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RATIONAL APPROXIMATION ON COMPACT NOWHERE DENSE SETS

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RATIONAL APPROXIMATION ON COMPACT NOWHERE DENSE SETS

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
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By
Christopher Mattingly
Lexington, Kentucky

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ABSTRACT OF DISSERTATION

RATIONAL APPROXIMATION ON COMPACT NOWHERE DENSE SETS

For a compact, nowhere dense set X in the complex plane, $\mathbb{C}$, define $R^p(X)$ as the closure of the rational functions with poles off X in $L^p(X, dA)$. It is well known that for $1 \leq p < 2$, $R^p(X) = L^p(X)$. Although density may not be achieved for $p > 2$, there exists a set X so that $R^p(X) = L^p(X)$ for p up to a given number greater than 2 but not after. Additionally, when $p > 2$ we shall establish that the support of the annihiliating and representing measures for $R^p(X)$ lies almost everywhere on the set of bounded point evaluations of X.

KEYWORDS: uniform and L^p rational approximation, $q-$capacity, bounded point evaluations, representing measures.

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Date: April 25, 2012

RATIONAL APPROXIMATION ON COMPACT NOWHERE DENSE SETS

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Director of Dissertation: James E. Brennan

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Date: April 25, 2012

Dedicated to my family - my parents, brothers, sister and grandparents. Although you may not have understood why I've been in school this long, you have always supported, loved, and never doubted me.

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Chapter 1 Introduction

Let X be a compact set in the complex plane $\mathbb{C}$, and let $|X|$ stand for the area (that is dA or two-dimensional Lebesgue measure). Define $C(X)$ to be the space of all continuous functions on X endowed with the uniform norm, and let $R(X)$ be the closure in $C(X)$ of the rational functions with poles off X. It is an old problem to determine conditions on X so that $R(X) = C(X)$. An obvious necessary condition is that X have no interior, and so we shall adopt this hypothesis as a standing assumption.

In 1931, Hartogs and Rosenthal [17] proved that $R(X) = C(X)$ whenever $|X| = 0$, leaving open the question as to whether the rational functions are dense in $C(X)$ for every compact nowhere dense set X. By the end of the decade, Alice Roth [31] (cf. [8], [14], [16]) settled that question by constructing an example of a compact nowhere dense set X so that $R(X) \neq C(X)$, the so-called Swiss cheese. However, it wasn't until 1958 that Vitushkin (cf. [39]) established necessary and sufficient conditions for $R(X) = C(X)$ in terms of analytic capacity.

In the 1960's, more interest developed in a different aspect of rational approximation. For $p \geq 1$, let $L^p(X, dA)$ (or more generally $L^p(X)$) be the usual space of functions on X which are $p-$integrable with respect to the area measure dA. Then $R^p(X, dA)$ (or more generally $R^p(X)$) is the closure in the $L^p(dA)$ norm of the rational functions with poles off X. It is well known that if $1 \leq p < 2$, then $R^p(X) = L^p(X)$ (see Section 2.2).

Because the uniform norm is more restrictive than the L^p norm, it is clear that $R(X) \subset R^p(X)$. As a result of this containment property of the spaces, it follows easily that if $R(X) = C(X)$ then $R^p(X) = L^p(X)$ for all $p \geq 1$. Again, the questions arise as to what conditions are necessary in order that $R^p(X) = L^p(X)$, and is it possible that $R^p(X) = L^p(X)$ for some $p \geq 1$ without having $R(X) = C(X)$.

With regard to the latter question posed above, Sinanjan [33] (cf. [3], [6]) constructed a Swiss cheese to show that there exists a compact nowhere dense set X so that $R(X) \neq C(X)$, but nevertheless $R^p(X) = L^p(X)$ for all p, $1 \leq p < \infty$. So it is possible to have density in $L^p(X)$ without having density in $C(X)$. This paper seeks to build on those ideas and to show that it is possible to have density up to a certain point, but not beyond - more precisely, in certain instances it can happen that $R^p(X) = L^p(X)$ if $1 \leq p < p^*$ but not if $p \geq p^*$.

There is an obvious obstruction to the possibility that $R^p(X) = L^p(X)$. There may exist a point x_0 with the property that

$$|f(x_0)| \leq C\|f\|_{L^p(X)}$$

for every rational function f with poles off X and some fixed constant C. Such a point x_0 is referred to as a *bounded point evaluation* (or *bpe*) for $R^p(X)$. In that case, the map $f \to f(x_0)$ extends from $R(X)$ to a bounded linear functional on $R^p(X)$, and the Hahn-Banach theorem guarantees the existence of a function $k \in L^q(X)$, where $1/p + 1/q = 1$, such that

$$f(x_0) = \int_X fk\ dA$$

for all $f \in R(X)$. Thus, $(z - x_0)k(z)\ dA$ is a nontrivial annihilating measure for $R^p(X)$ and therefore $R^p(X) \neq L^p(X)$. In this way, when $p \geq 2$ it is possible to construct a compact nowhere dense set X such that $R^p(X) \neq L^p(X)$, but there will always be density when $p < 2$ (cf. [3]). In order that $R^p(X) = L^p(X)$ it is both necessary and sufficient that $R^p(X)$ have no bpe's if $p > 2$, but as Fernström [11] has shown this is not sufficient if $p = 2$. Hedberg [19] obtained a necessary and sufficient condition in terms of $q-$capacity for a point $x_0 \in X$ to be a bpe for $R^p(X)$ whenever $p > 2$. Later, this was extended to cover the case $p = 2$ by Fernström and Polking [13]. For a more extensive discussion of the history of these problems, see [27].

In Chapter 5 it will be shown that the annihilating measures and the representing measures for $R^p(X)$ where $p > 2$ are supported almost everywhere on the set of bpe's, thereby extending an earlier result of Øksendal [44] to the L^p case. The situation is much different when $p = 2$ by virtue of Fernström's example.

Chapter 2 Early Results on Rational Approximation

2.1 The Cauchy transform

In order to deal with questions of density, it will be convenient to argue by duality. If, for example, we wish to prove that $R(X) = C(X)$ it is enough to show that if μ is any measure of finite total variation supported on X and if $\mu \perp R(X)$ in the sense that

$$\int_X f \, d\mu = 0$$

for all $f \in R(X)$, then $\mu = 0$ as a measure. That will be the inescapable conclusion whenever it can be shown that the *Cauchy transform*

$$\widehat{\mu}(z) = \int \frac{d\mu_\zeta}{\zeta - z}$$

vanishes a.e.$-dA$. Similar remarks are valid for approximation in $L^p(X)$ with $d\mu = k \, dA$ and $k \in L^q(X)$ where $1/p + 1/q = 1$.

It is important to note at the outset that the Cauchy integral $\widehat{\mu}(z)$ exists and is finite a.e.$-dA$ in the plane. In fact the *Newtonian potential*

$$\widetilde{\mu}(z) = \int_X \frac{d|\mu_\zeta|}{|\zeta - z|}$$

is finite a.e.$-dA$, from which the assertion follows. To see this, choose $R > 0$ sufficiently large so that for any $\zeta \in X$, the disk B_R with center ζ and radius R contains X in its interior. Evidently,

$$\int_{B_R} \widetilde{\mu} \, dA_z = \int_{B_R} \int_X \frac{d|\mu_\zeta|}{|\zeta - z|} \, dA_z = \int_X \int_{B_R} \frac{dA_z}{|\zeta - z|} \, d|\mu_\zeta| \leq 2\pi R \, |\mu|(X) < \infty,$$

where $|\mu|$ denotes the total variation of μ. Therefore $\widetilde{\mu} < \infty$ a.e.$-dA$ on X, and since $\widetilde{\mu}$ is also finite off X, the integral $\widetilde{\mu}(z) < \infty$ a.e.$-dA$.

Theorem 2.1. *Let μ be a measure of finite total variation on X. If $\widehat{\mu} = 0$ a.e.$-dA$ on $\mathbb{C}$, then $\mu = 0$ as a measure.*

Proof. The proof presented here is due to Beurling (cf. [40, p. 75]) and applies Fubini's theorem to the Cauchy integral over any rectangle R in $\mathbb{C}$ where $|\mu| = 0$ on ∂R:

$$\int_{\partial R} \widehat{\mu}\, dz = \int_{\partial R} \int_X \frac{d\mu_\zeta}{\zeta - z} dz = \int_X \int_{\partial R} \frac{dz}{\zeta - z} d\mu_\zeta = 0$$

However, by Cauchy's theorem

$$\frac{1}{2\pi i} \int_{\partial R} \frac{dz}{z - \zeta} = \chi_R(\zeta)$$

and so

$$\frac{-1}{2\pi i} \int_X \int_{\partial R} \frac{dz}{\zeta - z} d\mu_\zeta = \int_X \chi_R(\zeta) d\mu_\zeta = \mu(R \cap X) = 0.$$

This can be done with enough rectangles to show that $\mu = 0$ as a measure: Suppose E is any compact subset of X and let U be any neighborhood of E. Cover E with rectangles $\{R_i\}$ so that μ places no mass on ∂R_i and $\cup R_i \subset U$. Then $|\mu(\cup R_i)| \leq \sum |\mu(R_i)| = 0$, since $\mu(R_i) = 0$ for each i. Hence $\mu(E) = \lim_{U \downarrow E} \mu(\cup R_i) = 0$. □

As a corollary, let us recall a theorem of Hartogs and Rosenthal [17] from 1931 in which we can illustrate the use of the Cauchy transform.

Corollary 2.2 (Hartogs and Rosenthal)**.** *If $|X| = 0$, then $R(X) = C(X)$.*

Proof. Let μ be a measure on X with $\mu \perp R(X)$. It follows that

$$\widehat{\mu}(z) = \int_X \frac{d\mu_\zeta}{\zeta - z} = 0$$

whenever $z \in \mathbb{C} \setminus X$, and so $\widehat{\mu} = 0$ a.e.$-dA$. Hence $\mu = 0$ as a measure by Theorem 2.1. Thus μ not only annihilates the rational functions, but all continuous functions as well, and so $R(X) = C(X)$. □

2.2 L^p approximation

In order to study approximation in $L^p(X)$ we can argue along lines similar to those outlined above. In this case, let $k \in L^q(X)$ where $1/p + 1/q = 1$, and assume that $\int_X fk \, dA = 0$ for all $f \in R(X)$. Hence,

$$\widehat{k}(z) = \int_X \frac{k(\zeta)}{\zeta - z} \, dA_\zeta = 0$$

whenever $z \in \mathbb{C} \setminus X$. Our problem is to determine whether $\widehat{k}$ enjoys sufficient continuity at points of X to ensure that $\widehat{k} = 0$ a.e.$-dA$ on X. If so we can conclude that $R^p(X) = L^p(X)$.

Theorem 2.3. *If $1 \leq p < 2$ then $R^p(X) = L^p(X)$ for any compact nowhere dense set X.*

Here the theorem is a consequence of the fact that $\widehat{k}$ is a continuous function whenever $k \in L^q(X)$ for $q > 2$. That in turn follows easily from the fact that translation is a continuous operator on L^q (cf. [32, p. 3]). A more precise description of the degree of continuity enjoyed by $\widehat{k}$ is contained in the following:

Lemma 2.4. *If $k \in L^q(X)$ for $q > 2$, then $|\widehat{k}(z_1) - \widehat{k}(z_2)| \leq C|z_1 - z_2|^{1-2/q}$.*

Proof of lemma. Let $k \in L^q(X)$ for $q > 2$ and let x_1, x_2 be any pair of points in the plane. Then

$$\left|\widehat{k}(x_1) - \widehat{k}(x_2)\right| \leq |x_1 - x_2| \int \frac{|k(z)|}{|z - x_1||z - x_2|} \, dA.$$

Define $R = \frac{1}{2}|x_1 - x_2|$, and let D_1 and D_2 be the disks of radius R centered at x_1 and x_2 respectively. We will proceed in two parts: first by considering z in either D_1 or D_2, and then by considering z outside $D = D_1 \cup D_2$.

Case 1: Without loss of generality, assume $z \in D_1$. We have $|z - x_2| \geq \frac{1}{2}|x_1 - x_2|$ on D_1, and so

$$\begin{aligned} |x_1 - x_2| \int_{D_1} \frac{|k(z)|}{|z - x_1||z - x_2|}\, dA &\leq |x_1 - x_2| \int_{D_1} \frac{2|k(z)|}{|z - x_1||x_1 - x_2|}\, dA \\ &\leq 2\|k\|_q \left(\int_{D_1} \frac{1}{|z - x_1|^p}\, dA \right)^{1/p}. \end{aligned}$$

Using polar coordinates centered at x_1 inside the parenthesis,

$$\int_{D_1} \frac{1}{|z - x_1|^p}\, dA = \int_0^{2\pi} \int_0^R \frac{1}{r^p}\, r\, dr\, d\theta = \frac{2\pi}{2 - p}\, R^{2-p}.$$

Recall that $1/p = 1 - 1/q$ and $R = \frac{1}{2}|x_1 - x_2|$ which gives

$$|x_1 - x_2| \int_{D_1} \frac{|k(z)|}{|z - x_1||z - x_2|}\, dA \leq C|x_1 - x_2|^{1-2/q},$$

where C is a constant that depends only on q. Similar reasoning gives the same bound for the contribution from integrating over D_2.

Case 2: Consider what happens when $z \notin D$. Since $ab \leq \frac{1}{2}(a^2 + b^2)$ for all real numbers a, b, we have:

$$|x_1 - x_2| \int_{\mathbb{C}\setminus D} \frac{|k(z)|}{|z - x_1||z - x_2|}\, dA \leq |x_1 - x_2| \int_{\mathbb{C}\setminus D} \left(\frac{|k(z)|}{|z - x_1|^2} + \frac{|k(z)|}{|z - x_2|^2} \right) dA$$

For the first term, we estimate that

$$\int_{\mathbb{C}\setminus D} \frac{|k(z)|}{|z - x_1|^2}\, dA \leq \int_{\mathbb{C}\setminus D_1} \frac{|k(z)|}{|z - x_1|^2}\, dA \leq \|k\|_q \left(\int_{\mathbb{C}\setminus D_1} \frac{1}{|z - x_1|^{2p}}\, dA \right)^{1/p}.$$

Again, we can use polar coordinates to estimate the integral inside the parenthesis yielding

$$\int_{\mathbb{C}\setminus D_1} \frac{1}{|z - x_1|^{2p}}\, dA = \int_0^{2\pi} \int_R^{\infty} \frac{1}{r^{2p}}\, r\, dr = \frac{2\pi}{2p - 2}\, R^{2-2p}.$$

Using similar reasoning for the second integral, we obtain

$$|x_1 - x_2| \int_{\mathbb{C}\setminus D} \frac{|k(z)|}{|z - x_1||z - x_2|}\, dA \leq 2R(CR^{2/p-2}) = C|x_1 - x_2|^{1-2/q},$$

where C is a constant that depends only on q.

Combining the two cases gives a bound for the integral over the entire plane, and completes the proof of the lemma. $\square$

If we are to construct a compact nowhere dense set X so that $R^p(X) \neq L^p(X)$, we must first ensure that $R(X) \neq C(X)$. Consider, therefore, a set X obtained by removing countably many disjoint open disks D_j from the closed unit disk $\overline{D}$ in such a way that:

1. $\overline{D_j} \subset \text{int}(\overline{D})$ for each $j = 1, 2, \ldots$
2. $\overline{D_j} \bigcap \overline{D_k} = \emptyset$ when $j \neq k$
3. $X = \overline{D} \setminus (\bigcup_{j=1}^{\infty} D_j)$ has no interior.
4. $\sum_j r_j < \infty$ where r_j is the radius of D_j.

Such a set X is now known as a *Swiss cheese*. It was first employed by Alice Roth [31] in 1938 to produce a compact nowhere dense set with $R(X) \neq C(X)$, and rediscovered by Mergelyan [28] in a similar context more than a decade later.

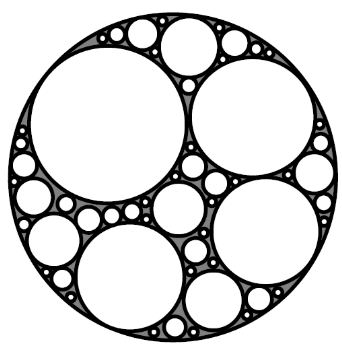

Figure 2.1: Alice Roth's Swiss cheese

By construction, X is compact and has no interior. Setting $d\mu = dz$ on ∂D, and $d\mu = -dz$ on ∂D_j for each j, then for any rational function f,

$$\int_{\partial X} f \; d\mu = 0$$

and so $d\mu$ is a non-zero annihilating measure for $R(X)$ and $R(X) \neq C(X)$. By virtue of the Hartogs-Rosenthal theorem, it follows that $|X| > 0$. Therefore, $R^p(X)$ is nontrivial for each $p < \infty$ and we can ask if $R^p(X) = L^p(X)$. We know the answer if $p < 2$, but if $p \geq 2$ the problem is more subtle. In order to prove in any instance that $R^p(X) = L^p(X)$, we must show that if $k \in L^q(X)$ and $k \perp R^p(X)$ then

$$\widehat{k}(z) = \int_X \frac{k(\zeta)}{\zeta - z} \, dA_\zeta = 0$$

a.e.$-dA$ in $\mathbb{C}$. On the other hand, $\widehat{k} \equiv 0$ in $\mathbb{C} \setminus X$ is clear and if $p \geq 2$ (or equivalently $q \leq 2$), we must determine whether $\widehat{k}$ retains sufficient residual continuity to ensure that $\widehat{k} = 0$ a.e.$-dA$ on X.

Chapter 3 Sobolev Spaces and Capacity

3.1 Sobolev Spaces

Throughout this chapter, X will be a compact nowhere dense subset of $\mathbb{C}$. And, unless otherwise stated, p and q will denote conjugate indices, that is $1/p + 1/q = 1$. The differential operators ∂ and $\overline{\partial}$ are defined as follows:

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

The residual continuity that we seek for $\widehat{k}$ can be found in the inherent properties of Sobolev spaces. The *Sobolev space* W_1^q is defined as the space of functions in L^q whose first-order real partial derivatives are also in L^q. We will present some background which leads to the importance of Sobolev spaces in this investigation.

The following generalized Cauchy formula apparently first appeared in the work of Pompeiu in 1912 and 1913 (cf. [30]). It seems to have then lied relatively dormant until it reappeared in the 1950's in the work of Mergelyan and Vitushkin on approximation in the plane by analytic functions, and in the work of Dolbeault and Grothendieck in several variables.

Lemma 3.1. *Let $\Omega \subset \mathbb{C}$ be a region bounded by finitely many smooth curves, and let g be a continuously differentiable function defined in a neighborhood of $\overline{\Omega}$. Then, for every $z \in \Omega$:*

$$g(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{g(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{\pi}\int_{\Omega} \frac{\overline{\partial} g(\zeta)}{\zeta - z}\, dA_{\zeta}.$$

Proof. Fix a point $z \in \Omega$ and choose $\varepsilon > 0$ so that the disk $D_\varepsilon = \{\zeta : |\zeta - z| \le \varepsilon\}$ is contained in Ω. Let $\Omega_\varepsilon = \Omega \setminus D_\varepsilon$ so that $\dfrac{g(\zeta)}{\zeta - z}\, d\zeta$ is a smooth $1-$form on $\overline{\Omega}_\varepsilon$. By the

Gauss-Green theorem

$$\frac{1}{2i}\int_{\partial\Omega_\varepsilon}\frac{g(\zeta)}{\zeta - z}\,d\zeta = \int_{\Omega_\varepsilon}\overline{\partial}\left(\frac{g(\zeta)}{\zeta - z}\right)\,dA_\zeta = \int_{\Omega_\varepsilon}\frac{\overline{\partial}g(\zeta)}{\zeta - z}\,dA_\zeta.$$

Letting $\varepsilon \to 0$, we conclude by dominated convergence that

$$\int_{\Omega_\varepsilon}\frac{\overline{\partial}g(\zeta)}{\zeta - z}\,dA_\zeta \to \int_{\Omega}\frac{\overline{\partial}g(\zeta)}{\zeta - z}\,dA_\zeta,$$

since for a suitable constant M

$$\left|\frac{\overline{\partial}g}{\zeta - z}\right| \leq \frac{M}{|\zeta - z|} \in L^1(\Omega, dA).$$

Setting $\zeta = z + \varepsilon e^{i\theta}$ for $0 \leq \theta \leq 2\pi$ on the portion of $\partial\Omega_\varepsilon$ corresponding to the circle $|\zeta - z| = \varepsilon$, we get

$$\int_{|\zeta - z| = \varepsilon}\frac{g(z)}{\zeta - z}\,d\zeta = i\int_0^{2\pi} g(z + \varepsilon e^{i\theta})\,d\theta \to 2\pi i g(z)$$

as $\varepsilon \to 0$, from which the lemma follows by collecting terms. □

As a consequence of the lemma, we easily obtain a representation formula for functions of compact support which is particularly useful.

Corollary 3.2. *If φ is a continuously differentiable function of compact support in $\mathbb{C}$, then*

$$\varphi(z) = -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\overline{\partial}\varphi(\zeta)}{\zeta - z}\,dA_\zeta$$

for all $z \in \mathbb{C}$.

To prove the corollary, one only has to apply Lemma 3.1 to φ on a large disk.

Suppose that $k \in L^1(X)$ and extend k to the entire plane by setting $k = 0$ in $\mathbb{C} \setminus X$. According to Corollary 3.2, if φ is any continously differentiable function of compact support, then

$$\begin{aligned}
\int_{\mathbb{C}}\widehat{k}(z)\overline{\partial}\varphi(z)\,dA_z &= \int_{\mathbb{C}}\left(\int\frac{k(\zeta)}{\zeta - z}\,dA_\zeta\right)\overline{\partial}\varphi(z)\,dA_z \\
&= \int\left(\int_{\mathbb{C}}\frac{\overline{\partial}\varphi(z)}{\zeta - z}\,dA_z\right)k(\zeta)\,dA_\zeta \\
&= \int -\pi\varphi(\zeta)k(\zeta)\,dA_\zeta
\end{aligned}$$

And so, $\overline{\partial}\widehat{k} = -\pi k$ as a distribution. Assuming further that $k \in L^q(X)$ for $q > 1$, and that $k \perp R^p(X)$, then $\widehat{k}$ has compact support and it follows from the Calderon-Zygmund theorem on the continuity of singular integral operators that $\partial\widehat{k}$ also exists as a distribution and is in $L^q(X)$ (cf. [7]; [34, p. 35]; and [37, p. 72, Thm 1.36]). As a result, the real partial derivatives of $\widehat{k}$ exist as distributions and

$$\|\nabla\widehat{k}\|_q \le C\|\overline{\partial}\widehat{k}\|_q = C\pi\|k\|_q$$

provided $q > 1$ and $k \perp R^p(X)$. And therefore $\widehat{k}$ belongs to W_1^q.

In Lemma 2.4, it was shown that if $k \in L^q(X)$ for $q > 2$ then $\widehat{k}$ is Hölder continuous. In fact for $q > 2$, every element $f \in W_1^q$ admits a precise Hölder continuous representative with exponent $1 - 2/q$ (cf. [43, p. 61]). On the other hand, $\widehat{k} \in L^q$ for any $q \ge 1$ and is therefore approximately continuous a.e.$-dA$. That is, for a.e.$-dA$ point $x_0 \in X$ there exists an exceptional set E with the property that

$$\frac{|B_r(x_0) \cap E|}{|B_r(x_0)|} \to 0$$

as $r \to 0$ and so that

$$f(x_0) = \lim_{z \to x_0,\ z \notin E} f(z)$$

(cf. [10]). Here, $B_r(x_0)$ denotes the disk with center at x_0 and radius r. However, we need a finer measure of the continuity enjoyed by $\widehat{k}$ when $q \le 2$, and that continuity is best described in terms of capacity. There will be no loss in generality if we assume that $q < 2$.

3.2 Sobolev and Potential Theoretic Capacities

For $1 < q < 2$, define the Sobolev *q-capacity* of a compact set $X \subset \mathbb{C}$ by

$$\Gamma_q(X) = \inf \int |\nabla u|^q \, dA,$$

where the infimum is taken over all infinitely differentiable functions u of compact support with $u \equiv 1$ on X. For an arbitrary set E, define

$$\Gamma_q(E) = \sup \Gamma_q(X),$$

where the supremum is taken over all compact sets $X \subset E$. All Borel sets are capacitable in the sense that it is also true that

$$\Gamma_q(E) = \inf \Gamma_q(G),$$

where the infimum is taken over all open sets $G \supset E$. We say that a property holds $q-$*quasieverywhere* if it holds everywhere except on a set of $q-$capacity zero.

It is often useful to have a different, but equivalent, definition of capacity. The potential theoretic q-capacity of a Borel set E is defined by

$$C_q(E)^{1/q} = \sup_{\nu} \nu(E),$$

where the supremum is taken over all positive measures ν concentrated on E for which $\|\widetilde{\nu}\|_p \leq 1$.

These two capacities are equivalent in that there exists a constant $K > 0$ so that

$$K^{-1}\Gamma_q(E) \leq C_q(E) \leq K\Gamma_q(E)$$

for every E. This (and similar equivalences) will be denoted by writing $C_q \approx \Gamma_q$. More information on these capacities, as well of proofs of the following can be found in the books [1] and [21] (cf. also [4], [5], [18], [20]):

1. if Φ is a contraction, $C_q(\Phi E) \leq KC_q(E)$ where K is a constant depending only on q [1, p. 140]

2. $C_q(B_r) \approx C_q(\text{diam } B_r) \approx r^{2-q}$ for any disk B_r of radius r and $1 < q < 2$

3. C_q is countably subadditive

For any $\lambda > 0$, we have a weak-type inequality similar to Tchebyschev's inequality for L^1 functions:

$$\Gamma_q\{z \in \mathbb{C} : |\widehat{k}(z)| > \lambda\} \leq \frac{1}{\lambda^q} \int |\nabla \widehat{k}|^q \, dA$$

and this is key to obtaining the substitute for approximate continuity promised above. If $\widehat{k}_j = \widehat{k} * \chi_j$ is a sequence of mollifiers obtained by convolving $\widehat{k}$ with a C^∞ approximate identity χ_j, $j = 1, 2, 3, \ldots$, it is well-known that

$$\|\widehat{k}_j - \widehat{k}\|_q \to 0 \text{ and } \|\nabla \widehat{k}_j - \nabla \widehat{k}\|_q \to 0.$$

Passing to a subsequence if necessary, we can arrange that $\widehat{k}_j \to \widehat{k}$ uniformly off open sets of arbitrarily small q-capacity (cf. [9, p. 354] and [42, p. 124]). Hence, given any $\varepsilon > 0$ there exists an open set U so that $\Gamma_q(U) < \varepsilon$ and $\widehat{k}$ is continuous in the complement of U. Functions with this property are said to be $q-$*quasicontinuous*. Every W_1^q function agrees a.e.$-dA$ with a quasicontinuous representative. If $q > 2$, then $\widehat{k}$ is actually continuous as we have seen.

In addition to quasicontinuity, there is a pointwise notion more closely resembling approximate continuity which is also enjoyed by W_1^q functions, called *fine continuity*. A function h that is defined $q-$q.e. is said to be $q-$*finely continuous* at x_0 if there exists a set E that is thin in a potential theoretic sense at x_0 and

$$\lim_{z \to x_0,\ z \notin E} h(z) = h(x_0).$$

The precise sense in which E is understood to be thin is this: If $1 < q < 2$ a set E is $q-$*thin* at x_0 if and only if

$$\int_0 \left(\frac{\Gamma_q(E \cap B_r(z_0))}{r^{2-q}} \right)^{p-1} \frac{dr}{r} < \infty.$$

If E is not thin at x_0, then it is said to be *thick* there. It can be shown that every $q-$quasicontinuous function is $q-$finely continuous $q-$q.e. (cf. [1, p. 177]). Because C_q is countably subadditive ([1, p. 126]) and $\Gamma_q \approx C_q$, it follows that E is thick at x_0

whenever

$$\limsup_{r\to 0} \frac{\Gamma_q(E\cap B_r(x_0))}{r^{2-q}} > 0,$$

which is more in line with the aforementioned condition describing approximate continuity.

3.3 Analytic Capacity

Sobolev and potential theoretic $q-$capacities are set functions designed to measure the size of the exceptional sets associated with functions in the Sobolev space W_1^q, and therefore to measure the size of those associated with the Cauchy integral $\widehat{k}$ for a function $k \in L^q$. For this reason, $q-$capacity is especially useful in studying questions of approximation in the $L^p(dA)$ norm. However, in order to be able to present an accurate picture of the differences between the results in [6] and our work in Chapter 4 we need to have a corresponding understanding of the exceptional sets for the Cauchy integral $\widehat{\mu}$ of an arbitrary measure μ. And for this, we need to consider analytic capacity, a concept introduced by Ahlfors in 1947.

The *analytic capacity* of a compact set X, denoted $\gamma(X)$ is defined as

$$\gamma(X) = \sup |f'(\infty)|,$$

where the supremum is taken over all functions f analytic in $\widehat{\mathbb{C}} \setminus X$, where $\|f\|_\infty = \sup_{\widehat{\mathbb{C}}\setminus X} |f| \le 1$ and $f(\infty) = 0$. For a general set E, we define $\gamma(E) = \sup \gamma(X)$ where this supremum is taken over all compact sets $X \subset E$. There is, however, an equivalent capacity γ^+ which is more directly linked to the Cauchy integral. For a compact set X, let

$$\gamma^+(X) = \sup_\nu \nu(X),$$

where the supremum is over all positive measures ν supported on X so that $\widehat{\nu} \in L^\infty(\mathbb{C})$ and $\|\widehat{\nu}\|_\infty \le 1$. Since $\widehat{\nu}$ is analytic in $\widehat{\mathbb{C}} \setminus X$ and $\widehat{\nu}'(\infty) = \nu(X)$, the function $\widehat{\nu}$ which also vanishes at ∞ is admissible in the definition of γ and so

$$\gamma^+(X) \leq \gamma(X).$$

Again, if E is an arbitrary set in $\mathbb{C}$, we let

$$\gamma^+(E) = \sup_X \gamma^+(X),$$

where X is compact and $X \subset E$. Moreover, Tolsa [35] has shown that there exists an absolute constant $C > 0$ such that

$$\gamma^+(E) \leq \gamma(E) \leq C\gamma^+(E)$$

for all planar sets E, and therefore $\gamma \approx \gamma^+$. It follows that γ and γ^+ share the properties:

1. If $E_1, E_2, \ldots$ are Borel sets then

$$\gamma\left(\bigcup_n E_n\right) \leq C \sum_n \gamma(E_n),$$

with C being an absolute constant; that is, γ is *countably semiadditive*.

2. If μ is a complex measure and $\widehat{\mu}(x)$ is taken in the principal value sense, then for any $\lambda > 0$,

$$\gamma\{x \in \mathbb{C} : |\widehat{\mu}| > \lambda\} \leq \frac{C}{\lambda}|\mu|,$$

where $|\mu|$ denotes the total variation of the measure μ.

For an extensive survey of the properties of analytic capacity and its relation to problems in approximation theory, the reader is referred to [16] and [41] (cf. also [39]). Two of the more basic properties to be found are these:

(i) $\gamma(B_r) = r$ for every disk B_r of radius r

(ii) $\gamma(K) \leq \operatorname{diam}(K) \leq 4\gamma(K)$ if K is compact and connected.

Perhaps the major difference between analytic capacity and $q-$capacity, in-so-far as we are concerned, is that if Φ is a contraction and $1 < q \leq 2$, then

$$C_q(\Phi E) \leq kC_q(E),$$

where k is a constant depending only on q. But, in the case of analytic capacity no such constant k exists. In fact, Garnett [15] and Vitushkin [38] have constructed compact sets X with the property that $\gamma(X) = 0$, but $\gamma(\Phi X) > 0$. This phenomenom played a key role in [6].

3.4 Instability of Capacity

Let E be an arbitrary Borel measurable subset of the complex plane. It is a well-known fact and a classic theorem (cf. [10]) that Lebesgue measure is unstable in the sense that for almost every $x \in \mathbb{C}$, either

$$\lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1 \qquad \text{or} \qquad \lim_{r\to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 0.$$

In the late 1960's, Vitushkin [39] was able to show that analytic capacity enjoys a similar instability. He proved that for almost every $x \in \mathbb{C}$, either

(i) $\lim_{r\to 0} \frac{\gamma(B_r(x) \cap E)}{r} = 1$, or

(ii) $\lim_{r\to 0} \frac{\gamma(B_r(x) \cap E)}{r^2} = 0$.

Contrasting this with the case of Lebesgue density, one might have expected the $\gamma-$capacitary density to either be 0 or 1. However, since $\gamma(B_r) = r$, the second conclusion is a stronger statement.

Around the same time that Vitushkin's work appeared in [39], Lysenko and Pisarevskiĭ [22] proved that a similar instability holds for harmonic capacity (ie. $2-$capacity), although it was in $\mathbb{R}^3$. On the other hand, Hedberg [20] discovered that each of the $q-$capacities considered here are unstable in the sense that the following two relations are equivalent for every Borel set $E \subset \mathbb{C}$:

(a) $C_q(E \cap \Omega) = C_q(\Omega)$ for every open set Ω

(b) $\limsup_{r \to 0} \frac{C_q(B_r(x) \cap E)}{r^2} > 0$ for a.e. $x \in \mathbb{C}$.

Shortly thereafter, Fernström [12] obtained the correct analogue of Vitushkin's theorem by showing that the limit as $r \to 0$ in (b) actually exists, and by also proving that for almost every $x \in \mathbb{C}$, either

(i) $\lim_{r \to 0} \frac{C_q(B_r(x) \cap E)}{r^{2-q}} = 1$

(ii) $\lim_{r \to 0} \frac{C_q(B_r(x) \cap E)}{r^2} = 0$

Here again, the conclusion in (ii) is stronger than what might be expected. We shall take full advantage of that fact for the construction in Chapter 4.

3.5 Rational Approximation

Necessary and sufficient conditions for the rational functions to be dense in either $C(X)$ or in $L^p(X)$ were first obtained by Vitushkin (cf. [39]) in the case of uniform approximation, and later by Hedberg [20] for L^p approximation. In both cases, the condition is expressed in terms of an appropriate capacity:

Theorem 3.3 (Vitushkin)**.** *For a compact set X, the following are equivalent:*

(a) $R(X) = C(X)$

(b) $\limsup_{r \to 0} \frac{\gamma(B_r(x) \setminus X)}{r} > 0$ *for almost every $x \in X$.*

Theorem 3.4 (Hedberg)**.** *For a compact set X and $2 < p < \infty$, the following are equivalent:*

(a) $R^p(X) = L^p(X)$

(b) $\limsup_{r \to 0} \frac{C_q(B_r(x) \setminus X)}{r^{2-q}} > 0$ *for almost every $x \in X$.*

In both theorems, the implication $(b) \Rightarrow (a)$ depends largely on the continuity of the Cauchy transform of an annihilator. In Hedberg's theorem, for example, suppose that $k \in L^q(X)$ and that $k \perp R^p(X)$. By our earlier discussion, $\widehat{k}$ is $q-$finely continuous q.e., and by assumption vanishes identically off X. Since (b) ensures that $\mathbb{C} \setminus X$ is $q-$thick at a.e. point of X, it follows that $\widehat{k} = 0$ a.e. on X. Thus by Thm. 2.1, $k = 0$ a.e.$-dA$ and so $R^p(X) = L^p(X)$.

In Vitushkin's theorem, the implication $(b) \Rightarrow (a)$ can be obtained from the following lemma, which gives a kind of *lower semicontinuity* to the Cauchy transform $\widehat{\mu}$ of a compactly supported measure μ. The proof, which can be found in [5], depends on Tolsa's theorem that $\gamma \approx \gamma^+$.

Lemma 3.5 (Brennan). *Let μ be a finite, complex, compactly supported measure in $\mathbb{C}$, and let x_0 be any point where $\widetilde{\mu}(x_0) < \infty$. Suppose that E is a set with the property that for each $r > 0$ there is a relatively large subset $E_r \subset (E \cap B_r(x_0))$ on which $\widetilde{\mu}$ is bounded, that is*

(1) $\widetilde{\mu} \leq M_r < \infty$ on E_r,

(2) $\gamma(E_r) \geq \varepsilon\gamma(E \cap B_r(x_0))$ for some absolute constant ε.

If E is thick at x_0 in the sense that

$$\limsup_{r \to 0} \frac{\gamma(E \cap B_r(x_0))}{r} > 0,$$

then $|\widehat{\mu}(x_0)| \leq \limsup\limits_{z \to x_0,\ z \in E} |\widehat{\mu}(z)|$.

Going back to Vitushkin's theorem, let ν be any measure on X so that $\nu \perp R(X)$. Then, $\widehat{\nu} \equiv 0$ in $\mathbb{C} \setminus X$ and since (b) gives sufficient thickness, the lemma implies that for a.e. $x_0 \in X$

$$|\widehat{\nu}(x_0)| \leq \limsup_{z \to x_0,\ z \in \mathbb{C} \setminus X} |\widehat{\mu}(z)| = 0.$$

So $\widehat{\nu} = 0$ a.e.$-dA$ on X, and hence $R(X) = C(X)$.

In the next chapter, we will present several examples whose constructions depend on the fact that the capacitary density condition (b) in both the Vitushkin and Hedberg theorems can be replaced using the instability of capacity by a stronger condition. In particular, for Hedberg's theorem, the instability of $q-$capacity allows us to conclude that if for a.e. $x \in \mathbb{C} \setminus X$

$$\limsup_{r \to 0} \frac{C_q(B_r(x) \setminus X)}{r^2} > 0$$

then $R^p(X) = L^p(X)$.

Chapter 4 Construction of Compact Nowhere Dense Sets

From the preceding discussion, there are compact nowhere dense sets X for which each one of following two possibilities have been realized:

(1) $R^p(X) = L^p(X)$ for $1 \leq p < 2$, but $R^p(X) \neq L^p(X)$ if $p \geq 2$

(2) $R^p(X) = L^p(X)$ for $1 \leq p < \infty$, but $R(X) \neq C(X)$

As was shown in Thm. 2.3, for a compact nowhere dense set, density is guaranteed for $1 \leq p < 2$. To ensure that property (1) is satisfied, it is sufficient to construct a Swiss cheese X which has a bpe for $R^2(X)$ at some point $x_0 \in X$ (cf. [3, p. 301]). In the second case, (2), the difficulties are more subtle, but together these two examples provide motivation for the main theorem of this chapter.

Theorem 4.1. *Fix p^* with $2 < p^* < \infty$. There exists a compact nowhere dense set X in the plane so that*

(i) $R^p(X) = L^p(X)$ *for* $1 \leq p < p^*$

(ii) $R^p(X) \neq L^p(X)$ *if* $p \geq p^*$.

In their 2011 paper, Brennan and Militzer [6] constructed a set which satisfies (2). There are some important differences between the construction in [6] and the construction of the set promised in Theorem 4.1. For example, the argument in [6] depends in an essential way on the fact that $q-$capacity C_q and analytic capacity γ behave in fundamentally different ways under a contraction. In order to provide some background and to contrast the arguments involved, we shall first recall the line of reasoning in [6] and later return to the proof of Theorem 4.1.

The argument in [6] begins with the construction of a planar Cantor set as follows: Let Q be the closed unit square, split Q into sixteen congruent squares of side length 1/4 and choose the four corner squares, that is those squares which contain a vertex of Q. Apply the same procedure to each of the four squares obtained in the first step, and continue in this manner. At the $n-$th stage, there are 4^n closed squares $Q_j^n, j = 1, 2, \ldots 4^n$, each having side length $1/4^n$. For each n, define

$$E_n = \bigcup_{j=1}^{4^n} Q_j^n$$

and let

$$K = \bigcap_{n=1}^{\infty} E_n.$$

The set K is known as the *corner quarters Cantor set*. The orthogonal projection of K onto the line $2y = x$ covers an interval of length $3/\sqrt{5}$, and therefore of length greater than $\frac{1}{2}\text{diam}(Q)$. Garnett [15] has shown that $\gamma(K) = 0$. A similar, but more complicated example of this kind was first obtained by Vitushkin [38].

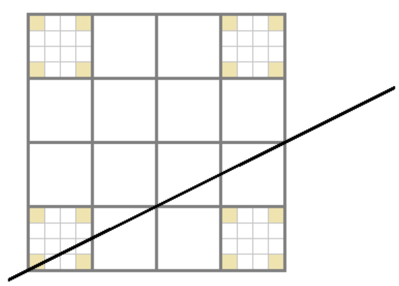

Figure 4.1: The second iteration in the corner quarters Cantor set, and the line $2y = x$

Now use the Cantor sets constructed above in a new procedure. Decompose Q into 4 congruent squares $S_j^1, j = 1, 2, 3, 4$. In each square S_j^1, construct another Cantor set K_j^1 similar to K with a scaling factor of 1/4. Let $K_1 = \cup_j K_j^1$. Continue the process by decomposing Q into 4^n congruent squares S_j^n, in each of which a Cantor set K_j^n similar to K is contructed. Thus we obtain a sequence of Cantor sets $K_1, K_2, \ldots$ with $K_n = \cup_j K_j^n$ and

(i) $\gamma(K_n) = 0$

(ii) $E = \cup K_n$ is dense in Q

(iii) $\Lambda(\text{proj}(K_j^n)) > \frac{1}{2}\text{diam}(S_j^n)$.

Where $\text{proj}(K_j^n)$ denotes the orthogonal projection of K_j^n onto the line $2y = x$, and $\Lambda(\text{proj}(K_j^n))$ denotes the $1-$dimensional Hausdorff measure or length of the projection. It follows from Tolsa's theorem on the countable semiadditivity of analytic capacity that $\gamma(E) = 0$, and so $|E| = 0$ also.

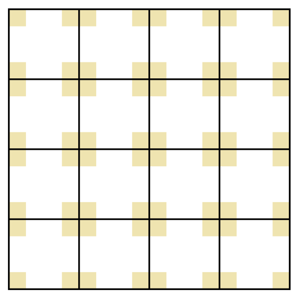

Figure 4.2: An iteration of the Cantor sets in K_2.

Choose a compact set X_0 lying in the interior of Q so that $|X_0| > 0$ and $E \cap X_0 = \emptyset$. Let r_1 be small enough that $\{z : \text{dist}(z, X_0) < r_1\}$ lies inside Q. Since K_1 is a compact totally disconnected set with $\gamma(K_1) = 0$, it is possible to cover K_1 by finitely many open rectangles with sides parallel to the coordinate axes, having mutually disjoint closures, and so that their union Ω_1 satisfies $\gamma(\Omega_1) < \frac{1}{2}r_1$. Next, choose $r_2 < r_1$ so that $\{z : \text{dist}(z, X_0) < r_2\}$ does not meet $\overline{\Omega}_1$. In a completely analogous fashion, cover $K_2 \setminus \overline{\Omega}_1$ by open rectangles whose union Ω_2 satisfies

(i) $\gamma(\Omega_2) < \dfrac{1}{2^2}r_2$

(ii) $\gamma(\Omega_1 \cup \Omega_2) < C\left(\dfrac{r_1}{2} + \dfrac{r_2}{2^2}\right) < Cr_1$, where C is an absolute constant guaranteed by Tolsa's theorem. Continuing in this way, we arrive at a sequence of numbers $r_j \downarrow 0$ and a sequence of open sets $\Omega_1, \Omega_2, \ldots$ so that

(a) $E \subset \bigcup_j \Omega_j$

(b) $X_0 \subset Q \setminus (\bigcup_j \Omega_j)$

(c) $\gamma(\Omega_j) < \frac{1}{2^j} r_j$

(d) $\gamma(\Omega_1 \cup \ldots \cup \Omega_j) < \frac{C}{2^{j-1}} r_j$ for all $j = 1, 2, \ldots$

Setting $X = Q \setminus (\cup_j \Omega_j)$ we obtain a compact nowhere dense set with the desired properties, that is $R(X) \neq C(X)$, but $R^p(X) = L^p(X)$ for all p, $1 \leq p < \infty$.

For each point $x \in X_0$, we have

$$\frac{\gamma(B_{r_j}(x) \setminus X)}{r_j} \leq \frac{C}{2^{j-1}}$$

for all $j = 1, 2, \ldots$ with C an absolute constant. Thus, at each point of X_0 the lower capacitary density of $\mathbb{C} \setminus X$ is zero. By the instability of capacity,

$$\lim_{r \to 0} \frac{\gamma(B_r(x) \setminus X)}{r} = 0$$

at a.e.$-dA$ point of X_0, and so by Vitushkin's theorem (Thm. 3.3), $R(X) \neq C(X)$.

Again, for a.e.$-dA$ point $x \in X$ and r sufficiently small,

$$\Lambda(\text{proj}(B_r(x) \setminus X)) \geq Cr,$$

where C is an absolute constant. Since $q-$capacity decreases modulo a multiplicative constant under a contraction, for a fixed $q < 2$ this implies that $C_q(B_r(x) \setminus X) \geq Cr^{2-q}$. Therefore by Hedberg's theorem (Thm. 3.4), it follows that $R^p(X) = L^p(X)$ for all p.

The preceding discussion highlights the subtleties involved in ensuring that property (2) holds. However, as we return to our proof of Theorem 4.1, it should be pointed out that a different approach is required to cut off the density in L^p at some specific value greater than 2.

Proof of Theorem 4.1. Begin with a constant p^* where $2 < p^* < \infty$. Let q^* be the dual exponent to p^*, that is $q^* = p^*/(p^* - 1)$. We shall construct a compact set X with the property that either

$$\limsup_{r\to 0} \frac{C_q(B_r(x) \setminus X)}{r^2} > 0 \qquad \text{or} \qquad \lim_{r\to 0} \frac{C_q(B_r(x) \setminus X)}{r^2} = 0$$

for a.e. $x \in X$, depending on whether $q > q^*$ or $q \leq q^*$, respectively; or equivalently whether $p < p^*$ or $p \geq p^*$. The desired result will then be an immediate consequence of Hedberg's Theorem 3.4.

Start with the closed unit square $Q = [0, 1] \times [0, 1]$. We shall place a grid of squares inside of Q consisting of lines parallel to the coordinate axes. Let δ_1 be the side length of a generic square grid in Q. At each vertex of the grid, remove a much smaller disk Δ_{α_1} of radius $\delta_1^{\alpha_1}$, where $\alpha_1 > 0$ has yet to be determined. Form the set

$$X_1 = Q \setminus \bigcup \Delta_{\alpha_1},$$

where the union is taken over the entire family of deleted disks. Since any disk B_{δ_1} of

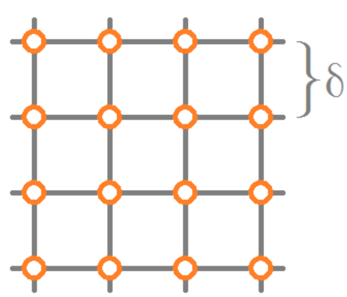

Figure 4.3: A grid of side length δ, with disks of radius δ^α removed

radius δ_1 meets at least one, and at most four Δ_{α_1}'s, it follows from the subadditivity of $q-$capacity that

$$\frac{C_q(B_{\delta_1} \setminus X_1)}{\delta_1^2} \approx \frac{(\delta_1^{\alpha_1})^{2-q}}{\delta_1^2} = \delta_1^{\alpha_1(2-q)-2}.$$

If $2 > q_1 > q^*$ and α_1 is chosen so that

$$\frac{2}{2-q^*} < \alpha_1 < \frac{2}{2-q_1},$$

then we can choose δ_1 sufficiently small so that

(1) $\dfrac{C_{q_1}(B_{\delta_1}(x) \setminus X_1)}{\delta_1^2} > 1/2$

(2) $\dfrac{C_{q^*}(B_{\delta_1}(x) \setminus X_1)}{\delta_1^2} < \varepsilon/2$

for an arbitrary, but fixed, $\varepsilon > 0$ and every $x \in X_1$.

Now we will iterate the process. Pick a sequence

$$2 > q_1 > q_2 > \ldots > q^*$$

so that $q_j \downarrow q^*$, or equivalently, $p_j \uparrow p^*$. Let $r_1 > 0$ be small enough that $\{z \in X_1 : \text{dist}(z, \partial X_1) \leq r_1\}$ is the union of mutually disjoint closed annuli surrounding each of the first generation disks Δ_{α_1}. Choose a second generation grid of side length δ_2 and fix α_2 so that

$$\frac{2}{2-q^*} < \alpha_2 < \frac{2}{2-q_2}.$$

We may assume that δ_2 is sufficiently small to ensure that by subadditivity the total q^*-capacity of the union of all disks Δ_{α_2} of radius $\delta_2^{\alpha_2}$ at points of the new grid does not exceed

$$\frac{1}{\delta_2^2}(\delta_2^{\alpha_2})^{2-q^*} = \delta_2^{\alpha_2(2-q^*)-2} < \frac{\varepsilon}{4}\,\delta_1^2.$$

Now remove from X_1 those disks Δ_{α_2} which do not meet $\{z \in X_1 : \text{dist}(z, \partial X_1) \leq r_1\}$ and set

$$X_2 = X_1 \setminus \bigcup \Delta_{\alpha_2},$$

where again the union is over all deleted disks.

Taking δ_2 even smaller if necessary, we can arrange that the inequalities

$$(3)\quad \frac{C_{q_2}(B_{\delta_2}(x)\setminus X_2)}{\delta_2^2} > 1/2 \qquad \textit{and} \qquad \frac{C_{q_1}(B_{\delta_2}(x)\setminus X_2)}{\delta_2^2} > 1/2$$

$$(4)\quad \frac{C_{q^*}(B_{\delta_2}(x)\setminus X_2)}{\delta_2^2} < \varepsilon/4$$

are also satisfied simultaneously for all $x \in X_2$. At this stage, inequalities (1) and (2) are essentially preserved, except that (2) is now replaced by

$$(2')\quad \frac{C_{q^*}(B_{\delta_1}(x)\setminus X_2)}{\delta_1^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \text{ for all } x \in X_2.$$

Continuing in this manner, we obtain a descending sequence of compact sets $X_1 \supset X_2 \supset \ldots$ together with sequences $q_j \downarrow q^*$ and $\delta_j \downarrow 0$ so that whenever $k \geq j$

$$(5)\quad \frac{C_{q_n}(B_{\delta_j}(x)\setminus X_k)}{\delta_j^2} > 1/2,\ n = 1, 2, \ldots, j$$

$$(6)\quad \frac{C_{q^*}(B_{\delta_j}(x)\setminus X_k)}{\delta_j^2} < \frac{\varepsilon}{2^j} + \ldots + \frac{\varepsilon}{2^k}$$

for all $x \in X_k$. Now, define the set

$$X = \bigcap_{n=1}^{\infty} X_n.$$

Since C_q is a capacity in the Choquet sense and $(B \setminus X_k) \uparrow (B \setminus X)$ for any disk B,

$$C_q(B \setminus X) = \lim_{n\to\infty} C_q(B \setminus X_n)$$

for any $q, 1 \leq q < 2$ (cf. [29, p. 262] and [1, p. 29]). In particular, it follows that

$$(7)\quad \frac{C_{q_n}(B_{\delta_j}(x)\setminus X)}{\delta_j^2} > 1/2 \text{ whenever } j \geq n$$

$$(8)\quad \frac{C_{q^*}(B_{\delta_j}(x)\setminus X)}{\delta_j^2} < \frac{\varepsilon}{2^j} + \frac{\varepsilon}{2^{j+1}} + \ldots = \frac{\varepsilon}{2^{j-1}}$$

for all $x \in X$. Letting $j \to \infty$ it follows from the instability of capacity that

(a) $\limsup_{r\to 0} \dfrac{C_{q_n}(B_r(x)\setminus X)}{r^2} > 1/2, \ n = 1, 2, \ldots$

(b) $\lim_{r\to 0} \dfrac{C_{q^*}(B_r(x)\setminus X)}{r^2} = 0$

for almost every $x \in X$.

In view of property (a), there is a sequence $p_n \uparrow p^*$ for which $R^{p_n}(X) = L^{p_n}(X)$, $n = 1, 2, \ldots$, and therefore $R^p(X) = L^p(X)$ for all $p < p^*$. Property (b), on the other hand, implies that $R^p(X) \neq L^p(X)$ for any $p \geq p^*$.

$\square$

Chapter 5 Support of Representing Measures

As noted in the introduction, when x_0 is a bpe for $R^p(X)$, then the Hahn-Banach theorem guarantees the existence of a function $k \in L^q(X)$ with the property that

$$f(x_0) = \int_X fk \, dA$$

for all rational functions having no poles on X. In this way, every $f \in R^p(X)$ admits a representative that is precisely defined at all bounded point evaluations. A measure $k\,dA$ with the reproducing property indicated above will be referred to as a *representing measure* for x_0.

Bounded point evaluations play a role in L^p approximation similar to the role played by peak points in uniform approximation. Recall that a point $x_0 \in X$ is a *peak point* for $R(X)$ if there exists a function $f \in R(X)$ so that $f(x_0) = 1$, but $|f(z)| < 1$ for all $z \neq x_0$. According to a theorem of Bishop [2], $R(X) = C(X)$ if and only if almost-every point of X is a peak point for $R(X)$. This is strikingly similar to Brennan's criterion (cf. [3]) to the effect that if $p > 2$, then $R^p(X) = L^p(X)$ if and only if almost no point of X is a bounded point evaluation for $R^p(X)$. Our goal here is to describe the support sets of both the annihilating and representing measures for $R^p(X)$ when $p > 2$. The results in this chapter were originally motivated by a paper of Øksendal (cf. [44, Thm. 1.3]), in which he showed that if $\mu \perp R(X)$ then $|\mu| = 0$ a.e. on the set of peak points for $R(X)$.

Theorem 5.1. *If $p > 2$ and $R^p(X) \neq L^p(X)$ then the supports of both the annihilating measures and the representing measures for $R^p(X)$ are contained almost everywhere in the set of bounded point evaluations for $R^p(X)$.*

The proof of the theorem will make use of two lemmas:

Lemma 5.2. *If* $f \in L^q(X)$ *and* $1 < q < 2$, *then* $\dfrac{f(z)}{z-\zeta} \in L^q(X)$ *for a.e.*$-dA$ $\zeta \in X$.

Proof. Choose R large enough so that $f = 0$ outside the disk $|z| < R$. Then for $\zeta \in X$:

$$\begin{aligned}
\int_X \left(\int_X \left| \frac{f(x)}{z-\zeta} \right|^q dA_\zeta \right) dA_z &= \int_X \left(\int_X \frac{dA_z}{|z-\zeta|^q} \right) |f(x)|^q \, dA_\zeta \\
&\leq \int_X \left(\int_{|z|\leq R} \frac{dA_z}{|z-\zeta|^q} \right) |f(\zeta)|^q \, dA_\zeta \\
&\leq \int_X \left(\int_{|z-\zeta|\leq R} \frac{dA_z}{|z|^q} \right) |f(\zeta)|^q \, dA_\zeta \\
&\leq \int_X \left(\int_{|z|\leq 2R} |z|^{-q} \, dA_z \right) |f(\zeta)|^q \, dA_\zeta \\
&\leq \frac{2\pi}{2-q} (2R)^{2-q} \int_X |f(\zeta)|^q \, dA_\zeta < \infty.
\end{aligned}$$

Therefore

$$\int_X \left| \frac{f(x)}{z-\zeta} \right|^q dA_\zeta < \infty$$

for almost every $\zeta \in X$. □

Lemma 5.3. *Each function in* $W_1^q(\Omega)$ *has a representative which is absolutely continuous on almost all lines parallel to the coordinate axes. Moreover, the distributional gradient of a function in* W_1^q *coincides almost everywhere with the usual gradient computed pointwise.*

The proof of this lemma can be found in [23, p. 8] (cf. also [43, p. 44]).

Proof of Theorem 5.1. Fix $p > 2$, and let X be a compact, nowhere dense set in the plane, and P be the set of non-bounded point evaluations for $R^p(X)$. Take $k \perp R^p(X)$, and let $\widehat{k}$ represent the usual Cauchy transform of k. We showed in Section 2.1 that $\widehat{k}$ converges absolutely a.e.$-dA$ in X.

Choose $x_0 \in P$ and assume that $\widehat{k}$ converges absolutely at x_0 and that $\widehat{k}(x_0) \neq 0$. Then for any rational function φ, the function $\dfrac{\varphi(z) - \varphi(x_0)}{z - x_0}$ is also rational, and so

$$\int \frac{\varphi(z) - \varphi(x_0)}{z - x_0}\, k(z)\, dA_z = 0.$$

Then

$$\varphi(x_0) \int \frac{k(z)}{z - x_0}\, dA_z = \int \frac{\varphi(z)}{z - x_0}\, k(z)\, dA_z.$$

And since $\widehat{k}(x_0) \neq 0$, we have

$$\varphi(x_0) = \frac{1}{\widehat{k}(x_0)} \int \frac{k(z)}{z - x_0}\, \varphi(z)\, dA_z.$$

But by Lemma 5.2, for a.e. x_0, we have $\dfrac{k(z)}{z - x_0} \in L^q$. This would mean that x_0 is a bpe for $R^p(X)$ as

$$|\varphi(x_0)| \leq \frac{1}{|\widehat{k}(x_0)|} \left(\int \left| \frac{k(z)}{z - x_0} \right|^q \, dA_z \right)^{1/q} \left(\int |\varphi(z)|^p \, dA_z \right)^{1/p} < C \|\varphi\|_p$$

for some absolute constant C and any rational function φ. This is a contradiction, and so we must have that $\widehat{k} = 0$ a.e.$-dA$ on P.

Since $\widehat{k} \in W_1^q$, by Lemma 5.3 $\widehat{k}$ is absolutely continuous on almost all lines parallel to the coordinate axes and its distributional derivatives coincide almost everywhere with the usual derivatives computed pointwise. Additionally, almost-every point of Lebesgue area density 1 is a point of positive linear density in the direction of both coordinate axes. Thus we use the fact that $\widehat{k} = 0$ almost-everywhere to obtain for $z = x + iy$ that $\dfrac{\partial \widehat{k}}{\partial x} = \dfrac{\partial \widehat{k}}{\partial y} = 0$ almost-everywhere. Finally, we can conclude that $\overline{\partial}\, \widehat{k} = -\pi k = 0$ almost-everywhere on P. Hence, $k = 0$ a.e.$-dA$ on P.

This shows that any annihilating measure for $R^p(X)$ has its support almost-everywhere in the set of bpe's for $R^p(X)$. However, if $f \in L^q(dA)$ and $f\ dA$ is a representing measure on $R^p(X)$ for a point x_0, then $(z - x_0)f(z)\ dA$ is an annihilating measure for $R^p(X)$, and so the support of f must also be contained almost everywhere in the set of bounded point evaluations for $R^p(X)$. $\square$

Remark 5.4. *Theorem 5.1 cannot be extended to $p = 2$ due to an example by Fernström (cf. [11]). In this paper, he showed that there exists a compact set X with no bpe and yet $R^2(X) \neq L^2(X)$. In light of this example, any annihilating measure on $R^2(X)$ could not have its support on the set of bpe's.*

In [36] Tolsa and Verdera raised a question which is pertinent to the preceding discussion: If μ is a finite compactly supported Borel measure in the plane, and if $\widehat{\mu}$ vanishes $\mu-$a.e. on its support, must $\mu = 0$ as a measure? At that time they were able to give a positive answer in two important special cases, the most relevant here being the case μ is absolutely continuous with respect to area; that is, $\mu = k\, dA$ with $k \in L^1$. Their argument will give the corresponding conclusion in the proof of Lemma 5.3, but when $k \in L^q$ for $q > 1$, the reasoning presented here is more transparent.

Subsequently, Mel'nikov, Poltoratski and Vol'berg [26]) showed that there is a large class of continuous measures for which the conclusion is false; that is, for which $\widehat{\mu} = 0$ a.e.$-d\mu$, but $\mu \neq 0$. The situation in general is still not fully understood.

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