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THE GENERALIZED BURNSIDE AND REPRESENTATION RINGS

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ABSTRACT OF DISSERTATION

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A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Eric B. Kahn
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Director: Dr. Marian Anton, Professor of Mathematics
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ABSTRACT OF DISSERTATION

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Making use of linear and homological algebra techniques we study the linearization map between the generalized Burnside and rational representation rings of a group G. For groups G and H, the generalized Burnside ring is the Grothendieck construction of the semiring of $G \times H$-sets with a free H-action. The generalized representation ring is the Grothendieck construction of the semiring of rational $G \times H$-modules that are free as rational H-modules. The canonical map between these two rings mapping the isomorphism class of a G-set X to the class of its permutation module is known as the linearization map. For p a prime number and H the unique group of order p, we describe the generators of the kernel of this map in the cases where G is an elementary abelian p-group or a cyclic p-group. In addition we introduce the methods needed to study the Bredon homology theory of a G-CW-complex with coefficients coming from the classical Burnside ring.

KEYWORDS: algebraic topology, Burnside ring, representation ring, elementary abelian p-group, cyclic p-group

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Chapter 1 Introduction

The topic of this dissertation lies within the subject of algebraic topology. In a broad sense of the field, algebraic topology is concerned with studying invariants of topological spaces up to homeomorphism using algebraic tools; thus it is a bridge between the larger disciplines of algebra and topology. Some of these invariants of interest are ordinary homology and cohomology theories, extraordinary homology and cohomology theories, and homotopy theory. In addition these theories can be used to study differing types of topological spaces, ranging from CW, simplicial, or singular complexes to manifolds and Lie groups.

We have restricted ourselves to a problem concerned with CW-complexes. It turns out this is not a significant restriction as many of the interesting topological spaces considered in mathematics are in fact CW-complexes such as a sphere or the complex projective plane. In particular we are concerned with the specific CW-complex known as the classifying space BG whose existence was given by J. Milnor [12, 13]. By studying the geometry of vector bundles over BG, we introduce the generalized cohomology theory known as K-theory and associate to BG the ring $K^*(BG)$. In the case G is a Lie group, Atiyah also showed [4] we can replace $K^*(BG)$ with a completion of the ring of complex representations $R_{\mathbb{C}}(G)$. Another generalized cohomology theory, stable cohomotopy, can be defined by considering $[BG, pt.]$, the stable homotopy classes of maps from BG to a point. Analogously $[BG, pt]$ can be replaced by a completion of the Burnside ring $A(G)$ [11].

The Burnside ring $A(G)$ of a group G is the free abelian group generated by conjugacy classes of subgroups $L < G$ with the product given by a double coset formula. The representation ring $R(G)$ of the same group is the free abelian group generated by isomorphism classes of minimal left ideal of kG, the group ring of G over a base field k and the product is given by the tensor product. There is a linearlization map $A(G) \to R(G)$ by sending a subgroup $L < G$ to the permutation representation kG/L, the vector space on the cosets of L with G permuting the basis. The image of this linearlization map is the virtual permutation representations of G. A classical result is that any representation of a finite p-group G is such a virtual permutation representation of $k = \mathbb{Q}$. In this dissertation we compute the kernel of the linearization map associated with generalized versions of $A(G)$ and $R(G)$ where G is an elementary abelian or cyclic p-group and $k = \mathbb{Q}$. The motivation for studying $A(G)$ and $R(G)$ and their generalizations comes from the topological considerations explained above. These algebraic objects occur as important invariants in algebraic topology.

Chapter 2 is primarily concerned in Section 2.1 and Section 2.2 with developing the functorial tools necessary to construct the Burnside and rational representation rings. We then explicitly define these rings developing examples of each. In particular, Section 2.5 describes a ring homomorphism between these spaces and contains the main classical results of Ritter and Segal [17, 20] and Tornehave [23].

The main concerns of our work is to generalize the results of Tornehave. Section 3.2 develops the necessary algebraic structures, the generalized Burnside module A' and representation module R' and describes a homomorphism f' from A' to R'. An analogue of the Ritter-Segal result due to Anton [1], Theorem 3.2.7 is at the end of this section. Section 3.3 is concerned with the situation for a prime p, G is an elementary abelian p-group and $H = \mathbb{Z}_p$. Calculations of the ranks of the generalized Burnside and representation modules occur in Proposition 3.3.1 and Proposition 3.3.2. The main result of this dissertation, Theorem 3.3.7 gives a description of the kernel of the map f' through the visual tool of a commutative diagram. An example and illustration follows in Section 3.3.4

In Chapter 4 we states a conjecture of how to generalize Tornhave's results for arbitrary p-groups G, Conjecture 1. We prove the conjecture for elementary abelian groups in Theorem 4.1.1 using the main theorem of Chapter 3. Additionally we show the conjecture holds for all cyclic p-groups in Theorem 4.1.2.

We conclude with Chapter 5 where we return to the construction of classical Burnside ring and examine a functorial property of the construction. In particular we want to view the Burnside ring as a Bredon functor. Some basic calculations are carried out in Theorem 5.2.2 to demonstrate the necessary techniques.

As an appendix, we offer the essential background knowledge to understand where this problems fits into algebraic topology. References for further reading concerning the general theory are also given here.

Chapter 2 Preliminaries

2.1 Grothendieck Construction

The Grothendieck construction is a method used to enhance some predetermined structure on a given set. Depending on the original makeup, the resulting new set contains differing amounts of increased structure. One basic type of set to consider is that of an abelian semi-group which is a set endowed with an associative and commutative bilinear operation. Given an abelian semi-group X, the Grothendieck construction creates a specific group $\mathcal{G}(X)$ containing X and an artificial zero element. In this case the zero element was not part of the original set and thus had to be created as part of the construction. If on the other hand the abelian semi-group under consideration has a natural zero element it is commonly referred to as an abelian monoid and the resulting group $\mathcal{G}(X)$ will have a natural zero element. In the scenario that an abelian monoid has a second bilinear operation with unit that distributes over the first one, then we call the set a semi-ring. Upon performing the Grothendieck construction on a semi-ring, the resulting object $\mathcal{G}(X)$ is in fact a ring.

We begin our concern of the subject with the case of an abelian monoid and our constructions will agree with those of Lang [10, p. 39-40]. Given an abelian monoid M with operation *, let F be the free abelian group generated by M, and B be the subgroup generated by all elements of the type

$$[x * y] - [x] - [y]$$

for $x, y \in M$. Then forming the quotient group $\mathcal{G}(M) = F/B$ there is a canonical map $\gamma : M \to F/B$ which preserves the operation * in the sense:

$$\gamma(x * y) = \gamma(x) + \gamma(y).$$

Suppose f is any other map from M into an abelian group A preserving the $*$ operation. Then there is a unique homomorphism $f_* : \mathcal{G}(M) \to A$ which extends f such that $f_* \circ \gamma = f$. If particular, the Grothendieck construction $\mathcal{G}(M)$ is universal in the above sense.

To show that $\mathcal{G}$ is a functor we need to assign to each operation preserving map $g : X \to Y$ between abelian monoids, a group homomorphism $\mathcal{G}(g) : \mathcal{G}X \to \mathcal{G}Y$. Since $\mathcal{G}Y$ is an abelian group we naturally gain a map $\gamma \circ g$ from X to $\mathcal{G}Y$ which preserves the operations. We thus define $\mathcal{G}(g) = (\gamma \circ g)_*$ and see that $\mathcal{G}$ is in fact a functor from the category of abelian monoids to the category of abelian groups.

Definition 2.1.1. We say $\mathcal{G}$ is the Grothendieck construction assigning to each abelian moniod X an abelian group $\mathcal{G}X$ in the above universal manner.

If in fact an abelian monoid M is a semiring with two binary operations * and #, we can perform the Grothendieck construction relative to the first operation of the monoid $(M, *)$ to gain an abelian group $\mathcal{G}(M)$. In this case the left distributivity property

$$x\#(y * z) = (x\#y) * (x\#z)$$

of the semiring induces a distributive property on $\mathcal{G}(M)$. Thus $\mathcal{G}(M)$ is an abelian group with a left distributive binary operation induced by #. Similarly this induced operation in $\mathcal{G}(M)$ is right distributive. As the operation # is also associative in the semiring M, we gain an associative ring structure on $\mathcal{G}(M)$.

One common example of the Grothendieck construction is that of the integers. The natural numbers $\mathbb{N}$ form an abelian monoid under standard addition with 0 as a zero element. The Grothendieck construction then artificially introduces the additive inverses of the natural numbers to form the integers $\mathbb{Z}$. Many of the rings and modules that will be discussed are Grothendieck constructions.

2.2 Homological Algebra

In a vast oversimplification, homological algebra is the study of categories and functors. The definition of a functor and many examples with applications will become present throughout this text and thus it may be expected that a framework of basic and necessary results be laid. The basic definitions described in this section agree with those by Vermani [25].

By a category $\mathcal{C}$ we mean a collection of objects denoted $obj\mathcal{C}$ and for every pair of objects $A, B \in \mathcal{C}$, a set of morphisms $Mor(A, B)$ such that $Mor(A, A) \neq \emptyset$ and together with a law of composition of morphisms. One example of a category is the category of abelian groups denoted $\mathcal{A}b$. As expected, the objects of this category are abelian groups and the morphisms are group homomorphisms. Another example of a category is for a ring R, the category of left R-modules denoted ${}_R\mathcal{M}$ with the objects be all left R-modules and for modules A, B, the set of morphisms from A to B is the set of all R-homomorphisms, $Hom_R(A, B)$. In ${}_R\mathcal{M}$, a module M with no proper R-submodules is call simple.

Definition 2.2.1. An R-module M is called a finitely generated free R-module if there is a finite subset of elements $X \subset M$ and a map α from X to M such that for any map f and left R-module A where $f : X \to A$, there exists a unique R-homomorphism $g : M \to A$ such that $f = g\alpha$.

We say R is semisimple if any finitely generated R-module M is R-isomorphic to a direct sum of simple R-modules. If I is a maximal left ideal of a semisimple ring R, then the factor module R/I is isomorphic to a minimal left ideal of R. This factor module will contain no non-trivial submodules and so it is a simple module. Two minimal left ideals are said to be equivalent if and only if they are isomorphic as left R-modules. The isomorphism classes of finitely generated R-modules form a set T with a binary operation $*$ induced by the direct sum. We come to the following conclusion concerning the structure of R-modules for semisimple rings R.

Proposition 2.2.2. *If R is a semisimple ring then the Grothendieck construction $\mathcal{G}(T)$ is the free abelian group on the set of isomorphism classes of minimal left ideals $I < R$.*

Given two R-modules A and B and an R-homomorphism α from A to B, the cokernel of α is defined by $coker(\alpha) = B/\alpha(A)$ and the kernel of α is defined to be the module $ker(\alpha)$ which is all elements of A which map to 0 under α. Given a second R-homomorphism β from B into another R-module C, we form what is known as a sequence of R-modules:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C.$$

The sequence is said to be exact if the composition $\beta \circ \alpha$ is zero. A sequence of R-modules may be extended to include any finite number of modules and homomorphisms or may be extending infinitely to the left, the right, or both directions to form a sequence:

$$\cdots A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \cdots .$$

We say the sequence is exact at A_i if the composition $f_{i+1} \circ f_i = 0$. An exact sequence of the form

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is called a short exact sequence. The following lemma will be necessary later in this text.

Lemma 2.2.3. *Consider the short sequence of finitely generated free $\mathbb{Z}$-modules*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

with α injective and β surjective. If the cokernel of α is a free module and the rank of the image of α equals the rank of the kernel of β, then the sequence is exact.

Proof. Since $Im(\alpha) \subset Ker(\beta)$ and $Coker(\alpha)$ is free, we have the free $\mathbb{Z}$-submodule $Ker(\beta)/Im(\alpha) \subset B/Im(\alpha)$. But the rank of the image of α equals the rank of the kernel of β so that $Ker(\beta)/Im(\alpha)$ is torsion. Therefore $Ker(\beta)/Im(\alpha) = 0$. □

In addition to Lemma 2.2.3 which is used explicitly, the notion of a functor is fundamental to the concepts addressed. Given two categories $\mathcal{C}\,\&\,D$, we want a functor to associate objects and morphisms between them. Denote by $F : \mathcal{C} \to \mathcal{D}$ a map which takes objects and morphisms of $\mathcal{C}$ to objects and morphisms of $\mathcal{D}$ respectively such that $F(\mathbb{I}_A) = \mathbb{I}_{F(A)}$ and for morphisms $f \in Mor(A, B)$ and $g \in Mor(B, C)$ in $\mathcal{C}$, the associated unique morphisms in $\mathcal{D}$, $F(f) \in Mor(A, B)$ and $F(g) \in Mor(B, C)$ are such that the unique morphism $F(gf) = F(g)F(f) \in Mor(F(A), F(C))$ is well defined. In other words, the mapping F of morphisms must respect compositions.

Definition 2.2.4. A map F between two categories satisfying the conditions above is called a covariant functor. If rather than the above composition, to morphisms $f \in Mor(A, B)$ and $g \in Mor(B, C)$ the functor F associates maps $F(f) \in Mor(B, A)$ and $F(g) \in Mor(C, B)$, and the composition to the unique map $F(fg) = F(f)F(g) \in Mor(C, A)$, then we say F is a contravariant functor.

For example, suppose X is a left R-module and for every left R-module A define $T(A) = Hom_R(X, A)$. Thus to every object A of ${}_R\mathcal{M}$, T associates an abelian group. In addition if A, B are R-modules and $f \in Hom_R(A, B)$, define for all R-homomorphisms α from X to A, the group homomorphism:

$$T(f) = Hom(1, f) : Hom_R(X, A) \to Hom_R(X, B)$$

by $T(f)(\alpha) = f\alpha$. It is immediate that T is a covariant functor from the category of left R-modules to the category of abelian groups.

2.3 Burnside Ring

If a group G acts on a finite set X by permutations, we call X a G-set. A G-orbit of a set X, denoted Gx, is the subset of X generated by the single element $x \in X$ under the G-action. Any G-set can be decomposed into G-orbits of the form G/L where L is the stabilizer subgroup of that particular orbit. Such a decomposition is unique up to G-isomorphism.

Given two G-sets X and Y, we can form the disjoint union $X \coprod Y$. The G-actions from X and Y induce an action on $X \coprod Y$ making it into a G-set. Since the disjoint union of two sets is trivially a closed and associative binary operation, the set of finite G-sets forms an abelian monoid. In addition, the group G acts on the Cartesian product $X \times Y$ diagonally. Thus the Cartesian product is a second closed, associative binary operation which distributes over the disjoint union thus turning the set of finite G-sets into a semi-ring. In particular, define S to be the set of isomorphism classes of finite G-sets with the semiring operations:

$$[X] + [Y] = [X \coprod Y] \text{ and } [X][Y] = [X \times Y].$$

Definition 2.3.1. The Burnside ring, $A(G)$, is the Grothendieck construction of the above semiring S.

Suppose in $A(G)$, we have a sum $\sum \pm[G/L_i] = 0$. Collecting terms with positive coefficients together and those with negative coefficients together we see

$$\sum [G/L_j] = \sum [G/L_k].$$

Thus in S we have $\coprod G/L_j \cong \coprod G/L_k$. However the decomposition of any G-set into G-orbits is unique up to isomorphism so the original sum must be empty after a cancelation of terms. Thus the set of isomorphism classes of left coset spaces $\{[G/L_i]\}$ forms a linearly independent set in $A(G)$. In addition since any G-set is decomposable into orbits, these left cosets span $A(G)$.

In fact, $A(G)$ is a free $\mathbb{Z}$-module with a basis given by the set of isomorphism classes of left coset spaces $[G/L]$. First note that $[G/L] = [G/K]$ if and only if L is conjugate to K. Indeed, if $[G/L] = [G/K]$ then there is a map $G/L \to G/K$ given by $L \mapsto gK$ which is a G-isomorphism. If $l \in L$ then $lgK = gK$ so $L \subset gKg^{-1}$ and thus equal as cardinalities are equal. For the other direction, if $L = gKg^{-1}$, then the map sending H to $Lg = gK$ is a G-isomorphism.

The multiplicative ring structure of $A(G)$ can be expressed in terms of the above basis by the formula:

$$[G/L][G/K] = \sum_g [G/(L \cap gKg^{-1})]$$

where the sum runs over representatives g of double cosets LgK of G. This is due to the fact that a group element $x \in G$ is in the stabilizer of an element $(eL, gK) \in G/L \times G/K$ if and only if $x \in L \cap gKg^{-1}$.

For each subgroup L of G we define an induction map $L \uparrow: A(L) \to A(G)$ by sending an L-set X to the G-set $G \times_L X$ where $gl \times x = g \times lx$ for all (g, l, x) in $G \times L \times X$. This definition extends to induction maps $L/C \uparrow: A(L/C) \to A(G)$ via the pullback map $A(L/C) \to A(L)$ where L/C is a subquotient of G. The induction maps are $\mathbb{Z}$-linear but do not preserve the product.

Example 2.3.2. For a prime number p, the Burnside ring $A(\mathbb{Z}_p)$ is easily computed. The only subgroups of $\mathbb{Z}_p$ are $\mathbb{Z}_p$ and 0 so a basis of $A(\mathbb{Z}_p)$ is $\{[\mathbb{Z}_p/0] = [\mathbb{Z}_p], [\mathbb{Z}_p/\mathbb{Z}_p] = [0]\}$. For the multiplicative structure, $[0]$ acts as the multiplicative identity and:

$$[\mathbb{Z}_p][\mathbb{Z}_p] = \sum_{m \in \mathbb{Z}_p} \left[\frac{\mathbb{Z}_p}{0 \cap (m + 0 - m)} \right] = p[\mathbb{Z}_p].$$

Thus we gain a natural map $\mathbb{Z}[x] \to A(\mathbb{Z}_p)$ which takes 1 to $[0]$ and x to $[\mathbb{Z}_p]$. The map is clearly surjective with kernel $(x^2 - px)$ so we have the isomorphism:

$$\frac{\mathbb{Z}[x]}{(x^2 - px)} \cong A(\mathbb{Z}_p).$$

2.4 Representation Ring

Representation theory is concerned with studying abstract groups as groups of matrices. This can be done both by looking at homomorphisms from an abstract group into a general linear group or by studying characters. Many results from fields different from representation theory have elegant proofs using its techniques; thus making representation theory an indispensable tool to mathematicians of varied fields. We primarily concern ourselves with permutation representations and the associated ring that they generate. Although we begin with an arbitrary field of characteristic 0, we will mainly work with the field of rational numbers, $K = \mathbb{Q}$. The construction of representations and definition of the representation ring will agree with the standard texts, one written by Serre [21] and the other written by Curtis and Reiner [6].

For a finite group G and field K of characteristic 0, we say V is a G-module over K if G acts on a finite dimensional K-vector space V by K-automorphisms. In particular, given a G-set X define $K[X]$ to be the K-vector space spanned by the elements of X. Then the G action on X extends by linearity to a G-module structure on $K[X]$ over K. Letting G act by left translations on itself, $K[G]$ is a G module over K. We call $K[G]$ with this action the regular representation of G over K. The group multiplication on G induces a ring structure on $K[G]$, the group ring of G over K, and any G-module over K is simply a $K[G]$-module. The following result is due to Maschke [6, page 88].

Theorem 2.4.1. *The ring KG is semisimple if the order of G is invertible in K.*

The proof is essentially the following. Since K is taken to be a field of characteristic zero, the order of a finite group G is always invertible. Given a submodule W of a KG-module V, we can always form a subvector space U over K such that $W \oplus U \cong V$. The function

$$p(x) = \frac{1}{|G|} \sum_{g \in G} g^{-1} f(gx)$$

defines a G-homomorphism from V onto U where f is the projection of V onto W. The kernel of p is a G-submodule of V and KG-modules, $V \cong W \oplus ker(p)$. If we repeat the process we can decompose V into a direct sum of simple modules.

Given two $K[G]$-modules U and V, we can form the direct sum $X \oplus Y$. The G-actions from X and Y induce an action on $X \oplus Y$ that is commutative with the scalar product of K, making it into a $K[G]$-module. Since the direct sum of two $K[G]$-modules is trivially a closed and associative binary operation, the set of finitely generated $K[G]$-modules forms an abelian monoid. In addition, the group G acts on the tensor product $X \otimes Y$ diagonally. Thus the tensor product is a second closed, associative binary operation which distributes over the direct sum thus turning the set of finitely generated $K[G]$-modules into a semi-ring. In particular, let T be the semiring of isomorphism classes of finitely generated $\mathbb{Q}[G]$-modules with respect to the semiring operations:

$$[M] + [N] = [M \oplus N] \text{ and } [M][N] = [M \otimes N].$$

Definition 2.4.2. The rational representation ring of G, $R(G)$, is the Grothendieck construction of the above semiring T.

We say that a $\mathbb{Q}$-linear map $f : V \to W$ between two G-modules over $\mathbb{Q}$ is a G-map if f commutates with the G-action. A corollary of Maschke is that any G-module V over $\mathbb{Q}$ is decomposable into a direct sum of irreducible G-modules and that this decomposition is unique up to G-isomorphism. With this in mind the following result defines a group structure on the representation ring that reflects the G-isomorphism classes of non-zero minimal left ideals of $\mathbb{Q}[G]$. Thus choosing a set of representatives for these classes is equivalent to finding a decomposition of $\mathbb{Q}[G]$.

Theorem 2.4.3. *The representation ring $R(G)$ is the free abelian group with basis the G-isomorphism classes of non-zero minimal left ideals of $\mathbb{Q}[G]$.*

Proof. Suppose V is an irreducible G-module. Then there is a maximal left ideal of the group ring $\mathbb{Q}[G]$ such that $V \cong \mathbb{Q}[G]/I$. By Maschke's Theorem, there is also a left ideal J of $\mathbb{Q}[G]$ such that $\mathbb{Q}[G]$ is G-isomorphic with the direct sum $I \oplus J$. In particular J is G-isomorphic to V and thus J is a minimal left ideal by the irreducibility of V. □

To explicitly define the multiplicative structure of the representation ring $R(G)$ it is convenient to introduce the notion of characters. However the additional information of a basis decomposition for products is not necessary for our work and we will simply use the tensor product of $\mathbb{Q}[G]$-modules as the multiplicative structure.

For each subgroup L of G we define an induction map $L \uparrow : R(L) \to R(G)$ by sending an $\mathbb{Q}[L]$-module M to the $\mathbb{Q}[G]$-module $\mathbb{Q}[G] \times_{\mathbb{Q}[L]} X$. This map extends to subquotients in the same manner as in the Burnside ring scenario.

Example 2.4.4. Again if we consider $G = \mathbb{Z}_p$ for p a prime then the additive structure of the representation ring $R(G)$ is easily determined. Let ξ_p be a primitive p-root of unity. To determine a representative set of irreducible $\mathbb{Z}_p$-modules over $\mathbb{Q}$, define $V_0 = \mathbb{Q}$ with trivial $\mathbb{Z}_p$-action and $V_1 = \mathbb{Q}(\xi_p)$ with the action of $1 \in \mathbb{Z}_p$ defined by the square matrix:

$$A = (a_{i,j}) \text{ where } a_{i,i+1} = 1,\ a_{p-1,j} = -1, \text{ and } a_{i,j} = 0 \text{ otherwise.}$$

It is clear these modules are irreducible and by a dimension argument, $\mathbb{Q}[\mathbb{Z}_p] = \mathbb{Q} \oplus \mathbb{Q}(\xi_p)$ is a complete decomposition into minimal left ideals. Thus V_0, V_1 generate $R_{\mathbb{Q}}(\mathbb{Z}_p)$ as a free abelian group.

2.5 Linearization Map

At this point there are some obvious similarities and connections between the Burnside ring and the representation ring. Both rings occur naturally through the Grothendieck construction and there is an external group G whose possible actions on sets or vector spaces effects their structure. Particularly the induction maps are defined in identical manners with respect to subgroups $L < G$. We thus are led to prove the following result concerning both induction maps.

Proposition 2.5.1. *Both induction maps are injective.*

Proof. Let N be the Grothendieck construction of either of the monoids S or T defining the Burnside and representation rings for a group G. For a subquotient L/C of G, let M be the Grothendieck construction of the same monoid S or T and the induction map is defined by a homomorphism $L/C \uparrow : M \to N$. The Grothendieck construction of M consists of formal differences $[X]-[Y]$ of elements in M such that $[X]-[Y]=[X']-[Y']$ if and only if

$$[X+Y'+Z]=[X'+Y+Z]$$

for some $[Z]$ in M. In particular, if $L/C\uparrow[X]-L/C\uparrow[Y]=0$ then

$$[L/C\uparrow X+V]=[L/C\uparrow Y+V]$$

for some $[V]$ in N. By restricting the G-structure to an L-structure we have a restriction map $L\downarrow$ such that $L\downarrow(L/C\uparrow[X])=[\tilde{G}:L][X]$. In particular,

$$[\tilde{G}:L][X]+L\downarrow[V]=[\tilde{G}:L][Y]+L\downarrow[V].$$

Since each element of M has a unique decomposition into a sum of irreducible elements, we conclude that $[X]=[Y]$ proving the injectivity of the induction map. □

In addition, the Burnside and representation rings are related by a natural ring homomorphism $f:A(G)\to R(G)$ sending a G-set X to the permutation $\mathbb{Q}[G]$-module $\mathbb{Q}[X]$. This map f is a linearization map and it is immediate that f commutes with the inductions.

Definition 2.5.2. The Burnside kernel $N(G)$ is the kernel of the linearization map f between $A(G)$ and $R(G)$.

In the case G is a finite p-group, the map f is well understood. In this instance the follow property of the linearization map f was proven by Ritter [17] and Segal [20].

Theorem 2.5.3. *If G is a finite p-group for any prime p, then the map $f:A(G)\to R(G)$ is surjective.*

Proof. Take V to be an irreducible, nontrivial $\mathbb{Q}G$-module. Since by passing to $G/ker(V)$ we can assume V is also faithful, it is a theorem of Roquette [18] that all abelian normal subgroups of G are cyclic. Using this fact Tornehave [23] showed we have a subquotient B/C of G, a B/C representation W over $\mathbb{Q}$ where $V=\uparrow_{B/C}W$. Since our induction process is natural, figure 2.1 commutes. Thus V is in the image of f. As f is a ring homomorphism, we extend the argument linearly and see that f is surjective. □

The above proof for Theorem 2.5.3 is a sketch of the argument from Tornehave [23]. In particular when G is an elementary abelian p-group of rank 2, Laitenan explicitly described the kernel in [9].

Figure 2.1: Surjectivity of f for p-groups

$$\begin{array}{ccc} A(G) & \xrightarrow{f} & R(G) \\ {\scriptstyle B/C\uparrow}\big\uparrow & & {\scriptstyle B/C\uparrow}\big\uparrow \\ A(B/C) & \overset{f}{\twoheadrightarrow} & R(B/C) \end{array}$$

Theorem 2.5.4. *If p is a prime and $G = \mathbb{Z}_p \times \mathbb{Z}_p$ then $N(G)$ is infinite cyclic generated by:*

$$[G/0] - \sum [G/C] + p[G/G]$$

where C runs through the non-trivial cyclic subgroups.

Following Laitenan's work, Tornehave attempted to describe the kernel for arbitrary p-groups. Although we do not see the generators of the kernel directly, he did prove the following theorem in [23] that describes the kernel in terms of inductions from subquotients L/K of G.

Theorem 2.5.5. *If p is a prime and G is a finite p-group then:*

$$N(G) = \sum_{L/K} G \uparrow N(L/K)$$

where the sum is taken over all subquotients L/K of G isomorphic to one of the following groups:

$$\begin{array}{l} \mathbb{Z}_p \times \mathbb{Z}_p \\ D_{2^n} \text{ for } p = 2 \\ M(p) \text{ for } p \neq 2. \end{array}$$

Combining the Ritter-Segal and Tornehave results we have a well understood short exact sequence:

$$0 \to N(G) \to A(G) \xrightarrow{f} R(G) \to 0. \tag{2.1}$$

Chapter 3 The Relative Version

3.1 Introduction

Our stated goal is to develop analogous results to Tornehave [23] in a more general setting. We first develop the objects of interest in section 3.2.1 and section 3.2.2. Instead of considering sets with a single group action on the left to form the Burnside ring, we will consider sets with group actions on both the left and the right to form a new algebraic object. For finite groups G and H, the Grothendieck construction of the set of isomorphism class of finite H-free, $G \times H$-sets is denoted $A(G, H)$. This set $A(G, H)$ is a free abelian group in addition to being endowed with an $A(G)$-module structure. Additionally rather than consider $\mathbb{Q}[G]$-modules and forming the representation ring, we will consider bimodules to create another algebraic object denoted $R(G, H)$. In the case where H is the trivial group we regain the classical Burnside and classical representation rings. We finish section 3.2 by constructing a relative version of the linearization map from the classical case.

The main result, Theorem 3.3.7, is found in section 3.3.3 and for a prime p, describes the kernel of the relative linearization map in terms of induced subquotients in the case where G is an elementary abelian p-group and H is the cyclic group of order p. Section 3.3.2 develops algorithms to determine the ranks of the classical Burnside and representation rings and also the relative analogues. We then proceed to prove the main result in section 3.3.3 based on the prior rank calculations. We conclude the chapter by calculating the kernel for the example $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and offer diagrams that illustrate the methods.

3.2 The Relative Burnside and Representation Modules

3.2.1 The Relative Burnside Module

If $\tilde{G} = G \times H$ is a direct product of two finite groups then a $\tilde{G}$-set can be thought of with G acting on the left and H on the right. Given two $\tilde{G}$-sets X and Y, we can form the disjoint union $X \coprod Y$, and the $\tilde{G}$-actions from X and Y induce an action on $X \coprod Y$ making it into a $\tilde{G}$-set. Since the disjoint union of two sets is trivially a closed and associative binary operation, the set of finite $\tilde{G}$-sets forms a monoid. Unlike the classical case of Section 2.3, there is no natural product between $\tilde{G}$-sets. However given given a G-set X and a $\tilde{G}$-set Y we can form the set $X \times Y$ where G

acts diagonally on the left and H acts on only Y on the right. Let S' be the set of isomorphism classes of finite H-free $\tilde{G}$-sets with respect to disjoint union. Since the disjoint union is commutative for isomorphism class of $\tilde{G}$-sets, S' is an abelian monoid. The Grothendieck construction of S' gives an abelian group $\mathcal{G}(S')$. In addition to the group structure, $\mathcal{G}(S')$ is an $A(G)$-module where as sets:

$$[X][Y] = [X \times Y]$$

and the G-action is diagonal on the left while the H-action affects only Y on the right.

Definition 3.2.1. The relative Burnside module $A(G, H)$ is the Grothendieck construction of the monoid S'.

The relative Burnside module $A(G, H)$ is clearly a subgroup of the classical Burnside ring $A(\tilde{G})$ with respect to the common additive operation. A natural question to ask is whether or not $A(G, H)$ is a natural free subgroup of $A(\tilde{G})$?

Proposition 3.2.2. *If G and H are finite groups then $A(G, H)$ is the free abelian group with a basis given by the twisted products $[G \times_\rho H]$ where the pairs (K, ρ) runs through conjugacy class representatives of subgroups K in G and homomorphisms $\rho : K \to H$.*

Proof. Let $K < G$ and ρ be a homomorphism from K to H. Consider the $\tilde{G}$-set

$$S_\rho = G \times_\rho H = \frac{G \times H}{(gk, h) \sim (g, \rho(k)h)}$$

for $g \in G$, $h \in H$, and $k \in K$ and let $(x, y) \in S_\rho$. If (x, y) is fixed by H then for all $h' \in H$ we have:

$$(x, y) = (x, y)h' = (x, yh')$$

with the identification $(gk, h) = (g, \rho(k)h)$. This is equivalent to $hh' = h$ for all $h \in H$ which implies $h' = e$ and so $S_\rho \in A(G; H)$. It is clear that such a set is completely determined by the pair (K, ρ).

Choose an element of $A(G, H)$ with a single $\tilde{G}$-orbit, say $(\tilde{G})x_0$ and define the set

$$K = \{k \in G | \exists\, h \in H \text{ such that } kx_0 = x_0 h\}$$

and the map

$$\rho : K \to H \text{ by } \rho(k) = h \text{ if } kx_0 = x_0 h.$$

Now ρ is well defined as H acts freely on the orbit and is a group homomorphism. Using our relations we see that $G \times_\rho H = \tilde{G}x_0$. Extending this argument linearly we gain H-free, $\tilde{G}$ sets with multiple orbits. Hence the twisted products S_ρ span the $A(G, H)$.

In addition, note that for each $K < G$ and $\rho : K \to H$ we have:

$$G \times_\rho H = \frac{G \times H}{W}$$

where $W = \{(k, \rho(k^{-1})) : k \in K\}$. Since this set of elements is a subset of the basis for the classical Burnside ring, two elements:

$$G \times_\rho H = \frac{G \times H}{W} \text{ and } G \times_\psi H = \frac{G \times H}{U}$$

are in the same isomorphism class if T and K (the domains of ρ and ψ respectively) are conjugate and the maps ρ and ψ commute with the conjugation. In other words, $K = T^g$ and $\psi = \rho \circ c_g$ and so we have one representative for each isomorphism class. Also if $\sum[G/W_k] = 0$ then:

$$\sum[G \times H/W_i] = \sum[G \times H/W_j] \text{ so } \left[\prod G \times H/W_i\right] \cong \left[\prod G \times H/W_j\right].$$

Since the decomposition into orbits is unique, the original sum must be empty and the set of all S_ρ is linearly independent. □

Example 3.2.3. Let p be a prime number, $G = \mathbb{Z}_p$, and $H = \mathbb{Z}_p$, then the relative Burnside module $A(G; H)$ is easily computable. Since G is abelian, for any subgroup $K < G$ we have $K^g = K$ and thus each conjugacy class of subgroups consists of a single element. Also, the only possibilities for K are 0 or G. In the case of $K = 0$, the only homomorphism $\rho : K \to H$ is the zero homomorphism. In this situation,

$$G \times_\rho H = G \times H/0 = G \times H.$$

If $K = G$, then either $\rho \equiv 0$ or $\rho(1) = m$ for $m \not\equiv 0 \bmod p$. In these cases,

$$G \times_0 H = G \times H/G = H$$

and

$$G \times_m H = G \times H/ < (1, -m) >$$

where $< (1, -m) >$ is the cyclic subgroup of $G \times H$ generated by the element $(1, -m)$. So a basis of twisted products for $A(G; H)$ is given by the set:

$$\{[G \times H], [G \times H/ < (1, m) >] \mid m \not\equiv 0 \in \mathbb{Z}_p\}.$$

To conclude this section on the construction of the relative Burnside module we develop induction maps similar to the classical case. The relative induction maps $L/C \uparrow: \tilde{A}(L/C) \to A(G,H)$ are defined by the usual induction $L/C \uparrow$ restricted to the submodule $\tilde{A}(L/C)$ made of those elements of $A(L/C)$ that land in $A(G,H)$ where L/C is a subquotient of $\tilde{G}$. The abuse of notation concerning the induction map is made irrelevant as the target space will indicate whether we are dealing with the classical or relative case.

3.2.2 The Relative Representation Module

Similarly, we construct a relative representation module. If $\tilde{G} = G \times H$ is a direct product of two finite groups then a $\mathbb{Q}[\tilde{G}]$-module can be thought of as a $\mathbb{Q}[G] - \mathbb{Q}[H]$ bimodule. Given two $\mathbb{Q}[\tilde{G}]$-modules M and N, we can form the direct sum $M \oplus N$. The $\mathbb{Q}[\tilde{G}]$-module structure from X and Y induce a module structure on $X \oplus Y$ making it into a $\mathbb{Q}[\tilde{G}]$-module. Since the direct sum of two modules is trivially a closed and associative binary operation, the set of finite $\mathbb{Q}[\tilde{G}]$-modules forms a monoid. Unlike the classical case of Section 2.4, there is no natural product between $\mathbb{Q}[\tilde{G}]$-modules. However given given a $\mathbb{Q}[G]$-module M and a $\mathbb{Q}[\tilde{G}]$-module N we can form the set $X \times Y$ where $\mathbb{Q}[G]$ acts diagonally on the left and $\mathbb{Q}[H]$ acts on only N from the right. Let T' be the monoid of isomorphism classes of finite $\mathbb{Q}[H]$-free $\mathbb{Q}[\tilde{G}]$-modules with respect to direct sum. Since we are only concerned with the isomorphism classes of $\mathbb{Q}[\tilde{G}]$-modules, the direct sum is a commutative operation and and T' is in fact an abelian monoid. The Grothendieck construction of T' gives an abelian group $\mathcal{G}(T')$. In addition to the group structure, $\mathcal{G}(T')$ is an $R(G)$-module where as sets:

$$[M][N] = [M \times N]$$

and the $\mathbb{Q}[G]$-module structure is diagonal on the left while the $\mathbb{Q}[H]$-structure affects only N on the right.

Definition 3.2.4. The relative rational representation module $R(G,H)$ is the Grothendieck construction of the monoid T'.

The relative induction maps $L/C \uparrow: \tilde{R}(L/C) \to R(G,H)$ are defined in the same manner as the relative induction maps for the relative Burnside modules. We do this by the usual induction $L/C \uparrow$ restricted to the submodule $\tilde{R}(L/C)$ made of those elements of $R(L/C)$ that land in $R(G,H)$ where L/C is a subquotient of $\tilde{G}$. Again the abuse of notation is easily overcome by looking at the context of the induction maps.

3.2.3 The Relative Linearization Map

We again see some immediate similarities both between the relative Burnside and relative representation modules and also between the classical and relative cases. Both modules occur naturally through the Grothendieck construction and their structure is completely determined by the possible actions of an external group $G \times H$ where H acts freely on sets or vector spaces. In particular we notice that the induction maps for the relative Burnside and representation modules are defined to be restrictions of the classical inductions.

Proposition 3.2.5. *Both relative induction maps are injective.*

The proof of Proposition 3.2.5 is nearly identical to the argument in the classical case of Proposition 2.5.1. The only difference is we now take M to be either the monoid S' or T'.

In addition, the natural ring homomorphism $f : A(\tilde{G}) \to R(\tilde{G})$ restricts to a module homomorphism f' from the relative Burnside module $A(G,H)$ to the relative representation module $R(\tilde{G})$. The map f' sends an isomorphism class of an H-free, $\tilde{G}$ set X to the $\mathbb{Q}[\tilde{G}]$-module $\mathbb{Q}[X]$.

Definition 3.2.6. We call the map f' the relative linearization map and call its kernel the relative Burnside kernel denoted $N(G,H)$.

In the instance that p is a prime, G is a finite p-group and $H = \mathbb{Z}_p$, the map f' can again be described by generators and relations. In particular Theorem 3.2.7 was proven by Anton [1].

Theorem 3.2.7. *If p is a prime number, G is a finite p-group and H a group of order p, then the image of f' agrees with $R(G,H)$.*

Thus we can now view f' as a group homomorphism between the relative Burnside and representation modules. Again it is immediate that the map f' commutes with the relative inductions. Theorem 3.2.7 also implies that we have an analogous result to Ritter and Segal's Theorem 2.5.3. Thus if we can understand the generators of $N(G,H)$ for particular p-groups G, it is possible to gain a complete understanding of the short exact sequence:

$$0 \to N(G,H) \to A(G,H) \xrightarrow{f'} R(G,H) \to 0. \tag{3.1}$$

3.3 The Relative Burnside Kernel for Elementary Abelian Groups

3.3.1 Notations

For the remainder of Chapter 3, we will take the following notational conventions to simplify notation. For a prime p, let G be the elementary abelian p-group of dimension n and H the group of order p so that $\tilde{G} = G \times H = \mathbb{Z}_p^{n+1}$. In the classical setting we denote the Burnside ring, representation ring, and Burnside kernel by

$$A = A(\tilde{G}),\ R = R(\tilde{G}),\ N = N(\tilde{G}).$$

Defining $A_k \subset A$ to be the set generated by all $[\tilde{G}/L]$ with $L \subset \tilde{G}$ of dimension k, we gain the following decomposition of the Burnside ring

$$A = A_0 \oplus A_1 \oplus ... \oplus A_{n+1}.$$

Likewise we denote the relative Burnside and representation modules and the relative kernel by

$$A' = A(G,H),\ R' = R(G,H),\ N' = N(G,H).$$

Defining $A'_k = A_k \cap A'$, we gain a decomposition of A' similar to that of A.

3.3.2 Rank calculations

Let $G(k,n)$ denote the number of k-dimensional subspaces of the n-dimensional vector space $\mathbb{Z}_p^n$. Then as a consequence from Stanley [22, p. 28], we gain the following formula.

$$G(k,n) = \prod_{j=1}^{k} \frac{p^{n-j+1} - 1}{p^j - 1}$$

Proposition 3.3.1. *The ranks a_k and a'_k of A_k and A'_k are given by the formulas*

$$a_k = G(k, n+1),\ a'_k = p^k G(k,n).$$

Proof. The basis elements $[\tilde{G}/L]$ for A_k are in one-to-one correspondence with the k-dimensional subspaces $L < \tilde{G} = \mathbb{Z}_p^{n+1}$. Hence, we get the first formula.

The basis elements $[G \times_\rho H]$ for A'_k are in one-to-one correspondence with pairs (K, ρ) with $K < G$ a k-dimensional subspace and $\rho : K \to H$ a homomorphism. Given K, ρ is uniquely determined by its kernel and an automorphism of its image. If K is k-dimensional, the kernel of ρ is either K or some $(k-1)$-dimensional subspace of K. In the later case the image admits $(p-1)$ automorphisms. Hence, for a given

k-dimensional K there are $(p-1)G(k-1,k)+1$ different ρ's. For a given dimension k the number of pairs (K,ρ) is thus given by the calculation:

$$G(k,n)[(p-1)G(k-1,k)+1] = G(k,n)[(p-1)\sum_{i=0}^{k-1} p^i + 1] = p^k G(k,n)$$

and we have the second formula. □

Let ξ denote a primitive p-root of unity and $F = \mathbb{Q}(\xi)$ be the associated cyclotomic field. For each $s \in \mathbb{Z}_p^{n+1}$ let F_s be the $\mathbb{Q}[\tilde{G}]$-module F obtained by letting the i^{th} canonical generator of $\tilde{G}$ act on F via the automorphism sending ξ to ξ^{s_i} where s_i is the i^{th} coordinate of s.

Proposition 3.3.2. *The ranks r and r' of R and R' are given by the formulas*

$$r = G(1,n+1)+1, \;\; r' = G(1,n+1).$$

Proof. With the above notations, two isomorphism classes are equal $[F_s]=[F_t]$ if and only if $t=s=0$ or $t=us$ for some unit u in $\mathbb{Z}_p$. In the later case we say that s and t represent the same point $[s]=[t]$ in the projective n-space P^n over $\mathbb{Z}_p$. With this observation $[F_s]$ indexed by $[s]\in P^n$ and the trivial module $[\mathbb{Q}]$ form a basis for R. Thus we get the first formula.

For the second formula we claim that a basis for R' is given by the elements

$$[F_{s'\times 1}]+[\mathbb{Q}], \quad [F_{t\times 0}]-(p-1)[\mathbb{Q}]$$

indexed by $s'\in\mathbb{Z}_p^n$ and $[t]\in P^{n-1}$. Let $\mathcal{B}$ denote the set of these elements and $\mathcal{M}$ the $\mathbb{Z}$-module generated by $\mathcal{B}$. Since $F_{0\times 1}+\mathbb{Q}=\mathbb{Q}[H]$ it follows that by forgetting the G-action, the elements:

$$[F_{s'\times 1}]+[\mathbb{Q}], \; [F_{t\times 0}]+(p-1)[F_{0\times 1}], \text{ and } (p-1)[F_{0\times 1}]+(p-1)[\mathbb{Q}]$$

are all represented by the $\mathbb{Q}[H]$-free modules $\mathbb{Q}[H]$ or $(p-1)\mathbb{Q}[H]$. Thus $\mathcal{M}\subset R'$ and it is immediate that $\mathcal{B}$ is a linearly independent set. Now by inspection $R/\mathcal{M}$ is the free module generated by $[\mathbb{Q}]$ and $m[\mathbb{Q}]\in R'$ implies $m=0$. Thus the rank of R' equals the rank of $\mathcal{M}$. In particular Lemma 2.2.3 applies to the sequence:

$$0\to\mathcal{M}\to R\to R/R'\to 0$$

implying that $R'=\mathcal{M}$. □

From the Propositions 3.3.1 and 3.3.2 and the short exact sequence (2.1) we deduce the following result.

Corollary 3.3.3. *The ranks b and b' of N and N' are given by the formulas*

$$b = \sum_{k=0}^{n-1} G(k, n+1), \; b' = \sum_{k=0}^{n} p^k G(k, n) - G(1, n+1).$$

3.3.3 The main theorem

It is convenient to identify each basis element $[\tilde{G}/L]$ of A where $L < \tilde{G}$ with the projective subspace $(L) \subset P^n$ generated by L. Also, let e denote the distinguished vector $(0, ..., 0, 1) \in \mathbb{Z}_p^{n+1}$. Then we have the following characterization for the basis elements of A'_k in terms of projective subspaces.

Lemma 3.3.4. *The submodule $A'_k \subset A'$ is the free abelian group on the set of projective subspaces $(L) \subset P^n$ with $L < \tilde{G}$ of dimension k not containing e.*

Proof. It is easy to see that the basis elements $[G \times_\rho H]$ of A' associated with a pair (K, ρ) is of the form $[\tilde{G}/L]$ where $K < G$, $\rho : K \to H$ is a homomorphism, and $L = \{(k, \rho(k)) | k \in K\}$ is a linear subspace of $\tilde{G}$ not containing e.

Conversely, let $(L) \subset P^n$ with $L < \tilde{G}$ of dimension k not containing e and define K to be the image of the canonical projection $\tilde{G} \to G$. If (g, h) is an element in L which maps to 0 under the projection, then $g = 0$. This would imply $he \in L$ so $h = 0$. Thus the projection induces an isomorphism $L \cong K$. Let $\alpha : K \to L$ be the inverse and define $\rho : K \to H$ by composing α with the canonical projection $\tilde{G} \to H$. We can then check that $[\tilde{G}/L] = [G \times_\rho H]$. □

Given L a subspace of codimension at least 2 in $\tilde{G}$ we define $L^* < \tilde{G}$ to be a distinguished subspace such that the following two conditions are both satisfied:

1. L^* contains L and L^*/L has rank 2.
2. If L does not contain the distinguished vector e and has codimension at least 3 then L^* also does not contain e.

Now we observe that L^* always exists subject to the two conditions. In particular, if L has codimension exactly 2 then $L^* = \tilde{G}$ is the only choice without violating condition 2.

Definition 3.3.5. For each such L define

$$t(L) = (L) - \sum (C) + p(L^*)$$

where the sum is over all proper subspaces $L < C < L^*$.

In particular, define M_{n-1} to be the set of all (L) with $L < \tilde{G}$ an $(n-1)$-dimensional subpace where $e \notin L$. By Lemma 3.3.4, this set is also a basis for A'_{n-1}.

Definition 3.3.6. Let A''_{n-1} be the submodule of A'_{n-1} generated by all those differences $(L) - (L')$ of elements in M_{n-1} that are subject to the relation

$$(L + \mathbb{Z}_p e) = (L' + \mathbb{Z}_p e).$$

Theorem 3.3.7. *The rank of A''_{n-1} is $G(1,n)(p^{n-1}-1)$ and we have the following commutative diagram of short exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A'_0 \oplus A'_1 \oplus \ldots \oplus A'_{n-2} \oplus A''_{n-1} & \xrightarrow{t'} & A' & \xrightarrow{f'} & R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_0 \oplus A_1 \oplus \ldots \oplus A_{n-2} \oplus A_{n-1} & \xrightarrow{t} & A & \xrightarrow{f} & R & \longrightarrow & 0 \end{array}$$

where the vertical arrows are all inclusions and t', f' are the restrictions of t, f.

Proof. Given (L) in M_{n-1} we can write $t(L)$ in the form:

$$t(L) = (L) - (C_0) - \sum(C) + p(\tilde{G})$$

where $C_0 = L + \mathbb{Z}_p e$ and the sum is taken over all $L < C < \tilde{G}$ not containing e. By Definition 3.3.6, if $(L) - (L')$ is a generator of A''_{n-1} then

$$(L + \mathbb{Z}_p e) = (L' + \mathbb{Z}_p e) = (C_0)$$

and we deduce that $t((L) - (L'))$ is in A' so all maps in the diagram are well defined. Also Theorem 2.5.5 and Theorem 3.2.7 prove that f and f' are surjective.

From Section 2.5 we know that the kernel of f is generated by the induced kernels $L/C \uparrow N(L/C)$ where $L/C \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In particular, by applying $L/C \uparrow$ to the generator of Theorem 2.5.4 with $G = L/C$ we deduce that $N(\tilde{G})$ is generated by elements of the form

$$[\tilde{G}/C] - \sum[\tilde{G}/D] + p[\tilde{G}/L]$$

where L/C is any subquotient of $\tilde{G}$ isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ and the sum runs over all proper subgroups $C < D < L$. By Definition 3.3.5 the above elements with L/C replaced by L^*/L generate the image of t so that the composition $f \circ t$ and $f' \circ t'$ are both zero.

We are left to prove the injectivity of the map t and the inclusions of the kernels of f and f' inside the images of t and t' respectively. Under the map t, each basis element

(L) of A_i is mapped to an element inside $A_i \oplus A_{i+1} \oplus A_{i+2}$ whose first component is again (L). Therefore the matrix representation of t is upper triangular with cokernel $A_n \oplus A_{n+1}$ which is free. Hence t, and therefore t', are injective.

Regarding the exactness at A observe that by Proposition 3.3.1 and the injectivity of t it follows that the rank of t is the sum $G(k, n+1)$ for $k = 0, 1, ..., n-1$. The same sum by Corollary 3.3.3 is the rank of the kernel of f. Since the cokernel of t is a free module we conclude by Lemma 2.2.3 that the bottom sequence is exact at A.

To determine exactness at A' we must first determine the rank of A''_{n-1}. As calculated in Proposition 3.3.1, the rank of A'_{n-1} which is also the order of M_{n-1} is equal to $a'_{n-1} = p^{n-1}G(n-1, n)$. We observe that M_{n-1} breaks into $G(n-1, n)$ equivalence classes relative to the equivalence relation $(L) \sim (L')$ if and only if

$$L + \mathbb{Z}_p e = L' + \mathbb{Z}_p e.$$

By Lemma 3.3.4 and Proposition 3.3.1 with n replaced by $n-1$, each n-subspace containing e contains $p^{n-1}G(n-1, n-1)$ subspaces of dimension $n-1$ not containing e. This product gives the number of elements in any of the equivalence classes. Hence, since each equivalence class produces exactly $p^{n-1}-1$ basis elements for A''_{n-1} and there are $a'_{n-1}p^{n-1}$ equivalence classes, we conclude that the rank of A''_{n-1} is given by the following formula:

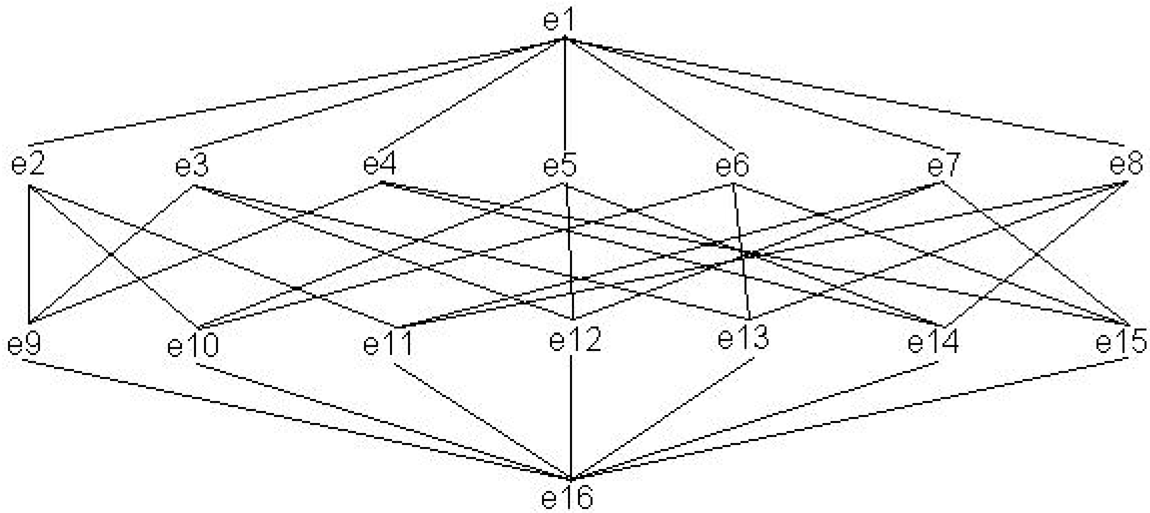
$$a'_{n-1}p^{n-1}(p^{n-1}-1) = G(1, n)(p^{n-1}-1).$$

Combining this with the fact that t' is injective it follows that the image of t' has rank equal to the kernel of f'. Moreover when considering the generators of A''_{n-1}, if we allow any given basis element $(K) \in A(n-1)$ to play the role of an (L) in the difference $(L) - (L')$ at most once, then we see that the matrix of t' will be upper triangular as t' maps a difference $(L) - (L')$ to an element inside $A_{n-1} \oplus A_n \oplus A_{n+1}$ with first component $(L) - (L')$. Therefore the cokernel of t' is a free module and by Lemma 2.2.3, the top row is exact. □

3.3.4 An illustration for $n = 2$ and $p = 2$

Order $\mathbb{Z}_2$ such that $0 < 1$ and order $\mathbb{Z}_2^3$ lexicographically. Then for $n = p = 2$ we gain a labeling of the basis, $\{e_i\}$, of $A(\mathbb{Z}_2^3)$ such that $e_1 < e_2 < ... < e_{16}$. With this labeling of the basis of $A(\mathbb{Z}_2^3)$, the subgroup lattice of $\tilde{G}$ can be represented by the graph E in figure 3.1 and offers a visual description of the relationship between basis elements e_i and e_j.

Figure 3.1: Basis lattice for a classical Burnside ring.



Theorem 3.3.7 implies that we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_0' \oplus A_1'' & \xrightarrow{t'} & A' & \xrightarrow{f'} & R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_0 \oplus A_1 & \xrightarrow{t} & A & \xrightarrow{f} & R & \longrightarrow & 0 \end{array}$$

We see using our basis that

$$A_0 = A_0' = \mathbb{Z}e_1,\ A_1 = \sum_{i=2}^{8} \mathbb{Z}e_i \text{ and } A_1' = \mathbb{Z}(e_3 - e_4) + \mathbb{Z}(e_5 - e_6) + \mathbb{Z}(e_7 - e_8).$$

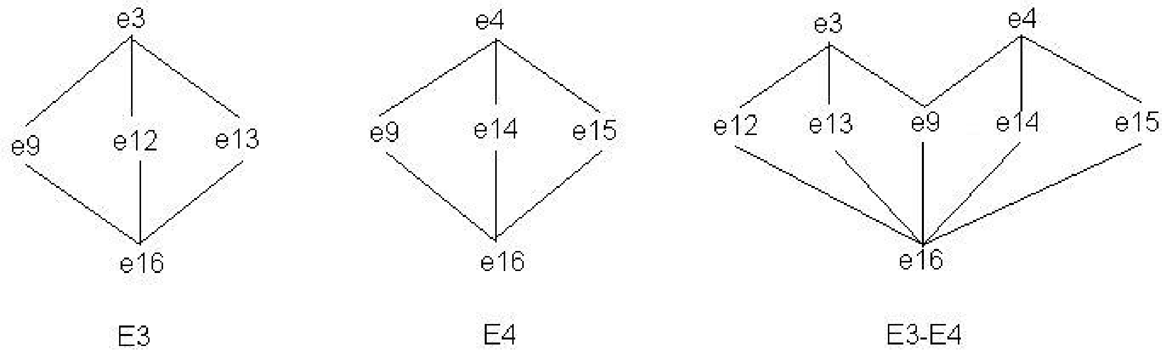
Hence t is well defined on A_1 while we define:

$$t(e_1) = e_1 - e_3 - e_5 - e_7 + 2e_{12}.$$

Define the subgraph E_i to be the full subgraph of E where the vertices are the terms occurring in $t(e_i)$. Then the image $t'(e_i - e_j)$ is associated with the subgraph $E_i - E_j$ whose vertices are those in E_i and E_j. For example, if $i = 3$ and $j = 4$ these subgraphs are described in figure 3.2.

Conversely, given a subgraph E_i the image $t(e_i)$ is uniquely determined by taking a weighted sum of the vertices of E_i. Moreover, given a subgraph $E_i - E_j$, the image

Figure 3.2: Construction of a basis element for a relative Burnside module.



$t(e_i - e_j)$ is also uniquely determined by the vertices of $E_i - E_j$. It follows that the kernel of f is generated by all of the subgraphs E_i for $i = 1, 2, ..., 8$ and the kernel of f' is generated by all the non-singular subgraphs E_1, $E_3 - E_4$, $E_5 - E_6$, $E_7 - E_8$.

Chapter 4 Applications

4.1 A Conjecture for Arbitrary p-Groups

We would like to develop a description for the kernel $N(G, H)$ in certain cases analogous to that given by Tornehave in [23] for $N(G)$ where G was an arbitrary finite p-group. One possible situation where this should be possible is the case for a prime p, we consider $H = \mathbb{Z}_p$ and G to be any finite p-group. We thus want to build the kernel $N(G, H)$ using induced kernels whose basis elements maintain free H-actions. To accomplish this construction, define $\tilde{N}(L/C)$ for L/C a subquotient of $\tilde{G} = G \times H$ to be the intersection of $N(L/C)$ with the submodule $\tilde{A}(L/C)$ of $A(L/C)$ that lands inside $A(G, H)$ under the induction $L/C \uparrow$ of Section 3.2.1.

Conjecture 1. *Let p be a prime, G any finite p-group, and $H \cong \mathbb{Z}_p$. Then*

$$N(G, H) = \sum L/C \uparrow \tilde{N}(L/C)$$

where the sum is taken over subquotients L/C of $\tilde{G}$ isomorphic to $T \times H$ where T is the elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$, the dihedral group, or the nonabelian group of order p^3 and exponent p.

In the case that H is a point, we recover the classical Tornehave result. As the relative kernel is an additive subgroup of the classical one, if $N(G, H)$ is going to be induced from subquotients their only possible forms are $\mathbb{Z}_p \times \mathbb{Z}_p$, D_n, or $M(p)$. For G elementary abelian or cyclic this conjecture can readily be checked using Theorem 3.3.7 and rank arguments.

Theorem 4.1.1. *Let p be any prime, G be an elementary abelian p-group, and $H \cong \mathbb{Z}_p$. Then*

$$N(G, H) = \sum L/C \uparrow \tilde{N}(L/C)$$

with the sum taken over all subquotients $L/C \cong \mathbb{Z}_p^3$.

Proof. From Theorem 3.3.7 we know that the image of t generates the kernel $N(G, H)$. If $(L) \in A_i'$ with $0 \leq i \leq n-2$, then there exists subgroups $L < L^* < B < \tilde{G}$ such that $B/L \cong \mathbb{Z}_p^3$ where L^* is the distinguished element used to define t in Definition 3.3.5. In addition, regardless of our choice of B,

$$t((L)) \in B/L \uparrow \tilde{N}(B/L).$$

If $(L) - (L') \in \tilde{A}_{n-1}$, let

$$C = L + \mathbb{Z}_p e = L' + \mathbb{Z}_p e, \; D = L \cap L'.$$

We see immediately that $\tilde{G}/D \cong \mathbb{Z}_p^3$ and also that $t((L) - (L'))$ is an element of $\tilde{G}/D \uparrow \tilde{N}(\tilde{G}/D)$. Hence we conclude that

$$N(G, H) \subset \sum L/C \uparrow \tilde{N}(L/C).$$

The converse is immediate. □

Theorem 4.1.2. *Let p be any prime, G the cyclic p-group with order p^k, and $H \cong \mathbb{Z}_p$. Then f' is an isomorphism between $A(G, H)$ and $R(G, H)$.*

Proof. Let $\tilde{G} = G \times H$ and ξ be the primitive p^k-root of unity. Since G is cyclic, easily the rank of $A(G, H)$ is equal to $kp + 1$ as G has $k + 1$ subgroups and for a nontrivial subgroup $K < G$, there are p homomorphisms $\rho : K \to H$.

To determine the rank of $R(G, H)$, let $F_{\nu,\phi} = \mathbb{Q}(\xi^{p^{k-\nu}}, \xi^{\phi p^{k-1}})$ be the $\mathbb{Q}[\tilde{G}]$-module with the generators of G and H acting by multiplication by $\xi^{p^{k-\nu}}$ and $\xi^{\phi p^{k-1}}$ respectively where $\nu = 0, 1, ..., k$ and $\phi = 0, 1, ..., p - 1$. Then the irreducible $\mathbb{Q}[\tilde{G}]$-modules as seen from the decomposition of the group ring $\mathbb{Q}[\tilde{G}]$ are listed in Table 4.1 and we conclude that the rank of $R(\tilde{G})$ is $kp + 2$.

Table 4.1: Irreducible $\mathbb{Q}G$-modules

$F_{0,0}$	$=$	$\mathbb{Q}$
$F_{0,1}$	$=$	$\mathbb{Q}(\xi^{p^{k-1}})$
$F_{\nu,\phi}$	with	$\nu = 1, ..., k$ and $\phi = 0, 1, ..., p - 1$.

For $[M], [M'] \in R(\tilde{G})$, define $[M] \equiv [M']$ if we have $[M] - [M'] \in R(G, H)$. Using this relation, from [1] we immediately gain the following equivalences found in Table 4.2.

Table 4.2: Equivalence relationships for $R(G, H)$

$[F_{\nu,\phi}]$	$\equiv$	$-p^{\nu-1}[\mathbb{Q}]$	for $\nu = 1, ..., k$ and $\phi = 1, ..., p - 1$
$[F_{0,1}]$	$\equiv$	$-[\mathbb{Q}]$	
$[F_{\nu,0}]$	$\equiv$	$p^{\nu-1}(p - 1)[\mathbb{Q}]$	for $\nu = 1, ..., k$.

The equivalences from Table 4.2 imply that the rank of $R(\tilde{G})/R(G, H)$ is less than or equal to 1. In addition, the facts f' is surjective, the rank of $A(G, H)$ is $kp+1$, and the rank of $R(\tilde{G})$ is $kp+2$, imply that the rank of $R(\tilde{G})/R(G, H)$ is at least 1. Thus the rank of $R(\tilde{G})/R(G,H)$ is exactly 1 which implies the rank of $R(G,H) = kp+1$. As $A(G,H)$ is a free module, the rank of $A(G,H)$ is equal to the rank of $R(G,H)$, and f' is a surjection, we see that f' is an isomorphism.

$\square$

As a corollary, Conjecture 1 is true for G a cyclic p-group.

Chapter 5 A Bredon Homology Theory

For this chapter we let G denote an arbitrary discrete group and we return to the study of the classical Burnside ring. One can view the Burnside ring as a functor between categories in multiple ways. In particular, we discussed in Chapter 2 the effects A has on the objects of the category of groups and thus only need to construct a map $A(\phi)$ between Burnside rings. Suppose we have an H-set X and a group homomorphism ϕ between groups G and H. Using the induction maps, A can be viewed as a covariant functor by defining the image of $[X]$ to be $[G \times_\phi X]$. However the inductions do not commute with the multiplicative structure so A is a functor from the category of finite groups to that of abelian groups. A more straight forward construction is to view A as a contravariant functor from the category of finite groups in to the category of rings. This viewpoint is more natural as we immediately gain a G-action on X via $g \cdot x = \phi(g)x$ which induces a homomorphism from $A(H)$ to $A(G)$. When we discuss the Burnside functor $\mathcal{A}$, it is with this contravariant viewpoint.

Section 5.1 focuses on defining the necessary tools to construct a specific Bredon homology theory. We discuss the definition for a model of the classifying space $\underline{E}G$ where G is a finite group and construct an example of such a model. We also discuss an alternative definition in Proposition 5.1.3 and offer a proof of the equivalence. Section 5.2 focuses on constructing the specific Bredon homology groups for $G = \mathbb{Z}_2$ which we do in Theorem 5.2.2. The proof of this theorem displays the general techniques that will be helpful in future calculations.

5.1 Classifying Spaces

Definition 5.1.1. A G-CW-complex is a CW-complex with a continuous left action. Such a space is called proper if all of the point stabilizers are finite.

Definition 5.1.2. A model for $\underline{E}G$ is a proper G-CW complex X such that for any proper G-CW complex Y there is a unique G-map $Y \to X$, up to G-homotopy equivalence.

In other words, we say X is a model if it is a terminal object in the homotopy category of proper G-spaces. To show the existence of such a space we will construct a model given an arbitrary discrete group G following the method of Valette [24, page 88]. Let M be the zero dimensional G-CW complex given by the disjoint union of all left cosets G/H for $H < G$ and where H is finite. Define $M(n) = *_n M$ by the n-fold join and let $X = \cup_n M(n)$ with the obvious inclusions $M(n) \to M(n+1)$.

We must show that our constructed space is the desired terminal object, to do this we follow the argument on [14, page 6]. Let $H < G$ be finite. We want to show that

$$X^H = (\cup_n M(n))^H = \cup_n M(n)^H \simeq pt.$$

Now $M(n) = *_n M$ so $M(n)^H = (*_n M)^H = *_n(M^H)$ which is $n-2$ connected due to the fact that for any two spaces Y and Z, $Y * Z = S(Y \wedge Z)$ and to the Freudenthal Suspension theorem. So given a map $S^j \to X^H$, S^j is compact so it's image under the map is contained in some $M(n)^H$ which is j-connected for $n > j+2$. Thus for $i \leq j$, $\pi_i(X^H) = 0$ so X^H is contractible.

The following proposition is well known. However we offer a different proof as both Valette [24] and Mislin [14] refer to using obstruction theory.

Proposition 5.1.3. *A proper G-CW complex X is a model for $\underline{E}G$ if and only if every subcomplex of fixed points X^H is contractible for each finite $H < G$.*

Proof. We begin with the sufficient direction. Let X be a proper G-space such that $X^H \simeq pt$ for all finite $H < G$ and Y be a G-CW complex. Then the 0-cells of Y can be collected into orbits $G/H_i \times e_i^0$. Since X^H is contractible, for each orbit we can define an H-map $e_i^0 \mapsto e_i^{0'}$ and this map gives us a G-map on each orbit and thus on the 0-skeleton of Y.

Assume we have a G-map $Y^{n-1} \to X$ and we want to extend this map to Y^n. Collect the n-cells of Y into orbits $G/H_i \times e_i^n = G/H \times e^n$. On the boundary of e^n we have a $G - map$

$$\partial e^n \to X^H.$$

Since X^H is contractible there is a nullhomotopy $\partial e_n \times I \to X^H$ which is constant on $\partial e_n \times 1$. Therefore it factorizes through the cone:

$$C = \partial e_n \times I / \partial e_n \times 1$$

and since C is homeomorphic to e^n we gain a map $e^n \to X^H$. This map can be extended to a G-map by:

$$G/H \times e^n \to X^H \subset X.$$

Since the maps $G/H \times e^n \to X^H$ and $Y^{n-1} \to X$ agree on the boundary, we gain a G-map $Y^n \to X$ so by definition, X is a model of $\underline{E}G$. This direction of the proof in conjunction with the construction of $\cup_n M(n)$ also proves the existence of a terminal object in our category.

To prove the necessary condition suppose X is a terminal object in the homotopy category of proper G-spaces and $H < G$ is finite. We know there exists a G-map $f : G/H \to X$ since X is terminal and H acts trivially on the image of f. So $Im(f)$ is nonempty.

Consider the G-maps $G/H \times X^H \to X$ defined by:

$$\phi_1 : (gH, x) \mapsto gx$$

$$\phi_2 : (gH, x) \mapsto gx_0$$

where $x_0 \in X^H$ and the G-structure on the domain is $d(gH, x) = (dgH, x)$. Upon restriction to $H/H \times X^H$ we see that $\phi_1 = \mathbb{I}$ and $\phi_2 = c_{x_0}$. However the maps ϕ_1 and ϕ_2 are homotopic in X since up to homotopy there must be a unique G-map $G/H \times X^H \to X$. Thus upon restriction to H we see $\mathbb{I} \simeq c$ in X^H since the maps are now $H/H \times X^H \to X^H$ and thus, X^H is contractible. □

5.2 Bredon Homology

We now discuss the definition for the Bredon homology of a G-CW-complex X found in [19] using the Burnside functor to give the coefficient groups. A Bredon module $\mathcal{A}$ is a covariant functor from the orbit space of a group G into the category of abelian groups. If we define $\mathcal{A}(G/L) = A(L)$ where $A(L)$ represents the Burnside ring for the group L, then we see immediately that $\mathcal{A}$ is a Bredon module. To define a homology theory we must now specify the chain complexes and the differential maps.

For chain complexes define

$$C_d = \oplus(\mathcal{A}(G/S_\alpha) \otimes e_\alpha) = \oplus(A(S_\alpha) \otimes e_\alpha)$$

where the sum is over the stabilizer subgroups S_α of d-cell orbit representatives e_α. If ge' is a $(d-1)$-cell in the boundary of e then $S_\alpha^g \subset S'$ which defines a map ϕ from G/S_α to G/S'. As A is a contravariant functor, this induces a map $A(\phi)$ between Burnside rings yielding a differential δ_d from C_d to C_{d-1}. In particular the map agrees with the classical induction maps from the Burnside ring of a stabilizer subgroup of a d-cell $A(S_\alpha)$ to the Burnside ring of a stabilizer subgroup of a boundary cell of dimension $d-1$ $A(S_{S'})$. Then the differential map sends $m \otimes e_\alpha \in A(S_\alpha)$ to the alternating sum of $S' \uparrow m \otimes e_\alpha \in A(S')$ where the inductions are taken over the different stabilizing subgroups of the $(d-1)$-boundary cells.

Definition 5.2.1. The Bredon homology groups $H_i(X;\mathcal{A})$ are the homology groups associated to (C_*, δ_*).

Theorem 5.2.2. *Let $G = \mathbb{Z}_2$, $X = \underline{E}G$, and denote the Burnside functor by $\mathcal{A}$. Then the first two Bredon homology groups of G are:*

$$\begin{aligned} H_0(X,\mathcal{A}) &= \mathbb{Z} \\ H_1(X,\mathcal{A}) &= \mathbb{Z}^2 \end{aligned}$$

Proof. To first compute $H_0(X;\mathcal{A})$ we need the chain complexes C_0 and C_1. For the 0-cells let e_1 and e_3 be the orbit representative with stabilizer subgroups $S_1 = e$ and $S_3 = \mathbb{Z}_2$. We then see that the 1-cell representative are e_{11}, e_{12}, e_{13}, e_{31}, and e_{33} with stabilizer subgroups

$$S_{11} = S_{12} = S_{13} = S_{31} = e, \text{ and } S_{33} = \mathbb{Z}_2.$$

We define the chains by the formula

$$C_d = \oplus_\alpha \mathcal{A}(G/S_\alpha) \otimes e_\alpha.$$

Therefore using the Burnside functor we have:

$$\begin{aligned} C_0 &= \mathcal{A}(G/S_1) \otimes e_1 \oplus \mathcal{A}(G/S_3) \otimes e_3 \\ &= A(S_1) \otimes e_1 \oplus A(S_3) \otimes e_3 \\ &= \mathbb{Z}[e/e] \otimes e_1 \oplus \mathbb{Z}[\mathbb{Z}_2/e] \otimes e_3 \oplus \mathbb{Z}[\mathbb{Z}_2/\mathbb{Z}_2] \otimes e_3 \\ &= \mathbb{Z}^3 \end{aligned}$$

and

$$\begin{aligned} C_1 &= A(S_{11}) \otimes e_{11} \oplus A(S_{12}) \otimes e_{12} \oplus A(S_{13}) \otimes e_{13} \oplus A(S_{31}) \otimes e_{31} \oplus A(S_{33}) \otimes e_{33} \\ &= \mathbb{Z}[e/e] \otimes e_{11} \oplus \mathbb{Z}[e/e] \otimes e_{12} \oplus \mathbb{Z}[e/e] \otimes e_{13} \oplus \mathbb{Z}[e/e] \otimes e_{31} \oplus \mathbb{Z}[\mathbb{Z}_2/\mathbb{Z}_2] \otimes e_{33} \\ &\quad \oplus \mathbb{Z}[\mathbb{Z}_2/e] \otimes e_{33} \\ &= \mathbb{Z}^6. \end{aligned}$$

So $d_1 : \mathbb{Z}^6 \to \mathbb{Z}^3$ and we want to compute the image of d_1. We will do this by looking at the image of the generators of each component of C_1.
If $i = j = 1$ then

$$d([e/e] \otimes e_{11}) = \uparrow [e/e] \otimes e_1 - \uparrow [e/e] \otimes e_1 = 0.$$

If $i = 1$ and $j = 2$ then

$$\begin{aligned} d([e/e]) \otimes e_{12} &= S_{12} \uparrow [e/e] \otimes e_1 - S_{12} \uparrow [e/e] \otimes e_2 \\ &= S_1 \otimes_{S_{12}} ([e/e] \otimes e_1) - S_2 \otimes_{S_{12}} ([e/e] \otimes e_2) \\ &= [e/e] \otimes e_1 - [e/e] \otimes ge_1 \\ &= 0. \end{aligned}$$

If $i = 1$ and $j = 3$ then

$$\begin{aligned} d([e/e]) \otimes e_{13} &= S_{13} \uparrow [e/e] \otimes e_1 - S_{13} \uparrow [e/e] \otimes e_3 \\ &= e \otimes_e ([e/e] \otimes e_1) - \mathbb{Z}_2 \otimes_e ([e/e] \otimes e_3) \\ &= [e/e] \otimes e_1 - [\mathbb{Z}_2/e] \otimes e_3. \end{aligned}$$

If $i = 3$ and $j = 1$ then

$$\begin{aligned} d([e/e]) \otimes e_{31} &= S_{31} \uparrow [e/e] \otimes e_3 - S_{31} \uparrow [e/e] \otimes e_1 \\ &= [\mathbb{Z}_2/e] \otimes e_3 - [e/e] \otimes e_1. \end{aligned}$$

And lastly if $i = j = 3$ we for $H < G$

$$d([G/H] \otimes e_{33}) = S_{33} \uparrow [G/H] \otimes e_3 - S_{33} \uparrow [G/H] \otimes e_3 = 0.$$

So d_1 maps the generators of $A(S_{11})$, $A(S_{12})$, and $A(S_{33})$ to zero, it maps the generator of $A(S_{13})$ to $(1, -1, 0)$ and the generator of $A(S_{31})$ to $(-1, 1, 0)$. Thus we see the Bredon homology to be:

$$H_0(X, M) = \mathbb{Z}^3 / < (1, -1, 0) > \cong \mathbb{Z}^2.$$

To determine d_2 we need the 2-cells representatives and their stabilizer subgroups. The representative 2-cells will be:

$$e_{111}, e_{112}, e_{113}, e_{121}, e_{131}, e_{211}, e_{311}, e_{123}, e_{132}, e_{312}, e_{331}, e_{313}, e_{133}, e_{333}$$

with stabilizer subgroups $S_{333} = \mathbb{Z}_2$ and all other stabilizers $S_{ijk} = e$.

If $i = j = k = 3$ then

$$\begin{aligned} d(m \otimes e_{333}) &= S_{333} \uparrow m \otimes e_{33} - S_{333} \uparrow m \otimes e_{33} + S_{333} \uparrow m \otimes e_{33} \\ &= S_{333} \uparrow m \otimes e_{33} \\ &= (\mathbb{Z}_2 \otimes_{\mathbb{Z}_2} m) \otimes e_{33} \\ &= m \otimes e_{33}. \end{aligned}$$

If $i = j = 3$ and $k \neq 3$ then

$$\begin{aligned} d(m \otimes e_{33k}) &= S_{33k} \uparrow m \otimes e_{3k} - S_{33k} \uparrow m \otimes e_{3k} + S_{33k} \uparrow m \otimes e_{33} \\ &= (\mathbb{Z}_2 \otimes_e m) \otimes e_{33}. \end{aligned}$$

If $i = 3$ and j, $k \neq 3$ then

$$d(m \otimes e_{3jk}) = S_{3jk} \uparrow m \otimes e_{jk} - S_{3jk} \uparrow m \otimes e_{3k} + S_{3jk} \uparrow m \otimes e_{3j}$$

This breaks into two case. If $j = k$ then the last two terms cancel and we see:

$$d(m \otimes e_{3jk}) = m \otimes e_j k.$$

If $j \neq k$ then

$$d(m \otimes e_{3jk}) = m \otimes e_j k - m \otimes g e_{3k} + m \otimes e_{3k} = m \otimes e_{jk}.$$

Lastly if $i, j, k \neq 3$ then

$$\begin{aligned} d(m \otimes e_{ijk}) &= S_{ijk} \uparrow m \otimes e_{jk} - S_{ijk} \uparrow m \otimes e_{ik} + S_{ijk} \uparrow m \otimes e_{ij} \\ &= m \otimes e_{jk} - m \otimes e_{ik} + m \otimes e_{ij}. \end{aligned}$$

Again this breaks into two cases. If $i = j$ then the first two terms cancel and

$$d(m \otimes e_{ijk}) = m \otimes e_i k.$$

If $i = k$ then no terms cancel and

$$d(m \otimes e_{ijk}) = -m \otimes e_{ii} + 2m \otimes e_{ij}.$$

So the image of d_2 is

$$Im(d_2) = A(S_{33}) \oplus A(S_{12}) \oplus A(S_{11}) = < (a, b, 0, 0, c) >$$

and the kernel of d_1 is

$$ker(d_1) = < (a, b, d, d, c) > .$$

Thus we gain the following Bredon homology group:

$$H_1(X; \mathcal{M}) = \frac{< (a, b, d, d, c) >}{< (a, b, 0, 0, c) >} \cong \mathbb{Z}.$$

□

Appendix: Generalized Cohomology Theories and Ω Spectra

This appendix is meant to give the background information necessary to understand the geometric motivation to this problem. It will be broken down into five parts: what are generalized cohomology theories and Ω-spectra, how does a cohomology theory induce an Ω spectra, how does an Ω spectra induce a cohomology theory, and how these objects directly relate to our studies.

Generalized Cohomology Theory

A generalized cohomology theory, H^q, is a sequence of contravariant functors which associates to a pair of CW-complexes (X, Y), abelian groups while satisfying certain axioms. In addition to the axioms, there must also be a group homomorphism δ from the group $H^{q-1}(Y, \emptyset)$ to $H^q(X, Y)$; the function δ is called the coboundary operator. An original set of six axioms describing a generalized cohomology theory was worked out by Eilenberg and Steenrod [7], however the following three axioms of Hatcher [8, page 202] are equivalent.

To be precise, a contravariant functor H^q from pairs of CW-complexes to abelian groups is a generalized cohomology theory if it has a coboundary homomorphism δ from $H^n(A, pt)$ to $H^{n+1}(X, A)$ satisfying the following three axioms.

1. If f and g are homotopic maps between pairs (X, Y) and (A, B) then $f^* = g^*$.
2. If i and j are the inclusion maps, then we have a long exact sequence:
$$\cdots \to^{i^*} H^{q-1}(Y) \to^{\delta} H^q(X, Y) \to^{j^*} H^q(X) \to^{i^*} H^q(Y) \to \cdots$$
3. For a wedge sum $X = \vee_\alpha X_\alpha$ and inclusion maps i_α from X_α to X, every induced map:
$$i_\alpha^* : H^q(X) \to \prod H^n(X_\alpha)$$
is an isomorphism.

Ω Spectra

A CW-spectrum E is composed of a sequence of CW-spaces E_n and cellular maps $\epsilon_n : \sum E_n \to E_{n+1}$. Often it is said in the case where the adjoint map $\epsilon' : E_n \to \Omega E_{n+1}$ are weak homotopy equivalences, E is an Ω-spectrum. These definitions of both CW

and Ω spectra agree with those of Petrović [16]. In addition if we require the spaces E_n of an Ω-spectra to be connected we see the adjoint maps ϵ' are in fact homotopy equivalences due to Whitehead [26]. For our purposes we define an Ω-spectra to be a sequence of connected CW-spaces E_n and cellular homotopy equivalences

$$\epsilon_n : E_n \to \Omega E_{n+1}.$$

For a common example consider the Eilenberg-Mac Lane spaces $K(G,n)$ and define $E_n = K(G,n)$. Then since the only nonzero homotopy group is $\pi_k(E_n) = G$ for $k = n$, it follows $E_n \simeq \Omega E_{n+1}$ and the sequence of $K(G,n)$ forms an Ω-spectra.

Cohomology to Ω-Specrtum

If h^* is a generalized cohomology theory, then we can define what is called a reduced cohomology theory by $\widetilde{h}^*(X) = h^*(X, pt)$. Using Brown representability [5] we then have $\widetilde{h}^n(X) = [X, E_n]$ where the E_n are connected, basepointed, CW-spaces. This allows us to construct the following sequence of cohomology groups:

$$h^n(X, pt) \cong h^{n+1}(CX, X) \cong h^{n+1}(SX, C'X) \cong h^{n+1}(SX, pt).$$

The first isomorphism is a consequence of the long exact sequence of the pair (CX, X) and the homotopy invariance axiom while the second isomorphism is due to excision on the pair $(SX, C'X)$ where $C'X$ represents the lower cone of the suspension. The last isomorphism follows from the long exact sequence of the pair $(SX, C'X)$ and the fact $C'X$ is contractible. This composition of isomorphisms shows the cohomology theory is stable under suspension and

$$[X, E_n] \cong [SX, E_{n+1}] \cong [X, \Omega_0 E_{n+1}]$$

when X is connected. Since E_n form an Ω-spectrum and E_n are all connected spaces, the above composition of isomorphisms is induced by a homotopy equivalence ϵ' : $E_n \to \Omega_0 E_{n+1}$.

In the case X is not connected, the suspension SX is and so:

$$\widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(SX) = [SX, E_{n+1}] \cong [X, \Omega E_{n+1}].$$

It was shown by Milnor that ΩE_{n+1} is weakly equivalent to a CW-complex F_{n+1} so we can conclude $\widetilde{h}^n(X) \cong [X, F_{n+1}]$. In addition,

$$[X, F_n] \cong \widetilde{h}^n(X) \cong \widetilde{h}^{n+1}(SX) \cong [X, \Omega F_{n+1}]$$

which is induced by the homotopy equivalence ϵ from F_n to ΩF_{n+1}. Thus h is described by the Ω-spectrum F_n.

Ω-Specrtum to Cohomology

Suppose E_n is an Ω-spectrum and we want to demonstrate that $[X, E_n]$ satisfies the axioms for a cohomology theory. The main difficulties arises in proving the long exact sequence of the CW-pair (X, A).

A basepointed map f from X to Y defines a class in $[X, Y]$ and through composition induces a map f^* from $[Y, E_n]$ to $[X, E_n]$. Since $[Y, E_n]$ is defined on homotopy classes of maps, f^* depends only on the basepoint of Y and the homotopy class of f which shows the homotopy axiom holds.

To see the wedge axiom holds, we note that a map $f \in [\vee X_\alpha, E_n]$ is defined by component maps f_α with domain X_α. This implies $[\vee X_\alpha, E_n] \cong \prod [X_\alpha, E_n]$.

Lastly, consider the long exact sequence

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \cdots$$

If we fix a space K then we gain a sequences:

$$[A, K] \leftarrow [X, K] \leftarrow [X/A, K] \leftarrow [\Sigma A, K] \leftarrow [\Sigma X, K] \leftarrow \cdots \tag{1}$$

with the maps defined by composition. The spaces $[\Sigma^i Y, Z]$ are groups for $i > 0$ and abelian for $i > 1$ with the maps between groups all group homomorphisms.

We claim this sequence is exact since we can view $[X/A, K] \cong [X \cup CA, K]$ and see a map f from X to K goes to 0 if and only if it extends to $X \cup CA$. Thus if we replace K with the space E_n we can extend sequence 1 to the left to form the following long exact sequence.

$$[A, E_{n+1}] \leftarrow [X, E_{n+1}] \leftarrow [X/A, E_{n+1}] \leftarrow [A, E_n] \leftarrow [X, E_n] \leftarrow \cdots$$

Thus we gain a long exact sequence of abelian groups and homomorphisms with the naturality of a map $(X, A) \to (Y, B)$ coming from the naturality of a cofibration.

Connections to the Burnside Kernel

The Stable Transfer Map

Given a CW-complex X, define a spectrum with the nth space equal to $S^n X$ for $n \geq 0$ with the obvious maps ϕ_n from SE_n to E_{n+1}. For $n \leq 0$ we take E_n to equal a point. Then the maps ϕ_n are homotopy equivalences for all n and we say such a spectrum is the suspension spectrum of X. We denote such a spectra by $\Sigma^\infty X$. In particular, the sphere spectrum S has as its nth term the sphere S^n. Given a CW-complex X, the stable cohomotopy groups of X are defined by:

$$\pi_s^i(X) = \lim_{\rightarrow} < X, S^{n+i} >=< X, \Omega^n S^{n+i} > .$$

In general these groups are difficult to compute. However if X is the 1-sphere then $\pi_s^i(S^1) = \pi_{n+1}^s(S^{n+i})$. Thus in the case $i = 1$, we have $\pi_s^1(X)$ is in fact the stable homotopy group $\pi_i^s(S^i) = \mathbb{Z}$. Also if $i > 1$, then $\pi_s^i(X) = 0$.

Given any finite G-set T, one can form the covering space:

$$T \to T \times_G EG \to BG.$$

For any cell σ in BG we can associate the disjoint union of cells, $\sum_i \sigma_i$, that map to σ under the projection. This induces the transfer map τ from $H_*(BG)$ to $H_*(T \times_G EG)$ by taking a homology class $[z]$ to the element $\sum[z_i]$. We want to form a similar map between $\widetilde{H}_*(\Sigma^\infty BG)$ and $\widetilde{H}_*(\Sigma^\infty(T \times_G EG))$. By Mayer-Vietoris and the definition of the suspension spectrum, we see $\widetilde{H}_*(BG) \cong \widetilde{H}_*(\Sigma^\infty BG)$ and $\widetilde{H}_*(T \times_G EG) \cong \widetilde{H}_*(\Sigma^\infty(T \times_G EG))$. Thus we now have a map from $\widetilde{H}_*(\Sigma^\infty BG)$ to $\widetilde{H}_*(\Sigma^\infty(T \times_G EG))$ which is induced by a map at the level of spectra.

Thus given our finite G-set T, we have the composition of maps:

$$\Sigma^\infty BG \to \Sigma^\infty(T \times_G EG) \to \Sigma^\infty point$$

which defines a map $\alpha(T)$ from BG to $\Omega^\infty\Sigma^\infty$. Thus we in fact have a homomorphism

$$\alpha : A(G) \to [BG, \Omega^\infty\Sigma^\infty] = \pi_s^0(BG).$$

The map α extends to an isomorphism

$$\hat{\alpha} : \hat{A}(G) \to \pi_s^0(BG)$$

where $\hat{A}(G)$ is the completion of the Burnside ring with respect to the IG-adic topology. This isomorphism is known as the Segal Conjecture and was proven by Lewis, McClure, and May [11].

Complex K-theory

Let X be a topological space. A family of vector spaces over X is a triple (V, π, X) often denoted by just V, where V is a topological space and π is a continuous surjection from V to X where $\pi^{-1}(x) = V_x$ is a vector space for all $x \in X$. If a family has the property that for each $x \in X$ there is an open neighborhood $U \subset X$ containing

X such that $V|U \cong U \times \mathbb{C}^n$ for some n, then we call the family V a vector bundle. A homomorphism between vector bundles V and W is a map γ that restricts to a linear transformation between all vector spaces $V_x \to W_x$. The Grothendieck completion of the set of isomorphism classes of vector bundles over a compact Hausdorff, topological space X is denoted $K^0(X)$ and the assignment from X to $K^0(X)$ is a contravariant functor. The ring structure on $K^0(X)$ is given by internal Whitney sum and tensor products. Another K group, K^1 can be defined as the kernel of the homomorphism from $K^0(X \times S^1)$ to $K^0(X)$ induced by the inclusion map. These constructions of the rings K^0 and K^1 agree those of Park [15] and the higher K-groups are defined via the Bott periodicity theorem.

The analogous definition of Atiyah [2] for complex K-theory quickly defines functors K^n for any natural number n. For a compact Hausdorff X and $n \geq 1$, define $K^{-n}(X)$ to be the set of homotopy classes of maps $[X, \Omega^n BU]$. This implies that K is in fact a homotopy invariant functor. Given a map $f : X \to Y$, we gain a map $K^n(f)$ from $K^n(Y)$ to $K^n(X)$ via composition. In addition we can define the relative groups $K^{-n}(X,Y)$ to be $\widetilde{K}^{-n}(X/Y) = \widetilde{K}^0(S^n(X/Y))$ where $\widetilde{K}(Z)$ is the kernel of the map from $K(Z)$ to $K(z_0)$ induced by the inclusion. Finally there is an exact sequence, of the form:

$$\cdots K^{-2}(Y) \to K^{-1}(X,Y) \to K^{-1}(X) \to K^{-1}(Y) \to K^0(X,Y) \to K^0(X) \to K^0(Y).$$

Thus it is easy to check that the sequence K^n forms a generalized cohomology theory as described in Section 5.2. In addition, Bott periodicity states that $\Omega^2 U \sim U$ and so $K^n(X) \cong K^{n+2}(X)$ for all integers n which takes our long exact sequence and turns it into a commutative square. In the case X is locally compact, we easily gain identical results by taking the compactification of X and defining $K(X) = \widetilde{K}(X^+)$.

In the case we are working with a G-space X, we say X is free if every element x has a trivial stabilizer subgroup. When X is a free G-space we can form the space X^* of all pairs (x, sx) and the function τ from X^* to G such that $\tau(x, x')x = x'$ is called the translation function. A G-space X is called principal if the action is free and the translation function is continuous. A principal G-bundle is a G-bundle (X, π, B) with X a principal G-space.

In particular let G be a topological group, the Milnor construction yields the classifying space $BG = (G * G * \cdots)/G$. Then we gain a principal G-bundle $\xi = (EG, \pi, BG)$ with fibers equal to G. In addition, given a G-module M over $\mathbb{C}$ we can define a vector bundle $M \times_G \xi = (M \times_G EG, \pi, BG)$ with fiber M. This induces a map β between the representation ring of complex G-modules and the ring of complex vector bundles with the inverse limit topology, $\mathfrak{K}^0(BG)$.

In the case G is a compact Lie group, the map β extends to an isomorphism

$$\hat{\beta} : \hat{R_{\mathbb{C}}}(G) \to \mathfrak{K}^0(BG)$$

where $\hat{R_{\mathbb{C}}}(G)$ is the completion of the complex representation ring with respect to the IG-adic topology. The fact that β extends to an isomorphism was originally proven by Atiyah in [4] to be true for finite groups G and later for all compact Lie groups by Atiyah and Segal [3].

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