sigma)} + |q(\eta)|^2})\Big)^{b/2} \\
\leq & C\langle h^{-(1-\sigma)}(\xi - \eta)\rangle^{2b}.
\end{aligned} \tag{2.23}$$

which gives us ϕ is an order function.

To prove ϕ is a symbol, we need to estimate the norm of derivatives. We estimate the first order partial derivatives as follows

$$\begin{aligned}
|\frac{\partial}{\partial \xi_j}\phi(\xi)| = & \big|\frac{b}{2}(h^{2(1-\sigma)} + |q(\xi)|^2)^{b/2-1}\frac{\partial}{\partial \xi_j}(q(\xi)\bar{q(\xi)})\big| \\
= & |\frac{b}{2}|\big|(h^{2(1-\sigma)} + |q(\xi)|^2)^{-1}\big((2\xi_j + 2\mu_j)\overline{q(\xi)} + \overline{(2\xi_j + 2\mu_j)}q(\xi)\big)\big||\phi(\xi)| \\
\leq & C\big|(h^{2(1-\sigma)} + |q(\xi)|^2)^{-1/2}|2\xi_j + 2\mu_j|\big||\phi(\xi)| \\
\leq & Ch^{-(1-\sigma)}|\phi(\xi)|
\end{aligned} \tag{2.24}$$

Similarly, we can calculate higher partial derivatives, and get $\phi \in S^{\phi}_{0,1-\sigma}$. □

Lemma 2.8. *If $a \in S^{\varphi}_{\sigma_1,\sigma_2}$, $b \in S^{\psi}_{\sigma_1,\sigma_2}$, then $ab \in S^{\varphi\psi}_{\sigma_1,\sigma_2}$.*

Proof. By the product rule,

$$
\begin{aligned}
|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)b(x,\xi;h)| &= |\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} C(\alpha_1,\beta_1,\alpha_2,\beta_2)\partial_x^{\alpha_1}\partial_\xi^{\beta_1} a \ \partial_x^{\alpha_2}\partial_\xi^{\beta_2} b| \\
&\le |\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} C(\alpha_1,\beta_1,\alpha_2,\beta_2) h^{-|\alpha_1|\sigma_1-|\beta_1|\sigma_2}\varphi h^{-|\alpha_2|\sigma_1-|\beta_2|\sigma_2}\phi| \\
&\le C(\alpha,\beta)h^{-|\alpha|\sigma_1-|\beta|\sigma_2}\varphi\phi
\end{aligned}
\tag{2.25}
$$

Thus $ab \in S^{\varphi\psi}_{\sigma_1,\sigma_2}$. ☐

The lemma below shows that if A and B are semiclassical pseudodifferential operators, then the composition $A \circ B$ is a semiclassical pseudodifferential operator and gives information about the symbol.

Lemma 2.9. *Let $a \in S^{\varphi}_{\sigma_1,\sigma_2}$ and $b \in S^{\psi}_{\tilde\sigma_1,\tilde\sigma_2}$, with $\sigma_2+\tilde\sigma_1 \le 1$, $c(x,\xi)$ be the symbol of operator $A \circ B$ with $A = Op_h(a)$ and $B = Op_h(b)$, then $c \in S^{\varphi\psi}_{\max(\sigma_1,\tilde\sigma_1),\max(\sigma_2,\tilde\sigma_2)}$ and for each $M = 1, 2, \ldots$ we have*

$$
c(x,\xi) = \sum_{|\alpha|<M} \frac{h^{|\alpha|}}{\alpha!}\partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi) + h^{M(1-\sigma_2-\tilde\sigma_1)} S^{\varphi\psi}_{\sigma_1,\tilde\sigma_2} \tag{2.26}
$$

Proof. Our argument follows the proof in Stein [20, page 320 - 323], with obvious changes to handle the parameter h and the order functions.

We first assume that a and b have compact support in the x and ξ variables. By definition

$$
Af(x) = (2\pi)^{-n}\int a(x,h\eta)e^{i\eta(x-y)}f(y)\,dyd\eta
$$

$$
Bf(y) = (2\pi)^{-n}\int b(y,h\xi)e^{i\xi(y-z)}f(z)\,dzd\xi.
$$

So we have

$$
ABf(x) = (2\pi)^{-2n}\int a(x,h\eta)e^{i\eta\cdot(x-y)}b(y,h\xi)e^{i\xi\cdot(y-z)}f(z)\,dzd\xi dyd\eta.
$$

Since $e^{i\eta(x-y)}e^{i\xi(y-z)} = e^{i\xi(x-z)}e^{i(\eta-\xi)(x-y)}$, we can write

$$
ABf(x) = (2\pi)^{-n}\int e^{i\xi\cdot(x-z)}c(x,h\xi)f(z)\,dzd\xi
$$

where

$$
\begin{aligned}
c(x,h\xi) =& (2\pi)^{-n}\int e^{i(\eta-\xi)\cdot(x-y)}a(x,h\eta)b(y,h\xi)\,dyd\eta \\
=& (2\pi)^{-n}\int e^{i\eta\cdot(x-y)}a(x,h(\eta+\xi))b(y,h\xi)\,dyd\eta.
\end{aligned}
$$

If we replace $h\xi$ with ξ in the definition of $c(x, h\xi)$, we have

$$c(x,\xi) = (2\pi)^{-n} \int e^{i\eta\cdot(x-y)} a(x, \xi + h\eta) b(y, \xi)\, dy d\eta \tag{2.27}$$

or equivalently

$$c(x,\xi) = (2\pi)^{-n} \int e^{i\eta\cdot x} a(x, \xi + h\eta) \hat{b}(\eta, \xi)\, d\eta \tag{2.28}$$

where $\hat{b}$ denotes the Fourier transform of b in the first variable.

We choose an arbitrary point x_0 in $\mathbf{R}^n$ and we will compute $c(x,\xi)$ for x with $|x - x_0| < \frac{1}{2}h^{\tilde{\sigma}_1}$. Let $\chi(x) \in C_c^\infty(\mathbf{R}^n)$, where $\chi(x) = 1$, when $x \in B(x_0, h^{\tilde{\sigma}_1})$, and $\chi(x) = 0$, when x outside of $B(x_0, 2h^{\tilde{\sigma}_1})$. Define $b_0(x,\xi) = b(x,\xi)\chi(x)$, and $b_\infty(x,\xi) = b(x,\xi)(1-\chi(x))$, define $c_0(x,\xi)$ and $c_\infty(x,\xi)$ as

$$c_0(x,\xi) = (2\pi)^{-n} \int e^{i\eta\cdot(x-y)} a(x, \xi + h\eta) b_0(y, \xi)\, dy d\eta$$

$$c_\infty(x,\xi) = (2\pi)^{-n} \int e^{i\eta\cdot(x-y)} a(x, \xi + h\eta) b_\infty(y, \xi)\, dy d\eta.$$

Thus $c(x,\xi) = c_0(x,\xi) + c_\infty(x,\xi)$.

We first consider $c_0(x,\xi)$. Replacing b by b_0 in (2.28), we have

$$c_0(x,\xi) = (2\pi)^{-n} \int e^{i\eta\cdot x} a(x, \xi + h\eta) \hat{b}_0(\eta, \xi)\, d\eta.$$

Apply Taylor's formula to $a(x, \xi + h\eta)$, to obtain

$$a(x, \xi + h\eta) = \sum_{|\alpha| < M} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)(h\eta)^\alpha + R_M(x, \xi, \eta, h).$$

For each α, we obtain

$$(2\pi)^{-n} \int e^{i\eta\cdot x} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi)(h\eta)^\alpha \hat{b}_0(\eta,\xi)\, dy d\eta = \frac{h^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b_0(x,\xi).$$

Then for the remainder part, we know

$$\begin{aligned} |R_M(x,\xi,\eta,h)| &\le C_M \sup_{t\in[0,1],|\alpha|=M} \Big\{ |\partial_{\xi'}^\alpha a(x,\xi')| : \xi' = \xi + th\eta \Big\} |h\eta|^M \\ &\le C_M h^{M(1-\sigma_2)} |\eta|^M \sup_{t\in[0,1],|\alpha|=M} \Big\{ |\varphi(x,\xi')| : \xi' = \xi + th\eta \Big\} \\ &\le C_M h^{M(1-\sigma_2)} |\eta|^M \langle h^{1-\sigma_2}\eta \rangle^{N_2} \varphi(x,\xi) \end{aligned}$$

and since b_0 is in the symbol class $S^\phi_{\tilde{\sigma}_1,\tilde{\sigma}_2}$, we have

$$|\hat{b}_0(\eta,\xi)| = |\int \frac{e^{-iy\eta}}{\langle h^{\tilde{\sigma}_1}\eta \rangle^{2M_1}} (1 - h^{2\tilde{\sigma}_1}\Delta_y)^{M_1} b_0(y,\xi) dy| \le C_{M_1} \frac{h^{\tilde{\sigma}_1 n}}{\langle h^{\tilde{\sigma}_1}\eta \rangle^{2M_1}} |\psi(x,\xi)|.$$

So the remainder integral will be bounded by

$$C_{M,M_1}h^{M(1-\sigma_2)}|\varphi(x,\xi)||\psi(x,\xi)|\int\frac{|\eta|^M\langle h^{1-\sigma_2}\eta\rangle^{N_2}h^{\tilde\sigma_1 n}}{\langle h^{\tilde\sigma_1}\eta\rangle^{2M_1}}\,d\eta.$$

If we choose M_1 large enough so the integral in this estimate is integrable, this bound will be less then

$$C_M h^{M(1-\sigma_2-\tilde\sigma_1)}|\varphi(x,\xi)||\psi(x,\xi)|.$$

Further, if we apply $\partial_x^\alpha\partial_\xi^\beta$ to the remainder, the remainder will be bounded by $C_M h^{M(1-\sigma_2-\tilde\sigma_1)}h^{-\sigma_1|\alpha|-\tilde\sigma_2|\beta|}|\varphi(x,\xi)||\psi(x,\xi)|$, and we can conclude the remainder is in $h^{M(1-\sigma_2-\tilde\sigma_1)}S^{\varphi\psi}_{\sigma_1,\tilde\sigma_2}$.

Above all we have

$$c_0(x,\xi)=\sum_{|\alpha|<M}\frac{h^{|\alpha|}}{\alpha!}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi)+h^{M(1-\sigma_2-\tilde\sigma_1)}S^{\varphi\psi}_{\sigma_1,\tilde\sigma_2}$$

Next we deal with $c_\infty(x,\xi)$. Since

$$(-\Delta)_\eta^{M_1}e^{i\eta\cdot(x-y)}=(-1)^{M_1}|x-y|^{2M_1}e^{i\eta(x-y)}$$

$$(1-h^{2\tilde\sigma_1}\Delta_y)^{M_2}e^{i\eta\cdot(x-y)}=\langle h^{\tilde\sigma_1}\eta\rangle^{2M_2}e^{i\eta(x-y)}$$

$$|x-y|>\frac{1}{2}h^{\tilde\sigma_1}$$

then,

$$\begin{aligned}|c_\infty(x,\xi)|&=\Big|\int e^{i\eta\cdot(x-y)}\frac{\Delta_\eta^{M_1}a(x,\xi+h\eta)}{\langle h^{\tilde\sigma_1}\eta\rangle^{2M_2}}(1-h^{2\tilde\sigma_1}\Delta_y)^{M_2}\Big\{\frac{b_\infty(y,\xi)}{|x-y|^{2M_1}}\Big\}dyd\eta\Big|\\
&\le C_{M_1,M_2}\Big|\int\frac{h^{2M_1(1-\sigma_2-\tilde\sigma_1)}|\varphi(x,\xi+h\eta)|}{\langle h^{\tilde\sigma_1}\eta\rangle^{2M_2}}(1-h^{2\tilde\sigma_1}\Delta_y)^{M_2}\\
&\quad\times\Big\{\frac{b_\infty(y,\xi)}{(h^{-\tilde\sigma_1}|x-y|)^{2M_1}}\Big\}dyd\eta\Big|\\
&\le C_{M_1,M_2}\Big|\int\frac{h^{2M_1(1-\sigma_2-\tilde\sigma_1)}\langle h^{-\sigma_2}h\eta\rangle^{N_2}|\varphi(x,\xi)|}{\langle h^{\tilde\sigma_1}\eta\rangle^{2M_2}}\\
&\quad\times\frac{|\psi(y,\xi)|}{(h^{-\tilde\sigma_1}|x-y|)^{2M_1}}dyd\eta\Big|\\
&\le C_{M_1,M_2}h^{2M_1(1-\sigma_2-\tilde\sigma_1)}\int\frac{\langle h^{-\sigma_2}h\eta\rangle^{N_2}|\varphi(x,\xi)|}{\langle h^{\tilde\sigma_1}\eta\rangle^{2M_2}}\\
&\quad\times\frac{\langle h^{-\tilde\sigma_1}(x-y)\rangle^{N_1}|\psi(x,\xi)|}{(h^{-\tilde\sigma_1}|x-y|)^{2M_1}}dyd\eta\end{aligned}$$

If we choose M_1, M_2 large enough, we get this integral is less then

$$\begin{aligned}&C_{M_1}h^{2M_1(1-\sigma_2-\tilde\sigma_1)}|\varphi(x,\xi)||\psi(x,\xi)|\\
&\qquad\le C_M h^{M(1-\sigma_2-\tilde\sigma_1)}|\varphi(x,\xi)||\psi(x,\xi)|\end{aligned}$$

Thus, we know that

$$c(x,\xi) = \sum_{|\alpha|<M} \frac{h^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi) + h^{M(1-\sigma_2-\tilde{\sigma}_1)} S^{\varphi\psi}_{\sigma_1,\tilde{\sigma}_2}$$

Finally, we consider the case when a and b do not have compact support. We choose a function $\gamma(x,\xi) \in C_c^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, with $\gamma(0) = 1$, then we can use same method to get (2.27) and (2.28) by replacing a and b with a_τ and b_τ respectively, where $a_\tau(x,\xi) = a(x,\xi) \cdot \gamma(\tau x, \tau\xi)$ and $b_\tau(x,\xi) = b(x,\xi) \cdot \gamma(\tau x, \tau\xi)$. Note that a_τ and b_τ are in the same symbol class as a and b respectively, $0 < \tau < 1$. We let $\tau \to 0$ to get the symbol c. Let $a_\tau(x,\xi) = a(x,\xi) \cdot \gamma(\tau x, \tau\xi)$ and $b_\tau(x,\xi) = b(x,\xi) \cdot \gamma(\tau x, \tau\xi)$, and define

$$C_\tau = A_\tau \circ B_\tau.$$

We have proven that c_τ, which is the symbol of C_τ, satisfies the formula (2.26), with c_τ, a_τ and b_τ replacing c, a and b respectively, uniformly in τ. And what we did above show that c_τ converges pointwise to some limit c. By the continuity properties (see Stein [20, pp. 232 - 233]), we have $C = A \circ B$. Thus, this theorem is proved. □

We use L^2_r to denote the weighted L^2 space with the norm $\|f\|_{L^2_r} = \|\langle\cdot\rangle^r f(\cdot)\|_2$. We need use this space in the following Proposition.

Proposition 2.10. *Let $a \in S_{\sigma_1,\sigma_2}$ with σ_1, $\sigma_2 \in [0,1]$, $\sigma_1 + \sigma_2 \leq 1$, $A = Op_h(a)$ then*

(1) A is bounded on L^2.

(2) A is bounded on $X^b_{\mu,h,\sigma}$, for any $1 > \sigma \geq \sigma_1$, $b \in (-1,1)$.

(3) Suppose A is bounded and invertible bounded in L^2, then

$$\|A\langle x\rangle^r f\|_{L^2} \approx \|\langle x\rangle^r Af\|_{L^2},$$

for any real number r.

Proof. For part (1), If $\sigma_1 = \sigma_2$, the proof can be found in [7]. If $\sigma_1 \neq \sigma_2$, since

$$\|a(x,hD)\|_{L^2\to L^2} = \|a(sx, s^{-1}hD)\|_{L^2\to L^2}$$

for $s \in \mathbf{R}$. So we can choose $s = h^{\frac{\sigma_1-\sigma_2}{2}}$, so that $a(sx, s^{-1}hD) \in S^1_{\frac{\sigma_1+\sigma_2}{2},\frac{\sigma_1+\sigma_2}{2}}$, which is an operator bounded on L^2. Thus $a(x,hD)$ is bounded on L^2.

For (2), we let $\varphi(\xi) = (h^{2(1-\sigma)} + |q(\xi)|^2)^{b/2} \in S^\varphi_{0,1-\sigma}$ which is from Lemma 2.7. The estimate (2) is equivalent to proving that $\varphi(hD) \circ a(x,hD) \circ \varphi(hD)^{-1}$ is bounded on L^2. Now we consider two operators $\varphi(hD)$ and $a(x,hD)\varphi(hD)^{-1}$. Since $(1-\sigma)+\sigma_1 \leq 1$, then by using Lemma 2.9, we know that the composition of these two operators is a semiclassical pseudodifferential operator with symbol in $S_{\sigma,1-\sigma}$, and this operator is bounded on L^2 by part (1) of this Proposition.

Finally for part (3), by Salo [18, Propostion 2.2], we know that the operator norm $\|A\|_{L^2_r \to L^2_r} \leq C$ for any r. Thus

$$\begin{aligned} \|A\langle x\rangle^r f\|_{L^2} &= \|A\langle x\rangle^r A^{-1}\langle x\rangle^{-r}\langle x\rangle^r Af\|_{L^2} \\ &\leq C\|A^{-1}\langle x\rangle^{-r}\langle x\rangle^r Af\|_{L^2} \\ &\leq C\|\langle x\rangle^{-r}\langle x\rangle^r Af\|_{L^2} \\ &\leq C\|\langle x\rangle^r Af\|_{L^2}. \end{aligned} \tag{2.29}$$

Similarly,we obtain $\|\langle x\rangle^r Af\|_{L^2} \leq C\|A\langle x\rangle^r f\|_{L^2}$. □

Chapter 3 Existence of CGO solution in $\dot{X}^{1/2}_{\mu,h}$ spaces

We need some notation before we come to the main topic. Let $\Delta = \sum D_j \cdot D_j$, $\Delta_\zeta = \Delta + 2\zeta \cdot D$ and $D_\zeta = D + \zeta$, where $D = (D_1, D_2, \cdots, D_n)$.

For each $\zeta \in \mathbf{C}^n$ with $\zeta \cdot \zeta = 0$, we want to find CGO solutions of the equation $H_{W,p}u = 0$ with $u = e^{ix\cdot\zeta}(1+v)$. Substituting u in $H_{W,p}u = 0$, we obtain that v solves the equation

$$(\Delta_\zeta + 2W \cdot D_\zeta + (W \cdot W + D \cdot W + p))v = -(2W \cdot \zeta + W \cdot W + D \cdot W + p).$$

Let $G = W \cdot W + D \cdot W + p$ and $f = -(2W \cdot \zeta + G)$, we have

$$(\Delta_\zeta + 2W \cdot D_\zeta + G)v = f. \tag{3.1}$$

Let $\mu = h\zeta$ with $|\mu| = \sqrt{2}$, $h = \sqrt{2}/|\zeta|$. After multiplying h^2 on both sides of (3.1), we obtain an equivalent equation

$$(Q(hD) + 2hW \cdot (hD + \mu) + h^2 G)v = h^2 f. \tag{3.2}$$

where $Q(hD)$ is the semiclassical pseudodifferential operator with symbol $q(\xi) = \xi \cdot \xi + 2\mu \cdot \xi$.

Now we assume $W \in C^\lambda$ for some $\lambda > 1/2$, $m(x)$ is a standard mollifier, $m_\kappa(x) = \frac{1}{\kappa^n} m(\frac{x}{\kappa})$, and

$$\begin{aligned} &W^\sharp = W * m_\kappa \\ &W^\flat = W - W^\sharp \\ &\|W^\flat\|_\infty \le Ch^{\sigma_0\lambda} \\ &\|\partial_x^\alpha W^\sharp\|_\infty \le Ch^{\sigma_0(\lambda-|\alpha|)}, \qquad |\alpha| \ge 1 \end{aligned} \tag{3.3}$$

where $\kappa = h^{-\sigma_0}$ and $\sigma_0 \in (0, 1/2)$. Then our equation (3.2) becomes

$$(Q + 2hW^\sharp \cdot (hD + \mu) + 2hW^\flat \cdot (hD + \mu) + h^2 G)v = h^2 f. \tag{3.4}$$

The reason we decompose W is that the term $Q + 2hW^\sharp \cdot (hD + \mu)$ in (3.4) is the main term compared with the remaining terms. This will be clear after we finish the following Lemma 3.3 and Theorem 3.4.

First we present a lemma from Haberman and Tataru [10, Lemma 2.1],

Lemma 3.1. *Let w_1, w_2 be nonnegative weights, and ϕ be a rapidly decreasing function, then*

$$\|\phi * v\|_{L^2_{w_2}} \le C \min\left\{\sup_\xi \sqrt{J(\xi,\eta)\, d\eta}, \sup_\eta \sqrt{J(\eta,\xi)\, d\xi}\right\} \|v\|_{L^2_{w_1}}$$

where

$$J(\xi,\eta) = |\phi(\xi-\eta)| \frac{w_2(\xi)}{w_1(\eta)}.$$

For symbols with some special properties, the following Lemma gives that the pseudodifferential operator is bounded on the Bourgain space $\dot{X}^{1/2}_{\mu,h}$.

Lemma 3.2. *Suppose that $\sigma_0+\sigma \leq 1$. If we assume symbol $a \in S_{\sigma_0,\sigma}$ and $(a-1)\langle x\rangle \in S_{\sigma_0,\sigma}$, then for h small enough, we have*

$$\|A\|_{\dot{X}^{1/2}_{\mu,h},\dot{X}^{1/2}_{\mu,h}} \leq Ch^{-\sigma_0/2} \tag{3.5}$$

If in addition $|a|$ is bounded away from 0 and $\sigma_0+\sigma < 1$, then we have the same estimate for the operator A^{-1},

$$\|A^{-1}\|_{\dot{X}^{1/2}_{\mu,h},\dot{X}^{1/2}_{\mu,h}} \leq Ch^{-\sigma_0/2} \tag{3.6}$$

Proof. For any u, since $A=(A-I)+I$ and estimate (2.12) and (2.14) in Proposition (2.2), we have

$$\begin{aligned}\|Au\|_{\dot{X}^{1/2}_{\mu,h}} &\leq \|(A-I)u\|_{\dot{X}^{1/2}_{\mu,h}} + \|Iu\|_{\dot{X}^{1/2}_{\mu,h}} \\ &\leq \|(A-I)\langle x\rangle\langle x\rangle^{-1}u\|_{X^{1/2}_{\mu,h,\sigma_0}} + \|u\|_{\dot{X}^{1/2}_{\mu,h}} \\ &\leq C\|\langle x\rangle^{-1}u\|_{X^{1/2}_{\mu,h,\sigma_0}} + \|u\|_{\dot{X}^{1/2}_{\mu,h}} \\ &\leq Ch^{-\sigma_0/2}\|\langle x\rangle^{-1}u\|_{X^{1/2}_{\mu,h}} + \|u\|_{\dot{X}^{1/2}_{\mu,h}}\end{aligned} \tag{3.7}$$

By estimate (2.9) in Proposition 2.2, we obtain

$$\begin{aligned}&\leq Ch^{-\sigma_0/2}\|u\|_{\dot{X}^{1/2}_{\mu,h}} + \|u\|_{\dot{X}^{1/2}_{\mu,h}} \\ &\leq Ch^{-\sigma_0/2}\|u\|_{\dot{X}^{1/2}_{\mu,h}}\end{aligned} \tag{3.8}$$

Thus we have proven estimate (3.5). For estimate (3.6), we consider an approximate right-inverse of A of the form

$$I+\sum_{j=0}^{N} h^j B_j$$

with $B_j = Op_h(b_j)$ and $b_j \in S^{\langle x\rangle^{-1}}_{\sigma_0,\sigma}$. In fact, $(a-1) \in S^{\langle x\rangle^{-1}}_{\sigma_0,\sigma}$. And

$$A\circ(I+\sum_{j=0}^{N} h^j B_j) = I + h^{N(1-\sigma_0-\sigma)}Op_h(b_{N+1}), \tag{3.9}$$

with $b_{N+1} \in S^{<x>^{-1}}_{\sigma_0,\sigma}$. Then for N large enough, we have that A has a right inverse on $\dot{X}^{1/2}_{\mu,h}$, and by a similar argument, we can find a left inverse, so A is invertible.

To find b_j, we use the symbol calculus to formally compute the symbol of the composite operator

$$
\begin{aligned}
A\circ(I+\sum_{j=0}^{N}h^jB_j) =&Op_h\big(a(1+b_0)\\
&+h\sum_{|\alpha|=1}\partial_\xi^\alpha aD_x^\alpha b_0+hab_1\\
&+h^2\sum_{|\alpha|=2}\frac{1}{\alpha!}\partial_\xi^\alpha aD_x^\alpha b_0+\sum_{|\alpha|=1}\partial_\xi^\alpha aD_x^\alpha b_0+h^2ab_2\\
&+\cdots\\
&+h^N\sum_{|\alpha|=N}\frac{1}{\alpha!}\partial_\xi^\alpha aD_x^\alpha b_0+\cdots+h^Nab_N\\
&+h^{N(1-\sigma_0-\sigma)}Op_h(b_{N+1})\big),
\end{aligned}
\tag{3.10}
$$

where $b_{N+1}\in S_{\sigma_0,\sigma}^{\langle x\rangle^{-1}}$ by Lemma 2.9. Solving for b_j to make all but the remainder term vanish gives

$$
\begin{aligned}
b_0&=\frac{1}{a}-1,\\
b_1&=-\frac{1}{a}(\sum_{|\alpha|=1}\partial_\xi^\alpha aD_x^\alpha b_0)\\
b_2&=-\frac{1}{a}(\sum_{|\alpha|=2}\frac{1}{\alpha!}\partial_\xi^\alpha aD_x^\alpha b_0+\sum_{|\alpha|=1}\partial_\xi^\alpha aD_x^\alpha b_1)\\
&\vdots\\
b_N&=-\frac{1}{a}(\sum_{|\alpha|=N}\frac{1}{\alpha!}\partial_\xi^\alpha aD_x^\alpha b_0+\cdots+\sum_{|\alpha|=1}\partial_\xi^\alpha aD_x^\alpha b_{N-1})\\
.
\end{aligned}
\tag{3.11}
$$

Since $I+\sum_{j=1}^N h^jB_j$ and $\big(I+h^{N(1-\sigma_0-\sigma)}Op_h(b_{N+1})\big)^{-1}$is bounded on $\dot{X}^{1/2}_{\mu,h}$ with norm $Ch^{-\sigma_0/2}$ and C respectively, thus $A^{-1}=(I+\sum_{j=1}^N h^jB_j)\circ\big(I+h^{N(1-\sigma_0-\sigma)}Op_h(b_{N+1})\big)^{-1}$ is bounded and $\|A^{-1}\|_{\dot{X}^{1/2}_{\mu,h},\dot{X}^{1/2}_{\mu,h}}\le Ch^{-\sigma_0/2}$. □

Then we prove the following lemma.

Lemma 3.3. *Suppose that h is small, $\delta>0$ is small and $W\in C_c^\lambda(\mathbf{R}^n)$ with $\lambda>1/2$. If g satisfies $\langle x\rangle^{1/2+\delta}g\in\dot{X}^{-1/2}_{\mu,h}$, then we have a solution v to the equation*

$$\big(Q(hD)+2hW^\sharp\cdot(hD+\mu)\big)v=g \tag{3.12}$$

with

$$\|v\|_{\dot{X}^{1/2}_{\mu,h}}\le Ch^{-\sigma_0}\|\langle x\rangle^{1/2+\delta}g\|_{X^{-1/2}_{\mu,h}}. \tag{3.13}$$

Furthermore, there is only one solution to (3.12) in the space $\dot{X}^{1/2}_{\mu,h}$.

Proof. From Salo [18, Lemma 4.1], we know there exists σ which satisfies $1/2 > \sigma > \sigma_0$, $\theta = \sigma - \sigma_0$, and symbols a,b,r_0,$\langle x\rangle r_0 \in S_{\sigma_0,\sigma}$ which give intertwining operators that satisfy

$$(Q + 2hW^\sharp \cdot (hD + \mu))A = BQ + h^{1+\tau} R_0 \tag{3.14}$$

where $\tau = \min\{\theta, 1 - 2\sigma\}$, and $a = e^{i\phi}$ with

$$|\partial_x^\alpha \partial_\xi^\beta \phi(x, \xi)| \le C_{\alpha,\beta} h^{-\sigma_0|\alpha| - \sigma|\beta|} \langle x\rangle^{-1}.$$

Here, the support of r_0 is contained in the set $\{(x, \xi) : |x| < Mh^{-\theta}\}$. Next we show A and B are bounded and invertible on X^b_{μ,h,σ_0}, for $-1 < b < 1$ and $\sigma_0 \in [0, 1)$, and when h is small enough, the norms are bounded uniformly in h. This follows from Lemma 2.9, since $\frac{1}{a} = e^{-i\phi} \in S_{\sigma_0,\sigma}$, then

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(1/a) = I + h^{1-\sigma_0-\sigma}\,\mathrm{Op}_h(d)$$

where $d \in S_{\sigma_0,\sigma}$. Since $\mathrm{Op}_h(d)$ is bounded on X^b_{μ,h,σ_0}, then operator $I + h^{1-\sigma_0-\sigma}\,\mathrm{Op}_h(d)$ is invertible on X^b_{μ,h,σ_0} if h is small enough. So A is invertible with norm of the inverse uniformly bounded in h. We can use the same method to prove this property for B.

Using the intertwining operators, equation (3.12) becomes

$$(BQ + h^{1+\tau} R_0)A^{-1} v = g. \tag{3.15}$$

Furthermore, (3.12) is equivalent to

$$(I + h^{1+\tau} AQ^{-1}B^{-1}R_0A^{-1})v = AQ^{-1}B^{-1}g. \tag{3.16}$$

Now we prove that

$$\|AQ^{-1}\|_{\dot X^{-1/2}_{\mu,h},\dot X^{1/2}_{\mu,h}} \le Ch^{-\sigma_0/2}, \tag{3.17}$$

which is equivalent to

$$\|A\|_{\dot X^{1/2}_{\mu,h},\dot X^{1/2}_{\mu,h}} \le Ch^{-\sigma_0/2}. \tag{3.18}$$

By the estimate of symbol a after (3.14), we have

$$(a - 1)\langle x\rangle \in S_{\sigma_0,\sigma}.$$

Thus we can apply Lemma 3.2 to A and obtain (3.18).

Thus, from estimate (3.17), we obtain

$$\begin{aligned} &\|h^{1+\tau} AQ^{-1}B^{-1}R_0A^{-1}v\|_{\dot X^{1/2}_{\mu,h}} \\ &\qquad \le Ch^{1+\tau-\sigma_0/2}\|B^{-1}R_0A^{-1}v\|_{\dot X^{-1/2}_{\mu,h}}. \end{aligned} \tag{3.19}$$

By estimate (2.10) in Proposition 2.2 and part (3) of Proposition 2.10 we have

$$\begin{aligned} &\le Ch^{1+\tau-\sigma_0/2}\|\langle x\rangle^{1/2+\delta}B^{-1}R_0A^{-1}v\|_{X^{-1/2}_{\mu,h}} \\ &\le Ch^{1/2+\tau-\sigma_0/2}\|\langle x\rangle^{1/2+\delta}B^{-1}R_0A^{-1}v\|_{L^2} \\ &\le Ch^{1/2+\tau-\sigma_0/2}\|B^{-1}\langle x\rangle^{1/2+\delta}R_0\langle x\rangle^{1/2+\delta}\langle x\rangle^{-1/2-\delta}A^{-1}v\|_{L^2} \end{aligned} \tag{3.20}$$

By the boundedness of B^{-1} in L^2 and part (3) of Proposition 2.10, we obtain

$$\leq Ch^{1/2+\tau-\sigma_0/2}\|\langle x\rangle^{1/2+\delta}R_0\langle x\rangle^{1/2+\delta}A^{-1}\langle x\rangle^{-1/2-\delta}v\|_{L^2} \tag{3.21}$$

Using that the support of $r_0(\cdot,\cdot,h)$ is contained in $\{(x,\xi):|x|<Mh^{-\theta}\}$ and that $\langle x\rangle r_0 \in S^0_{\sigma_0,\sigma}$, it follows that

$$\|\langle x\rangle^{1/2+\delta}R_0\langle x\rangle^{1/2+\delta}\|_{L^2\to L^2} \leq Ch^{-2\theta\delta}. \tag{3.22}$$

Finally, the estimate (3.22), the boundedness of A^{-1} followed by the estimate (2.11), gives the bound

$$\begin{aligned} &\leq Ch^{1/2+\tau-\sigma_0/2-2\theta\delta}\|\langle x\rangle^{-1/2-\delta}v\|_{L^2} \\ &\leq Ch^{\tau-\sigma_0/2-2\theta\delta}\|v\|_{\dot X^{1/2}_{\mu,h}}. \end{aligned} \tag{3.23}$$

Since δ could be any small positive number, we can choose suitable τ, σ_0 to make the power of h positive in (3.23) with the requirement $\tau=\min\{\theta,1-2\sigma\}$, $\sigma=\sigma_0+\theta$. For example, if we let $\sigma_0=\theta$ be some small number and $\delta<1/4$, then $\sigma=2\sigma_0$, $\tau=\sigma_0$, thus $\tau-\sigma_0/2-2\theta\delta>\sigma_0-\sigma_0/2-\sigma_0/2=0$.

Then by the contraction mapping theorem, there exists a solution v for equation (3.12), and the solution satisfies

$$\begin{aligned} \|v\|_{\dot X^{1/2}_{\mu,h}} \leq& C\|AQ^{-1}B^{-1}g\|_{\dot X^{1/2}_{\mu,h}} \\ \leq& Ch^{-\sigma_0/2}\|B^{-1}g\|_{\dot X^{-1/2}_{\mu,h}} \\ \leq& Ch^{-\sigma_0/2}\|B^{-1}\langle x\rangle^{1/2+\delta}g\|_{X^{-1/2}_{\mu,h}} \\ \leq& Ch^{-\sigma_0}\|\langle x\rangle^{1/2+\delta}g\|_{X^{-1/2}_{\mu,h}} \end{aligned} \tag{3.24}$$

For the uniqueness, suppose we have two solutions of (3.12), v_1 and v_2, which lie in $\dot X^{1/2}_{\mu,h}$. Using the intertwining operators, we obtain

$$(BQ+h^{1+\tau}R_0)A^{-1}(v_1-v_2)=0. \tag{3.25}$$

Since Lemma 3.2 gives that A^{-1} is invertible on $\dot X^{1/2}_{\mu,h}$, it suffices to show that $\tilde v = A^{-1}(v_1-v_2)$ which satisfies

$$(BQ+h^{1+\tau}R_0)\tilde v=0, \qquad \tilde v\in X^{1/2}_{\mu,h},$$

is zero. We will show that $\langle\cdot\rangle^{1/2+\delta}R_0\tilde v$ is in L^2 and then since the equation $Q\tilde v=f$ has a unique solution in $\dot X^{1/2}_{\mu,h}$ when f is in the weighted L^2-space, $L^2_{1/2+\delta}$, we have

$$(I+h^{1+\tau}Q^{-1}B^{-1}R_0)\tilde v=0. \tag{3.26}$$

Thus by (3.22) and (2.11),

$$\begin{aligned} \|\langle x\rangle^{1/2+\delta}R_0\tilde v\|_{L^2} &= \|\langle x\rangle^{1/2+\delta}R_0\langle x\rangle^{1/2+\delta}\langle x\rangle^{-1/2-\delta}\tilde v\|_{L^2} \\ &\leq Ch^{-2\theta\delta}\|\langle x\rangle^{-1/2-\delta}\tilde v\|_{L^2} \\ &\leq Ch^{-2\theta\delta-1/2}\|\tilde v\|_{\dot X^{1/2}_{\mu,h}}. \end{aligned}$$

Finally, we use that the operator B^{-1} is bounded on the weighted L^2-space $L^2_{1/2+\delta}$, (2.10), and that $Q^{-1}: \dot{X}^{-1/2}_{\mu,h} \to \dot{X}^{1/2}_{\mu,h}$ and we have

$$h^{1+\tau}\|Q^{-1}B^{-1}R_0\tilde{v}\|_{\dot{X}^{1/2}_{\mu,h}} \le Ch^{\tau-2\theta\delta}\|\tilde{v}\|_{\dot{X}^{1/2}_{\mu,h}}.$$

As $\delta > 0$, may be arbitrarily small, it follows that the equation (3.26) has only the solution $\tilde{v} = 0$. □

Our next theorem gives the existence of CGO solutions to $H_{W,p}u = 0$.

Theorem 3.4. *Suppose $W \in C^\lambda$, $\lambda > 1/2$, $p = p_1 + div p_2$, p_1, $p_2 \in C^{\tilde{\lambda}}$, $\tilde{\lambda} > 0$; Then for each ζ large enough, the equation $H_{W,p}u = 0$ has a unique CGO solution $u = e^{ix\cdot\zeta}(1 + v)$ with v in $\dot{X}^{1/2}_{\mu,h}$, and*

$$\|v\|_{\dot{X}^{1/2}_{\mu,h}} \le Ch^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta}f\|_{X^{-1/2}_{\mu,h}} \le h^{\tilde{\lambda}-\sigma_0}$$

for some $\sigma_0 > 0$ and for any $\delta > 0$, and f is defined (3.1).

Proof. We know that $H_{W,p}u = 0$ is equivalent to the following equation for v

$$(Q + 2hW^\sharp \cdot (hD + \mu) + 2hW^\flat \cdot (hD + \mu) + h^2G)v = h^2 f. \tag{3.27}$$

Furthermore, as in Lemma 3.3 equation (3.16), the equation (3.27) is equivalent to

$$\begin{aligned}&(I + h^{1+\tau}AQ^{-1}B^{-1}R_0A^{-1} + 2hAQ^{-1}B^{-1}W^\flat(hD + \mu) + h^2AQ^{-1}B^{-1}G)v \\ &= -h^2AQ^{-1}B^{-1}f.\end{aligned} \tag{3.28}$$

In the argument leading up to (3.23) in Lemma 3.3, we have proved

$$\begin{aligned}&\|h^{1+\tau}AQ^{-1}B^{-1}R_0A^{-1}v\|_{\dot{X}^{1/2}_{\mu,h}} \\ &\qquad \le Ch^{\tau-\sigma_0/2-2\theta\delta}\|v\|_{\dot{X}^{1/2}_{\mu,h}}\end{aligned} \tag{3.29}$$

with suitable τ, σ_0 to make the power of h positive.

Now for the second remainder term, similarly, by equation (3.17) and the definition of the space $\dot{X}^{-1/2}_{\mu,h}$ and $X^{1/2}_{\mu,h}$, we have

$$\begin{aligned}&\|hAQ^{-1}B^{-1}W^\flat(hD + \mu)v\|_{\dot{X}^{1/2}_{\mu,h}} \\ &\qquad \le Ch^{1-\sigma_0/2}\|B^{-1}W^\flat(hD + \mu)v\|_{\dot{X}^{-1/2}_{\mu,h}} \\ &\qquad \le Ch^{1-\sigma_0/2}\|\langle x\rangle^{1/2+\delta}B^{-1}W^\flat(hD + \mu)v\|_{X^{-1/2}_{\mu,h}},\end{aligned} \tag{3.30}$$

by part 3 of Proposition 2.10 and $\|W^\flat\|_\infty \le Ch^{\sigma_0\lambda}$, $W^\flat$ is supported in a ball $B(0, M)$, we obtain (3.30) is less then

$$\begin{aligned}&Ch^{1/2-\sigma_0/2}\|B^{-1}\langle x\rangle^{1/2+\delta}W^\flat(hD + \mu)v\|_{L^2} \\ &\qquad \le Ch^{1/2-\sigma_0/2}\|W^\flat(hD + \mu)v\|_{L^2} \\ &\qquad \le Ch^{1/2-\sigma_0/2}\|W^\flat\|_\infty\|(hD + \mu)\langle x\rangle^{-1/2-\delta}v\|_{L^2} \\ &\qquad \le Ch^{1/2-\sigma_0/2+\sigma_0\lambda}\|(hD + \mu)\langle x\rangle^{-1/2-\delta}v\|_{L^2},\end{aligned} \tag{3.31}$$

because $|h\xi+\mu|\le h^{-1/2}(h+|q(h\xi)|)^{1/2}$, we obtain

$$\begin{aligned}&\le Ch^{-\sigma_0/2+\sigma_0\lambda}\|\langle x\rangle^{-1/2-\delta}v\|_{X^{1/2}_{\mu,h}}\\&\le Ch^{\sigma_0(\lambda-1/2)}\|v\|_{\dot X^{1/2}_{\mu,h}}.\end{aligned}\tag{3.32}$$

Finally, for the last term, by same argument of $AQ^{-1}B^{-1}$ and $\|G\|_{\dot X^{1/2}_{\mu,h},\dot X^{-1/2}_{\mu,h}} < h^{-2+\tilde\lambda}$ from Haberman and Tataru [10, Theorem 2.1], then

$$\begin{aligned}h^2\|AQ^{-1}B^{-1}Gv\|_{\dot X^{1/2}_{\mu,h}} &\le h^{2-\sigma_0/2}\|Gv\|_{\dot X^{-1/2}_{\mu,h}}\\&\le h^{\tilde\lambda-\sigma_0/2}\|v\|_{\dot X^{1/2}_{\mu,h}}\end{aligned}\tag{3.33}$$

thus, if we choose σ_0 small enough, the power of h will be positive.

Above all, by contraction mapping theorem, there exists a solution v for equation (3.1). And v satisfies

$$\begin{aligned}\|v\|_{\dot X^{1/2}_{\mu,h}} \le&Ch^2\|AQ^{-1}B^{-1}f\|_{\dot X^{1/2}_{\mu,h}}\\\le&Ch^{2-\sigma_0/2}\|B^{-1}f\|_{\dot X^{-1/2}_{\mu,h}}\\\le&Ch^{2-\sigma_0/2}\|B^{-1}\langle x\rangle^{1/2+\delta}f\|_{X^{-1/2}_{\mu,h}}\\\le&Ch^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta}f\|_{X^{-1/2}_{\mu,h}}\end{aligned}\tag{3.34}$$

Since $f=-(2W\cdot\zeta+G)$, we have

$$\begin{aligned}\|v\|_{\dot X^{1/2}_{\mu,h}} \le&Ch^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta}(2W\cdot\zeta+G)\|_{X^{-1/2}_{\mu,h}}\\\le&Ch^{2-\sigma_0}(\|\langle x\rangle^{1/2+\delta}W\cdot\zeta\|_{X^{-1/2}_{\mu,h}}+\|\langle x\rangle^{1/2+\delta}G\|_{X^{-1/2}_{\mu,h}})\\\le&Ch^{2-\sigma_0}(h^{-3/2}+h^{-2+\tilde\lambda})\\\le&Ch^{\tilde\lambda-\sigma_0}\end{aligned}\tag{3.35}$$

The uniqueness of the solution is similar to our earlier proofs. □

Note that we can pick σ_0 arbitrarily small.

Chapter 4 Existence of solutions in semiclassical spaces

Now we still consider equation

$$H_{W,p}(e^{ix\cdot\zeta}(1+v)) = 0 \tag{4.1}$$

or equivalently

$$(Q(hD) + 2hW(hD+\mu) + h^2G)v = h^2 f \tag{4.2}$$

with $G = W \cdot W + D \cdot W + p$, $p = p_1 + \operatorname{div} p_2 = p^\sharp + p^\flat$ with $p^\sharp$ smooth and $p^\flat \in \operatorname{div} C^{\tilde{\lambda}}(\mathbf{R}^n; \mathbf{C}^n)$, $f = -(2W \cdot \zeta + G)$.

But instead of working in the X-spaces, this time we discuss the solution in semiclassical Sobolev spaces $H^t_{\rho,h}$, which are defined by $\{f : \|f\|_{H^t_{\rho,h}} = \|\langle hD\rangle^t \langle x\rangle^\rho f\|_2 < \infty\}$, for any real number t, ρ.

Like Lemma 3.3, we also have an estimate for operator $Q(hD) + 2hW^\sharp \cdot (hD+\mu)$ here, where $W^\sharp$ is as defined in Chapter 3.

Lemma 4.1. *Equation*

$$\big(Q(hD) + 2hW^\sharp \cdot (hD+\mu)\big)v = g \tag{4.3}$$

has a unique solution $v \in H^1_{\rho,h}$, for $\rho \in (-1, 0)$. Furthermore, we have

$$\|v\|_{H^1_{\rho,h}} \le Ch^{-1}\|g\|_{H^{-1}_{\rho+1}} \le Ch^{-1}\|g\|_{L^2_{\rho+1}} \tag{4.4}$$

Proof. By (3.14) and following the argument in the previous section, equation (4.3) becomes

$$(I + h^{1+\tau}AQ^{-1}B^{-1}R_0A^{-1})v = AQ^{-1}B^{-1}g. \tag{4.5}$$

where the symbols of A, B, R_0 satisfy a, b, r_0, $\langle x\rangle r_0 \in S_{\sigma_0,\sigma}$.

From Salo [18, Proposition 2.2], we can see that the operators in $Op_h(S_{\sigma_0,\sigma})$ are bounded in the space $H^t_{\rho,h}$ for $-1 < \rho < 0$ and any real number t. And in Salo [18, Propositon 4.1], consider the case when $s = 1$, the operator norm of the operator Q^{-1} from $L^2_{\rho+1}$ to $H^1_{\rho,h}$ is bounded by Ch^{-1}.

Thus, for the second term in the equation (4.5), and we have

$$\begin{aligned} &\|h^{1+\tau}AQ^{-1}B^{-1}R_0A^{-1}v\|_{H^1_{\rho,h}} \\ &\qquad \le Ch^{1+\tau}\|Q^{-1}B^{-1}R_0A^{-1}v\|_{H^1_{\rho,h}}, \\ &\qquad \le Ch^{\tau}\|B^{-1}R_0A^{-1}v\|_{L^2_{\rho+1}} \end{aligned} \tag{4.6}$$

since $\langle x\rangle r_0 \in S^0_{\sigma_0,\sigma}$, then we obtain

$$\begin{aligned} &\le Ch^{\tau}\|R_0A^{-1}v\|_{L^2_{\rho+1}} \\ &\le Ch^{\tau}\|A^{-1}v\|_{L^2_{\rho}} \\ &\le Ch^{\tau}\|v\|_{L^2_{\rho}} \\ &\le Ch^{\tau}\|v\|_{H^1_{\rho,h}} \end{aligned} \tag{4.7}$$

So by the contraction mapping theorem, there exists solution to (4.5). Since $\|\cdot\|_{H^{-1}_{\rho+1}} \le C\|\cdot\|_{L^2_{\rho+1}}$, thus the operator norm of Q^{-1} is a map from $H^{-1}_{\rho+1}$ to $H^1_{\rho+1}$ is bounded by Ch^{-1}, then the solution satisfies the following estimate

$$\begin{aligned} \|v\|_{H^1_{\rho,h}} &\le C\|AQ^{-1}B^{-1}g\|_{H^1_{\rho,h}} \\ &\le C\|Q^{-1}B^{-1}g\|_{H^{-1}_{\rho+1}} \\ &\le Ch^{-1}\|B^{-1}g\|_{H^{-1}_{\rho+1}} \\ &\le Ch^{-1}\|g\|_{H^{-1}_{\rho+1}}. \end{aligned} \tag{4.8}$$

For the uniqueness, suppose we have two solutions v_1, $v_2 \in H^1_{\rho,h}$ of equation (4.3). Apply the intertwining operators, we have then

$$(BQ + h^{1+\tau}R_0)A^{-1}(v_1 - v_2) = 0. \tag{4.9}$$

If we following the argument in Lemma 3.3, we can see A^{-1} is invertible on $H^1_{\rho,h}$. Let $\tilde{v} = A^{-1}(v_1 - v_2)$, if we can prove $\tilde{v} = 0$, the uniqueness is obtained, where $\tilde{v}$ satisfies

$$(BQ + h^{1+\tau}R_0)\tilde{v} = 0. \quad \tilde{v} \in H^1_{\rho,h} \tag{4.10}$$

Since the operator norm of Q^{-1} from $L^2_{\rho+1}$ to $H^1_{\rho,h}$ is bounded by Ch^{-1} and $\|R_0\tilde{v}\|_{L^2_{\rho+1}} \le \|\langle x\rangle^{-1}R_0\tilde{v}\|_{L^2_\rho} \le \|\tilde{v}\|_{L^2_\rho} \le \|\tilde{v}\|_{H^1_{\rho,h}}$, thus we have

$$(I + h^{1+\tau}Q^{-1}B^{-1}R_0)\tilde{v} = 0. \tag{4.11}$$

Because

$$\begin{aligned} \|h^{1+\tau}Q^{-1}B^{-1}R_0\tilde{v}\|_{H^1_{\rho,h}} \le& h^\tau\|B^{-1}R_0\tilde{v}\|_{L^2_{\rho+1}} \\ \le& h^\tau\|R_0\tilde{v}\|_{L^2_{\rho+1}} \\ \le& h^\tau\|\tilde{v}\|_{H^1_{\rho,h}}. \end{aligned} \tag{4.12}$$

Then equation (4.11) has only solution $\tilde{v} = 0$. □

Theorem 4.2. *Equation (4.2) has a unique solution in $H^1_{\rho,h}$ for $-1 < \rho < 0$ and satisfies*

$$\|v\|_{H^1_{\rho,h}} \le Ch\|f\|_{H^{-1}_{\rho+1}}$$

Proof. Let $K = \big(Q + 2W^\sharp \cdot (hD + \mu)\big)^{-1}$, equation (4.2) becomes

$$(I + 2hKW^\flat \cdot (hD + \mu) + h^2KG)v = h^2Kf. \tag{4.13}$$

or in more detail

$$\begin{aligned} \big(I + 2hKW^\flat \cdot (hD + \mu) + h^2K(W \cdot W + D\cdot W^\sharp + p^\sharp) + h^2K(D \cdot W^\flat + p^\flat)\big)v \\ = -h^2AQ^{-1}B^{-1}f. \end{aligned} \tag{4.14}$$

where a, b, r_0, $\langle x\rangle r_0 \in S^0_{\sigma_0,\sigma}$. Now we consider the terms on the left side of equation (4.14). By Lemma 4.1, we have

$$\begin{aligned} &\|hKW^\flat(hD+\mu)v\|_{H^1_{\rho,h}} \\ &\qquad \le C\|W^\flat(hD+\mu)v\|_{L^2_{\rho+1}} \\ &\qquad \le C\|W^\flat\|_\infty\|(hD+\mu)v\|_{L^2_\rho} \\ &\qquad \le C\|W^\flat\|_\infty\|v\|_{H^1_{\rho,h}} \end{aligned} \tag{4.15}$$

and since $W\cdot W + D\cdot W^\sharp + p^\sharp$ has compact support, we obtain

$$\begin{aligned} &\|h^2K(W\cdot W + D\cdot W^\sharp + p^\sharp)v\|_{H^1_{\rho,h}} \\ &\qquad \le Ch\|(W\cdot W + D\cdot W^\sharp + p^\sharp)v\|_{L^2_{\rho+1}} \\ &\qquad \le Ch\|(W\cdot W + D\cdot W^\sharp + p^\sharp)\|_\infty\|v\|_{L^2_\rho} \\ &\qquad \le Ch\|(W\cdot W + D\cdot W^\sharp + p^\sharp)\|_\infty\|v\|_{H^1_{\rho,h}} \\ &\qquad \le Ch\|v\|_{H^1_{\rho,h}}. \end{aligned} \tag{4.16}$$

Since $\|\phi(x)\|_{H^{-1}_{\rho+1}} \le h^{-1}\|\phi(x)\|_{H^1_{\rho,h}}$ for any compact supported function $\phi(x)$, thus

$$\begin{aligned} &\|h^2K(D\cdot W^\flat + p^\flat)v\|_{H^1_{\rho,h}} \\ &\qquad \le Ch\|(D\cdot W^\flat + p^\flat)v\|_{H^{-1}_{\rho+1}} \\ &\qquad \le C\|D\cdot W^\flat + (\mathrm{div})^{-1}p^\flat\|_\infty\|v\|_{H^1_{\rho,h}} \end{aligned} \tag{4.17}$$

Above all and by the contraction mapping theorem, there exists a solution to (4.14), and the solution v satisfies that

$$\begin{aligned} \|v\|_{H^1_{\rho,h}} &\le Ch^2\|AQ^{-1}B^{-1}f\|_{H^1_{\rho,h}} \\ &\le Ch\|f\|_{H^{-1}_{\rho+1}} \end{aligned} \tag{4.18}$$

For the uniqueness, it follows similar argument in Lemma 4.1.

□

Chapter 5 From $\Lambda_{W,p}$ to boundary value of CGO solution

In this section, we show the boundary value of solution of $H_{W,p}u = 0$ is determined by the DN map. We begin by giving several equivalent characterizations of the CGO solution. This argument follows closely the work of Nachman [14] and Salo [18].

Proposition 5.1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with $C^{1,1}$ boundary. Suppose $W \in C^{\lambda}$, $p = p_1 + div p_2$, with p_1, $p_2 \in C^{\tilde{\lambda}}$, λ, $\tilde{\lambda} \in (1/2, 1)$. If 0 is not a Dirichlet eigenvalue of $H_{W,p}$ in Ω, let $\zeta \in \mathbf{C}^n$ with $\zeta \cdot \zeta = 0$, then the following four problems are equivalent.*

$$
\begin{aligned}
&(DE)\begin{cases} H_{W,p}u = 0 \ \ in\ \mathbf{R}^n \\ u = e^{i\zeta\cdot x}(1+\omega) \ \ with\ \omega \in \dot{X}^{1/2}_{\mu,h} \end{cases}\\
&(IE)\begin{cases} u + G_\zeta \circ (2W\cdot Du + (W\cdot W + D\cdot W + p)u) = e^{i\zeta\cdot x} \in \mathbf{R}^n \\ u \in H^1_{loc}(\mathbf{R}^n) \end{cases}\\
&(EP)\begin{cases} i)\ \Delta u = 0 \in \Omega' \\ ii)\ u \in H^1(\Omega'_R) \ for\ any\ R > R_0 \\ iii)\ u\ satisfies\ (5.2)\ for\ a.e.\ x \in \mathbf{R}^n \\ iv)\ \frac{\partial u}{\partial \nu^+} = \Lambda_{W,p}(u_+)\ \ on\ \partial\Omega \end{cases}\\
&(BE)\begin{cases} (\frac{1}{2}I + S_\zeta\Lambda_{W,p} - B_\zeta)f = e^{i\zeta\cdot x} \ \ on\ \partial\Omega \\ f \in H^{1/2}(\partial\Omega). \end{cases}
\end{aligned}
\tag{5.1}
$$

We first show the differential equation (DE) and the integral equation (IE) are equivalent. Now define the Green function G_ζ by

$$G_\zeta = e^{i\zeta\cdot x} g_\zeta e^{-i\zeta\cdot x}$$

where g_ζ is the fundamental solution to $\Delta_\zeta = e^{-i\zeta\cdot x}\Delta e^{i\zeta\cdot x}$ such that $\Delta_\zeta^{-1} f = g_\zeta * f$ for f in the Schwartz class. Then

$$\Delta G_\zeta = \zeta^2 G_\zeta + 2e^{i\zeta\cdot x}\zeta\cdot Dg_\zeta + e^{i\zeta\cdot x}\Delta g_\zeta = e^{i\zeta\cdot x}\Delta_\zeta g_\zeta = \delta_0$$

where δ_0 is the Dirac measure at 0. So we can let $G_\zeta = G_0 + H_\zeta$ where $G_0(x) = c_n|x|^{2-n}$ is the fundamental solution of Δ, $c_n = \frac{1}{n(n-2)\alpha(n)}$, $\alpha(n)$ is the volume of unit ball in $\mathbf{R}^n$ and H_ζ is a global harmonic function.

Lemma 5.2. *Suppose we have the same conditions as Proposition 5.1. Then u is a solution of (DE) if and only if u is a solution of (IE). Also, a solution of (DE) is unique if and only if u is a solution of (IE) is unique.*

Proof. Let $u = e^{i\zeta\cdot x}(1+\omega)$ solve (DE) where $\omega = \Delta_\zeta^{-1} f$ with $f \in \dot{X}^{-1/2}_{\mu,h}$. Substitute u into $H_{W,p}u = 0$, we have

$$(\Delta_\zeta + 2W\cdot D_\zeta + (W\cdot W + D\cdot W + p))(1 + \Delta_\zeta^{-1} f) = 0.$$

Since $\Delta_\zeta(1+\Delta_\zeta^{-1}f) = f$, apply Δ_ζ^{-1} on both sides; then

$$\omega + \Delta_\zeta^{-1}(2W\cdot D_\zeta(1+\omega) + (W\cdot W + D\cdot W + p)(1+\omega)) = 0.$$

Next we add one on both sides and multiply $e^{i\zeta\cdot x}$, and we obtain (IE).

If u solves (IE), write $u = e^{i\zeta\cdot x}u_0$; then u_0 solves

$$u_0 + \Delta_\zeta^{-1}(2W\cdot D_\zeta u_0 + (W\cdot W + D\cdot W + p)u_0) = 1.$$

If we apply Δ_ζ to both sides, we have $H_{W,p}u = 0$.

The uniqueness part is obtained by noting that if u_1 and u_2 solve (DE) then u_1 and u_2 solve (IE), and vice versa. □

Now, we show that (IE) and the exterior problem (EP) are equivalent. We use the notation $\Omega' = \mathbf{R}^n \setminus \bar\Omega$ and $\Omega'_R = B(0,R)\setminus\bar\Omega$, where $R > R_0$ and $\bar\Omega \subseteq B(0,R_0)$. Let u_+ (resp. u_-) for the restriction of u to $\partial\Omega$ from the exterior (resp. interior), and $\frac{u}{\partial\nu_+}$ (resp. $\frac{u}{\partial\nu_-}$) for the value of $\nabla u\cdot\nu$ on $\partial\Omega$ from the exterior (resp. interior), where ν is the outward unit normal to $\partial\Omega$. We also write $G_\zeta(x,y) = G_\zeta(x-y)$.

We want to obtain that a solution of (IE) satisfies the radiation condition

$$\int_{|y|=R}(G_\zeta(x,y)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial G_\zeta(x,y)}{\partial\nu(y)}\,dS(y) \to e^{i\zeta\cdot x} \tag{5.2}$$

for a.e. $x\in\mathbf{R}^n$ as $R\to\infty$. In order to apply Green's identity, we define a smooth approximation of G_ζ by $G_\zeta^\varepsilon = G_0^\varepsilon + H_\zeta$, where

$$G_0^\varepsilon = c_n(\varepsilon^2 + |x|^2)^{\frac{2-n}{2}}.$$

In fact $\Delta G_\zeta^\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$ where

$$\varepsilon(x) = \frac{1}{\alpha(n)}(1+|x|^2)^{-\frac{n+2}{2}}$$

and $\int\varphi(x)\,dx = 1$. Then $\Delta G_\zeta^\varepsilon$ is an approximation of the identity.

We need a lemma on regularity properties of solution $H_{W,p}u = 0$ and of $\Lambda_{W,p}$.

Lemma 5.3. *Under the conditions of Proposition 5.1, the operator $P_{W,p}$, which maps $f\in H^{1/2}(\Omega)$ to the solution u of $H_{W,p}u = 0$ in Ω with $u|_{\partial\Omega} = f$ is bounded $H^{1/2}(\partial\Omega)\to H^1(\Omega)$. Further, we have $\Lambda_{W,p} : H^{1/2}(\partial\Omega)\to H^{-1/2}(\partial\Omega)$, and*

$$\Lambda_{W,p}f = \frac{\partial u}{\partial\nu}|_{\partial\Omega}.$$

Proof. The operator $H_{W,p}$, written in nondivergence form, satisfies the assumption of [8]. This gives that u is in $H^1(\Omega)$ if $f\in H^{1/2}(\partial\Omega)$, and the solution operator $P_{W,p}$ is bounded.

For the $\Lambda_{W,p}$ part, we claim that if $W \in L^n_\Omega(\mathbf{R}^n;\mathbf{C}^n)$ and $D\cdot W \in L^{n/2}(\mathbf{R}^n;\mathbf{C})$, then for any $v \in W^{1,n/(n-1)}(\Omega)$ one has

$$\int_\Omega (W\cdot Dv + (D\cdot W)v)\,dx = 0. \tag{5.3}$$

This statement means that $W\cdot\nu = 0$ on $\partial\Omega$, in some weak sense. Then we take $W_j \in C^\infty_c(\mathbf{R}^n;\mathbf{C}^n)$ to be the convolution approximation of W so that $W_j \to W$ in L^n and $D\cdot W_j \to D\cdot W$ in $L^{n/2}$, and we take an extension of v in $W^{1,n/(n-1)}(\mathbf{R}^n)$. If the supports of W_j and W are contained in $B(0,R)$, then

$$\begin{aligned}\int_\Omega (W\cdot Dv + (D\cdot W)v\,dx &= \lim_{j\to\infty}\int_{B(0,R)}(W_j\cdot Dv + (D\cdot W_j)v)\,dx \\ &= \lim_{j\to\infty}\frac{1}{i}\int_{\partial B(0,R)}(W_j\cdot\nu)v\,dS = 0.\end{aligned} \tag{5.4}$$

Let $f,\ g \in H^{1/2}(\Omega)$ and $u_f = P_{W,p}f$ and $e_g \in H^1(\Omega)$ with $e_g|_{\partial\Omega} = g$. Use the definition of $\Lambda_{W,p}$ to obtain

$$\langle \frac{\partial u_f}{\partial\nu}|_{\partial\Omega)}, g\rangle = \int_\omega (\nabla u_f\cdot\nabla e_g + (2W\cdot Du_f + (W\cdot W + D\cdot W + p)u_f)e_g)\,dx. \tag{5.5}$$

Now $u_f e_g \in W^{2,1}(\Omega) \subseteq W^{1,n/(n-1)}(\Omega)$. Using (5.3) with $v = u_f e_g$ and substituting to (5.5) gives $\frac{\partial u_f}{\partial\nu}|_{\partial\Omega} = \Lambda_{w,p}f$. □

Lemma 5.4. *Assume the conditions of Proposition (5.1). If u is a solution of (IE), then $u|_{\Omega'}$ is a solution of (EP). Conversely, if u is a solution of (EP), then there is a unique extension of u to $\mathbf{R}^n$ so that $\tilde u$ is a solution of (IE). The uniqueness also holds for (IE) and (EP).*

Proof. If u solves (IE). By Lemma 5.2, we have $H_{W,p}u = 0$ and $u = e^{ix\cdot\zeta}(1+\omega)$ with $\omega \in \dot X^{1/2}_{\mu,h}$, which gives us (EP) i) - ii). To show iii), for fixed x, let $R > |x|$ and $R > R_0$, and write

$$\begin{aligned}&\int_{|y|=R}(G_\zeta(x,y)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial G_\zeta(x,y)}{\partial\nu(y)}\,dS(y) \\ &\quad= -\int_{B(0,R)}(G^\varepsilon_\zeta(x,y)\frac{\partial u}{\partial\nu}(y) - u(y)\Delta_y G^\varepsilon_\zeta(x,y))\,dy \\ &\quad= \int_{B(0,R)} u\Delta_y G^\varepsilon_\zeta(x,y)\,dy + \int_{B(0,R)} G^\varepsilon_\zeta(x,y)(2W\cdot Du + (W\cdot W + D\cdot W + p)u)\,dy \\ &\quad= (\Delta G^\varepsilon_\zeta * u\chi_{B(0,R)})(x) + (G^\varepsilon_\zeta(2W\cdot Du + (W\cdot W + D\cdot W + p)u))(x)\end{aligned} \tag{5.6}$$

since W and p supported inside $B(0,R)$, a solution of (IE) is harmonic and in $C^\infty(\mathbf{R}^n\setminus\Omega)$. As $\varepsilon\to 0$, the first term converges to $u(x)$, a.e. x. The second term converges to $(G_\zeta * (2W\cdot Du + (W\cdot W + D\cdot w + p)u))(x)$ for a.e. x. This gives us (EP) iii). By

Lemma 5.3, since $u \in H^1$ and 0 is not a Dirichlet eigenvalue of $H_{W,p}$, we obtain (EP) iv).

If u solve (EP). We use Lemma 5.3 and let $v = P_{W,p}u_+$ and define $\tilde{u}(x) = u(x)$ for $x \in \Omega'$ and $\tilde{u}(x) = v(x)$ for $x \in \Omega$. By (EP) i) we have $H_{W,p}\tilde{u} = 0$. The uniqueness part follows from the facts that if u_1 and u_2 solve (IE) then $u_1|_{\Omega'}$ and $u_2|_{\Omega'}$ solve (EP), and if u_1 and u_2 solve (EP) then $\tilde{u_1}$ and $\tilde{u_2}$ solve (IE). □

To prove (EP) and (BE), we need to use the single layer potential S_ζ, double layer potential D_ζ, and boundary layer potential B_ζ, which are defined by

$$\begin{aligned} S_\zeta f(x) &= \int_{\partial\Omega} G_\zeta(x,y) f(y)\, dS(y) \qquad (x \in \mathbf{R}^n \setminus \partial\Omega), \\ D_\zeta f(x) &= \int_{\partial\Omega} \frac{\partial G_\zeta(x,y)}{\partial\nu(y)} f(y)\, dS(y) \qquad (x \in \mathbf{R}^n \setminus \partial\Omega), \\ B_\zeta f(x) &= \int_{\partial\Omega} \frac{\partial G_\zeta(x,y)}{\partial\nu(y)} f(y)\, dS(y) \qquad (x \in \partial\Omega). \end{aligned} \tag{5.7}$$

Lemma 5.5. *Under the condition of Proposition 5.1, if u is a solution of (EP), then $f = u|_{\partial\Omega}$ is a solution of (BE). Conversely, if f is a solution of (BE), then*

$$u = e^{i\zeta\cdot x} - S_\zeta\Lambda_{W,p}f + D_\zeta f \tag{5.8}$$

is a solution of (EP), with $u_+ = f$. Also, solutions of (EP) are unique if and only if solutions of (BE) are unique.

Proof. Suppose u solves (EP). If we let $f = u_+$ on $\partial\Omega$. Then $f \in H^{3/2}(\partial\Omega)$. If $x \in \Omega'$ and $R > |x|$, we have

$$\begin{aligned} &-\int_{\Omega'_R} (G^\varepsilon_\zeta(x,y)\Delta u(y) - u(y)\Delta_y G^\varepsilon_\zeta(x,y))\, dy \\ &\qquad = (\int_{|y|=R} - \int_{\partial\Omega})(G_\zeta(x,y)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial G_\zeta(x,y)}{\partial\nu(y)}\, dS(y) \end{aligned} \tag{5.9}$$

Let $\varepsilon \to 0$ and use (EP) i),iii) - iv),we have f solves (BE).

If f satisfies (BE) and define u by (5.8)in Ω'. Then u satisfies (EP) i) - iii). We need to show (EP) iv). Let $R \to \infty$, we can obtain iv).

Since (5.8) gives a correspondence between solutions of (BE) and (EP), this gives the equivalence of uniqueness for the two problems. □

Lemma 5.6. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a bounded domain with $C^{1,1}$ boundary. Then the operator $S_\zeta\Lambda_{W,p} - B_\zeta - \frac{1}{2}I : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is compact.*

Proof. Suppose $f \in H^{1/2}(\partial\Omega)$ and $u = P_{W,p}f$. Then for $x \in \Omega$, we have

$$\begin{aligned} &-\int_{\Omega} (G^\varepsilon_\zeta(x,y)\Delta u(y) - u(y)\Delta_y G^\varepsilon_\zeta(x,y))\, dy \\ &\qquad = \int_{\partial\Omega} (G^\varepsilon\zeta(x,y)\frac{\partial u}{\partial\nu}(y) - u(y)\frac{\partial G^\varepsilon_\zeta(x,y)}{\partial\nu(y)}\, dS(y). \end{aligned} \tag{5.10}$$

Let $\varepsilon \to 0$ we obtain

$$u(x) = \int_\Omega G_\zeta(x,y)(2W\cdot D + (W\cdot W)u\,dy + \langle D\cdot W + p, u\rangle = (S_\zeta\Lambda_{W,p} - D_\zeta)f(x)$$

a.e. in Ω. If let $x \to \partial\Omega$ nontangentially, then we get

$$\begin{aligned}(S_\zeta\Lambda_{W,p} - B_\zeta - \frac{1}{2}I)f(x) \\ = R\int_\Omega G_\zeta(x,y)\big(2W\cdot D + (W\cdot W)\big)P_{W,p}f(y)\,dy + \langle D\cdot W + p, P_{W,p}f(y)\rangle\end{aligned}$$

which can be written as

$$S_\zeta\Lambda_{W,p} - B_\zeta - \frac{1}{2}I = RG_\zeta MP_{W,p}$$

where R is the trace $H^1(\Omega) \to H^{1/2}(\Omega)$, $G_\zeta : H^{-1}(\Omega) \to H^1(\Omega)$ is the map that restricts $G_\zeta\tilde{u} = e^{i\zeta\cdot x}\Delta_\zeta^{-1}e^{-i\zeta\cdot x}\tilde{u}$ to Ω with $\tilde{u}$ be the extension by zero of $u \in L^2(\Omega)$ to $\mathbf{R}^n$. The map $M : H^1\Omega \to H^{-1}(\Omega)$ maps u to $2W\cdot Du + (W\cdot W + D\cdot W + p)u$ since $|\langle pu, v\rangle| \le \|u\|_{H^1}\|v\|_{H^1}$. Since W, p_1 and p_2 are Hölder continuous, the composition is compact. □

Proposition 5.7. *Suppose the conditions of Theorem 1.3, then there exists* $C = C(n, W, p, \Omega)$ *such that for any* $|\zeta| \ge C$*, each of the four problems (DE), (IE), (EP), (BE) has a unique solution.*

Proof. If we can show the problem (DE) has a unique solution, then the other three problems will also have a unique solution. We know problem (DE) has a unique solution from Lemma 3.4. □

Chapter 6 Recovery of curl W

Let $u_\zeta = e^{i\zeta\cdot x}(\omega_0+\omega)$ be the solution of $H_{W,p}u_\zeta = 0$ with

$$\begin{aligned} &\omega_0 = e^{i\chi_\zeta \phi^\sharp},\\ &\chi_\zeta(x) = \chi(x/|\zeta|^\theta),\\ &\phi^\sharp(x) = N_\mu^{-1}(-\mu\cdot W^\sharp). \end{aligned} \tag{6.1}$$

where function $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi = 1$ in $B(0,M/2)$, $\chi = 0$ outside $B(0,M)$, and $\overline{\Omega} \in B(0,M/2)$, $W^\sharp$ is as defined in (3.3); the operator $N_\mu = \mu\cdot\nabla$, and when $\mu = \gamma_1 + i\gamma_2$ where $|\gamma_j| = 1$ and $\gamma_1\cdot\gamma_2 = 0$,

$$N_\mu^{-1} f(x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1+iy_2} f(x - y_1\gamma_1 - y_2\gamma_2)\, dy_1 dy_2.$$

Lemma 6.1. *The function $\omega \in H^1_{\rho,h}$ for $-1<\rho<0$ and $\|\omega\|_{H^1_{\rho,h}} = o(1)$ as $h\to 0$.*

Proof. Substitute u_ζ into $H_{W,p}u_\zeta = 0$ and we get,

$$(\Delta_\zeta + 2W\cdot D_\zeta + G)\omega = -f, \tag{6.2}$$

or equivalently

$$(Q + 2hW\cdot(hD+\mu) + h^2G)\omega = -h^2 f, \tag{6.3}$$

where $G = W\cdot W2 + D\cdot W + p$ and

$$\begin{aligned} f &= (\Delta_\zeta + 2W\cdot D_\zeta + G)\omega_0\\ &= e^{i\chi_\zeta\phi^\sharp}\Big[i\chi_\zeta\Delta\phi^\sharp + 2iD\chi_\zeta\cdot D\phi^\sharp + i(\Delta\chi_\zeta)\phi^\sharp + (\chi_\zeta\nabla\phi^\sharp + (\nabla\chi_\zeta)\phi^\sharp)^2\\ &\quad + 2\zeta\cdot(\nabla\chi_\zeta)\phi^\sharp + 2\zeta\cdot(\chi_\zeta\nabla\phi^\sharp) + 2W\cdot(\nabla\chi_\zeta)\phi^\sharp + 2W\cdot(\chi_\zeta\nabla\phi^\sharp)\\ &\quad + 2W^\sharp\cdot\zeta + 2W^\flat\cdot\zeta + G\Big], \end{aligned} \tag{6.4}$$

Since

$$\zeta\cdot\nabla\phi^\sharp + W^\sharp\cdot\zeta = 0$$

and $W^\sharp = \chi_\zeta W^\sharp$, which can cancel two terms of f. By Theorem 4.2, we have that

$$\begin{aligned} \|\omega\|_{H^1_{\rho,h}} &\le Ch\|f\|_{H^{-1}_{\rho+1}}\\ &\le Ch\Big[\|\chi_\zeta\Delta\phi^\sharp\|_{L^2_{\rho+1}} + \|\nabla\chi_\zeta\cdot\nabla\phi^\sharp\|_{L^2_{\rho+1}} + \|(\Delta\chi_\zeta)\phi^\sharp\|_{L^2_{\rho+1}}\\ &\quad + \||\chi_\zeta\nabla\phi^\sharp|^2\|_{L^2_{\rho+1}} + \||(\nabla\chi_\zeta)\phi^\sharp|^2\|_{L^2_{\rho+1}} + |\zeta|^{1-\theta}\|\Big(\nabla\chi(x/|\zeta|^\theta)\Big)\phi^\sharp\|_{L^2_{\rho+1}}\\ &\quad + \|W\cdot(\nabla\chi_\zeta)\phi^\sharp\|_{L^2_{\rho+1}} + \|W\cdot(\chi_\zeta\nabla\phi^\sharp)\|_{L^2_{\rho+1}} + \|W^\flat\cdot\zeta\|_{L^2_{\rho+1}}\\ &\quad + h\|Ge^{i\chi_\zeta\phi^\sharp}\|_{H^{-1}_{\rho,h}}\Big]. \end{aligned} \tag{6.5}$$

The first nine terms in the brackets, are $o(h^{-1})$ as in Salo [18]. For the tenth term, since $G = W \cdot W + D \cdot W + p$, we use the same argument as in (4.16) and (4.17), to show it is bounded by $\|Ge^{i\chi_\zeta \phi^\sharp}\|_{H^{-1}_{\rho,h}} \leq o(h^{-\sigma_0})$. Combining with the h outside the brackets, we conclude that $\|\omega\|_{H^1_{\rho,h}} = o(1)$, as $h \to 0$. □

Now if $\xi \in \mathbf{R}^n$ with $|\xi|^2$ not a Dirichlet eigenvalue of Δ, then for any $\zeta \in \mathbf{C}^n$ satisfies $\zeta \cdot \zeta = 0$, $|\zeta| \geq C > 0$, $Re\,\zeta \perp \xi$ and $Im\,\zeta \perp \xi$, we define

$$\begin{aligned} t_{W,p}(\xi, \zeta) &= \langle (\Lambda_{W,p} - \Lambda_{0,-|\xi|^2})(u_\zeta|_{\partial\Omega}), e^{-ix\cdot(\xi+\zeta)}|_{\partial\Omega}\rangle \\ &= \int_\Omega e^{-ix\cdot\xi}(2(\zeta \cdot W)u_0 + W \cdot Du_0 + (\xi \cdot W + |\xi|^2)u_0)dx + \langle e^{-ix\cdot\xi}p, u_0\rangle. \end{aligned} \tag{6.6}$$

where $u_\zeta = e^{i\zeta\cdot x}u_0$ satisfies $H_{W,p}u_\zeta = 0$.
Replacing u_0 with $\omega + \omega_0$, we get

$$R_{W,p}(\xi, \mu) = \lim_{h\to 0} h\, t_{W,p}(\xi, \zeta) = 2\int e^{-ix\cdot\xi}e^{i\phi}(\mu \cdot W)dx$$

where $\phi = N_\mu^{-1}(-\mu \cdot W)$. We use one lemma in Salo's paper [18][Lemma 6.2]

Lemma 6.2. *One has*

$$R_{W,p}(\xi, \mu) = 2\int e^{-ix\cdot\xi}(\mu \cdot W)dx.$$

To recover $\operatorname{curl} W$, we need recover $D_jW_k - D_kW_j$ for any $j \neq k$. For any μ with $|\xi|^2$ not a Dirichlet eigenvalue of Δ, we let $\mu_1 = \frac{\xi_j e_k - \xi_k e_j}{|\xi_j e_k - \xi_k e_j|}$; then $\mu_1 \cdot \xi = 0$. Find an unit vector $\mu_2 \in \mathbf{R}^n$ with $\mu_2 \cdot \xi = \mu \cdot \mu_1 = 0$, now if we let $\mu = \mu_1 + i\mu_2$, from Lemma 6.2, we know we can recover $R_{W,q}(\xi, \mu)$ and $R_{W,p}(\xi, \bar{\mu})$, since $R_{W,p}(\xi, \mu) + R_{W,p}(\xi, \bar{\mu})$ determines

$$\int e^{-ix\cdot\xi}(\mu + \bar{\mu}) \cdot W dx = (D_jW_k - D_kW_j)\hat{\ }(\xi).$$

Thus we can recover $D_jW_k - D_kW_j$ from this Fourier transform.

Chapter 7 Recovery of p

In this chapter, we show how to recover p from the D-N map. This is similar to what we did in the last chapter, but we estimate the CGO solution in Bourgain type spaces. In the previous chapter, we recovered $\operatorname{curl} W$; now we need to construct a certain W from $\operatorname{curl} W$. Given a vector field $W = (W_1, \dots, W_n)$, we consider the one form $\sum_{i=1}^n W_i dx_i$. Applying the exterior derivative, we obtain

$$d\big(\sum_{i=1}^n W_i dx_i\big) = \sum_{i<j} \partial_{x_i} W_j dx_i \wedge dx_j.$$

Thus, the coefficients on right-hand side are the components of $\operatorname{curl} W$. This implies that finding a solution $\widetilde{W}$ of $\operatorname{curl} \widetilde{W} = \operatorname{curl} W$ is equivalent to finding a solution of $d \sum \widetilde{W}_i dx_i = d \sum_{i=1}^n W_i dx_i$. For the next result, we work with the differential form $\sum_{i=1}^n W_i dx_i$ (which we still denote by W) and the exterior derivative d, rather than the vector W and the operator curl.

Since we assume that W is compactly supported in Ω, we may use a partition of unity to reduce to the case where W is supported in a ball. According to Mitrea, Mitrea and Monniaux [13, Theorem 4.1], we have operators J_1 and J_2 so that

$$W = J_2(dW) + d(J_1 W). \tag{7.1}$$

Since $d^2(J_1 W) = 0$, $\widetilde{W} = J_2(dW)$ will be give us a solution of $d\widetilde{W} = dW$ as desired. The coefficients of $J_l(\omega)$, $l = 1, 2, \cdots, n$ are defined by

$$J_l u(x) = \int_\Omega \int_1^\infty (t-1)^{n-l} t^{l-1} \varphi(y + t(x-y))(x-y) \vee u(y)\, dt dy$$

with $\varphi \in C_c^\infty(B(0,R))$ with $\int \varphi = 1$ and $1 \le l \le n$. Let

$$T_{l,j} f(x) = \int_\Omega \int_1^\infty (t-1)^{n-l} t^{l-1} \varphi(y + t(x-y))(x_j - y_j) f(y)\, dt dy.$$

We can write

$$T_{l,j}(f(x)) = \int_\Omega k_{l,j}(x, x-y) f(y)\, dy$$

for $1 \le m \le n$, where $k_{l,j}$ is the kernel supported in a ball $\{(x,y) : |(x,y)| < R\}$ and satisfies

$$|\partial_x^\alpha \partial_y^\beta k_{l,j}(x,y)| \le \frac{C}{|y|^{n-1+|\beta|}} \tag{7.2}$$

Finally, to compute $J_2(dW)$, we want to consider the map $T_{\ell,j}(\partial_m u)$; and we need to show that this maps into $C^\lambda(B)$ when u is in $C_c^\lambda(B)$.

Lemma 7.1. *If $u \in C_c^\infty(B)$, then*

$$T_{l,j}(\partial_{y_m} u(x)) = -\int_{\mathbf{R}^n} \partial_{y_m} k_{l,j}(x, x-y)(u(y) - u(x))\, dy \tag{7.3}$$

Proof. This is a result of using integration by parts and the fact u is compactly supported. □

If $u \in C_c^\lambda(B)$, we may find $u_j \in C_c^\infty(B)$ with $u_i \to u$ uniformly and $\|u_i\|_{C^\lambda} \leq C\|u\|_{C^\lambda}$. Then we may write

$$\begin{aligned} &T_{l,j}(\partial_{y_m} u)(x) \\ &\qquad = -\lim_{i\to\infty} \int \partial_{y_m} k_{l,j}(x, x-y)(u_i(y) - u_i(x))\, dy \end{aligned} \tag{7.4}$$

Theorem 7.2. *If $u \in C_c^\lambda(\Omega)$, then*

$$\|T_{l,j}\partial_{y_m} u\|_{C^\lambda} \leq c\|u\|_{C^\lambda}.$$

Proof. We easily have

$$\|T_{l,j}\partial_{y_m} u\|_\infty \leq C\|u\|_{C^\lambda}.$$

Then we can write

$$\begin{aligned} &T_{l,j}\partial_m u(y) - T_{l,j}\partial_m u(x) \\ &\quad = \int \partial_{z_m} k_{l,j}(y, y-z)(u(z) - u(y)) - \partial_{z_m} k_{l,j}(x, x-z)(u(z) - u(x))\, dz \end{aligned} \tag{7.5}$$

We set $\bar{x} = 1/2(x+y)$, $s = 10|x-y|$, we use number 10 here to make sure we can have a suitable distance between z and $\bar{x}$. Then

$$\begin{aligned} T_{l,j}\partial_m(y) - T_{l,j}\partial_m u(x) = &\int_{|\bar{x}-z|<d} \partial_{z_m} k_{l,j}(y, y-z)(u(z) - u(y))\, dz \\ &- \int_{|\bar{x}-z|<d} \partial_{z_m} k_{l,j}(x, x-z)(u(z) - u(x))\, dz \\ &+ \int_{|\bar{x}-z|>d} (\partial_{z_m} k_{l,j}(y, y-z) - \partial_{z_m} k_{l,j}(x, x-z)) \\ &\quad \times (u(z) - u(y))dz \\ &+ (u(x) - u(y)) \int_{|\bar{x}-z|>d} \partial_{z_m} k_{l,j}(x, x-z)\, dz \\ =& I + II + III + IV. \end{aligned} \tag{7.6}$$

We have $I + II \leq Cs^\lambda$. For III, we have the estimate

$$|\partial_{z_m} k_{l,j}(y, y-z) - \partial_{z_m} k_{l,j}(x, x-z))| \leq C\frac{|x-y|}{|z-\bar{x}|^{n+1}}$$

for $|z - \bar{x}| \geq s$. Then

$$\begin{aligned} III &\leq Cs \int_s^\infty r^{-n-1} r^{n-1+\lambda}\, dr \\ &\leq Cs \cdot s^{\lambda-1} \\ &\leq Cs^\lambda. \end{aligned} \tag{7.7}$$

Finally, for IV, observe Green's identity gives

$$\int_{|\bar{x}-z|>s} \partial_{z_m} k_{l,j}(x, x-z)\, dz = \int_{|\bar{x}-z|=s} \nu_m k_{l,j}(x, x-z)\, dz$$

and using (7.2), we have

$$|\int_{|\bar{x}-z|=d} \nu_m k_{l,j}(x, x-z)\, dz| \leq C.$$

Thus $IV \leq Cs^\lambda$. $\square$

Now we can recover $\operatorname{curl} W$.

Theorem 7.3. *Suppose the conditions in Theorem 1.3 holds; then we can construct* $\widetilde{W} \in C^\lambda$ *and compactly supported with* $curl \widetilde{W} = curl W$ *and* $\widetilde{W}|_{\partial\Omega} = 0$.

Proof. As outlined earlier, $\widetilde{W} = J_2(dW)$, the Theorem follows from the mapping properties of J_2 proved above. $\square$

Next we show how to recover p. Let $\xi \in \mathbf{R}^n \setminus \{0\}$ and γ_1, γ_2 be two unit vectors with $\{\xi, \gamma_1, \gamma_2\}$ an orthogonal set. For $s = 1/h$ we define

$$\begin{aligned} \zeta_1 &= -\frac{\xi}{2} + s\sqrt{1 - \frac{|\xi|^2}{4s^2}}\gamma_1 + is\gamma_2 \\ \zeta_2 &= -\frac{\xi}{2} - s\sqrt{1 - \frac{|\xi|^2}{4s^2}}\gamma_1 - is\gamma_2. \end{aligned} \tag{7.8}$$

Let $u_\zeta = e^{i\zeta\cdot x}(\omega_0 + \omega)$ be the solution of $H_{W,p} u_\zeta = 0$ with

$$\begin{aligned} &\omega_0 = e^{i\chi_\zeta \phi^\sharp}, \\ &\chi_\zeta(x) = \chi(x/|\zeta|^\theta), \\ &\phi^\sharp(x) = N_\mu^{-1}(-\mu \cdot W^\sharp), \end{aligned} \tag{7.9}$$

where the function $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi = 1$ in $B(0, M/2)$, $\chi = 0$ outside $B(0, M)$, and $\overline{\Omega} \in B(0, M/2)$, $|\partial_x^\alpha W^\sharp(x)| < Ch^{-|\alpha|\sigma}$. Note that this σ is not the σ we mentioned before. In Chapter 3, $\sigma = \sigma_0 + \theta$, but here, we choose σ independent of σ_0. Let $\mu_j = \zeta_j/s = \mu_j^1 + \mu_j^2$, $j = 1, 2$. We show the following lemma.

Lemma 7.4. *Suppose* $W \in C^\lambda$ *and* $p = p_1 + div p_2$, p_1, $p_2 \in C^{\tilde{\lambda}}$, *with* $\lambda > 1/2$ *and* $\tilde{\lambda} > 0$. *Then there exists a sequence pair of* $\zeta_j = h^{-1}\mu_j$, *j=1,2, defined in (7.8), such that* $\|\omega_1\|_{\dot{X}^{1/2}_{\mu_j,h}} = o(h^{1-\sigma-\varepsilon})$, *and* $\|\omega_2\|_{\dot{X}^{1/2}_{\mu_j,h}} = o(h^{1-\sigma-\varepsilon})$ *as* $h \to 0$, *where* $\sigma = \frac{1}{2(1+\lambda)}$, *and* $\varepsilon > 0$ *can be arbitrarily small and independent of* h, σ.

Proof. Substituting u_ζ in $H_{W,p} u_\zeta = 0$, we get

$$(\Delta_\zeta + 2W \cdot D_\zeta + G)\omega = -f, \tag{7.10}$$

or equivalently

$$(Q + 2hW \cdot (hD + \mu) + h^2 G)\omega = -h^2 f, \tag{7.11}$$

with $G = W \cdot W + D \cdot W + p$, and

$$\begin{aligned} f &= (\Delta_\zeta + 2W \cdot D_\zeta + G)\omega_0 \\ &= e^{i\chi_\zeta \phi^\sharp}\Big[i\chi_\zeta \Delta\phi^\sharp + 2iD\chi_\zeta \cdot D\phi^\sharp + i(\Delta\chi_\zeta)\phi^\sharp + (\chi_\zeta \nabla\phi^\sharp + (\nabla\chi_\zeta)\phi^\sharp)^2 \\ &\quad + 2\zeta \cdot (\nabla\chi_\zeta)\phi^\sharp + 2\zeta \cdot (\chi_\zeta \nabla\phi^\sharp) + 2W \cdot (\nabla\chi_\zeta)\phi^\sharp + 2W \cdot (\chi_\zeta \nabla\phi^\sharp) \\ &\quad + 2W^\sharp \cdot \zeta + 2W^\flat \cdot \zeta + G\Big]. \end{aligned} \tag{7.12}$$

Since

$$2\zeta \cdot \nabla\phi^\sharp + 2W^\sharp \cdot \zeta = 0 \tag{7.13}$$

which removes two terms from f, we have

$$\begin{aligned} f =& (\Delta_\zeta + 2W \cdot D_\zeta + G)\omega_0 \\ =& e^{i\chi_\zeta \phi^\sharp}\Big[i\chi_\zeta \Delta\phi^\sharp + 2iD\chi_\zeta \cdot D\phi^\sharp + i(\Delta\chi_\zeta)\phi^\sharp \\ & + (\chi_\zeta \nabla\phi^\sharp + (\nabla\chi_\zeta)\phi^\sharp)^2 + 2\zeta \cdot (\nabla\chi_\zeta)\phi^\sharp \\ & + 2W \cdot (\nabla\chi_\zeta)\phi^\sharp + 2W \cdot (\chi_\zeta \nabla\phi^\sharp) + 2W^\flat \cdot \zeta + G\Big]. \end{aligned} \tag{7.14}$$

Then

$$\begin{aligned} \|\omega\|_{\dot X^{1/2}_{\mu_j,h}} \leq& Ch^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta} f\|_{X^{-1/2}_{\mu_j,h}} \\ \leq& Ch^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta} e^{i\chi_\zeta \phi^\sharp}\Big[i\chi_\zeta \Delta\phi^\sharp + 2iD\chi_\zeta \cdot D\phi^\sharp + i(\Delta\chi_\zeta)\phi^\sharp \\ & + (\chi_\zeta \nabla\phi^\sharp + (\nabla\chi_\zeta)\phi^\sharp)^2 + 2\zeta \cdot (\nabla\chi_\zeta)\phi^\sharp \\ & + 2W \cdot (\nabla\chi_\zeta)\phi^\sharp + 2W \cdot (\chi_\zeta \nabla\phi^\sharp) + 2W^\flat \cdot \zeta + G\Big]\|_{X^{-1/2}_{\mu_j,h}}. \end{aligned} \tag{7.15}$$

Next we estimate each term, since $|\partial^\alpha \phi^\sharp(x)| \leq Ch^{-\sigma|\alpha|}\langle x_T\rangle^{-1}\chi_{B(0,M)}(x_\perp)$, where x_T is the projection of x to $\mathrm{span}\{\gamma_1, \gamma_2\}$ and $x_\perp = x - x_T$. Then for the first term

$$\begin{aligned} & h^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta} e^{i\chi_\zeta \phi^\sharp} i\chi_\zeta \Delta\phi^\sharp\|_{X^{-1/2}_{\mu_j,h}} \\ &\leq Ch^{3/2-\sigma_0}\|\langle x\rangle^{1/2+\delta} e^{i\chi_\zeta \phi^\sharp} i\chi_\zeta \Delta\phi^\sharp\|_{L^2} \\ &\leq Ch^{3/2-\sigma_0}\Big(\int_{\mathbf{R}^n} \langle x\rangle^{2(1/2+\delta)}(\chi_\zeta(x))^2|\Delta\phi^\sharp(x)|^2\,dx\Big)^{1/2} \\ &\leq Ch^{3/2-\sigma_0}h^{-2\sigma}\Big(\int_{|x_T|\leq Mh^{-\theta}, |x_\perp|\leq M} \langle x\rangle^{2(1/2+\delta)}\langle x_T\rangle^{-2}\,dx\Big)^{1/2} \\ &\leq Ch^{3/2-\sigma_0-2\sigma}\Big(\int_{|x_T|\leq Mh^{-\theta}} \langle x_T\rangle^{-1+2\delta}\,dx_T\Big)^{1/2} \\ &\leq Ch^{3/2-\sigma_0-2\sigma-\theta(1/2+\delta)}. \end{aligned} \tag{7.16}$$

This term has the worst behavior among the first four terms of (7.15) since the derivatives of χ_ζ bring decay and the other terms only have first derivatives of $\phi^\sharp$. For the fifth term, we have

$$\begin{aligned} h^{2-\sigma_0}\|\langle x\rangle^{1/2+\delta}e^{i\chi_\zeta\phi^\sharp}2\zeta\cdot(\nabla\chi_\zeta)\phi^\sharp\|_{X^{-1/2}_{\mu_j,h}} \\ &\le Ch^{2-\sigma_0}h^{-1+\theta}h^{-1/2}\|\langle x\rangle^{1/2+\delta}e^{i\chi_\zeta\phi^\sharp}\phi^\sharp\|_{L^2} \\ &\le Ch^{1/2-\sigma_0+\theta}\|\langle x\rangle^{1/2+\delta}\phi^\sharp\|_{L^2} \\ &\le Ch^{1/2-\sigma_0+\theta}h^{-\theta(1/2+\delta)} \\ &\le Ch^{1/2-\sigma_0+\theta(1/2-\delta)}. \end{aligned} \tag{7.17}$$

The estimates for the sixth, seventh, and eighth terms are $Ch^{3/2-\sigma_0+\theta}$, $Ch^{3/2-\sigma_0-\sigma}$, and $Ch^{1/2-\sigma_0+\sigma\lambda}$, respectively. For the ninth term, by Haberman and Tataru's paper [10, Lemma 3.1, p.10], we can find a sequence (μ_j, h_j) with h_j tending to zero so that

$$\|G\|_{X^{-1/2}_{\mu_j,h_j}} = o(h^{-1+\frac{1}{2}\tilde{\lambda}}).$$

So the estimate of the ninth term is $Ch^{1-\sigma_0-\frac{\sigma}{2}+\frac{1}{2}\tilde{\lambda}}$.

Now consider the worst three terms: $h^{3/2-\sigma_0-2\sigma-\theta(1/2+\delta)}$, $h^{1/2-\sigma_0+\theta(1/2-\delta)}$ and $h^{1/2-\sigma_0+\sigma\lambda}$. Let $3/2-\sigma_0-2\sigma-\theta(1/2+\delta) = 1/2-\sigma_0+\theta(1/2-\delta)$. We find that $\theta = 1-2\sigma$ and $h^{3/2-\sigma_0-2\sigma-\theta(1/2+\delta)} = h^{1/2-\sigma_0+\theta(1/2-\delta)} = h^{1-\sigma-\sigma_0-\delta(1-2\sigma)}$, since σ_0, δ can be arbitrarily small. Now we let $1-\sigma = 1/2-+\sigma\lambda$; we have $\sigma = \frac{1}{2(1+\lambda)}$. Thus if we make a summary, choosing $\theta = 1-2\sigma$ and $\sigma = \frac{1}{2(1+\lambda)}$, then for any $\varepsilon > 0$,

$$\|\omega\|_{\dot{X}^{1/2}_{\mu_j,h}} \le Ch^{1-\sigma-\varepsilon}.$$

□

Let u_{ζ_1}, u_{ζ_2} satisfy $H_{W,p}u_{\zeta_1} = 0$, $H_{-W,0}u_{\zeta_2} = 0$, respectively, and take the forms

$$u_{\zeta_1} = e^{i\zeta_1\cdot x}(e^{i\phi_1^\sharp} + \omega_1)$$

$$u_{\zeta_2} = e^{i\zeta_2\cdot x}(e^{-i\phi_2^\sharp} + \omega_2)$$

Define a scattering transform

$$\tilde{t}(\xi) = \langle(\Lambda_{W,p} - \Lambda_{-W,0})(u_{\zeta_1}|_{\partial\Omega}), v_{\zeta_2}|_{\partial\Omega}\rangle = \langle pu_{\zeta_1}, u_{\zeta_2}\rangle$$

Substituting u_{ζ_1} and u_{ζ_2}, we get

$$\begin{aligned} \tilde{t}(\xi) &= \langle e^{-ix\cdot\xi}p(e^{i\phi_1^\sharp} + \omega_1), (e^{-i\phi_2^\sharp} + \omega_2)\rangle \\ &= \langle e^{-ix\cdot\xi}pe^{i\phi_1^\sharp}, e^{-i\phi_2^\sharp}\rangle + \langle e^{-ix\cdot\xi}pe^{i\phi_1^\sharp}, \omega_2\rangle + \langle e^{-ix\cdot\xi}p\omega_1, e^{-i\phi_2^\sharp}\rangle + \langle e^{-ix\cdot\xi}p\omega_1, \omega_2\rangle \end{aligned}$$

For the first term, since

$$\zeta_1\cdot\nabla\phi_1^\sharp + W^\sharp\cdot\zeta_1 = 0,$$

$$\zeta_2 \cdot \nabla\phi_2^\sharp - W^\sharp \cdot \zeta_2 = 0,$$

and $\zeta_1 + \zeta_2 = -\xi = s(\mu_1 + \mu_2)$ as $s \to \infty$ or $\mu_1 = -\mu_2 - h\xi$. Furthermore,

$$\phi_1^\sharp = N_{\mu_1}^{-1}(\mu_1 \cdot W^\sharp) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}(\mu_1 \cdot W^\sharp)(x - y_1\mu_1^1 - y_2\mu_1^2)\, dy_1 dy_2$$

and

$$\phi_2^\sharp = N_{\mu_2}^{-1}(-\mu_2 \cdot W^\sharp) = \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}(-\mu_2 \cdot W^\sharp)(x + y_1\mu_2^1 + y_2\mu_2^2)\, dy_1 dy_2.$$

So

$$\begin{aligned}
\phi_1^\sharp - \phi_2^\sharp &= \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}(\mu_1 \cdot W^\sharp)(x - y_1\mu_1^1 - y_2\mu_1^2)\, dy_1 dy_2 \\
&+ \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}(\mu_2 \cdot W^\sharp)(x + y_1\mu_2^1 + y_2\mu_2^2)\, dy_1 dy_2 \\
&= \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}((-\mu_2 - h\xi) \cdot W^\sharp)(x - y_1\mu_1^1 - y_2\mu_1^2)\, dy_1 dy_2 \\
&+ \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}(\mu_2 \cdot W^\sharp)(x + y_1\mu_2^1 + 2\mu_2^2)\, dy_1 dy_2 \\
&= \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{1}{y_1 + iy_2}((-h\xi) \cdot W^\sharp)(x - y_1\mu_1^1 - y_2\mu_1^2)\, dy_1 dy_2 \\
&+ \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{(\mu_2 \cdot W^\sharp)(x + y_1\mu_2^1 + y_2\mu_2^2) - (\mu_2 \cdot W^\sharp)(x - y_1\mu_1^1 - y_2\mu_1^2)}{y_1 + iy_2}\, dy_1 dy_2 \\
&= I + II.
\end{aligned} \tag{7.18}$$

Because $\partial_x^\alpha W^\sharp \le Ch^{-\sigma|\alpha|}$ and $|(x + y_1\mu_2^1 + y_2\mu_2^2) - (x - y_1\mu_1^1 - y_2\mu_1^2)| \le Ch|y_1|$, then $I \le Ch$ and $II \le Ch^{1-\sigma}$, similarly, we can prove $|\nabla(\phi_1^\sharp - \phi_2^\sharp)| \le Ch^{1-2\sigma}$.

It follows that

$$\langle e^{-ix\cdot\xi} p e^{i\phi_1^\sharp}, e^{-i\phi_2^\sharp}\rangle \to \hat{p}(\xi)$$

as $s \to \infty$.

Next the second term: if we consider $e^{i\phi_1^\sharp}$ as a symbol, from Lemma 3.2, we have

$$\begin{aligned}
|\langle e^{-ix\cdot\xi} p e^{i\phi_1^\sharp}, \omega_2\rangle &\le C\|e^{-ix\cdot\xi} p e^{i\phi_1^\sharp}\|_{\dot{X}_{\mu,h}^{-1/2}} \|\omega_2\|_{\dot{X}_{\mu,h}^{1/2}} \\
&\le Ch^{-\sigma/2}\|e^{-ix\cdot\xi} p\|_{\dot{X}_{\mu,h}^{-1/2}} \|\omega_2\|_{\dot{X}_{\mu,h}^{1/2}} \\
&\le Ch^{-\sigma/2} h^{-1+\frac{1}{2}\tilde{\lambda}} h^{1-\sigma-\varepsilon} \\
&\le Ch^{\frac{1}{2}\tilde{\lambda} - \frac{3}{2}\sigma - \varepsilon} \\
&\le Ch^{1/2(\tilde{\lambda} - \frac{3}{2(1+\lambda)}) - \varepsilon}
\end{aligned} \tag{7.19}$$

where the estimate of $\|e^{-ix\cdot\xi} p e^{i\phi_1^\sharp}\|_{\dot{X}_{\mu,h}^{-1/2}}$ leads to an average estimate found in Haberman and Tataru [10][Lemma 3.1]. The third term is similar to the second.

Finally we look at the last term,

$$\begin{aligned}|\langle e^{-ix\cdot\xi} p\omega_1, \omega_2\rangle| &\le Ch^{-2+\tilde{\lambda}} \|\omega_1\|_{\dot{X}^{1/2}_{\mu,h}} \|\omega_2\|_{\dot{X}^{1/2}_{\mu,,h}} \\ &\le Ch^{-2+\tilde{\lambda}}\, h^{1-\sigma-\varepsilon}\, h^{1-\sigma-\varepsilon} \\ &\le Ch^{\tilde{\lambda}-\frac{1}{1+\lambda}-2\varepsilon},\end{aligned} \tag{7.20}$$

where the estimate of the operator $e^{-ix\cdot\xi}p$ is follow to Haberman and Tataru [10][Theorem 2.1].

Above all, if $\tilde{\lambda} - \frac{3}{2(1+\lambda)} > 0$, or equivalently $\tilde{\lambda}(1+\lambda) > \frac{3}{2}$, we can recover $\hat{p}$, which is the Fourier transform of p. Then we could construct our electric potential p. If we let $\tilde{\lambda}$ approaches 1, then λ can approaches to 1/2 from above, and if we let λ approaches to 1, $\tilde{\lambda}$ converges to 3/4 from above. To make a summary, we can say that for any $\lambda \in (1/2, 1)$, $\tilde{\lambda} \in (3/4, 1)$ and $\tilde{\lambda}(1+\lambda) > \frac{3}{2}$, p can be determined by DN map.

Chapter 8 Future work

In the previous chapter, we recovered p by using CGO solutions of the form $u = e^{ix\cdot\zeta}(e^{i\phi^\sharp} + \omega)$. The estimate (7.19) requires $\tilde{\lambda} - \frac{3}{2(1+\lambda)} > 0$, the estimate (7.20) requires $\tilde{\lambda} - \frac{1}{1+\lambda} > 0$. We do not expect this result to be sharp and in future work we would like to recover p for a larger range of λ and $\tilde{\lambda}$.

Other interesting questions include studying the stability of the recovery process for non-smooth potentials. This would extend the work of Leo Tzou [25]. We can also consider extending Haberman's methods for magnetic Schrödinger operators to the case when the magnetic potential W is not small. Finally, there is much interesting work for the inverse boundary value problem in the case when we only have data on part of the boundary. The inverse boundary value problem for the magnetic Schrödinger operator was studied by Chung [6]. Another area of investigation is to study this partial data problem when the potentials are not smooth.

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