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Probabilistic Models for Order-Picking Operations with Multiple in-the-Aisle Pick Positions

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by

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Abstract

The development of probability density functions (pdfs) for travel time of a narrow aisle lift truck (NALT) and an automated storage and retrieval (AS/R) machine is the focus of the dissertation. The multiple in-the-aisle pick positions (MIAPP) order picking system can be modeled as an M/G/1 queueing problem in which storage and retrieval requests are the customers and the vehicle (NALT or AS/R machine) is the server. Service time is the sum of travel time and the deterministic time to pick up and deposit a pallet (T_{PD}).

Our first contribution is the development of travel time pdfs for retrieval operations in an MIAPP order picking system supported by a narrow aisle lift truck (MIAPP-NALT); storage operations are assumed to occur when order picking is not being performed. A rectilinear travel metric is used for the NALT; pdfs are derived and finite population queueing and infinite population queueing models are used to analyze the retrieval operations under stochastic conditions.

Our second contribution is the development of travel time pdfs for retrieval operations in an MIAPP order picking system supported by an AS/R machine (MIAPP-AS/RS); storage operations are assumed to occur when order picking is not being performed. A Chebyshev travel metric is used for the AS/R machine. For the MIAPP-AS/RS operation, pick positions are located at floor level and on a mezzanine. The pdfs for four scenarios are derived and finite population and infinite population queueing models are used to analyze the retrieval operation under stochastic conditions.

Our final contribution is the development of travel time pdfs for storage and retrieval operations in an MIAPP-NALT system with two classes of stock keeping units (skus): fast movers and slow movers. A rectilinear travel metric is used and two levels of pick positions are considered. Non-preemptive priority queueing and non-priority queueing models are used to analyze storage

and retrieval requests in the MIAPP-NALT system. Retrieval requests are given a higher priority than storage requests; alternately, storage and retrieval requests are served using a first come, first serve (FCFS) discipline.

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Dedication

To my beloved parents

Mr. Huanyun Liu & Mrs. Xiuchun Qiao

and

To my younger sister

Mrs. Jinghong Liu

for their enormous personal sacrifices and unconditional loves.

This humble study is a sign of my love to you!

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Chapter 2

Liu, J. Liao, H. and White, J. A., Queueing analysis of the replenishment of multiple in-the-aisle pick positions, submitted to *IISE Transactions*. Under Review

Chapter 1

Introduction

The focus of the dissertation is the development of probabilistic models of a multiple in-the-aisle pick positions (MIAPP) order-picking operation within a distribution center. Underlying the effort is the need to derive the probability density functions (pdfs) for aisle-captive material handling equipment operating in a storage/retrieval (S/R) aisle supporting MIAPP order-picking operations. Specifically, in the dissertation, we consider two types of material handling equipment, a narrow aisle lift truck (NALT) and an automated storage and retrieval (AS/R) machine performing retrieval operations. In Chapters 2 and 3, for retrieval operations, the vehicle (NALT or AS/R machine) travels from the position where it just finished a retrieval operation to a random reserve-storage location to pick up a pallet and then travels to the pick positon requiring replenishment and deposits it at the pick position needing replenishment. In Chapter 4, we expand our coverage to include storage operations, as well as retrieval operations; the NALT travels from the position where it just finished a storage or retrieval operation to the input/output (I/O) point to pick up a pallet and then travels to a random reserve-storage location for the pallet to be stored and deposits it. From the pdfs for travel times by the material handling equipment and the deterministic value of pick up and deposit (T_{PD}) times, the Laplace transform of the service time distribution (sum of travel time and T_{PD} time) is derived; from the pdf, the expected value, second moment and variance for service time are obtained.

In Chapters 2 and 3, we consider a finite number of pick positions, as well as an infinite number of pick positions. For the case of a finite number of pick positions, the retrieval operation in the order picking application is modeled as an M/G/1/N queue. For the case of an infinite number of pick positions, the retrieval operation is modeled as an $M/G/1/\infty$ queue. For the M/G/1/N queue, Takács' formulas (Takács,1960) are used to determine the steady state performance of the multiple in-the-aisle pick position (MIAPP) problem, based on Laplace transforms of service time pdfs. Pollaczek-Khinchine formulas (Gross, et al, 2008) are used for the $M/G/1/\infty$ queue by employing derivations of the expected value and variance for service time.

In Chapter 2, the material handling equipment used for MIAPP replenishment is an aisle-captive narrow-aisle lift truck (NALT). Based on a rectilinear travel metric for the NALT, derivations of pdfs of travel time are developed for four scenarios: 1) a finite number of pick positions with dedicated storage; 2) a finite number of pick positions with random storage; 3) an infinite number of pick positions with dedicated storage; and 4) an infinite number of pick positions with random storage. The MIAPP-NALT operation is modeled, first, as a finite population queue M/G/1/N; then, when the number of pick positions is sufficiently large, the MIAPP-NALT problem is approximated using an $M/G/1/\infty$ queueing model.

In Chapter 3, the material handling equipment used for MIAPP replenishment is an automated storage and retrieval (AS/R) machine. Based on the Chebyshev travel metric for the aisle-captive AS/R machine, derivations of the pdfs of travel time for an aisle-captive AS/R machine are developed for four scenarios: 1) a finite number of pick positions with dedicated storage; 2) a finite number of pick positions with random storage; 3) an infinite number of pick positions with dedicated storage; and 4) an infinite number of pick positions with random storage. The MIAPP-AS/RS operation is modeled, first, as a finite population queue, M/G/1/N; then, when the number of pick positions is sufficiently large, the MIAPP-AS/R problem is approximated using an $M/G/1/\infty$ queueing model.

In Chapter 4, the problem studied in Chapter 2 is extended to include both storage and retrieval of unit loads. In addition, two classes of skus are considered: fast movers and slow movers. Also, giving non-preemptive priority to retrieval requests is compared with using first come, first serve (FCFS) for storage and retrieval requests in the MIAPP-NALT operation.

Specifically, unit loads are assumed to arrive at an input point at the end of the S/R aisle and the NALT stores unit loads in reserve-storage regions; in addition, the NALT replenishes pick positions. Dedicated storage, random storage and two class-based storage policies are considered; pdfs, expected values, second moments and variances of service times are obtained for the two queueing models. For a storage operation, the NALT will travel from a random floor-level location to the I/O point to pick up a pallet and then travel to a random reserve-storage location and store the pallet, and then lower the forks to floor level in preparation for the next operation. For a retrieval operation, the NALT will travel from a floor-level random location to a random reserve-storage location for the next pick position needing to be replenished, pick up a pallet from the reserve-storage location and travel to the pick position requiring replenishment and deposit the pallet. After the retrieval operation, the NALT does nothing if the pick position being replenished is at floor level. The NALT will lower its fork to floor level if the pick positon just replenished is at the mezzanine level.

From a queueing perspective, the NALT must serve two types of customers: those arriving at the input point (storage requests) and those arriving at pick positions (retrieval requests). To avoid delays in case picking, non-preemptive priority is given to retrieval requests. We assume the number of pick positions is sufficiently large and the traffic intensity for the replenishment function is relatively small, such that the MIAPP-NALT operation can be modeled accurately as an $M/G/1/\infty$ queueing problem.

Each chapter in the dissertation includes a literature review, a case study in which the analytical models are used in determining the performance measures of the operation under study; also, each chapter provides insights and practical considerations of the research, as well as a closing section providing a summary, conclusions, and recommendations for further study. Detailed derivations of pdfs are provided in end-of-chapter appendixes. Because there is considerable duplication in the literature reviews, a comprehensive reference list is provided at the end of the dissertation.

Chapter 5 includes a summary of the dissertation and conclusions drawn. Recommendations for further study are also provided.

Chapter 2

Contribution 1: A Submitted Paper on "Queueing Analysis of the Replenishment of Multiple in-the-Aisle Pick Positions"

Abstract

Case-picking operation with multiple in-the-aisle pick positions (MIAPP) is modeled as an M/G/1 queueing system. In such a system, cases are picked manually from pallets located at the bottom or floor level of storage racks. Rectilinear travel is considered for an aisle-captive narrow-aisle lift truck (NALT) that replenishes the floor level of a rack by retrieving a pallet load from an upper level of the storage. From a queueing perspective, the NALT is the server and the order-picking positions in need of replenishment are the customers. It is assumed that replenishment requests from order-picking positions (homogeneous customers) occur at a Poisson rate. The corresponding probability density functions of service times are derived and their Laplace transforms are obtained, leading to steady-state performance measures of the system. In many situations, arrivals of replenishment requests from individual pick positions may not follow a Poisson process, and the order-picking operation may consist of heterogeneous customers. However, a simulation study indicates that a finite population queueing model yields accurate performance measures. Interestingly, when the number of pick positions is sufficiently large to justify an MIAPP operation, the time between consecutive replenishment requests within a storage/retrieval (S/R) aisle approximately follows an exponential distribution. The analytical results obtained from the finite population queueing model are compared with the simulation results. In addition, the analytical results from the finite population model and an infinite population model are compared when dedicated storage is used (random storage will reach the same conclusion). A numerical example

is provided to illustrate the use of the analytical results in practice.

Keywords: Case-level order picking, rectilinear travel, general service times, Poisson distributed arrivals, Laplace transform

2.1. Introduction

Order-picking operations are a major activity within a warehouse, representing approximately 55 percent of operating costs (Drury, 1988). Order picking can be categorized by what is picked: unit or pallet load, case or carton and item or each. The most labor intensive is item picking; next, is case picking. In this paper, we focus on a specific type of case picking, which is performed manually from the floor level of double-deep selective pallet rack installed on both sides of an order-picking aisle. The order-picking aisle is on one side of the storage rack containing pallet loads of cases; replenishment of the pallet loads occurs on the opposite side of the storage rack in a storage/retrieval (S/R) aisle. Overall, the arrangement is called a multiple in-the-aisle pick position (MIAPP) case-picking operation.

Order pickers place cases on a conveyor located in the middle of the order-picking aisle. When the last case of a pallet is picked, a replenishment request occurs and the empty pallet is removed. On the opposite side of the rack, an aisle-captive narrow-aisle lift truck (NALT) replenishes the floor levels of the rack on both sides of the S/R aisle by retrieving full pallets from the upper levels of the storage rack. From the perspective of the NALT, the two pallet positions on both sides of the S/R aisle can be considered the same pick position, and the time to travel to the pick position is the same regardless of which side initiates the replenishment request.

From a queueing perspective, the MIAPP-NALT operation is modeled as an M/G/1 queue with the NALT being the server and the pick positions in the double-deep rack being the customers. The service performed by the NALT is the replenishment of a deposit position. Therefore, it does not matter if the order pickers are placing cases on a manually operated pallet truck, an automated guided pallet truck, or a conveyor; the model of the MIAPP-NALT queueing system depends only on what happens with the NALT in the S/R aisle.

In observing MIAPP case-picking operations, we realized the time required for a pallet load of cases to be depleted is unlikely to be exponentially distributed. Typically, case pickers removed a few cases from a pallet and then walked to another pick position to remove a few cases. As such, unlike the shape of an exponential distribution, the distribution of the time required to deplete a pallet has a shape like a gamma distribution. For the depletion time to be exponentially distributed, full-pallet picks and almost-full-pallet picks would have to occur with very high frequency; in such situations, an MIAPP design would be impractical.

From a modeling perspective, it is convenient to assume that the arrivals of replenishment requests from an individual pick position follow a Poisson process. Moreover, it is convenient to assume identical pick positions, even though they may not have the same arrival rates. We made these assumptions, keeping in mind the robustness of the finite population model. Specifically, the probability mass function for the number of customers in the system is known to be "… valid for any finite-source system with exponential service independent of the nature of the distribution of time to breakdown, as long as the lifetimes are independent …" from Gross, et al (2008). Although service times are not exponentially distributed, we anticipated that they can be well approximated using an Erlang distribution, which leads to the M/G/1/N queueing model that provides reasonable results for an MIAPP-NALT operation.

Simulations of the order-picking process revealed that although the time required to deplete a pallet load of cases at a particular pick position is not exponentially distributed and the time differs among different pick positions, the finite population model yields performance measures nearly identical to those obtained from a simulation model in which customers are not identical and the interarrival times for customers are not exponentially distributed. Furthermore, simulation results showed the time between consecutive requests for pallet load replenishment within an S/R aisle can be reasonably represented by an exponential distribution. Therefore, the MIAPP-NALT retrieval operation can be modeled quite accurately as an M/G/1 queueing system.

Because the locations of replenishment requests occur from a random pick position, the time required for the NALT to perform replenishments is a random variable. Two storage policies are considered: dedicated storage and random storage. In the case of dedicated storage, we assume identical *x*-coordinates of a pick position and the locations of reserve storage for the pick position; we also assume the *y*-coordinate is uniformly distributed between zero and the upper limit of the storage region; in the case of random storage, we assume storage locations are uniformly distributed over a continuous representation of the storage region.

In practice, the NALT also stores full pallets in the upper levels of the storage racks; it retrieves pallets from an input station located at the end of the S/R aisle. To simplify modeling, we assume the NALT stores full pallets in the upper levels of the storage rack during a period of time when order picking is not being performed, such as in the evening or during lunch periods or break periods when no demands for replenishment occur.

In contrast to an AS/R machine, the NALT has the flexibility of serving multiple S/R aisles. However, in our paper we restrict the NALT to a single S/R aisle. We do so because it simplifies modeling, it provides an alternative to using an aisle-captive AS/R machine to support a MIAPP operation, and economic reasons exist for limiting the use of the NALT to a single aisle.

If the NALT is assigned to multiple S/R aisles, assumptions are required in developing the probability density function (pdf) for NALT travel time. For example, if the NALT is assigned to two adjacent S/R aisles, the methodology we develop for obtaining the pdf can be used by assuming all storage locations are equally likely to be visited by the NALT, the NALT can enter and exit the S/R aisle from only one end of the S/R aisle and by including a time component for the NALT to change aisles.

In the case of an MIAPP-AS/RS operation, the S/R machine is limited to a single aisle. Economic justifications of such operations are typically based on harsh conditions, such as freezer storage conditions, expensive or limited land on which the distribution center is located, and high density storage coupled with rapid throughput requirements. However, many of the benefits can be obtained more economically for traditional applications by reducing the vertical storage dimension and replacing the AS/R machine with a NALT. Regardless of how the pick positions are replenished (AS/R machine or NALT), comparable material handling equipment can be used to transport unit loads from receiving to the MIAPP installation and transport unit loads from the MIAPP installation to order consolidation and shipping.

Finally, economic reasons exist for restricting the NALT to narrow aisles. We have observed instances where an NALT performed functions that could be performed by less expensive material handling equipment. Moreover, in designing the MIAPP-NALT system, if the aisle-captive NALT is utilized more than 50 percent then it cannot be assigned to more than one S/R aisle if workload is distributed equally over the S/R aisles (Note: for safety reasons, two NALTs are not allowed to operate in the same S/R aisle). If the NALT utilization is low when assigned to a single S/R aisle, consideration should be given to assigning the NALT to adjacent aisles; our model can still be used by treating the length of the S/R aisle to be slightly more than twice the length of the actual S/R aisle.

During the time period when the NALT performs only replenishments, if it becomes idle, its dwell point is at the location of the last replenishment. Hence, NALT travel occurs rectilinearly from a pick position on the floor level to a storage location in an upper level within the rack and, then, rectilinearly to a pick position requiring replenishment on the floor level.

Although, with some material handling equipment, the vehicle can travel horizontally while simultaneously raising or lowering its forks, the NALT we consider, based on safety considerations, is permitted to travel horizontally only when the forks are at floor level. Therefore, movement of the forks on the NALT is modeled using a rectilinear travel metric.

Service time for the NALT consists of two components: travel time and the time to pick up a unit load and to deposit the unit load. For one retrieval operation, the time to pick up a unit load and to deposit the unit load equals the sum of two P/D times. Generally, P/D time (time for the NALT to pick up or to deposit a unit load) is relatively small compared to travel time and we learned its distribution had little impact on the queueing results. Therefore, we consider P/D time to be deterministic. To simplify the development of the model of the MIAPP-NALT system, we assume the time to pick up a unit load equals the time to deposit a unit load. Therefore, a deterministic value is used for the sum of two P/D times.

Our analysis is based on an assumption of independent and identically distributed pick positions. Therefore, each pick position has the same arrival rate. Insofar as the NALT is concerned, the arrival rate for the finite population is based on the time between depositing a unit load at the pick position and a replenishment request being generated by the same pick position. With double-deep pallet flow rack, the interarrival time for a particular pick position is the length of time the most recently replenished pallet load remains in the pick position, before gravity moves it to the pick position.

As noted, our simulation results indicated that the finite population queueing model can be used to represent the NALT activities in the S/R aisle; and the time between replenishment requests within an S/R aisle can be reasonably approximated by an exponential distribution. Therefore, when the number of pick positions is large, an infinite population approximation can be incorporated in the queueing model. To understand when such an approximation can be used, we compare the performance measures of M/G/1/N (N denotes population size) and M/G/1/∞ models for various traffic intensities and values of N. Among the performance measures considered is the expected number of customers either waiting for service or being served, which is the expected number of empty deposit positions in the replenishment aisle. As soon as the deposit position is empty, a replenishment request occurs; the time between the replenishment request and when the NALT initiates travel to retrieve a unit load for the pick position is the waiting time for the customer; and the time required for the NALT to travel to the retrieval point, pick up the unit load, travel to the pick position and deposit the unit load is the service time for the customer.

Drawing on results in the paper, designers of an MIAPP-NALT system will be able to answer such questions as

(1) Is dedicated storage or random storage better, based on a consideration of throughput versus storage space requirements for the system?

(2) How accurately will the infinite population queueing model approximate the finite population queueing model?

(3) Are the expected values for waiting time and number of replenishment requests waiting satisfactory for the given case-picking application?

(4) How long and tall should the picking aisle be, based on queueing performance measures?

The remainder of the paper is organized as follows. In Section 2, the literature related to an MIAPP-NALT system is reviewed; the literature review concludes with a list of contributions of this paper. In Section 3, notation and key assumptions underlying the research are provided. In Section 4, we derive the probability density functions (pdfs) and associated expected values and variances for service time; we do so for both dedicated and random storage, as well as for a finite and an infinite number of pick positions in the S/R aisle. For a finite number of pick positions, we also obtain the Laplace transforms for the pdfs. In Section 5, assuming demands for replenishment occur in a Poisson fashion, we employ Laplace transforms of the pdfs of service times to obtain the steady-state performance measures for M/G/1/N queues using the formulas provided by Takács (1960) and included in Saaty (1961) and Sztrik (2016), among others; next, for the M/G/1/∞ queueing model, we apply the Pollaczek-Khinchine formulas to obtain the steady-state performance measures. In Section 6, based on computational results, we validate the M/G/1/N queueing model using simulation with both homogeneous and heterogeneous customers and we determine conditions when results from the M/G/1/∞ model closely approximate results from the M/G/1/N model; we also evaluate the impact on performance measures of dedicated versus random storage policies, and share useful insights gained. Finally, in Section 7, conclusions are drawn and recommendations for further research are provided.

2.2. Literature review

Numerous papers addressing unit load storage and retrieval, case picking, and item picking have been published. However, few include queueing formulations of order-picking operations. In general, papers employing queueing formulations make simplifying assumptions regarding the service time distribution. In sharp contrast to the Chebyshev travel metric, we found a limited number of papers addressing the pdf for the rectilinear travel metric; in those papers, the expected time required to travel rectilinearly from a fixed point to a random point within a specified spatial region was calculated.

We focus principally on literature related to the MIAPP system we address. As a result, we only include a review of literature incorporating Chebyshev travel when a queueing analysis is involved. The literature review is organized as follows: the papers dealing with rectilinear travel are considered first; next, we review MIAPP literature. Finally, because the queueing literature is so vast, we limit its review to papers related to storage/retrieval operations.

Rectilinear metric: Research papers based on rectilinear travel tend to address facility location problems, such as those considered by Francis et al. (1992). A paper by Miyagawa (2010) is typical of facility location problems involving the calculation of the expected time required to travel rectilinearly from a fixed point to a random point and to the nearest point within a diamond or square lattice pattern. His paper is an extension of other facility location problems in which rectilinear travel occurs between a fixed point and a pattern of random points. Early work on the problem is considered by Larson and Odoni (1981), including the derivation of the pdf from the centroid of a square area to a random point.

Research performed on high density storage systems can incorporate aspects of rectilinear and Chebyshev travel. Specifically, when a shuttle or storage/retrieval device travels within a storage lane it travels rectilinearly; the S/R device might then connect to an AS/R machine to move between storage lanes; in the latter case, Chebyshev travel occurs. Early work on high density storage systems was performed by Gue (2006) and Gue and Kim (2007). Subsequently Uludag (2014) introduced another high density storage system called Grid Pick, which has performance characteristics similar to a miniload AS/RS, but with rectilinear movement.

MIAPP literature: Ramtin and Pazour (2014) appear to have been the first to develop an analytical model of the MIAPP-AS/RS system; they calculated throughput performance and the length-to-height configuration of an AS/RS aisle to minimize expected travel time. Ramtin and Pazour (2015) explored different operating policies, demand profiles, and shape factors for an MIAPP-AS/RS; in addition to calculating expected travel time and system throughput, they also developed a dedicated storage assignment model for stock used to replenish order-picking positions. In contrast to their work, we analyze an MIAPP operation in which replenishment of pick positions is performed using a NALT; likewise, we extend their work by employing a queueing model of the MIAPP operation, deriving the Laplace transform for service time and obtaining performance characteristics of the M/G/1/N queueing system. In addition, we use an $M/G/1/\infty$ queueing model to approximate the M/G/1/N model by calculating the service time mean and variance.

Queueing models of storage/retrieval operations: Bozer and White (1990) introduced the first analytical stochastic analysis of a mini-load AS/RS modeled as a two-server closed queueing network. Lee (1997) performed a stochastic analysis of a unit load AS/RS with a single-server, infinite population queueing model including two types of customers. Park et al. (1999) modeled the miniload AS/RS as an infinite population, two-stage cyclic queueing system with limited capacity and obtained closed form expressions for system performance measures. Vlasiou et al. (2004) modeled a system consisting of two carousel conveyors and one AS/R machine as an infinite population queueing system and analyzed its performance. Hur et al. (2004) analyzed a unit-load AS/RS by using an $M/G/1/\infty$ queuing model with two types of customers; Hur and Nam (2006) analyzed the same $M/G/1/\infty$ queue with different arrival rates for two types of customers. Bozer and Cho (2005) developed closed-form analytical results to evaluate AS/RS performance under stochastic demand with an M/G/1/∞ queue having two types of customers. Malmborg (2002) used a state transition diagram to compare the autonomous vehicle storage and retrieval system (AVS/RS) with the AS/RS. Using a state transition diagram, Malmborg (2003) analyzed interleaving storage and retrieval operations in an AVS/RS. Kuo et al. (2007) analyzed an AVS/RS using an M/G/c/∞ queueing model based on 12 scenarios for vehicle travel time. Fukunari and Malmborg (2009) analyzed the AVS/R system by including interleaving of storage and retrieval requests. Heragu, et al. (2011) used an infinite population open queueing network to compare the AS/RS with the AVS/RS and developed a manufacturing system performance analyzer to quickly evaluate alternate configurations of the systems. Zhang et al. (2009) analyzed the AVS/RS using a non-Poisson arrival rate and non-exponential service time queueing model. Cai et al. (2013) employed a semi-open queueing model of the AVS/RS in which customers are paired; they obtained queueing performance measures using Markov chains. Cai et al. (2014) analyzed a semi-open queue with two stages for the AVS/R system in which a customer from a finite population and a customer from an infinite population must be coupled, much like a dual command operation. Roy et al. (2015) used a multi-class semi-open queuing network to analyze the AVS/RS and determine the best dwell-point policy and cross-aisle location.

Contributions of the paper: In comparison with previous research, we believe this work is the first 1) to model the MIAPP system as a queueing system; 2) to derive pdfs for rectilinear travel from a fixed point to a random point and, then, to a fixed point (finite population), as well as from a random point to a random point and, then, to a random point (infinite population); 3) to employ queueing models to analyze the impact of a storage policy on order-picking productivity; 4) to analyze conditions justifying the use of an infinite population approximation for the M/G/1/*N* queueing system; and 5) to compare space requirements in the MIAPP system with dedicated versus random storage. Another contribution of the paper is a demonstration of the robustness of the M/G/1/N queueing model; specifically, we show quite accurate results can be obtained from the homogeneous customer model when customers are heterogeneous and arrival rates for individual customers are not Poisson distributed.

In addition, we believe our results will be beneficial for other research involving rectilinear travel from a fixed point to a uniformly distributed random point to a fixed point, as well as rectilinear travel from a uniformly distributed random point to a uniformly distributed random point to a uniformly distributed random point within a rectangle-shaped region. In both cases, the "travel legs" are statistically dependent. The lack of statistical independence does not hinder researchers who are only interested in expected values; however, if researchers are interested in travel time variances, then our results will be quite beneficial.

As noted, the preponderance of research on the design of material handling systems has been based on deterministic analyses. In those cases where random variation is acknowledged, expected values are used for design purposes. Therefore, if random variation occurs and the design is based on expected values, then the resulting design is not influenced by variation inherent in the operation. Admittedly, the queueing performance measures we employ in the paper are, themselves, expected values; however, at least they are based an incorporation of variation in the queueing models. Although expected value analysis can provide information on NALT utilization, queueing analysis provides information on the expected time a pick position waits to be replenished, the expected number of pick positions waiting for replenishment, NALT utilization, and other performance measures.

2.3. Notation and assumptions

The following notation is used in formulating the queueing models:

B : breadth or width of the rack (or length of an S/R aisle);

H : height of the rack;

N : number of pick positions on both sides of the rack in an S/R aisle;

V_h : horizontal speed of the NALT;

V_v : vertical speed of the NALT;

t_h : time for the NALT to travel horizontally a distance of B, B/V_h;

t_v : time for the NALT to travel vertically a distance of H, H/V_v;

b : shape factor, the ratio of t_v and t_h;

$T_{P/D}$: time for the NALT to pick up or to deposit a full pallet;

T_{PD}: time for the NALT to pick up and to deposit a full pallet, equals $T_{P/D} + T_{P/D}$;

T : travel time for the NALT to perform one replenishment;

T_s : service time, time to perform a retrieval operation, equals $T + T_{PD}$;

λ : arrival rate of replenishment requests for each pick position;

$\bar{\lambda}$: effective arrival rate of replenishment requests for a finite population queue;

μ : service rate of replenishment requests, $1/T_s$;

L_q : expected number of empty deposit positions waiting for replenishment to begin;

L : expected number of empty deposit positions in the queueing system (number waiting plus being served);

ρ : traffic intensity, λ/μ for an infinite population;

$\bar{\rho}$: traffic intensity, $\bar{\lambda}/\mu$ for a finite population;

W_q : expected time for a pick position to wait for replenishment to begin;

W : expected time for a pick position waiting for replenishment to be completed (waiting time plus service time);

$P_{i,j}$: probability the NALT travels from pick position i to pick position j.

In addition to assumptions stated previously, our analysis of the MIAPP-NALT system is based on the following assumptions: a) $t_h \geq t_v$; b) storage locations are uniformly distributed over a continuous storage region; and c) acceleration and deceleration of the NALT are neglected when traveling horizontally and vertically. For assumption a), it is highly unlikely for a case-picking operation to be designed without $t_h \geq t_v$, because expanding an S/R aisle horizontally is generally less expensive than expanding it vertically; in cases where it is necessary for $t_h < t_v$, it is likely an AS/R machine will be preferred to a NALT. Regarding assumption b), although we can model storage locations discretely, doing so complicates significantly the analysis. Assumption c) is also a modeling convenience, because incorporating acceleration/deceleration adds considerable complexity to the derivation of probability density functions.

Our results are intended to provide design insights and to facilitate first-cut design decisions. Before an MIAPP-NALT system is installed, a detailed design must be performed. Indeed, we anticipate a detailed simulation model will be used, allowing the incorporation of discrete locations for reserve storage, acceleration/deceleration losses, and deviations from other assumptions we made.

To apply M/G/1/N and M/G/1/∞ queueing models, the pdf for NALT travel time is needed. We employ the approach Bozer and White (1984) used in developing travel time models for the AS/RS. Specifically, based on constant travel velocities, we calculate the time required for the NALT to travel horizontally the length of the S/R aisle (t_h) and the time required for the NALT to travel vertically from floor level to the top of the storage rack (t_v). We normalize the storage region by letting b equal t_v/t_h. Thereafter, we measure time in multiples of t_h. To simplify calculations, we assume t_h equals 1.0. In the event, t_h does not equal 1.0, values for W_q and W obtained from our models must be multiplied by t_h, respectively, to obtain accurate time-based results. Likewise, arrival rates used in the models must be based on normalized time units.

From the definition of T_s, the expected value and variance for service time are as follow:

$$E[T_s] = E[T] + T_{PD}, \tag{2.1}$$

and

$$V[T_s] = V[T]. \tag{2.2}$$

To facilitate classifying the various scenarios and cases considered, we let F denote a finite population, I denote an infinite population, D denote a dedicated storage policy, and R denote a random storage policy. Thus, FDE$[T]$ denotes the expected value of NALT travel time for a finite population model with dedicated storage and IR denotes the scenario for an infinite population queue with random storage.

2.4. Probability density functions and Laplace transforms

The NALT travels from a pick position to the storage location for a unit load, then travels to a pick position requiring replenishment. The same travel sequence occurs when the NALT is performing a replenishment after being idle or completing a replenishment at a pick position.

In this section, we model travel of the NALT when dedicated storage and random storage are used. Figure 1 portrays storage rack with 8 levels and 25 columns of storage. The floor (bottom) level of the rack contains the pick positions. Blue dashed lines define the continuous region within which the forks on the NALT travel. Based on the travel region, we establish coordinates shown in Figure 2. Specifically, horizontal travel occurs between the coordinates (0, 0) and (1, 0); vertical travel occurs between the coordinates (0, 0) and (0, b).

As the red dashed line in Figure 1 illustrates, for dedicated storage, the NALT travels from a pick position (the pick position previously replenished) to a storage location above the pick position requiring replenishment to retrieve a unit load and delivers it to the pick position initiating the replenishment request; therefore, the storage location and pick position have the same *x*-coordinate. For random storage, the NALT travels to a uniformly distributed random storage location in the rack to retrieve a unit load and delivers it to the pick position initiating the replenishment request.

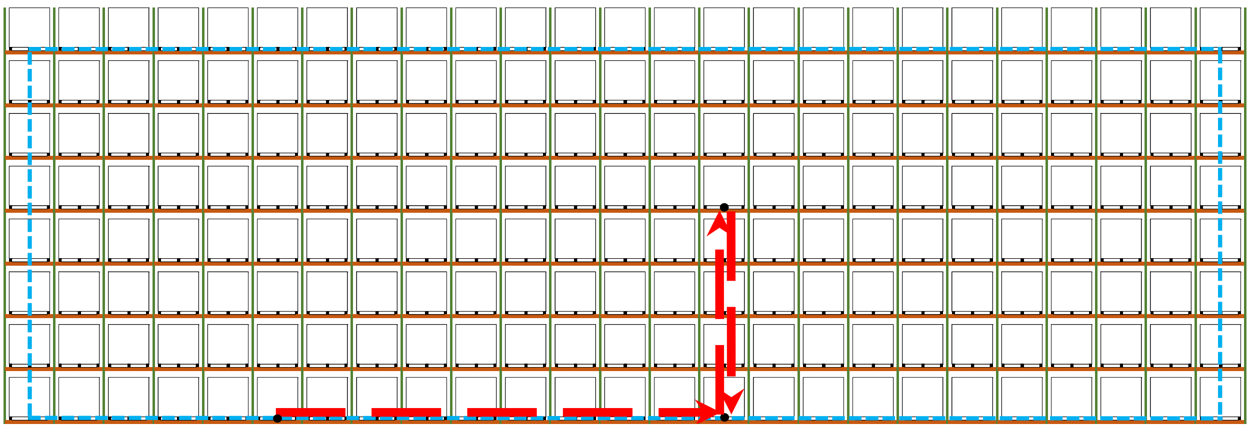

Figure 1. Storage rack front view.

With a finite number of pick positions at the floor (bottom) level of the rack, the NALT travels from one of N pick positions on the floor level of the rack to a uniformly distributed vertical storage location within the rack and, then, to one of N pick positions on the floor level of the rack. With an infinite number of pick positions, the NALT travels from a uniformly distributed point at the floor level to a random storage location within the rack and, then, to another random point at the floor level (the starting point can be the same as the ending point because the next pick position requesting replenishment can be on the opposite side of the replenishment aisle from the previously replenished pick position).

With two storage policies (dedicated and random) and either a finite or an infinite number of pick positions, there are four distinct scenarios (FD, FR, ID and IR defined in Section 3) to be considered. As a result, four sets of pdfs are developed for the travel time required for the NALT to perform a retrieval operation. For the FD and FR scenarios, Laplace transforms, expected values and variances for the two sets of pdfs are obtained. For the ID and IR scenarios, only expected values and variances are obtained because Pollaczek-Khinchine formulas are used to obtain the performance measures for the infinite population queueing models. Service time, T_s, is the sum of T and T_{PD}. For the FC and FR scenarios, we obtain the Laplace transform for service time. For the ID and IR scenarios, we obtain the expected value and variance for service time.

2.4.1. Probability density functions for a finite population and a dedicated storage policy

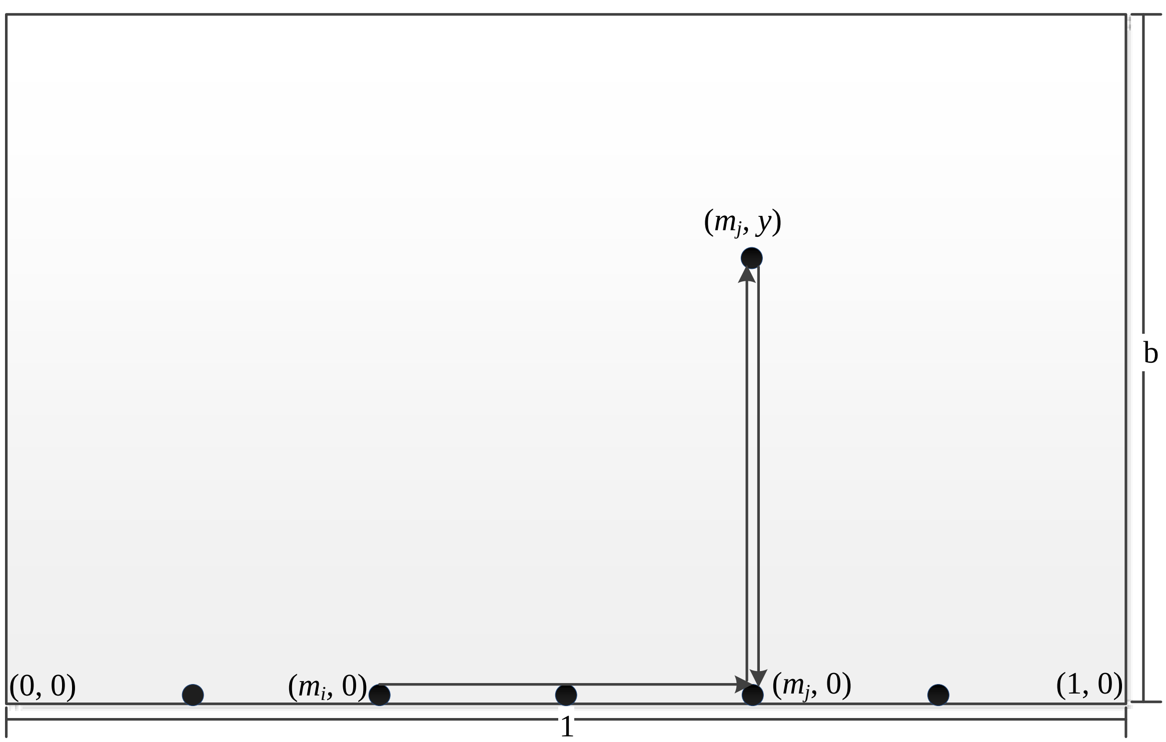

Figure 2. Travel for a finite population and a dedicated storage policy.

For the FD scenario, to perform a replenishment when the replenishment load is located immediately above the pick position to be replenished (dedicated storage), as Figure 2 illustrates, NALT travel originates at a pick position, $(m_i, 0)$; next, the NALT travels rectilinearly to a uniformly distributed storage point (m_j, y) within the rack and, then, travels rectilinearly to another pick position $(m_j, 0)$ where $y \in \text{unif}(0, b)$. Letting T_{ij} denote the total travel time,

$$T_{ij} = |m_i - m_j| + 2y.$$

Letting $m_{[1]} = min(m_i, m_j)$ and, $m_{[2]} = max(m_i, m_j)$ then

$$T_{ij} = m_{[2]} - m_{[1]} + 2y.$$

Because $y \in \text{unif}(0, b)$, the pdf for y is $f(y) = \frac{1}{b}$. Therefore, the cumulative distribution function (cdf) for T_{ij} is

$$F(t_{ij}) = Pr(m_{[2]} - m_{[1]} + 2y \le t_{ij}) = Pr\left(y \le \frac{t_{ij} + m_{[1]} - m_{[2]}}{2}\right) = \int_0^{\frac{t_{ij}+m_{[1]}-m_{[2]}}{2}} f(y)dy$$

$$= \frac{t_{ij}+m_{[1]}-m_{[2]}}{2b} \qquad t_{ij} \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}. \tag{2.3}$$

From Equation (3), the pdf is

$$f(t_{ij}) = F'(t_{ij}) = \frac{1}{2b} \qquad t_{ij} \in \left[m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}\right]. \tag{2.4}$$

From Equation (4), the expected value of T_{ij} is

$$E[T_{ij}] = \int t_{ij} f(t_{ij}) dt_{ij} = m_{[2]} - m_{[1]} + b \tag{2.5}$$

and the variance of T_{ij} is obtained as follows:

$$E[T_{ij}{}^2] = \int t_{ij}{}^2 f(t_{ij}) dt_{ij} = \frac{4b^2}{3} + 2b(m_{[2]} - m_{[1]}) + (m_{[2]} - m_{[1]})^2 \tag{2.6}$$

$$V[T_{ij}] = E[T_{ij}{}^2] - (E[T_{ij}])^2 = \frac{b^2}{3}. \tag{2.7}$$

Because the NALT serves both sides of the S/R aisle, with N pick positions, there are N^2 possible combinations of pick position i and pick position j. Let $P_{ij} = \frac{1}{N^2}$ be the probability the NALT travels from pick position i to replenish pick position j. Because T_{ij} is the time for the NALT to travel from pick positon i to replenish pick position j, the travel time to perform a replenishment is

$$T = \sum_{i=1}^{N} \sum_{j=1}^{N} T_{ij} P_{ij}. \tag{2.8}$$

From Equation (2.8), the expected value and variance for T are obtained as follows:

$$E[T] = \sum_{i=1}^{N} \sum_{j=1}^{N} E\left[T_{ij}\right] P_{ij}, \tag{2.9}$$

$$E[T^2] = \sum_{i=1}^{N} \sum_{j=1}^{N} E\left[{T_{ij}}^2\right] P_{ij}, \tag{2.10}$$

and

$$V[T] = E[T^2] - (E[T])^2. \tag{2.11}$$

The expected service time is

$$E[T_s] = E[T] + T_{PD} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E[T_{ij}] + 2c. \tag{2.12}$$

From the pdf in Equation (4), the Laplace transform of the pdf for T_{ij} is

$$L\left[f\left(t_{ij}\right)\right] = \int e^{-st_{ij}} f(t_{ij})\, dt_{ij} = \frac{e^{s(m_{[1]}-m_{[2]})}(1-e^{-2bs})}{2bs}. \tag{2.13}$$

From Equation (8), the Laplace transform of the pdf for T is

$$L[f(t)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{e^{s(m_{[1]}-m_{[2]})}(1-e^{-2bs})}{2bs} P_{ij} = \frac{(1-e^{-2bs})}{2bN^2 s} \sum_{i=1}^{N} \sum_{j=1}^{N} e^{s(m_{[1]}-m_{[2]})}. \tag{2.14}$$

For the FD scenario (finite population with dedicated storage), the Laplace transform of the deterministic T_{PD} is

$$L[T_{PD}] = e^{-2cs}. \tag{2.15}$$

Therefore, the Laplace transform of the pdf for T_s is

$$L[f(t_s)] = L[T_{PD}]L[f(t)] = \frac{e^{-2cs}(1-e^{-2bs})}{2bN^2 s} \sum_{i=1}^{N} \sum_{j=1}^{N} e^{s(m_{[1]}-m_{[2]})}. \tag{2.16}$$

2.4.2. Probability density functions for a finite population and a random storage policy

For the FR scenario, as Figure 3 illustrates, NALT travel originates at a pick position, $(m_i, 0)$; next, it travels rectilinearly to a random storage point (x, y) within the rack and then travels rectilinearly to another pick position $(m_j, 0)$ where $x \in \text{unif}(0,1)$, $y \in \text{unif}(0,b)$. Therefore, $T_{ij} = |x - m_i| + y + |x - m_j| + y = |x - m_i| + |x - m_j| + 2y$.

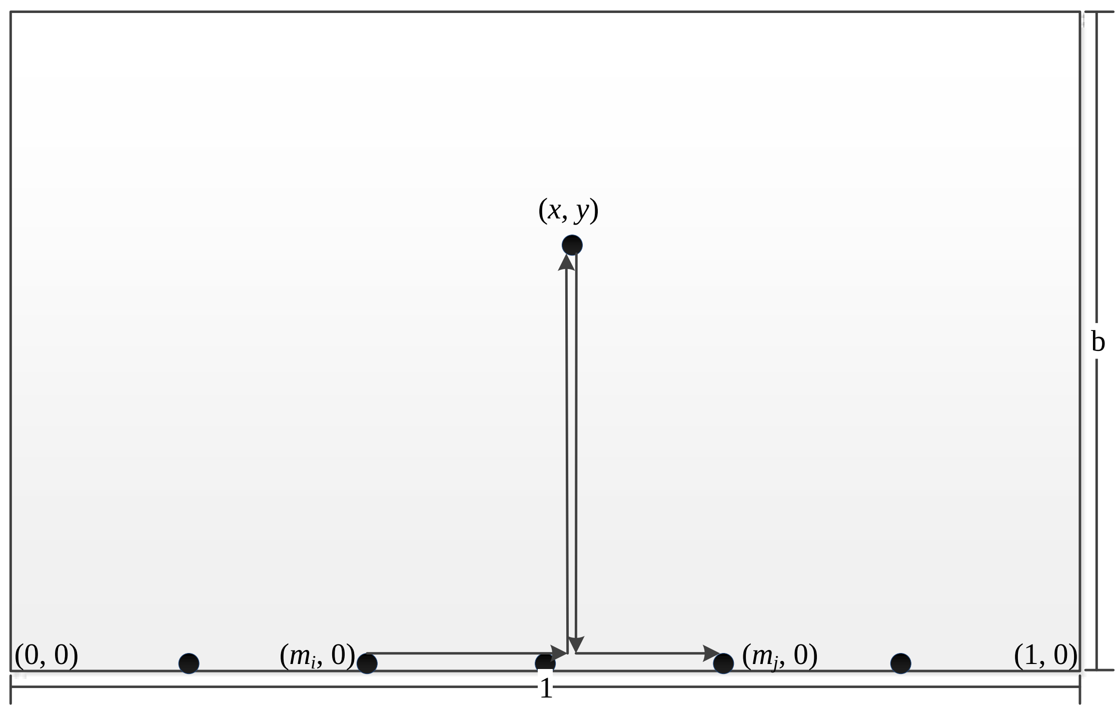

Figure 3. Travel for a finite population and a random storage policy.

Letting $m_{[1]} = min(m_i, m_j)$ and $m_{[2]} = max(m_i, m_j)$, the pdfs will vary according to the relative values of $m_{[1]}$, $m_{[2]}$ and b. (Derivations of the pdfs are provided in Appendix A.)

When $m_{[1]} \geq 1 - m_{[2]} \geq b$ and $m_{[1]} - b \geq 1 - m_{[2]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ 2 - m_{[2]} - m_{[1]}] \\ \frac{4b+2-m_{[2]}-m_{[1]}-t}{4b} & t \in (2 - m_{[1]} - m_{[2]}, 2 + 2b - m_{[1]} - m_{[2]}] \\ \frac{1}{2} & t \in (\ 2 + 2b - m_{[1]} - m_{[2]}, m_{[1]} + m_{[2]}] \\ \frac{2b+m_{[1]}+m_{[2]}-t}{4b} & t \in [m_{[1]} + m_{[2]},\ 2b+m_{[1]} + m_{[2]}] \end{cases} . \quad (2.17)$$

When $m_{[1]} \geq 1 - m_{[2]} \geq b$ and $m_{[1]} - b < 1 - m_{[2]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ 2 - m_{[1]} - m_{[2]}] \\ \frac{4b+2-m_{[1]}-m_{[2]}-t}{4b} & t \in (2 - m_{[1]} - m_{[2]}, m_{[1]} + m_{[2]}] \\ \frac{2b+2-1-t}{2b} & t \in (\ m_{[1]} + m_{[2]}, 2 + 2b - m_{[1]} - m_{[2]}] \\ \frac{2b+m_{[1]}+m_{[2]}-t}{4b} & t \in [2 + 2b - m_{[1]} - m_{[2]},\ 2b + m_{[1]} + m_{[2]}] \end{cases} . \tag{2.18}$$

When $1 - m_{[2]} \geq m_{[1]} \geq b$ and $1 - m_{[2]} - b \geq m_{[1]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ m_{[1]} + m_{[2]}] \\ \frac{4b+m_{[1]}+m_{[2]}-t}{4b} & t \in (m_{[1]} + m_{[2]}, 2b + m_{[1]} + m_{[2]}] \\ \frac{1}{2} & t \in (\ 2b + m_{[1]} + m_{[2]}, 2 - m_{[1]} - m_{[2]}] \\ \frac{2b+2-m_{[1]}-m_{[2]}-t}{4b} & t \in [2 - m_{[1]} - m_{[2]},\ 2 + 2b - m_{[1]} - m_{[2]}] \end{cases} \tag{2.19}$$

When $1 - m_{[2]} \geq m_{[1]} \geq b$ and $1 - m_{[2]} - b < m_{[1]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ m_{[1]} + m_{[2]}] \\ \frac{4b+m_{[1]}+m_{[2]}-t}{4b} & t \in \left(m_{[1]} + m_{[2]}, 2 - m_{[1]} - m_{[2]}\right] \\ \frac{2b+1-t}{2b} & t \in (\ 2 - m_{[1]} - m_{[2]}, 2b + m_{[1]} + m_{[2]}] \\ \frac{2b+2-m_{[1]}-m_{[2]}-t}{4b} & t \in [2b + m_{[1]} + m_{[2]},\ 2 + 2b - m_{[1]} - m_{[2]}] \end{cases} . \tag{2.20}$$

When $b \geq m_{[1]} \geq 1 - m_{[2]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [c,\ 2 - m_{[1]} - m_{[2]}] \\ \frac{2+t-m_{[1]}-m_{[2]}}{4b} & t \in [2 - m_{[1]} - m_{[2]},\ m_{[1]} + m_{[2]}] \\ \frac{1}{2b} & t \in (m_{[1]} + m_{[2]}, 2b + m_{[2]} - m_{[1]}] \\ \frac{2b+1-t}{2b} & t \in (\ 2b + m_{[2]} - m_{[1]}, 2 + 2b - m_{[1]} - m_{[2]}] \\ \frac{2b+m_{[1]}+m_{[2]}-t}{4b} & t \in [2 + 2b - m_{[1]} - m_{[2]}, 2b + m_{[1]} + m_{[2]}] \end{cases} . \tag{2.21}$$

When $b \geq 1 - m_{[2]} \geq m_{[1]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [c,\ m_{[1]} + m_{[2]}] \\ \frac{t+m_{[1]}+m_{[2]}}{4b} & t \in [m_{[1]} + m_{[2]},\ 2 - m_{[1]} - m_{[2]}] \\ \frac{1}{2b} & t \in (2 - m_{[1]} - m_{[2]}, 2b + m_{[2]} - m_{[1]}] \\ \frac{2b+1-t}{2b} & t \in (\ 2b + m_{[2]} - m_{[1]}, 2b + m_{[1]} + m_{[2]}] \\ \frac{2b+2-m_{[1]}-m_{[2]}-t}{4b} & t \in [2b + m_{[1]} + m_{[2]}, 2 + 2b - m_{[1]} - m_{[2]}] \end{cases} . \qquad (2.22)$$

Although there are six different sets of pdfs, because of the relative values of $m_{[1]}$, $m_{[2]}$ and b, the formulas for the expected value, variance and the Laplace transform for the NALT travel time are identical for the six sets of pdfs. From the pdfs, the expected value, second moment and variance are:

$$E[T_{ij}] = \int t_{ij} f(t_{ij})\, dt_{ij} = 1 + b - m_{[1]} - m_{[2]} + {m_{[1]}}^2 + {m_{[2]}}^2, \qquad (2.23)$$

$$E[{T_{ij}}^2] = \int {t_{ij}}^2 f(t_{ij})\, dt_{ij} = \frac{1}{3}(4 + 6b + 4b^2 - 6m_{[2]} - 6bm_{[2]} + 3{m_{[2]}}^2 + 6b{m_{[2]}}^2 + 2{m_{[2]}}^3 - 6m_{[1]} - 6bm_{[1]} + 6m_{[1]}m_{[2]} - 6{m_{[2]}}^2 m_{[1]} + 3{m_{[1]}}^2 + 6b{m_{[1]}}^2 + 6m_{[2]}{m_{[1]}}^2 - 2{m_{[1]}}^3), \qquad (2.24)$$

and

$$V[T_{ij}] = E[{T_{ij}}^2] - \left(E[{T_{ij}}^2]\right)^2 = 1/3\,(1 + b^2 + 8{m_{[2]}}^3 - 3{m_{[2]}}^4 - 6{m_{[1]}}^2 + 12m_{[2]}{m_{[1]}}^2 + 4{m_{[1]}}^3 - 3{m_{[1]}}^4 - 6{m_{[2]}}^2(1 + {m_{[1]}}^2)). \qquad (2.25)$$

From the pdfs in Equations (17) through (22), the Laplace transform of the pdf for T_{ij} is

$$L[f(t_{ij})] = \int e^{-st_{ij}} f(t_{ij})\, dt_{ij} = \frac{1}{4bs^2}(e^{-2bs} - 1)(e^{-s(m_{[1]}+m_{[2]})} + e^{-2s}e^{s(m_{[1]}+m_{[2]})} + 2(-1 + (m_{[1]} - m_{[2]})s)e^{s(m_{[1]}-m_{[2]})}), \qquad (2.26)$$

and the Laplace transform of the pdf for T (from Equation (8)) is

$$L[f(t)] = \sum_{i=1}^{N}\sum_{j=1}^{N} L\left[f\left(t_{ij}\right)\right]P_{ij} = \frac{(e^{-2bs}-1)}{4bN^2s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{e^{-s\left(m_{[1]}+m_{[2]}\right)} + e^{-2s}e^{s\left(m_{[1]}+m_{[2]}\right)}$$

$$+2\left[-1+(m_{[1]}-m_{[2]})\right]se^{s(m_{[1]}-m_{[2]})}\}. \quad (2.27)$$

For the FR scenario, from Equation (15) and (27), the Laplace transform of the pdf for T_s is

$$L[f(t_s)] = L[T_{PD}]L[f(t)] = \frac{e^{-2cs}(e^{-2bs}-1)}{4bN^2s^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\{e^{-s\left(m_{[1]}+m_{[2]}\right)} + e^{-2s}e^{s\left(m_{[1]}+m_{[2]}\right)}$$

$$+2\left[-1+(m_{[1]}-m_{[2]})\right]se^{s(m_{[1]}-m_{[2]})}\}. \quad (2.28)$$

2.4.3. Probability density functions for an infinite population and a dedicated storage policy

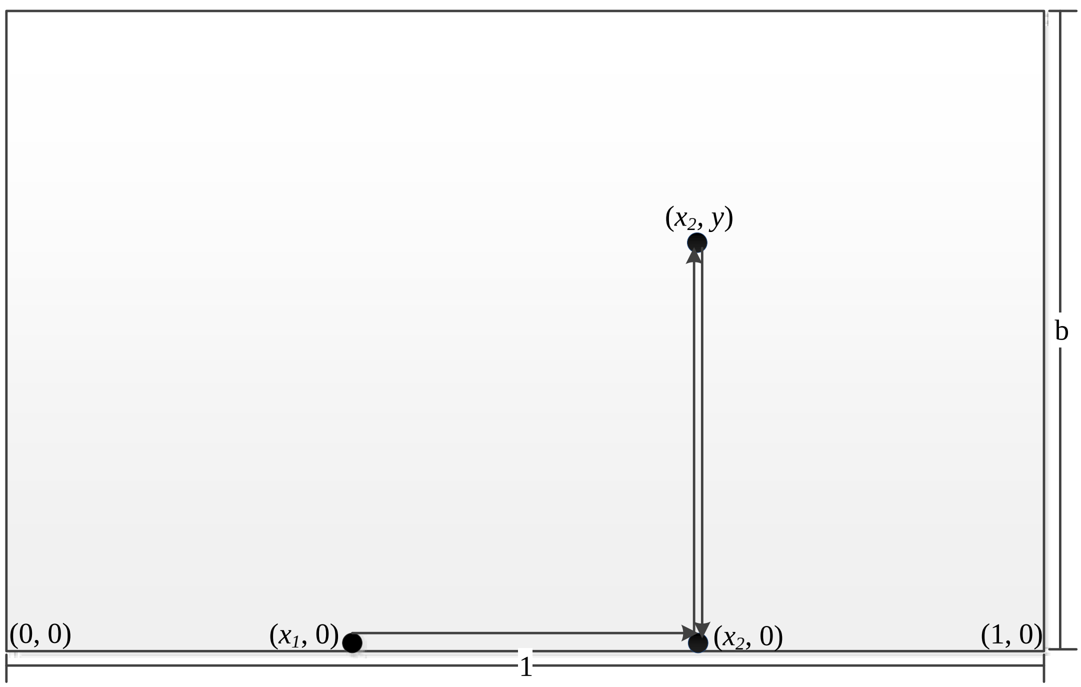

Figure 4. Travel for an infinite population and a dedicated storage policy.

For the ID scenario, as Figure 4 illustrates, NALT travel originates at a random pick position, $(x_1, 0)$, from which the NALT travels rectilinearly to a random storage point (x_2, y) within the rack and, then, travels rectilinearly to another random pick position $(x_2, 0)$ where $x_1, x_2 \in$

unif$(0,1)$ and y ∈ unif$(0,b)$. Therefore, $T = |x_2 - x_1| + 2y$.

From Appendix B, the pdfs are:

Case 1, $b \leqslant 0.5$,

$$f(t) = \begin{cases} \frac{(2-t)t}{2b} & t \in [0,2b] \\ 2+2b-2t & t \in (2b,1], \\ \frac{(t-1-2b)^2}{2b} & t \in (1,1+2b] \end{cases} \tag{2.29}$$

Case 2, $b > 0.5$

$$f(t) = \begin{cases} \frac{(2-t)t}{2b} & t \in [0,1] \\ \frac{1}{4b} & t \in (1,2b]. \\ \frac{(t-1-2b)^2}{2b} & t \in (2b,1+2b] \end{cases} \tag{2.30}$$

Although the pdfs for both cases differ, the formulas for the expected value and variance for Case 1 are identical to the formulas for the expected value and variance for Case 2:

$$E[T] = \int tf(t)\,dt = \frac{1}{3} + b, \tag{2.31}$$

$$E[T^2] = \int t^2 f(t)\,dt = \frac{1}{6}(1+4b+8b^2), \tag{2.32}$$

and

$$V[T] = E[T^2] - (E[T])^2 = \frac{1}{18}(1+6b^2). \tag{2.33}$$

For the ID scenario, the expected value and variance for service time are:

$$E[T_s] = E[T] + T_{PD} = \frac{1}{3} + b + 2c, \tag{2.34}$$

$$V[T_s] = \frac{1}{18}(1+6b^2). \tag{2.35}$$

2.4.4. Probability density functions for an infinite population and a random storage policy

For the IR scenario, as Figure 5 illustrates, NALT travel originates at a random pick position, $(x_1, 0)$; next, the NALT travels rectilinearly to a random storage point (x, y) within the rack and then travels rectilinearly to a random replenishment point $(x_2, 0)$ where $x, x_1, x_2 \in \text{unif}(0, 1)$ and $y \in \text{unif}(0, b)$. Letting T denote total travel time, $T = T_1 + T_2$, where T_1 is horizontal travel time and T_2 is vertical travel time. Therefore, $T_1 = |x_1 - x| + |x_2 - x|$, and $T_2 = 2y$.

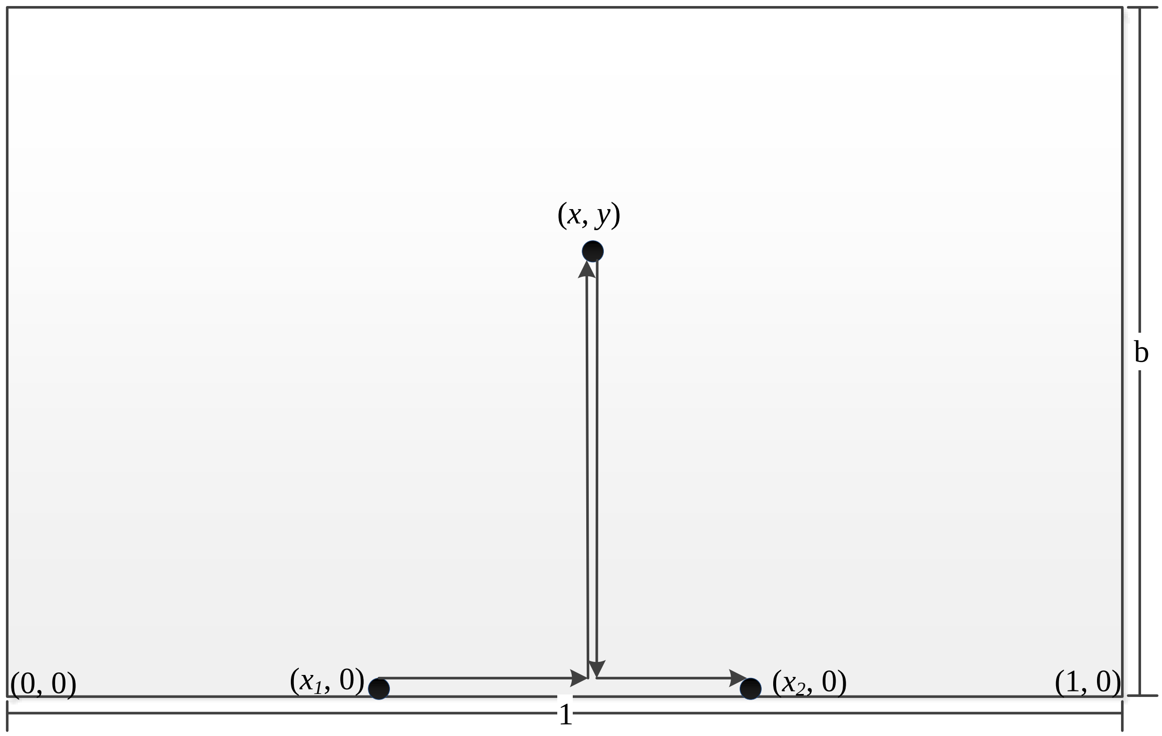

Figure 5. Travel for an infinite population and a random storage policy.

From Appendix C, the pdf of T_1 is

$$f(t_1) = \begin{cases} -\frac{7}{2}{t_1}^2 + 4t_1 & t_1 \in [0,1] \\ \frac{1}{2}{t_1}^2 - 2t_1 + 2 & t_1 \in (1,2] \end{cases}. \tag{2.36}$$

The cdf for vertical travel time is

$$F(t_2) = Pr(2y \leq t_2) = Pr(y \leq t_2/2) = \int_0^{t_2/2} \frac{1}{b} dt = \frac{t_2}{2b}. \tag{2.37}$$

From the cdf in Equation (37), the pdf of vertical travel time can be expressed as

$$f(t_2) = \frac{1}{2b} \qquad t_2 \in [0, 2b]. \tag{2.38}$$

From the pdfs in Equation (36) and Equation (38), the expected value and variance for travel time are obtained as follows:

$$E[T] = E[T_1] + E[T_2] = \int t_1 f(t_1)\, dt_1 + \int t_2 f(t_2)\, dt_2 = E[T_1] + E[T_2] = \frac{2}{3} + b, \quad (2.39)$$

$$V[T_1] = \left(E\left[{T_1}^2\right]\right) - (E[T_1])^2 = \int {t_1}^2 f(t_1)\, dt_1 - \left(\int t_1 f(t_1)\, dt_1\right)^2 = \frac{11}{90}, \quad (2.40)$$

$$V[T_2] = \left(E\left[{T_2}^2\right]\right) - (E[T_2])^2 = \int {t_2}^2 f(t_2)\, dt_2 - \left(\int t_2 f(t_2)\, dt_2\right)^2 = \frac{b^2}{3}, \quad (2.41)$$

and

$$V[T] = V[T_1] + V[T_2] = \frac{11}{90} + \frac{b^2}{3}. \quad (2.42)$$

For the IR scenario, the expected value and variance for service time are:

$$E[T_s] = E[T] + T_{PD} = \frac{2}{3} + b + 2c, \quad (2.43)$$

and

$$V[T_s] = \frac{11}{90} + \frac{b^2}{3}. \quad (2.44)$$

2.5. The replenishment queueing model

In the MIAPP-NALT system, travel of the NALT originates at a pick positon and ends after replenishing another pick position. Assuming demands for replenishment from individual pick positions occur in a Poisson fashion, based on results from the previous section, the MIAPP-NALT system with a finite number of pick positions is modeled as an M/G/1/N queue. When the interarrival time for replenishment requests is exponentially distributed and the number of customers is large, an M/G/1/∞ queueing model is used.

2.5.1. Finite population queueing

From the work of Takács (1960) contained in Sztrik (2016), the server's idle probability is:

$$P_0 = \left[1 + \frac{N\lambda}{\mu} \sum_{i=0}^{N-1} \binom{N-1}{i} B_i\right]^{-1}, \tag{2.45}$$

where

$$B_i = \begin{cases} 1 & i = 0 \\ \prod_{j=1}^{i} \left(\frac{1-L(j\lambda)}{L(j\lambda)}\right) & i = 1, 2, \dots, N-1 \end{cases} \tag{2.46}$$

and $L(s)$ is the Laplace-Stieltjes transform of the service time density function from Section 4.

Therefore, for the FD scenario:

$$L(j\lambda) = \frac{e^{-2c(j\lambda)}\left(1-e^{-2b(j\lambda)}\right)}{2bN^2(j\lambda)} \sum_{i=1}^{N} \sum_{j=1}^{N} e^{(j\lambda)\left(m_{[1]}-m_{[2]}\right)}. \tag{2.47}$$

For the FR scenario:

$$L(j\lambda) = \frac{e^{-2c(j\lambda)}(e^{-2b(j\lambda)}-1)}{4bN^2(j\lambda)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \{e^{-(j\lambda)\left(m_{[1]}+m_{[2]}\right)} + e^{-2(j\lambda)} e^{(j\lambda)\left(m_{[1]}+m_{[2]}\right)}$$

$$+2\left[-1 + \left(m_{[1]} - m_{[2]}\right)\right](j\lambda) e^{(j\lambda)\left(m_{[1]}-m_{[2]}\right)}\}. \tag{2.48}$$

From Equation (45), the steady-state performance measures for the M/G/1/N queueing model are:

$$\overline{\lambda} = (1 - P_0)\mu, \tag{2.49}$$

$$W = \frac{N}{\overline{\lambda}} - \frac{1}{\lambda}, \tag{2.50}$$

$$L = \overline{\lambda} W, \tag{2.51}$$

$$W_q = W - \frac{1}{\mu}, \tag{2.52}$$

$$L_q = \overline{\lambda} W_q. \tag{2.53}$$

2.5.2. Infinite population queueing

For the M/G/1/∞ queueing model, the following Pollaczek-Khinchine formulas are used to obtain the performance measures for the MIAPP-NALT system Gross et al. (2008):

$$L_q = \frac{\lambda^2 V[T_s] + \rho^2}{2(1-\rho)}, \tag{2.54}$$

$$L = \rho + \frac{\lambda^2 V[T_s] + \rho^2}{2(1-\rho)}, \tag{2.55}$$

$$W_q = \frac{\lambda(V[T_s] + 1/\mu^2)}{2(1-\rho)}, \tag{2.56}$$

$$W = \frac{1}{\mu} + \frac{\lambda(V[T_s] + 1/\mu^2)}{2(1-\rho)}. \tag{2.57}$$

2.6. Model validation, parameter estimation, computational results, and insights

In this section, we use a simulation model to validate the M/G/1/N queueing model in Section 5.1. Specifically, we compare the results obtained for the expected number of empty deposit locations at pick positions (L) obtained from the analytical model with the results obtained from the simulation model when pick positions have the same arrival rates and when the arrival rates differ. Next, we examine the distribution of times between consecutive replenishment requests within the S/R aisle obtained from the simulation model. In section 6.2, we consider the connection between population size and traffic intensity for an infinite population queueing approximation to be reasonably accurate in calculating the performance measures for the MIAPP-NALT system. In section 6.3, we compare dedicated and random storage policies from queueing models. In the last section, we provide important insights and address practical considerations obtained from the results.

To illustrate how the queueing models can be applied, suppose there are 3,000 SKUs for which case picking is performed and three alternative designs are under consideration: 30 S/R aisles with

50 pick positions on each side of the aisle (N=100); 20 S/R aisles with 75 pick positions on each side of the aisle (N=150); and 15 S/R aisles, with 100 pick positions on each side of the aisle (N=200). If spacing between the centerlines of pick positions is 4 feet, the length of the S/R aisle (rack), L, is $4(N/2 - 1)$. If the storage rack consists of 8 levels, with spacing of 5 feet between levels, the distance from the floor level to the bottom of the top level, H, is $5(8 - 1)$ or 35 feet. If the horizontal velocity of the NALT, V_h, is 255 feet per minute, it takes $4(N/2 - 1)/255$ minutes for the NALT to travel between the pick positions at opposite ends of the S/R aisle. If the vertical velocity of the NALT, V_v, is 50 feet per minute, it takes 35/50 or 0.70 minutes to travel from the pick position to the upper level of the storage rack. Normalizing the storage region by dividing both the maximum horizontal time and the maximum vertical time by the maximum horizontal time, for N equal to 100, 150, and 200, the time required to travel the length of the S/R aisle, t_h, is 0.7686 minutes, 1.1608 minutes, and 1.5529 minutes, respectively, and the time required to travel the height of the S/R aisle, t_v, measured in a normalized time unit, is 0.70/0.7686 or 0.9107, 0.70/1.1608 or 0.6030, and 0.70/1.5529 or 0.4508 for N equal to 100, 150, and 200, respectively. Therefore, b is equal to 0.9107, 0.6030, and 0.4508 for N equal to 100, 150, and 200, respectively.

2.6.1. Simulation model

The simulation model of the MIAPP-NALT operation differs from the analytical finite queueing model in several ways. First, replenishment requests are based on the time required for an order picker to deplete a full pallet of cases. So, no particular distribution is assumed. The random variables in the simulation include a Bernoulli distribution used to determine if a particular SKU is included in the pick list given to an order picker, a discrete distribution for the number of cases of the SKU to be picked, and a discrete distribution for the number of cases on a full pallet for a particular SKU. The pick time and the time to walk from one pick location to an adjacent pick location are deterministic. In addition, SKUs included on the pick list are in sequential order, not a random order; this is an example of actual practice differing from an assumption of the M/G/1/N analytical model. The latter assumes replenishment requests are served on a first-come, first-served basis and the replenishment requests are statistically independent. By picking SKUs in sequential order, rather than randomly, lower-numbered SKUs are picked first.

Two simulation models were employed for an MIAPP-NALT operation for the optional design above with 150 pick positions, 75 on each side of the S/R aisle: one based on homogeneous customers and the other based on heterogeneous customers. The homogeneous customer simulation model employed a Bernoulli probability of 0.03, one or two cases being equally likely to be picked, and 42 cases on a full pallet, the same distribution for number of cases picked per SKU and order pickers were assumed to walk at a speed of 255 feet per minute. The average time required to deplete a pallet from the simulation was used to establish the arrival rate for the M/G/1/*N* model. To create twelve traffic intensities for the heterogeneous customer simulation and twelve traffic intensities for the homogeneous customer simulation, as shown in Table 1, the number of order pickers and the pick time for a case were changed. For the homogeneous case and for each traffic intensity, Figure 6 provides the expected number of empty deposit positions obtained from 50 replications; each replication ran for 500 hours, following a "warm up period" of 50 hours. Notice, how closely the M/G/1/N results approximate the simulation results.

Table 1. Combination of number of pickers and time to pick a case for homogeneous and heterogeneous customers.

homogeneous customers		heterogeneous customers	
# pickers	time to pick a case (in seconds)	# pickers	time to pick a case (in seconds)
3	4.50	2	4.90
4	6.00	2	3.50
4	5.00	2	3.20
4	4.00	3	6.00
4	3.50	3	5.00
4	3.00	3	4.80
5	5.00	3	4.60
5	4.80	3	4.40
5	4.60	3	4.20
5	4.40	3	4.00
5	4.30	3	3.90
5	4.25	3	3.80

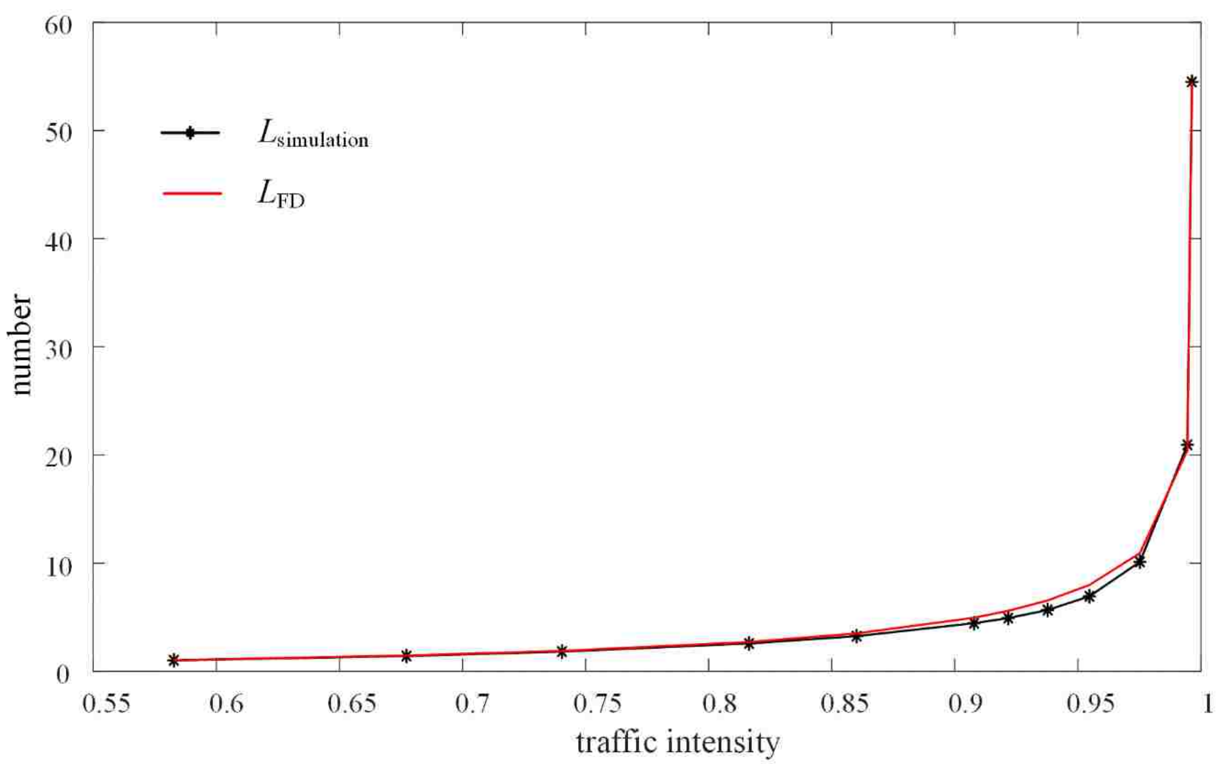

Figure 6. For twelve traffic intensities considered, the expected number of empty deposit positions obtained from the finite population queueing model and from simulation with homogeneous customers.

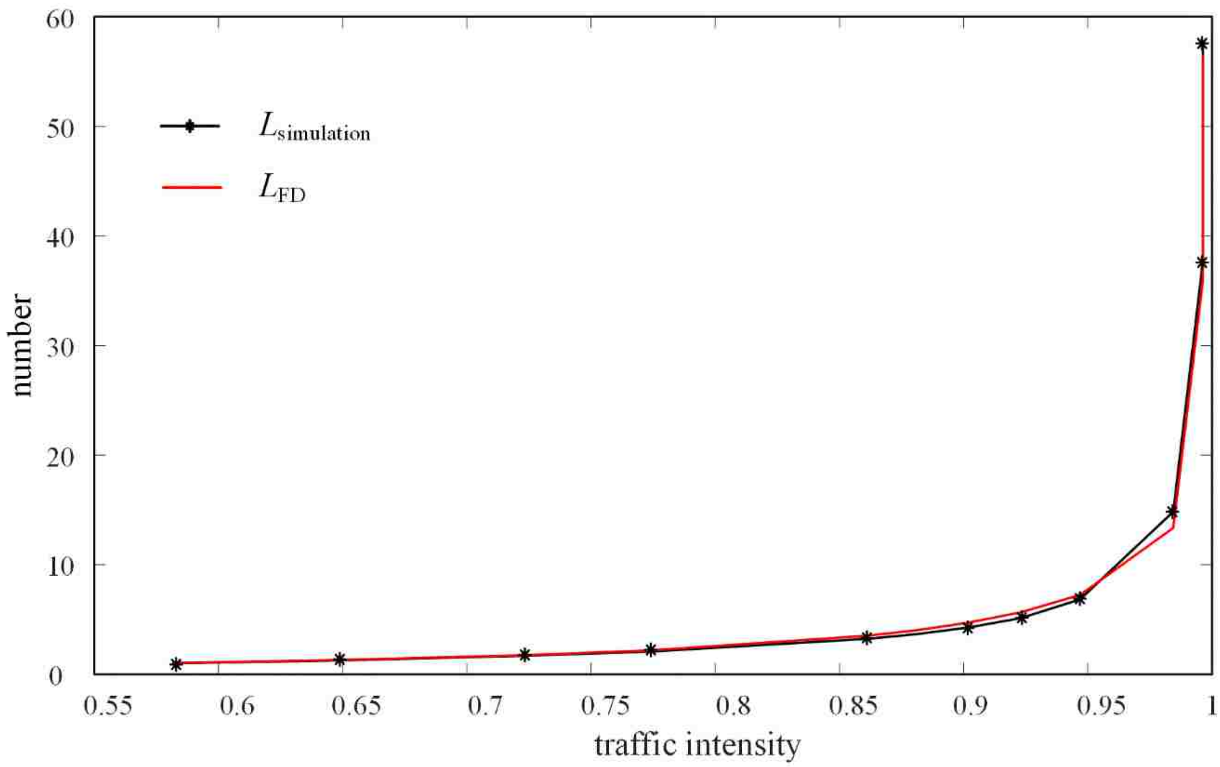

Figure 7. For twelve traffic intensities considered, the expected number of empty deposit positions obtained from the finite population queueing model and from simulation with heterogeneous customers.

For the heterogeneous customer case, SKUs were randomly assigned to one of four groups; the Bernoulli probabilities for the groups were 0.01, 0.03, 0.05, and 0.07. Five pallet configurations were included, with the number of cases per pallet being 24, 30, 36, 42, and 48. The following distributions for number of cases picked were assumed: equally likely to be either one or two cases picked; one case picked with probability 0.20, two cases picked with probability 0.20, three cases picked with probability 0.30, and four cases picked with probability 0.30; one case picked with probability 0.50, two cases picked with probability 0.30, three cases picked with probability 0.15, and four cases picked with probability 0.05; and equally likely to be one, two, three or four cases picked. The distributions were randomly allocated to the 150 skus. Table 1 provides the number of order pickers and the time required to pick a case. The same walking speed was used for order pickers, 255 feet per minute. The results obtained from the simulation model and analytical model are provided in Figure 7. Again, even though the analytical model assumes homogeneous customers, the results indicate an M/G/1/N model with homogeneous customers can still be used to provide reasonable estimates for the performance measures for heterogeneous customers.

The distribution of times between consecutive requests for replenishment of heterogeneous customers, obtained from the simulation, is provided in Figure 8. The average of the data in Figure 8 is 1.425 minutes; the standard deviation of the data is 1.366 minutes. Because the mean and standard deviation are equal for an exponential distribution, by the method of moments we concluded the distribution of the time between consecutive "arrivals" can be reasonably approximated by an exponential distribution, which fits the assumption of an M/G/1/∞ queueing model. The latter raises the issue of how accurately an infinite population model approximates the performance of an MIAPP-NALT operation.

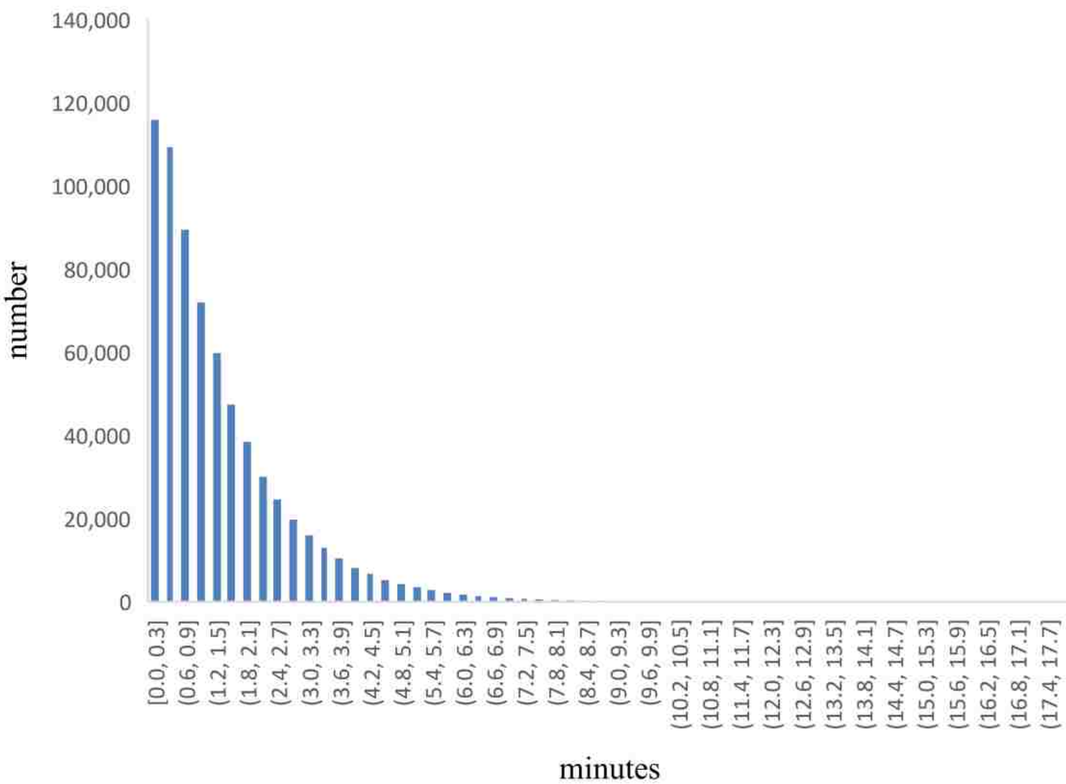

Figure 8. Histogram of replenishment interarrival times from simulation results for heterogeneous customers.

2.6.2. Approximating the finite population model with an infinite population model

Employing an infinite population approximation for the MIAPP-NALT system requires the selection of an arrival rate. For a finite population, the arrival rate for an individual pick position is the reciprocal of the expected time between replenishment of the pick position and the next replenishment request by the pick position. For an infinite population, the arrival rate is the reciprocal of the expected time between two consecutive replenishment requests, regardless of the sources of the replenishment requests. When approximating a finite population retrieval operation with an infinite population model, what should be the arrival rate? Based on a dedicated storage policy (the same conclusion can be reached with a random storage policy) we chose L, the expected number of pick positions waiting to be replenished, to assess the accuracy of the approximation. Specifically, we let the arrival rate for an infinite population queue equal the effective arrival rate for the finite population queue. Continuing the illustration of the application of the queueing models, the size of the finite population (N) is set equal to 100, 150 and 200.

The expected values and variances of service times for finite and infinite populations for three population sizes (100, 150 and 200) are given in Table 2 for both dedicated and random storage. Included in service time is the deterministic time to perform pick up and deposit of the unit load (0.30 minute, converted to normalized time, with the conversion dependent on t_h). Based on the parameter values in Table 2, values of L from queueing models in Section 5 are provided in Table 3, Table 4 and Table 5 for the respective population sizes and FD and ID scenarios, with a Poisson arrival rate ranging from 0.001 per minute to 0.009 per minute for each pick position and the arrival rate for the ID scenario equaling the effective arrival rate from the FD scenario. (Because the pdf for the finite population case differs from the pdf for the infinite population case, when the same arrival rate is used for both cases, the traffic intensities will differ.)

Table 2. Normalized service time parameters for the dedicated (random) storage queueing model.

N	Finite population		Infinite population	
	Expected Value	Variance	Expected Value	Variance
100	1.6410 (1.9745)	0.3343 (0.4005)	1.6344 (1.9677)	0.3343 (0.3987)
150	1.1993 (1.5326)	0.1783 (0.2446)	1.1948 (1.5282)	0.1768 (0.2434)
200	0.9806 (1.3140)	0.1244 (0.1908)	0.9773 (1.3106)	0.1244 (0.1900)

Table 3. Results obtained from queueing models for FD & ID with N = 100.

individual arrival rate	$\bar{\lambda}$	$\bar{\rho}$	ρ	L_{FD}	L_{ID}	$\frac{\lvert L_{FD} - L_{ID}\rvert}{L_{FD}}$
0.0010	0.0768	0.1260	0.1255	0.1360	0.1356	0.3%
0.0014	0.1074	0.1762	0.1755	0.1972	0.1965	0.3%
0.0018	0.1380	0.2264	0.2255	0.2632	0.2624	0.3%
0.0022	0.1685	0.2766	0.2754	0.3350	0.3343	0.2%
0.0026	0.1990	0.3266	0.3253	0.4140	0.4134	0.2%
0.0030	0.2294	0.3765	0.3750	0.5017	0.5014	0.1%
0.0034	0.2598	0.4263	0.4245	0.6003	0.6006	0.1%
0.0038	0.2900	0.4759	0.4740	0.7125	0.7140	0.2%
0.0042	0.3201	0.5253	0.5232	0.8421	0.8458	0.4%
0.0046	0.3501	0.5744	0.5721	0.9946	1.0021	0.8%
0.0050	0.3798	0.6232	0.6207	1.1774	1.1917	1.2%
0.0054	0.4092	0.6716	0.6688	1.4016	1.4283	1.9%
0.0058	0.4383	0.7193	0.7163	1.6834	1.7332	3.0%
0.0062	0.4668	0.7660	0.7629	2.0478	2.1428	4.6%
0.0066	0.4944	0.8114	0.8081	2.5328	2.7210	7.4%
0.0070	0.5208	0.8547	0.8512	3.1972	3.5893	12.3%
0.0074	0.5453	0.8949	0.8912	4.1286	4.9956	21.0%
0.0078	0.5669	0.9303	0.9265	5.4468	7.4890	37.5%
0.0082	0.5844	0.9590	0.9551	7.2828	12.3683	69.8%
0.0086	0.5968	0.9794	0.9754	9.7118	22.7348	134.1%
0.0090	0.6042	0.9915	0.9874	12.6610	44.6431	252.6%

Table 4. Results obtained from queueing models for FD & ID with N = 150.

individual arrival rate	$\bar{\lambda}$	$\bar{\rho}$	ρ	L_{FD}	L_{ID}	$\frac{\lvert L_{FD} - L_{ID}\rvert}{L_{FD}}$
0.0010	0.1738	0.2085	0.2077	0.2391	0.2383	0.3%
0.0012	0.2085	0.2501	0.2492	0.2965	0.2956	0.3%
0.0014	0.2432	0.2916	0.2906	0.3584	0.3574	0.3%
0.0016	0.2778	0.3332	0.3319	0.4256	0.4246	0.2%
0.0018	0.3124	0.3746	0.3732	0.4990	0.4981	0.2%
0.0020	0.3469	0.4160	0.4145	0.5801	0.5793	0.1%
0.0022	0.3813	0.4573	0.4556	0.6704	0.6699	0.1%
0.0024	0.4157	0.4986	0.4967	0.7719	0.7722	0.0%
0.0026	0.4500	0.5397	0.5377	0.8877	0.8891	0.2%
0.0028	0.4842	0.5807	0.5785	1.0213	1.0248	0.3%
0.0030	0.5183	0.6215	0.6192	1.1781	1.1850	0.6%
0.0032	0.5521	0.6621	0.6597	1.3653	1.3781	0.9%
0.0034	0.5857	0.7024	0.6998	1.5934	1.6166	1.5%
0.0036	0.6190	0.7423	0.7396	1.8777	1.9197	2.2%
0.0038	0.6518	0.7816	0.7787	2.2415	2.3188	3.4%
0.0040	0.6838	0.8201	0.8171	2.7205	2.8677	5.4%
0.0042	0.7149	0.8573	0.8541	3.3707	3.6644	8.7%
0.0044	0.7443	0.8926	0.8893	4.2792	4.9018	14.5%
0.0046	0.7712	0.9248	0.9214	5.5772	6.9907	25.3%
0.0048	0.7943	0.9526	0.9491	7.4410	10.8829	46.3%
0.0050	0.8123	0.9741	0.9705	10.0501	18.9128	88.2%

Table 5. Results obtained from queueing models for FD & ID with N = 200.

individual arrival rate	$\bar{\lambda}$	$\bar{\rho}$	ρ	L_{FD}	L_{ID}	$\frac{\lvert L_{FD} - L_{ID}\rvert}{L_{FD}}$
0.0010	0.3100	0.3040	0.3030	0.3783	0.3773	0.3%
0.0012	0.3718	0.3646	0.3634	0.4816	0.4804	0.2%
0.0014	0.4335	0.4251	0.4237	0.6006	0.5995	0.2%
0.0016	0.4951	0.4855	0.4838	0.7407	0.7399	0.1%
0.0018	0.5565	0.5457	0.5439	0.9097	0.9100	0.0%
0.0020	0.6177	0.6057	0.6037	1.1201	1.1227	0.2%
0.0022	0.6785	0.6654	0.6631	1.3918	1.4000	0.6%
0.0024	0.7389	0.7245	0.7221	1.7594	1.7811	1.2%
0.0026	0.7983	0.7828	0.7802	2.2863	2.3431	2.5%
0.0028	0.8562	0.8396	0.8367	3.0975	3.2573	5.2%
0.0030	0.9110	0.8933	0.8903	4.4551	4.9696	11.5%
0.0032	0.9595	0.9409	0.9377	6.9089	8.9115	29.0%
0.0034	0.9958	0.9765	0.9732	11.4050	20.8853	83.1%
0.0036	1.0142	0.9946	0.9912	18.5798	63.9817	244.4%
0.0038	1.0192	0.9994	0.9960	27.2963	140.7327	415.6%
0.0040	1.0197	1.0000	0.9966	35.8368	164.3204	358.5%
0.0042	1.0198	1.0000	0.9966	43.6490	165.8978	280.1%
0.0044	1.0198	1.0000	0.9966	50.7557	165.9465	227.0%
0.0046	1.0198	1.0000	0.9966	57.2446	165.9474	189.9%
0.0048	1.0198	1.0000	0.9966	63.1927	165.9474	162.6%
0.0050	1.0198	1.0000	0.9966	68.6650	165.9474	141.7%

The approximation error resulting from using an infinite population model versus a finite population model equals $\lvert L_{FD} – L_{ID}\rvert/L_{FD}$. Arbitrarily, we concluded, when the approximation error for an population is less than 5%, an infinite population approximation is reasonable; depending on the situation, a different percentage may be more appropriate. Based on a 5% error benchmark, from Table 3, when the effective traffic intensity for the finite population, $\bar{\rho}$, is less than approximately 0.78, an infinite population approximation for the MIAPP-NALT is reasonably accurate. Likewise, from Table 4, when the traffic intensity for the finite population is less than approximately 0.80, an infinite population approximation for the MIAPP-NALT is reasonably accurate. Finally, from Table 5, when the traffic intensity for the finite population is less than approximately 0.82, an infinite population approximation for the MIAPP-NALT is reasonably accurate. (As population size increases, the greater the upper bound on traffic intensity in order for an infinite population approximation to be reasonably accurate for the MIAPP-NALT system.)

2.6.3. Queueing model results

To continue our demonstration of how our models can be used in designing an MIAPP-NALT system, suppose the arrival rate is 0.004 per minute for each pick position. For each value of number of pick positions (N), converting the arrival rate based on the appropriate normalized time unit for each pick position and applying the analytical models in Section 5, we obtain the performance measures given in Table 6 for four scenarios. To allow for comparisons among population sizes in Table 6, the results for waiting time in queue (W_q) and waiting time in system (W) are expressed in minutes, not normalized time units.

For each scenario, Equation (49) the arrival rate for the infinite population model is the same as the effective arrival rate for the finite population model. When the number of pick positions equals 100, the traffic intensities are 50 percent for dedicated storage cases and 60 percent for random storage cases. When the number of pick positions equals 150, the resulting traffic intensities are 82 percent for dedicated storage cases and 98 percent for random storage cases. When the number of pick positions equals 200, the resulting traffic intensities are approximately equal to 100 percent for all dedicated and random storage cases. As Table 6 indicates, values of the performance measures obtained from the finite and infinite population queueing models with 100 pick positions are very nearly the same because the traffic intensity is relatively low. However, performance measures values obtained from the finite and infinite population queueing models will differ significantly when traffic intensity is very high, such as the case with 200 pick positions, confirming the conclusion in Section 6.2 regarding the accuracy of an infinite population queueing approximation.

Table 6. Performance measures for four scenarios with W_q and W measured in minutes.

N	Queue Results	P_0	L_q	L	W_q	W
100	FD	0.4994	0.2742	0.7748	0.6908	1.9521
	FR	0.3995	0.4780	1.0785	1.2081	2.7258
	ID	0.5029	0.2762	0.7732	0.6979	1.9541
	IR	0.4016	0.4918	1.0903	1.2430	2.7554
150	FD	0.1799	1.9004	2.7205	3.2259	4.6180
	FR	0.0167	10.8393	11.8226	19.6112	21.3903
	ID	0.1829	2.0506	2.8677	3.4808	4.8677
	IR	0.0196	27.1152	28.0956	49.0586	50.8325
200	FD	0.0000	34.8368	35.8368	53.0522	54.5750
	FR	0.0000	76.4824	77.4824	156.0641	158.1046
	ID	0.0034	163.3239	164.3204	248.7218	250.2395
	IR	0.0026	215.5936	216.5911	439.9237	441.9590

Clearly, designing the MIAPP-NALT system with 200 pick positions assigned to a NALT is not feasible. Likewise, designing the system with only 100 pick positions assigned to a NALT results in the NALT being busy only 50 percent of the time when dedicated storage is used and only 60 percent of the time when random storage is used. Among the three design alternatives, the preferred alternative is the assignment of 150 pick positions to the NALT; resulting in an 82 percent utilization of the NALT when dedicated storage is used and a 98 percent utilization when random storage is used.

When 150 pick positions are assigned to the NALT, there are significant differences in the performance measures when using dedicated storage versus random storage. Notice, with the finite population and dedicated storage (FD) scenario, approximately 2.72 pick positions are waiting for replenishment to be performed (2.72 is the sum of 1.90 waiting for replenishment to begin and 0.82 in the process of being replenished); whereas, 11.82 pick positions are waiting for replenishment to be performed when random storage is used. Thus, there is a 335 percent increase in the average number of pick positions waiting for replenishment when using random storage versus dedicated storage. Because the mean and variance for service time are greater with random storage than for dedicated storage (see Table 2), the values of the performance measures will be greater.

The comparison of dedicated and random storage is based on the same amount of storage space for each, which imposes a significant penalty regarding the number of empty pick positions (L) when random storage is used. One reason for such a large penalty with number of pick positions equal to 150 is the traffic intensity for random storage is 98 percent, but only 82 percent for dedicated storage. (The traffic intensity differences are due to the differences in expected service times with dedicated and random storage.) Depending on the performance level required, random storage might not be acceptable when the number of pick positions equal 150.

As noted, a reason for the large difference in the value of L when using random storage is the assumption of equal storage space for both storage policies. With a population of 150 pick positions, we determined the value of shape factor (b) for random storage that yields the same value for expected number of empty deposit positions as obtained with dedicated storage. The results were as follow: when b equals 0.603 for dedicated storage, b must equal approximately 0.272 for random storage to yield the same value for expected number of empty deposit positions. Hence, for the parameter values used in our example, unless random storage reduces storage space by (0.603 – 0.272)/0.603 or 55 percent, then dedicated storage yields a smaller number of empty pick positions values than does random storage.

2.6.4. Insights and practical considerations

From Sections 6.1 6.2, 6.3 and 6.4, the following insights emerged:

a. Depending on how much storage space is required for random storage, a significant penalty in replenishment productivity can result from using random storage instead of dedicated storage. For the parameter values we used, with 150 pick positions in an S/R aisle, and an arrival rate of 0.004 per minute per pick position, the expected number of empty deposit positions in the aisle equals 2.72 with dedicated storage; whereas, the expected number of empty deposit positions equals 11.82 with random storage.

   Two insights result. First, the expected values are nearly the same with dedicated storage, but they are quite different with random storage. This is due, primarily, to the difference in traffic intensity with random storage. Recall, the infinite population approximation is less accurate as traffic intensity increases. With random storage, it is 98 percent. Therefore, we know the infinite approximation is not accurate, allowing us to ignore the value obtained for the IR scenario. The second insight relates to the difference in the results, based on the storage policy (FD versus FR). With the same amount of space required for both storage policies, the expected number of empty deposit positions increases 335 percent when switching from dedicated to random storage.

   A decision on using dedicated storage versus random storage should be based on the ratio of the space requirements for the two storage policies and the traffic intensity for the operation. The greater the traffic intensity the greater the need to locate replenishment stock as close as possible to the pick position.

b. Based on the parameter values we used, when traffic intensity is less than 0.82 and the number of pick positions is greater than or equal to 150 for the MIAPP-NALT system, using either a finite or an infinite population model will yield roughly the same values for the performance measures; otherwise, we do not recommend using the results from the infinite population model. Therefore, based on the results in Table 6, with a traffic intensity of 98 percent, we do not recommend using an infinite population approximation when random storage is used; however, using an infinite population approximation when dedicated storage is used yields reasonably accurate results for the performance measures of the MIAPP-NALT system.

c. The robustness of the finite population queueing model was demonstrated by comparing the results from the model with those obtained from a simulation of an MIAPP-NALT operation when the time required to deplete a pallet load of cases is not exponentially distributed and when pick positions have different arrival rates. Furthermore, the results obtained from the M/G/1/N queueing model were nearly identical to the simulation results when the simulation model was based on SKUs on a pick list being picked sequentially, whereas the M/G/1/N model is based on SKUs being picked first-come, first-served and the pick positions being statistically independent.

d. It is worthwhile to compare the results obtained by using only expected values for service times and replenishment rates with those obtained by using queueing models. For the MIAPP-NALT system with 3,000 SKUs, based on an arrival rate of 0.004 per minute for each pick position, in an 8-hour period, expected demands of 192, 288, and 384 replenishments occur, respectively, with 100, 150, and 200 pick positions in an S/R aisle. Converting the data in Table 2 to the expected number of replenishments in an 8-hour period with dedicated storage, we obtain values of 380.58, 344.78, and 315.22 for number of pick positions equal to 100, 150, and 200, respectively; therefore, expected NALT utilization is 192/380.58 or 50.45 percent, 288/344.78 or 83.53 percent, and 384/315.22 or 121.82 percent for 100, 150, and 200 pick positions in an S/R aisle, respectively. With random storage, the expected number of replenishments performed in an 8-hour period is 316.32, 269.81, and 235.26 for number of pick positions equal to 100, 150, and 200, respectively; therefore, expected NALT utilization is 192/316.32 or 60.70 percent, 288/269.81 or 106.74 percent, and 384/235.26 or 163.22 percent for 100, 150, and 200 pick positions in an S/R aisle, respectively. (Clearly, based on expected values, assigning 150 pick positions to an S/R aisle is not feasible with random storage and assigning 200 pick positions is not feasible with either dedicated or random storage.) Whereas, the NALT utilization resulting from queueing analysis, with dedicated storage, is 50.06 percent, 82.01 percent, and 100 percent; with random storage, the NALT utilization is 60.05 percent, 98.33 percent, and 100 percent for 100, 150, and 200 pick positions in an S/R aisle, respectively. (As noted previously, assigning 150 pick positions to an S/R aisle is acceptable with either dedicated storage or random storage in the queueing analysis, but problematic with random storage in the expected-value analysis; assigning 200 pick positions to an S/R aisle is not feasible for either storage policy.)

For the parameter values we used in the comparison, NALT utilization values are not significantly different when using expected-value analysis versus queueing analysis. However, because the latter provides values for several performance measures, the design of the MIAPP-NALT system can be based on more than NALT utilization. (Of course, the expected value analysis could not have been performed without the data in Table 2, which are products of the density functions derived in Section 4.)

e. For the designer of an MIAPP-NALT system, the results presented in the paper can be used in the following way: letting the number of SKUs determine the number of pick positions, consider a range of alternative designs with differing lengths of the replenishment aisle, number of pick positions per aisle and storage heights within the aisle. For each alternative, calculate the cost of the MIAPP-NALT design, including the number and cost of the NALTs, the cost of the storage racks and building, and the cost of pick positions waiting to be served. Alternatively, the design can be based on an aspiration level, such as the expected number of pick positions waiting to be served being less than a specified value.

f. Recall, no storages were performed by the NALT during times when replenishments were being performed. We assumed the NALT stored full pallets in the upper levels of the storage rack during a period of time when order picking is not performed. Assuming the NALT picks up unit loads at coordinate point (0,0), travels rectilinearly to a uniformly distributed point, stores the unit load and returns to (0,0), then in normalized time it requires $1 + b + T_{PD}$ time units on average to perform a storage. In the long term, the number of unit loads stored must equal the number of unit loads replenished. With 150 pick positions per S/R aisle and 0.004 replenishments per minute for the FDC scenario, an effective normalized arrival rate of 0.684 for replenishment requests converts to an arrival rate of 0.589 per minute. Therefore, during an 8-hour period of case picking, on average, 8(60)(0.589) or 282 replenishments will be performed. Based on an expected storage time of 2.1608 minutes (not normalized time units), it will take the NALT 10.18 hours to perform the expected number of storages. Hence, over a 24-hour period, the NALT should be able to perform the replenishments and the storages required for the assigned S/R aisle.

Looking beyond the numbers, three practical considerations emerged from the research.

a. If the demand for replenishment within an S/R aisle is less than needed to justify restricting the NALT to a single S/R aisle, the NALT can be assigned to two adjacent S/R aisles. In such a case, if the NALT can enter and exit only one end of the S/R aisle, our model can be used by inserting a “middle zone” in an S/R aisle with a length equal to twice the length of a single S/R aisle plus a “middle zone”; the “middle zone” contains no pick positions and its length is the distance the NALT travels from one S/R aisle to another assigned S/R aisle.

b. The results obtained for random storage were based on each storage location being equally likely to be selected for retrieval of a pallet load of cases when replenishing a pick position. In practice, a quasi-random storage policy can be used, instead of a “pure” random storage policy. Specifically, instead of determining the storage location for reserve pallet loads randomly, store the pallet in the open position closest to the pick position for which the pallet is destined. Using a closest-open-location policy reduces storage space requirements when compared with dedicated storage and reduces expected replenishment time when compared with random storage. Although our model cannot handle this situation, we believe it is worthy of further study. Therefore, we include it as a recommendation in the next section.

c. Even though interarrival times for individual pick positions in an MIAPP-NALT operation may not be exponentially distributed or identically distributed, our research indicates the M/G/1/N queueing model yields very accurate results regardless of the traffic intensity. Even though the infinite population model is computationally simpler to use than is the finite population model, the finite population model is recommended in determining the design parameters for an actual installation. In addition, we recommend a detailed simulation model incorporating discrete locations for reserve storage, acceleration/deceleration losses, and deviations from other assumptions we have made be used to finalize the design of the MIAPP-NALT system.

2.7. Summary, conclusions, and recommendations

In modeling the MIAPP-NALT operation as a queueing system, pdfs were developed for NALT travel time for four scenarios: FD, FR, ID and IR. For each scenario, formulas for the expected value and variance for NALT travel time were derived. For finite population scenarios, the Laplace transform for service time were obtained. The MIAPP-NALT system was modeled as a finite population queue (M/G/1/N) and infinite population queue (M/G/1/∞); then, the MIAPP-NALT system was approximated using an M/G/1/∞ queueing model. For the M/G/1/N queue, Takács' formulas were used to determine the steady-state performance measures of the MIAPP-NALT operation using Laplace transforms of the service time pdfs. Pollaczek-Khinchine formulas were used to determine the steady-state performance measures for the M/G/1/∞ model by using expected values and variances for service times obtained in Section 4.

Based on the research results obtained, we drew the following conclusions: the M/G/1/N queueing model can be used to model the performance of an MIAPP-NALT operation when times to deplete pallets at pick positions are neither exponentially distributed nor identically distributed; expected values obtained from the probability density functions we derived can be used to design an MIAPP-NALT system, but modeling the system as an M/G/1 queue provides greater design insights than relying solely on expected values for demands for replenishment and service times; depending on the traffic intensity and the number of pick positions in an S/R aisle, a significant penalty in order-picking productivity can result when using random storage versus dedicated storage unless random storage significantly reduces the amount of storage space required; for the parameter values we considered, an infinite population queueing approximation can be used to obtain reasonably accurate results when traffic intensity is less than approximately 80 percent and approximately 150 pick positions are assigned to an S/R aisle; and the smaller the population size, the smaller the traffic intensity must be for a reasonable infinite population approximation.

For future research, a variety of extensions to the MIAPP-NALT system can be considered; including, but not limited to, using the NALT to replenish double-deep pallet rack or flow racks in single or multiple aisles. As mentioned, a closest-open-location variation of the random storage policy merits further attention. In addition, replenishing pick positions based on a prioritization of pick positions or class-based storage in an MIAPP-NALT system can be explored. Yet another logical extension to the MIAPP-NALT system is incorporating mezzanines in the models developed; doing so introduces computational, not theoretical challenges we chose to limit our research to floor-level order picking to shorten the paper and simplify modeling notation. An interesting extension of our queueing model is to incorporate people performing case picking in the model, resulting in a queueing network; the designer will not only decide the length of the S/R aisle, but also the number of people assigned to the pick positions in a picking aisle. Finally, interleaving storage and retrieval operations performed by the NALT could prove to be a fertile area for further research.

References

Bozer, Y.A. and Cho, M. (2005) Throughput performance of automated storage/retrieval systems under stochastic demand, *IIE Transactions*, **37**(4), 367-378.

Bozer, Y.A. and White, J. A. (1984) Travel–time models for automated storage/retrieval systems, *IIE Transactions*, **16**(4), 329-338.

Bozer, Y.A. and White, J.A. (1990) Design and performance models for end-of-aisle order picking systems, *Management Science*, **36**(7), 852-866.

Cai, X., Heragu, S.S. and Liu, Y. (2013) *Handbook of stochastic models and analysis of manufacturing system operations*, Springer, 29-71.

Cai, X., Heragu, S.S. and Liu, Y. (2014) Modeling and evaluating the AVS/RS with tier-to-tier vehicles using a semi-open queueing network, *IIE Transactions*, **46**(9), 905-927.

Drury, J. (1988). Towards more efficient order picking, IMM monograph No. 1. Report, The Institute of Materials Management, Cranfield, U.K.

Francis, R.L., McGinnis, L.F., Jr. and White, J.A. (1992) *Facility Layout and Location: an Analytical Approach*, 2nd edition, Upper Saddle River (NJ): Prentice-Hall., 189-198, 258-280.

Fukunari, M. and Malmborg, C.J. (2009) A network queuing approach for evaluation of performance measures in autonomous vehicle storage and retrieval systems, *European Journal of Operational Research*, **193**(1), 152-167.

Gross, D., Shortle, J.F., Thompson, J.M. and Harris, C.M. (2008) *Fundamentals of Queueing Theory*, 4th ed., Hoboken (NJ): John Wiley & Sons, Inc.

Gue, K.R. (2006) Very high density storage systems, *IIE Transactions*, **38**(1), 79-90.

Gue, K.R. and Kim, B.S. (2007) Puzzle-based storage systems, *Naval Research Logistics*, **54**(8). 556-567.

Heragu, S.S, Cai X., Krishnamurthy, A. and Malmborg, C.J. (2011) Analytical models for analysis of automated warehouse material handling systems, *International Journal of Production Research*, **49**(22), 6833-6861.

Hur, S., Lee, Y.H., Lim, S.Y. and Lee, M.H. (2004) A performance estimation model for AS/RS by M/G/1 queuing system, *Computers & Industrial Engineering*, **46**(2), 233-241.

Hur, S. and Nam, J. (2006) Performance analysis of automatic storage/retrieval systems by stochastic modelling, *International Journal of Production Research*, **44**(8), 1613-1626.

Kuo, P., Krishnamurthy, A. and Malmborg, C.J. (2007) Design models for unit load storage and retrieval systems using autonomous vehicle technology and resource conserving storage and dwell point policies, *Applied Mathematical Modelling*, **31**(10), 2332-2346.

Larson, R.C. and Odoni, A.R. (1981) *Urban Operations Research*, Englewood Cliffs (NJ): Prentice-Hall, 138-139.

Lee, H.F. (1997) Performance analysis for automated storage/retrieval systems, *IIE Transactions*, **29**(1), 15-28.

Malmborg, C.J. (2002) Conceptualizing tools for autonomous vehicle storage and retrieval systems, *International Journal of Production Research*, **40**(8), 1807-1822.

Malmborg, C.J. (2003) Interleaving dynamics in autonomous vehicle storage and retrieval systems, *International Journal of Production Research*, **41**(5), 1057-1069.

Miyagawa, M., (2010) Distributions of rectilinear deviation distance to visit a facility, *European Journal of Operational Research*, **205**(1), 106-112.

Park, B.C., Frazelle, E.H. and White, J.A. (1999) Buffer sizing models for end-of-aisle order picking systems, *IIE Transactions*, **31**(1), 31-38.

Ramtin, F. and Pazour, J.A. (2014) Analytical models for an automated storage and retrieval system with multiple in-the-aisle pick positions, *IIE Transactions*, **46**(9), 968-986.

Ramtin, F. and Pazour, J.A. (2015) Production allocation problem for an AS/RS with multiple in-the-aisle pick positions, *IIE Transactions*, **47**(1), 1-18.

Roy D., Krishnamurthy, A., Heragu, S. and Malmborg, C.J. (2015) Queuing models to analyze dwell-point and cross-aisle location in autonomous vehicle-based warehouse systems, *European Journal of Operational Research*, **242**(1), 72-87.

Saaty, T.L. (1961) *Elements of Queueing Theory with Applications*, New York (NY): McGraw-Hill Book Company, 323-329.

Sztrik, J. (2016) *Basic Queueing Theory: Foundations of System Performance Modeling*, Globe Edit, 160. (URL: https://www.amazon.com/Basic-Queueing-Theory-Foundations-Performance /dp/3639734718/ref=sr_1_1?s=books&ie=UTF8&qid=1513713761&sr=1-1&Keywords=Basic+Queueing+Theory%3A+Foundations+of+System+Performance+Modeling)

Takács, L. (1960) *Stochastic Processes, Problems and Solutions*, Methuen & Co Ltd (NY): John Wiley & Sons, Inc., 73-74

Uludag, O. (2014) *GridPick: A High Density Puzzle Based Order Picking System with Decentralized control*, Ph.D. thesis, Auburn University.

Vlasiou, M., Adan, I.J.B.F. and Wessels, J. (2004) A Lindley-type equation arising from a carousel problem, *Journal of Applied Probability*, **41**(4), 1171-1181.

Zhang, L., Krishnamurthy, A, Malmborg, C.J. and Heragu, S. (2009) Variance-based approximation of transaction waiting times in autonomous vehicle storage and retrieval systems, *European Journal of Industrial Engineering*, **3**(2), 146-169.

Appendix

Appendix A: Derivation of probability density function for travel time with a finite population and random storage

The expression for the NALT travel time is:

$$T_{ij} = |x - m_i| + y + |x - m_j| + y = |x - m_i| + |x - m_j| + 2y$$

where $x \in \text{unif}[0,1]$ and $y \in \text{unif}[0,b]$.

Let $m_{[1]} = min(m_i, m_j)$ and $m_{[2]} = max(m_i, m_j)$, then

$$T_{ij} = \begin{cases} 2x + 2y - m_{[1]} - m_{[2]} & x \geq m_{[2]} \geq m_{[1]} \\ 2y + m_{[2]} - m_{[1]} & m_{[1]} \leq x < m_{[2]} \\ -2x + 2y + m_{[1]} + m_{[2]} & x < m_{[1]} \leq m_{[2]} \end{cases}.$$

Because x and y are statistically independent, $f(x,y) = f(x)f(y) = \frac{1}{b}$.

Case 1: $x \geq m_{[2]} \geq m_{[1]}$, $T = 2x + 2y - m_{[1]} - m_{[2]}$,

Case 1a: $1 - m_{[2]} \geq b$. When $\in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}]$, the cdf is

$$Pr\left(m_{[2]} - m_{[1]} \leq T \leq t | \text{Case 1a}\right) = \int_0^{\frac{\left(t - m_{[2]} + m_{[1]}\right)}{2}} \int_{m_{[2]}}^{\frac{\left(t + m_{[1]} + m_{[2]}\right)}{2} - y} f(x,y)dxdy =$$

$$\frac{(-m_{[2]} + m_{[1]} + t)^2}{8b}. \quad \text{(A1)}$$

When $\in [2b + m_{[2]} - m_{[1]},\ 2 - m_{[1]} - m_{[2]}]$, the cdf is

$$Pr(T \leq t | \text{Case 1a}) = Pr\left(2x + 2y - m_{[1]} - m_{[2]} \leq t\right) = \int_0^b \int_{m_{[2]}}^{\frac{\left(t + m_{[1]} + m_{[2]}\right)}{2} - y} f(x,y)dxdy =$$

$$\frac{1}{2}(m_{[1]} - m_{[2]} - b - t). \quad \text{(A2)}$$

When $t \in [2 - m_{[1]} - m_{[2]},\ 2 + 2b - m_{[1]} - m_{[2]}]$, the cdf is

$$Pr(T \leq t | \text{Case 1a}) = Pr\left(2x + 2y - m_{[1]} - m_{[2]} \leq t\right) = \int_0^{\frac{\left(t + m_{[1]} + m_{[2]}\right)}{2} - 1} \int_{m_{[2]}}^1 f(x,y)dxdy +$$

$$\int_{\frac{(t+m_{[1]}+m_{[2]})}{2}-1}^{b}\int_{m_{[2]}}^{\frac{(t+m_{[1]}+m_{[2]})}{2}-y} f(x,y)dxdy = \frac{1}{8b}\left[4b\left(m_{[1]} - m_{[2]} + t\right) - 4b^2 - (m_{[1]} + m_{[2]+t-1})^2\right] \quad \text{(A3)}$$

Case 1b: $1 - m_{[2]} < b$. When $t \in [m_{[2]} - m_{[1]},\ 2 - m_{[1]} - m_{[2]}]$, the cdf is

$$Pr\left(m_{[2]} - m_{[1]} \leq T \leq t|\text{Case 1b}\right) = Pr\left(2x + 2y - m_{[1]} - m_{[2]} \leq t\right) =$$

$$\int_{0}^{\frac{(t-m_{[2]}+m_{[1]})}{2}}\int_{m_{[2]}}^{\frac{(t+m_{[1]}+m_{[2]})}{2}+y} f(x,y)dxdy = \frac{(m_{[1]}-m_{[2]}+t)^2}{8b}. \quad \text{(A4)}$$

When $t \in [2 - m_{[1]} - m_{[2]},\ 2b + m_{[2]} - m_{[1]}]$, the cdf is

$$Pr(T \leq t|\text{Case 1b}) = Pr\left(2x + 2y - m_{[1]} - m_{[2]} \leq t\right)$$

$$= \int_{0}^{\frac{(t+m_{[1]}+m_{[2]})}{2}-1}\int_{m_{[2]}}^{1} f(x,y)dxdy$$

$$+\int_{\frac{(t+m_{[1]}+m_{[2]})}{2}-1}^{\frac{(t-m_{[2]}+m_{[1]})}{2}}\int_{m_{[2]}}^{\frac{(t+m_{[1]}+m_{[2]})}{2}-y} f(x,y)dxdy$$

$$= \frac{(m_{[2]}-1)(m_{[1]}+t-1)}{2b}. \quad \text{(A5)}$$

When $t \in \left[2b + m_{[2]} - m_{[1]},\ 2 + 2b - m_{[1]} - m_{[2]}\right]$, the cdf is

$$Pr(T \leq t|\text{Case 1b}) = Pr\left(2x + 2y - m_{[1]} - m_{[2]} \leq t\right)$$

$$= \int_{0}^{\frac{(t+m_{[1]}+m_{[2]})}{2}-1}\int_{m_{[2]}}^{1} f(x,y)dxdy + \int_{\frac{(t+m_{[1]}+m_{[2]})}{2}-1}^{b}\int_{m_{[2]}}^{\frac{(t+m_{[1]}+m_{[2]})}{2}-y} f(x,y)dxdy =$$

$$\frac{4b\left(m_{[1]}-m_{[2]}+t\right)-4b^2-(m_{[1]}+m_{[2]}+t-2)^2}{8b}. \quad \text{(A6)}$$

Case 2: $m_{[1]} < x < m_{[2]}$, $T = 2y + m_{[2]} - m_{[1]}$. When $\in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}]$, the cdf is

$$Pr\left(m_{[2]} - m_{[1]} \leq T \leq t|\text{Case 2}\right) = Pr\left(2y + m_{[2]} - m_{[1]} \leq t\right) =$$

$$\int_{m_{[1]}}^{m_{[2]}} \int_0^{\frac{\left(t+m_{[1]}-m_{[2]}\right)}{2}} f(x,y)dydx = \frac{(m_{[2]}-m_{[1]})(m_{[1]}-m_{[2]}+t)}{2b}. \quad \text{(A7)}$$

Case 3: When $x < m_{[1]} < m_{[2]}$, $T = 2y - 2x + m_{[1]} + m_{[2]}$.

Case 3a: $m_{[1]} \le b$. When $t \in [m_{[2]} - m_{[1]},\ m_{[1]} + m_{[2]}]$, the cdf is

$$Pr\left(m_{[2]} - m_{[1]} \le T \le t \middle| \text{Case 3a}\right) = Pr\left(2y - 2x + m_{[1]} + m_{[2]} \le t\right) =$$

$$\int_0^{\frac{(m_{[1]}-m_{[2]}+t)}{2}} \int_{\frac{\left(m_{[2]}+m_{[1]}-t\right)}{2}+y}^{m_{[1]}} f(x,y)dxdy = \frac{(-m_{[2]}+m_{[1]}+t)^2}{8b}. \quad \text{(A8)}$$

When $\in [m_{[2]} + m_{[1]},\ 2b + m_{[2]} - m_{[1]}]$, the cdf is

$$Pr(T \le t|\text{Case 3a}) = Pr\left(2y - 2x + m_{[1]} + m_{[2]} \le t\right) = \int_0^{\frac{t-m_{[1]}-m_{[2]}}{2}} \int_0^{m_{[1]}} f(x,y)dxdy +$$

$$\int_{\frac{t-m_{[1]}-m_{[2]}}{2}}^{\frac{t+m_{[1]}-m_{[2]}}{2}} \int_{\frac{m_{[1]}+m_{[2]}-t}{2}+y}^{m_{[1]}} f(x,y)dxdy = \frac{m_{[1]}(t-m_{[2]})}{2b}. \quad \text{(A9)}$$

When $\in [2b + m_{[2]} - m_{[1]},\ 2b + m_{[1]} + m_{[2]}]$, the cdf is

$$Pr(T \le t|\text{Case 3a}) = Pr\left(2y - 2x + m_{[1]} + m_{[2]} \le t\right)$$

$$= \frac{4b\left(m_{[1]}-m_{[2]}+t\right)-4b^2-(m_{[1]}+m_{[2]}-t)^2}{8b}. \quad \text{(A10)}$$

Case 3b: $m_{[1]} > b$. When $\in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}]$, the cdf is

$$Pr\left(m_{[2]} - m_{[1]} \le T \le t \middle| \text{Case 3b}\right) = Pr\left(2y - 2x + m_{[1]} + m_{[2]} \le t\right) =$$

$$\int_0^{\frac{t+m_{[1]}-m_{[2]}}{2}} \int_{\frac{m_{[1]}+m_{[2]}-t}{2}+y}^{m_{[1]}} f(x,y)dxdy = \int_0^{\frac{t-m_{[1]}-m_{[2]}}{2}} \int_0^{m_{[1]}} f(x,y)dxdy +$$

$$\int_{\frac{t-m_{[1]}-m_{[2]}}{2}}^{b} \int_{\frac{m_{[1]}+m_{[2]}-t}{2}+y}^{m_{[1]}} f(x,y)dxdy = \frac{(-m_{[2]}+m_{[1]}+t)^2}{8b}. \quad \text{(A11)}$$

When $\in [2b + m_{[2]} - m_{[1]},\ m_{[1]} + m_{[2]}]$, the cdf is

$$Pr(T \le t|\text{Case 3b}) = Pr\left(2y - 2x + m_{[1]} + m_{[2]} \le t\right) = \int_0^b \int_{\frac{m_{[1]}+m_{[2]}-t}{2}+y}^{m_{[1]}} f(x,y)dxdy =$$

$$\frac{1}{2}(m_{[1]} - m_{[2]} + t - b). \quad \text{(A12)}$$

When $\in [m_{[1]} + m_{[2]},\ m_{[1]} + m_{[2]} + 2b]$, the cdf is

$$Pr(T \leq t|\text{Case 3b}) = Pr\left(2y - 2x + m_{[1]} + m_{[2]} \leq t\right) = \int_0^{\frac{t-m_{[2]}-m_{[1]}}{2}} \int_0^{m_{[1]}} f(x,y)dxdy + \int_{\frac{t-m_{[2]}-m_{[1]}}{2}}^{b} \int_{\frac{m_{[1]}+m_{[2]}-t}{2}+y}^{m_{[1]}} f(x,y)dxdy = \frac{4b\left(m_{[1]}-m_{[2]}+t\right)-4b^2-\left(m_{[1]}+m_{[2]}-t\right)^2}{8b}. \tag{A13}$$

When $m_{[1]} \geq 1 - m_{[2]} \geq b$ and $m_{[1]} - b \geq 1 - m_{[2]}$, from Equations (A1), (A2), (A3), (A7), (A11), (A12) and (A13), the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ 2 - m_{[2]} - m_{[1]}] \\ \frac{4b+2-m_{[2]}-m_{[1]}-t}{4b} & t \in (2 - m_{[1]} - m_{[2]}, 2 + 2b - m_{[1]} - m_{[2]}] \\ \frac{1}{2} & t \in (\ 2 + 2b - m_{[1]} - m_{[2]}, m_{[1]} + m_{[2]}] \\ \frac{2b+m_{[1]}+m_{[2]}-t}{4b} & t \in [m_{[1]} + m_{[2]},\ 2b+m_{[1]} + m_{[2]}] \end{cases}.$$

Likewise, when $m_{[1]} \geq 1 - m_{[2]} \geq b$ and $m_{[1]} - b < 1 - m_{[2]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ 2 - m_{[1]} - m_{[2]}] \\ \frac{4b+2-m_{[1]}-m_{[2]}-t}{4b} & t \in (2 - m_{[1]} - m_{[2]}, m_{[1]} + m_{[2]}] \\ \frac{2b+2-1-t}{2b} & t \in (\ m_{[1]} + m_{[2]}, 2 + 2b - m_{[1]} - m_{[2]}] \\ \frac{2b+m_{[1]}+m_{[2]}-t}{4b} & t \in [2 + 2b - m_{[1]} - m_{[2]},\ 2b+m_{[1]} + m_{[2]}] \end{cases}.$$

When $1 - m_{[2]} \geq m_{[1]} \geq b$ and $1 - m_{[2]} - b \geq m_{[1]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ m_{[1]} + m_{[2]}] \\ \frac{4b+m_{[1]}+m_{[2]}-t}{4b} & t \in (m_{[1]} + m_{[2]}, 2b + m_{[1]} + m_{[2]}] \\ \frac{1}{2} & t \in (\ 2b + m_{[1]} + m_{[2]}, 2 - m_{[1]} - m_{[2]}] \\ \frac{2b+2-m_{[1]}-m_{[2]}-t}{4b} & t \in [2 - m_{[1]} - m_{[2]},\ 2 + 2b - m_{[1]} - m_{[2]}] \end{cases}.$$

When $1 - m_{[2]} \geq m_{[1]} \geq b$ and $1 - m_{[2]} - b < m_{[1]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [m_{[2]} - m_{[1]},\ 2b + m_{[2]} - m_{[1]}] \\ 1 & t \in (2b + m_{[2]} - m_{[1]},\ m_{[1]} + m_{[2]}] \\ \frac{4b + m_{[1]} + m_{[2]} - t}{4b} & t \in (m_{[1]} + m_{[2]}, 2 - m_{[1]} - m_{[2]}] \\ \frac{2b + 1 - t}{2b} & t \in (\ 2 - m_{[1]} - m_{[2]}, 2b + m_{[1]} + m_{[2]}] \\ \frac{2b + 2 - m_{[1]} - m_{[2]} - t}{4b} & t \in [2b + m_{[1]} + m_{[2]},\ 2 + 2b - m_{[1]} - m_{[2]}] \end{cases}.$$

When $b \geq m_{[1]} \geq 1 - m_{[2]}$, from Equations (A4), (A5), (A6), (A7), (A8), (A9) and (A10), the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [c,\ 2 - m_{[1]} - m_{[2]}] \\ \frac{2 + t - m_{[1]} - m_{[2]}}{4b} & t \in [2 - m_{[1]} - m_{[2]},\ m_{[1]} + m_{[2]}] \\ \frac{1}{2b} & t \in (m_{[1]} + m_{[2]}, 2b + m_{[2]} - m_{[1]}] \\ \frac{2b + 1 - t}{2b} & t \in (\ 2b + m_{[2]} - m_{[1]}, 2 + 2b - m_{[1]} - m_{[2]}] \\ \frac{2b + m_{[1]} + m_{[2]} - t}{4b} & t \in [2 + 2b - m_{[1]} - m_{[2]}, 2b + m_{[1]} + m_{[2]}] \end{cases}.$$

Likewise, when $b \geq 1 - m_{[2]} \geq m_{[1]}$, the pdf is

$$f(t) = \begin{cases} \frac{t}{2b} & t \in [c,\ m_{[1]} + m_{[2]}] \\ \frac{t + m_{[1]} + m_{[2]}}{4b} & t \in [m_{[1]} + m_{[2]},\ 2 - m_{[1]} - m_{[2]}] \\ \frac{1}{2b} & t \in (2 - m_{[1]} - m_{[2]}, 2b + m_{[2]} - m_{[1]}] \\ \frac{2b + 1 - t}{2b} & t \in (\ 2b + m_{[2]} - m_{[1]}, 2b + m_{[1]} + m_{[2]}] \\ \frac{2b + 2 - m_{[1]} - m_{[2]} - t}{4b} & t \in [2b + m_{[1]} + m_{[2]}, 2 + 2b - m_{[1]} - m_{[2]}] \end{cases}.$$

Appendix B: Derivation of probability density function for travel time with an infinite population and dedicated storage

The expression for the NALT travel time is:

$$T = |x_1 - x_2| + 2y = \begin{cases} x_1 - x_2 + 2y & x_1 \geq x_2 \\ x_2 - x_1 + 2y & x_1 < x_2 \end{cases},$$

where $x_1, x_2 \in \text{unif}[0, 1]$ and $y \in \text{unif}[0, b]$.

Because x_1, x_2 and y are statistically independent, $f(x_1, x_2, y) = f(x_1)f(x_2)f(y) = \frac{1}{b}$

Case 1: $b \leq 0.5$,

Case 1a: $x_1 \geq x_2$. When $t \in [0, 2b]$, the cdf is

$Pr(0 \leq T \leq t|\text{Case 1a}) = Pr(x_1 - x_2 + 2y \leq t, x_1 \geq x_2)$

$$= \int_0^t \int_0^{x_1} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1 + \int_t^1 \int_{x_1-t}^{x_1} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1$$

$$= \frac{(3-t)t^2}{12b}. \quad \text{(B1)}$$

When $t \in (2b, 1]$, , the cdf is

$Pr(T \leq t|\text{Case 1a}) = Pr(x_1 - x_2 + 2y \leq t, x_1 \geq x_2)$

$$= \int_0^{t-2b} \int_0^{x_1} \int_0^1 \frac{1}{b} dy dx_2 dx_1 + \int_{t-2b}^1 \int_{x_1+2b-t}^{x_1} \int_0^1 \frac{1}{b} dy dx_2 dx_1$$

$$+ \int_{t-2b}^t \int_0^{x_1+2b-t} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1 + \int_t^1 \int_{x_1-t}^{x_1+2b-t} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1$$

$$= \frac{1}{6}(6t + 6bt - 3t^2 - 4b^2 - 6b). \quad \text{(B2)}$$

When $t \in (1, 1 + 2b]$, , the cdf is

$Pr(T \leq t|\text{Case 1a}) = Pr(x_1 - x_2 + 2y \leq t, x_1 \geq x_2)$

$$= \int_0^{t-2b} \int_0^{x_1} \int_0^b \frac{1}{b} dy dx_2 dx_1 + \int_{t-2b}^1 \int_{x_1+2b-t}^{x_1} \int_0^b \frac{1}{b} dy dx_2 dx_1$$

$$+ \int_{t-2b}^1 \int_0^{x_1+2b-t} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1$$

$$= \frac{1}{12b}[12b^2(t-1) - 8b^3 + (t-1)^3 - 6bt(t-2)]. \quad \text{(B3)}$$

Case 1b: $x_1 < x_2$

Because of the symmetry of x_1 and x_2, the cdf for Case 1a is identical to the cdf for Case1b.

Case 2: $b > 0.5$.

Case 2a: $x_1 \geq x_2$. When $t \in [0, 1]$, , the cdf is

$Pr(0 \leq T \leq t|\text{Case 2a}) = Pr(x_1 - x_2 + 2y \leq t, x_1 \geq x_2) =$

$$\int_0^t \int_0^{x_1} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1 + \int_t^1 \int_{x_1-t}^{x_1} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1$$

$$= \frac{(3-t)t^2}{12b}. \qquad \text{(B4)}$$

When $t \in (1, 2b]$, , the cdf is

$$Pr(T \leq t|\text{Case 2a}) = Pr(x_1 - x_2 + 2y \leq t, x_1 \geq x_2)$$

$$= \int_0^1 \int_0^{x_1} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1$$

$$= \frac{3t-1}{12b}. \qquad \text{(B5)}$$

When $t \in (2b, 1 + 2b]$, , the cdf is

$$Pr(T \leq t|\text{Case 2a}) = Pr(x_1 - x_2 + 2y \leq t, x_1 \geq x_2) = \int_0^{t-2b} \int_0^{x_1} \int_0^{b} \frac{1}{b} dy dx_2 dx_1$$

$$+ \int_{t-2b}^1 \int_{x_1+2b-t}^{x_1} \int_0^b \frac{1}{b} dy dx_2 dx_1 + \int_{t-2b}^1 \int_0^{x_1+2b-t} \int_0^{\frac{1}{2}(t-x_1+x_2)} \frac{1}{b} dy dx_2 dx_1$$

$$= \frac{1}{12b}[12b^2(t-1) - 8b^3 + (t-1)^3 - 6bt(t-2)]. \qquad \text{(B6)}$$

Case 2b: $x_1 < x_2$. Because of the symmetry of x_1 and x_2, the cdf for Case 2a is identical to the cdf for Case 2b.

For Case 1, the cdf for Case 1a is identical to the cdf for Case 1b. Therefore, from the cdf in Equation (B1) when $t \in [0, 2b]$, the cdf is

$$F(t) = 2\frac{(3-t)t^2}{12b} = \frac{(3-t)t^2}{6b}. \qquad \text{(B7)}$$

From the cdf in Equation (B2) when t $\in (2b, 1]$, the cdf is

$$F(t) = 2\frac{(3-t)t^2}{12b}$$

$$= 2\frac{1}{6}(6t + 6bt - 3t^2 - 4b^2 - 6b). = \frac{1}{3}(6t + 6bt - 3t^2 - 4b^2 - 6b). \qquad \text{(B8)}$$

From the cdf in Equation (B3) when $t \in (1, 1+2b]$, the cdf is

$$F(t) = 2\frac{1}{12b}[12b^2(t-1) - 8b^3 + (t-1)^3 - 6bt(t-2)]$$

$$= \frac{1}{6b}[12b^2(t-1) - 8b^3 + (t-1)^3 - 6bt(t-2)]. \tag{B9}$$

Therefore, the pdf for Case 1 (b < 0.5) is

$$f(t) = \begin{cases} \frac{(2-t)t}{2b} & t \in [0, 2b] \\ 2 + 2b - 2t & t \in (2b, 1]. \\ \frac{(t-1-2b)^2}{2b} & t \in (1, 1+2b] \end{cases}$$

For Case 2, the cdf for Case 2a is identical to the cdf for Case 2b. Therefore, from the cdf in Equation (B4) when $t \in [0, 1]$, the cdf is

$$F(t) = 2\frac{(3-t)t^2}{12b} = \frac{(3-t)t^2}{6b}. \tag{B10}$$

From the cdf in Equations (B5) when $t \in (1, 2b]$, the cdf is

$$F(t) = 2\frac{3t-1}{12b} = \frac{3t-1}{6b}. \tag{B11}$$

From the cdf in Equation (B6) when $t \in (2b, 1+2b]$, the cdf is

$$F(t) = 2\frac{1}{12b}[12b^2(t-1) - 8b^3 + (t-1)^3 - 6bt(t-2)]$$

$$= \frac{1}{6b}[12b^2(t-1) - 8b^3 + (t-1)^3 - 6bt(t-2)]. \tag{B12}$$

Therefore, the pdf for Case 2 (b > 0.5) is

$$f(t) = \begin{cases} \frac{(2-t)t}{2b} & t \in [0, 1] \\ \frac{1}{4b} & t \in (1, 2b]. \\ \frac{(t-1-2b)^2}{2b} & t \in (2b, 1+2b] \end{cases}$$

Appendix C: Derivation of probability density function for travel time with an infinite population and random storage

The expression for the NALT horizontal travel time is:

$$T_1 = |x_1 - x| + |x_2 - x| = \begin{cases} x_1 + x_2 - 2x & x \le x_1 \le x_2 \\ x_1 + x_2 - 2x & x \le x_2 \le x_1 \\ x_2 - x_1 & x_1 \le x \le x_2 \\ 2x - x_1 - x_2 & x_1 \le x_2 \le x \\ x_1 - x_2 & x_2 \le x \le x_1 \\ 2x - x_1 - x_2 & x_2 \le x_1 \le x \end{cases}.$$

The cdf of T_1 is

$Pr(T_1 \le t_1) = Pr(x_1 + x_2 - 2x \le t_1, x \le x_1 \le x_2) + Pr(x_1 + x_2 - 2x \le t_1, x \le x_2 \le x_1) + Pr(x_2 - x_1 \le t_1, x_1 \le x \le x_2) + Pr(2x - x_1 - x_2 \le t_1, x_1 \le x_2 \le x) + Pr(x_1 - x_2 \le t_1, x_2 \le x \le x_1) + Pr(2x - x_1 - x_2 \le t_1, x_2 \le x_1 \le x)$.

Because of symmetry, we have

$Pr(x_1 + x_2 - 2x \le t_1, x \le x_1 \le x_2) = Pr(x_1 + x_2 - 2x \le t_1, x \le x_2 \le x_1)$,

$Pr(x_2 - x_1 \le t_1, x_1 \le x \le x_2) = Pr(x_1 - x_2 \le t_1, x_2 \le x \le x_1)$ *and*

$Pr(2x - x_1 - x_2 \le t_1, x_1 \le x_2 \le x) = Pr(2x - x_1 - x_2 \le t_1, x_2 \le x_1 \le x)$.

Therefore, instead of deriving six cdfs, we need only consider three cdfs. However, we can reduce the problem further by letting $A = 1 - x_1$, $B = 1 - x$, and $C = 1 - x_2$. Therefore,

$$\begin{aligned} Pr(x_1 + x_2 - 2x \le t_1, x \le x_1 \le x_2) &= Pr(1 - A + 1 - C - 2(1 - B) \le t_1, C \le A \le B) \\ &= Pr(2B - A - C \le t_1, C \le A \le B), \end{aligned}$$

which reduces to

$$\begin{aligned} Pr(x_1 + x_2 - 2x \le t_1, x \le x_1 \le x_2) &= Pr(x_1 + x_2 - 2x \le t_1, x \le x_2 \le x_1) \\ &= Pr(2x - x_1 - x_2 \le t_1, x_1 \le x_2 \le x) \\ &= Pr(2x - x_1 - x_2 \le t_1, x_2 \le x_1 \le x). \end{aligned}$$

Based on the result, we need to derive only two cdfs. When $x_1 \le x \le x_2$, $T_1 = x_2 - x_1$, and $t \in [0, 1]$

$$Pr(T_1 \le t_1 | \, x_1 \le x \le x_2) = Pr(x_2 - x_1 \le t_1, x_1 \le x \le x_2) = \int_0^{1-t_1} \int_{x_1}^{x_1+t_1} \int_{x_2}^{x_2} 1dxdx_2dx_1$$

$$= -\frac{1}{3}{t_1}^3 + \frac{1}{2}{t_1}^2. \tag{C1}$$

When $x_1 \le x_2 \le x$, $T_1 = 2x - x_1 - x_2$, and $t_1 \in [0, 1]$

$$Pr(T_1 \le t_1 | \, x_1 \le x_2 \le x) = Pr(2x - x_1 - x_2 \le t_1, x_1 \le x_2 \le x)$$

$$= \int_0^{1-t} \int_{x_1}^{x_1+t_1} \int_{x_1}^{1/2(x_1+x_2+t_1)} 1dxdx_2dx_1 + \int_{1-t}^{1-t_1/2} \int_{x_1}^{-x_1+2-t_1} \int_{x_2}^{1/2(x_1+x_2+t_1)} 1dxdx_2dx_1$$

$$+ \int_{1-t_1/2}^{1} \int_{-x_2+2-t_1}^{x_2} \int_{x_2}^{1} 1dxdx_1dx_2$$

$$= -\frac{1}{8}{t_1}^3 + \frac{1}{4}{t_1}^2. \tag{C2}$$

When $x_1 \le x_2 \le x$, $T_1 = 2x - x_1 - x_2$, and $t_1 \in (1,2]$

$$Pr(T_1 \le t_1 | \, x_1 \le x_2 \le x) = Pr(2x - x_1 - x_2 \le t_1, x_1 \le x_2 \le x)$$

$$= \int_0^{1-t_1/2} \int_{x_1}^{-x_1+2-t_1} \int_{x_2}^{1/2(x_1+x_2+t_1)} 1dxdx_2dx_1 + \int_0^{1-t_1/2} \int_{-x_1+2-t_1}^{1} \int_{x_2}^{1} 1dxdx_2dx_1$$

$$+ \int_{1-t_1/2}^{1} \int_{x_1}^{1} \int_{x_2}^{1} 1dxdx_2dx_1$$

$$= \frac{{t_1}^3 - 6{t_1}^2 + 12t_1 - 4}{24}. \tag{C3}$$

From the definition of the cdf,

$F(t_1) = P(T < t_1) = 2Pr(x_2 - x_1 \le t_1, x_1 \le x \le x_2) + 4\,Pr(2x - x_1 - x_2 \le t_1, x_1 \le x_2 \le x)$.

From Equations (C1) and (C2), when $x_1 \le x_2 \le x$, $T_1 = 2x - x_1 - x_2$, and $t_1 \in [0, 1]$, the cdf is

$$F(t_1) = -\frac{7}{6}{t_1}^3 + 2{t_1}^2. \tag{C4}$$

Likewise, when $t_1 \in (1, 2]$, from Equation (C1), $F(1|\ x_1 \le x \le x_2) = \frac{2}{6}$ and from Equation (C3) and $F(1)$, the cdf is

$$F(t_1) = F(1|\ x_1 \le x \le x_2) + 4\left(\frac{t^3 - 6t^2 + 12t - 4}{24}\right)$$

$$= \frac{1}{6}{t_1}^3 - {t_1}^2 + 2t_1 - \frac{1}{3}. \tag{C5}$$

From the cdf in Equations (C4) and (C5), the pdf of horizontal travel time is

$$f(t_1) = \begin{cases} -\frac{7}{2}{t_1}^2 + 4t_1 & t_1 \in [0,1] \\ \frac{1}{2}{t_1}^2 - 2t_1 + 2 & t_1 \in (1,2] \end{cases}.$$

Certification of Student Work

UNIVERSITY OF ARKANSAS **Certification of Student Work**

College of Engineering
Department of Industrial Engineering

MEMORANDUM

TO: University of Arkansas Graduate School

FROM: Haitao Liao, Professor, John and Mary Lib White Endowed Systems Integration Chair
John A. White, Chancellor Emeritus and Distinguished Professor (retired)

DATE: July 25, 2019

SUBJECT: Certification of Student Effort and Contribution

As co-directors of Mr. Jingming Liu's doctoral dissertation, we certify Mr. Liu contributed more than 51 percent of the work included in the chapter entitled, "A Working Paper on "Queueing analysis of the replenishment of multiple in-the-aisle pick positions" contained in the doctoral dissertation entitled, "Probabilistic Models for Order-Picking Operations with Multiple in-the-Aisle Pick Positions".

Chapter 3

Contribution 2: A Working Paper on "Stochastic Analysis of an Automated Storage and Retrieval System with Multiple in-the-Aisle Pick Positions"

Abstract

This study is focused on an automated storage and retrieval system with multiple in-the-aisle pick positions (MIAPP-AS/RS). The pick positions are located at both the floor and mezzanine levels. Previous research showed that the replenishment requests in such an order picking system can be modeled as a homogenous Poisson process. In this paper, the replenishment process of such a system is handled as an M/G/1 queueing problem, where replenishment of full pallet loads of cases is performed by an aisle-captive automated storage and retrieval (AS/R) machine. In the queueing formulation, the AS/R machine is the server, and the pick positions requiring replenishments are the customers. The service time is the sum of travel time, pick-up time and deposit time of the AS/R machine. For the travel times of different travel paths, the corresponding probability density functions are derived based on a Chebyshev travel metric. For a given number of pick positions, the MIAPP-AS/RS retrieval operation is modeled as a finite population queueing problem, and an infinite population approximation model is considered when the number of pick positions is sufficiently large. Moreover, dedicated and random storage policies for reserve storage are compared, and the conditions in favor of each policy are provided. A case study demonstrates how our results can be applied in guiding the design of an MIAPP-AS/RS.

Keywords: MIAPP-AS/RS, order picking, Chebyshev travel, Poisson distributed arrivals, general service time queue, Laplace transform

3.1. Introduction

The MIAPP design problem studied in this work was motivated by a visit to a distribution center for food products. Within a freezer, the AS/R machine replenished pick positions at which case picking of frozen food products was performed manually. A similar operation was analyzed by Ramtin and Pazour (2014); they developed expected travel time models of an aisle-captive AS/R machine used to replenish multiple in-the-aisle pick positions at both floor and mezzanine levels in a case-picking operation. In contrast to their work, we model the MIAPP-AS/RS problem as a queueing problem, in which the aisle-captive AS/R machine is the server and the pick positions in need of replenishments are the customers. A similar problem was addressed by Liu, et al. (2019); they analyzed a queueing model of an MIAPP operation, in which replenishment was performed using a narrow-aisle lift truck.

In the MIAPP operation, case picking is performed manually from double-deep selective pallet rack installed on both sides of a case-picking aisle. Order pickers move along the picking aisle, retrieving cases from pallets and placing them on an automated guided vehicle (AGV). Once all cases for an individual order have been picked and placed on the AGV, the AGV departs and a new AGV arrives, with a new pick list. When the last case of a pallet at one pick position (first pallet location) is picked, the empty pallet is removed from the pick position, the pallet load at the second pallet location moves forward automatically, and a replenishment request occurs.

On the opposite side of the selective pallet rack, an AS/R machine replenishes pick positions located on the bottom levels of two vertical sections of the rack. In this paper, we assume the bottom levels of the two vertical sections provide pick positions at the floor level and at the mezzanine level; further, we assume that the mezzanine is located at a mid-point in the vertical dimension of the storage rack and we assume upper levels of the storage rack within each section are reserve-storage locations. The AS/R machine replenishes pick positions by retrieving a full pallet from a reserve-storage location and depositing it at the pick position requiring replenishment. Four travel patterns are considered for the AS/R machine: (1) from a floor-level pick position to a floor-level reserve-storage location, then to a floor-level pick position; (2) from a floor-level pick position to a mezzanine-level reserve-storage location, then to a mezzanine-level pick position; (3) from a mezzanine-level pick position to a floor-level reserve-storage location, then to a floor-level pick position; and (4) from mezzanine-level pick position to a mezzanine-level reserve-storage location, then to a mezzanine-level pick position.

The research conducted by Liu, et al. (2019) revealed that the interarrival time for replenishment requests from a particular pick position are not exponentially distributed; however, the robustness of the finite population queueing model is such that, not only are quite accurate results obtained by assuming arrivals are Poisson distributed, but also the results obtained are quite accurate when customers do not have the same arrival rates in the MIAPP-ASRS. Indeed, in the latter case, using an average arrival rate the results obtained from the M/G/1/N model were nearly identical to the simulation results considering heterogeneous customers. Furthermore, a simulation study performed by Liu, et al. (2019) showed that the time between consecutive replenishment requests is approximately exponentially distributed for both homogeneous and heterogeneous customers in an MIAPP operation.

With random demands for replenishment requests from pick positions, the MIAPP-AS/RS retrieval operation is modeled as a queueing system with service time being the sum of AS/R travel time and pick-up time and deposit time (T_{PD}). We assume T_{PD} is deterministic. When there are relatively few pick positions, the MIAPP-AS/RS retrieval operation can be modeled as a finite population queue (M/G/1/N). When the number of pick positions is sufficiently large, an infinite population approximation can be used to obtain performance measures for the MIAPP-AS/RS retrieval operation.

Typically, the input station is located at the end of the replenishment aisle at floor level or at mezzanine level. Pallet loads arrive via conveyor at the input station and the AS/R machine performs storage operations; in performing only storages, the S/R travel path is from the input station to a reserve-storage location and then back to the input station. We assume the AS/R machine only performs storages during a period of time when order picking is not being performed. Therefore, replenishment and storage operations are not performed sequentially during the same period of time, thus eliminating dual-command operations (traveling from the input station to a reserve-storage location, traveling to another reserve-storage location, and traveling to a pick position needing to be replenished). In this paper, we only address the retrieval operation.

When the AS/R machine performs only replenishments, if it becomes idle, we assume the dwell point is at the pick position where it last performed a replenishment. The AS/R machine travels horizontally and vertically simultaneously; therefore, the Chebyshev metric is involved in calculating the time required for the AS/R machine to travel from a pick position to a reserve-storage location for the pick position requiring replenishment and, then, travel to the pick position requiring replenishment. Two reserve-storage policies are considered: dedicated- and random- (shared-) storage policies. With dedicated storage, the reserve-storage locations for a pick position are the rack openings in the vertical rack section immediately above the pick position; with random storage, the reserve storage locations are the rack openings in the vertical rack section over the entire width of the storage rack.

Drawing on conclusions in the paper, designers of an MIAPP-AS/RS can answer such questions as

(1) Can an infinite population approximation be used to obtain reasonably accurate results for the performance measures?

(2) For the reserve-storage space and throughput requirements of the system, which storage policy is preferred, random storage or dedicated storage?

(3) Are the expected values obtained for the queueing performance measures satisfactory for the current design?

(4) What dimension of the MIAPP-AS/RS will be most cost effective based on the expected values for the performance measures?

The remainder of this paper is organized as follows. In Section 2, the research literature related to the problem under study is reviewed; the literature review concludes with the contributions of our research. In Section 3, key notations and assumptions in the research are provided. In Section 4, for replenishment travel paths by the AS/R machine, we derive probability density functions for both dedicated and random storage policies, as well as for both a finite and an infinite number of pick positions in an S/R aisle. The expected value and variance for service time are obtained for the finite and infinite population cases. For the finite population case, we also obtain the Laplace transforms for the service time probability density functions. In Section 5, assuming incoming demands for replenishment occur in a Poisson fashion, we employ Laplace transforms of service time probability density functions to obtain steady state performance measures for M/G/1/N queues using formulas based on a result due to Takács (1960); next, we apply Pollaczek-Khinchine formulas to obtain steady state performance measures for M/G/1/∞ queues. In Section 6, a finite population queueing model is used to obtain the steady state performance measures of the MIAPP-AS/RS and we also employ an infinite population queueing approximation; next, we use an infinite population pdf instead of a finite population pdf for the finite population queue to obtain steady state performance measures; third, dedicated- and random-storage policies are compared using queueing performance measures; finally, we conclude the section with insights and practical considerations. In Section 7, we summarize our research, share conclusions drawn from the research and provide recommendations for further research.

3.2. Literature review

Numerous papers addressing unit-load storage and retrieval, case picking, and item picking have been published. In a recent paper, Liu et al. (2019) reviewed the literature related to an MIAPP-NALT operation. The principal difference in this paper and Liu et al. (2019) is the usage of an AS/R machine instead of a narrow aisle lift truck to perform replenishments for pick positions and consideration of a mezzanine level of pick positons. Rather than duplicate the literature they reviewed, we focus principally on research papers related to the MIAPP-AS/RS retrieval operation we address. The literature review is organized as follows: after considering papers dealing with Chebyshev travel, we review queueing papers related to storage/retrieval and order picking.

Chebyshev Metric: Automated storage/retrieval systems (AS/RS) have received considerable attention in the research literature, addressing such topics as system configuration (including two-dimension and three-dimension models), travel time estimation, storage policy and dwell-point strategies. See, for example, Rouwenhorst et al., 2000; De Koster et al., 2007; Gu et al., 2007; Roodbergen and Vis, 2009; Gu et al., 2010; Gagliardi et al., 2011. Models used to calculate the expected AS/R travel time for single command (SC) travel, travel between (TB), and dual command (DC) travel were developed by several researchers (Graves et al., 1977; Bozer and White, 1984; Scharfstein (1990); Foley and Frazelle, 1991; Change et al., 1995; Kouvelis and Papanicolaou, 1995; Sarker and Babu, 1995; Hu et al., 2005). Although most researchers assume constant velocity for the S/R machine, several researchers have considered explicitly acceleration and deceleration (Chang, et al., 1995; Hwang and Lee, 1990; Wen et al., 2001). Ashayeri et al. (2002) proposed an exact geometry-based analytical model to determine the cycle time for the AS/R machine. Chen et al. (2003) developed a travel time model for an AS/RS application for a container port with two types of platforms and studied two dwell point policies. Sari et al. (2005) developed travel time models for an AS/R machine coupled with a flow rack. Ghomri et al. (2009) developed new models for SC and DC cycle times of a multi-aisle AS/RS; the closed form models were compared to other more complicated models. Parikh and Meller (2010) developed a travel time model for a man-on-board S/R order picking system. Lerher et al. (2010) developed travel time models for a double-deep AS/RS. Sari and Bessnouci (2012) developed travel time models for a combination of flow rack and an AS/R machine with two dwell points. Kouloughli and Sari (2014) developed analytical AS/RS cycle time models to optimize a three dimension multi-aisle AS/RS in which the AS/R machine can travel among different aisles. With a double-deep storage rack and dual shuttle-based AS/R machine in Xu et al. (2015), a dual-shuttle AS/RS was analyzed. Bortolini et al. (2015) derived travel time models for an AS/RS with a three-class-based storage policy. Yang et al. (2015) designed an optimal multi-deep AS/RS by considering acceleration and deceleration characteristics in minimizing operation time under a random storage policy; Yang et al. (2017) extended their research by considering a full turnover-based storage policy. Hao et al. (2015) studied the SC operation of a three dimension AS/RS with a lower mid-point I/O; Xu et al. (2017) extended their work by considering the DC operation.

A variety of storage assignment policies have been studied in conjunction with the AS/RS: random storage; closest-open-location storage; class-based storage; full-turnover-based storage; and dedicated storage (Hausman et al., 1976; Graves et al., 1977; Schwarz et al., 1978;

Goetschalckx and Ratliff, 1990; Kouvelis and Papanicolaou, 1995; Van den Berg, 1999; Park, 2006; Park et al., 2003; Gagliardi et al., 2012). Dallari et al. (2000) analyzed man-on-board AS/RS performance under different storage policies. Park et al. (2006) analyzed miniload AS/RS system performance when the storage rack contains high turnover and low turnover areas and the pick time distribution is either exponential or deterministic. De Koster et al. (2008) and Yu and De Koster (2009) analyzed the optimal rack dimensions of a compact multi-deep (3D) AS/RS under different storage policies. Meghelli-Gaouar and Sari (2010) studied the flow rack AS/RS with a two class-based storage policy. Cardin et al. (2012) investigated the flow rack AS/RS with a class-based storage policy.

Park (1999) analyzed the optimal dwell point policy for an AS/RS assuming square-in-time racks and a dedicated storage policy. Van den Berg (2002) developed an analytical expression to determine the optimal dwell point of the AS/R machine. Meller and Mungwattana (2005) investigated the magnitude of the benefit of different dwell point strategies for the AS/RS. Sari et al. (2007) studied the effect of P/D station positions and conveyor positions. Hale et al. (2008) proposed a closed form model for the location of the dwell point in an AS/RS. Tanaka and Araki (2009) investigated AS/RS routing with multiple input and output points under a random storage policy. Xu et al. (2017) developed a dual-command cycle time model with a lower mid-point I/O dwell point policy.

Queueing models of storage/retrieval: Chow (1986) obtained the system performance for an AS/RS used in assembly lines by modeling it as an M/G/1/∞ queue with a FCFS discipline; other service disciplines were examined using simulation. Bozer and White (1990) analyzed a mini-load AS/RS using a two-server closed queueing network. De Koster (1994) employed a Jackson queueing network to analyze a zoned-pick-and-pass order-picking system to estimate throughput.

Lee (1997) used an M/M/1 queueing model to obtain the steady state performance measures for a unit-load AS/RS performing SC and DC operations. Park (1999) obtained steady state probabilities and system throughput for an end-of-aisle order picking system by modeling it as a two–state cyclic queueing system with a limited capacity. Koh et al. (2005) extended the work of Park (1999) by assuming an order picker serving multiple aisles within the end-of-aisle order picking system. Vlasiou et al. (2004) used a queueing model to analyze the performance of a system consisting of two carousel conveyors and one AS/R machine. Hur et al. (2004) used an M/G/1 queueing model with two types of customers to analyze a unit-load AS/RS, assuming the SC and DC operations have the same travel time distributions; Bozer and Cho (2005) obtained closed-form analytical results for an AS/RS with two types of customers using an M/G/1 queueing model. Hur and Nam (2006) analyzed the same M/G/1 queue with different arrival rates for two types of customers. Yu and De Koster (2009) used a network queue to analyze the impact of order batching and picking area zoning in a pick-and-pass order picking system. Pan and Wu (2012) employed a GI/G/1 closed network queue to analyze the throughput of a picker-to-parts warehousing system with multiple order pickers, including aisle congestion.

Malmborg (2002) used a state-transition diagram to compare the AS/RS and autonomous vehicle storage and retrieval system (AVS/RS). Then, Malmborg (2003) used a state-transition diagram to analyze interleaving effects between storage and retrieval operations. Kuo et al. (2007) analyzed the AVS/RS with an M/G/c queueing model, based on 12 service scenarios for the vehicle and probability distributions for vertical travel time. Kuo et al. (2008) used a queueing network approach to analyze the performance of the AVS/RS under a class-based storage policy. Fukunari and Malmborg (2008) approximated the performance of an AVS/RS with queueing approximations and compared the AVS/RS with a crane-based AS/RS. Fukunari and Malmborg

(2009) analyzed the AVS/R system by including interleaving of storage and retrieval requests with a network queueing approach to estimate the utilization rate of an AVS/R system. Heragu, et al. (2009) used an open queueing network model to compare AS/RS with AVS/RS and developed a manufacturing system performance analyzer to quickly evaluate alternate configurations of the systems. Zhang et al. (2009) analyzed the AVS/RS using a G/G/c queueing model to obtain the accurate waiting time of the incoming customers. Roy et al. (2012) used a multi-class semi-open queueing network to analyze the AVS/RS and determine the best dwell-point policy, cross-aisle location, configuration of aisle and columns, allocation of resources to zones, and vehicle dispatching rules. Roy et al. (2015) extended the work of Roy et al. (2012) to determine the best dwell-point policy and cross-aisle location. Ekren (2013) used semi-open queuing network to model the AVS/RS and obtained approximate performance results. Ekren (2014) used a matrix-geometric method to solve the queuing network employed by Ekren (2013). Cai et al. (2013) employed a semi-open queueing model to analyze an AVS/RS in which customers must be paired; using Markov chains, they obtained queueing performance measures. Cai et al. (2014) also analyzed a semi-open queue with two stages for the AVS/R system.

Ramtin and Pazour (2014) were the first to develop an analytical model of the MIAPP-AS/RS operation; they calculated throughput performance and the length-to-height configuration of an AS/RS aisle to minimize expected travel time. Ramtin and Pazour (2015) explored different operating policies, demand profiles, and shape factors for an MIAPP-AS/RS; in addition to calculating expected travel time and system throughput, they also developed a dedicated storage-assignment optimization model for stock used to replenish order-picking positions. Liu, et al. (2019) conducted a general service time queueing analysis for a MIAPP order picking system in which pick positions are only on the floor level and they are replenished by a narrow aisle lift truck;

a rectilinear metric was used to calculate the travel time for the narrow aisle lift truck.

Contributions of the paper: In comparison with previous research, we believe we are the first 1) to model the MIAPP-AS/RS as a queueing system; 2) to derive probability density functions for Chebyshev travel from a fixed point to a random point and, then, to a different fixed point (finite population), as well as from a random point to a random point and, then, to a random point (infinite population); 3) to employ a finite population queueing model (M/G/1/N) to analyze the impact of a storage policy on a retrieval operation using the Chebyshev travel metric; 4) to analyze conditions justifying the use of an infinite population queueing approximation for the M/G/1/N queueing model of an MIAPP-AS/RS; 5) to obtain very accurate performance measures for the M/G/1/N queueing model of an MIAPP-AS/RS using an infinite population pdf instead of a finite population pdf and (6) to compare dedicated storage and random storage space requirements in an MIAPP-AS/RS.

We believe our results will be beneficial for other research involving Chebyshev travel from a fixed point to a random point to a different fixed point, as well as Chebyshev travel from a random point to a random point to a random point. In both cases, the "travel legs" are statistically dependent. The lack of statistical independence does not hinder researchers who are only interested in expected values; however, if researchers are interested in travel time variances, then our results will be quite beneficial.

3.3. Notation and Assumptions

The following notation is used in formulating the queueing models:

B: breadth or width of the rack (or length of the S/R aisle);

H: height of the rack;

N: number of pick positions , including both levels and both sides of an S/R aisle;

V_h: horizontal travel speed of the AS/R machine;

V_v: vertical travel speed of the AS/R machine;

t_h: time for the AS/R machine to travel a distance of B horizontally, B/V_h;

t_v: time for the AS/R machine to travel a distance of H vertically, H/V_v;

b: shape factor, the ratio of t_v and t_h;

f: floor level pick position;

m: mezzanine level pick position;

ff: travel path from floor-level pick position to floor-level pick position;

fm: travel path from floor-level pick position to mezzanine-level pick position;

mf: travel path from mezzanine-level pick position to floor-level pick position;

mm: travel path from mezzanine-level pick position to mezzanine-level pick position;

p_i: time-based horizontal coordinate location of pick position i;

p_j: time-based horizontal coordinate location of pick position j;

d: time-based distance required for the AS/R machine to travel horizontally between p_i and p_j , $d = |p_i - p_j|$;

T_{PD}: time for the AS/R machine to pick up and deposit a full pallet;

T: travel time for the AS/R machine to perform a replenishment;

T_s: time to perform a retrieval operation or service time, equals $T + T_{PD}$;

λ: arrival rate of replenishment requests for each pick position;

$\bar{\lambda}$: effective arrival rate of replenishment requests for a finite population queue;

μ: service rate of replenishment requests, $1/T_s$;

$\bar{\rho}$: Traffic intensity, $\bar{\lambda}/\mu$ for a finite population;

ρ: for a finite population; λ/μ for an infinite population;

L_q: expected number of pick positions waiting for replenishment to begin;

L: expected number of pick positions in the queueing system (number of customers waiting plus being served);

W_q: expected time for a pick position to wait for replenishment to begin;

W: expected time for a pick position waiting for replenishment to be completed (waiting time plus service time);

$P_{i,j}$: probability the AS/R machine travels from pick position i to pick position j.

To apply both finite population and infinite population M/G/1 queueing models, the probability density functions of travel times of an AS/R machine must be derived. Based on constant horizontal and vertical travel speeds, following the approach of Bozer and White (1984), we convert the storage area to a time-dimension space by calculating the time for the AS/R machine to travel horizontally the length of the S/R aisle (t_h) and the time required for the AS/R machine to travel vertically from floor level to the top of the storage rack (t_v). Assuming $t_h \geq t_v$, the storage area of the rack is normalized into time-based space with dimensions $1 \times b$, where horizontal travel time is 1 and vertical travel time is b by letting b equal t_v/t_h. Thereafter, we can convert normalized time to minutes by multiplying normalized time units by t_h. If t_h is not 1.0, the expected value and variance obtained from our models are multiplied by t_h and t_h^2, respectively, to obtain accurate time-based results.

From the definition of T_s, because T_{PD} is a constant and T and T_{PD} are statistically independent, the expected value and variance expressions for service time are:

$$E[T_s] = E[T] + T_{PD} \tag{1}$$

and

$$V[T_s] = V[T] \tag{2}$$

To differentiate various scenarios involving either a finite population or an infinite population under either a dedicated- or a random-storage policy, F is used to denote a finite population, I is used to denote an infinite population, D is used to denote a dedicated-storage policy, and R is used to denote a random-storage policy. For example, FDE[T] represents the expected travel time of the AS/R machine with a finite population under a dedicated-storage policy.

3.4. Probability density functions

The AS/R machine travels from a pick position to a reserve-storage location containing a unit load for the next pick position needing to be replenished, then travels to the pick position requiring replenishment. The same travel pattern repeats if the AS/R machine is idle or has just completed a replenishment at a pick position.

In this section, we model travel paths of the AS/R machine for both dedicated-storage and random-storage policies. Figure 1 shows a storage rack with pick positions located at the floor level and at the mezzanine level, located at a mid-point vertically in the storage rack. Above each pick position are 9 reserve storage locations. Along the S/R aisle are 35 columns of storage, with each column representing a pick position at floor level or mezzanine level. The solid horizontal black line represents the mezzanine; the dashed black lines enclose the region within which the AS/R machine shuttle travels. In treating the travel region as a continuous space, we establish coordinates shown in Figure 2. Specifically, horizontal travel occurs between the coordinates (0, 0) and (1, 0); vertical travel occurs between the coordinates (0, 0) and (0, b). The mezzanine level extends from coordinate (0, b/2) to (1, b/2).

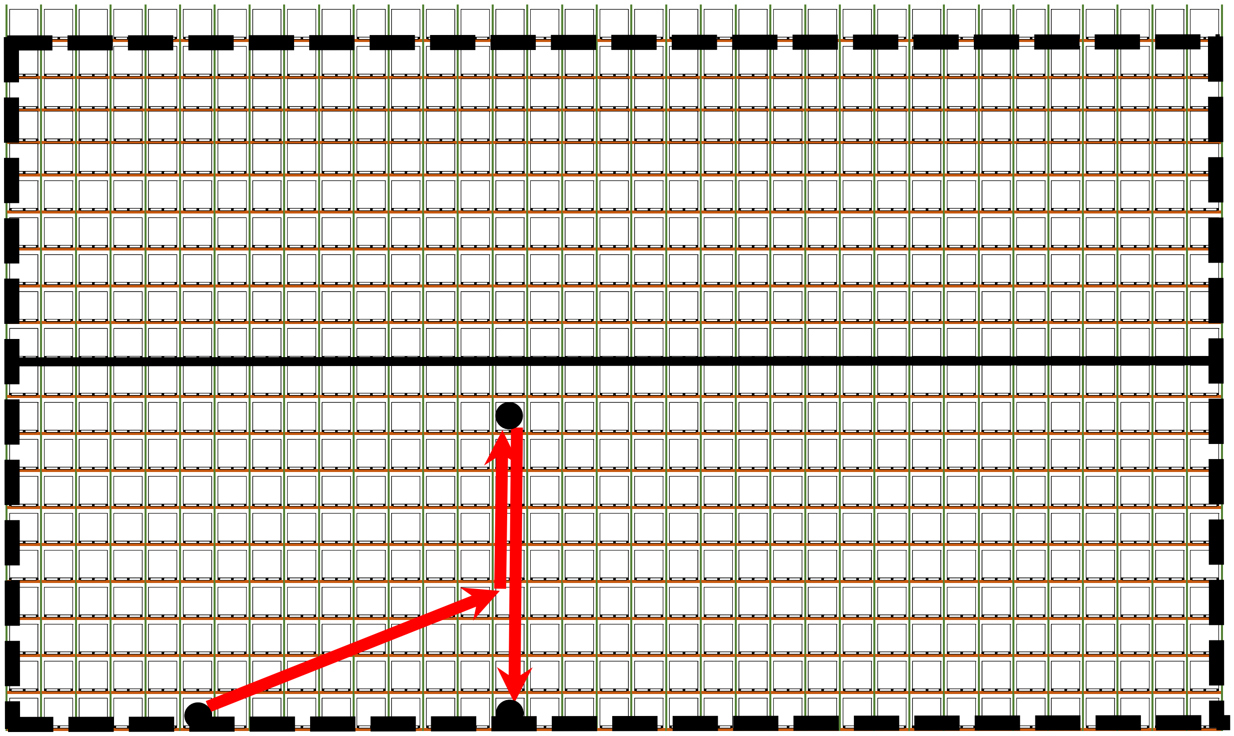

Figure 1. Front view of a storage rack.

In Figure 1, the black dots and the red lines depict the travel of the AS/R machine from a pick position at floor level to a reserve-storage location for a floor-level pick position and to the pick position requiring replenishment. With dedicated storage, the reserve-storage location is immediately above the pick position to be replenished; therefore, the reserve-storage location and the pick position requiring replenishment share the same *x*-coordinate. For random storage, the AS/R machine travels to a random reserve-storage location in the vertical section containing the SKUs for the pick position to be replenished and, then to the pick position requiring replenishment. To simplify derivations, we assume storage locations are uniformly distributed over the entire reserve-storage region. In doing so, we recognize the results obtained are approximations because the discrete storage locations are ignored and pick positions are treated as points on a line.

With a finite number of pick positions on each picking level of the storage rack, the AS/R machine travels from one pick position on a level (bottom or mezzanine level) of the rack to a random reserve-storage location within the rack and, then, to another pick position on the level of the rack at which the replenishment request is initiated. With an infinite approximation for the number of pick positions, the AS/R machine travels from a random point at one of the two picking levels to a random reserve-storage location within a vertical section containing SKUs for the next replenishment position, and, then, to another random point representing the pick position requiring replenishment at the bottom level of one of the two vertical sections (the starting point and the ending point can be at the same location because the two pick positions requiring replenishment can be located on opposite sides of the S/R aisle).

Four distinct scenarios (FD, FR, ID and IR) are considered. Therefore, four sets of pdfs are developed for the travel time of the AS/R machine. For the four scenarios, we obtain expected values, variances and Laplace transforms of the travel time pdfs.

3.4.1 Probability density functions for a finite population and a dedicated storage policy

For the FD Scenario, to perform a replenishment when the replenishment load is located immediately above the pick position to be replenished (dedicated storage), as Figure 2 illustrates, AS/R machine travel originates at pick position *i*, with coordinates $(p_i, 0)$; next, the AS/R machine travels via the Chebyshev metric to a random storage point at coordinates (p_j, y) within the rack and, then, travels via the Chebyshev metric to pick position j at coordinates $(p_j, 0)$ where y $\in$ unif$[0, b]$. Let T_{ij} denote the total travel time, $T_{ij} = max(|p_i - p_j|, y) + y = max(|p_i - p_j|, y) + y$.

Three alternative travel paths are considered. The travel path for the AS/R machine from a pick position on one level (*f* or *m*) to a dedicated storage location for a pick position on the same level (*f* or *m*) and to a pick position on the same level (*f* or *m*) is denoted Path *ff* or Path *mm*. The pdf for Path *ff* is the same as the pdf for Path *mm*. With Path *fm*, the AS/R machine travels from the floor level to a mezzanine-level dedicated storage location and to a mezzanine-level pick position. With Path *mf*, the AS/R machine travels from the mezzanine level to a floor-level dedicated storage location and to a floor-level pick position.

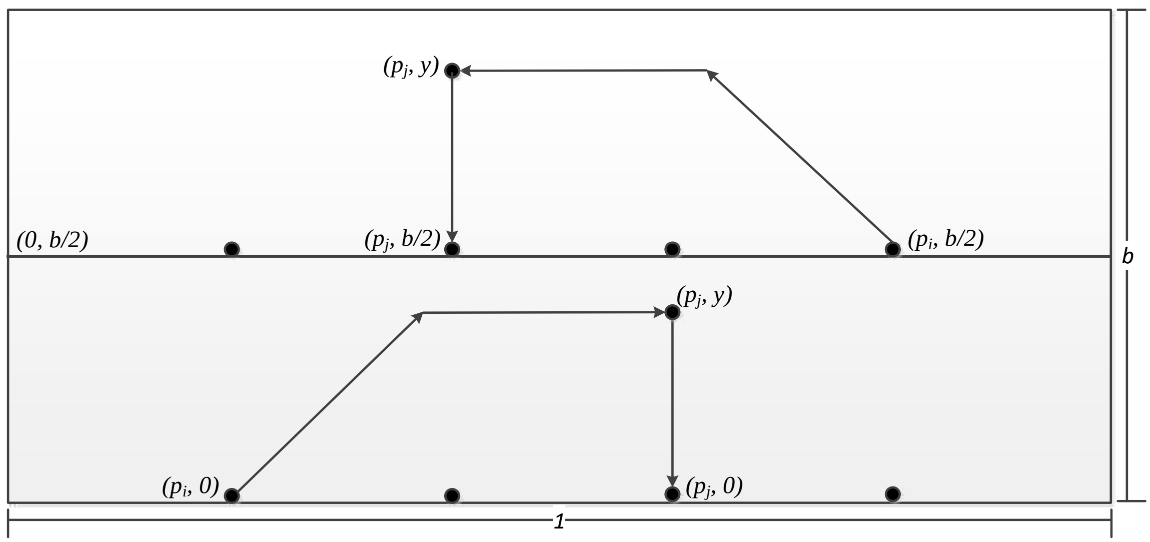

Figure 2. AS/R machine travel for Path *ff* (floor to floor) and Path *mm* (mezzanine to mezzanine).

Letting $p_{[1]} = min(p_i, p_j)$, $m_{[2]} = max(p_i, p_j)$ and $d = p_{[2]} - p_{[1]}$, then from Appendix A, there are two Conditions for Path *ff* and Path *mm* in the FD Scenario: $b/2 \leq d$ and $\mathrm{b}/2 > d$. When $b/2 \leq d$, the pdf for the FD Scenario is

$$f(t_{ij}) = 2/b \qquad\qquad t_{ij} \in [d, b/2 + d]. \tag{3}$$

When $\mathrm{b}/2 > d$, the pdf for the FD Scenario is

$$f(t_{ij}) = \begin{cases} 2/b & t_{ij} \in [d, 2d] \\ 1/b & t_{ij} \in (2d, b] \end{cases}. \tag{4}$$

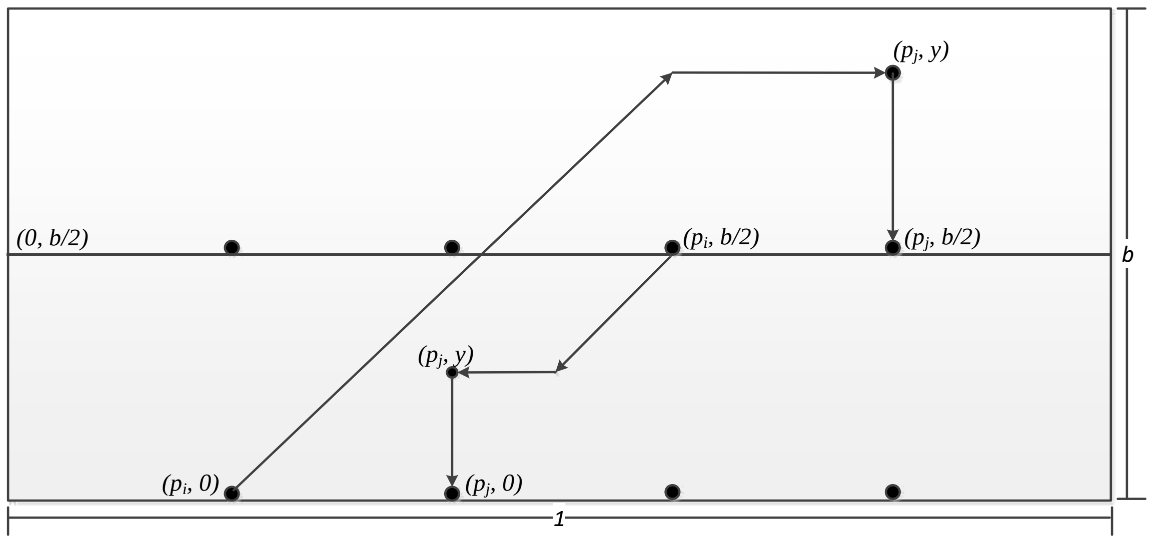

Figure 3. AS/R machine travel for Path *fm* (floor to mezzanine) and Path *mf* (mezzanine to floor).

As Figure 3 illustrates, from a pick position at floor level $(p_i, 0)$ to a reserve-storage location for a mezzanine level pick position (p_j, y) and to the pick position $(p_j, b/2)$ requesting replenishment. For AS/R travel for Path *fm*, letting $p_{[1]} = min(p_i, p_j)$ and, $p_{[2]} = max(p_i, p_j)$, from Appendix B, there are three Conditions: $b \le d$, $b/2 \le d < b$ and $d < b/2$.

Condition 1: $b \le d$, the pdf is

$$f(t_{ij}) = 2/b \qquad t_{ij} \in [d, b/2 + d]. \tag{5}$$

Condition 2: $b/2 \le d < b$, the pdf is

$$f(t_{ij}) = \begin{cases} 2/b & t_{ij} \in [d, 2d - b/2] \\ 1/b & t_{ij} \in (2d - b/2, 3b/2] \end{cases}. \tag{6}$$

Condition 3: $d < b/2$, the pdf is

$$f(t_{ij}) = 1/b, \qquad t_{ij} \in [b/2, 3b/2]. \tag{7}$$

As Figure 3 illustrates, from a pick position at mezzanine level $(p_i, b/2)$ to a reserve-storage location for a floor-level pick position (p_j, y) and to the pick position requesting replenishment $(p_j, 0)$. For AS/R travel for Path *mf*, letting $m_{[1]} = min(m_i, m_j)$ and, $m_{[2]} = max(m_i, m_j)$, from Appendix C, there are two Conditions: $b/2 \le d$ and $b/2 > d$.

Condition 1: $b/2 \leq d$, the pdf is

$$f(t_{ij}) = F'(t_{ij}) = 2/b \qquad\qquad t_{ij} \in [d, b/2 + d]. \tag{8}$$

Condition 2: $d < \mathrm{b}/2$, the pdf for the FD Scenario is

$$f(t_{ij}) = \begin{cases} 1 - 2d/b & t_{ij} = b/2 \\ 2/b & t_{ij} \in (b/2, b/2 + d] \end{cases}. \tag{9}$$

3.4.2 Probability density functions for a finite population and a random storage policy

For Path *ff* or Path *mm* (travel within the same level), with a random storage policy, because of the relative values of p_i, p_j and *b*/2, many combinations must be considered in deriving the pdfs. For convenience, we separate the rack into two parts (left part and right part) by a solid vertical line at the x-coordinate, $(p_i + p_j)/2$. As illustrated in Figure 4, we use contour lines to derive the pdfs. From the left portion of Figure 4, as illustrated by the shadowed areas with different colors and hatching, there are three regions. The shape of the contour line is different for each region and the cumulative density function (cdf) can be easily obtained using the contour lines in Figure 4. For example, in the left portion of Figure 4, travel time for a random point on the dashed contour line is C; therefore, the cdf, $F(t_{ij} \leq C)$, is obtained by dividing the area enclosed by the dashed contour line and line $x = (p_i + p_j)/2$ by the area of the entire rectangle. By the definition of a contour line, the calculation procedure works because travel time for the AS/R machine will be less than or equal to C for all random points in the region enclosed between the dashed contour line and the line $x = (p_i + p_j)/2$.

Depending on the locations of the pair of pick positions, the time required to travel between the x-coordinates, $(p_i + p_j)/2$ and p_i, can be 1) less than *b*/2, the maximum vertical travel time or 2) greater than or equal to *b*/2. For the first instance, there can be two regions; for the second instance, there can be another two regions. Considering both the left and right portions, there are 4^2, or 16, pdfs to derive. By considering only one portion of the storage region, there are 2 + 2, or 4, pdfs to derive.

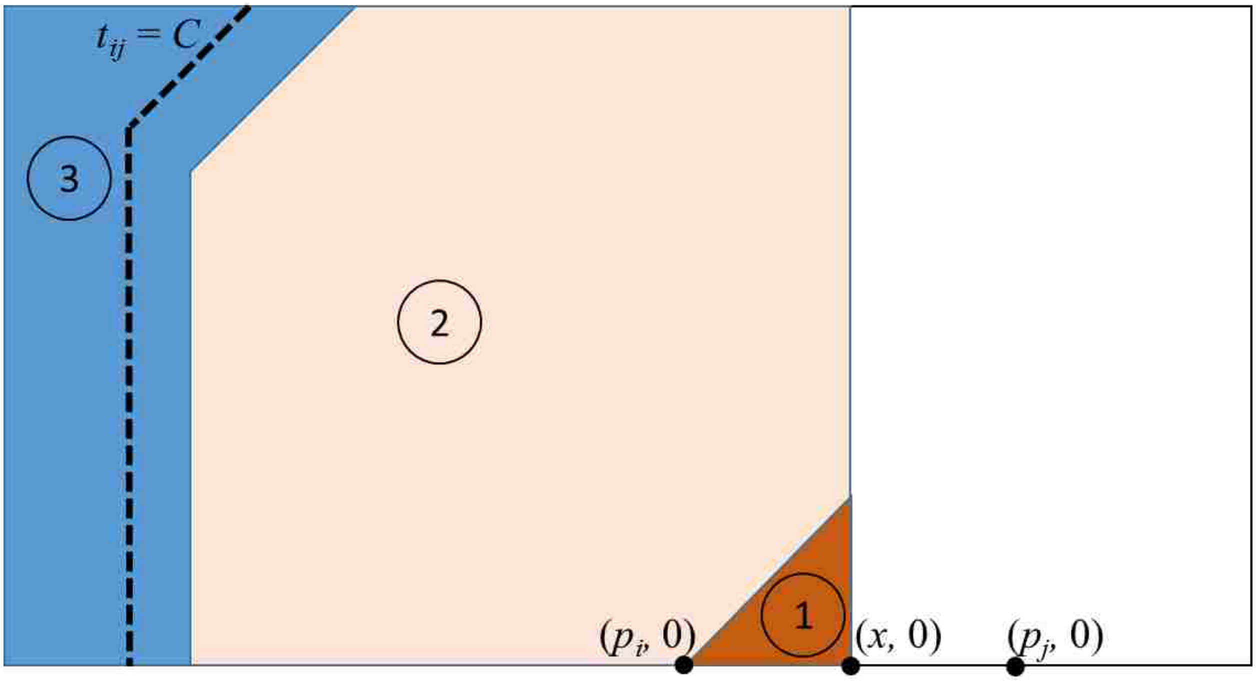

Figure 4. Travel within the same storage level for a finite population and a random storage policy

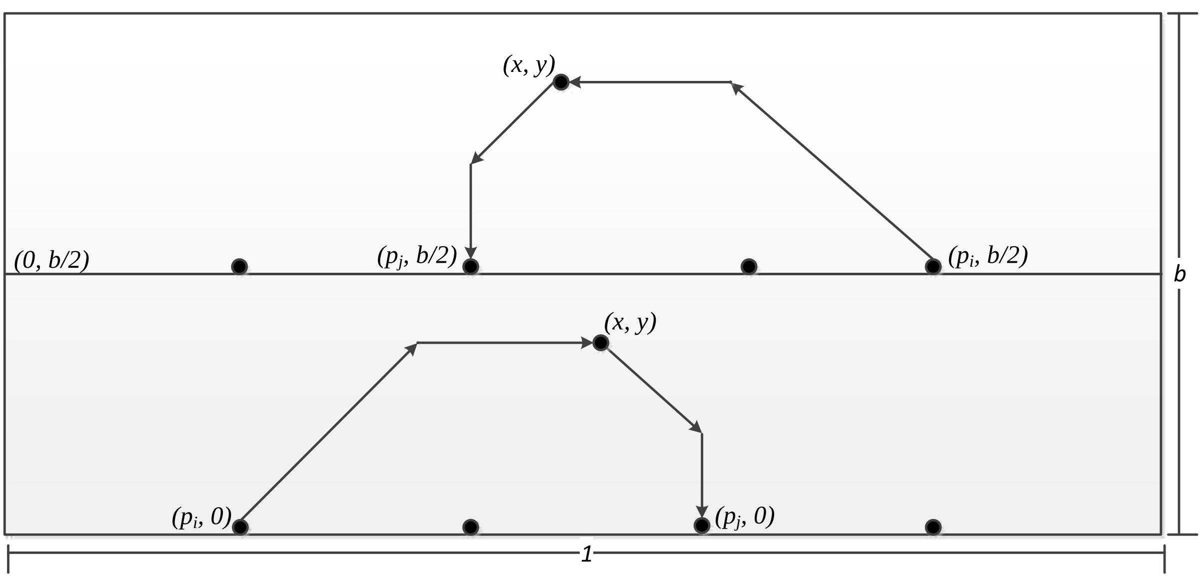

Figure 5. Path *ff* and Path *mm* for the FR Scenario

Because the travel patterns for travel paths *ff* and travel path *mm* are the same, we only need to derive the probability density function (pdf) for path *ff*. As Figure 5 illustrates, travel occurs from a pick position at a floor level $(p_i, 0)$ to a random reserve-storage location for a floor-level pick position (x, y) and to the pick position requesting replenishment $(p_j, 0)$. Let $d = p_{[2]} - p_{[1]}$. For the part to the left of $\mathrm{x} = (p_{[1]} + p_{[2]})/2$ in Figure 5: $t_{min} = min((p_{[1]} + p_{[2]})/2, b/2)$ and $t_{max} = max\left((p_{[1]} + p_{[2]})/2, b/2\right)$. For the part to the right of $x = (p_{[1]} + p_{[2]})/2:$ $t_{min} = min(1 - (p_{[1]} + p_{[2]})/2, b/2)$ and $t_{max} = max(1 - (p_{[1]} + p_{[2]})/2, b/2)$.

Because of symmetry, the left and right parts will have the same pdf. For the FR Scenario, there are four Conditions to consider. From Appendix D, the pdfs for the four Conditions for Path *ff* are:

Condition 1: $b \geq d$ and $t_{max} \geq t_{min} + d/2$

$$f(t_{ij}) = \begin{cases} d^2/(4b) & t_{ij} = d \\ t/b & d < t_{ij} \leq 2t_{min} \\ (d + 4t_{min} - t)/(2b) & 2t_{min} < t_{ij} \leq d + 2t_{min} \\ t_{min}/b & d + 2t_{min} < t_{ij} \leq 2t_{max} \end{cases}. \tag{10}$$

Condition 2: $b \geq d$ and $t_{max} < t_{min} + d/2$

$$f(t_{ij}) = \begin{cases} d^2/(4b) & t_{ij} = d \\ t/b & d < t_{ij} \leq 2t_{min} \\ (d + 4t_{min} - t)/(2b) & 2t_{min} < t_{ij} \leq 2t_{max} \\ (d + 2t_{min} + 2t_{max} - 2t)/b & 2t_{max} < t_{ij} \leq d/2 + t_{min} + t_{max} \end{cases}. \tag{11}$$

Condition 3: $b < d$ and $t_{max} \geq (b + d)/2$

$$f(t_{ij}) = \begin{cases} (2d - b)/4 & t_{ij} = d \\ (d + 2b - t_{ij})/(2b) & d < t_{ij} \leq d + b \\ 1/2 & d + b < t_{ij} \leq 2t_{max} \end{cases}. \tag{12}$$

Condition 4: $b < d$ and $t_{max} < (b + d)/2$

$$f(t_{ij}) = \begin{cases} (2d - b)/4 & t_{ij} = d \\ (d + 2b - t_{ij})/(2b) & d < t_{ij} \le 2t_{max}. \\ (b + d + 2t_{max} - 2t_{ij})/b & 2t_{max} < t_{ij} \le d/2 + b/2 + t_{max} \end{cases} \quad (13)$$

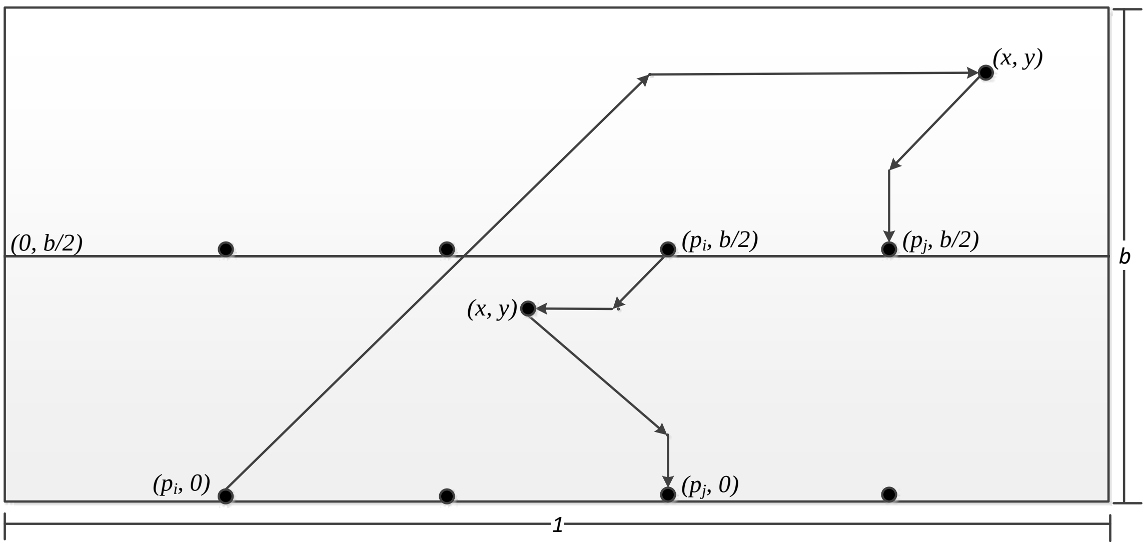

Figure 6. Path *fm* and Path *mf* for the FR Scenario

As Figure 6 illustrates for the travel path *fm*, travel occurs from a pick position at floor level $(p_i, 0)$ to a random reserve-storage location for a mezzanine-level pick position (x, y) and to the pick position $(p_j, b/2)$ requesting replenishment. For Path *fm* in Figure 6, the rack is separated into a left part and a right part by a vertical line at the coordinate $x = p_{[2]}$. We will consider the left part and right part for all possible conditions. Combining the probability density functions for the left part and the right part, we obtain the probability density function for one particular condition. For the path *fm*, we have

$$T_{ij} = max(|p_i - x|, y) + max(|p_j - x|, y - b/2).$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$, $d = p_{[2]} - p_{[1]}$,

Conditions 1 through 5 in the derivation process apply to the right part and Conditions 6 through 10 in the derivation process apply to the left part when $d < b/2$. Conditions 11 through 14 in the derivation process apply to the right part and Conditions 15 through 18 in the derivation process apply to the left part when $b/2 \leq d \leq 3b/2$. Condition 19 and condition 20 in the derivation process apply to the right part and Conditions 21 through 23 in the derivation process apply to the left part when $b/2 \leq d \leq 3b/2$.

From Appendix E, the pdfs for all the Conditions for Path *fm* are:

Condition 1: $d < b/2$ and $b \leq 1 - p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b - d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in (b - d, 3b/2] \\ (3b - d - t_{ij})/(2b) & t_{ij} \in (3b/2, 2b - d] \\ 1/2 & t_{ij} \in \left(2b - d, 2 - p_{[1]} - p_{[2]}\right] \end{cases}. \quad (14)$$

Condition 2: $d < b/2$ and $3b/4 + d/2 \leq 1 - p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b - d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in (b - d, 3b/2] \\ (3b - d - t_{ij})/(2b) & t_{ij} \in \left(3b/2, 2 - p_{[1]} - p_{[2]}\right] \\ (2b - 2(t_{ij} - 1 + p_{[2]}))/b & t_{ij} \in \left(2 - p_{[1]} - p_{[2]}, b + 1 - p_{[2]}\right] \end{cases}. \quad (15)$$

Condition 3: $d < b/2$ and $b/2 + d \leq 1 - p_{[1]} < 3b/4 + d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b - d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in \left(b - d, 2 - p_{[1]} - p_{[2]}\right] \\ (8 + b - 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(2 - p_{[1]} - p_{[2]},\ 3b/2\right] \\ (2b - 2(t_{ij} - 1 + p_{[2]}))/b & t_{ij} \in \left(3b/2, b + 1 - p_{[2]}\right] \end{cases}. \quad (16)$$

Condition 4: $d < b/2$ and $b/2 \leq 1 - p_{[1]} < b/2 + d$, the pdf is ,

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b - d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in \left(b - d, 2 - p_{[1]} - p_{[2]}\right] \\ (8 + b - 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(2 - p_{[1]} - p_{[2]}, b/2 + 2 - 2p_{[2]}\right] \\ (1 - p_{[2]})/b & t_{ij} \in \left(b/2 + 2 - 2p_{[2]}, 3b/2\right] \end{cases}. \quad (17)$$

Condition 5: $d < b/2$ and $1 - p_1 < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in \left[b/2, b/2 + 1 - p_{[2]}\right] \\ (8 + b - 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(b/2 + 1 - p_{[2]},\ b/2 + 2 - 2p_{[2]}\right]. \\ (1 - p_{[2]})/b & t_{ij} \in \left(b/2 + 2 - 2p_{[2]}, 3b/2\right] \end{cases} \quad (18)$$

Condition 6: $d < b/2$ and $b \le p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b + d] \\ (4t_{ij} - b + 2d)/(4b) & t_{ij} \in (b + d, 3b/2] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in (\ 3b/2, 2b + d]. \\ 1/2 & t_{ij} \in \left(2b + d, 2p_{[1]} + d\right] \end{cases} \quad (19)$$

Condition 7: $d < b/2$ and $3b/4 - d/2 \le p_{[1]} < b$, the pdf is, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b + d] \\ (4t_{ij} - b + 2d)/(4b) & t_{ij} \in (b + d, 3b/2] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in \left(\ 3b/2, 2p_{[1]} + d\right]. \\ 2(b + p_{[2]} - t_{ij})/b & t_{ij} \in \left(2p_{[1]} + d, b + p_{[2]}\right] \end{cases} \quad (20)$$

Condition 8: $d < b/2$ and $b/2 \le p_{[1]} < 3b/4 - d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b + d] \\ (4t_{ij} - b + 2d)/(4b) & t_{ij} \in \left(b + d, 2p_{[1]} + d\right] \\ (b + 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(2p_{[1]} + d, 3b/2\right]. \\ 2(b + p_{[2]} - t_{ij})/b & t_{ij} \in \left(3b/2, b + p_{[2]}\right] \end{cases} \quad (21)$$

Condition 9: $d < b/2$ and $b/2 - d \le p_{[1]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in \left[b/2, b/2 + p_{[2]}\right] \\ (b + 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(b/2 + p_{[2]}, 3b/2\right]. \\ 2(b + p_{[2]} - t_{ij})/b & t_{ij} \in \left(3b/2, b + p_{[2]}\right] \end{cases} \quad (22)$$

Condition 10: $d < b/2$ and $p_{[1]} < b/2 - d$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b/2 + p_{[2]}] \\ (b + 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in (b/2 + p_{[2]}, 2p_{[2]} + b/2]. \\ p_{[2]}/b & t_{ij} \in (2p_{[2]} + b/2, 3b/2] \end{cases} \tag{23}$$

Condition 11: $b/2 \le d < 3b/2$ and $b/2 + d \le 1 - p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 3b/2] \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (3b/2, b + d] \\ 1/2 & t_{ij} \in (b + d, 2 - 2p_{[1]} - d] \end{cases}. \tag{24}$$

Condition 12: $b/2 \le d < 3b/2$ and $3b/4 + d/2 \le 1 - p_{[1]} < b/2 + d$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 3b/2] \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (3b/2, 2 - 2p_{[1]} - d] \\ (b + 2 - 2p_{[1]} - 2t_{ij})/b & t_{ij} \in (2 - 2p_{[1]} - d, b/2 + 1 - p_{[1]}] \end{cases}. \tag{25}$$

Condition 13: $b/2 \le d < 3b/2$ and $b \le 1 - p_{[1]} < 3b/4 + d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 2 - 2p_{[1]} - d] \\ (4 - b - d - 4p_{[1]} - t_{ij})/(2b) & t_{ij} \in (2 - 2p_{[1]} - d, 3b/2] \\ (b + 2 - 2p_{[1]} - 2t_{ij})/b & t_{ij} \in (3b/2, b/2 + 1 - p_{[1]}] \end{cases}. \tag{26}$$

Condition 14: $b/2 \le d < 3b/2$ and $1 - p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 2 - 2p_{[1]} - d] \\ (4 - b - d - 4p_{[1]} - t_{ij})/(2b) & t_{ij} \in (2 - 2p_{[1]} - d, 2 - 2p_{[1]} - b/2] \\ (4 - b - 2d - 4p_{[1]})/(4b) & t_{ij} \in (2 - 2p_{[1]} - b/2, 3b/2] \end{cases}. \tag{27}$$

Condition 15: $b/2 \leq d < 3b/2$ and $b \leq p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij} - b - d)/(2b) & t_{ij} \in (d, 3b/2] \\ 1 & t_{ij} \in (3b/2, b+d] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in (b+d, 2b+d] \\ 1/2 & t_{ij} \in \left(2b+d, 2p_{[1]} + d\right] \end{cases}. \tag{28}$$

Condition 16: $b/2 \leq d < 3b/2$ and $b/2 \leq p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij} - b - d)/(2b) & t_{ij} \in (d, 3b/2] \\ 1 & t_{ij} \in (3b/2, b+d] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in \left(b+d, 2p_{[1]} + d\right] \\ 2(b + d + p_{[1]} - t_{ij})/b & t_{ij} \in \left(2p_{[1]} + d, b + p_{[1]} + d\right] \end{cases}. \tag{29}$$

Condition 17: $b/2 \leq \ d < 3b/2$ and $b - d \leq p_{[1]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij} - b - d)/(2b) & t_{ij} \in (d, 3b/2] \\ 1 & t_{ij} \in \left(3b/2, b/2 + d + p_{[1]}\right] \\ 2(b + d + p_{[1]} - t_{ij})/b & t_{ij} \in \left(b/2 + d + p_{[1]}, b + d + p_{[1]}\right] \end{cases}. \tag{30}$$

Condition 18: $b/2 \leq d < 3b/2$ and $p_{[1]} < b - d$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij} - b - d)/(2b) & t_{ij} \in \left(d, b/2 + d + p_{[1]}\right] \\ (b + 3d + 4p_{[1]} - t_{ij})/(2b) & t_{ij} \in \left(b/2 + d + p_{[1]}, 3b/2\right] \\ 2(b + d + p_{[1]} - t_{ij})/b & t_{ij} \in \left(3b/2, b + d + p_{[1]}\right] \end{cases}. \tag{31}$$

Condition 19: $3b/2 < d$ and $b/2 + d \leq 1 - p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (d, b+d] \\ 1/2 & t_{ij} \in \left(b+d, 2 - 2p_{[1]} - d\right] \end{cases}. \tag{32}$$

Condition 20: $3b/2 < d$ and $1 - p_{[1]} < b/2 + d$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ (2b + d - t_{ij})/(2b) & t_{ij} \in \left(d, 2 - 2p_{[1]} - d\right]. \\ \left(b + 2 - 2p_{[1]} - 2t_{ij}\right)/b & t_{ij} \in \left(2 - 2p_{[1]} - d, b/2 + 1 - p_{[1]}\right] \end{cases} \tag{33}$$

Condition 21: $3b/2 < d$ and $b \le p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ 1 & t_{ij} \in (d, b + d] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in (b + d, 2b + d]. \\ 1/2 & t_{ij} \in \left(2b + d, 2p_{[1]} + d\right] \end{cases} \tag{34}$$

Condition 22: $3b/2 < d$ and $b/2 \le p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ 1 & t_{ij} \in (d, b + d] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in \left(b + d, 2p_{[1]} + d\right]. \\ 2\left(b + d + p_{[1]} - t_{ij}\right)/b & t_{ij} \in \left(2p_{[1]} + d, b + d + p_{[1]}\right] \end{cases} \tag{35}$$

Condition 23: $3b/2 < d$ and $p_{[1]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ 1 & t_{ij} \in \left(d, b/2 + d + p_{[1]}\right]. \\ 2\left(b + d + p_{[1]} - t_{ij}\right)/b & t_{ij} \in \left(b + d, 2 - 2p_{[1]} - d\right] \end{cases} \tag{36}$$

As Figure 6 illustrates for the travel path *mf*, travel occurs from a pick position at mezzanine level $(p_i, b/2)$ to a random reserve-storage location for a floor-level pick position (x, y) and to the pick position requesting replenishment $(p_j, 0)$. For Path *mf* in Figure 6, from symmetry, the pdf for travel to the left of $x = max(p_j, p_j)$ will be the same as that for travel to the right of $x = max(p_j, p_j)$. For the FR Scenario, there are six Conditions to consider.

From Appendix F, there are six Conditions to consider in deriving the pdf for Path *mf* with the FR Scenario.

Condition 1: $d \leq b/2$ and $b/2 \leq 1 - p_{[2]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (b^2 - 4d^2)/(8b) & t = b/2 \\ (3b - 2t_{ij})/(2b) & t \in (b/2, b-d] \\ (2b + d - t_{ij})/(2b) & t \in (b-d, b+d] \\ 1/2 & t \in (b+d, 2+d-2p_{[2]}] \end{cases}. \tag{37}$$

Condition 2: $d \leq b/2$ and $b/2 - d \leq 1 - p_{[2]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (b^2 - 4d^2)/(8b) & t = b/2 \\ (3b - 2t_{ij})/(2b) & t \in (b/2, b-d] \\ (2b + d - t_{ij})/(2b) & t \in (b-d, 2+d-2p_{[2]}] \\ (2 + b - 2\mathrm{p}_{[1]} - 2t_{ij})/b & t \in (2+d-2p_{[2]}, b/2+1-p_{[1]}] \end{cases}. \tag{38}$$

Condition 3: $d \leq b/2$ and $b/4 - d/2 \leq 1 - p_{[2]} < b/2 - d$, the pdf is

$$f(t_{ij}) = \begin{cases} (b^2 - 4d^2)/(8b) & t = b/2 \\ (3b - 2t_{ij})/(2b) & t \in (b/2, 2-p_{[1]}-p_{[2]}] \\ 2(2 + b - p_{[1]} - p_{[2]} - 2t_{ij})/b & t \in (2-p_{[1]}-p_{[2]}, b/2+1-p_{[2]}] \\ (2 + b - 2\mathrm{p}_{[1]} - 2t_{ij})/b & t \in (b/2+1-p_{[2]},\ b/2+1-p_{[1]}] \end{cases}. \tag{39}$$

Condition 4: $d \leq b/2$ and $1 - p_{[2]} < b/4 - d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (-4 + 2b + 4p_{[1]} - bp_{[1]} - 2p_{[1]}^2 + 4p_{[2]} - bp_{[2]} - 2p_{[2]}^2)/(2b) & t = b/2 \\ 2(2 + b - p_{[1]} - p_{[2]} - 2t_{ij})/b & t \in (b/2, b/2+1-p_{[2]}] \\ (2 + b - 2\mathrm{p}_{[1]} - 2t_{ij})/b & t \in (b/2+1-p_{[2]},\ b/2+1-p_{[1]}] \end{cases}. \tag{40}$$

Condition 5: $d > b/2$ and $1 - p_{[2]} \geq b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (2d - b)/4 & t_{ij} = d \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (d, b+d] \\ 1/2 & t_{ij} \in (b+d, 2-p_{[1]}-p_{[2]}] \end{cases}. \tag{41}$$

Condition 6: $d > b/2$ and $1 - p_{[2]} \geq b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (2d-b)/4 & t = d \\ (2b+d-t_{ij})/(2b) & t \in (d, 2-p_{[1]}-p_{[2]}] \\ (b+2-2p_{[1]}-2t_{ij})/b & t \in (2-p_{[1]}-p_{[2]},\ 1-p_{[1]}+b/2] \end{cases}. \tag{42}$$

3.4.3 Probability density functions for an infinite population and a dedicated storage policy

For AS/R travel for Path *ff* and Path *mm*, the travel pattern is similar to that in Figure 2, except travel is from and to random points, not fixed points. With a dedicated storage policy, travel is from (x_1, 0) to (x_2, y) to (x_2, 0). Therefore,

$$T = \max(|x_2 - x_1|, y) \begin{cases} x_2 - x_1 + y & x_2 \geq x \text{ and } x_2 - x_1 \geq y \\ 2y & x_2 \geq x \text{ and } x_2 - x_1 < y \\ x_1 - x_2 + y & x_2 < x \text{ and } x_1 - x_2 \geq y \\ 2y & x_2 < x \text{ and } x_1 - x_2 < y \end{cases}$$

where $x_1, x_2 \in \text{unif}[0,1]$ and $y \in \text{unif}[0,b]$.

$$f(x_1, x_2, y) = f(x_1)f(x_2)f(y) = \frac{2}{b}$$

For AS/R travel for Path *ff* and *mm*, the pdf is

$$f(t) = \begin{cases} (12-7t)t/(4b) & t \in [0,b] \\ 2 + b/2 - 2t & t \in (b,1] \\ (2+b-2t)^2/(2b) & t \in (1, 1+b/2] \end{cases}. \tag{43}$$

For AS/R travel for Path *fm*, with the ID Scenario, the travel pattern is similar to the finite population case in Figure 3, except travel is from and to random locations. Here, there are two Conditions to consider: when $b \leq 2/3$ and when $b > 2/3$.

When $b \leq 2/3$, the pdf is

$$f(t) = \begin{cases} (b^2 + 4b(-2+t) + 4(12-7t)t)/(16b) & t \in [b/2, 3\,b/2] \\ (4+b-4t)/2 & t \in (3\,b/2, 1] \\ (2+b-2t)^2/(2b) & t \in (1, 1+b/2] \end{cases}. \tag{44}$$

When $b > 2/3$, the pdf is

$$f(t) = \begin{cases} (b^2 + 4b(-2+t) + 4(12-7t)t)/(16b) & t \in [b/2,1] \\ (b^2 + 4b(-2+t) + 4(8-4t+t^2))/(16b) & t \in (1, 3\,b/2]. \\ (2+b-2t)^2/(2b) & t \in (3\,b/2, 1+b/2] \end{cases} \tag{45}$$

For AS/R travel for Path *mf*, the travel pattern is similar to the finite population case in Figure 3, except travel is from and to random locations; the pdf is

$$f(t) = \begin{cases} b(6-b)/12 & t = b/2 \\ (4+b-4t)/2 & t \in (b/2, 1]. \\ (2+b-2t)^2/(2b) & t \in (1, 1+b/2] \end{cases} \tag{46}$$

3.4.4 Probability density functions for an infinite population and a random storage policy

For AS/R travel for Path *ff* and *mm*, with a random storage policy, travel is from (x_1, 0) to (x, y) to (x_2, 0). The travel pattern is similar to that shown in Figure 5, except travel is from and to pick positions having random locations. Travel time is $T = max(|x_1 - x|, y) + max(|x_2 - x|, y)$ where x, x_1, and $x_1 \in$ unif$[0, 1]$ and $y \in$ unif$[0, b]$. From Appendix G, the pdf for the IR Scenario is:

$$f(t) =$$

$$\begin{cases} (60t^2 - 43t^3)/(12b) & t \in [0, b] \\ (2b^2 + 24t - 21t^2)/6 & t \in (b, 1] \\ (-8 + 12b + 6b^2 + 2b^3 + 24t - 12bt - 6b^2t - 24t^2 + 3bt^2 + 8t^3)/(6b) & t \in (1, 1+b/2] \\ (-2+t)^2/2 & t \in (1+b/2, 2] \end{cases}$$

. (47)

For AS/R travel for Path *fm* with random storage, the travel pattern is similar to one of the travel patterns in Figure 6, except travel is from and to randomly located pick positions.

When $b \leq 1/2$

$f(t) =$

$$\begin{cases} (20(4-3t)t^2 - b^2(4+3t) + 4bt(-8+9t))/(16b) & t \in [b/2, b] \\ (-8b^3 + 4(60-43t)t^2 + 3b^2(-4+5t) + 12bt(-8+7t))/(48b) & t \in (b, 3b/2] \\ t(24 + 12b - 27t + t^2/b)/6 & t \in (3b/2, 2b] \\ (8b^2 + 3(8-7t)t)/6 & t \in (2b, 1] \\ (8b^3 + 3b(-2+t)^2 - 12b^2(-1+t) + 4(-1+t)^3)/(6b) & t \in (1, 1+b] \\ (-2+t)^2/2 & t \in (1+b, 2] \end{cases}. \quad (48)$$

When $1/2 < b \leq 2/3$

$f(t) =$

$$\begin{cases} (20(4-3t)t^2 - b^2(4+3t) + 4bt(-8+9t))/(16b) & t \in [b/2, b] \\ (-8b^3 + 4(60-43t)t^2 + 3b^2(-4+5t) + 12bt(-8+7t))/(48b) & t \in (b, 3b/2] \\ t(24 + 12b - 27t + t^2/b)/6 & t \in (3b/2, 1] \\ (-4 + 12b^2 + 12t - 12t^2 + 5t^3 - 3b(-4+4t+t^2))/(6b) & t \in (1, 2b] \\ (8b^3 + 3b(-2+t)^2 - 12b^2(-1+t) + 4(-1+t)^3)/(6b) & t \in (2b, 1+b] \\ (-2+t)^2/2 & t \in (1+b, 2] \end{cases}. \quad (49)$$

When $2/3 < b \leq 1$

$f(t) =$

$$\begin{cases} (20(4-3t)t^2 - b^2(4+3t) + 4bt(-8+9t))/(16b) & t \in [b/2, b] \\ (-8b^3 + 4(60-43t)t^2 + 3b^2(-4+5t) + 12bt(-8+7t))/(48b) & t \in (b, 1] \\ 3b^2(-8+9t) - 8b^3 - 12b(-2+t)^2 + 4(-12 + 60t - 36t^2 + 5t^3)/(48b) & t \in (1, 3b/2] \\ (-4 + 12b^2 + 12t - 12t^2 + 5t^3 - 3b(-4+4t+t^2))/(6b) & t \in (3b/2, 2b] \\ (8b^3 + 3b(-2+t)^2 - 12b^2(-1+t) + 4(-1+t)^3)/(6b) & t \in (2b, 1+b] \\ (-2+t)^2/2 & t \in (1+b, 2] \end{cases}.$$

(50)

For AS/R travel for *mf* with random storage, the travel pattern is similar to the travel pattern in Figure 6, except travel is from and to randomly located pick positions; the pdf is

$$f(t) = \begin{cases} (32b^2 - 7b^3)/192 & t = b/2 \\ (24bt + 6b^2t - 27bt^2 + 2t^3)/(6b) & t \in (b/2, b] \\ (2b^2 + 24t - 21t^2)/6 & t \in (b, 1]. \\ (-8 + 12b + 6b^2 + 2b^3 + 24t - 12bt - 6b^2t - 24t^2 + 3bt^2 + 8t^3)/(6b) & t \in (1, 1 + b/2] \\ (4 - 4t + t^2)/2 & t \in (1 + b/2, 2] \end{cases} \tag{51}$$

3.5. The replenishment queueing model

In the MIAPP-AS/RS system, travel of the AS/R machine originates at a pick position and ends after replenishing another pick position. Letting demands for replenishment occur in a Poisson fashion, based on the results from the previous section, the MIAPP-AS/R system with a finite number of pick positions is modeled as an M/G/1/N queue. When conditions justify the use of an infinite population approximation, then an M/G/1/∞ queueing model is used.

3.5.1 Finite population queueing

Drawing on the work of Takács (1960) contained in Sztrik (2016), the probability of an idle server is obtained as follows:

$$P_0 = \left[1 + \frac{N\lambda}{\mu} \sum_{i=0}^{N-1} \binom{N-1}{i} B_i\right]^{-1}, \tag{52}$$

where

$$B_i = \begin{cases} 1 & i = 0 \\ \prod_{j=1}^{i} \left(\frac{1-L(j\lambda)}{L(j\lambda)}\right) & i = 1, 2, \dots, N-1 \end{cases}, \tag{53}$$

and L(s) is the Laplace-Stieltjes transform of the service time density function.

The steady state performance measures for the M/G/1/N queue are:

$$\bar{\lambda} = (1 - P_0)\mu, \tag{54}$$

$$W = \frac{N}{\bar{\lambda}} - \frac{1}{\lambda}, \tag{55}$$

$$L = \bar{\lambda} W, \tag{56}$$

$$W_q = W - \frac{1}{\mu}, \tag{57}$$

$$L_q = \bar{\lambda} W_q. \tag{58}$$

For the FD Scenario, Laplace transform of service time is

$$L(j\lambda) = e^{-c(j\lambda)} L[f(t)], \tag{59}$$

where $L[f(t)]$ is the Laplace transform of the AS/R machine travel time pdf in Section 4.

3.5.2 Infinite population queueing

The performance measures for the MIAPP-AS/RS for the M/G/1/∞ queue are obtained using the Pollaczek-Khinchine formulas (Gross et al. 2008):

$$L_q = \frac{\lambda^2 Var[T_S] + \rho^2}{2(1-\rho)}, \tag{60}$$

$$L = \rho + \frac{\lambda^2 Var[T_S] + \rho^2}{2(1-\rho)}, \tag{61}$$

$$W_q = \frac{\lambda(Var[T_S] + 1/\mu^2)}{2(1-\rho)}, \tag{62}$$

$$W = \frac{1}{\mu} + \frac{\lambda(Var[T_S] + 1/\mu^2)}{2(1-\rho)}. \tag{63}$$

3.6. Parameter estimation and insights

In this section, the connection between population size and traffic intensity for an accurate infinite population queueing approximation is analyzed using a performance measure for the MIAPP-

AS/RS retrieval operation. Next, we compare the performance measure of the finite population queue using the Laplace transform of the pdf for an infinite population case with the performance measure of the finite population queue using the Laplace transform of the pdf for a finite population case; we do so because of the large number of pdf cases to be considered with a finite population. Then, we compare finite population with dedicated storage (FD) and finite population with random storage (FR) scenarios to analyze the reserve-storage space requirements for dedicated and random storages; specifically, we determine the reduction in the vertical dimension of the system for random storage to yield the same value of the performance measure as obtained with dedicated storage. Finally, we share insights and practical considerations based on the results obtained.

Suppose the MIAPP-AS/RS contains 6,000 SKUs and we assign each SKU to one pick position. There are three potential designs: 20 S/R aisles with 300 pick positions (*N*) in each aisle, i.e., 75 pick positions on each side of the floor level or the mezzanine level; 15 S/R aisles with 400 pick positions in each aisle, i.e., 100 pick positions on each side of the floor level or the mezzanine level; 12 S/R aisles with 500 pick positions in each aisle, i.e., 125 pick positions on each side of the floor level or the mezzanine level. We assume the horizontal distance between the centerlines of two adjacent pick positions is 4 feet and the vertical distance between two adjacent levels is 5 feet; therefore, with 20 vertical pallet positions for the rack, the vertical distance from the bottom level to the top level of the rack is 100 feet. For the design with 300 pick positions, the length of the S/R aisle (rack) (*B*) is 4×75 or 300 feet. For the design with 400 pick positions, *B* is 400 feet. For the design with 500 pick positions, *B* is 500 feet. Assuming the horizontal travel speed (V_h) is 400 feet per minute and the vertical travel speed (V_v) is 150 feet per minute for the S/R machine, t_h, the time to travel a distance of *B* is 300/400, 400/400 and 500/400, respectively, for the three alternative designs with *N* equal to 300, 400 and 500; the three designs share the same vertical travel time t_v with a value of 100/150 or 0.67. Normalizing the storage region by dividing the maximum vertical travel time by the maximum horizontal travel time because we assume $t_h \geq t_v$, with N equal to 300, t_h is 0.75 and t_v is 0.67; therefore, the shape factor b is 0.67/0.75 or 0.89. With N equal to 400, t_h is 1.00 and t_v is 0.67; therefore, the shape factor b is 0.67. With N equal to 500, t_h is 1.25 and t_v is 0.67; therefore, the shape factor b is 0.53.

3.6.1 Infinite population approximation

Liu et al. (2019) found that the finite population queueing model yields very accurate results for the replenishment of the MIAPP-NALT retrieval operation. However, because of the large number of pdfs involved with the finite population model of an MIAPP-AS/RS retrieval operation, employing the infinite population queueing model greatly simplifies the calculations of queueing performance measures. In examining the accuracy of an infinite population approximation to the finite population operation, we based the infinite population queueing results on an arrival rate equal to the effective arrival rate for the finite population queue. Although we based the calculation on dedicated storage, the same conclusion results when the calculation is based on random storage.

For the alternative designs with population sizes (N) equal to 300, 400 and 500, service time expected values and variances are provided in Table 1. Values of the expected number of empty deposit positions (L) for the FD and ID scenarios are provided in Table 2, Table 3 and Table 4. The individual arrival rate with N equal to 300 ranges from 0.0025 to 0.0045; the individual arrival rate with N equal to 400 ranges from 0.0020 to 0.0030; and the individual arrival rate with N equal to 500 ranges from 0.0013 to 0.0023. The queueing models in Section 5 are used to obtain values for L for the finite population and infinite population queues.

Based on the performance measure, L, the error introduced by employing an infinite population approximation equals $|L_{\mathrm{FD}}\text{-}L_{\mathrm{ID}}|/L_{\mathrm{FD}}$. Arbitrarily deeming a 5 percent approximation error a reasonable approximation, from the results in Table 2, Table 3 and Table 4, an infinite population approximation is reasonable when the effective traffic intensity is less than 0.832 with N equal to 300, less than 0.858 with N equal to 400, and less than 0.880 with N equal to 500. From the infinite population approximation results for the three alternative designs, the larger the population size, the larger the traffic intensity can be for a reasonable infinite population approximation.

Table 1. Normalized service time parameters for the dedicated (random) storage queueing model

N	Finite population		Infinite population	
	Expected Value	Variance	Expected Value	Variance
300	1.0872 (0.8569)	0.0758 (0.0988)	1.0872 (1.2569)	0.0758 (0.0988)
400	0.8787 (0.7793)	0.0560 (0.0951)	0.8787 (1.0793)	0.0560 (0.1192)
500	0.7588 (0.7409)	0.0508 (0.0986)	0.7588 (0.9809)	0.0508 (0.1141)

Table 2. Results obtained from queueing models for FD & ID with N = 300.

λ	$\bar{\lambda}$	$\bar{\rho}$	ρ	L_{FD}	L_{ID}	$\frac{\lvert L_{FD} - L_{ID}\rvert}{L_{FD}}$
0.0025	0.5604	0.6093	0.6093	1.108	1.115	0.6%
0.0026	0.5826	0.6335	0.6335	1.207	1.216	0.8%
0.0027	0.6048	0.6576	0.6576	1.318	1.330	0.9%
0.0028	0.6270	0.6817	0.6817	1.443	1.458	1.1%
0.0029	0.6491	0.7057	0.7057	1.586	1.606	1.3%
0.0030	0.6711	0.7296	0.7296	1.750	1.777	1.6%
0.0031	0.6930	0.7534	0.7534	1.941	1.978	1.9%
0.0032	0.7148	0.7771	0.7772	2.167	2.219	2.4%
0.0033	0.7365	0.8007	0.8007	2.438	2.513	3.1%
0.0034	0.7579	0.8241	0.8241	2.768	2.878	4.0%
0.0035	0.7792	0.8471	0.8471	3.178	3.345	5.3%
0.0036	0.8000	0.8698	0.8698	3.698	3.962	7.1%
0.0037	0.8204	0.8919	0.8919	4.371	4.809	10.0%
0.0038	0.8400	0.9133	0.9133	5.264	6.031	14.6%
0.0039	0.8586	0.9335	0.9335	6.473	7.902	22.1%
0.0040	0.8756	0.9520	0.9520	8.134	10.996	35.2%
0.0041	0.8904	0.9681	0.9681	10.424	16.616	59.4%
0.0042	0.9024	0.9811	0.9811	13.529	28.094	107.7%
0.0043	0.9109	0.9903	0.9903	17.557	55.045	213.5%
0.0044	0.9160	0.9959	0.9959	22.431	129.440	477.1%
0.0045	0.9185	0.9986	0.9986	27.865	375.792	1248.6%

Table 3. Results obtained from queueing models for FD & ID with N = 400.

λ	$\bar{\lambda}$	$\bar{\rho}$	ρ	L_{FD}	L_{ID}	$\frac{\lvert L_{FD} - L_{ID}\rvert}{L_{FD}}$
0.00200	0.7969	0.700	0.700	1.563	1.577	0.9%
0.00205	0.8166	0.718	0.718	1.677	1.695	1.1%
0.00210	0.8362	0.735	0.735	1.804	1.827	1.3%
0.00215	0.8558	0.752	0.752	1.947	1.975	1.5%
0.00220	0.8754	0.769	0.769	2.108	2.144	1.7%
0.00225	0.8948	0.786	0.786	2.291	2.338	2.0%
0.00230	0.9142	0.803	0.803	2.503	2.563	2.4%
0.00235	0.9335	0.820	0.820	2.748	2.829	2.9%
0.00240	0.9527	0.837	0.837	3.036	3.145	3.6%
0.00245	0.9717	0.854	0.854	3.378	3.529	4.5%
0.00250	0.9905	0.870	0.870	3.790	4.005	5.7%
0.00255	1.0091	0.887	0.887	4.294	4.606	7.3%
0.00260	1.0272	0.903	0.903	4.919	5.389	9.6%
0.00265	1.0449	0.918	0.918	5.706	6.441	12.9%
0.00270	1.0619	0.933	0.933	6.712	7.910	17.8%
0.00275	1.0780	0.947	0.947	8.017	10.060	25.5%
0.00280	1.0928	0.960	0.960	9.722	13.393	37.8%
0.00285	1.1059	0.972	0.972	11.954	18.923	58.3%
0.00290	1.1170	0.981	0.981	14.840	28.877	94.6%
0.00295	1.1255	0.989	0.989	18.475	48.587	163.0%
0.00300	1.1314	0.994	0.994	22.860	92.104	302.9%

Table 4. Results obtained from queueing models for FD & ID with N = 500.

λ	$\bar{\lambda}$	$\bar{\rho}$	ρ	L_{FD}	L_{ID}	$\frac{\lvert L_{\text{FD}} - L_{\text{ID}} \rvert}{L_{\text{FD}}}$
0.00130	0.8106	0.615	0.615	1.145	1.150	0.4%
0.00135	0.8416	0.639	0.639	1.247	1.253	0.5%
0.00140	0.8726	0.662	0.662	1.360	1.368	0.6%
0.00145	0.9036	0.686	0.686	1.489	1.499	0.7%
0.00150	0.9344	0.709	0.709	1.636	1.649	0.8%
0.00155	0.9653	0.732	0.732	1.805	1.823	1.0%
0.00160	0.9960	0.756	0.756	2.003	2.028	1.2%
0.00165	1.0266	0.779	0.779	2.239	2.273	1.5%
0.00170	1.0571	0.802	0.802	2.522	2.572	2.0%
0.00175	1.0875	0.825	0.825	2.871	2.944	2.6%
0.00180	1.1176	0.848	0.848	3.309	3.422	3.4%
0.00185	1.1473	0.871	0.871	3.874	4.057	4.7%
0.00190	1.1765	0.893	0.893	4.624	4.936	6.7%
0.00195	1.2050	0.914	0.914	5.654	6.225	10.1%
0.00200	1.2322	0.935	0.935	7.117	8.255	16.0%
0.00205	1.2575	0.954	0.954	9.265	11.774	27.1%
0.00210	1.2797	0.971	0.971	12.483	18.712	49.9%
0.00215	1.2974	0.984	0.984	17.259	34.917	102.3%
0.00220	1.3091	0.993	0.993	23.960	81.992	242.2%
0.00225	1.3151	0.998	0.998	32.411	260.498	703.7%
0.00230	1.3172	1.000	1.000	41.829	1187.264	2738.4%

As noted, because of the large number of pdfs for the finite population queue, it is far less cumbersome to formulate the Laplace transform based on the pdf for an infinite population, especially for random storage. As shown in Table 5 with the individual arrival rate (λ) equal to 0.0025 for a dedicated storage, the value of L obtained for a finite population queue by using the Laplace transform for an infinite population pdf is almost identical to the value of L obtained by using the Laplace transform for the finite population pdf. As population size increases, the accuracy in using the infinite population pdf to calculate the finite population queueing performance measure increases. Because MIAPP-AS/RS operations usually have a large number of pick positions, using an infinite population pdf for the finite population queueing model can yield quite accurate results for the performance measures.

Table 5. Comparison for the Laplace transform between FD pdf and ID pdf.

N	Finite pdf *L*	Infinite pdf *L*	MAPE
300	1.108	1.106	0.124%
400	3.790	3.786	0.113%
500	78.285	78.290	0.006%

3.6.2 Queueing results for dedicated and random storages

To analyze the impact on the expected number of empty deposit positions (*L)* when using dedicated storage versus random storage, we let the arrival rate for an individual pick position (λ) be 0.0020, 0.0025, 0.0030, 0.0035 and 0.0040. Values for *L* are shown in Table 6 for dedicated storage and random storage. Arbitrarily deeming a value of *L* less than 10 percent of the population size to be reasonable for the order picking process, with λ equal to 0.002, only the 500 pick positions design with random storage is unacceptable. With λ equal to 0.0025, the 500 pick positions design with dedicated storage or random storage is unacceptable compared to the other two alternative designs. With λ equal to 0.003, the three acceptable designs are 300 pick positions with dedicated storage or random storage and 400 pick positions with dedicated storage. With λ equal to 0.0035, only the 300 design with dedicated or random storage is acceptable. With λ equal to 0.004, only the design with dedicated storage and 300 pick positions is acceptable.

Table 6. Performance measures for dedicated storage and random storage.

Queue Results	Individual Arrival Rates									
	0.0020		0.0025		0.0030		0.0035		0.0040	
N	FD*L*	FR*L*	FD*L*	FR*L*	FD*L*	FR*L*	FD*L*	FR*L*	FD*L*	FR*L*
300	0.73	0.95	1.11	1.57	1.75	3.03	3.18	9.09	8.13	34.86
400	1.56	3.45	3.79	30.10	22.86	91.16	74.84	135.28	115.49	168.37
500	7.12	92.22	78.29	173.78	148.57	228.15	198.77	266.98	236.43	296.11

For a given application, calculations should be performed based on the actual arrival rates for empty deposit positions. However, based on the results in Table 6, there are significant differences in the value of the chosen performance measure when using dedicated versus random storage. As anticipated, dedicated storage yields smaller values than random storage for the performance measures when the same amount of reserve-storage space is used for both storage policies. However, random storage will require less storage space than dedicated storage. The space requirement for random storage is based on the maximum of the sum of the storage requirements of the SKUs; whereas, the space requirement for dedicated storage is based on the sum of the maximum storage requirements of the SKUs (Tompkins, et al, 2010). How much must the reserve-storage space for random storage be reduced to yield the same value of L obtained when using dedicated storage? We address this question in Section 3.6.3.

3.6.3 Storage space requirement for dedicated storage versus random storage

To address the question raised in the previous section, we consider the case with population size equal to 300 and λ equal to 0.004 for dedicated storage; as before, there are 20 vertical storage levels. Therefore, the shape factor is 0.89. From the finite population queueing model in Section 5, L equals 8.13. To obtain the same value of L for random storage, the shape factor must be approximately 0.26. Therefore, the storage space required for random storage must be reduced by 70 percent, which is highly unlikely to occur for an MIAPP-AS/RS. Further, a value of 0.26 for the shape factor translates to a vertical storage height of 29 feet. With a storage height of only 29 feet, one would not use an AS/R machine to perform retrieval operations; a narrow-aisle lift truck would be more appropriate to support this MIAPP retrieval operation. Further, there would be no need for a mezzanine with a storage height of 29 feet.

The comparison of reserve-storage requirements for dedicated and random storage raises a number of interesting questions not addressed in our research. For example, multiple mezzanines can be used, allowing a reduction in the length of the S/R aisles. However, our models are only accurate for two vertical levels of pick positions. Given the tedious work required in deriving pdfs for two vertical levels of storage for the finite population case, we cannot recommend pursuing the derivation of pdfs for, say, four vertical levels of pick positions. On the other hand, based on the accuracy of using the Laplace transform for the infinite population pdf to calculate the performance measures for a finite population queue, we believe four vertical levels of storage can be modeled quite accurately by developing pdfs based on infinite populations.

3.6.4 Insights and practical considerations

From Sections 6.1, 6.2 and 6.3, the following insights and practical considerations are shared:

a. The larger the population size, the larger the traffic intensity can be for an infinite population approximation to be acceptable based on a given aspiration level. Compared to the finite population queueing model, the infinite population queueing model is easier to apply in obtaining values of the performance measures. However, when traffic intensity is high, with the infinite population approximation, values obtained for the performance measures can be unrealistic; for example, having a value of L greater than the size of the population. Therefore, based on our parameter values, an infinite population approximation is recommended when traffic intensity is less than approximately 0.88 for N equal to 500, less than 0.86 for N equal to 400, and less than 0.83 for N equal to 300.

b. Because the number of pick positions is usually large for the MIAPP-AS/RS and the next pick position requiring replenishment is equally likely to be any pick position in one aisle for the MIAPP-AS/RS, using the Laplace transform of the infinite pdf instead of the Laplace transform for the finite population pdf can yield very accurate results for performance measures when using Takács's result for the finite population queueing model.

c. Using the queueing results, the dimensions of the MIAPP-AS/RS with dedicated storage or random storage can be determined. Depending on the traffic intensity, dedicated storage can yield significantly smaller values of the performance measures compared to random storage. However, dedicated storage requires more space for reserve storage compared to random storage. Therefore, the system designer must consider tradeoffs between replenishment efficiency and reserve-storage space utilization.

   For random storage, we assumed every reserve storage space was equally likely to be used to replenish a given pick position. An extension of our research is to employ a quasi-random storage policy in which reserve storage is assigned to the open storage slot closest to the pick position. Alternatively, instead of having the same x-coordinate as the pick position to be replenished, a replenishment zone can be defined for each pick position, within which reserve storage is located. We believe the performance measures will be reduced significantly from those obtained with "pure random" storage. Our recommendations for future research include a consideration of a quasi-random storage policy.

d. The queueing model can be used to obtain easily the number of empty deposit positions. However, a detailed simulation model, combining order picker and AS/R machine activities will yield more accurate system performance measures, especially when pick positions have different arrival rates.

e. This research only considers the AS/R machine retrieval operation by assuming storage operations occur when no order picking is being conducted. We assumed the reserve-storage space is adequate for the order picking process and no “stock outs” occur during order picking.

3.7. Conclusions and recommendations

To model the MIAPP-AS/RS retrieval operation as a queueing system, travel time probability density functions (pdfs) were derived for four travel paths for both dedicated storage and random storage. For each path, pdfs were derived. The MIAPP-AS/RS retrieval operation was first modeled as a finite population queue (M/G/1/N), then, the pdfs for an infinite population queue (M/G/1/∞) were used to approximate the finite population queue. For the finite population queue, the steady state performance measure was obtained using Equations 52 thru 59, based on Tak*á*cs’s results by applying the Laplace transform of the pdfs for the four travel paths. For the infinite population queue, the steady state performance measures can be obtained using Pollaczek-Khinchine formulas, Equations 60 thru 63, based on expected values and variances for service times obtained for the four paths in Section 4.

In comparison with expected-value analysis used to determine the utilization of the S/R machine, the queueing results in Section 6 can be used to determine the values of the performance measures, as well as the utilization of the AS/R machine. Additionally, Takács’s results can be used to obtain the probability mass function for the number of empty deposit positions. Further, by establishing an aspiration level for the expected number of pick positions waiting to be replenished, the results in Section 6 can be used to determine if the design meets the established aspiration level.

A large number of cases must be considered in developing the pdf for the finite population; doing so can be quire cumbersome. To simplify calculations for the finite population queueing model, the Laplace transform of the pdf for service time based on an infinite population can be used with Takács's results for the finite population queue. To justify economically the use of an MIAPP-AS/RS, a large number of pick positions is generally required; therefore, using the Laplace transform of the pdf for service time for an infinite population, the results obtained from Takács's results will closely approximate those that would be obtained using the Laplace transform of the pdf for service time for a finite population.

Comparing dedicated storage to random storage, when the same amount of reserve-storage space is used for dedicated and random storage, we observed dedicated storage yields significantly smaller values for the performance measures than random storage. The exception is when traffic intensity is quite low. (It is unlikely an MIAPP-AS/RS design will be used for very low traffic intensity applications.) The results in Section 6 can be used to establish an upper bound on the relative reserve-storage requirements for random storage to yield the same replenishment performance as dedicated storage.

For future research, several extensions can be considered for the MIAPP-AS/RS. One potential extension is to use a quasi-random storage policy by storing replenishment loads in the open storage position closest to the pick position destined for the replenishment load. Another extension is to assign SKUs to pick position locations based on a class-based or turnover-based storage policy. A third extension is to model both the AS/R machine and the order-picking workers in the MIAPP-AS/RS as a queueing network. A fourth research extension is to consider the storage and retrieval operations in the MIAPP-AS/RS and analyze interleaving of storage and retrieval operations.

References

Ashayeri, J. and Heuts, R.M. (2002) A geometrical approach to computing expected cycle times for zone-based storage layouts in AS/RS, International Journal of Production Research, 40(17), 4467-4483.

Bortolini, M., Accorsi, R., Gamberi, M., Manzini, R. and Regattieri, A. (2015) Optimal design of AS/RS storage systems with three-class-based assignment strategy under single and dual command operations, The International Journal of Advanced Manufacturing Technology, 79(9-12), 1747-1759.

Bozer, Y.A. and Cho, M. (2005) Throughput performance of automated storage/retrieval systems under stochastic demand, IIE Transactions, 37(4), 367-378.

Bozer, Y.A. and White, J. A. (1984) Travel–time models for automated storage/retrieval systems, IIE Transactions, 16(4), 329-338.

Bozer, Y.A. and White, J.A. (1990) Design and performance models for end-of-aisle order picking systems, Management Science, 36(7), 852-866.

Cai, X., Heragu, S.S. and Liu, Y. (2013) Handbook of stochastic models and analysis of manufacturing system operations, Springer, 29-71.

Cai, X., Heragu, S.S. and Liu, Y. (2014) Modeling and evaluating the AVS/RS with tier-to-tier vehicles using a semi-open queueing network, IIE Transactions, 46(9), 905-927.

Cardin, O., Castagna, P., Sari, Z. and Meghelli, N. (2012) Performance evaluation of in-deep class storage for flow rack AS/RS, International Journal of Production Research, 50(23), 6775-6791.

Change, D.T., Wen, U.P. and Lin, J.T. (1995) The impact of acceleration/deceleration on travel time models for automated storage/retrieval systems, IIE Transactions, 27(1), 108-111.

Chen, C., Huang, S.-Y., Hsu, W.-J., Toh, A.C. and Loh, C.K. (2003) Platform-based AS/RS for container storage, Proceedings of International Conference on Robotics and Automation 2003, Taipei, Taiwan, 181-187.

Chow W.M. (1986) An Analysis of Automated Storage and Retrieval Systems in Manufacturing Assembly Lines, IIE Transactions, 18(2), 204-214.

Dallari, F., Marchet, G. and Ruggeri, R. (2000) Optimization of man-on-board automated storage/retrieval systems, Integrated Manufacturing Systems, 11(2), 87-93.

De Koster, R. (1994) Performance approximation of pick-to-belt order picking systems, European Journal of Operational Research, 72(3), 558-573.

De Koster, R., Le-Anh, T. and Yu, Y. (2008) Optimal storage rack design for a 3-dimensional compact AS/RS, International Journal of Production Research, 46(6), 1495-1514.

De Koster, R., Le-Duc, T. and Roodbergen, K.J. (2007) Design and control of warehouse order picking: a literature review, European Journal of Operational Research, 182(2), 481-501.

Ekren B.Y., Heragu, S.S, Krishnamurthy, A. and Malmborg, C.J. (2013), An approximate solution for semi-open queueing network model of an autonomous vehicle storage and retrieval system, IEEE Transactions on Automation Science and Engineering, 10(1), 205-215.

Ekren B.Y., Heragu, S.S, Krishnamurthy, A. and Malmborg, C.J. (2014), Matrix-geometric solution for semi-open queuing network model of autonomous vehicle storage and retrieval system, Computers & Industrial Engineering, 68, 78-86.

Foley, R.D. and Frazelle, E.H. (1991) Analytical results for miniload throughput and distribution of dual command travel time, IIE Transactions, 23(3), 273-281.

Fukunari M. and Malmborg C.J. (2008) An efficient cycle time model for autonomous vehicle storage and retrieval systems, International Journal of Production Research, 46(12), 3167-3184.

Fukunari M. and Malmborg C.J. (2009) A network queuing approach for evaluation of performance measures in autonomous vehicle storage and retrieval systems, European Journal of Operational Research, 193(1), 152-167.

Gagliardi, J.P., Renaud, J. and Ruiz, A. (2011) Models for automated storage and retrieval systems: a literature review, International Journal of Production Research, 50(3), 879-892.

Gagliardi, J.P., Renaud, J. and Ruiz, A. (2012), On storage assignment policies for unit-load automated storage and retrieval systems, International Journal of Production Research, 50(24), 7110-7125.

Ghomri, L., Sari, Z., Guezzen, A.H. and Sari, T. (2009) Continuous models for single and dual cycle times of a multi aisle automated storage and retrieval system, Proceedings of the 13th IFAC Symposium on Information Control Problems in Manufacturing, 3-5 June, Moscow, Russia, 1061-1066.

Goetschalckx, M. and Ratliff, H.D. (1990) Research on warehouse design and performance evaluation: a comprehensive review, Management Science, 36(9), 1120-1132.

Graves, S.C., Hausman, W.H. and Schwarz, L.B. (1977) Storage-retrieval interleaving in automatic warehousing systems, Management Science, 23(9), 935-945.

Gross, D., Shortle, J.F., Thompson, J.M. and Harris, C.M. (2008) Fundamentals of Queueing Theory, 4th ed., Hoboken (NJ): John Wiley & Sons, Inc., 222.

Gu, J.X., Goetschalckx, M. and McGinnis, L.F. (2007) Research on warehouse operation: a comprehensive review, European Journal of Operational Research, 177(1), 1-21.

Gu, J.X., Goetschalckx, M. and McGinnis, L.F. (2010) Research on warehouse design and performance evaluation: a comprehensive review, European Journal of Operational Research, 203(3), 539-549.

Hale, T.S., Huq, F. and Pujari, N.A. (2008) Closed from models for dwell point locations in automated storage carousel systems, International Journal of Production Research, 46(4), 1089-1098.

Hao, J., Yu, Y. and Zhang, L.L. (2015), Optimal design of a 3D compact storage system with the I/O port at the lower mid-point of the storage rack, International Journal of Production Research, 53(17), 5153-5173.

Hausman, W.H., Schwarz, L.B. and Graves, S.C. (1976) Optimal storage assignment in automatic warehousing systems, Management Science, 22(6), 629-638.

Heragu, S.S., Cai, X., Krishnamurthy, A. and Malmborg, C.J. (2009) Analysis of autonomous vehicle storage and retrieval system by open queueing network, Proceedings of the 5th Annual IEEE Conference on Automation Science and Engineering, 22-25 August, Bangalore, Indian, 455-459.

Hu, Y.H., Huang, S.Y., Chen, C., Hsu, W.J., Toh, A.C., Loh, C.K. and Song, T. (2005) Travel time analysis of a new automated storage and retrieval system, Computers & Operations Research, 32(6), 1515-1544.

Hur, S., Lee, Y.H., Lim, S.Y. and Lee, M.H. (2004) A performance estimation model for AS/RS by M/G/1 queuing system, Computers & Industrial Engineering, 46(2), 233-241.

Hur, S. and Nam, J. (2006) Performance analysis of automatic storage/retrieval systems by stochastic modelling, International Journal of Production Research, 44(8), 1613-1626.

Hwang, H. and Lee, S.B. (1990) Travel-time models considering the performance measures of the storage and retrieval machine, International Journal of Production Research, 28(10), 1779-1789.

Koh, S.G., Kwon, H.M. and Kim, Y.J. (2005) An analysis of the end-of-aisle order picking system: Multi-aisle served by a single order picker, International Journal of Production Economics, 98(2), 162-171.

Kouloughli, S. and Sari, Z. (1995) Multi-aisle AS/RS dimensions optimization for cycle time minimization, The International Journal of Advanced Manufacturing Technology, 79(1-4), 675-692.

Kouvelis, P. and Papanicolaou, V. (1995) Expected travel time and optimal boundary formulas for a two-class-based automated storage/retrieval system, International Journal of Production Research, 33(10), 2889-2905.

Kuo, P.H., Krishnamurthy, A. and Malmborg, C.J. (2007) Design models for unit load storage and retrieval systems using autonomous vehicle technology and resource conserving storage and dwell point policies, Applied Mathematical Modelling, 31(10), 2332-2346.

Kuo, P.H., Krishnamurthy, A. and Malmborg, C.J. (2008) Performance modelling of autonomous vehicle storage and retrieval systems using class-based storage policies, International Journal of Computer Applications in Technology, 31(3-4), 238-248.

Lee, H.F. (1997) Performance analysis for automated storage and retrieval systems, IIE Transactions, 29(1), 15-28.

Lerher, T., Potrcě, I., Šraml, M. and Tollazzi, T. (2010) Travel time models for automated warehouses with aisle transferring storage and retrieval machine, European Journal of Operational Research, 205(3), 571-583.

Liu, J., Liao, H. and White, J.A. (2019) Queueing analysis of the replenishment of multiple in-the-aisle pick positons, working paper.

Malmborg, C.J. (2002) Conceptualizing tools for autonomous vehicle storage and retrieval systems, International Journal of Production Research, 40(8), 1807-1822.

Malmborg, C.J. (2003) Interleaving dynamics in autonomous vehicle storage and retrieval systems, International Journal of Production Research, 41(5), 1057-1069.

Meghelli-Gaouar, N. and Sari, Z. (2010) Assessment of performance of a class-based storage in a flow-rack AS/RS, Journal of Studies on Manufacturing, 1(2-3), 100-107.

Meller, R.D. and Mungwattana, A. (2005) AS/RS dwell-point strategy selection at high system utilization: a simulation study to investigate the magnitude of the benefit, International Journal of Production Research, 43(24), 5217-5227.

Pan, J.C.H and Wu, M.H (2012) Throughput analysis for order picking system with multiple pickers and aisle congestion considerations, Computers & Operations Research, 39(7), 1661-1672.

Parikh, P.J. and Meller, R.D. (2010) A travel-time model for person-onboard order picking systems, European Journal of Operational Research, 200(2), 385-394.

Park, B.C. (1999) Optimal dwell point policies for automated storage/retrieval systems with dedicated storage, IIE Transactions, 31(10), 1011-1013.

Park, B.C. (2006) Performance of automated storage/retrieval systems with non-square-in-time racks and two-class storage, International Journal of Production Research, 44(6), 1107-1123.

Park, B.C., Frazelle, E.H. and White, J.A. (1999) Buffer sizing models for end-of-aisle order picking systems, IIE Transactions, 31(1), 31-38.

Park, B.C., Foley, R.D. and Frazelle, E.H. (2006) Performance of miniload systems with two-class storage, European Journal of Operational Research, 170(1), 144-155.

Park B.C., Foley R.D., White J.A. and Frazelle E.H. (2003) Dual command travel times and miniload system throughput with turnover based storage, IIE Transactions, 35(4), 343–355.

Ramtin, F. and Pazour, J.A. (2014) Analytical models for an automated storage and retrieval system with multiple in-the-aisle pick positions, IIE Transactions, 46(9), 968-986.

Ramtin, F. and Pazour, J.A. (2015) Production allocation problem for an AS/RS with multiple in-the-aisle pick positions, IIE Transactions, 47(1), 1-18.

Roodbergen, K.J. and Vis, I.F.A. (2009) A survey of literature on automated storage and retrieval systems, European Journal of Operational Research, 194(2), 343-362.

Rouwenhorst, B., Reuter, B., Stockrahm, V., Van Houtum, G.J., Mantel, R.J. and Zijm, W.H.M. (2000) Warehouse design and control: framework and literature review, European Journal of Operational Research, 122(3), 515-533.

Roy D., Krishnamurthy, A., Heragu, S.S. and Malmborg, C.J. (2012), Performance analysis and design trade-offs in warehouses with autonomous vehicle technology, IIE Transactions, 44(12), 1045-1060

Roy D., Krishnamurthy, A., Heragu, S.S. and Malmborg, C.J. (2015), Queuing models to analyze dwell-point and cross-aisle location in autonomous vehicle-based warehouse systems, European Journal of Operational Research, 242 (1), 72-87

Sari, Z. and Bessnouci, N.H. (2012) Design & modeling of a single machine flow rack AS/RS, Proceedings of International Material Handling Research Colloquium, Gardanne, France.

Sari, Z., Grassman, S. and Ghouali, N. (2007) Impact of pickup/delivery stations and restoring conveyor locations on retrieval time models of flow-rack automated storage and retrieval systems, Production Planning & Control, 18(2), 105-116.

Sari, Z., Saygin, C. and Ghouali, N. (2005) Travel time models for flow-rack automated storage and retrieval systems, International Journal of Advanced Manufacturing Technology, 25(9-10), 979-987.

Sarker, B.R. and Babu, P.S. (1995) Travel time models in automated storage/retrieval systems: a critical review, International Journal of Production Economics, 40(2-3), 173-184.

Schwarz, L.B., Graves, S.C. and Hausman, W.H. (1978) Scheduling policies for automatic warehousing systems: simulation results, AIIE Transactions, 10(3), 260-270.

Scharfstein, D.O., (1990) Analytical performance measures for the miniload automated storage/retrieval systems. Unpublished MS thesis, Georgia Institute of Technology, Atlanta, Georgia.

Takács, L. (1960) Stochastic Processes, Problems and Solutions, Methuen & Co Ltd (NY): John Wiley & Sons, Inc., 73-74

Tanaka, S. and Araki, M. (2009) Routing problem under the shared storage policy for unit-load automated storage and retrieval systems with separate input and output points, International Journal of Production Research, 47(9), 2391-2408.

Tompkins, J.A., White, J.A., Bozer, Y.A., and Tanchoco, J.M.A. (2010) Facilities Planning, 4th edition, John Wiley & Sons, Inc. (NY), 418-422.

Van den Berg, J.P. (1999) A literature survey on planning and control of warehousing systems, IIE Transactions, 31(8), 751-762.

Van den Berg, J.P. (2002) Analytic expressions for the optimal dwell point in an automated storage/retrieval system, International Journal of Production Economics, 76(1), 13-25.

Vlasiou, M., Adan, I.J.B.F. and Wessels, J. (2004) A Lindley-type equation arising from a carousel problem, Journal of Applied Probability, 41(4), 1171-1181.

Wen, U.P., Chang, D.T., and Chen, S.P. (2001), The impact of acceleration/deceleration on travel-time models in class-based automated S/R systems, 33(7), 599-608.

Xu, X., Gong, Y., Fan, X., Shen, G. and Zou, B. (2017) Travel-time model of dual-command cycles in a 3D compact AS/RS with lower mid-point I/O dwell point policy, International Journal of Production Research, 56(4), 1620-1641.

Xu, X., Gong, X., Shen, G., Yu, Y. and Huang, W. (2015) Travel time analysis for the double-deep dual-shuttle AS/RS, International Journal of Production Research, 53(3), 757-773.

Yang, P., Miao L., Xue, Z. and Qin, L. (2015) Optimal storage rack design for a multi-deep compact AS/RS considering the acceleration/deceleration of the storage and retrieval machine, International Journal of Production Research, 53(3), 929-943.

Yang, P., Yang, K., Qi, M., Miao L. and Ye, B. (2017) Designing the optimal multi-deep AS/RS storage rack under full turnover-based storage policy based on non-approximate speed model of S/RS machine, Transportation Research Part E, 104, 113-130.

Yu, Y. and De Koster, R. (2009) Designing an optimal turnover-based storage rack for a 3D compact automated storage and retrieval system, International Journal of Production Research, 47(6), 1551-1571.

Yu, Y. and De Koster, R. (2009) The impact of order batching and picking area zoning on order picking system performance, European Journal of Operational Research, 198(2), 480-490.

Zhang, L., Krishnamurthy, A, Malmborg, C.J. and Heragu, S. (2009) Variance-based approximation of transaction waiting times in autonomous vehicle storage and retrieval systems, *European Journal of Industrial Engineering*, **3**(2), 146-169.

Appendix

Appendix A: Derivation of probability density function for path *ff* or *mm* for travel time with a finite population and dedicated storage

Because the travel time pdf for path *ff* is the same as the travel time pdf for path *mm*, we only need to derive the travel time pdf for path *ff*. For the path *ff*, we have

$$T_{ij} = max(|p_i - p_j|, y) + y$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$ and $d = p_{[2]} - p_{[1]}$, then

$$T_{ij} = max(|p_i - p_j|, y) + y = max(d, y) + y$$

Because $y \in unif[0, b/2]$, the pdf for y is $f(y) = 2/b$

Condition 1: $b/2 \le d$

With $b/2 \le d$, which means $|p_i - p_j| \ge y$, therefore, $T_{ij} = d + y$, the probability for T_{ij} is

$P_r\left(d \le T_{ij} \le t_{ij}\right) = Pr\left(d \le d + y \le t_{ij}\right) = Pr\left(0 \le y \le t_{ij} - d\right) = \int_0^{t_{ij}-d} f(y)dy = \frac{2(t_{ij}-d)}{b}$,

and take the derivative of $P_r\left(d \le T_{ij} \le t_{ij}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}\right) = \frac{d}{dt_{ij}} P_r\left(d \le T_{ij} \le t_{ij}\right) = \frac{2}{b}, \qquad t_{ij} \in \left[d,\ \frac{b}{2} + d\right]. \tag{A1}$$

Condition 2: $b/2 > d$

Case 1: with $y \le d$, which means $|p_i - p_j| \ge y$, therefore, $T_{ij} = d + y$, the probability for T_{ij} is

$P_r\left(d \le T_{ij} \le t_{ij} | \text{Case 1}\right) = P_r\left(d \le d + y \le t_{ij}\right) = P_r\left(0 \le y \le t_{ij} - d\right) = \int_0^{t_{ij}-d} f(y)dy = \frac{2(t_{ij}-d)}{b}$ and take the derivative of $P_r\left(d \le T_{ij} \le t_{ij} | \text{Case 1}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 1}\right) = \frac{d}{dt_{ij}} P_r\left(d \le T_{ij} \le t_{ij} | \text{Case 1}\right) = \frac{2}{b}, \qquad t_{ij} \in [d,\ 2d]. \tag{A2}$$

Case 2: with $y > d$, which means $|p_i - p_j| < y$, therefore, $T_{ij} = 2y$, the probability for T_{ij} is

$P_r\left(2d < T_{ij} \le t_{ij} | \text{Case 2}\right) = Pr\left(d < 2y \le t_{ij}\right) == \int_d^{t_{ij}/2} f(y)dy = \frac{2(t_{ij}/2-d)}{b}$

and take the derivative of $P_r(2d < T_{ij} \le t_{ij}|\text{Case 2})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{d}{dt_{ij}} P_r(2d < T_{ij} \le t_{ij}|\text{Case 2}) = \frac{1}{b}, \qquad t_{ij} \in [2d, 2b]. \tag{A3}$$

In summary, from the derivations above,

when $b/2 \le d$, from Equation (A1), the pdf is

$$f(t_{ij}) = 2/b, \qquad t_{ij} \in [d, b/2 + d]. \tag{A4}$$

When $b/2 > d$, from Equations (A2) and (A3), the pdf is

$$f(t_{ij}) = \begin{cases} 2/b, & t_{ij} \in [d, 2d] \\ 1/b, & t_{ij} \in (2d, b] \end{cases}. \tag{A5}$$

Appendix B: Derivation of probability density function for path *fm* travel time with a finite population and dedicated storage

For the path *fm*, we have

$$T_{ij} = max(|p_i - p_j|, y + b/2) + y$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$ and $d = p_{[2]} - p_{[1]}$, then

$$T_{ij} = max(|p_i - p_j|, y + b/2) + y = max(d, y + b/2) + y$$

Because $y \in unif[b/2, b]$, the pdf for y is $f(y) = 2/b$.

Condition 1: $b \le d$

With $b \le d$, which means $d \ge y + b/2$, therefore, $T_{ij} = d + y$, the probability for T_{ij} is

$$P_r(d \le T_{ij} \le t_{ij}) = P_r(d \le d + y \le t_{ij}) = P_r(0 \le y \le t_{ij} - d) = \int_0^{t_{ij}-d} f(y)dy = \frac{2(t_{ij}-d)}{b}$$

and take the derivative of $P_r(d \le T_{ij} \le t_{ij})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}) = \frac{d}{dt_{ij}} P_r(d \le T_{ij} \le t_{ij}) = \frac{2}{b}, \qquad t_{ij} \in \left[d, \ \frac{b}{2} + d\right]. \tag{B1}$$

Condition 2: $b/2 \leq d < b$

Case 1: with $d \geq y + b/2$, which means $T_{ij} = d + y$, the probability for T_{ij} is

$P_r\left(d \leq T_{ij} \leq t_{ij} | \text{Case 1}\right) = P_r\left(d \leq d + y \leq t_{ij}\right) = P_r\left(0 \leq y \leq t_{ij} - d\right) = \int_0^{t_{ij}-d} f(y)dy = \frac{2(t_{ij}-d)}{b}$

and take the derivative of $P_r\left(d \leq T_{ij} \leq t_{ij} | \text{Case 1}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 1}\right) = \frac{d}{dt_{ij}} P_r\left(d \leq T_{ij} \leq t_{ij} | \text{Case 1}\right) = \frac{2}{b}, \qquad t_{ij} \in \left[d,\ 2d - \frac{b}{2}\right]. \qquad \text{(B2)}$$

Case 2: with $d < y + b/2$, which means $T_{ij} = 2y + b/2$, the probability for T_{ij} is

$P_r\left(2d - b/2 \leq T_{ij} \leq t_{ij} | \text{Case 2}\right) = P_r\left(2d - b/2 < 2y + b/2 \leq t_{ij}\right) = P_r\left(d - b/2 \leq y \leq t_{ij}/2 - b/4\right) = \int_{d-b/2}^{t_{ij}/2-b/4} f(y)dy = \frac{2}{b}\left(\frac{t_{ij}}{2} + \frac{b}{4} - d\right)$

and take the derivative of $P_r\left(2d - b/2 \leq T_{ij} \leq t_{ij} | \text{Case 2}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 2}\right) = \frac{d}{dt_{ij}} P_r\left(2d - b/2 \leq T_{ij} \leq t_{ij} | \text{Case 2}\right) = \frac{1}{b}, \qquad t_{ij} \in \left[2d - \frac{b}{2},\ \frac{3b}{2}\right]. \text{(B3)}$$

Condition 3: $d < b/2$

With $d < b/2$, which means $d < y + b/2$ therefore, $T_{ij} = 2y + b/2$, the probability for T_{ij} is

$P_r\left(b/2 \leq T_{ij} \leq t_{ij}\right) = P_r\left(b/2 \leq 2y + b/2 \leq t_{ij}\right) = P_r\left(0 \leq y \leq t_{ij}/2 - b/4\right) = \int_0^{t_{ij}/2-b/4} f(y)dy = \frac{2(t_{ij}/2-b/4)}{b}$

and take the derivative of $P_r\left(b/2 \leq T_{ij} \leq t_{ij}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}\right) = \frac{d}{dt_{ij}} P_r\left(b/2 \leq T_{ij} \leq t_{ij}\right) = \frac{1}{b}, \qquad t_{ij} \in \left[\frac{b}{2}, \frac{3b}{2}\right]. \qquad \text{(B4)}$$

In summary, from the derivations above,

when $b \leq d$, from Equation (B1), the pdf is

$$f\left(t_{ij}\right) = 2/b, \qquad t_{ij} \in [d, b/2 + d]. \qquad \text{(B5)}$$

When $b/2 \leq d < b$, from Equations (B2) and (B3), the pdf is

$$f(t_{ij}) = \begin{cases} 2/b, & t_{ij} \in [d, 2d - b/2] \\ 1/b, & t_{ij} \in (2d - b/2, 3b/2] \end{cases}. \tag{B6}$$

When $d < b/2$, from Equation (B4), the pdf is

$$f(t_{ij}) = 1/b, \qquad t_{ij} \in [b/2, 3b/2]. \tag{B7}$$

Appendix C: Derivation of probability density function for path *mf* travel time with a finite population and dedicated storage

For the path *mf*, we have

$$T_{ij} = max(|p_i - p_j|, b/2 - y) + y$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$ and $d = p_{[2]} - p_{[1]}$, then

$$T_{ij} = max(|p_i - p_j|, b/2 - y) + y = max(d, b/2 - y) + y$$

Because $y \in unif[0, b/2]$, the pdf for y is $f(y) = 2/b$

Condition 1: $b/2 \leq d$

With $b/2 \leq d$, which means $d \geq b/2 - y$, therefore, $T_{ij} = y + d$, the probability for T_{ij} is

$P_r(d \leq T_{ij} \leq t_{ij}) = P_r(d \leq y + d \leq t_{ij}) = P_r(0 \leq y \leq t_{ij} - d) = \int_0^{t_{ij}-d} f(y)dy = 2(t_{ij} - d)/b$

and take the derivative of $P_r(d \leq T_{ij} \leq t_{ij})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}) = \frac{d}{dt_{ij}} P_r(d \leq T_{ij} \leq t_{ij}) = \frac{2}{b}, \qquad t_{ij} \in \left[d,\ \frac{b}{2} + d\right]. \tag{C1}$$

Condition 2: with $d < b/2$

Case 1: with $d < b/2 - y$, which means $T_{ij} = b/2$, the probability and pdf for T_{ij} are the same and they are

$$f(t_{ij} = b/2 \big| \text{Case 1}) = P_r(t_{ij} = b/2 | \text{Case 1}) = \frac{2}{b}\left(\frac{b}{2} - d\right) = 1 - \frac{2d}{b}. \tag{C2}$$

Case 2: with $d \geq b/2 - y$, which means $T_{ij} = d + y$, the probability for T_{ij} is

$P_r\left(b/2 < T_{ij} \leq t_{ij}|\text{Case } 2\right) = Pr\left(b/2 < d + y \leq t_{ij}\right) = P\left(b/2 - d < y \leq t_{ij} - d\right) =$

$\int_{b/2-d}^{t_{ij}-d} f(y)dy = \frac{2}{b}(t_{ij} - d - \frac{b}{2} + d) = \frac{2}{b}\left(t_{ij} - \frac{b}{2}\right).$

and take the derivative of $P_r\left(b/2 < T_{ij} \leq t_{ij}|\text{Case } 2\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case } 2\right) = \frac{d}{dt_{ij}} P_r\left(d \leq T_{ij} \leq t_{ij}|\text{Case } 2\right) = \frac{2}{b}, \qquad t_{ij} \in \left[d, \ \frac{b}{2} + d\right]. \tag{C3}$$

In summary, from the derivations above, when $b/2 \leq d$, from Equation (C1), the pdf is

$$f\left(t_{ij}\right) = 2/b, \qquad t_{ij} \in [d, \ b/2 + d]. \tag{C4}$$

When $d < b/2$, from Equation (C2) and (C3), the pdf is

$$f\left(t_{ij}\right) = \begin{cases} 1 - 2d/b, & t_{ij} = b/2 \\ 2/b, & t_{ij} \in (b/2, b/2 + d] \end{cases}. \tag{C5}$$

Appendix D: Derivation of probability density function for path *ff* or *mm* travel time with a finite population and random storage

For the travel path *ff* in scenario FR, the rack is separated into two parts (left part and right part) by a vertical line at the x-coordinate, $\left(p_i + p_j\right)/2$. From symmetry, each part will have the same number of pdfs. After combining the pdfs for the left and right parts, we obtain the pdf for a specific condition. The following derivation process is based on all conditions for the right part, based on contour lines described in the paper.

$$T_{ij} = max(|p_i - x|, y) + max(|p_j - x|, y).$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$, $d = p_{[2]} - p_{[1]}$,

$t_{min} = min\left[1 - \left(p_{[1]} + p_{[2]}\right)/2, b/2\ \right]$ and

$t_{max} = max\left[1 - \left(p_{[1]} + p_{[2]}\right)/2, b/2\right].$

Because $x \in unif[0, 1]$, the pdf for *x* is $f(x) = 1$, $y \in unif[0, b/2]$, the pdf for *y* is $f(y) = 2/b$.

Condition 1: with $d \leq b$ and $t_{max} \geq t_{min} + d/2$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r\left(t_{ij} = d|\text{Case 1}\right) = \frac{d^2}{4b}. \tag{D1}$$

Case 2: with $d < t_{ij} \leq 2t_{min}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \left(d^2/8 + \left(t_{ij} - d\right)^2/4 + t_{ij}(t_{ij} - d)/2\right)/b$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{t_{ij}}{b}. \tag{D2}$$

Case 3: with $2t_{min} < t_{ij} \leq d + 2t_{min}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left(\frac{1}{4}{t_{min}}^2 - \frac{1}{8}d^2 + \frac{1}{2}t_{min}\left(t_{ij} - 2t_{min}\right) - \frac{1}{8}\left(t_{ij} - 2t_{min}\right)^2\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{1}{2b}\left(d + 4t_{min} - t_{ij}\right). \tag{D3}$$

Case 4: with $d + 2t_{min} < t_{ij} \leq 2t_{max}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right) = \frac{2}{b}\left(t_{min}\left(t_{min} + \frac{1}{2}d\right) + \frac{1}{2}t_{min}(t_{ij} - d - 2t_{min})\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right) = \frac{t_{min}}{b}. \tag{D4}$$

Condition 2: $d \leq b$ and $t_{max} < t_{min} + d/2$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 1 and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r\left(t_{ij} = d|\text{Case 1}\right) = \frac{d^2}{4b}. \tag{D5}$$

Case 2: with $d < t_{ij} \leq 2t_{min}$, the pdf is the same as $d < t_{ij} \leq 2t_{min}$ in Condition 1, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{t}{b}. \tag{D6}$$

Case 3: with $2t_{min} < t_{ij} \leq 2t_{max}$, the pdf is the same as $2t_{min} < t_{ij} \leq d + 2t_{min}$ in Condition 1, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{1}{2b}\left(d + 4t_{min} - t_{ij}\right). \tag{D7}$$

Case 4: with $2t_{max} < t_{ij} \leq d/2 + t_{min} + t_{max}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right) = \frac{2}{b}\left(t_{min}t_{max} - \frac{1}{2}(\frac{1}{2}d + t_{min} + t_{max} - t_{ij})^2\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right) = \frac{1}{b}\left(d + 2t_{min} + 2t_{max} - 2t_{ij}\right). \tag{D8}$$

Condition 3: $b < d$ and $t_{max} \geq (b + d)/2$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 2 and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r\left(t_{ij} = d|\text{Case 1}\right) = \frac{1}{4}(2d - b). \tag{D9}$$

Case 2: with $d < t_{ij} \leq d + b$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{2}{b}\left(\frac{1}{4}b(d - b/2) + \frac{1}{4}(d + 2b - t_{ij})(t_{ij} - d) + \frac{1}{8}(t_{ij} - d)^2\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{1}{2b}\left(d + 2b - t_{ij}\right). \tag{D10}$$

Case 3: with $d + b < t_{ij} \leq 2t_{max}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left(\frac{1}{2}bt_{max} + \frac{1}{4}b(t_{ij} - d - b)\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{1}{2}. \tag{D11}$$

Condition 4: $b < d$ and $t_{max} < (b+d)/2$

Case 1: with $t_{ij} = d$: the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 3 and they are

$$f(t_{ij} = d|\text{Case 1}) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{4}(2d - b). \quad \text{(D12)}$$

Case 2: with $d < t_{ij} \leq 2t_{max}$, the pdf is the same as $d < t_{ij} \leq d + b$ in Condition 3, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{2b+d-t_{ij}}{2b} \quad \text{(D13)}$$

Case 3: with $2t_{max} < t_{ij} \leq b/2 + d/2 + t_{max}$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 3}) = \frac{2}{b}\left(\frac{1}{2}bt_{max} - \frac{1}{2}(t_{max} + b/2 + d/2 - t_{ij})^2\right)$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 3})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{d}{dt_{ij}}P_r(T_{ij} \leq t_{ij}|\text{Case 3}) = \frac{1}{b}(b + d + 2t_{max} - 2t_{ij}). \quad \text{(D14)}$$

In summary, from the derivations above,

Condition 1: when $b \geq d$ and $t_{max} \geq t_{min} + d/2$, from Equations (D1), (D2), (D3) and (D4), the pdf is

$$f(t_{ij}) = \begin{cases} d^2/(4b), & t_{ij} = d \\ t_{ij}/b, & d < t_{ij} \leq 2t_{min} \\ (d + 4t_{min} - t_{ij})/(2b), & 2t_{min} < t_{ij} \leq d + 2t_{min} \\ t_{min}/b, & d + 2t_{min} < t_{ij} \leq 2t_{max} \end{cases}. \quad \text{(D15)}$$

Condition 2: when $b \geq d$ and $t_{max} < t_{min} + d/2$, from Equations (D5), (D6), (D7) and (D8), the pdf is

$$f(t_{ij}) = \begin{cases} d^2/(4b), & t_{ij} = d \\ t_{ij}/b, & d < t_{ij} \leq 2t_{min} \\ (d + 4t_{min} - t_{ij})/(2b), & 2t_{min} < t_{ij} \leq d + 2t_{min} \\ t_{min}/b, & d + 2t_{min} < t_{ij} \leq 2t_{max} \end{cases}. \quad \text{(D16)}$$

Condition 3: when $b < d$ and $t_{max} \geq (b + d)/2$, from Equations (D9), (D10) and (D11), the pdf is

$$f(t_{ij}) = \begin{cases} (2d - b)/4, & t_{ij} = d \\ (d + 2b - t_{ij})/(2b), & d < t_{ij} \leq d + b. \\ 1/2, & d + b < t_{ij} \leq 2t_{max} \end{cases} \quad \text{(D17)}$$

Condition 4: when $b < d$ and $t_{max} < (b + d)/2$, from Equations (D12), (D13) and (D14), the pdf is

$$f(t_{ij}) = \begin{cases} (2d - b)/4, & t_{ij} = d \\ (d + 2b - t_{ij})/(2b), & d < t_{ij} \leq 2t_{max}. \\ (b + d + 2t_{max} - 2t_{ij})/b, & 2t_{max} < t_{ij} \leq d/2 + b/2 + t_{max} \end{cases} \quad \text{(D18)}$$

Appendix E: Derivation of probability density function for path *fm* travel time with a finite population and random storage

The rack is separated into a left part and a right part by a vertical line at the coordinate $x = p_{[2]}$. We consider the left and right parts for all possible conditions. Combining the pdfs for the left and right parts, we obtain the pdf for a particular condition. For the path *fm*, we have

$$T_{ij} = max(|p_i - x|, y) + max(|p_j - x|, y - b/2).$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$, $d = p_{[2]} - p_{[1]}$,

Because $x \in unif[0, 1]$, the pdf for *x* is $f(x) = 1$, $y \in unif[b/2, b]$, the pdf for *y* is $f(y) = 2/b$. Conditions 1 through 5 apply for the right part and Conditions 6 through 10 apply for the left part when $d < b/2$. Conditions 11 through 14 apply for the right part and Conditions 15 through 18 apply for the left part when $b/2 \leq d \leq 3b/2$. Condition 19 and condition 20 apply for right part and Conditions 21 through 23 apply for the left part when $b/2 \leq d \leq 3b/2$.

Condition 1: $d < b/2$ and $b \leq 1 - p_{[1]}$

Case 1: with $b/2 \leq t_{ij} \leq b - d$, the probability for T_{ij} is

$$P_r\left(b/2 \leq T_{ij} \leq t_{ij} | \text{Case 1}\right) = \frac{2}{b}\left[\frac{1}{2}\left(\frac{1}{2}\left(t_{ij} - \frac{b}{2}\right)\right)^2 + \frac{1}{2}\left(\frac{\sqrt{2}}{2}\left(t_{ij} - \frac{b}{2}\right)\right)^2\right]$$

and take the derivative of $P_r\left(b/2 \leq T_{ij} \leq t_{ij} | \text{Case 1}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 1}\right) = \frac{d}{dt_{ij}} P_r\left(b/2 \leq T_{ij} \leq t_{ij} | \text{Case 1}\right) = \frac{3}{4\text{b}}\left(2t_{ij} - b\right). \tag{E1}$$

Case 2: with $b - d < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case 2}\right) = \frac{2}{b}\left[\frac{1}{2}\left(\frac{1}{2}\left(t_{ij} - \frac{b}{2}\right)\right)^2 + \frac{\sqrt{2}}{4}\left(\sqrt{2}\left(t_{ij} - \frac{b}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{b}{2} - d\right)\right)\left(\frac{b}{2} - d\right) + \frac{1}{2}\left(\frac{1}{2}\left(t_{ij} - (b - d)\right)\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case 2}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 2}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case 2}\right) = \frac{1}{4b}\left(4t_{ij} - b - 2d\right). \tag{E2}$$

Case 3: with $3b/2 < t_{ij} \leq 2b - d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case 3}\right) = \frac{2}{b}\left[\frac{b}{2}\ (b - d) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\left(2b - d - t_{ij}\right)\right)^2 - \frac{2b-d-t_{ij}}{4}\left(b - \frac{2b-d-t_{ij}}{2}\right)\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 3}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case 3}\right) = \frac{1}{2b}\left(3b - d - t_{ij}\right). \tag{E3}$$

Case 4: with $2b - d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case 4}\right) = \frac{2}{b}\left[\frac{b}{2}\ \left(1 - p_{[2]}\right) - \frac{b}{2}\left(\frac{2-p_{[1]}-p_{[2]}-t_{ij}}{2}\right)\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case 4}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 4}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case 4}\right) = \frac{1}{2}. \tag{E4}$$

Condition 2: $d < b/2$ and $3b/4 + d/2 \leq 1 - p_{[1]} < b$

Case 1: with $b/2 \leq t_{ij} \leq b - d$, the pdf for T_{ij} is the same as $b/2 \leq t_{ij} \leq b - d$ in Condition 1, therefore, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{3}{4\text{b}}(2t_{ij} - b). \tag{E5}$$

Case 2: with $b - d < t_{ij} \leq 3b/2$, the pdf for T_{ij} is the same as $b - d < t_{ij} \leq 3b/2$ in Condition 1, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{1}{4b}(4t_{ij} - b - 2d). \tag{E6}$$

Case 3: with $3b/2 < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$, the pdf for T_{ij} is the same as $3b/2 < t_{ij} \leq 2b - d$ in Condition 1. Therefore, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{1}{2b}(3b - d - t_{ij}). \tag{E7}$$

Case 4: with $2 - p_{[1]} - p_{[2]} < t_{ij} \leq b + 1 - p_{[2]}$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \frac{2}{b}\left[\frac{b}{2}\left(1 - p_{[2]}\right) - \frac{1}{2}\left(b + 1 - p_{[2]} - t_{ij}\right)^2\right]$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 4})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{d}{dt_{ij}}P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \frac{1}{b}\left(2b - 2\left(t_{ij} - 1 + p_{[2]}\right)\right). \tag{E8}$$

Condition 3: $d < b/2$ and $b/2 + d \leq 1 - p_{[1]} < 3b/4 + d/2$

Case 1: with $b/2 \leq t_{ij} \leq b - d$, the pdf for T_{ij} is the same as $b/2 \leq t_{ij} \leq b - d$ in Condition 2, therefore, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{3}{4b}(2t_{ij} - b). \tag{E9}$$

Case 2: with $b - d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$, the pdf for T_{ij} is the same as the $b - d < t_{ij} \leq 3b/2$ in Condition 2, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{1}{4b}(4t_{ij} - b - 2d). \tag{E10}$$

Case 3: with $2 - p_{[1]} - p_{[2]} < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 3}) = \frac{2}{b}\left[\frac{1}{2}\left(t_{ij} - \frac{b}{2}\right)(1 - p_{[2]}) - \frac{1}{2}\left(1 - p_{[2]} - \frac{1}{2}\left(t_{ij} - \frac{b}{2}\right)\right)^2\right].$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 3})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{d}{dt_{ij}}P_r(T_{ij} \leq t_{ij}|\text{Case 3}) = \frac{1}{4b}(8 + b - 8p_{[2]} - 2t_{ij}). \tag{E11}$$

Case 4: with $3b/2 < t_{ij} \leq b + 1 - p_{[2]}$, the pdf for T_{ij} is the same as $2 - p_{[1]} - p_{[2]} < t_{ij} \leq b + 1 - p_{[2]}$ in Condition 2, therefore, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{1}{b}\left(2b - 2(t_{ij} - 1 + p_{[2]})\right). \tag{E12}$$

Condition 4: $d < b/2$ and $b/2 \leq 1 - p_{[1]} < b/2 + d$

Case 1: with $b/2 \leq t_{ij} \leq b - d$, the pdf for T_{ij} is the same as $b/2 \leq t_{ij} \leq b - d$ in Condition 3, therefore, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{3}{4\text{b}}(2t_{ij} - b). \tag{E13}$$

Case 2: with $b - d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$, the pdf for T_{ij} is the same as $b - d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$ in Condition 3, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{1}{4b}(4t_{ij} - b - 2d). \tag{E14}$$

Case 3: with $2 - p_{[1]} - p_{[2]} < t_{ij} \leq b/2 + 2 - 2p_{[2]}$ the pdf for T_{ij} is the same as the $2 - p_{[1]} - p_{[2]} < t_{ij} \leq 3b/2$ in Condition 3, therefore, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{1}{4b}(8 + b - 8p_{[2]} - 2t_{ij}). \tag{E15}$$

Case 4: with $b/2 + 2 - 2p_{[2]} < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \frac{2}{b}\left[\frac{1}{2}\left(t_{ij} - \frac{b}{2}\right)(1 - p_{[2]})\right]$$

and take the derivative of $P_r(T_{ij} \le t_{ij}|\text{Case 4})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{d}{dt_{ij}} P_r(\, T_{ij} \le t_{ij}|\text{Case 4}) = F'(t_{ij}) = \frac{1}{b}(1 - p_{[2]}). \tag{E16}$$

Condition 5: $d < b/2$ and $b/4 + d/2 \le 1 - p_{[1]} < b/2$

Case 1: with $b/2 \le t_{ij} \le b/2 + 1 - p_{[2]}$, the pdf for T_{ij} is the same as $b/2 \le t_{ij} \le b - d$ in Condition 4, therefore, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{3}{4b}(2t_{ij} - b). \tag{E17}$$

Case 2: with $b/2 + 1 - p_{[2]} < \mathrm{t} \le \mathrm{b}/2 + 2 - 2p_{[2]}$, the pdf for T_{ij} is the same as the $2 - p_{[1]} - p_{[2]} < t_{ij} \le b/2 + 2 - 2p_{[2]}$ in Condition 4, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{1}{4b}(8 + b - 8p_{[2]} - 2t_{ij}). \tag{E18}$$

Case 3: with $b/2 + 2 - 2p_{[2]} < t_{ij} \le 3b/2$, the pdf for T_{ij} is the same as the $b/2 + 2 - 2p_{[2]} \le t_{ij} \le 3b/2$ in Condition 4, therefore, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{1}{b}(1 - p_{[2]}). \tag{E19}$$

Condition 6: $d < b/2$ and $b \le p_{[1]}$

Case 1: with $b/2 \le t_{ij} \le b + d$, the probability for T_{ij} is

$$P_r(\, b/2 \le T_{ij} \le t_{ij}|\text{Case 1}) = \frac{2}{b}\left[\frac{1}{2}\left(\frac{t_{ij} - b/2}{2}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{2}(t_{ij} - b/2)}{2}\right)^2\right]$$

and take the derivative of $P_r(b/2 \le T_{ij} \le t_{ij}|\text{Case 1})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{d}{dt_{ij}} P_r(\, b/2 < T_{ij} \le t_{ij}|\text{Case 1}) = \frac{3}{4b}(2t_{ij} - b). \tag{E20}$$

Case 2: with $b + d < t_{ij} \le 3b/2$, the probability for T_{ij} is

$$P_r(T_{ij} \le t_{ij}|\text{Case 2}) = \frac{2}{b}\left[\frac{1}{2}\left(\frac{t_{ij} - b/2}{2}\right)^2 + \frac{1}{2}\left(\frac{\sqrt{2}(t_{ij} - b/2)}{2}\right)^2 - \frac{1}{2}\left(\frac{t_{ij} - (b+d)}{2}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \le t_{ij} | \text{Case } 2\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 2\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij} | \text{Case } 2\right) = \frac{1}{4b}\left(4t_{ij} - b + 2d\right). \tag{E21}$$

Case 3: with $3b/2 < T_{ij} \le 2b + d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \le t_{ij} | \text{Case } 3\right) = \frac{2}{b}\left[\frac{b}{4}\left(t_{ij} + d\right) - \frac{1}{2}\left(\frac{2b+d-t_{ij}}{2}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \le t_{ij} | \text{Case } 3\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 3\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij} | \text{Case } 3\right) = \frac{1}{2b}\left(3b + d - t_{ij}\right). \tag{E22}$$

Case 4: with $2b + d < T_{ij} \le 2p_{[1]} + d$, the pdf for T_{ij} is the same as the $2b - d < t_{ij} \le 2 - p_{[1]} - p_{[2]}$ in Condition 1, therefore, the pdf is

$$f\left(t_{ij} | \text{Case } 4\right) = \frac{1}{2}. \tag{E23}$$

Condition 7: $d < b/2$ and $3b/4 - d/2 \le p_{[1]} < b$

Case 1: with $b/2 \le t_{ij} \le b + d$, the pdf for T_{ij} is the same as $b/2 \le t_{ij} \le b + d$ in Condition 6, therefore, the pdf is

$$f\left(t_{ij} | \text{Case } 1\right) = \frac{3}{4\text{b}}\left(2t_{ij} - b\right). \tag{E24}$$

Case 2: with $b + d < t_{ij} \le 3b/2$, the pdf for T_{ij} is the same as $b + d < t_{ij} \le 3b/2$ in Condition 6, therefore, the pdf is

$$f\left(t_{ij} | \text{Case } 2\right) = \frac{1}{4b}\left(4t_{ij} - b + 2d\right). \tag{E25}$$

Case 3: with $3b/2 < t_{ij} \le 2p_{[1]} + d$, the pdf for T_{ij} is the same as the case $3b/2 \le 2b + d$ in Condition 6, therefore, the pdf is

$$f\left(t_{ij} | \text{Case } 3\right) = \frac{1}{2b}\left(3b + d - t_{ij}\right). \tag{E26}$$

Case 4: with $2p_{[1]} + d < t_{ij} \leq b + p_{[2]}$, the probability for T_{ij} is

$P_r\left(T_{ij} \leq t_{ij} | \text{Case 4}\right) = \frac{2}{b}\left[\frac{b}{2}p_{[2]} - \frac{1}{2}\left(b + p_{[2]} - t_{ij}\right)^2\right]$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case 4}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 4}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case 4}\right) = \frac{2}{b}\left(b + p_{[2]} - t_{ij}\right). \tag{E27}$$

Condition 8: $d < b/2$ and $b/2 \leq p_{[1]} < 3b/4 - d/2$

Case 1: with $b/2 \leq t_{ij} \leq b + d$, the pdf for T_{ij} is the same as $b/2 < t_{ij} \leq b + d$ in Condition 7, therefore, the pdf is

$$f\left(t_{ij} | \text{Case 1}\right) = \frac{3}{4b}\left(2t_{ij} - b\right). \tag{E28}$$

Case 2: with $b + d < t_{ij} \leq 2p_{[1]} + d$, the pdf for T_{ij} is the same as the case $b + d < t_{ij} \leq 3b/2$ in Condition 7, therefore, the pdf is

$$f\left(t_{ij} | \text{Case 2}\right) = \frac{1}{4b}\left(4t_{ij} - b + 2d\right). \tag{E29}$$

Case 3: with $2p_{[1]} + d < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$P_r\left(T_{ij} \leq t_{ij} | \text{Case 3}\right) = \frac{2}{b}\left[\frac{1}{2}\, p_{[2]}\left(t_{ij} - \frac{b}{2}\right) - \frac{1}{2}\left(p_{[2]} - \frac{1}{2}\left(t_{ij} - \frac{b}{2}\right)\right)^2\right].$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 3}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case 3}\right) = \frac{1}{4b}\left(b + 8p_{[2]} - 2t_{ij}\right). \tag{E30}$$

Case 4: with $3b/2 < t_{ij} \leq b + p_{[2]}$, the pdf for T_{ij} is the same as the case $2p_{[1]} + d < t_{ij} \leq b + p_{[2]}$ in Condition 7, therefore, the pdf is

$$f\left(t_{ij} | \text{Case 4}\right) = \frac{2}{b}\left(b + p_{[2]} - t_{ij}\right). \tag{E31}$$

Condition 9: $d < b/2$ and $b/2 - d \leq p_{[1]} < b/2$

Case 1: with $b/2 < t_{ij} \leq b/2 + p_{[2]}$, the pdf for T_{ij} is the same as the case $b/2 < t_{ij} \leq b + d$ in Condition 8, therefore, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{3}{4b}(2t_{ij} - b). \tag{E32}$$

Case 2: with $b/2 + p_{[2]} < t_{ij} \leq 3b/2$, the pdf for T_{ij} is the same as the case $2p_{[1]} + d < t_{ij} \leq 3b/2$ in Condition 8, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{1}{4b}(b + 8p_{[2]} - 2t_{ij}). \tag{E33}$$

Case 3: with $3b/2 < t_{ij} \leq b + p_{[2]}$, the pdf for T_{ij} is the same as $3b/2 < t_{ij} \leq b + p_{[2]}$ in Condition 8, therefore, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{2}{b}(b + p_{[2]} - t_{ij}). \tag{E34}$$

Condition 10: $d < b/2$ and $p_{[1]} < b/2 - d$

Case 1: with $b/2 < t_{ij} \leq b/2 + p_{[2]}$, the pdf for T_{ij} is the same as $b/2 < t_{ij} \leq b/2 + p_{[2]}$ in Condition 9, therefore, the pdf is

$$f(t_{ij}|\text{Case 1}) = \frac{3}{4b}(2t_{ij} - b). \tag{E35}$$

Case 2: with $b/2 + p_{[2]} < t_{ij} \leq b/2 + 2p_{[2]}$, the pdf for T_{ij} is the same as the $b/2 + p_{[2]} < t_{ij} \leq 3b/2$ in Condition 9, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{1}{4b}(b + 8p_{[2]} - 2t_{ij}). \tag{E36}$$

Case 3: with $b/2 + 2p_{[2]} < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right) = \frac{2}{b}\left[\frac{1}{2}\, p_{[2]}\left(t_{ij} - \frac{b}{2}\right)\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 3\right) = \frac{d}{dt_{ij}} P_r\left(\, T_{ij} \leq t_{ij} | \text{Case } 3\right) = \frac{p_{[2]}}{b}. \quad \text{(E37)}$$

Condition 11: $b/2 \leq d < 3b/2$ and $b/2 + d \leq 1 - p_{[1]}$

Case 1: with $t_{ij} = d$, the pdf and probability for T_{ij} are the same and they are

$$f\left(t_{ij} = d | \text{Case } 1\right) = P_r\left(t_{ij} = d | \text{Case } 1\right) = \frac{2}{b}\left[\frac{1}{2}\left(\frac{1}{2}\left(d - \frac{b}{2}\right)\right)^2\right] = \frac{1}{16b}(2d - b)^2. \quad \text{(E38)}$$

Case 2: with $d < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case } 2\right) = \frac{2}{b}\left[\left(\frac{1}{2}\left(t_{ij} - d\right) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right)^2 - \frac{1}{2}\left(\frac{1}{2}\left(d - \frac{b}{2}\right)\right)^2\right] = \frac{1}{16b}\left(b^2 + 4bd - 4d^2 - 8bt_{ij} + 8{t_{ij}}^2\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case } 2\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 2\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case } 2\right) = -\frac{1}{2} + \frac{t_{ij}}{b}. \quad \text{(E39)}$$

Case 3: with $3b/2 < t_{ij} \leq b + d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right) = \frac{2}{b}\left[\frac{b}{2}\left(\frac{1}{2}\left(t_{ij} - d\right) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right) - \frac{1}{2}\left(\frac{1}{2}\left(b + d - t_{ij}\right)\right)^2\right] = \frac{1}{4b}\left(-2b^2 - 2b\left(d - 2t_{ij}\right) - \left(d - t_{ij}\right)^2\right)$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 3\right) = \frac{d}{dt_{ij}} P_r\left(\, T_{ij} \leq t_{ij} | \text{Case } 3\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \quad \text{(E40)}$$

Case 4: with $b + d < t_{ij} \leq 2 - 2p_{[1]} - d$, the pdf for T_{ij} is the same as $2b - d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$ in Condition 1, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \tfrac{1}{2}. \tag{E41}$$

Condition 12: $b/2 \le d < 3b/2$ and $3/4 + d/2 \le 1 - p_1 < b/2 + d$

Case 1: with $t_{ij} = d$, the pdf and probability for T_{ij} are the same as $t_{ij} = d$ in Condition 11, therefore, the pdf and the probability for T_{ij} are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r\left(t_{ij} = d|\text{Case 1}\right) = \tfrac{1}{16b}(2d - b)^2. \tag{E42}$$

Case 2: with $d < t_{ij} \le 3b/2$, the pdf for T_{ij} is the same as $d < t_{ij} \le 3b/2$ in Condition 11, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = -\tfrac{1}{2} + \tfrac{t_{ij}}{b}. \tag{E43}$$

Case 3: with $3b/2 < t_{ij} \le 2 - 2p_{[1]} - d$, the pdf for T_{ij} is the same as $3b/2 < t_{ij} \le b + d$ in Condition 11, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \tfrac{1}{2b}\left(2b + d - t_{ij}\right). \tag{E44}$$

Case 4: with $2 - 2p_{[1]} - d < t_{ij} \le b/2 + 1 - p_{[1]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \le t_{ij}|\text{Case 4}\right) = \tfrac{2}{b}\left[\tfrac{b}{2}\left(1 - p_{[1]} - \left(\tfrac{b}{4} + \tfrac{d}{2}\right)\right) - \tfrac{1}{2}\left(\tfrac{b}{2} + 1 - p_{[1]} - t_{ij}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \le t_{ij}|\text{Case 4}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \tfrac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij}|\text{Case 4}\right) = \tfrac{1}{b}\left(b + 2 - 2p_{[1]} - 2t_{ij}\right). \tag{E45}$$

Condition 13: $b/2 \le d < 3b/2$ and $b \le 1 - p_{[1]} < 3b/4 + d/2$

Case 1: with $t_{ij} = d$, the pdf and probability for T_{ij} are the same as $t_{ij} = d$ in Condition 12 and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r\left(t_{ij} = d|\text{Case 1}\right) = \tfrac{1}{16b}(2d - b)^2. \tag{E46}$$

Case 2: with $d < t_{ij} \le 2 - 2p_{[1]} - d$, the pdf for T_{ij} is the same as $d < t_{ij} \le 3b/2$ in Condition 12, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = -\frac{1}{2} + \frac{t_{ij}}{b}. \tag{E47}$$

Case 3: with $2 - 2p_{[1]} - d < t_{ij} \le 3b/2$, the probability for T_{ij} is

$$P_r(T_{ij} \le t_{ij}|\text{Case 3}) = \frac{2}{b}\left[\left(\frac{1}{2}(t_{ij} - d) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right)\left(1 - p_{[1]} - \left(\frac{b}{4} + \frac{d}{2}\right)\right) - \frac{1}{2}\left(1 - p_{[1]} - \left(\frac{b}{4} + \frac{d}{2}\right) - \frac{1}{2}(t_{ij} - d)\right)^2\right]$$

and take the derivative of $P_r(T_{ij} \le t_{ij}|\text{Case 3})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{d}{dt_{ij}} P_r(T_{ij} \le t_{ij}|\text{Case 3}) = \frac{1}{2b}(4 - b - d - 4p_{[1]} - t_{ij}). \tag{E48}$$

Case 4: with $3b/2 < t_{ij} \le b/2 + 1 - p_{[1]}$, the pdf for T_{ij} is the same as the $2 - 2p_{[1]} - d < t_{ij} \le b/2 + 1 - p_{[1]}$ in Condition 12, therefore, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{1}{b}(b + 2 - 2p_{[1]} - 2t_{ij}). \tag{E49}$$

Condition 14: $b/2 \le d < 3b/2$ and $1 - p_{[1]} < b$

Case 1: with $t_{ij} = d$, the pdf and probability for T_{ij} are the same as $t_{ij} = d$ in Condition 13, therefore, the pdf and the probability for T_{ij} are

$$f(t_{ij} = d|\text{Case 1}) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{16b}(2d - b)^2. \tag{E50}$$

Case 2: with $d < t_{ij} \le 2 - 2p_{[1]} - d$, the pdf for T_{ij} is the same as $d < t_{ij} \le 2 - 2p_{[1]} - d$ in Condition 13, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = -\frac{1}{2} + \frac{t_{ij}}{b}. \tag{E51}$$

Case 3: with $2 - 2p_{[1]} - d < t_{ij} \le 2 - 2p_{[1]} - b/2$, the pdf for T_{ij} is the same as $2 - 2p_{[1]} - d < t_{ij} \le 3b/2$ in Condition 13, therefore, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{1}{2b}\left(4 - b - d - 4p_{[1]} - t_{ij}\right). \quad \text{(E52)}$$

Case 4: with $2 - 2p_{[1]} - b/2 < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \frac{2}{b}\left[\left(\frac{1}{2}(t_{ij} - d) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right)\left(1 - p_{[1]} - \left(\frac{b}{4} + \frac{d}{2}\right)\right)\right]$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 4})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{d}{dt_{ij}}P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \frac{1}{4b}\left(4 - b - 2d - 4p_{[1]}\right). \quad \text{(E53)}$$

Condition 15: $b/2 \leq d < 3b/2$ and $b \leq p_{[1]}$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same and they are

$$f(t_{ij} = d|\text{Case 1}) = P_r(t_{ij} = d|\text{Case 1}) = \frac{2}{b}\left(\frac{1}{2}\left(\frac{1}{2}\left(d - \frac{b}{2}\right)\right)^2\right) = \frac{1}{16b}(2d - b)^2. \quad \text{(E54)}$$

Case 2: with $d < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 2}) = \frac{2}{b}\left[\frac{1}{2}\left(\frac{1}{2}(t_{ij} - d) + (t_{ij} - d) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right)\left(\frac{1}{2}(t_{ij} - d) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right)\right]$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 2})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{d}{dt_{ij}}P_r(T_{ij} \leq t_{ij}|\text{Case 2}) = \frac{3t_{ij} - b - d}{2\text{b}}. \quad \text{(E55)}$$

Case 3: with $3b/2 < t_{ij} \leq b + d$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 3}) = \frac{2}{b}\left[\frac{b}{2}\left(t_{ij} - d + \frac{1}{2}\left(d - \frac{b}{2}\right)\right) - \frac{1}{2}\left(\frac{b}{2}\right)^2\right]$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 3})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 3}) = \frac{d}{dt_{ij}}P_r(T_{ij} \leq t_{ij}|\text{Case 3}) = 1. \quad \text{(E56)}$$

Case 4: with $b + d < t_{ij} \leq 2b + d$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \frac{2}{b}\left[\frac{b}{2}\left(\frac{1}{2}(t_{ij} - (b + d)) + \frac{3}{4}b + \frac{d}{2}\right) - \frac{1}{2}\left(\frac{b}{2} - \frac{1}{2}(t_{ij} - (b + d))\right)^2\right]$$

and take the derivative of $P_r(T_{ij} \le t_{ij}|\text{Case 4})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{d}{dt_{ij}} P_r(\, T_{ij} \le t_{ij}|\text{Case 4}) = \frac{1}{2b}(3b + d - t_{ij}). \qquad \text{(E57)}$$

Case 5: with $2b + d < t_{ij} \le 2p_{[1]} + d$, the pdf for T_{ij} is the same as $b + d < t_{ij} \le 2 - 2p_{[1]} - d$ in Condition 11, therefore, the pdf is

$$f(t_{ij}|\text{Case 5}) = \frac{1}{2}. \qquad \text{(E58)}$$

Condition 16: $b/2 \le d < 3b/2$ and $b/2 \le p_{[1]} < b$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 15 and they are

$$f(t_{ij} = d|\text{Case 1}) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{16b}(2d - b)^2. \qquad \text{(E59)}$$

Case 2: with $d < t_{ij} \le 3b/2$, the pdf for T_{ij} is the same as $d < t_{ij} \le 3b/2$ in Condition 15, therefore, the pdf is

$$f(t_{ij}|\text{Case 2}) = \frac{3t_{ij} - b - d}{2\text{b}}. \qquad \text{(E60)}$$

Case 3: with $3b/2 < t_{ij} \le b + d$, the pdf for T_{ij} is the same as $3b/2 < t_{ij} \le b + d$ in Condition 15, therefore, the pdf is

$$f(t_{ij}|\text{Case 3}) = 1. \qquad \text{(E61)}$$

Case 4: with $b + d < t_{ij} \le 2p_{[1]} + d$, the pdf for T_{ij} is the same as $b + d < t_{ij} \le 2b + d$ in Condition 15, therefore, the pdf is

$$f(t_{ij}|\text{Case 4}) = \frac{1}{2b}(3b + d - t_{ij}). \qquad \text{(E62)}$$

Case 5: with $2p_{[1]} + d < t_{ij} \le b + p_{[1]} + d$, the probability for T_{ij} is

$$P_r(T_{ij} \le t_{ij}|\text{Case 5}) = \frac{2}{b}\left[\frac{b}{2}\left(p_{[1]} + \frac{b}{4} + \frac{d}{2}\right) - \frac{1}{2}\left(b + p_{[1]} + d - t_{ij}\right)^2\right]$$

and take the derivative of $P_r(T_{ij} \le t_{ij}|\text{Case 5})$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 5}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij}|\text{Case 5}\right) = \frac{2}{b}\left(b + d + p_{[1]} - t_{ij}\right). \tag{E63}$$

Condition 17: $b/2 \le d < 3b/2$ and $b - d \le p_{[1]} < b/2$

Case 1: with $t_{ij} = d,$ the pdf and probability for T_{ij} are the same as $t_{ij} = d$ in Condition 16 and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{16b}(2d - b)^2. \tag{E64}$$

Case 2: with $d < t_{ij} \le 3b/2,$ the pdf for T_{ij} is the same as $d < t_{ij} \le 3b/2$ in Condition 16, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{3t_{ij} - b - d}{2b}. \tag{E65}$$

Case 3: with $3b/2 < t_{ij} \le b/2 + d + p_{[1]}$, the pdf for T_{ij} is the same as $3b/2 < t_{ij} \le b + d$ in Condition 16, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = 1. \tag{E66}$$

Case 4: with $b/2 + d + p_{[1]} < t_{ij} \le b + d + p_{[1]}$, the pdf for T_{ij} is the same as $2p_{[1]} + d < t_{ij} \le b + p_{[1]} + d$ in Condition 16, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \frac{2}{b}\left(b + d + p_{[1]} - t_{ij}\right). \tag{E67}$$

Condition 18: $b/2 \le d < 3b/2$ and $p_{[1]} < b - d$

Case 1: with $t_{ij} = d,$ the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 17 and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{16b}(2d - b)^2. \tag{E68}$$

Case 2: with $d < t_{ij} \le b/2 + d + p_{[1]}$, the pdf for T_{ij} is the same as $d < t_{ij} \le 3b/2$ in Condition 17, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{3t_{ij}-b-d}{2b}. \tag{E69}$$

Case 3: with $b/2 + d + p_{[1]} < t_{ij} \leq 3b/2$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left[\left(p_{[1]} + \frac{b}{4} + \frac{d}{2}\right)\left(\frac{1}{2}\left(t_{ij} - d\right) + \frac{1}{2}\left(d - \frac{b}{2}\right)\right) - \frac{1}{2}\left(p_{[1]} + \frac{b}{4} + \frac{d}{2} - \frac{1}{2}\left(t_{ij} - d\right)\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{1}{2b}\left(b + 3d + 4p_{[1]} - t_{ij}\right). \tag{E70}$$

Case 4: with $3b/2 < t_{ij} \leq b + d + p_{[1]}$, the pdf for T_{ij} is the same as $b/2 + d + p_{[1]} < t_{ij} \leq b + d + p_{[1]}$ in Condition 17, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \frac{2}{b}\left(b + d + p_{[1]} - t_{ij}\right). \tag{E71}$$

Condition 19: $3b/2 < d$ and $b/2 + d \leq 1 - p_{[1]}$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r(t_{ij} = d|\text{Case 1}) = \frac{2}{b}\left[\frac{1}{4}b\left(\frac{d}{2} - \frac{3}{4}b + \frac{d}{2} - \frac{b}{4}\right)\right] = \frac{1}{2}(d - b). \tag{E72}$$

Case 2: with $d < t_{ij} \leq b + d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{2}{b}\left[\frac{b}{2}\left(\frac{d}{2} - \frac{b}{4} + \frac{1}{2}(t_{ij} - d)\right) - \frac{1}{2}\left(\frac{b}{2} - \frac{1}{2}(t_{ij} - d)\right)^2\right].$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \tag{E73}$$

Case 3: with $b + d < t_{ij} \leq 2 - 2p_{[1]} - d$, the pdf for T_{ij} is the same as $2b - d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$ in Condition 1, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{1}{2}. \tag{E74}$$

Condition 20: $3b/2 < d$ and $1 - p_{[1]} < b/2 + d$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 19, therefore, the pdf and the probability for T_{ij} are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{2}(d - b). \tag{E75}$$

Case 2: with $d < t_{ij} \le 2 - 2p_{[1]} - d$, the pdf for T_{ij} is the same as $d < t_{ij} \le b + d$ in Condition 19, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \tag{E76}$$

Case 3: with $2 - 2p_{[1]} - d < t_{ij} \le b/2 + 1 - p_{[1]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \le t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left[\frac{b}{2}\left(1 - p_{[1]} - \frac{b}{4} - \frac{d}{2}\right) - \frac{1}{2}\left(\frac{b}{2} + 1 - p_{[1]} - t_{ij}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \le t_{ij}|\text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij}|\text{Case 3}\right) = \frac{1}{b}\left(b + 2 - 2p_{[1]} - 2t_{ij}\right). \tag{E77}$$

Condition 21: $3b/2 < d$ and $b \le p_{[1]}$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r(t_{ij} = d|\text{Case 1}) = \frac{2}{b}\left[\frac{1}{4} b\left(\frac{}{2} - \frac{3}{4} b + \frac{d}{2} - \frac{b}{4}\right)\right] = \frac{1}{2}(d - b). \tag{E78}$$

Case 2: with $d < t_{ij} \le b + d$, the pdf is the same as $3b/2 < t_{ij} \le b + d$ in Condition 15, therefore, the pdf for T_{ij} is

$$f\left(t_{ij}|\text{Case 2}\right) = 1. \tag{E79}$$

Case 3: with $b + d < t_{ij} \le 2b + d$, the pdf is the same as $b + d < t_{ij} \le 2b + d$ in Condition 15, therefore, the pdf for T_{ij} is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{1}{2b}\left(3b + d - t_{ij}\right). \tag{E80}$$

Case 4: with $2b + d < t_{ij} \leq 2p_{[1]} + d$, the pdf is the same as $2b + d < t_{ij} \leq 2p_{[1]} + d$ in Condition 15, therefore, the pdf for T_{ij} is

$$f(t_{ij}|\text{Case 4}) = \tfrac{1}{2}. \tag{E81}$$

Condition 22: $3b/2 < d$ and $b/2 \leq p_{[1]} < b$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 21, therefore, the pdf is

$$f(t_{ij} = d|\text{Case 1}) = P_r(t_{ij} = d|\text{Case 1}) = \tfrac{1}{2}(d - b). \tag{E82}$$

Case 2: with $d < t_{ij} \leq b + d$, the pdf is the same as $d < t_{ij} \leq b + d$ in Condition 21, therefore, the pdf for T_{ij} is

$$f(t_{ij}|\text{Case 2}) = 1. \tag{E83}$$

Case 3: with $b + d < t_{ij} \leq 2p_{[1]} + d$, the pdf is the same as $b + d < t_{ij} \leq 2b + d$ in Condition 21, therefore, the pdf for T_{ij} is

$$f(t_{ij}|\text{Case 3}) = \tfrac{1}{2b}(3b + d - t_{ij}). \tag{E84}$$

Case 4: with $2p_{[1]} + d < t_{ij} \leq b + d + p_{[1]}$, the probability for T_{ij} is

$$P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \tfrac{2}{b}\left[\tfrac{b}{2}\left(\tfrac{b}{4} + \tfrac{d}{2} + p_{[1]}\right) - \tfrac{1}{2}\left(b + p_{[1]} + d - t_{ij}\right)^2\right]$$

and take the derivative of $P_r(T_{ij} \leq t_{ij}|\text{Case 4})$ with respect to T_{ij}, the pdf is

$$f(t_{ij}|\text{Case 4}) = \tfrac{d}{dt_{ij}} P_r(T_{ij} \leq t_{ij}|\text{Case 4}) = \tfrac{2}{b}\left(b + d + p_{[1]} - t_{ij}\right). \tag{E85}$$

Condition 23: $3b/2 < d$ and $p_{[1]} < b/2$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 22 and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r(t_{ij} = d|\text{Case 1}) = \frac{1}{2}(d-b). \qquad \text{(E86)}$$

Case 2: with $d < t_{ij} \leq b/2 + d + p_{[1]}$, the pdf is the same as $d < t_{ij} \leq b + d$ in Condition 22, therefore, the pdf for T_{ij} is

$$f\left(t_{ij}|\text{Case 2}\right) = 1. \qquad \text{(E87)}$$

Case 3: with $b/2 + d + p_{[1]} < t_{ij} \leq b + d + p_{[1]}$, the pdf is the same as $2p_{[1]} + d < t_{ij} \leq b + d + p_{[1]}$ in Condition 22, therefore, the pdf for T_{ij} is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left(b + d + p_{[1]} - t_{ij}\right). \qquad \text{(E88)}$$

In summary, from the derivations above,

Condition 1: $d < b/2$ and $b \leq 1 - p_{[1]}$, the pdf is

$$f\left(t_{ij}\right) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b-d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in (b-d, 3b/2] \\ (3b - d - t_{ij})/(2b) & t_{ij} \in (\,3b/2, 2b-d] \\ 1/2 & t_{ij} \in \left(2b-d, 2 - p_{[1]} - p_{[2]}\right] \end{cases}. \qquad \text{(E89)}$$

Condition 2: $d < b/2$ and $3b/4 + d/2 \leq 1 - p_{[1]} < b$, the pdf is

$$f\left(t_{ij}\right) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b-d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in (b-d, 3b/2] \\ (3b - d - t_{ij})/(2b) & t_{ij} \in \left(\,3b/2, 2 - p_{[1]} - p_{[2]}\right] \\ (2b - 2(t_{ij} - 1 + p_{[2]}))/b & t_{ij} \in \left(2 - p_{[1]} - p_{[2]}, b + 1 - p_{[2]}\right] \end{cases}. \qquad \text{(E90)}$$

Condition 3: $d < b/2$ and $b/2 + d \leq 1 - p_{[1]} < 3b/4 + d/2$, the pdf is

$$f\left(t_{ij}\right) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b-d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in \left(b-d, 2 - p_{[1]} - p_{[2]}\right] \\ (8 + b - 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(2 - p_{[1]} - p_{[2]},\ 3b/2\right] \\ (2b - 2(t_{ij} - 1 + p_{[2]}))/b & t_{ij} \in \left(3b/2, b + 1 - p_{[2]}\right] \end{cases}. \qquad \text{(E91)}$$

Condition 4: $d < b/2$ and $b/2 \le 1 - p_{[1]} < b/2 + d$, the pdf is ,

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b-d] \\ (4t_{ij} - b - 2d)/(4b) & t_{ij} \in \left(b-d, 2 - p_{[1]} - p_{[2]}\right] \\ (8 + b - 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(2 - p_{[1]} - p_{[2]}, b/2 + 2 - 2p_{[2]}\right] \\ (1 - p_{[2]})/b & t_{ij} \in \left(b/2 + 2 - 2p_{[2]}, 3b/2\right] \end{cases}. \quad \text{(E92)}$$

Condition 5: $d < b/2$ and $1 - p_1 < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in \left[b/2, b/2 + 1 - p_{[2]}\right] \\ (8 + b - 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(b/2 + 1 - p_{[2]},\ b/2 + 2 - 2p_{[2]}\right] \\ (1 - p_{[2]})/b & t_{ij} \in \left(b/2 + 2 - 2p_{[2]}, 3b/2\right] \end{cases}. \quad \text{(E93)}$$

Condition 6: $d < b/2$ and $b \le p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b+d] \\ (4t_{ij} - b + 2d)/(4b) & t_{ij} \in (b+d, 3b/2] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in (3b/2, 2b+d] \\ 1/2 & t_{ij} \in \left(2b + d, 2p_{[1]} + d\right] \end{cases}. \quad \text{(E94)}$$

Condition 7: $d < b/2$ and $3b/4 - d/2 \le p_{[1]} < b$, the pdf is, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b+d] \\ (4t_{ij} - b + 2d)/(4b) & t_{ij} \in (b+d, 3b/2] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in \left(3b/2, 2p_{[1]} + d\right] \\ 2(b + p_{[2]} - t_{ij})/b & t_{ij} \in \left(2p_{[1]} + d, b + p_{[2]}\right] \end{cases}. \quad \text{(E95)}$$

Condition 8: $d < b/2$ and $b/2 \le p_{[1]} < 3b/4 - d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b+d] \\ (4t_{ij} - b + 2d)/(4b) & t_{ij} \in \left(b+d, 2p_{[1]} + d\right] \\ (b + 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in \left(2p_{[1]} + d, 3b/2\right] \\ 2(b + p_{[2]} - t_{ij})/b & t_{ij} \in \left(3b/2, b + p_{[2]}\right] \end{cases}. \quad \text{(E96)}$$

Condition 9: $d < b/2$ and $b/2 - d \le p_{[1]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b/2 + p_{[2]}] \\ (b + 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in (b/2 + p_{[2]}, 3b/2]. \\ 2(b + p_{[2]} - t_{ij})/b & t_{ij} \in (3b/2, b + p_{[2]}] \end{cases} \quad \text{(E97)}$$

Condition 10: $d < b/2$ and $p_{[1]} < b/2 - d$, the pdf is

$$f(t_{ij}) = \begin{cases} 3(2t_{ij} - b)/(4b) & t_{ij} \in [b/2, b/2 + p_{[2]}] \\ (b + 8p_{[2]} - 2t_{ij})/(4b) & t_{ij} \in (b/2 + p_{[2]}, 2p_{[2]} + b/2]. \\ p_{[2]}/b & t_{ij} \in (2p_{[2]} + b/2, 3b/2] \end{cases} \quad \text{(E98)}$$

Condition 11: $b/2 \le d < 3b/2$ and $b/2 + d \le 1 - p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 3b/2] \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (3b/2, b + d]. \\ 1/2 & t_{ij} \in (b + d, 2 - 2p_{[1]} - d] \end{cases} \quad \text{(E99)}$$

Condition 12: $b/2 \le d < 3b/2$ and $3b/4 + d/2 \le 1 - p_{[1]} < b/2 + d$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 3b/2] \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (3b/2, 2 - 2p_{[1]} - d]. \\ (b + 2 - 2p_{[1]} - 2t_{ij})/b & t_{ij} \in (2 - 2p_{[1]} - d, b/2 + 1 - p_{[1]}] \end{cases} \quad \text{(E100)}$$

Condition 13: $b/2 \le d < 3b/2$ and $b \le 1 - p_{[1]} < 3b/4 + d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (b - 2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in (d, 2 - 2p_{[1]} - d] \\ (4 - b - d - 4p_{[1]} - t_{ij})/(2b) & t_{ij} \in (2 - 2p_{[1]} - d, 3b/2]. \\ (b + 2 - 2p_{[1]} - 2t_{ij})/b & t_{ij} \in (3b/2, b/2 + 1 - p_{[1]}] \end{cases} \quad \text{(E101)}$$

Condition 14: $b/2 \le d < 3b/2$ and $1 - p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ -1/2 + t_{ij}/b & t_{ij} \in \left(d, 2-2p_{[1]}-d\right] \\ (4-b-d-4p_{[1]}-t_{ij})/(2b) & t_{ij} \in \left(2-2p_{[1]}-d, 2-2p_{[1]}-b/2\right] \\ (4-b-2d-4p_{[1]})/(4b) & t_{ij} \in \left(2-2p_{[1]}-b/2, 3b/2\right] \end{cases}. \tag{E102}$$

Condition 15: $b/2 \le d < 3b/2$ and $b \le p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij}-b-d)/(2b) & t_{ij} \in (d, 3b/2] \\ 1 & t_{ij} \in (3b/2, b+d] \\ (3b+d-t_{ij})/(2b) & t_{ij} \in (b+d, 2b+d] \\ 1/2 & t_{ij} \in \left(2b+d, 2p_{[1]}+d\right] \end{cases}. \tag{E103}$$

Condition 16: $b/2 \le d < 3b/2$ and $b/2 \le p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij}-b-d)/(2b) & t_{ij} \in (d, 3b/2] \\ 1 & t_{ij} \in (3b/2, b+d] \\ (3b+d-t_{ij})/(2b) & t_{ij} \in \left(b+d, 2p_{[1]}+d\right] \\ 2(b+d+p_{[1]}-t_{ij})/b & t_{ij} \in \left(2p_{[1]}+d, b+p_{[1]}+d\right] \end{cases}. \tag{E104}$$

Condition 17: $b/2 \le d < 3b/2$ and $b - d \le p_{[1]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij}-b-d)/(2b) & t_{ij} \in (d, 3b/2] \\ 1 & t_{ij} \in \left(3b/2, b/2+d+p_{[1]}\right] \\ 2(b+d+p_{[1]}-t_{ij})/b & t_{ij} \in \left(b/2+d+p_{[1]}, b+d+p_{[1]}\right] \end{cases}. \tag{E105}$$

Condition 18: $b/2 \leq d < 3b/2$ and $p_{[1]} < b - d$, the pdf is

$$f(t_{ij}) = \begin{cases} (b-2d)^2/(16b) & t_{ij} = d \\ (3t_{ij} - b - d)/(2b) & t_{ij} \in \left(d, b/2 + d + p_{[1]}\right] \\ (b + 3d + 4p_{[1]} - t_{ij})/(2b) & t_{ij} \in \left(b/2 + d + p_{[1]}, 3b/2\right] \\ 2(b + d + p_{[1]} - t_{ij})/b & t_{ij} \in \left(3b/2, b + d + p_{[1]}\right] \end{cases}. \quad \text{(E106)}$$

Condition 19: $3b/2 < d$ and $b/2 + d \leq 1 - p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ (2b + d - t_{ij})/(2b) & t_{ij} \in (d, b+d] \\ 1/2 & t_{ij} \in \left(b + d, 2 - 2p_{[1]} - d\right] \end{cases}. \quad \text{(E107)}$$

Condition 20: $3b/2 < d$ and $1 - p_{[1]} < b/2 + d$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ (2b + d - t_{ij})/(2b) & t_{ij} \in \left(d, 2 - 2p_{[1]} - d\right] \\ \left(b + 2 - 2p_{[1]} - 2t_{ij}\right)/b & t_{ij} \in \left(2 - 2p_{[1]} - d, b/2 + 1 - p_{[1]}\right] \end{cases}. \quad \text{(E108)}$$

Condition 21: $3b/2 < d$ and $b \leq p_{[1]}$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ 1 & t_{ij} \in (d, b+d] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in (b+d, 2b+d] \\ 1/2 & t_{ij} \in \left(2b + d, 2p_{[1]} + d\right] \end{cases}. \quad \text{(E109)}$$

Condition 22: $3b/2 < d$ and $b/2 \leq p_{[1]} < b$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ 1 & t_{ij} \in (d, b+d] \\ (3b + d - t_{ij})/(2b) & t_{ij} \in \left(b + d, 2p_{[1]} + d\right] \\ 2\left(b + d + p_{[1]} - t_{ij}\right)/b & t_{ij} \in \left(2p_{[1]} + d, b + d + p_{[1]}\right] \end{cases}. \quad \text{(E110)}$$

Condition 23: $3b/2 < d$ and $p_{[1]} < b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (d-b)/2 & t_{ij} = d \\ 1 & t_{ij} \in \left(d, b/2 + d + p_{[1]}\right]. \\ 2\left(b + d + p_{[1]} - t_{ij}\right)/b & t_{ij} \in \left(b + d, 2 - 2p_{[1]} - d\right] \end{cases} \tag{E111}$$

Appendix F: Derivation of probability density function for path *mf* travel time with a finite population and random storage

The rack is separated into a left part and a right part by a vertical line at the coordinate $x = p_{[2]}$. We consider the left and right parts for all possible conditions. Because of symmetry, the left and right parts have the same travel time pdfs; therefore, we only need to derive the pdfs for the right part. For the path *mf*, we have

$$T_{ij} = max(|p_i - x|, b/2 - y) + max(|p_j - x|, y).$$

Letting $p_{[1]} = min(p_i, p_j)$, $p_{[2]} = max(p_i, p_j)$, $d = p_{[2]} - p_{[1]}$,

Because $x \in unif[0,1]$, the pdf for *x* is $f(x) = 1$, $y \in unif[0, b/2]$, the pdf for *y* is $f(y) = 2/b$.

Condition 1: $d \leq b/2$ and $b/2 \leq 1 - p_{[2]}$

Case 1: with $t_{ij} = b/2$, the pdf and the probability for T_{ij} are the same and they are

$$f\left(t_{ij} = b/2|\text{Case } 1\right) = P_r\left(t_{ij} = b/2|\text{Case } 1\right) = \frac{2}{b}\left[\frac{1}{\left(2\sqrt{2}\right)^2}(b + 2d)(b - 2d)\right] = \frac{1}{8b}(b^2 - 4d^2). \tag{F1}$$

Case 2: with $b/2 < t_{ij} \leq b - d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case } 2\right) = \frac{2}{b}\left[\frac{1}{8}(b + 2d)(2t_i - b) + \frac{1}{8}(b - 2d)\left(2t_{ij} - b\right) - \frac{1}{16}(2t_{ij} - b)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case } 2\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case } 2\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case } 2\right) = \frac{1}{2b}\left(3b - 2t_{ij}\right). \tag{F2}$$

Case 3: with $b - d < t_{ij} \leq b + d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left[\frac{1}{4}(b^2 - d^2) + \frac{1}{4}\left(b - 2d + \frac{1}{2}(t_{ij} - (b - d))\right)\left(t_{ij} - (b - d)\right) + \frac{\sqrt{2}}{4}\left(2\sqrt{2}d - \frac{\sqrt{2}}{2}\left(t_{ij} - (b - d)\right)\right)(t_{ij} - (b - d))\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \quad \text{(F3)}$$

Case 4: with $b + d < t_{ij} \leq 2 + d - 2p_{[2]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right) = \frac{2}{b}\left[\frac{b}{2}\left(\frac{b}{2} + \frac{d}{2} + \frac{1}{2}\left(t_{ij} - (b - d)\right)\right)\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 4}\right) = \frac{d}{dt_{ij}}P_r\left(T_{ij} \leq t_{ij}|\text{Case 4}\right) = \frac{1}{2} \quad \text{(F4)}$$

Condition 2: $d \leq b/2$ and $b/2 - d \leq 1 - p_{[2]} < b/2$

Case 1: with $t_{ij} = b/2$, the pdf and the probability for T_{ij} are the same as $t_{ij} = b/2$ in Condition 1 and they are

$$f\left(t_{ij} = b/2|\text{Case 1}\right) = P_r\left(t_{ij} = b/2|\text{Case 1}\right) = \frac{1}{8b}(b^2 - 4d^2). \quad \text{(F5)}$$

Case 2: with $b/2 < t_{ij} \leq b - d$, the pdf is the same as $b/2 < t_{ij} \leq b - d$ in Condition 1, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{1}{2b}\left(3b - 2t_{ij}\right). \quad \text{(F6)}$$

Case 3: with $b - d < t_{ij} \leq 2 + d - 2p_{[2]}$, the pdf is the same as $b - d < t_{ij} \leq b + d$ in Condition 1, therefore, the pdf for T_{ij} is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \quad \text{(F7)}$$

Case 4: with $2 + d - 2p_{[2]} < t_{ij} \leq b/2 + 1 - p_{[1]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case } 4\right) = \frac{2}{b}\left[\frac{b}{2}\left(1 - \frac{1}{2}(p_{[1]} + p_{[2]})\right) - \frac{1}{2}\left(1 + \frac{1}{2}(b + d) - \frac{1}{2}\left(p_{[1]} + p_{[2]}\right) - t_{ij}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case } 4\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 4\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case } 4\right) = \frac{1}{b}\left(2 + b - 2p_{[1]} - 2t_{ij}\right). \quad \text{(F8)}$$

Condition 3: $d \leq b/2$ and $b/4 - d/2 \leq 1 - p_{[2]} < b/2 - d$

Case 1: with $t_{ij} = b/2$, the pdf and the probability for T_{ij} are the same as $t_{ij} = b/2$ in Condition 2, therefore, the pdf and the probability for T_{ij} are

$$f\left(t_{ij} = b/2 | \text{Case } 1\right) = P_r\left(t_{ij} = b/2 | \text{Case } 1\right) = \frac{1}{8b}(b^2 - 4d^2). \quad \text{(F9)}$$

Case 2: with $b/2 < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$, the pdf is the same as $b/2 < t_{ij} \leq b - d$ in Condition 2, therefore, the pdf for T_{ij} is

$$f\left(t_{ij} | \text{Case } 2\right) = \frac{1}{2b}\left(3b - 2t_{ij}\right). \quad \text{(F10)}$$

Case 3: with $2 - p_{[1]} - p_{[2]} < t_{ij} \leq b/2 + 1 - p_{[2]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right) = \frac{2}{b}\left[\frac{b}{2}\left(1 - \frac{1}{2}(p_{[1]} + p_{[2]})\right) - \frac{1}{2}\left(1 - p_{[1]} + \frac{b}{2} - t_{ij}\right)^2 - \frac{1}{2}\left(1 - p_{[2]} - \frac{b}{2} - t_{ij}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case } 3\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij} | \text{Case } 3\right) = \frac{2}{b}\left(2 + b - p_{[1]} - p_{[2]} - 2t_{ij}\right). \quad \text{(F11)}$$

Case 4: with $b/2 + 1 - p_{[2]} < t_{ij} \leq b/2 + 1 - p_{[1]}$, the pdf is the same as $2 + d - 2p_{[2]} < t_{ij} \leq \mathrm{b}/2 + 1 - p_{[1]}$ in Condition 2, therefore, the pdf is

$$f\left(t_{ij} | \text{Case } 4\right) = \frac{1}{b}\left(2 + b - 2\mathrm{p}_{[1]} - 2t_{ij}\right). \quad \text{(F12)}$$

Condition 4: $d \leq b/2$ and $1 - p_{[2]} < b/4 - d/2$

Case 1: with $t_{ij} = b/2$, the pdf and the probability for T_{ij} are the same and they are

$$f\left(t_{ij} = b/2|\text{Case 1}\right) = P_r\left(t_{ij} = b/2|\text{Case 1}\right) = \frac{2}{b}\left[\frac{b}{2}\left(1 - \frac{1}{2}(p_{[1]} + p_{[2]})\right) - \frac{1}{2}\left(1 - p_{[1]}\right)^2 - \frac{1}{2}\left(1 - p_{[2]}\right)^2\right] = \frac{1}{2b}(-4 + 2b + 4p_{[1]} - bp_{[1]} - 2p_{[1]}^2 + 4p_{[2]} - bp_{[2]} - 2p_{[2]}^2) \quad \text{(F13)}$$

Case 2: with $b/2 < t_{ij} \leq b/2 + 1 - p_{[2]}$, the pdf is the same as $2 - p_{[1]} - p_{[2]} < t_{ij} \leq b/2 + 1 - p_{[2]}$ in Condition 3, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{2}{b}\left(2 + b - p_{[1]} - p_{[2]} - 2t_{ij}\right). \quad \text{(F14)}$$

Case 3: with $b/2 + 1 - p_{[2]} < t_{ij} \leq b/2 + 1 - p_{[1]}$, the pdf is the same as $b/2 + 1 - p_{[2]} < t_{ij} \leq b/2 + 1 - p_{[1]}$ in Condition 3, therefore, the pdf is

$$f\left(t_{ij}|\text{Case 3}\right) = \frac{1}{b}\left(2 + b - 2p_{[1]} - 2t_{ij}\right). \quad \text{(F15)}$$

Condition 5: $b/2 < d$ and $1 - p_{[2]} \geq b/2$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same and they are

$$f\left(t_{ij} = d|\text{Case 1}\right) = P_r\left(t_{ij} = d|\text{Case 1}\right) = \frac{2}{b}\left(\frac{b}{4}(d - \frac{b}{2})\right) = \frac{1}{4}(2d - b). \quad \text{(F16)}$$

Case 2: with $d < t_{ij} \leq b + d$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{2}{b}\left[\frac{1}{2}b\left(t_{ij} - d\right) - \frac{1}{8}(t_{ij} - d)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij}|\text{Case 2}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \leq t_{ij}|\text{Case 2}\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \quad \text{(F17)}$$

Case 3: with $b + d < t_{ij} \leq 2 - p_{[1]} - p_{[2]}$, the probability for T_{ij} is

$$P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right) = \frac{2}{b}\left[\frac{b}{2}\left(1 - \frac{1}{2}(p_{[1]} + p_{[2]})\right) - \frac{b}{4}\left(2 - p_{[1]} - p_{[2]} - t_{ij}\right)\right]$$

and take the derivative of $P_r\left(T_{ij} \leq t_{ij}|\text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 3}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij} | \text{Case 3}\right) = \frac{1}{2}. \tag{F18}$$

Condition 6: $b/2 < d$ and $1 - p_{[2]} < b/2$

Case 1: with $t_{ij} = d$, the pdf and the probability for T_{ij} are the same as $t_{ij} = d$ in Condition 5 and they are

$$f\left(t_{ij} = d | \text{Case 1}\right) = P_r\left(t_{ij} = d | \text{Case 1}\right) = \frac{1}{4}(2d - b). \tag{F19}$$

Case 2: with $d < t_{ij} \le 2 - p_{[1]} - p_{[2]}$, the pdf is the same as $d < t_{ij} \le b + d$ in Condition 5, therefore, the pdf is

$$f\left(t_{ij} | \text{Case 2}\right) = \frac{1}{2b}\left(2b + d - t_{ij}\right). \tag{F20}$$

Case 3: with $2 - p_{[1]} - p_{[2]} < t_{ij} \le 1 - p_{[1]} + b/2$, the probability for T_{ij} is

$$P_r\left(T_{ij} \le t_{ij} | \text{Case 3}\right) = \frac{2}{b}\left[\frac{b}{2}\left(1 - \frac{1}{2}(p_{[1]} + p_{[2]})\right) - \frac{1}{2}\left(1 - p_{[1]} + \frac{b}{2} - t_{ij}\right)^2\right]$$

and take the derivative of $P_r\left(T_{ij} \le t_{ij} | \text{Case 3}\right)$ with respect to T_{ij}, the pdf is

$$f\left(t_{ij} | \text{Case 3}\right) = \frac{d}{dt_{ij}} P_r\left(T_{ij} \le t_{ij} | \text{Case 3}\right) = \frac{1}{b}\left(b + 2 - 2p_{[1]} - 2t_{ij}\right). \tag{F21}$$

In summary, from the derivations above,

Condition 1: $d \le b/2$ and $b/2 \le 1 - p_{[2]}$, the pdf is

$$f\left(t_{ij}\right) = \begin{cases} (b^2 - 4d^2)/(8b) & t = b/2 \\ (3b - 2t_{ij})/(2b) & t \in (b/2, b - d] \\ (2b + d - t_{ij})/(2b) & t \in (b - d, b + d] \\ 1/2 & t \in (b + d, 2 + d - 2p_{[2]}] \end{cases}. \tag{F22}$$

Condition 2: $d \le b/2$ and $b/2 - d \le 1 - p_{[2]} < b/2$, the pdf is

$$f\left(t_{ij}\right) = \begin{cases} (b^2 - 4d^2)/(8b) & t = b/2 \\ (3b - 2t_{ij})/(2b) & t \in (b/2, b - d] \\ (2b + d - t_{ij})/(2b) & t \in (b - d, 2 + d - 2p_{[2]}] \\ (2 + b - 2\mathrm{p}_{[1]} - 2t_{ij})/b & t \in (2 + d - 2p_{[2]}, b/2 + 1 - p_{[1]}] \end{cases}. \tag{F23}$$

Condition 3: $d \leq b/2$ and $b/4 - d/2 \leq 1 - p_{[2]} < b/2 - d$, the pdf is

$$f(t_{ij}) = \begin{cases} (b^2 - 4d^2)/(8b) & t = b/2 \\ (3b - 2t_{ij})/(2b) & t \in (b/2, 2 - p_{[1]} - p_{[2]}] \\ 2(2 + b - p_{[1]} - p_{[2]} - 2t_{ij})/b & t \in (2 - p_{[1]} - p_{[2]}, b/2 + 1 - p_{[2]}] \\ (2 + b - 2\mathrm{p}_{[1]} - 2t_{ij})/b & t \in (b/2 + 1 - p_{[2]},\ b/2 + 1 - p_{[1]}] \end{cases}. \quad \text{(F24)}$$

Condition 4: $d \leq b/2$ and $1 - p_{[2]} < b/4 - d/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (-4 + 2b + 4p_{[1]} - bp_{[1]} - 2p_{[1]}^2 + 4p_{[2]} - bp_{[2]} - 2p_{[2]}^2)/(2b) & t = b/2 \\ 2(2 + b - p_{[1]} - p_{[2]} - 2t_{ij})/b & t \in (b/2, b/2 + 1 - p_{[2]}] \\ (2 + b - 2\mathrm{p}_{[1]} - 2t_{ij})/b & t \in (b/2 + 1 - p_{[2]},\ b/2 + 1 - p_{[1]}] \end{cases}. \quad \text{(F25)}$$

Condition 5: $d > b/2$ and $1 - p_{[2]} \geq b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (2d - b)/4 & t_{ij} = d \\ \left(2b + d - t_{ij}\right)/(2b) & t_{ij} \in (d, b + d] \\ 1/2 & t_{ij} \in (b + d, 2 - p_{[1]} - p_{[2]}] \end{cases}. \quad \text{(F26)}$$

Condition 6: $d > b/2$ and $1 - p_{[2]} \geq b/2$, the pdf is

$$f(t_{ij}) = \begin{cases} (2d - b)/4 & t = d \\ \left(2b + d - t_{ij}\right)/(2b) & t \in (d, 2 - p_{[1]} - p_{[2]}] \\ \left(b + 2 - 2p_{[1]} - 2t_{ij}\right)/b & t \in (2 - p_{[1]} - p_{[2]},\ 1 - p_{[1]} + b/2] \end{cases}. \quad \text{(F27)}$$

Appendix G: Derivation of probability density function for path *ff* or *mm* travel time with an infinite population and dedicated storage

$$T = max(|x_2 - x_1|, y) + y = \begin{cases} x_2 - x_1 + y & x_2 \geq x \text{ and } x_2 - x_1 \geq y & \text{Case 1} \\ 2y & x_2 \geq x \text{ and } x_2 - x_1 < y & \text{Case 2} \\ x_1 - x_2 + y & x_2 < x \text{ and } x_1 - x_2 \geq y & \text{Case 3} \\ 2y & x_2 < x \text{ and } x_1 - x_2 < y & \text{Case 4} \end{cases}$$

where $x_1, x_2 \in$ unif$[0, 1]$ and $y \in$ unif$[0, b/2]$, and $f(x_1, x_2, y) = f(x_1)f(x_2)f(y) = \frac{2}{b}$

For the four cases, by symmetry, Case 1 and Case 3 are identical and Case 2 and Case 4 are identical. Therefore, we only need to consider Case 1 and Case 2.

From the expression above, the value of T can be stated as follows:

Case 1: with $x_2 \geq x_1$ and $x_2 - x_1 \geq y$, then $T = x_2 - x_1 + y$

Case 1A: with $x_1 \geq x_2 - t/2$

Case 1A1: with $0 \leq t \leq b$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 1A1}) = \int_0^{t/2}\int_0^{x_2}\int_0^{x_2-x_1}\frac{2}{b}dydx_1dx_2 + \int_{t/2}^{1}\int_{x_2-t/2}^{x_2}\int_0^{x_2-x_1}\frac{2}{b}dydx_1dx_2 = \frac{(3-t)t^2}{12b}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 1A1})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1}) = \frac{d}{dt}P_r(0 < T \leq t|\text{Case 1A1}) = \frac{(2-t)t}{4b}. \tag{G1}$$

Case 1A2: with $b < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 1A2})$

$$= \int_0^{b/2}\int_0^{x_2}\int_0^{x_2-x_1}\frac{2}{b}ddx_1dx_2 + \int_{b/2}^{1}\int_{x_2-b/2}^{x_2}\int_0^{x_2-x_1}\frac{2}{b}dydxdx_2$$

$$+ \int_{b/2}^{t/2}\int_0^{x_2-b/2}\int_0^{b/2}\frac{2}{b}dydx_1dx_2 + \int_{t/2}^{1}\int_{x_2-t/2}^{x_2-b/2}\int_0^{b/2}\frac{2}{b}dydx_1dz$$

$$= \frac{(-3t^2 + 12t + b^2 - 6b)}{24}$$

and take the derivative of $P_r(T \leq t|\text{Case 1A2})$ with respect to T, the pdf is

$$f(t|\text{Case 1A2}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A2}) = \frac{2-t}{4}. \tag{G2}$$

Case 1B: with $x_1 < x_2 - t/2$

Case 1B1: with $0 \le t \le b$, the probability for T is

$P_r(0 \le T \le t|\text{Case 1B1}) = \int_{t/2}^{t}\int_{0}^{x_2-t/2}\int_{0}^{x_1-x_2+t}\frac{2}{b}dydx_1dx_2 +$

$\int_{t}^{1}\int_{x_2-t}^{x_2-t/2}\int_{0}^{x_1-x_2+t}\frac{2}{b}dydx_1dx_2 = \frac{(3-2t)t^2}{12b}$

and take the derivative of $P_r(0 \le T \le t|\text{Case 1B1})$ with respect to T, the pdf is

$$f(t|\text{Case 1B1}) = \frac{d}{dt}P_r(0 \le T \le t|\text{Case 1B1}) = \frac{(1-t)t}{2b}. \tag{G3}$$

Case 1B2: with $b < t \le 1$, the probability for T is

$P_r(T \le t|\text{Case 1B2}) = \int_{t/2}^{t-b/2}\int_{0}^{x_2-t/2}\int_{0}^{b/2}\frac{2}{b}dydx_1dx_2 + \int_{t-b/2}^{1}\int_{x_2+b/2-t}^{x_2-t/2}\int_{0}^{b/2}\frac{2}{b}dydx_1dx_2 +$

$\int_{t-b/2}^{t}\int_{0}^{x_2+b/2-t}\int_{0}^{x_1-x_2+t}\frac{2}{b}dydx_1dx_2 + \int_{t}^{1}\int_{x_2-t}^{x_2+b/2-t}\int_{0}^{x_1-x_2+t}\frac{2}{b}dydx_1dx_2 =$

$\frac{(-9t^2+12t+6bt-b^2-6b)}{24}$

and take the derivative of $P_r(T \le t|\text{Case 1B2})$ with respect to T, the pdf is

$$f(t|\text{Case 1B2}) = \frac{d}{dt}P_r(T \le t|\text{Case 1B2}) = \frac{2+b-3t}{4}. \tag{G4}$$

Case 1B3: with $1 < t \le 1 + b/2$, the probability for T is

$P_r(T \le t|\text{Case 1B3}) = \int_{t/2}^{t-b/2}\int_{0}^{x_2-t/2}\int_{0}^{b/2}\frac{2}{b}ydx_1dx_2 + \int_{t-b/2}^{1}\int_{x_2+b/2-t}^{x_2-t/2}\int_{0}^{b/2}\frac{2}{b}dydx_1dx_2 +$

$\int_{t-b/2}^{t}\int_{0}^{x_2+b/2-t}\int_{0}^{x_1-x_2+t}\frac{2}{b}dydx_1dx_2 = \frac{-b^3+6b^2(-1+t)+8(-1+t)^3+3b(4-3t)t}{24b}$

and take the derivative of $P_r(T \le t|\text{Case 1B3})$ with respect to T, the pdf is

$$f(t|\text{Case 1B3}) = \frac{d}{dt}P_r(T \le t|\text{Case 1B3}) = \frac{b^2+b(2-3t)+4(-1+t)^2}{4b}. \tag{G5}$$

Case 2: with $x_2 \ge x_1$ and $x_2 - x_1 < y$, which means $T = 2y$ and $T \in [0, b]$, the probability for T is

$P_r(\,0 \le T \le t|\text{Case }2) = \int_0^{1-t/2}\int_{x_1}^{x_1+t/2}\int_{x_2-x_1}^{t/2}\frac{2}{b}dydx_2dx_1 + \int_{1-t/2}^{1}\int_{x_1}^{1}\int_{x_2-x_1}^{t/2}\frac{2}{b}dydx_2dx_1 =$

$\frac{(6-t)t^2}{24b}$

and take the derivative of $P_r\left(0 \le T_{ij} \le t|\text{Case }2\right)$ with respect to T, the pdf is

$$f(t|\text{Case }2) = \frac{d}{dt}P_r\left(0 \le T_{ij} \le t|\text{Case }2\right) = \frac{(4-t)t}{8b}. \qquad \text{(G6)}$$

From the derivations above, the pdf is as follows:

$$f(t) = \begin{cases} (12-7t)t/(4b) & t \in [0,b] \\ 2+b/2-2t & t \in (b,1]. \\ (2+b-2t)^2/(2b) & t \in (1, 1+b/2] \end{cases} \qquad \text{(G7)}$$

Appendix H: Derivation of probability density function for path *fm* travel time with an infinite population and dedicated storage

$$T = max(|x_1 - x_2|, y + b/2) + y$$

$$= \begin{cases} x_1 - x_2 + y, & x_1 \ge x_2\ and\ x_1 - x_2 \ge y + b/2 & \text{Case 1} \\ 2y + b/2, & x_1 \ge x_2\ and\ x_1 - x_2 < y + b/2 & \text{Case 2} \\ x_2 - x_1 + y, & x_1 < x_2\ and\ x_2 - x_1 \ge y + b/2 & \text{Case 3} \\ 2y + b/2, & x_1 < x_2\ and\ x_2 - x_1 < y + b/2 & \text{Case 4} \end{cases}$$

where $x_1, x_2 \in$ unif$[0,1]$ and $y \in$ unif$[0, b/2]$.

$$f(x_1, x_2, y) = f(x_1)f(x_2)f(y) = \frac{2}{b}$$

For the four cases, by symmetry, Case 1 and Case 3 are identical and Case 2 and Case 4 are identical. Therefore, we only need to consider Case 1 and Case 2.

From the expression above, the value of T_{ij} can be stated as follows:

Case 1: $x_1 \ge x_2\ and\ x_1 - x_2 \ge y + b/2$, , then $T = x_1 - x_2 + y$.

Case 1A: with $b \le 2/3$.

Case 1A1: with $b/2 \le t \le 3b/2$, the probability for T is

$P_r(b/2 \le T \le t|\text{Case 1A1})$

$$= \int_0^{t/2-b/4}\int_0^{y+1-t}\int_{x_2+y+b/2}^{x_2+t-y}\frac{2}{b}dx_1dx_2dy$$

$$+ \int_0^{t/2-b/4}\int_{y+1-t}^{-y+1-b/2}\int_{x_2+y+b/2}^{1}\frac{2}{b}dx_1dx_2dy = \frac{(b-2\,t)^2(4-b-2\,t)}{32b}$$

and take the derivative of $P_r(b/2 \le T \le t|\text{Case 1A1})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1}) = \frac{d}{dt}P_r(b/2 \le T \le t|\text{Case 1A1}) = \frac{(b-2\,t)\,(-8+b+6\,t)}{16b}. \quad \text{(H1)}$$

Case 1A2: with $3b/2 < t \le 1$, the probability for T is

$P_r(T \le t|\text{Case 1A2})$

$$= \int_0^{b/2}\int_0^{y+1-t}\int_{x_2+y+b/2}^{x_2+t-y}\frac{2}{b}dx_1dx_2dy + \int_0^{b/2}\int_{y+1-t}^{-y+1-b/2}\int_{x_2+y+b/2}^{1}\frac{2}{b}dx_1dx_2dy$$

$$= -\mathrm{b} + \frac{b^2}{4} + t + \frac{bt}{4} - \frac{t^2}{2}$$

and take the derivative of $P_r(T \le t|\text{Case 1A2})$ with respect to T, the pdf is

$$f(t|\text{Case 1A2}) = \frac{d}{dt}P_r(T \le t|\text{Case 1A2}) = 1 + \frac{b}{4} - t. \quad \text{(H2)}$$

Case 1A3: with $1 < t \le 1 + b/2$, the probability for T is

$P_r(T \le t|\text{Case 1A3})$

$$= \int_{t-1}^{b/2}\int_0^{y+1-t}\int_{x_2+y+b/2}^{x_2+t-y}\frac{2}{b}dx_1dx_2dy + \int_0^{t-1}\int_0^{-y+1-b/2}\int_{x_2+y+b/2}^{1}\frac{2}{b}dx_1dx_2dy$$

$$+ \int_{t-1}^{b/2}\int_{y+1-t}^{-y+1-b/2}\int_{x_2+y+b/2}^{1}\frac{2}{b}dx_1dx_2dy$$

$$= \frac{1}{12b}(3b^3 + 3b^2(-4+t) + 4(-1+t)^3 - 6b(-2+t)t)$$

and take the derivative of $P_r(T \le t|\text{Case 1A3})$ with respect to T, the pdf is

$$f(t|\text{Case 1A3}) = \frac{d}{dt}P_r(T \le t|\text{Case 1A3}) = \frac{1}{4b}(2+b-2t)^2. \quad \text{(H3)}$$

Case 1B: with $b > 2/3$

Case 1B1: with $b/2 \le t \le 1$, the pdf is the same as $b/2 \le t \le 3b/2$ in Case 1A1, therefore, the pdf is

$$f(t|\text{Case 1B1}) = \frac{(b-2\,t)\,(-8+b+6\,t)}{16b}. \quad \text{(H4)}$$

Case 1B2: with $1 < t \le 3b/2$, the probability for T is

$$\begin{aligned}
&P_r(T \le t|\text{Case 1B2}) \\
&= \int_{t-1}^{t/2-b/4} \int_{0}^{y+1-t} \int_{x_2+y+b/2}^{x_2+t-y} \frac{2}{b}\,dx_1 dx_2 dy \\
&+ \int_{0}^{t-1} \int_{0}^{-y+1-b/2} \int_{x_2+y+b/2}^{1} \frac{2}{b}\,dx_1 dx_2 dy \\
&+ \int_{t-1}^{t/2-b/4} \int_{y+1-t}^{-y+1-b/2} \int_{x_2+y+b/2}^{1} \frac{2}{b}\,dx_1 dx_2 dy \\
&= \frac{1}{96b}\left(-3b^3 + 12b(-4+t) + 6b^2(2+t) + 8(-4+12t-6t^2+t^3)\right)
\end{aligned}$$

and take the derivative of $P_r(T \le t|\text{Case 1B2})$ with respect to T, the pdf is

$$f(t|\text{Case 1B2}) = \frac{d}{dt}P_r(T \le t|\text{Case 1B2}) = \frac{1}{16b}(-4+b+2t)^2. \quad \text{(H5)}$$

When $3b/2 < t \le 1 + b/2$, the pdf is the same as $1 < t \le 1 + b/2$ in Case 1A, therefore, the pdf is

$$f(t) = \frac{1}{4b}(2+b-2t)^2. \quad \text{(H6)}$$

Case 2: with $b > 2/3$, $x_1 \geq x_2$ and $x_1 - x_2 < y + b/2$, then $T = 2y + b/2$, the probability for T is

$$P_r(3b/2 < T \leq t|\text{Case 2})$$

$$= \int_0^{t/2-b/4} \int_0^{-y+1-b/2} \int_{x_2}^{x_2+y+b/2} \frac{2}{b} dx_1 dx_2 dy$$

$$+ \int_0^{t/2-b/4} \int_{-y+1-b/2}^{1} \int_{x_2}^{1} \frac{2}{b} dx_1 dx_2 dy$$

$$= \frac{1}{192b}(7b^3 - 12b(-4+t)t - 8(-6+t)t^2 - 6b^2(6+t))$$

and take the derivative of $P_r(3b/2 < T_{ij} \leq t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt} P_r(3b/2 < T_{ij} \leq t|\text{Case 2}) = \frac{1}{32b}(-b^2 - 4\,b\,(-2+t) - 4\,(-4+t)t). \quad \text{(H7)}$$

From the derivations above, the pdf is as follows:

When $b \leq 2/3$, the pdf is

$$f(t) = \begin{cases} (b^2 + 4b(-2+t) + 4(12-7t)t)/(16b), & t \in [b/2, 3\,b/2] \\ (4+b-4t)/2, & t \in (3b/2, 1]. \\ (2+b-2t)^2/(2b), & t \in (1, 1+b/2] \end{cases} \quad \text{(H8)}$$

When $b > 2/3$, the pdf is

$$f(t) = \begin{cases} (b^2 + 4b(-2+t) + 4(12-7t)t)/(16b), & t \in [b/2, 1] \\ (b^2 + 4b(-2+t) + 4(8-4t+t^2))/(16b), & t \in (1, 3b/2]. \\ (2+b-2t)^2/(2b), & t \in (3b/2, 1+b/2] \end{cases} \quad \text{(H9)}$$

Appendix I: Derivation of probability density function for path *mf* travel time with an infinite population and dedicated storage

$$T = max(|x_1 - x_2|, b/2 - y) + y$$

$$= \begin{cases} x_1 - x_2 + y, & x_1 \geq x_2 \text{ and } x_1 - x_2 \geq b/2 - y \qquad \text{Case1} \\ b/2, & x_1 \geq x_2 \text{ and } x_1 - x_2 < b/2 - y \qquad \text{Case2} \\ x_2 - x_1 + y, & x_1 < x_2 \text{ and } x_2 - x_1 \geq b/2 - y \qquad \text{Case3} \\ b/2, & x_1 < x_2 \text{ and } x_2 - x_1 < b/2 - y \qquad \text{Case4} \end{cases}$$

where $x_1, x_2 \in$ unif$[0, 1]$ and $y \in$ unif$[0, b/2]$.

$$f(x_1, x_2, y) = f(x_1)f(x_2)f(y) = \frac{2}{b}$$

For the four cases, by symmetry, Case 1 and Case 3 are identical and Case 2 and Case 4 are identical. Therefore, we only need to consider Case 1 and Case 2.

From the expression above, the value of T can be stated as follows:

Case 1: $x_1 \geq x_2 \; and \; x_1 - x_2 \geq b/2 - y,$, then $T = x_1 - x_2 + y$

Case 1A: with $b/2 \leq t \leq 1$, the probability for T is

$P_r(b/2 \leq T \leq t|\text{Case 1A})$

$$= \int_0^{b/2}\int_0^{y+1-t}\int_{x_2-y+b/2}^{x_2-y+t}\frac{2}{b}dx_1dx_2dy + \int_0^{b/2}\int_{y+1-t}^{y+1-b/2}\int_{x_2-y+b/2}^{1}\frac{2}{b}dx_1dx_2dy$$

$$= \frac{1}{4}(2t - b)(2 - t)$$

and take the derivative of $P_r(b/2 \leq T \leq t|\text{Case 1A})$ with respect to T, the pdf is

$$f(t|\text{Case 1A}) = \frac{d}{dt}P_r(b/2 \leq T \leq t|\text{Case 1A}) = \frac{1}{4}(4 + b - 4t). \qquad \text{(I1)}$$

Case 1B: with $1 < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 1B})$

$$= \int_{t-1}^{b/2} \int_{0}^{y+1-t} \int_{x_2-y+b/2}^{x_2-+t} \frac{2}{b} dx_1 dx_2 dy + \int_{0}^{t-1} \int_{0}^{y+1-b/2} \int_{x_2-y+b/2}^{1} \frac{2}{b} dx_1 dx_2 dy$$

$$+ \int_{t-1}^{b/2} \int_{y+1-t}^{y+1-b/2} \int_{x_2-y+b/2}^{1} \frac{2}{b} dx_1 dx_2 dy$$

$$= \frac{1}{12b} (3b^2(-2+t) + 4(-1+t)^3 - 6b(-2+t)t)$$

and take the derivative of $P_r(T \leq t|\text{Case 1B})$ with respect to T, the pdf is

$$f(t|\text{Case 1B}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1B}) = \frac{1}{4b}(2 + b - 2t)^2. \qquad \text{(I2)}$$

Case 2: $x_1 \geq x_2$ and $x_1 - x_2 < b/2 - y$, , then $T = b/2$, the pdf and the probability for T are the same and they are

$$f(t = b/2|\text{Case 2}) = F(t = b/2|\text{Case 2}) = \int_0^{b/2} \int_0^{y+1-b/2} \int_{x_2}^{x_2-y+b/2} \frac{2}{b} dx_1 dx_2 dy + \int_0^{b/2} \int_{y+1-b/2}^{1} \int_{x_2}^{1} \frac{2}{b} dx_1 dx_2 dy = \frac{1}{24} b(6-b). \qquad \text{(I3)}$$

From the derivations above, the pdf is as follows:

$$f(t) = \begin{cases} b(6-b)/12, & t = b/2 \\ (4+b-4t)/2, & t \in (b/2, 1]. \\ (2+b-2t)^2/(2b), & t \in (1, 1+b/2] \end{cases} \qquad \text{(I4)}$$

Appendix J: Derivation of probability density function for path *ff* or *mm* travel time with an infinite population and random storage

$T = max(|x_1 - x|, y) + max(|x_2 - x|, y)$

$$= \begin{cases} 2x-(x_1+x_2) & x-x_1 \geq y \text{ and } x-x_2 \geq y & \text{Case 1} \\ x_2-x_1 & x-x_1 \geq y \text{ and } x_2-x \geq y & \text{Case 2} \\ x-x_1+y & x-x_1 \geq y \text{ and } y \geq x-x_2 & \text{Case 3} \\ x-x_1+y & x-x_1 \geq y \text{ and } y \geq x_2-x & \text{Case 4} \\ x_1-x_2 & x_1-x \geq y \text{ and } x-x_2 \geq y & \text{Case 5} \\ x_1+x_2-2x & x_1-x \geq y \text{ and } x_2-x \geq y & \text{Case 6} \\ x_1-x+y & x_1-x \geq y \text{ and } y \geq x-x_2 & \text{Case 7} \\ x_1-x+y & x_1-x \geq y \text{ and } y \geq x_2-x & \text{Case 8} \\ x-x_2+y & y \geq x-x_1 \text{ and } x-x_2 \geq y & \text{Case 9} \\ x_2-x+y & y \geq x-x_1 \text{ and } x_2-x \geq y & \text{Case 10} \\ 2y & y \geq x-x_1 \text{ and } y \geq x-x_2 & \text{Case 11} \\ 2y & y \geq x-x_1 \text{ and } y \geq x_2-x & \text{Case 12} \\ x-x_2+y & y \geq x_1-x \text{ and } x-x_2 \geq y & \text{Case 13} \\ x_2-x+y & y \geq x_1-x \text{ and } x_2-x \geq y & \text{Case 14} \\ 2y & y \geq x_1-x \text{ and } y \geq x-x_2 & \text{Case 15} \\ 2y & y \geq x_1-x \text{ and } y \geq x_2-x & \text{Case 16} \end{cases}$$

where $x, x_1$and $x_2 \in$ unif$[0, 1]$ and $y \in$ unif$[0, b/2]$.

$$f(x, x_1, x_2, y) = f(x)f(x_1)f(x_2)f(y) = \frac{2}{b}$$

For the sixteen cases, by symmetry, Case 1 and Case 6 are identical, Case 2 and Case 5 are identical, Case 3, Case 8, and Case 9 are identical, Case 4, Case 7, and Case 13 are identical, Case 11 and Case 16 are identical, and Case 12 and Case 15 are identical. . Therefore, we only need to consider Cases 1, 2, 3, 4, 11 and 12.

From the expression above, the value of T can be stated as follows:

Case 1: $x - x_1 \geq y$ and $x - x_2 \geq y$

Case 1A: with $x_1 \geq x_2$

Case 1A1: with $0 \leq t \leq b$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 1A1}) = \int_0^{t/2}\int_0^{x}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx +$

$\int_{t/2}^{t}\int_{2x-t}^{x}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx + \int_{t/2}^{t}\int_{x-t/2}^{2x-t}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx +$

$\int_t^1\int_{x-t/2}^{x}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx = \frac{(2-t)t^3}{24b}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 1A1})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1}) = \frac{d}{dt}P_r(0 < T \leq t|\text{Case 1A1}) = \frac{(3-2t)t^2}{12b}. \quad \text{(J1)}$$

Case 1A2: with $b < t \leq 1$, the probability for T is

$P_r(T \leq t|\text{Case 1A2}) = \int_0^{b/2}\int_0^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx +$

$\int_{b/2}^{t-b/2}\int_{x-b/2}^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx + \int_{t-b/2}^{t}\int_{2x-t}^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx +$

$\int_{t-b/2}^{t}\int_{x-b/2}^{2x-t}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx + \int_t^1\int_{x-b/2}^{x}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx +$

$\int_{b/2}^{t/2}\int_0^{x-b/2}\int_0^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx + \int_{t/2}^{t-b/2}\int_{2x-t}^{x-b/2}\int_0^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx +$

$\int_{t/2}^{t-b/2}\int_{x-t/2}^{2x-t}\int_{2x-x_1-t}^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx + \int_{t-b/2}^{1}\int_{x-t/2}^{x-b/2}\int_{2x-x_1-t}^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx =$

$\frac{1}{24}(2-t)(b^2-3bt+3t^2)$

and take the derivative of $P_r(T \leq t|\text{Case 1A2})$ with respect to T, the pdf is

$$f(t|\text{Case 1A2}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A2}) = \frac{1}{24}(-b^2+6b(-1+t)+3(4-3t)t). \quad \text{(J2)}$$

Case 1A3: with $1 < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 1A3})$

$$= \int_0^{b/2}\int_0^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx + \int_{b/2}^{t-b/2}\int_{x-b/2}^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$+ \int_{t-b/2}^{1}\int_{2x-t}^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_0^{t-b}\int_{x_1+b/2}^{1/2(x_1+t)}\int_0^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dxdx_1+\int_{t/2}^{t-b/2}\int_{x-t/2}^{2x-t}\int_{2x-x_1-t}^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t-b/2}^{1}\int_{x-t/2}^{x-b/2}\int_{2x-x_1-t}^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{1}^{t-b/2}\int_{x-b/2}^{2x-t}\int_{2x-x_1-t}^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$=\frac{-b^3(-2+t)+2(-1+t)^4+3b^2(-2+t)t-3b(-2+t)t^2}{24b}$$

and take the derivative of $P_r(T\leq t|\text{Case 1A3})$ with respect to T, the pdf is

$$f(t|\text{Case 1A3})=\frac{d}{dt}P_r(T\leq t|\text{Case 1A3})=\frac{-b^3+6b^2(-1+t)+8(-1+t)^3+3b(4-3t)t}{24b}. \qquad \text{(J3)}$$

Case 1A4: with $1+b/2<t\leq 2$, the probability for T is

$P_r(T\leq t|\text{Case 1A4})$

$$=\int_0^{b/2}\int_0^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx+\int_{b/2}^{1}\int_{x-b/2}^{x}\int_0^{x_1}\int_0^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{b/2}^{t/2}\int_0^{x-b/2}\int_0^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx+\int_{t/2}^{1}\int_{2x-t}^{x-b/2}\int_0^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t/2}^{1}\int_{x-t/2}^{2x-t}\int_{2x-x_1-t}^{x_1}\int_0^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$=\frac{1}{192}(-32-24b+8b^2-b^3+96t-48t^2+8t^3)$$

and take the derivative of $P_r(T\leq t|\text{Case 1A4})$ with respect to T, the pdf is

$$f(t|\text{Case 1A4})=\frac{d}{dt}P_r(T\leq t|\text{Case 1A4})=\frac{1}{8}(-2+t)^2. \qquad \text{(J4)}$$

Case 1B: with $x_1 \geq x_2$

Because of the symmetry of x_1 and x_2, the pdf is the same as obtained for Case 1A.

Case 2: $x - x_1 \geq y$ and $x_2 - x \geq y$, then $T = x_2 - x_1$

Case 2A: with $x \leq (x_1 + x_2)/2$

Case 2A1: with $0 \leq t \leq b$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 2A1}) = \int_0^{1-t} \int_{x_1}^{x_1+t} \int_{x_1}^{1/2(x_1+x_2)} \int_0^{x-x_1} \frac{2}{b} dydxdx_2dx_1 +$

$\int_{1-t}^{1} \int_{x_1}^{1} \int_{x_1}^{1/2(x_1+x_2)} \int_0^{x-x_1} \frac{2}{b} dydxdx_2dx_1 = \frac{(4-3t)t^3}{48b}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 2A1})$ with respect to T, the pdf is

$$f(t|\text{Case 2A1}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 2A1}) = \frac{(1-t)t^2}{4b}. \tag{J5}$$

Case 2A2: with $b < t \leq 1$ and $x \leq x_1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 2A2})$

$$= \int_0^{1-t} \int_{x_1+b}^{x_1+t} \int_{x_1}^{x_1+b/2} \int_0^{x-x_1} \frac{2}{b} dydxdx_2dx_1$$

$$+ \int_{1-t}^{1-b} \int_{x_1+b}^{1} \int_{x_1}^{x_1+b/2} \int_0^{x-x_1} \frac{2}{b} dydxdx_2dx_1$$

$$+ \int_0^{1-b} \int_{x_1}^{x_1+b} \int_{x_1}^{1/2(x_1+x_2)} \int_0^{x-x_1} \frac{2}{b} dydxdx_2dx_1$$

$$+ \int_{1-b}^{1} \int_{x_1}^{1} \int_{x_1}^{1/2(x_1+x_2)} \int_0^{x-x_1} \frac{2}{b} dydxdx_2dx_1 = \frac{1}{48} b(-8b + 3b^2 - 6(-2+t)t)$$

and take the derivative of $P_r(T \leq t|\text{Case 2A2})$ with respect to T, the pdf is

$$f(t|\text{Case 2A2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 2A2}) = \frac{1}{4} b(1-t). \tag{J6}$$

Case 2A3: with $b < t \leq 1$ and $x > x_1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 2A3}) = \int_0^{1-t} \int_{x_1+b}^{x_1+t} \int_{x_1+b/2}^{1/2(x_1+x_2)} \int_0^{b/2} \frac{2}{b} dydxdx_2dx_1 +$

$\int_{1-t}^{1-b} \int_{x_1+b}^{1} \int_{x_1+b/2}^{1/2(x_1+x_2)} \int_0^{b/2} \frac{2}{b} dydxdx_2dx_1 = -\frac{1}{12}(-b+t)^2(-3+b+2t)$

and take the derivative of $P_r(T \leq t|\text{Case 2A3})$ with respect to T, the pdf is

$$f(t|\text{Case 2A3}) = \frac{d}{dt} P_r(T \leq t|\text{Case 2A3}) = \frac{1}{2}(b-t)(-1+t). \qquad \text{(J7)}$$

Case 2B: with $x > (x_1 + x_2)/2$

Case 2B1: with $x \geq x_2 - b/2$

Case 2B1A: with $0 \leq t \leq b$ and $x \leq x_1 + b/2$, the probability for T is

$$P_r(0 \leq T \leq t|\text{Case 2B1A}) = \int_0^{1-t} \int_{x_1}^{x_1+t} \int_{1/2(x_1+x_2)}^{x_2} \int_0^{x_2-x} \frac{2}{b} dydxdx_2dx_1 +$$

$$\int_{1-t}^{1} \int_{x_1}^{1} \int_{1/2(x_1+x_2)}^{x_2} \int_0^{x_2-x} \frac{2}{b} dydxdx_2dx_1 = \frac{(4-3t)t^3}{48b}$$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 2B1A})$ with respect to T, the pdf is

$$f(t|\text{Case 2B1A}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 2B1A}) = \frac{(1-t)t^2}{4b}. \qquad \text{(J8)}$$

Case 2B1B: with $b < t \leq 1$ and $x \leq x_1 + b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 2B1B}) = \int_0^{1-t} \int_{x_1+b}^{x_1+t} \int_{x_2-b/2}^{x_2} \int_0^{x_2-x} \frac{2}{b} dydxdx_2dx_1 +$$

$$\int_{1-t}^{1-b} \int_{x_1+b}^{1} \int_{x_2-b/2}^{x_2} \int_0^{x_2-x} \frac{2}{b} dydxdx_2dx_1 + \int_0^{1-b} \int_{x_1}^{x_1+b} \int_{1/2(x_1+x_2)}^{x_2} \int_0^{x_2-x} \frac{2}{b} dydxdx_2dx_1 +$$

$$\int_{1-b}^{1} \int_{x_1}^{1} \int_{1/2(x_1+x_2)}^{x_2} \int_0^{x_2-x} \frac{2}{b} dydxdx_2dx_1 = \frac{1}{48} b(-8b + 3b^2 - 6(-2+t)t)$$

and take the derivative of $P_r(T \leq t|\text{Case 2B1B})$ with respect to T, the pdf is

$$f(t|\text{Case 2B1B}) = \frac{d}{dt} P_r(T \leq t|\text{Case 2B1B}) = \frac{1}{4} b(1-t). \qquad \text{(J9)}$$

Case 2B2: with $b < t \leq 1$ and $x < x_2 - b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 2B2}) = \int_0^{1-t} \int_{x_1+b}^{x_1+t} \int_{1/2(x_1+x_2)}^{x_2-b/2} \int_0^{b/2} \frac{2}{b} dydxdx_2dx_1 +$$

$$\int_{1-t}^{1-b} \int_{x_1+b}^{1} \int_{1/2(x_1+x_2)}^{x_2-b/2} \int_0^{b/2} \frac{2}{b} dydxdx_2dx_1 = -\frac{1}{12}(-b+t)^2(-3+b+2t)$$

and take the derivative of $P_r(T \leq t|\text{Case 2B2})$ with respect to T, the pdf is

$$f(t|\text{Case 2B2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 2B2}) = \frac{1}{2}(b-t)(-1+t). \qquad \text{(J10)}$$

Case 3: $x - x_1 \geq y$ and $y \geq x - x_2$, then $T = x - x_1 + y$

Case 3A: with $0 \leq t \leq b/2$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 3A}) = \int_0^{t/2} \int_0^x \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx +$

$\int_{t/2}^1 \int_{x-t/2}^x \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx + \int_{t/2}^t \int_0^{x-t/2} \int_{2x-x_1-t}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx +$

$\int_t^1 \int_{x-t}^{x-t/2} \int_{2x-x_1-t}^x \int_{x-x_2}^{t+x_1-x} \frac{2}{b} dy dx_2 dx_1 dx = \frac{(2-t)t^3}{24b}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 3A})$ with respect to T, the pdf is

$$f(t|\text{Case 3A}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 3A}) = \frac{(3-2t)t^2}{12b}. \quad \text{(J11)}$$

Case 3B: with $b/2 < t \leq b$, the probability for T is

$P_r(T \leq t|\text{Case 3B}) = \int_0^{t-b/2} \int_0^x \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx +$

$\int_{t-b/2}^1 \int_{x+b/2-t}^x \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx + \int_{t-b/2}^{t/2} \int_0^{x+b/2-t} \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx +$

$\int_{t/2}^1 \int_{x-t/2}^{x+b/2-t} \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy x_2 dx_1 dx + \int_{t/2}^t \int_0^{x-t/2} \int_{2x-x_1-t}^x \int_{x-x_2}^{t+x_1-x} \frac{2}{b} dy dx_2 dx_1 dx +$

$\int_t^1 \int_{x-t}^{x-t/2} \int_{2x-x_1-t}^x \int_{x-x_2}^{t+x_1-x} \frac{2}{b} dy dx_2 dx_1 dx = \frac{(2-t)t^3}{24b}$

and take the derivative of $P_r(T \leq t|\text{Case 3B})$ with respect to T, the pdf is

$$f(t|\text{Case 3B}) = \frac{d}{dt} P_r(T \leq t|\text{Case 3B}) = \frac{(3-2t)t^2}{12b}. \quad \text{(J12)}$$

Case 3C: with $b < t \leq 1$, the probability for T is

$P_r(T \leq t|\text{Case 3C})$

$$= \int_0^{b/2} \int_0^x \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx + \int_{b/2}^1 \int_{x-b/2}^x \int_{x_1}^x \int_{x-x_2}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx$$

$$+ \int_{b/2}^{t/2} \int_0^{x-b/2} \int_{x-b/2}^x \int_{x-x_2}^{b/2} \frac{2}{b} dy dx_2 dx_1 dx$$

$$+\int_{t/2}^{1}\int_{x-t/2}^{x-b/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx+\int_{t/2}^{t-b/2}\int_{0}^{x-t/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t-b/2}^{1}\int_{x+b/2-t}^{x-t/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t-b/2}^{t}\int_{0}^{x+b/2-t}\int_{2x-x_1-t}^{x}\int_{x-x_2}^{t+x_1-x}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t}^{1}\int_{x-t}^{x+b/2-t}\int_{2x-x_1-t}^{x}\int_{x-x_2}^{t+x_1-x}\frac{2}{b}dydx_2dx_1dx=\frac{1}{24}b(2b-3t)(-2+t)$$

and take the derivative of $P_r(T\leq t|\text{Case 3C})$ with respect to T, the pdf is

$$f(t|\text{Case 3C})=\frac{d}{dt}P_r(T\leq t|\text{Case 3C})=\frac{1}{12}b(3+b-3t). \quad \text{(J13)}$$

Case 3D: with $1<t\leq 1+b/2$, the probability for T is

$P_r(T\leq t|\text{Case 3D})$

$$=\int_{0}^{b/2}\int_{0}^{x}\int_{x_1}^{x}\int_{x-x_2}^{x-x_1}\frac{2}{b}dydx_2dx_1dx+\int_{b/2}^{1}\int_{x-b/2}^{x}\int_{x_1}^{x}\int_{x-x_2}^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{b/2}^{t/2}\int_{0}^{x-b/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t/2}^{1}\int_{x-t/2}^{x-b/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t/2}^{t-b/2}\int_{0}^{x-t/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t-b/2}^{1}\int_{x+b/2-t}^{x-t/2}\int_{x-b/2}^{x}\int_{x-x_2}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t-b/2}^{1}\int_{0}^{x+b/2-t}\int_{2x-x_1-t}^{x}\int_{x-x_2}^{t+x_1-x}\frac{2}{b}dydx_2dx_1dx$$

$$=\frac{2b^3(-2+t)+2(-1+t)^4-3b^2(-2+t)t}{24b}$$

and take the derivative of $P_r(T \leq t|\text{Case 3D})$ with respect to T, the pdf is

$$f(t|\text{Case 3D}) = \frac{d}{dt} P_r(T \leq t|\text{Case 3D}) = \frac{(2+b-2t)^2(-1+b+t)}{12b}. \tag{J14}$$

Case 4: $x - x_1 \geq y$ and $y \geq x_2 - x$, then $T = x - x_1 + y$

Case 4A: with $0 \leq t \leq b$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 4A})$

$$= \int_0^{1-t} \int_{x_1+t/2}^{x_1+t} \int_x^{x_1+t} \int_{x_2-x}^{x_1-x+t} \frac{2}{b} dy dx_2 dx dx_1$$

$$+ \int_{1-t}^{1-t/2} \int_{x_1+t/2}^{1} \int_x^{1} \int_{x_2-x}^{x_1-x+t} \frac{2}{b} dy dx_2 dx dx_1$$

$$+ \int_0^{1-t} \int_{x_1}^{x_1+t/2} \int_x^{2x-x_1} \int_{x_2-x}^{x-x_1} \frac{2}{b} dy dx_2 dx dx_1$$

$$+ \int_{1-t}^{1} \int_{x_1}^{x_1/2+1/2} \int_x^{2x-x_1} \int_{x_2-x}^{x-x_1} \frac{2}{b} dy dx_2 dx dx_1$$

$$+ \int_{1-t/2}^{1} \int_{x-t/2}^{2x-1} \int_x^{1} \int_{x_2-x}^{x-x_1} \frac{2}{b} dy dx_2 dx_1 dx = \frac{(8-5t)t^3}{96b}$$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 4A})$ with respect to T, the pdf is

$$f(t|\text{Case 4A}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 4A}) = \frac{(6-5t)t^2}{24b}. \tag{J15}$$

Case 4B: with $b < t \leq 1$, the probability for T is

$P_r(T \leq t|\text{Case 4B})$

$$= \int_0^{1-t} \int_{x_1+t-b/2}^{x_1+t} \int_x^{x_1+t} \int_{x_2-x}^{x_1-x+t} \frac{2}{b} dy dx_2 dx dx_1$$

$$+ \int_{1-t}^{1+b/2-t} \int_{x_1+t-b/2}^{1} \int_x^{1} \int_{x_2-x}^{x_1-x+t} \frac{2}{b} dy dx_2 dx dx_1$$

$$+ \int_{b/2}^{t-b/2} \int_0^{x-b/2} \int_x^{x+b/2} \int_{x_2-x}^{b/2} \frac{2}{b} dy dx_2 dx_1 dx$$

$$+\int_{t-b/2}^{1-b/2}\int_{x+b/2-t}^{x-b/2}\int_{x}^{x+b/2}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+\int_{1-b/2}^{1}\int_{x+b/2-t}^{x-b/2}\int_{x}^{1}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{1-b}^{1}\int_{x_1}^{x_1/2+1/2}\int_{x}^{2x-x_1}\int_{x_2-x}^{x-x_1}\frac{2}{b}dydx_2dxdx_1+\int_{1-b/2}^{1}\int_{x-b/2}^{2x-1}\int_{x}^{1}\int_{x_2-x}^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$=\frac{1}{96}b(3b^2+4b(-4+t)-12(-2+t)t)$$

and take the derivative of $P_r(T\leq t|\text{Case 4B})$ with respect to T, the pdf is

$$f(\text{t}|\text{Case 4B})=\frac{d}{dt}P_r(T\leq t|\text{Case 4B})=\frac{1}{24}\text{b}(6+\text{b}-6\text{t}). \tag{J16}$$

Case 4C: with $1<t\leq 1+b/2$, the probability for T is

$P_r(T\leq t|\text{Case 4C})$

$$=\int_{0}^{1+b/2-t}\int_{x_1+t-b/2}^{1}\int_{x}^{1}\int_{x_2-x}^{x_1-x+t}\frac{2}{b}dydx_2dxdx_1$$

$$+\int_{b/2}^{1-b/2}\int_{0}^{x-b/2}\int_{x}^{x+b/2}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{1-b/2}^{t-b/2}\int_{0}^{x-b/2}\int_{x}^{1}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{t-b/2}^{1}\int_{x+b/2-t}^{x-b/2}\int_{x}^{1}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx$$

$$+\int_{0}^{1-b}\int_{x_1}^{x_1+b/2}\int_{x}^{2x-x_1}\int_{x_2-x}^{x-x_1}\frac{2}{b}dydx_2dxdx_1$$

$$+\int_{1-b}^{1}\int_{x_1}^{x_1/2+1/2}\int_{x}^{2x-x_1}\int_{x_2-x}^{x-x_1}\frac{2}{b}dydx_2dxdx_1$$

$$+\int_{1-b/2}^{1}\int_{x-b/2}^{2x-1}\int_{x}^{1}\int_{x_2-x}^{x-x_1}\frac{2}{b}dydx_2dx_1dx$$

$$=\frac{3b^4+4b^3(-4+t)+16b(-1+t)^3-8(-1+t)^4-12b^2(-2+t)t}{96b}$$

and take the derivative of $P_r(T \leq t|\text{Case 4C})$ with respect to T, the pdf is

$$f(t|\text{Case 4C}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4C}) = \frac{(2+b-2t)^3}{24b}. \tag{J17}$$

Case 11: $y \geq x - x_1$ and $y \geq x - x_2$, then $T = 2y$ and $T \in [0, b]$, the probability for T is

$P_r(0 < T \leq t|\text{Case 11})$

$$= \int_0^1 \int_{x_2}^{x_2+t/2} \int_{x_2}^{x} \int_{x-x_2}^{t/2} \frac{2}{b} dy dx_1 dx\, dx_2$$

$$+ \int_0^{1-t/2} \int_{x_2}^{x_2+t/2} \int_{x_2}^{x} \int_{x-x_2}^{t/2} \frac{2}{b} dy dx_1 dx\, dx_2 = \frac{(4-t)t^3}{96b}$$

and take the derivative of $P_r(0 < T \leq t|\text{Case 11})$ with respect to T, the pdf is

$$f(t|\text{Case 11}) = \frac{d}{dt} P_r(0 < T \leq t|\text{Case 11}) = \frac{(3-t)t^2}{24b}. \tag{J18}$$

Case 12: $y \geq x - x_1$ and $y \geq x_2 - x$, then $T = 2y$

Case 12A: with $x \geq (x_1 + x_2)/2$, then, $0 \leq t \leq b$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 12A})$

$$= \int_0^{1-t/2} \int_{x_1}^{x_1+t/2} \int_{1/2(x_1+x_2)}^{x_2} \int_{x-x_1}^{t/2} \frac{2}{b} dy dx\, dx_2 dx_1$$

$$+ \int_{1-t/2}^{1} \int_{x_1}^{1} \int_{1/2(x_1+x_2)}^{x_2} \int_{x-x_1}^{t/2} \frac{2}{b} dy dx\, dx_2 dx_1$$

$$+ \int_0^{1-t} \int_{x_1+t/2}^{x_1+t} \int_{1/2(x_1+x_2)}^{x_1+t/2} \int_{x-x_1}^{t/2} \frac{2}{b} dy dx\, dx_2 dx_1$$

$$+ \int_{1-t}^{1-t/2} \int_{x_1+t/2}^{1} \int_{1/2(x_1+x_2)}^{x_1+t/2} \int_{x-x_1}^{t/2} \frac{2}{b} dy dx\, dx_2 dx_1 = \frac{t^3(8-3t)}{192b}$$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 12A})$ with respect to T, the pdf is

$$f(t|\text{Case 12A}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 12A}) = \frac{(2-t)t^2}{16b}. \tag{J19}$$

Case 12B: with $< (x_1 + x_2)/2$, because of symmetry, the pdf is the same as the pdf in Case 12A.

In summary, from the derivations above, the pdf is as follows:

$f(t) =$

$$\begin{cases} \frac{1}{12b}(60t^2 - 43t^3) & 0 \le t \le b \\ \frac{1}{6}(2b^2 + 24t - 21t^2) & b < t \le 1 \\ \frac{1}{6b}(-8 + 12b + 6b^2 + 2b^3 + 24t - 12bt - 6b^2t - 24t^2 + 3bt^2 + 8t^3) & 1 < t \le 1 + b/2 \\ \frac{1}{2}(-2+t)^2 & 1 + b/2 < t \le 2 \end{cases}. \tag{J20}$$

Appendix K: Derivation of probability density function for path *fm* travel time with an infinite population and random storage

$$T = max(|x - x_1|, y + b/2) + max(|x - x_2|, y)$$

$$= \begin{cases} 2x - x_1 - x_2 & x \ge x_1, x \ge x_2, x - x_1 \ge y + b/2 \text{ and } x - x_2 \ge y & \text{Case 1} \\ x - x_1 + y & x \ge x_1, x \ge x_2, x - x_1 \ge y + b/2 \text{ and } x - x_2 < y & \text{Case 2} \\ x_2 - x_1 & x \ge x_1, x < x_2, x - x_1 \ge y + b/2 \text{ and } x_2 - x \ge y & \text{Case 3} \\ x - x_1 + y & x \ge x_1, x < x_2, x - x_1 \ge y + b/2 \text{ and } x_2 - x < y & \text{Case 4} \\ y + b/2 + x - x_2 & x \ge x_1, x \ge x_2, x - x_1 < y + b/2 \text{ and } x - x_2 \ge y & \text{Case 5} \\ 2y + b/2 & x \ge x_1, x \ge x_2, x - x_1 < y + b/2 \text{and } x - x_2 < y & \text{Case 6} \\ y + b/2 + x_2 - x & x \ge x_1, x < x_2, x - x_1 < y + b/2 \text{ and } x_2 - x \ge y & \text{Case 7} \\ 2y + b/2 & x \ge x_1, x < x_2, x - x_1 < y + b/2 \text{ and } x_2 - x < y & \text{Case 8} \\ x_1 - x_2 & x < x_1, x \ge x_2, x_1 - x \ge y + b/2 \text{ and } x - x_2 \ge y & \text{Case 9} \\ x_1 - x + y & x < x_1, x \ge x_2, x_1 - x \ge y + b/2 \text{ and } x - x_2 < y & \text{Case 10} \\ x_1 + x_2 - 2x & x < x_1, x < x_2, x_1 - x \ge y + b/2 \text{ and } x_2 - x \ge y & \text{Case 11} \\ x_1 - x + y & x < x_1, x < x_2, x_1 - x \ge y + b/2 \text{ and } x_2 - x < y & \text{Case 12} \\ y + b/2 + x - x_2 & x < x_1, x \ge x_2, x_1 - x < y + b/2 \text{ and } x - x_2 \ge y & \text{Case 13} \\ 2y + b/2 & x < x_1, x \ge x_2, x_1 - x < y + b/2 \text{ and } x - x_2 < y & \text{Case 14} \\ y + b/2 + x_2 - x & x < x_1, x < x_2, x_1 - x < y + b/2 \text{ and } x_2 - x \ge y & \text{Case 15} \\ 2y + b/2 & x < x_1, x < x_2, x_1 - x < y + b/2 \text{ and } x_2 - x < y & \text{Case 16} \end{cases}$$

where $x, x_1$and $x_2 \in$ unif[0, 1] and $y \in$ unif[0, $b/2$].

$f(x, x_1, x_2, y) = f(x)f(x_1)f(x_2)f(y) = \frac{2}{b}$

There are 16 cases to consider for deriving the pdfs. Because of symmetry, Case 1 has the same pdf as Case 11, Case 2 has the same pdf as Case 12, Case 3 has the same pdf as Case 9, Case 4 has the same pdf as Case 9, Case 5 has the same pdf as Case 15, Case 6 has the same pdf as Case 16, Case 7 has the same pdf as Case 13, Case 8 has the same pdf as Case 14. Therefore, we only need to consider the Cases 1, 2, 3, 4, 5, 6, 7 and 8 to finish the derivation process to get the pdf.

From the expression above, the value of T can be stated as follows:

Case 1: $x - x_1 \geq y + b/2$ and $x - x_2 \geq y$, then $T = 2x - x_1 - x_2$

Case 1A: with $x_1 \leq 2x - t$

Case 1A1: with $b \leq 1/2$

Case 1A1A: with $b/2 \leq t \leq b$, the probability for T is

$$P_r(b/2 \leq T \leq t|\text{Case 1A1A}) = \int_0^{(t-b/2)/2} \int_{y+b/2}^{-y+t} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$$

$$\int_0^{(t-b/2)/2} \int_{-y+t}^{1} \int_{x+y-t}^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy = \frac{(b-2t)^3(-16+5b+6t)}{768b}$$

and take the derivative of $P_r(b/2 \leq T \leq t|\text{Case 1A1A})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1A}) = \frac{d}{dt} P_r(b/2 \leq T \leq t|\text{Case 1A1A}) = \frac{(b-2t)^2(4-b-2t)}{32b}. \qquad \text{(K1)}$$

Case 1A1B: with $b < t \leq 3b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 1A1B}) = \int_0^{(t-b)/2} \int_{t/2}^{t-y-b/2} \int_0^{2x-t} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$$

$$\int_0^{(t-b)/2} \int_{t-y-b/2}^{t-y} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$$

$$\int_{(t-b)/2}^{(t-b/2)/2} \int_{y+b/2}^{t-y} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$$

$$\int_0^{(t-b/2)/2} \int_{t-y}^{1} \int_{x+y-t}^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy =$$

$$\frac{-3b^4+8(16-11t)t^3+32bt^2(-6+5t)-24b^2t(-4+5t)+8b^3(-2+5t)}{768b}$$

and take the derivative of $P_r(T \leq t|\text{Case 1A1B})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1B}) = \frac{d}{dt}P_r(T \le t|\text{Case 1A1B}) = \frac{5b^3+4(12-11t)t^2+12bt(-4+5t)-6b^2(-2+5t)}{96b}. \quad \text{(K2)}$$

Case 1A1C: with $3b/2 < t \le 2b$, the probability for T is

$P_r(T \le t|\text{Case 1A1C}) = \int_0^{(t-b)/2}\int_{t/2}^{t-y-b/2}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_0^{(t-b)/2}\int_{t-y-b/2}^{t-y}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_{(t-b)/2}^{b/2}\int_{y+b/2}^{t-y}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy + \int_0^{b/2}\int_{t-y}^{1}\int_{x+y-t}^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy =$

$\frac{-24b^4+48bt^2-5t^4-6b^2t(16+t)+4b^3(13+8t)}{96b}$

and take the derivative of $P_r(T \le t|\text{Case 1A1C})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1C}) = \frac{d}{dt}P_r(T \le t|\text{Case 1A1C}) = \frac{b^2}{3} + t - \frac{5t^3}{24b} - \frac{1}{8}b(8+t). \quad \text{(K3)}$$

Case 1A1D: with $2b < t \le 1$, the probability for T is

$P_r(T \le t|\text{Case 1A1D}) = \int_0^{b/2}\int_{t/2}^{t-y-b/2}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_0^{b/2}\int_{t-y-b/2}^{t-y}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy + \int_0^{b/2}\int_{t-y}^{1}\int_{x+y-t}^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy =$

$\frac{1}{48}(12b^3 + b^2(26-40t) + 6(4-3t)t^2 + 3bt(-16+15t))$

and take the derivative of $P_r(T \le t|\text{Case 1A1D})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1D}) = \frac{d}{dt}P_r(T \le t|\text{Case 1A1D}) = -\frac{5b^2}{6} + t - \frac{9t^2}{8} + b\left(-1 + \frac{15t}{8}\right). \quad \text{(K4)}$$

Case 1A1E: with $1 < t \le 1 + b/2$, the probability for T is

$P_r(T \le t|\text{Case 1A1E}) = \int_0^{b/2}\int_{t/2}^{t-y-b/2}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_0^{t-1}\int_{t-y-b/2}^{1}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_{t-1}^{b/2}\int_{t-y-b/2}^{t-y}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy + \int_{t-1}^{b/2}\int_{t-y}^{1}\int_{x+y-t}^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy =$

$\frac{1}{48b}(12b^4 + b^3(26-40t) + 4(-1+t)^4 - 6bt^2(-4+3t) + 3b^2t(-16+15t))$

and take the derivative of $P_r(T \leq t|\text{Case 1A1E})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1E}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A1E}) = \frac{1}{24\text{b}}(-20b^3 + 8(-1+t)^3 - 3bt(-8+9t) + 3b^2(-8+15t)). \quad \text{(K5)}$$

Case 1A1F: with $1 + b/2 < t \leq 1 + b$, the probability for T is

$$P_r(T \leq t|\text{Case 1A1F}) = \int_0^{t-1-b/2}\int_{t/2}^{1}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy + \int_{t-1-b/2}^{b/2}\int_{t/2}^{t-y-b/2}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy + \int_{t-1-b/2}^{b/2}\int_{t-y-b/2}^{1}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{48b}\left(12b^4 + 4(-1+t)^3(3+t) - 8b^3(-3+5t) + 3b^2(-4-12t+15t^2) - 6b(4-8t+3t^3)\right)$$

and take the derivative of $P_r(T \leq t|\text{Case 1A1F})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1F}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A1F}) = \frac{1}{24\text{b}}(-20b^3 + 9b^2(-2+5t) - 3b(-8+9t^2) + 8(2-3t+t^3)). \quad \text{(K6)}$$

Case 1A1G: with $1 + b < t \leq 2$, the probability for T is

$$P_r(T \leq t|\text{Case 1A1G}) = \int_0^{b/2}\int_{t/2}^{1}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{16}((2t-b)(2-t)^2)$$

and take the derivative of $P_r(T \leq t|\text{Case 1A1G})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1G}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A1G}) = \frac{1}{8}(2-t)(2+b-3t). \quad \text{(K7)}$$

Case 1A2: with $1/2 < b \leq 2/3$

Case 1A2A: with $b/2 < t \leq b$, the pdf is the same as $b/2 < b \leq b$ in Case 1A1A, therefore, the pdf is

$$f(t|\text{Case 1A2A}) = \frac{(b-2t)^2(4-b-2t)}{32b}. \quad \text{(K8)}$$

Case 1A2B: with $b < t \leq 3b/2$, then the pdf is the same as $b < t \leq 3b/2$ in Case 1A1B, therefore, the pdf is

$$f(t|\text{Case 1A2B}) = \frac{5b^3+4(12-11t)t^2+12bt(-4+5t)-6b^2(-2+5t)}{96b}. \tag{K9}$$

Case 1A2C: with $3b/2 < t \leq 1$, then the pdf is the same as $3b/2 < t \leq 2b$ in Case 1A1C, therefore, the pdf is

$$f(t|\text{Case 1A2C}) = \frac{b^2}{3} + t - \frac{5t^3}{24b} - \frac{1}{8}b(8+t). \tag{K10}$$

Case 1A2D: with $1 < t \leq 2b$, the probability for T is

$$P_r(T \leq t|\text{Case 1A2D}) = \int_0^{(t-b)/2}\int_{t/2}^{t-y-b/2}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$$

$$\int_0^{t-1}\int_{t-y-b/2}^{1}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$$

$$\int_{t-1}^{(t-b)/2}\int_{t-y-b/2}^{t-y}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$$

$$\int_{(t-b)/2}^{b/2}\int_{y+b/2}^{t-y}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy + \int_{t-1}^{b/2}\int_{t-y}^{1}\int_{x+y-t}^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy =$$

$$\frac{1}{96b}\left(8-24b^4-32t+48t^2+48bt^2-32t^3+3t^4-6b^2t(16+t)+4b^3(13+8t)\right)$$

and take the derivative of $P_r(T \leq t|\text{Case 1A2D})$ with respect to T, the pdf is

$$f(t|\text{Case 1A2D}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A2D}) = \frac{1}{24\text{b}}\left(-8+8b^3+24t+24bt-24t^2+3t^3-3b^2(8+t)\right). \tag{K11}$$

Case 1A2E: with $2b < t \leq 1 + b/2$, then the pdf is the same as $1 < t \leq 1 + b/2$ in Case 1A1E, therefore, the pdf is

$$f(t|\text{Case 1A2E}) = \frac{1}{24\text{b}}(-20b^3+8(-1+t)^3-3bt(-8+9t)+3b^2(-8+15t)). \tag{K12}$$

Case 1A2F: with $1 + b/2 < t \leq 1 + b$, then the pdf is the same as $1 + b/2 < t \leq 1 + b$ in Case 1A1F, therefore, the pdf is

$$f(t|\text{Case 1A2F}) = \frac{1}{24\text{b}}(-20b^3 + 9b^2(-2 + 5t) - 3b(-8 + 9t^2) + 8(2 - 3t + t^3)).\text{(K13)}$$

Case 1A2G: with $1 + b < t \leq 2$, then the pdf is the same as $1 + b < t \leq 2$ in Case 1A1G, therefore, the pdf is

$$f(t|\text{Case 1A2G}) = \frac{1}{8}(2 - t)(2 + b - 3t). \quad \text{(K14)}$$

Case 1A3: with $2/3 < b \leq 1$

Case 1A3A: with $b/2 < t \leq b$, then the pdf is the same as $b/2 < t \leq b$ in Case 1A2A, therefore, the pdf is

$$f(t|\text{Case 1A3A}) = \frac{(b-2t)^2(4-b-2t)}{32b}. \quad \text{(K15)}$$

Case 1A3B: with $b < t \leq 1$, then the pdf is the same as $b < t \leq 3b/2$ when $1/2 < b \leq 2/3$.

When $1 < t \leq 3b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 1A3B}) = \int_0^{(t-b)/2}\int_{t/2}^{t-y-b/2}\int_0^{2x-t}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy +$$

$$\int_{t/2}^{t-b/2}\int_{t-x-b/2}^{x-b/2}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dydx +$$

$$\int_{t-b/2}^{(t+b/2)/2}\int_0^{x-b/2}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dydx +$$

$$+\int_{(t+b/2)/2}^{1}\int_0^{t-x}\int_0^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dydx +$$

$$\int_{t-1}^{(t-b/2)/2}\int_{t-y}^{1}\int_{x+y-t}^{x-y-b/2}\int_{2x-x_1-t}^{x-y}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{768b}\big(-3b^4 + 32bt^2(-6 + 5t) -$$

$$24b^2t(-4 + 5t) + 8b^3(-2 + 5t) - 8(-8 + 32t - 48t^2 + 16t^3 + 3t^4)\big)$$

and take the derivative of $P_r(T \leq t|\text{Case 1A3B})$ with respect to T, the pdf is

$$f(t|\text{Case 1A3B}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1A3B}) = \frac{1}{96\text{b}}\big(5b^3 + 12bt(-4 + 5t) - 6b^2(-2 + 5t) -$$

$$4(8 - 24t + 12t^2 + 3t^3)\big). \quad \text{(K16)}$$

Case 1A3C: with $3b/2 < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 1A3C}) = \int_0^{(t-b)/2} \int_{t/2}^{t-y-b/2} \int_0^{2x-t} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$

$\int_{t/2}^{b} \int_{t-x-b/2}^{x-b/2} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dy dx +$

$\int_{b}^{t-b/2} \int_{t-x-b/2}^{b/2} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dy dx +$

$\int_{t-b/2}^{1} \int_0^{t-x} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dy dx + \int_{t-1}^{b/2} \int_{t-y}^{1} \int_{x+y-t}^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy =$

$\frac{1}{96b}(8 - 24b^4 - 32t + 48t^2 + 48bt^2 - 32t^3 + 3t^4 - 6b^2 t(16 + t) + 4b^3(13 + 8t))$

and take the derivative of $P_r(T \leq t|\text{Case 1A3C})$ with respect to T, the pdf is

$f(t|\text{Case 1A3C}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1A3C}) = \frac{1}{24b}\big(-8 + 8b^3 + 24t + 24bt - 24t^2 + 3t^3 -$

$3b^2(8 + t)\big).$ (K17)

Case 1A3D: with $1 + b/2 < t \leq 2b$, the probability for T is

$P_r(T \leq t|\text{Case 1A3D}) = \int_{t/2}^{1} \int_0^{t-x-b/2} \int_0^{2x-t} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dy dx +$

$\int_{t-1-b/2}^{(t-b)/2} \int_{t-y-b/2}^{1} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$

$\int_{(t-b)/2}^{b/2} \int_{y+b/2}^{1} \int_0^{x-y-b/2} \int_{2x-x_1-t}^{x-y} \frac{2}{b} dx_2 dx_1 dx dy = \frac{1}{96b}(-24 - 24b^4 + 64t - 48t^2 + 3t^4 +$

$48b(-1 + 2t) + 16b^3(3 + 2t) - 6b^2(4 + 12t + t^2))$

and take the derivative of $P_r(T \leq t|\text{Case 1A3D})$ with respect to T, the pdf is

$f(t|\text{Case 1A3D}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1A3D}) = \frac{1}{24b}\big(16 + 24b + 8b^3 - 24t + 3t^3 - 3b^2(6 +$

$t)\big).$ (K18)

Case 1A3E: with $2b < t \leq 1 + b$, then the pdf is the same as $1 + b/2 < t \leq 1 + b$ in Case 1A2F, therefore, the pdf is

$f(t|\text{Case 1A3E}) = \frac{1}{24\text{b}}(-20b^3 + 9b^2(-2 + 5t) - 3b(-8 + 9t^2) + 8(2 - 3t + t^3))$.(K19)

Case 1A3F: with $1+b<t\le 2$, then the pdf is the same as $1+b<t\le 2$ in Case 1A2G, therefore, the pdf is

$$f(t|\text{Case 1A3F}) = \frac{1}{8}(2-t)(2+b-3t). \tag{K20}$$

Case 1B: with $x_1 > 2x - t$

Case 1B1: with $0 \le b \le 2/3$

Case 1B1A: with $b < t \le 2b$, the probability for T is

$P_r(b<T\le t|\text{Case 1B}) = \int_0^{(t-b)/2}\int_{y+b/2}^{t/2}\int_0^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_0^{(t-b)/2}\int_{t/2}^{-y+t-b/2}\int_{2x-t}^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{96b}(t-b)^3(b+3t)$

and take the derivative of $P_r(b<T\le t|\text{Case 1B1A})$ with respect to T, the pdf is

$$f(t|\text{Case 1B1A}) = \frac{d}{dt}P_r(b<T\le t|\text{Case 1B1A}) = \frac{1}{8b}(t-b)^2t. \tag{K21}$$

Case 1B1B: with $2b < t \le 1 + b/2$, the probability for T is

$$\begin{aligned} &P_r(T\le t|\text{Case 1B1B}) \\ &= \int_0^{b/2}\int_{y+b/2}^{t/2}\int_0^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy \\ &+ \int_0^{b/2}\int_{t/2}^{-y+t-b/2}\int_{2x-t}^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy \\ &= \frac{1}{96}(-17b^3+48b^2t-42bt^2+12t^3) \end{aligned}$$

and take the derivative of $P_r(T\le t|\text{Case 1B1B})$ with respect to T, the pdf is

$$f(t|\text{Case 1B1B}) = \frac{d}{dt}P_r(T\le t|\text{Case 1B1B}) = \frac{1}{8}(4b^2-7bt+3t^2). \tag{K22}$$

Case 1B1C: with $1 + b/2 < t \le 1 + b$, the probability for T is

$P_r(T \le t|\text{Case 1B1C}) = \int_0^{b/2}\int_{y+b/2}^{t/2}\int_0^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_0^{t-1-b/2}\int_{t/2}^{1}\int_{2x-t}^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_{t-1-b/2}^{b/2}\int_{t/2}^{-y+t-b/2}\int_{2x-t}^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{96b}\big(-17b^4 - 32(-1+t)^3 + b^3(4 +$

$48t) - 6b^2(-4 + 4t + 7t^2) + 12b(4 - 8t + 4t^2 + t^3)\big)$

and take the derivative of $P_r(T \le t|\text{Case 1B1C})$ with respect to T, the pdf is

$f(t|\text{Case 1B1C}) = \frac{d}{dt}P_r(T \le t|\text{Case 1B1C}) = \frac{1}{8b}\big(4b^3 - 8(-1+t)^2 - b^2(2+7t) + b(-8 +$

$8t + 3t^2)\big)$. (K23)

Case 1B1D: with $1 + b < t \le 2$, the probability for T is

$$
\begin{aligned}
&P_r(T \le t|\text{Case 1B1D})\\
&= \int_0^{b/2}\int_{y+b/2}^{t/2}\int_0^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy\\
&+ \int_0^{b/2}\int_{t/2}^{1}\int_{2x-t}^{x-y-b/2}\int_0^{x-y}\frac{2}{b}dx_2dx_1dxdy\\
&= \frac{1}{96}\big(20b^2 - b^3 + 6b(-4 - 4t + t^2) - 4(8 - 12t + t^3)\big)
\end{aligned}
$$

and take the derivative of $P_r(T \le t|\text{Case 1B1D})$ with respect to T, the pdf is

$$f(t|\text{Case 1B1D}) = \frac{d}{dt}P_r(T \le t|\text{Case 1B1D}) = \frac{1}{8}(2 + t - b)(2 - t). \quad \text{(K24)}$$

Case 1B2: with $2/3 < b \le 1$

Case 1B2A: with $b < t \le 1 + b/2$, then the pdf is the same as $b < t \le 2b$ in Case 1B1A, therefore, the pdf is

$$f(t|\text{Case 1B2A}) = \frac{1}{8b}(t - b)^2 t. \quad \text{(K25)}$$

Case 1B2B: with $1 + b/2 < t \leq 2b$, the probability for T is

$$P_r(T \leq t|\text{Case 1B2B}) = \int_0^{(t-b)/2} \int_{y+b/2}^{t/2} \int_0^{x-y-b/2} \int_0^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$$

$$\int_0^{t-1-b/2} \int_{t/2}^{1} \int_{2x-t}^{x-y-b/2} \int_0^{x-y} \frac{2}{b} dx_2 dx_1 dx dy +$$

$$\int_{t-1-b/2}^{(t-b)/2} \int_{t/2}^{-y+t-b/2} \int_{2x-t}^{x-y-b/2} \int_0^{x-y} \frac{2}{b} dx_2 dx_1 dx dy = \frac{1}{96b}\big(32 + 4b^3 - b^4 + 6b^2(-2+t)^2 -$$

$$96t + 96t^2 - 32t^3 + 3t^4 - 8b(-6 + 12t - 6t^2 + t^3)\big)$$

and take the derivative of $P_r(T \leq t|\text{Case 1B2B})$ with respect to T, the pdf is

$$f(t|\text{Case 1B2B}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1B2B}) = \frac{1}{8b}(-2+t)(4 + b^2 - 2b(-2+t) - 6t +$$

$$t^2). \tag{K26}$$

Case 1B2C: with $2b < t \leq 1 + b$, then the pdf is the same as $1 + b/2 < t \leq 1 + b$ in Case 1B1C, therefore, the pdf is

$$f(t|\text{Case 1B2C}) = \frac{1}{8b}\big(4b^3 - 8(-1+t)^2 - b^2(2+7t) + b(-8 + 8t + 3t^2)\big). \tag{K27}$$

Case 1B2D: with $1 + b < t \leq 2$, then the pdf is the same as $1 + b < t \leq 2$ in Case 1B1D, therefore, the pdf is

$$f(t|\text{Case 1B2D}) = \frac{1}{8}(2 + t - b)(2 - t). \tag{K28}$$

Case 2: $x - x_1 \geq y + b/2$ and $x - x_2 < y$, then $T = x - x_1 + y$

Case 2A: with $0 \leq b \leq 2/3$

Case 2A1: with $b/2 \leq t \leq 3b/2$, the probability for T is

$$P_r(b/2 < T \leq t|\text{Case 2A1}) = \int_0^{(t-b/2)/2} \int_0^{y+1-t} \int_{x_1+y+b/2}^{x_1-y+t} \int_{x-y}^{x} \frac{2}{b} dx_2 dx dx_1 dy +$$

$$\int_0^{(t-b/2)/2} \int_{y+1-t}^{-y+1-b/2} \int_{x_1+y+b/2}^{1} \int_{x-y}^{x} \frac{2}{b} dx_2 dx dx_1 dy = \frac{1}{384b}(b-2t)^3(-4+b+2t)$$

and take the derivative of $P_r(b/2 < T \leq t|\text{Case 2A1})$ with respect to T, the pdf is

$$f(t|\text{Case 2A1}) = \frac{d}{dt} P_r(b/2 < T \leq t|\text{Case 2A1}) = \frac{1}{96b}(2t-b)^2(6-b-4t). \tag{K29}$$

Case 2A2: with $3b/2 < t \leq 1$, the probability for T is

$$P_r(T \leq t|\text{Case 2A2}) = \int_0^{b/2}\int_0^{y+1-t}\int_{x_1+y+b/2}^{x_1-y+t}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy +$$

$$\int_0^{b/2}\int_{y+1-t}^{-y+1-b/2}\int_{x_1+y+b/2}^{1}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy = \frac{1}{96}b(7b^2 - 12(-2+t)t + 4b(-7+2t))$$

and take the derivative of $P_r(T \leq t|\text{Case 2A2})$ with respect to T, the pdf is

$$f(t|\text{Case2A2}) = \frac{d}{dt}P_r(T \leq t|\text{Case2A2}) = \frac{1}{12}b(3+b-3t). \tag{K30}$$

Case 2A3: with $1 < t \leq 1 + b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 2A3}) = \int_{t-1}^{b/2}\int_0^{y+1-t}\int_{x_1+y+b/2}^{x_1-y+t}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy +$$

$$\int_0^{t-1}\int_0^{-y+1-b/2}\int_{x_1+y+b/2}^{1}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy +$$

$$\int_{t-1}^{b/2}\int_{y+1-t}^{-y+1-b/2}\int_{x_1+y+b/2}^{1}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy = \frac{1}{96b}(7b^4 + 8(-1+t)^4 - 12b^2(-2+t)t +$$

$$4b^3(-7+2t))$$

and take the derivative of $P_r(T \leq t|\text{Case2A3})$ with respect to T, the pdf is

$$f(t|\text{Case2A3}) = \frac{d}{dt}P_r(T \leq t|\text{Case2A3}) = \frac{1}{12b}(2+b-2t)^2(-1+b+t). \tag{K31}$$

Case 2B: with $2/3 < b \leq 1$

Case 2B1: with $b/2 \leq t \leq 1$, then the pdf is the same as $b/2 \leq t \leq 3b/2$ in Case 2A1, therefore, the pdf is

$$f(t|\text{Case 2B1}) = \frac{1}{96b}(2t-b)^2(6-b-4t). \tag{K32}$$

Case 2B2: with $1 < t \leq 3b/2$, the probability for T is

$$P_r(T \leq t|\text{Case2B2}) = \int_{t-1}^{b/2}\int_0^{y+1-t}\int_{x_1+y+b/2}^{x_1-+t}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy +$$

$$\int_0^{t-1}\int_0^{-y+1-b/2}\int_{x_1+y+b/2}^{1}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy +$$

$\int_{t-1}^{(t-b/2)/2}\int_{y+1-t}^{-y+1-b/2}\int_{x_1+y+b/2}^{1}\int_{x-y}^{x}\frac{2}{b}dx_2dxdx_1dy = \frac{1}{384b}(b^4+24b^2t+16b(-3+t)t^2-$

$4b^3(1+t)+16(2-8t+12t^2-6t^3+t^4))$

and take the derivative of $P_r(T \leq t|\text{Case2B2})$ with respect to T, the pdf is

$$f(t|\text{Case2B2}) = \frac{d}{dt}P_r(T \leq t|\text{Case2B2}) = \frac{1}{96b}(4t-2-b)(2t+b-4)^2. \qquad \text{(K33)}$$

Case 2B3: with $3b/2 < t \leq 1+b/2$, then the pdf is the same as $1 < t \leq 1+b/2$ in Case 2A3, therefore, the pdf is

$$f(t|\text{Case2B3}) = \frac{1}{12b}(2+b-2t)^2(-1+b+t). \qquad \text{(K34)}$$

Case 3: $x-x_1 \geq y+b/2$ and $x_2-x \geq y$, then $T = x_2-x_1$
Case 3A: with $0 \leq b \leq 2/3$

Case 3A1: with $b/2 \leq t \leq 3b/2$, the probability for T is

$$P_r(b/2 < T \leq t|\text{Case 3A1})$$

$$= \int_0^{1-t}\int_{x_1+b/2}^{x_1+t}\int_0^{(x_2-x_1-b/2)/2}\int_{x_1+y+b/2}^{x_2-y}\frac{2}{b}dxdydx_2dx_1$$

$$+\int_{1-t}^{1-b/2}\int_{x_1+b/2}^{1}\int_0^{(x_2-x_1-b/2)/2}\int_{x_1+y+b/2}^{x_2-y}\frac{2}{b}dxdydx_2dx_1$$

$$= \frac{1}{384b}(b-2t)^3(-8+b+6t)$$

and take the derivative of $P_r(b/2 < T \leq t|\text{Case 3A1})$ with respect to T, the pdf is

$$f(t|\text{Case 3A1}) = \frac{d}{dt}P_r(b/2 < T \leq t|\text{Case 3A1}) = \frac{1}{8b}(2t-b)^2(1-t). \qquad \text{(K35)}$$

Case 3A2: with $3b/2 < t \leq 1$, the probability for T is

$P_r(T \leq t|\text{Case 3A2}) = \int_0^{1-3b/2}\int_{x_1+b/2}^{x_1+3b/2}\int_0^{(x_2-x_1-b/2)/2}\int_{x_1+y+b/2}^{x_2-y}\frac{2}{b}dxdydx_2dx_1 +$

$\int_{1-3b/2}^{1-b/2}\int_{x_1+b/2}^{1}\int_0^{(x_2-x_1-b/2)/2}\int_{x_1+y+b/2}^{x_2-y}\frac{2}{b}dxdydx_2dx_1 +$

$\int_0^{1-b/2}\int_{x_1+3b/2}^{x_1+t}\int_0^{b/2}\int_{x_1+y+b/2}^{x_2-y}\frac{2}{b}dxdydx_2dx_1 +$

$\int_{1-t}^{1-3b/2}\int_{x_1+3b/2}^{1}\int_0^{b/2}\int_{x_1+y+b/2}^{x_2-y}\frac{2}{b}dxdydx_2dx_1 = \frac{1}{24}(13b^2-5b^3+12b(-2+t)t+4(3-2t)t^2)$

and take the derivative of $P_r(T \le t|\text{Case 3A2})$ with respect to T, the pdf is

$$f(t|\text{Case 3A2}) = \frac{d}{dt}P_r(T \le t|\text{Case 3A2}) = (t-b)(1-t). \qquad \text{(K36)}$$

Case 3B: with $2/3 < b \le 1$, with $b/2 < t \le 1$, then the pdf is the same as $b/2 \le t \le 3b/2$ in Case 2A, therefore, the pdf is

$$f(t|\text{Case 3B}) = \frac{1}{8b}(2t-b)^2(1-t). \qquad \text{(K37)}$$

Case 4: $x - x_1 \ge y + b/2$and $x_2 - x < y$, then $T = x - x_1 + y$

Case 4A: with $0 \le b \le 2/3$

Case 4A1: with $b/2 \le t \le 3b/2$, the probability for T is

$P_r(b/2 \le T \le t|\text{Case 4A1}) = \int_0^{1-t}\int_{x_1+b/2}^{x_1+(t+b/2)/2}\int^{2x-x_1-b/2}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dxdx_1 +$

$\int_{1-t}^{1-b/2}\int_{x_1+b/2}^{(x_1+1+b/2)/2}\int_x^{2x-x_1-b/2}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dxdx_1 +$

$\int_{1+b/4-t/2}^{1}\int_{x-(t+b/2)/2}^{2x-(1+b/2)}\int_x^1\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dx_1dx +$

$\int_0^{1-t}\int_{x_1+(t+b/2)/2}^{1}\int_x^{x_1+t}\int_{x_2-x}^{x_1-x+t}\frac{2}{b}dydx_2dxdx_1 +$

$\int_{1-t}^{1-(t+b/2)/2}\int_{x_1+(t+b/2)/2}^{1}\int_x^1\int_{x_2-x}^{x_1-x+t}\frac{2}{b}dydx_2dxdx_1 = \frac{1}{1536b}(b-2t)^3(-16+3b+10t)$

and take the derivative of $P_r(b/2 \le T \le t|\text{Case 4A1})$ with respect to T, the pdf is

$$f(t|\text{Case 4A1}) = \frac{d}{dt}P_r(b/2 \le T \le t|\text{Case 4A1}) = \frac{1}{192b}(2t-b)^2(12-b-10t). \qquad \text{(K38)}$$

Case 4A2: with $3b/2 < t \le 1$, the probability for T is

$P_r(T \le t|\text{Case 4A2}) = \int_0^{1-3b/2}\int_{x_1+b/2}^{x_1+b}\int_x^{2x-x_1-b/2}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dxdx_1 +$

$\int_{1-3b/2}^{1-b/2}\int_{x_1+b/2}^{(x_1+1+b/2)/2}\int_x^{2x-x_1-b/2}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dxdx_1 +$

$\int_{1-b/2}^{1}\int_{x-b}^{2x-(1+b/2)}\int_{x}^{1}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dx_1dx+\int_{b}^{t-b/2}\int_{0}^{x-b}\int_{x}^{x+b/2}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+$

$\int_{t-b/2}^{1-b/2}\int_{x+b/2-t}^{x-b}\int_{x}^{x+b/2}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+\int_{1-b/2}^{1}\int_{x+b/2-t}^{x-b}\int_{x}^{1}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+$

$\int_{0}^{1-t}\int_{x_1+t-b/2}^{x_1+t}\int_{x}^{x_1+t}\int_{x_2-x}^{x_1-x+t}\frac{2}{b}dydx_2dxdx_1+\int_{1-t}^{1+b/2-t}\int_{x_1+t-b/2}^{1}\int_{x}^{1}\int_{x_2-x}^{x_1-x+t}\frac{2}{b}dydx_2dxdx_1=$

$\frac{1}{24}b(3b^2+b(-7+t)-3(-2+t)t)$

and take the derivative of $P_r(T\le t|\text{Case 4A2})$ with respect to T, the pdf is

$$f(t|\text{Case 4A2})=\frac{d}{dt}P_r(T\le t|\text{Case 4A2})=\frac{1}{24}b(6+b-6t). \qquad \text{(K39)}$$

Case 4A3: with $1<t\le 1+b/2$, the probability for T is

$P_r(T\le t|\text{Case 4A3})=\int_{0}^{1-3b/2}\int_{x_1+b/2}^{x_1+b}\int_{x}^{2x-x_1-b/2}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dxdx_1+$

$\int_{1-3b/2}^{1-b/2}\int_{x_1+b/2}^{(x_1+1+b/2)/2}\int_{x}^{2x-x_1-b/2}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dxdx_1+$

$\int_{1-b/2}^{1}\int_{x-b}^{2x-(1+b/2)}\int_{x}^{1}\int_{x_2-x}^{x-x_1-b/2}\frac{2}{b}dydx_2dx_1dx+\int_{b}^{1-b/2}\int_{0}^{x-b}\int_{x}^{x+b/2}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+$

$\int_{1-b/2}^{t-b/2}\int_{0}^{x-b}\int_{x}^{1}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+\int_{t-b/2}^{1}\int_{x+b/2-t}^{x-b}\int_{x}^{1}\int_{x_2-x}^{b/2}\frac{2}{b}dydx_2dx_1dx+$

$\int_{0}^{1+b/2-t}\int_{x_1+t-b/2}^{1}\int_{x}^{1}\int_{x_2-x}^{x_1-x+t}\frac{2}{b}dydx_2dxdx_1=\frac{1}{24b}(3b^4+b^3(-7+t)+4b(-1+t)^3-$

$2(-1+t)^4-3b^2(-2+t)t)$

and take the derivative of $P_r(T\le t|\text{Case 4A3})$ with respect to T, the pdf is

$$f(t|\text{Case 4A3})=\frac{d}{dt}P_r(T\le t|\text{Case 4A3})=\frac{1}{24b}(2+b-2t)^3. \qquad \text{(K40)}$$

Case 4B: with $2/3<b\le 1$

Case 4B1: with $b/2<t\le 1$, then the pdf is the same as $b/2\le t\le 3b/2$ in Case 4A1, therefore, the pdf is

$$f(t|\text{Case 4B1})=\frac{1}{192b}(2t-b)^2(12-b-10t). \qquad \text{(K41)}$$

Case 4B2: with $1 < t \leq 3b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 4B2}) = \int_0^{1-b/2} \int_{x_1+b/2}^{(x_1+1+b/2)/2} \int_x^{2x-x_1-b/2} \int_{x_2-x}^{x-x_1-b/2} \frac{2}{b} dy dx_2 dx dx_1 +$$

$$\int_0^{1-(t+b/2)/2} \int_{(x_1+1+b/2)/2}^{x_1+(t+b/2)/2} \int_x^1 \int_{x_2-x}^{x-x_1-b/2} \frac{2}{b} dy dx_2 dx dx_1 +$$

$$\int_{1-(t+b/2)/2}^{1-b/2} \int_{(x_1+1+b/2)/2}^{1} \int_x^1 \int_{x_2-x}^{x-x_1-b/2} \frac{2}{b} dy dx_2 dx dx_1 +$$

$$\int_0^{1-(t+b/2)/2} \int_{x_1+(t+b/2)/2}^{1} \int_x^1 \int_{x_2-x}^{x_1-x+t} \frac{2}{b} dy dx_2 dx dx_1 = \frac{1}{1536b}(3b^4 - 24b^2(-4+t)t - 8b^3(2+$$

$$t) - 32b(-4+12t-6t^2+t^3) - 16(12-32t+24t^2-8t^3+t^4))$$

and take the derivative of $P_r(T \leq t|\text{Case 4B2})$ with respect to T, the pdf is

$$f(t|\text{Case 4B2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4B2}) = \frac{1}{192b}(4-b-2t)^3. \quad \text{(K42)}$$

Case 4B3: with $3b/2 < t \leq 1 + b/2$, then the pdf is the same as $1 \leq t \leq 1 + b/2$ in Case 4A3, therefore, the pdf is

$$f(t|\text{Case 4B3}) = \frac{1}{24b}(2+b-2t)^3. \quad \text{(K43)}$$

Case 5: $x - x_1 < y + b/2$ and $x - x_2 \geq y$, then $T = y + b/2 + x - x_2$

Case 5A: with $0 \leq b \leq 2/3$

Case 5A1: with $b/2 \leq t \leq b$, the probability for T is

$$P_r(b/2 < T \leq t|\text{Case 5A1}) = \int_0^{(t-b/2)/2} \int_y^{-y+t-b/2} \int_0^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$$

$$\int_0^{(t-b/2)/2} \int_{-y+t-b/2}^{y+b/2} \int_{y+x+\frac{b}{2}-t}^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$$

$$\int_0^{(t-b/2)/2} \int_{y+b/2}^{1} \int_{y+x+b/2-t}^{x-y} \int_{x-y-b/2}^{x} \frac{2}{b} dx_1 dx_2 dx dy = \frac{1}{192b}(2t-b)^2(4(1-t)t + 2b(5+t) -$$

$$3b^2)$$

and take the derivative of $P_r(b/2 < T \leq t|\text{Case 5A1})$ with respect to T, the pdf is

$$f(t|\text{Case 5A1}) = \frac{d}{dt} P_r(b/2 < T \leq t|\text{Case 5A1}) = \frac{1}{96b}(7b^3 + 8(3-4t)t^2 + 12bt(2+3t) -$$

$$6b^2(3+4t)). \quad \text{(K44)}$$

Case 5A2: with $b < t \leq 3b/2$, the probability for T is

$$P_r(T \leq t|\text{Case 5A2}) = \int_0^{(t-b)/2} \int_y^{y+b/2} \int_0^{x-y} \int_0^x \frac{2}{} dx_1 dx_2 dx dy +$$

$$\int_{(t-b)/2}^{(t-b/2)/2} \int_y^{-y+t-b/2} \int_0^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$$

$$\int_0^{(t-b)/2} \int_{y+b/2}^{-y+t-b/2} \int_0^{x-y} \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy +$$

$$\int_{(t-b)/2}^{(t-b/2)/2} \int_{-y+t-b/2}^{y+b/2} \int_{y+x+b/2-t}^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$$

$$\int_0^{(t-b)/2} \int_{-y+t-b/2}^{1} \int_{y+x+b/2-t}^{x-y} \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy +$$

$$\int_{(t-b)/2}^{(t-b/2)/2} \int_{y+b/2}^{1} \int_{y+x+b/2-t}^{x-y} \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy = \frac{1}{192b} (5b^4 - 8b(-3+t)t^2 - 8(-2+$$

$$t)t^3 + 12b^2 t(-3+2t) - 2b^3(-5+9t))$$

and take the derivative of $P_r(T \leq t|\text{Case 5A2})$ with respect to T, the pdf is

$$f(t|\text{Case 5A2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 5A2}) = \frac{1}{96b} (-9b^3 - 12b(-2+t)t + 8(3-2t)t^2 +$$

$$6b^2(-3+4t)). \tag{K45}$$

Case 5A3: with $3b/2 < t \leq 2b$, the probability for T is

$$P_r(T \leq t|\text{Case 5A3})$$

$$= \int_0^{(t-b)/2} \int_y^{y+b/2} \int_0^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$+ \int_{(t-b)/2}^{b/2} \int_y^{-y+t-b/2} \int_0^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$+ \int_0^{(t-b)/2} \int_{y+b/2}^{-y+t-b/2} \int_0^{x-y} \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$+ \int_{(t-b)/2}^{b/2} \int_{-y+t-b/2}^{y+b/2} \int_{y+x+b/2-t}^{x-y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$+\int_0^{(t-b)/2}\int_{-y+t-b/2}^{1}\int_{y+x+b/2-t}^{x-y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy$$

$$+\int_{(t-b)/2}^{b/2}\int_{y+b/2}^{1}\int_{y+x+b/2-t}^{x-y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy$$

$$=\frac{1}{96b}(43b^4-32bt^3+4t^4+12b^2t(6+5t)-4b^3(19+18t))$$

and take the derivative of $P_r(T\le t|\text{Case 5A3})$ with respect to T, the pdf is

$$f(t|\text{Case 5A3})=\frac{d}{dt}P_r(T\le t|\text{Case 5A3})=\frac{1}{12}(-9b^2-12t^2+2t^3/b+3b(3+5t)). \tag{K46}$$

Case 5A4: with $2b<t\le 1+b/2$, the probability for T is

$$P_r(T\le t|\text{Case 5A4})=\int_0^{b/2}\int_y^{y+b/2}\int_0^{x-y}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy+\int_0^{b/2}\int_{y+b/2}^{-y+t-b/2}\int_0^{x-y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy+\int_0^{b/2}\int_{-y+t-b/2}^{1}\int_{y+x+b/2-t}^{x-y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy=\frac{1}{96}b(-21b^2-b(76-56t)-36(-2+t)t)$$

and take the derivative of $P_r(T\le t|\text{Case 5A4})$ with respect to T, the pdf is

$$f(t|\text{Case 5A4})=\frac{d}{dt}P_r(T\le t|\text{Case 5A4})=\frac{1}{12}b(9+7b-9t). \tag{K47}$$

Case 5A5: with $1+b/2<t\le 1+b$, the probability for T is

$$P_r(T\le t|\text{Case 5A5})=\int_0^{b/2}\int_y^{y+b/2}\int_0^{x-y}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy+\int_0^{b/2}\int_{y+b/2}^{-y+t-b/2}\int_0^{x-y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy+\int_0^{b/2}\int_{-y+t-b/2}^{1}\int_{y+x+b/2-t}^{x-y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy=\frac{1}{192b}(-45b^4+16(-1+t)^4+8b^3(-21+16t)-24b^2(1-8t+4t^2))$$

and take the derivative of $P_r(T\le t|\text{Case 5A5})$ with respect to T, the pdf is

$$f(t|\text{Case 5A5}) = \frac{d}{dt}P_r(T \leq t|\text{Case 5A5}) = \frac{1}{3\text{b}}(1+b-t)^2(-1+2b+t). \qquad \text{(K48)}$$

Case 5B: with $2/3 < b \leq 1$

Case 5B1: with $b/2 \leq t \leq b$, then the pdf is the same as $b/2 \leq t \leq b$ in Case 5A1, therefore, the pdf is

$$f(t|\text{Case 5B1}) = \frac{1}{96b}(7b^3 + 8(3-4t)t^2 + 12bt(2+3t) - 6b^2(3+4t)). \qquad \text{(K49)}$$

Case 5B2: with $b < t \leq 3b/2$, then the pdf is the same as $b < t \leq 3b/2$ in Case 5A2, therefore, the pdf is

$$f(t|\text{Case 5B2}) = \frac{1}{96b}(-9b^3 - 12b(-2+t)t + 8(3-2t)t^2 + 6b^2(-3+4t)). \qquad \text{(K50)}$$

Case 5B3: with $3b/2 < t \leq 1 + b/2$, then the pdf is the same as $3b/2 < t \leq 2b$ in Case 5A3, therefore, the pdf is

$$f(t|\text{Case 5B3}) == \frac{1}{12}(-9b^2 - 12t^2 + 2t^3/b + 3b(3+5t)). \qquad \text{(K51)}$$

Case 5B4: with $1 + b/2 < t \leq 2b$, the probability for T is

$$P_r(T \leq t|\text{Case 5B4}) = \int_0^{(t-b)/2}\int_y^{y+b/2}\int_0^{x-y}\int_0^x \frac{2}{b}dx_1dx_2dxdy +$$

$$\int_{(t-b)/2}^{b/2}\int_y^{-y+t-b/2}\int_0^{x-y}\int_0^x \frac{2}{b}dx_1dx_2dxdy + \int_0^{t-1-b/2}\int_{y+b/2}^{1}\int_0^{x-y}\int_{x-y-b/2}^x \frac{2}{b}dx_1dx_2dxdy +$$

$$\int_{t-1-b/2}^{(t-b)/2}\int_{y+b/2}^{-y+t-b/2}\int_{x-y-b/2}^{x-y}\int_{x-y-b/2}^{x} \frac{2}{b}dx_1dx_2dxdy +$$

$$\int_{(t-b)/2}^{b/2}\int_{-y+t-b/2}^{y+b/2}\int_{y+x+b/2-t}^{x-y}\int_0^x \frac{2}{b}dx_1dx_2dxdy +$$

$$\int_{t-1-b/2}^{(t-b)/2}\int_{-y+t-b/2}^{1}\int_{y+x+b/2-t}^{x-y}\int_{x-y-b/2}^{x} \frac{2}{b}dx_1dx_2dxdy +$$

$$\int_{(t-b)/2}^{b/2}\int_{y+b/2}^{1}\int_{y+x+b/2-t}^{x-y}\int_{x-y-b/2}^{x} \frac{2}{b}x_1dx_2dxdy = \frac{1}{192b}(83b^4 - 64bt^3 - 8b^3(21+16t) +$$

$$24b^2(-1+8t+4t^2) + 8(2-8t+12t^2-8t^3+3t^4))$$

and take the derivative of $P_r(T \leq t|\text{Case 5B4})$ with respect to T, the pdf is

$f(t|\text{Case 5B4}) = \frac{d}{dt} P_r(T \le t|\text{Case 5B4}) = \frac{1}{6b}(-2 - 4b^3 + 6t - 6t^2 - 6bt^2 + 3t^3 + 6b^2(1 + t))$. K52)

Case 5B5: with $2b < t \le 1 + b$, then the pdf is the same as $1 + b/2 < t \le 1 + b$ in Case 5A5, therefore, the pdf is

$$f(t|\text{Case 5B5}) = \frac{1}{3\text{b}}(1 + b - t)^2(-1 + 2b + t). \quad \text{(K53)}$$

Case 6: $x - x_1 < y + b/2$ and $x - x_2 < y$, then $T = 2y + b/2$

with $b/2 \le t \le 3b/2$, the probability for T is

$$P_r(b/2 \le T \le t|\text{Case 6})$$

$$= \int_0^{(t-b/2)/2} \int_0^y \int_0^x \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_0^{y+b/2} \int_{x-y}^x \int_0^x \int_0^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$+ \int_0^{(t-b/2)/2} \int_{y+b/2}^1 \int_{x-y}^x \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$= \frac{1}{768b}(2t - b)^2(4b(4 - t) + 4(4 - t)t - 3b^2)$$

and take the derivative of $P_r(b/2 \le T \le t|\text{Case 6})$ with respect to T, the pdf is

$$f(t|\text{Case 6}) = \frac{d}{dt} P_r(b/2 \le T \le t|\text{Case 6}) = \frac{1}{96b}(-6b^2 + b^3 - 8(-3 + t)t^2). \quad \text{(K54)}$$

Case 7: $x - x_1 < y + b/2$ and $x_2 - x \ge y$, then $T = y + b/2 + x_2 - x$

Case 7A: with $0 \le b \le 2/3$

Case 7A1: with $b/2 \le t \le 3b/2$, the probability for T is $P_r(b/2 \le T \le t|\text{Case 7A1}) = \int_0^{(t-b/2)/2} \int_0^{y+b/2} \int_{x+y}^{x-y+t-b/2} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_0^{(t-b/2)/2} \int_{y+b/2}^{y+1+b/2-t} \int_{x+y}^{x-y+t-b/2} \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy + \int_0^{(t-b/2)/2} \int_{y+1+b/2-t}^{1-y} \int_{x+y}^1 \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy$

$$= \frac{1}{1536b}(2t-b)^2(3b^2+b(80-44t)+4(8-5t)t)$$

and take the derivative of $P_r(b/2 < T \le t|\text{Case 7A1})$ with respect to T, the pdf is

$$f(t|\text{Case 7A1}) = \frac{d}{dt}P_r(b/2 \le T \le t|\text{Case 7A1}) = \frac{1}{192b}(-7b^3+12b(4-3t)t+8(6-5t)t^2+6b^2(-6+7t)). \quad \text{(K55)}$$

Case 7A2: with $3b/2 < t \le 1$, the probability for T is

$$P_r(T \le t|\text{Case 7A2})$$

$$= \int_0^{b/2}\int_0^{y+b/2}\int_{x+y}^{x-y+t-b/2}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy$$

$$+ \int_0^{b/2}\int_{y+b/2}^{y+1+b/2-t}\int_{x+y}^{x-y+t-b/2}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy$$

$$+ \int_0^{b/2}\int_{y+1+b/2-t}^{1-y}\int_{x+y}^{1}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy$$

$$= \frac{1}{24}b(3b^2-9(-2+t)t+b(-19+7t))$$

and take the derivative of $P_r(T \le t|\text{Case 7A2})$ with respect to T, the pdf is

$$f(t|\text{Case 7A2}) = \frac{d}{dt}P_r(T \le t|\text{Case 7A2}) = \frac{1}{24}b(18+7b-18t). \quad \text{(K56)}$$

Case 7A3: with $1 < t \le 1+b/2$, the probability for T is

$$P_r(T \le t|\text{Case 7A3}) = \int_0^{b/2}\int_0^{y+1+b/2-t}\int_{x+y}^{x-y+t-b/2}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy + \int_0^{b/2}\int_{y+1+b/2-t}^{y+b/2}\int_{x+y}^{1}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy + \int_0^{b/2}\int_{y+b/2}^{1-y}\int_{x-y-b/2}^{x}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy = \frac{1}{24}(3b^3+4(-1+t)^3-9b(-2+t)t+b^2(-19+7t))$$

and take the derivative of $P_r(T \le t|\text{Case 7A3})$ with respect to T, the pdf is

$$f(t|\text{Case 7A3}) = \frac{d}{dt}P_r(T \le t|\text{Case 7A3}) = \frac{1}{24}(7b^2-18b(-1+t)+12(-1+t)^2). \quad \text{(K57)}$$

Case 7A4: with $1 + b/2 < t \leq 1 + b$, the probability for T is

$P_r(T \leq t|\text{Case 7A4}) = \int_{t+b/2-1}^{b/2} \int_0^{y+1+b/2-t} \int_{x+y}^{x-y+t-b/2} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{t+b/2-1} \int_0^{y+b/2} \int_{x+y}^1 \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_{t+b/2-1}^{b/2} \int_{y+1+b/2-t}^{y+b/2} \int_{x+y}^1 \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{b/2} \int_{y+b/2}^{1-y} \int_{x+y}^1 \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy = \frac{1}{192b}(39b^4 + 64b(-1+t)^3 - 16(-1+t)^4 +$

$32b^3(-5+2t) - 24b^2(1 - 8t + 4t^2))$

and take the derivative of $P_r(T \leq t|\text{Case 7A4})$ with respect to T, the pdf is

$$f(t|\text{Case 7A4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 7A4}) = \frac{1}{3b}(1 + b - t)^3. \quad \text{(K58)}$$

Case 7B: with $2/3 < b \leq 1$,

Case 7B1: with $b/2 \leq t \leq 1$, then the pdf is the same as $b/2 \leq t \leq 3b/2$ in Case 7A1, therefore, the pdf is

$$f(t|\text{Case 7B1}) = \frac{1}{192b}(-7b^3 + 12b(4-3t)t + 8(6-5t)t^2 + 6b^2(-6+7t)). \quad \text{(K59)}$$

Case 7B2: with $1 < t \leq 3b/2$, the probability for T is

$P_r(\, T \leq t|\text{Case 7B2}) = \int_0^{(t-b/2)/2} \int_0^{y+1+b/2-t} \int_{x+y}^{x-y+t-b/2} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{(1-b/2)/2} \int_{y+1+b/2-t}^{y+b/2} \int_{x+y}^1 \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_{(1-b/2)/2}^{(t-b/2)/2} \int_{y+1+b/2-t}^{1-y} \int_{x+y}^1 \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{(1-b/2)/2} \int_{y+b/2}^{1-y} \int_{x+y}^1 \int_{x-y-b/2}^x \frac{2}{b} dx_1 dx_2 dx dy = \frac{1}{1536b}(3b^4 + b^3(80 - 56t) + 24b^2 t(-12 +$

$7t) - 32b(-4 + 12t - 18t^2 + 7t^3) + 16(-4 + 24t^2 - 24t^3 + 7t^4))$

and take the derivative of $P_r(T \leq t|\text{Case 7B2})$ with respect to T, the pdf is

$$f(t|\text{Case 7B2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 7B2}) = \frac{1}{192b}\big(-7b^3 + 6b^2(-6+7t) + 8t(12 - 18t + 7t^2) - 12b(4 - 12t + 7t^2)\big). \quad \text{(K60)}$$

Case 7B3: with $3b/2 < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 7B3}) = \int_0^{b/2} \int_0^{y+1+b/2-t} \int_{x+y}^{x-y+t-/2} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{(1-b/2)/2} \int_{y+1+b/2-t}^{y+b/2} \int_{x+y}^{1} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_{(1-b/2)/2}^{b/2} \int_{y+1+b/2-t}^{1-y} \int_{x+y}^{1} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{(1-b/2)/2} \int_{y+b/2}^{1-y} \int_{x+y}^{1} \int_{x-y-b/2}^{x} \frac{2}{b} dx_1 dx_2 dx dy = \frac{1}{384b}\big(-16 - 33b^4 + 8b^3(-11 + 14t) -$

$72b^2(3 - 4t + 2t^2) + 32b(1 + 6t - 6t^2 + 2t^3)\big)$

and take the derivative of $P_r(T \leq t|\text{Case 7B3})$ with respect to T, the pdf is

$f(t|\text{Case 7B3}) = \frac{d}{dt} P_r(T \leq t|\text{Case 7B3}) = \frac{1}{24}(7b^2 - 18b(-1 + t) + 12(-1 +$

$t)^2)$. (K61)

Case 7B4: with $1 + b/2 < t \leq 1 + b$, the probability for T is

$P_r(T \leq t|\text{Case 7B4}) = \int_{t-1-b/2}^{b/2} \int_0^{y+1+b/2-t} \int_{x+y}^{-y+t-b/2} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{t-1-b/2} \int_0^{y+1-b} \int_{x+y}^{1} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_{t-1-b/2}^{b/2} \int_{y+1+b/2-t}^{y+1-b} \int_{x+y}^{1} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{(1-b/2)/2} \int_{y+1-b}^{y+b} \int_{x+y}^{1} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy + \int_{(1-b/2)/2}^{b/2} \int_{y+1-b}^{1-y} \int_{x+y}^{1} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy +$

$\int_0^{(1-b/2)/2} \int_{y+b/2}^{1-y} \int_{x+y}^{1} \int_{x-y-b/2}^{x} \frac{2}{b} dx_1 dx_2 dx dy = \frac{1}{384b}\big(-35b^4 + 8b^3(-13 + 16t) -$

$24b^2(11 - 16t + 8t^2) + 32b(-1 + 12t - 12t^2 + 4t^3) - 16(3 - 8t + 12t^2 - 8t^3 + 2t^4)\big)$

and take the derivative of $P_r(T \leq t|\text{Case 7B4})$ with respect to T, the pdf is

$$f(t|\text{Case 7B4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 7B4}) = \frac{1}{3b}(1 + b - t)^3. \quad \text{(K62)}$$

Case 8: $x - x_1 < y + b/2$ and $x_2 - x < y$, then $T = 2y + b/2$

Case 8A: with $0 \leq b \leq 2/3$, with $b/2 \leq t \leq 3b/2$, the probability for T is

$$P_r(b/2 < T \leq t|\text{Case 8A}) = \int_0^{(t-b/2)/2} \int_0^{y+b/2} \int_x^{x+y} \int_0^x \frac{2}{b} dx_1 dx_2 dx dy$$

$$+\int_0^{(t-b/2)/2}\int_{y+b/2}^{-y+1}\int_x^{x+y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy$$

$$+\int_0^{(t-b/2)/2}\int_{-y+1}^{1}\int_x^{1}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy$$

$=\frac{1}{1536b}(2t-b)^2(-3b^2+4b(8-3t)+4t(8-3t))$

and take the derivative of $P_r(b/2<T\le t|\text{Case 8A})$ with respect to T, the pdf is

$$f(t|\text{Case 8A})=\frac{d}{dt}P_r(b/2<T\le t|\text{Case 8A})=\frac{1}{32b}(-2+t)(b^2-4t^2). \qquad \text{(K63)}$$

Case 8B: with $2/3<b\le 1$

Case 8B1: with $b/2\le t\le 1$, the pdf is the same as $b/2\le t\le 3b/2$ in Case 8A, therefore, the pdf is

$$f(t|\text{Case 8B1})=\frac{1}{32b}(-2+t)(b^2-4t^2). \qquad \text{(K64)}$$

Case 8B2: with $1\le t\le 3b/2$, the probability for T is

$P_r(T\le t|\text{Case 8B2})=\int_0^{(t-b/2)/2}\int_0^{y+b/2}\int_x^{x+y}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy+$ $\int_{(1-b/2)/2}^{(t-b/2)/2}\int_0^{-y+1}\int_x^{x+y}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy+\int_0^{(1-b/2)/2}\int_{y+b/2}^{-y+1}\int_x^{x+y}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy+$ $\int_{(1-b/2)/2}^{(t-b/2)/2}\int_{-y+1}^{y+b/2}\int_x^{1}\int_0^{x}\frac{2}{b}dx_1dx_2dxdy+\int_0^{(1-b/2)/2}\int_{-y+1}^{1}\int_x^{1}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy+$ $\int_{(1-b/2)/2}^{(t-b/2)/2}\int_{y+b/2}^{1}\int_x^{1}\int_{x-y-b/2}^{x}\frac{2}{b}dx_1dx_2dxdy=\frac{1}{1536b}(32b^3-3b^4+24b^2(-4+t)t+16(4-$ $16t+24t^2-8t^3+t^4))$

and take the derivative of $P_r(T\le t|\text{Case 8B2})$ with respect to T, the pdf is

$f(t|\text{Case 8B2})=\frac{d}{dt}P_r(T\le t|\text{Case 8B2})=\frac{1}{96b}\big(3b^2(-2+t)+4(-4+12t-6t^2+$ $t^3)\big)$. (K65)

In summary, from the derivations above,

when $b \leq 2/3$, the pdf is

$$f(t) = \begin{cases} (b^2 + 4b(-2+t) + 4(12-7t)t)/(16b) & t \in [b/2, 3\,b/2] \\ (4+b-4t)/2 & t \in (3\,b/2, 1] \\ (2+b-2t)^2/(2b) & t \in (1, 1+b/2] \end{cases}. \quad \text{(K66)}$$

When $b > 2/3$, the pdf is

$$f(t) = \begin{cases} (b^2 + 4b(-2+t) + 4(12-7t)t)/(16b) & t \in [b/2, 1] \\ (b^2 + 4b(-2+t) + 4(8-4t+t^2))/(16b) & t \in (1, 3\,b/2] \\ (2+b-2t)^2/(2b) & t \in (3\,b/2, 1+b/2] \end{cases}. \quad \text{(K67)}$$

Appendix L: Derivation of probability density function for path *mf* travel time with an infinite population and random storage

$$T = max(|x - x1|, b/2 - y) + max(|x - x2|, y)$$

$$= \begin{cases} 2x - x_1 - x_2, & x \geq x_1, x \geq x_2, x - x_1 \geq b/2 - y \text{ and } x - x_2 \geq y & \text{Case 1} \\ x + y - x_1, & x \geq x_1, x \geq x_2, x - x_1 \geq b/2 - y \text{ and } x - x_2 < y & \text{Case 2} \\ x_2 - x_1, & x \geq x_1, x < x_2, x - x_1 \geq b/2 - y \text{ and } x_2 - x \geq y & \text{Case 3} \\ x - x_1 + y, & x \geq x_1, x < x_2 \quad x - x_1 \geq b/2 - y \text{ and } x_2 - x < y & \text{Case 4} \\ b/2 - y + x - x_2, & x \geq x_1, x \geq x_2, x - x_1 < b/2 - y \text{ and } x - x_2 \geq y & \text{Case 5} \\ b/2, & x \geq x_1, x \geq x_2, x - x_1 < b/2 - y \text{ and } x - x_2 < y & \text{Case 6} \\ b/2 - y + x_2 - x, & x \geq x_1, x < x_2, x - x_1 < b/2 - y \text{ and } x_2 - x \geq y & \text{Case 7} \\ b/2, & x \geq x_1, x < x_2, x - x_1 < b/2 - y \text{ and } x_2 - x < y & \text{Case 8} \\ x_1 - x_2, & x < x_1, x \geq x_2, x_1 - x \geq b/2 - y \text{ and } x - x_2 \geq y & \text{Case 9} \\ x_1 - x + y, & x < x_1, x \geq x_2, x_1 - x \geq b/2 - y \text{ and } x - x_2 < y & \text{Case 10} \\ x_1 + x_2 - 2x, & x < x_1, x < x_2, x_1 - x \geq b/2 - y \text{ and } x_2 - x \geq y & \text{Case 11} \\ x_1 - x + y, & x < x_1, x < x_2, x_1 - x \geq b/2 - y \text{ and } x_2 - x < y & \text{Case 12} \\ b/2 - y + x - x_2, & x < x_1, x \geq x_2, x_1 - x < b/2 - y \text{ and } x - x_2 \geq y & \text{Case 13} \\ b/2, & x < x_1, x \geq x_2, x_1 - x < b/2 - y \text{ and } x - x_2 < y & \text{Case 14} \\ b/2 - y + x_2 - x, & x < x_1, x < x_2, x_1 - x < b/2 - y \text{ and } x_2 - x \geq y & \text{Case 15} \\ b/2, & x < x_1, x < x_2, x_1 - x < b/2 - y \text{ and } x_2 - x < y & \text{Case 16} \end{cases}.$$

where x, x_1 and $x_2 \in$ unif[0, 1] and $y \in$ unif$[0, b/2]$.

$$f(x, x_1, x_2, y) = f(x)f(x_1)f(x_2)f(y) = \frac{2}{b}$$

Because of symmetry, Case 1 and Case 11 are identical, Case 2, Case 5, Case 12 and Case 15 are identical, Case 3 and Case 9 are identical, Case 4, Case 7, Case 10 and Case 13 are identical, Case 6 and Case 16 are identical, and Case 8 and Case 14 are identical. Therefore, we only need to consider the Cases 1, 2, 3, 4, 6 and 8 to complete the derivation process.

From the expression above, the value of T can be stated as follows:

Case 1: $x - x_1 \geq b/2 - y$ and $x - x_2 \geq y$, then $T = 2x - x_1 - x_2$

Case 1A: $x \leq x_2/2 + t/2$

Case 1A1: with $b/2 \leq t \leq b$, the probability for T is

$$P_r(b/2 \leq T \leq t|\text{Case 1A1}) = \int_0^{b-t} \int_{x_2/2+b/4}^{x_2/2+t/2} \int_{b/2-x}^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 + \int_{b-t}^{b/2} \int_{x_2/2+b/4}^{x_2/2+t/2} \int_{b/2-x}^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 + \int_{b/2}^{t} \int_{2x-t}^{x} \int_0^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx = \frac{1}{384b}(2t-b)^3(3b+2t)$$

and take the derivative of $P_r(b/2 \leq T \leq t|\text{Case 1A1})$ with respect to T, the pdf is

$$f(t|\text{Case 1A1}) = \frac{d}{dt} P_r(b/2 \leq T \leq t|\text{Case 1A1}) = \frac{1}{24b}(2t-b)^2(b+t). \tag{L1}$$

Case 1A2: with $b < t \leq 1$, the probability for T is

$$P_r(T \leq t|\text{Case 1A2}) = \int_0^{b-t} \int_{x_2/2+b/4}^{b/2} \int_{b/2-x}^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 + \int_0^{b/2} \int_{x_2+b/2}^{x_2/2+t/2} \int_0^{b/2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 + \int_{b/2}^{t-b/2} \int_{x-b/2}^{x} \int_0^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx + \int_{t-b/2}^{t} \int_{2x-t}^{x} \int_0^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx = \frac{1}{384}(-3b^3 + 32b^2 t - 72bt^2 + 48t^3)$$

and take the derivative of $P_r(T \leq t|\text{Case 1A2})$ with respect to T, the pdf is

$$f(t|\text{Case 1A2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1A2}) = \frac{1}{24}(2b^2 - 9bt + 9t^2). \tag{L2}$$

Case 1A3: with $1 < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 1A3}) = \int_0^{b-t} \int_{x_2/2+b/4}^{b/2} \int_{b/2-x}^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 +$

$\int_0^{b/2} \int_{x_2+b/2}^{x_2/2+t/2} \int_0^{b/2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 + \int_{b/2}^{t-b/2} \int_{x-b/2}^{x} \int_0^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx +$

$\int_{t-b/2}^{1} \int_{2x-t}^{x} \int_0^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx = \frac{1}{384b}\big(-3b^4 + 32b^3t - 72b^2t^2 - 64(-1 +$

$t)^3(1+t) + 16b(-4 + 12t - 12t^2 + 7t^3)\big)$

and take the derivative of $P_r(T \leq t|\text{Case 1A3})$ with respect to T, the pdf is

$f(t|\text{Case 1A3}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1A3}) = \frac{1}{24b}\big(2b^3 - 9b^2t - 8(-1+t)^2(1+2t) + 3b(4 -$

$8t + 7t^2)\big)$. (L3)

Case 1A4: with $1 + b/2 < t \leq 2$, the probability for T is

$P_r(T \leq t|\text{Case 1A4}) = \int_0^{b-t} \int_{x_2/2+b/4}^{b/2} \int_{b/2-x}^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 +$

$\int_{b/2}^{1} \int_{x-b/2}^{x} \int_0^{x-x_2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx + \int_0^{2-t} \int_{x_2+b/2}^{x_2/2+t/2} \int_0^{b/2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 +$

$\int_{2-t}^{1-b/2} \int_{x_2+b/2}^{1} \int_0^{b/2} \int_0^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 = \frac{1}{384}\big(16b^2 + b^3 + 24b(-4+t)t - 16(8 -$

$12t + t^3)\big)$

and take the derivative of $P_r(T \leq t|\text{Case 1A4})$ with respect to T, the pdf is

$$f(t|\text{Case 1A4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1A4}) = \frac{1}{8}(-2 + b - t)(-2 + t). \quad \text{(L4)}$$

Case 1B: with $x > x_1/2 + t/2$

Case 1B1: with $b/2 \leq t \leq b$, the probability for T is

$P_r(b/2 < T \leq t|\text{Case 1B1}) = \int_0^{b-t} \int_{x_2/2+t/2}^{b/2} \int_{x-x_2+b/2-t}^{x-x_2} \int_{2x-x_2-t}^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 +$

$\int_{b/2}^{1} \int_{x-b/2}^{x+b/2-t} \int_{x-x_2+b/2-t}^{x-x_2} \int_{2x-x_2-t}^{x+y-b/2} \frac{2}{b} dx_1 dy dx dx_2 +$

$\int_{b/2}^{t} \int_{x+b/2-t}^{2x-t} \int_0^{x-x_2} \int_{2x-x_2-t}^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx + \int_t^1 \int_{x+b/2-t}^{x} \int_0^{x-x_2} \int_{2x-x_2-t}^{x+y-b/2} \frac{2}{b} dx_1 dy dx_2 dx +$

$\int_{0}^{1-t}\int_{x_2+b/2}^{x_2+t}\int_{x-x_2+b/2-t}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydxdx_2 +$

$\int_{1-t}^{1-b/2}\int_{x_2+b/2}^{1}\int_{x-x_2+b/2-t}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydxdx_2 = \frac{1}{48b}(2t-b)^2(-b^2-2b(-3+t)-t^2)$

and take the derivative of $P_r(b/2 < T \le t|\text{Case 1B1})$ with respect to T, the pdf is

$f(t|\text{Case 1B1}) = \frac{d}{dt}P_r(b/2 < T \le t|\text{Case 1B1}) = \frac{1}{24b}(b^3+3b^2(-4+t)-6b(-4+t)t-$

$8t^3).$ (L5)

Case 1B2: with $b < t \le 1$, the probability for T is

$P_r(T \le t|\text{Case 1B2}) = \int_{t/2}^{t-b/2}\int_{0}^{2x-t}\int_{0}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx +$

$\int_{t-b/2}^{1}\int_{x+b/2-t}^{x-b/2}\int_{0}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx + \int_{t-b/2}^{t}\int_{x-b/2}^{2x-t}\int_{0}^{x-x_2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx +$

$\int_{t}^{1}\int_{x-b/2}^{x}\int_{0}^{x-x_2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx +$

$\int_{0}^{1-t}\int_{x_2+t-b/2}^{x_2+t}\int_{x-x_2+b/2-t}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydxdx_2 +$

$\int_{1-t}^{1+b/2-t}\int_{x_2+t-b/2}^{1}\int_{x-x_2+b/2-t}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydxdx_2 = \frac{1}{48}(b^3+b^2(6-8t)+6(4-$

$3t)t^2+3bt(-8+7t))$

and take the derivative of $P_r(T \le t|\text{Case 1B2})$ with respect to T, the pdf is

$$f(t|\text{Case 1B2}) = \frac{d}{dt}P_r(T \le t|\text{Case 1B2}) = -\frac{b^2}{6}+t-\frac{9t^2}{8}+\frac{1}{8}b(-4+7t). \quad \text{(L6)}$$

Case 1B3: with $1 < t \le 1+b/2$, the probability for T is

$P_r(T \le t|\text{Case 1B3}) = \int_{t/2}^{t-b/2}\int_{0}^{2x-t}\int_{0}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx +$

$\int_{t-b/2}^{1}\int_{x+b/2-t}^{x-b/2}\int_{0}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx + \int_{t-b/2}^{1}\int_{x-b/2}^{2x-t}\int_{0}^{x-x_2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydx_2dx +$

$\int_{0}^{1+b/2-t}\int_{x_2+t-b/2}^{1}\int_{x-x_2+b/2-t}^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydxdx_2 = \frac{1}{48b}(b^4+b^3(6-8t)+16(-1+$

$t)^3t+3b^2t(-8+7t)+b(8-24t+48t^2-26t^3))$

and take the derivative of $P_r(T \le t|\text{Case 1B3})$ with respect to T, the pdf is

$f(t|\text{Case 1B3}) = \frac{d}{dt} P_r(T \le t|\text{Case 1B3}) = \frac{1}{24b}\big(-4b^3 + 8(-1+t)^2(-1+4t) + 3b^2(-4+7t) - 3b(4 - 16t + 13t^2)\big).$ (L7)

Case 1B4: with $1 + b/2 < t \le 2$, the probability for T is

$$P_r(T \le t|\text{Case 1B4}) = \int_0^{2-t}\int_{x_2/2+t/2}^{1}\int_0^{b/2}\int_{2x-x_2-t}^{x+y-b/2}\frac{2}{b}dx_1dydxdx_2 = \frac{1}{16}(2t-b)(2-t)^2$$

and take the derivative of $P_r(T \le t|\text{Case 1B4})$ with respect to T, the pdf is

$$f(t|\text{Case 1B4}) = \frac{d}{dt} P_r(T \le t|\text{Case 1B4}) = \frac{1}{8}(2+b-3t)(2-t). \quad \text{(L8)}$$

Case 2: $x - x_1 \ge b/2 - y$ and $x - x_2 < y$, then $T = x + y - x_1$

Case 2A: with $b/2 \le t \le b$, the probability for T is

$$P_r(b/2 < T \le t|\text{Case 2A}) = \int_{b/4}^{b/2}\int_{-y+b/2}^{y}\int_0^{x+y-b/2}\int_0^{x}\frac{2}{b}dx_2dx_1dxdy + \int_{t/2}^{b/2}\int_{-y+b/2}^{-y+t}\int_0^{x+y-b/2}\int_0^{x}\frac{2}{b}dx_2dx_1dxdy + \int_0^{b/4}\int_{-y+b/2}^{-y+t}\int_0^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy + \int_{b/4}^{t/2}\int_{y}^{-y+t}\int_0^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy + \int_{t/2}^{b/2}\int_{-y+t}^{y}\int_{x+y-t}^{x+y-b/2}\int_0^{x}\frac{2}{b}dx_2dx_1dxdy + \int_0^{t/2}\int_{-y+t}^{1}\int_{x+y-t}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy + \int_{t/2}^{b/2}\int_{y}^{1}\int_{x+y-t}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{384b}(11b^4 - 64bt^3 + 16t^4 + 48b^2t(2+t) - 16b^3(3+2t))$$

and take the derivative of $P_r(b/2 < T \le t|\text{Case 2A})$ with respect to T, the pdf is

$$f(t|\text{Case 2A}) = \frac{d}{dt} P_r(b/2 < T \le t|\text{Case 2A}) = \frac{1}{12b}\big(-b^3 - 6bt^2 + 2t^3 + 3b^2(1+t)\big). \quad \text{(L9)}$$

Case 2B: with $b < t \le 1$, the probability for T is

$$P_r(T \le t|\text{Case 2B}) = \int_{b/4}^{b/2}\int_{-y+b/2}^{y}\int_0^{x+y-b/2}\int_0^{x}\frac{2}{b}dx_2dx_1dxdy + \int_0^{b/4}\int_{-y+b/2}^{-y+t}\int_0^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy + \int_{b/4}^{b/2}\int_{y}^{-y+t}\int_0^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy + \int_0^{b/2}\int_{-y+t}^{1}\int_{x+y-t}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{384}b(-5b^2 - b(48-32t) - 48(-2+t)t)$$

and take the derivative of $P_r(T \le t|\text{Case 2B})$ with respect to T, the pdf is

$$f(t|\text{Case 2B}) = \frac{d}{dt}P_r(T \le t|\text{Case 2B}) = \frac{1}{12}b(3+b-3t). \tag{L10}$$

Case 2C: with $1 < t \le 1+b$, the probability for T is

$$P_r(T \le t|\text{Case 2C}) = \int_{b/4}^{b/2}\int_{-y+b/2}^{y}\int_{0}^{x+y-b/2}\int_{0}^{x}\frac{2}{b}dx_2dx_1dxdy +$$

$$\int_{b/4}^{b/2}\int_{-x+b/2}^{x}\int_{0}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dydx + \int_{b/2}^{t-b/2}\int_{0}^{b/2}\int_{0}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dydx +$$

$$\int_{t-b/2}^{1}\int_{0}^{-x+t}\int_{0}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dydx + \int_{t-b/2}^{1}\int_{-x+t}^{b/2}\int_{0}^{x+y-b/2}\int_{x-y}^{x}\frac{2}{b}dx_2dx_1dydx =$$

$$\frac{1}{384b}(-5b^4 + 32(-1+t)^4 - 48b^2(-2+t)t + 16b^3(-3+2t))$$

and take the derivative of $P_r(T \le t|\text{Case 2C})$ with respect to T, the pdf is

$$f(t|\text{Case 2C}) = \frac{d}{dt}P_r(T \le t|\text{Case 2C}) = \frac{1}{12b}(2+b-2t)^2(-1+b+t). \tag{L11}$$

Case 3: $x - x_1 \ge b/2 - y$ and $x_2 - x \ge y$, then $T = x_2 - x_1$

with $b/2 \le t \le 1$, the probability for T is

$$P_r(b/2 < T \le t|\text{Case 3}) = \int_{0}^{1-t}\int_{x_1+b/2}^{x_1+t}\int_{0}^{b/2}\int_{b/2-y+x_1}^{x_2-y}\frac{2}{b}dxdydx_2dx_1 +$$

$$\int_{1-t}^{1-b/2}\int_{x_1+b/2}^{1}\int_{0}^{b/2}\int_{b/2-y+x_1}^{x_2-y}\frac{2}{b}dxdydx_2dx_1 = \frac{1}{48}(2t-b)^2(6-b-4t)$$

and take the derivative of $P_r(b/2 < T \le t|\text{Case 3})$ with respect to T, the pdf is

$$f(t|\text{Case 3}) = \frac{d}{dt}P_r(b/2 < T \le t|\text{Case 3}) = \frac{1}{2}(2t-b)(1-t). \tag{L12}$$

Case 4: $x - x_1 \ge b/2 - y$ and $x_2 - x < y$, then $T = x - x_1 + y$

Case 4A: with $b/2 \le t \le 1$, the probability for T is

$$P_r(b/2 < T \le t|\text{Case 4A}) = \int_{0}^{b/2}\int_{-y+b/2}^{-y+t}\int_{0}^{x+y-b/2}\int_{x}^{x+y}\frac{2}{b}dx_2dx_1dxdy +$$

$$\int_{0}^{b/2}\int_{-y+t}^{-y+1}\int_{x+y-t}^{x+y-b/2}\int_{x}^{x+y}\frac{2}{b}dx_2dx_1dxdy + \int_{0}^{b/2}\int_{-y+t}^{1}\int_{x+y-t}^{x+y-b/2}\int_{x}^{1}\frac{2}{b}dx_2dx_1dxdy = \frac{1}{96}b(b^2 +$$

$$4b(-3+t) - 12(-2+t)t)$$

and take the derivative of $P_r(b/2 < T \le t|\text{Case 4A})$ with respect to T, the pdf is

$$f(t|\text{Case 4A}) = \frac{d}{dt}P_r(b/2 < T \leq t|\text{Case 4A}) = \frac{1}{24}b(6 + b - 6t). \tag{L13}$$

Case 4B: with $1 < t \leq 1 + b/2$, the probability for T is

$P_r(T \leq t|\text{Case 4B}) = \int_0^{b/2}\int_{-y+b/2}^{-y+1}\int_0^{x+y-b/2}\int_x^{x+y}\frac{2}{b}dx_2dx_1dxdy +$

$\int_{1-b/2}^{t-b/2}\int_{-x+1}^{b/2}\int_0^{x+y-b/2}\int_x^1\frac{2}{b}dx_2dx_1dydx + \int_{t-b/2}^{1}\int_{-x+1}^{-x+t}\int_0^{x+y-b/2}\int_x^1\frac{2}{b}dx_2dx_1dydx +$

$\int_{t-b/2}^{1}\int_{-x+t}^{b/2}\int_{x+y-t}^{x+y-b/2}\int_x^1\frac{2}{b}dx_2dx_1dy\,dx = \frac{1}{96b}(b^4 + 4b^3(-3+t) + 16b(-1+t)^3 - 8(-1+$

$t)^4 - 12b^2(-2+t)t)$

and take the derivative of $P_r(T \leq t|\text{Case 4B})$ with respect to T, the pdf is

$$f(t|\text{Case 4B}) = \frac{d}{dt}P_r(T \leq t|\text{Case 4B}) = \frac{1}{24b}(2 + b - 2t)^3. \tag{L14}$$

Case 6: $x - x_1 < b/2 - y$ and $x - x_2 < y$, then $T = b/2$

When $t = b/2$, the pdf and the probability for T are the same. Specifically,

$f(t = b/2|\text{Case 6}) = F(t = b/2|\text{Case 6}) = \int_0^{b/2}\int_{x_2}^{x_2/2+b/4}\int_0^{x}\int_{x-x_2}^{b/2-(x-x_1)}\frac{2}{b}dydx_1dxdx_2 +$

$\int_{b/4}^{b/2}\int_0^{2x-b/2}\int_{2x-x_2-b/2}^{x}\int_{x-x_2}^{b/2-(x-x_1)}\frac{2}{b}dydx_1dx_2dx +$

$$\int_{b/2}^{1}\int_{x-b/2}^{x}\int_{2x-x_2-b/2}^{x}\int_{x-x_2}^{b/2-(x-x_1)}\frac{2}{b}dydx_1dx_2dx = \frac{1}{384}(16-3b)b^2. \tag{L15}$$

Case 8: $x - x_1 < b/2 - y$ and $x_2 - x < y$, then $T = b/2$

with $t = b/2$, the pdf and the probability for T are the same. Specifically,

$f(t = b/2|\text{Case 8}) = F(t = b/2|\text{Case 8}) = \int_0^{b/2}\int_{x_1}^{b/2+x_1}\int_x^{b/2+x_1}\int_{x_2-x}^{b/2-(x-x_1)}\frac{2}{b}dydx_2dxdx_1 +$

$$\int_{b/2}^{1}\int_{x_1}^{1}\int_x^{1}\int_{x_2-x}^{b/2-(x-x_1)}\frac{2}{b}dydx_2dxdx_1 = \frac{1}{96}(4-b)b^2. \tag{L16}$$

In summary, from the derivations above, the pdf is

$$f(t) = \begin{cases} b(6-b)/12 & t = b/2 \\ (4+b-4t)/2 & t \in (b/2, 1]. \\ (2+b-2t)^2/(2b) & t \in (1, 1+b/2] \end{cases} \tag{L17}$$

Certification of Student Work

UNIVERSITY OF ARKANSAS

College of Engineering
Department of Industrial Engineering

MEMORANDUM

TO: University of Arkansas Graduate School

FROM: Haitao Liao, Professor, John and Mary Lib White Endowed Systems Integration Chair
John A. White, Chancellor Emeritus and Distinguished Professor (retired)

DATE: July 25, 2019

SUBJECT: Certification of Student Effort and Contribution

As co-directors of Mr. Jingming Liu's doctoral dissertation, we certify Mr. Liu contributed more than 51 percent of the work included in the chapter entitled, "A Working Paper on "Stochastic analysis of an automated storage and retrieval system with multiple in-the-aisle pick positions" contained in the doctoral dissertation entitled, "Probabilistic Models for Order-Picking Operations with Multiple in-the-Aisle Pick Positions".

Chapter 4

Contribution 3: A Working Paper on "Class-based storage and retrieval in an MIAPP-NALT system in stochastic conditions"

Abstract

An M/G/1 queueing model is used to assess the performance of a manual case-picking operation having multiple in-the-aisle pick positions (MIAPP). A narrow aisle lift truck (NALT) is used to support the storage and retrieval operations. Manual order picking is performed at floor-level and mezzanine-level pick positions to fulfill customer orders. Unit loads of cases are stored in double-deep storage rack on both sides of the picking aisle. On the opposite sides of the storage rack are storage/retrieval (S/R) aisles, within which narrow aisle lift trucks travel rectilinearly in replenishing pick positions when they become empty; a NALT retrieves pallet loads of cases from reserve-storage locations in the S/R aisle and stores pallet loads of cases in reserve-storage locations within the storage rack. Three storage policies are considered: dedicated, random and class-based storage. Non-preemptive priority and non-priority M/G/1 queues are used to model the MIAPP-NALT order picking system. For the non-preemptive priority case, the NALT operations are modeled as an M/G/1 priority queue with retrieval requests having a higher priority than storage requests. In the non-priority case, S/R requests are served with a first come, first serve discipline. S/R requests are the customers and the NALT is the server in the M/G/1 queueing model. The arrivals of the two types of customers are assumed to occur at Poisson rates; service times of the server are the sum of NALT travel time and time to pick up and deposit a pallet load of cases.

Keywords: MIAPP order picking, Priority queue, General service time queue, Rectilinear travel, Storage policies

4.1. Introduction

In a multiple in-the-aisle pick positions case picking operation supported by a narrow aisle lift truck (MIAPP-NALT), order pickers select cases from pallets to fulfill orders containing different lines. Pallet loads of cases are placed in a double-deep pallet flow rack at each pick position. Depending on the system design, the order picker might place each case on a conveyor which transports it to an order accumulation point or each case might be placed on an automated guided vehicle on which the order is accumulated before being dispatched to shipping.

The narrow aisle lift truck (NALT) performs a storage operation in replenishing reserve-storage locations for floor-level pick and mezzanine-level pick positions: likewise, the NALT performs retrieval operations to replenish a pick position when the second position in the double-deep pallet flow rack becomes empty. Because the NALT is not allowed to lower or lift its forks while travelling in the S/R aisle, travel of the NALT is performed rectilinearly. When the NALT is idle, it remains in the S/R aisle at the point where it just completed an S/R operation; its forks will be at the floor-level position in the S/R aisle. To perform a storage operation, the NALT's forks are moved from a floor-level position to the floor-level input/output (I/O) point located at the left end of the S/R aisle, the forks lift a pallet load of cases and transport it to a random storage location in the reserve-storage region, the forks deposit the load in a reserve-storage location, and the forks are lowered to a floor-level position. (Hereafter, we abbreviate descriptions of the movement of the forks on the NALT, by stating "the NALT travels".) For a retrieval operation of a floor-level pick position, the NALT travels from a floor-level position to a random location in the reserve-storage region for the floor-level pick position requesting replenishment, picks up a pallet load, and transports it to the floor-level pick position requiring replenishment. To replenish a mezzanine-level pick position, the NALT travels from a floor-level position to a random location in the reserve-storage region for the mezzanine-level pick position requesting replenishment, picks up a pallet load and transports it to the mezzanine-level pick position requiring replenishment. Therefore, there are two types of operations for the NALT; one is a storage operation for the reserve-storage region and the other is a retrieval operation to replenish pick positons at floor level or mezzanine level.

The two operations of the NALT in the S/R aisle can be modeled as a queueing problem in which the random arrivals of the storage and retrieval requests are two types of customers and the NALT is the server; service time is the sum of the NALT travel time and the time to pick up and deposit a unit load (T_{PD}). We assume the time required for the NALT to pick up or deposit (P/D) a unit load is the same and is deterministic. Therefore, to perform both a pick up and deposit for a retrieval operation, the expected value for service time is the sum of two P/D times and the expected value for travel time; the variance for service time is the variance for travel time.

The paper is organized as follows. The relevant research literature is reviewed in Section 2; the literature review concludes by summarizing the contributions of this paper. In Section 3, notation and assumptions underlying the research are presented. In Section 4, drawing on derivations in Appendixes, expected values, second moments and overall variances for S/R service times are provided for dedicated, random and class-based storage policies. In Section 5, two queueing models are considered: M/G/1/NPRP/∞ and M/G/1/FCFS/∞ queues. In Section 6, S/R requests are assumed to occur in a Poisson fashion, the steady state operating characteristics for priority and non-priority queues are obtained and insights and practical considerations gained are shared. Finally, in Section 7, we summarize the research, provide conclusions drawn from the research and make recommendations for further research.

4.2. Literature Review

Many papers dealing with storage and retrieval operations in an order picking system have been published. However, applying a queueing model in an order-picking system attracted a limited attention. For the Chebyshev travel metric, several researchers employed queueing models by simplifying the assumptions regarding the service time distribution. Liu et al. (2019) analyzed a retrieval operation in an MIAPP-AS/RS. In sharp contrast to the Chebyshev travel metric, there are a limited number of papers addressing the travel time probability density function (pdf) in a rectilinear travel metric. Liu (2019) applied a general service time queueing model to analyze a retrieval operation in an MIAPP order picking system supported by a narrow aisle lift truck; other papers employed expected values for the time to travel rectilinearly from a fixed point to a random point within a specified spatial region. Our paper considers both retrieval and storage operations and extends the work of Liu (2019) which considered only retrieval operations.

Because we are focusing on an MIAPP-NALT system, we do not include in our review of the literature papers employing the Chebyshev travel metric unless queueing analysis is included. The organization of the literature review is as follows: papers employing the rectilinear travel metric are considered; next, MIAPP related literature is reviewed. The literature review concludes with papers employing queueing models of storage/retrieval operations.

Rectilinear metric: Research involving a rectilinear travel metric is very limited. Francis et al. (1992) was one of the first researchers to address facility location problems in which distance was measured using the rectilinear travel metric. Larson and Odoni (1981) derived the pdf for rectilinear distance from the centroid of a square area to a random point within the square area. In extending the work of Larson and Odoni (1981), Miyagawa (2010) calculated the expected distance from a random point to the nearest point within a diamond or square lattice pattern. Liu

(2019) derived two sets of travel time pdfs: one from a fixed point to a random point within a rectangular area and then to another fixed point; the other is from a random point on the bottom edge of a rectangle to a random point within a rectangular area and, then, to another random point on the bottom edge of the rectangle. Gue (2006) and Gue and Kim (2007) analyzed high density storage systems in which a shuttle or storage/retrieval device travels rectilinearly. Another high density storage system called grid pick is addressed by Uludag (2014).

MIAPP literature: Ramtin and Pazour (2014) are the first to analyze an MIAPP-AS/RS system by developing models of expected travel time to optimize the configuration of an S/R aisle. Ramtin and Pazour (2015) developed a polynomial algorithm to assign skus in a rack to minimize the expected travel time for a different rack shape and different sku demand curves. Liu et al. (2019) analyzed an MIAPP-NALT system using a finite population queueing model and an infinite population queueing model to assess the expected number of pick positions requiring replenishment when the arrival of replenishment requests is Poisson distributed. Similar research by Liu et al. (2019) also employed queueing models of replenishment requests in an MIAPP-AS/RS system. In contrast to their work, we analyze an MIAPP-NALT with two $M/G/1/\infty$ queueing models including two types of customers: retrieval requests and storage requests. The queueing models are a non-preemptive priority model giving retrieval requests a higher priority than storage requests; the other is a queueing model in which there is no priority and a first come, first serve discipline is employed for retrieval and storage requests.

Queueing models of storage/retrieval operations: Bozer and White (1990) analyzed a mini-load AS/RS by modeling it as a two-server queueing network. Lee (1997) analyzed storage and retrieval operations uaing an M/M/1 infinite population queueing model. Park et al. (1999) obtained expressions for steady state performance of a two-state network queueing system for a miniload AS/RS. Vlasiou et al. (2004) analyzed the performance for a system containing two carousel conveyors and an AS/R machine using an infinite population queueing model. Hur et al. (2004) used an M/G/1/∞ queuing model to analyze storage and retrieval operations in a unit load AS/RS; Hur and Nam (2006) extended the work of Hur et al. (2004) by considering different arrival rates for the storage and retrieval requests. Bozer and Cho (2005) used an M/G/1 queueing model to analyze storage and retrieval operations in an AS/RS having different operating policies. Malmborg (2002) compared the steady state performance of an autonomous vehicle storage and retrieval system (AVS/RS) with an AS/RS by employing a state transition diagram in a queueing model . Malmborg (2003) extended the work of Malmborg (2002) to analyze the interleaving of storage and retrieval operations in an AVS/RS. Kuo et al. (2007) employed an M/G/c/∞ queue to analyze an AVS/RS by considering a dozen travel time scenarios. The interleaving of storage and retrieval requests in an AVS/RS is addressed in the work of Fukunari and Malmborg (2009). Heragu, et al. (2011) applied an open queueing network to compare the performance of an AVS/RS with an AS/RS. Zhang et al. (2009) analyzed an AVS/RS with non-Poisson arrivals and non-exponential service times. Cai et al. (2013) studied an AVS/RS by employing Markov chains to obtain steady state performance measures in a semi-open queue having two types of customers which must be paired for service. Cai et al. (2014) analyzed an AVS/RS using a two-stage semi-open queueing model containing two types of customer: one from a finite population and the other from an infinite population; the two types of customers must be paired for service. Roy et al. (2015) analyzed optimal dwell-point policies and cross-aisle location for an AVS/RS using a multi-class, semi-open queueing network.

Contributions of the paper: In comparison to previous research, we are the first to: 1) treat retrieval and storage requests as two types of customers and give the retrieval request a higher priority in a priority queueing model; 2) derive travel time pdfs for rectilinear travel in an MIAPP-NALT system for both retrieval and storage operations; 3) employ a priority queueing model and a non-priority queueing model to show the benefits of giving priority to retrieval requests; 4) show how the skewness of an AB curve of skus affects the performance of an MIAPP-NALT system; and 5) analyze the locations of two classes of skus in a S/R aisle with an MIAPP-NALT system.

In addition, our results will benefit other research employing rectilinear travel when retrieval and storage operations are performed within a rectangle-shaped region. In contrast to models based on expected travel time in the warehouse, if researchers want to employ travel time pdfs, we believe our results will be quite beneficial. Most research on the design of material handling systems has been based on deterministic analyses. The limited number of stochastic analyses on storage/retrieval operations are based on expected values of operation times. Our research combines the random aspects of arrivals and services in queueing models to deliver results.

4.3. Notation

A: length of the S/R aisle;

B: height of the rack;

C: length of S/R aisle devoted to pick positions for class A skus;

C_1: length of S/R aisle devoted to floor-level pick positions for class A skus in Layout 3;

C_2: length of S/R aisle devoted to mezzanine-level pick positions for class A skus in Layout 3;

D: vertical distance between floor-level pick positions and mezzanine-level pick positions;

V_h: horizontal travel velocity of the NALT;

V_v: vertical travel velocity of the NALT;

a: time for the NALT to travel a distance of A horizontally, A/V_h;

b: time for the NALT to travel a distance of B vertically, B/V_v;

c: time for the NALT to travel a distance of C horizontally, C/V_h;

c_1: time for the NALT to travel a distance of C_1 horizontally, C_1/V_h;

c_2: time for the NALT to travel a distance of C_2 horizontally, C_2/V_h;

d: time for the NALT to travel a distance of D vertically, D/V_v;

x_i: time-based horizontal coordinate of random location i;

x_j: time-based horizontal coordinate of random location j;

x: time-based horizontal coordinate of a random location;

T_{PD}: time for the NALT to pick up and deposit a full pallet;

T_h: horizontal travel time for the NALT to perform an operation;

T_v: vertical travel time for the NALT to perform an operation;

T_{ij}: travel time for the NALT to perform an operation in travel path ij;

T_s: travel time to perform a storage operation;

T_r: travel time to perform a retrieval operation;

S_s: service time to perform a storage operation;

S_r: service time to perform a retrieval operation;

S: weighted service time to perform a storage operation or a retrieval operation;

λ_r: arrival rate of retrieval requests (class 1 customers);

λ_s: arrival rate of storage requests (class 2 customers);

λ: arrival rate of retrieval (class 1 customers) or storage (class 2 customers) requests, equals $\lambda = \lambda_{s+} \lambda_r$;

μ_r: service rate of retrieval requests (class 1 customers);

μ_s: service rate of storage requests (class 2 customers);

μ: weighted service rate of retrieval (class 1 customers) or storage (class 2 customers) requests;

ρ_r: traffic intensity of retrieval requests (class 1 customers);

ρ_s: traffic intensity of storage requests (class 2 customers);

ρ: weighted traffic intensity of retrieval (class 1 customers) or storage (class 2 customers) requests;

L_1: expected number of empty secondary positions in the gravity flow rack (class 1 customers);

L_2: expected number of storage requests being filled or waiting to be filled (class 2 customers);

L: expected number of retrieval or storage requests in the queueing system;

To calculate L for the M/G/1/NPPT and M/G/1/FCFS queues, the expected value, second moment and variance of travel times of a NALT must be derived. Based on given rack dimensions and constant travel speeds, we convert the storage area into a time-dimension space by calculating the maximum horizontal travel time a (A/ V_h) and maximum vertical travel time b (B/ V_v).

4.4. Expected value and second moment for the NALT travel time

To calculate L for the M/G/1/NPPT and M/G/1/FCFS queues, the expected value, second moment and variance of travel times of a NALT must be derived. Based on given rack dimensions and constant travel speeds, we convert the storage area into a time-dimension space by calculating the maximum horizontal travel time a (A/ V_h) and maximum vertical travel time b (B/ V_v).

Two types of operations exist for the NALT: a retrieval operation and a storage operation. For the retrieval operation, the NALT travels from a floor-level location to a random reserve-storage location for the pick position needing to be replenished, then travels to the pick position requiring replenishment. If the pick position is at floor level, then the NALT stops at the floor-level pick position after finishing the retrieval operation. If the pick position is at the mezzanine-level, the forks on the NALT move directly to a floor-level location to complete the retrieval operation. For the storage operation, the NALT travels from a floor-level location to the I/O point, travels to a reserve-storage location for the pallet, and the forks on the NALT move directly to a floor-level location to complete the storage operation. It does not matter what the previous NALT operation was or what the next NALT operation will be; the NALT starts from a floor-level location to perform the next operation.

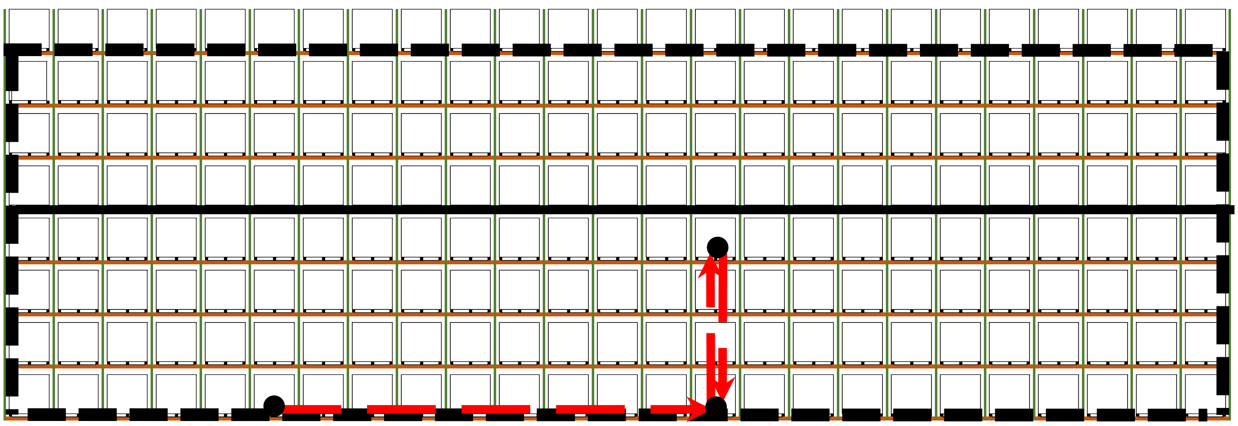

Figure 1. Front view of a storage rack.

In this section, travel time first and second moments for different travel paths of the NALT are presented for dedicated-storage and random-storage policies with one class or two classes of skus. Figure 1 shows a storage rack with the I/O point located at the lower left corner and the pick positions at floor level and mezzanine level, located at a mid-point vertically in the storage rack. There are 8 reserve storage locations above each pick position. For each S/R aisle, there are 35 columns of storage, with each column representing a pick position at floor level or mezzanine level. The solid horizontal black line denotes the mezzanine with pick positions; the dashed black lines enclose a region within which the NALT travels. We approximate the discrete travel region of the rack with a continuous space representation; we establish the time-based coordinates with two classes of skus (A and B) in Figure 2. In the time-based rectangular space, the NALT travels horizontally and vertically within the time-based rectangle having coordinates (0, 0), (0, *b*), (*a*, 0) and (*a*, *b*).

In Figure 1, the I/O point is at the lower left-hand corner and the black dots at floor level represent two pick positions. The red lines depict the travel path starting from a random floor-level pick location to a random reserve-storage location for a floor-level pick position requiring replenishment and then travel to the pick position. For dedicated storage, the reserve-storage location of a particular pick position is immediately above the pick position; therefore, the pick position and its reserve-storage locations share the same *x*-coordinate. For random storage, we assume the reserve-storage locations for pick positions within a class of SKUs are uniformly distributed over the entire reserve-storage region for the class. For dedicated storage, we assume the vertical reserve-storage locations for pick positions are uniformly distributed along the y-axis.

4.4.1. Travel time for Layout 1

For Layout 1 in Figure 2, class A skus represent fast movers and class B skus represent slow movers. B_1 denotes the part of class B skus located at floor level pick positions and B_2 denotes the part of class B skus located at mezzanine level pick positions. Nine different NALT travel paths are considered: path AA, path B_1B_1, path B_2B_2, path AB_1, path AB_2, path B_1A, path B_1B_2, path B_2A and path B_2B_1. Therefore, nine sets of travel time pdfs must be developed. For the nine travel paths, the expected values and second moments of travel time are obtained from the travel time pdfs.

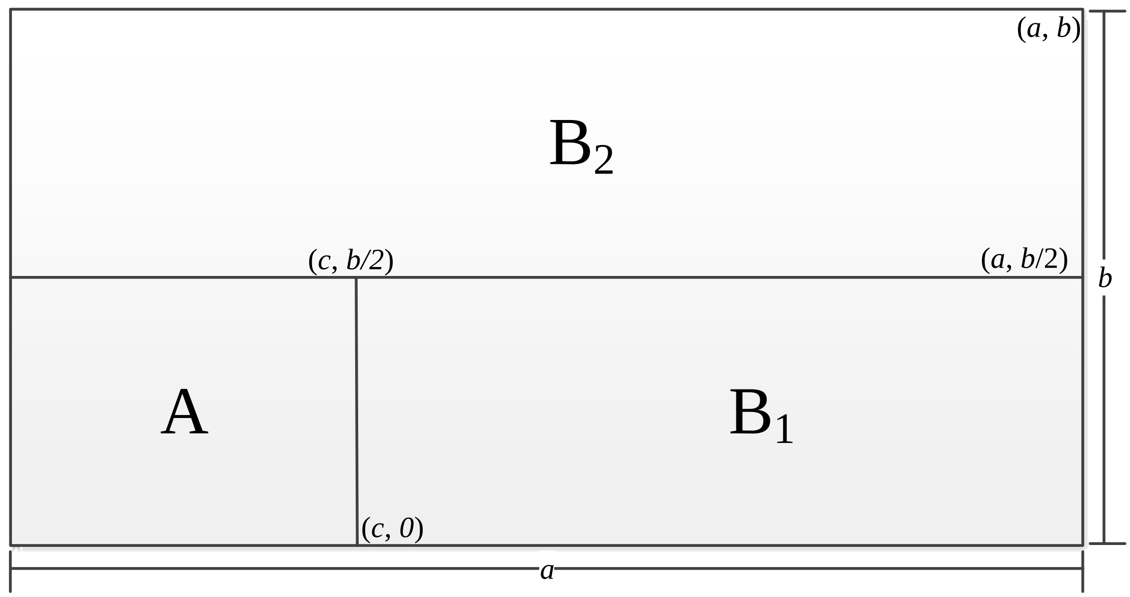

Figure 2: Layout 1 with class A at floor level

4.4.1.1. Vertical travel time

For storage operations and retrieval operations with dedicated storage and random storage within each class, two cases must be considered for vertical travel time: one for skus (class A and class B_1 in Figure 2) located at floor level and one for skus located at mezzanine level (class B_2).

First two moments for floor-level class skus:

From Appendix A with $d = b/2$, the first two moments for vertical travel time are:

$$E[T_v] = b/2 \tag{4.1}$$

$$E\left[{T_v}^2\right] = b^2/3 \tag{4.2}$$

First two moments for mezzanine-level class skus:

From Appendix B with $d = b/2$, the first two moments for vertical travel time are:

$$E[T_v] = 3b/2 \tag{4.3}$$

$$E\left[{T_v}^2\right] = 7b^2/3 \tag{4.4}$$

4.4.1.2. Storage operation horizontal travel time for Layout 1

First two moments for horizontal travel time for path AA or path B_2B_2:

From Appendix C, the first two moments for horizontal travel time for the travel path AA are:

$$E[T_h] = c \tag{4.5}$$

$$E\left[{T_h}^2\right] = 7c^2/6 \tag{4.6}$$

For the travel path B_2B_2, the first two moments for horizontal travel time are obtained by replacing c with a in the formulas for the first two moments for horizontal travel time for the travel path AA.

First two moments for Horizontal travel time for path B_1B_1:

From Appendix D, the first two moments for horizontal travel time are:

$$E[T_h] = a + c \tag{4.7}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(7a^2 + 10ac + 7c^2) \tag{4.8}$$

First two moments for horizontal travel time for path AB_1 or path B_1A:

From Appendix E, the first two moments for horizontal travel time for the travel path AB_1 or the travel path B_1A are:

$$E[T_h] = a/2 + c \tag{4.9}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + 5ac + 7c^2) \tag{4.10}$$

First two moments for horizontal travel time for path AB_2 or path B_2A:

From Appendix F, the first two moments for horizontal travel time for the travel path AB_2 or the travel path B_2A are:

$$E[T_h] = a/2 + c/2 \tag{4.11}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + 3ac + 2c^2) \tag{4.12}$$

First two moments for horizontal travel time for path B_1B_2 or path B_2B_1:

From Appendix G, the first two moments for horizontal travel time for the travel path B_1B_2 or the travel path B_1B_2 are:

$$E[T_h] = a + c/2 \tag{4.13}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(7a^2 + 5ac + 2c^2) \tag{4.14}$$

4.4.1.3. Retrieval operation horizontal travel time with dedicated storage for Layout 1

First two moments for horizontal travel time for path AA, path B_1B_1 or path B_2B_2:

From Appendix H, the first two moments for horizontal travel time for the travel path AA are

$$E[T_h] = c/3 \tag{4.15}$$

$$E\left[{T_h}^2\right] = c^2/6 \tag{4.16}$$

For travel path B_1B_1, the first two moments for horizontal travel time are obtained by replacing c with a-c in the formulas for the first two moments for horizontal travel time for travel path AA. For travel path B_2B_2, the first two moments for horizontal travel time are obtained by replacing c with a in the formulas for the first two moments for horizontal travel time for travel path AA.

First two moments for horizontal travel time for path AB_1 or path B_1A:

From Appendix I, the first two moments for horizontal travel time for the travel path AB_1 or the travel path B_1A are

$$E[T_h] = a/2 \tag{4.17}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 - ac + c^2) \tag{4.18}$$

First two moments for horizontal travel time for path AB_2, path B_2A, B_1B_2 or B_2B_1:

From Appendix J, the first two moments for horizontal travel time for the travel path AB_2 or the travel path B_2A are

$$E[T_h] = \frac{a}{2} - \frac{c}{2} + \frac{c^2}{3a} \tag{4.19}$$

$$E[T_h{}^2] = \frac{a^2}{3} - \frac{ac}{2} + \frac{c^2}{3} \tag{4.20}$$

For the travel path B_1B_2 or travel path B_2B_1, the first two moments for horizontal travel time are obtained by replacing c with *a-c* in the formulas for the first two moments for horizontal travel time for travel path AB_2.

4.4.1.4. Retrieval operation horizontal travel time with random storage for Layout 1

First two moments for horizontal travel time for path AA, B_1B_1 or B_2B_2:

From Appendix K, the first two moments for horizontal travel time for the travel path AA are

$$E[T_h] = \frac{2c}{3} \tag{4.21}$$

$$E[T_h{}^2] = \frac{17c^2}{30} \tag{4.22}$$

For the travel path B_1B_1, the first two moments for horizontal travel time are obtained by replacing *c* with *a-c* in the formulas for the first two moments for horizontal travel time for travel path AA. For the travel path B_2B_2, the first two moments for horizontal travel time are obtained by replacing *c* with *a* in the formulas for first two moments for horizontal travel time for travel path AA.

Horizontal travel time first two moments for path AB_1 or B_1A:

From Appendix L, the first two moments for horizontal travel time for the travel path AB_1 are

$$E[T_h] = \frac{1}{6}(5a - 2c) \tag{4.23}$$

$$E[T_h{}^2] = \frac{1}{6}(5a^2 - 5ac + 2c^2) \tag{4.24}$$

For the travel path B_1A, the first two moments for horizontal travel time are obtained by replacing *c* with *a-c* in the formulas for the first two moments for horizontal travel time for travel path AB_1.

First two moments for horizontal travel time for path AB_2 or B_1B_2:

From Appendix M, the first two moments for horizontal travel time for the travel path AB_2 are

$$E[T_h] = \frac{5a}{6} - \frac{c}{2} + \frac{c^2}{3a} \tag{4.25}$$

$$E\left[{T_h}^2\right] = \frac{5a^2}{6} - \frac{5ac}{6} + \frac{2c^2}{3} - \frac{c^3}{6a} + \frac{c^4}{15a^2} \tag{4.26}$$

For the travel path B_1B_2, the first two moments for horizontal travel time are obtained by replacing c with a-c in the formulas for the first two moments for horizontal travel time for travel path AB_2.

First two moments for horizontal travel time for path B_2A or B_2B_1:

From Appendix N, the first two moments for horizontal travel time for the travel path B_2A are

$$E[T_h] = \frac{a}{2} - \frac{c}{6} + \frac{c^2}{3a} \tag{4.27}$$

$$E\left[{T_h}^2\right] = \frac{a^2}{3} - \frac{ac}{6} + \frac{c^2}{6} + \frac{7c^3}{30a} \tag{4.28}$$

For the travel path B_2B_1, the first two moments for horizontal travel time are obtained by replacing c with a-c in the formulas for the first two moments for horizontal travel time for the travel path B_2A.

4.4.2. Travel time for Layout 2

For Layout 2 in Figure 3, fast movers (class A skus) are located at the leftmost part of the rack and slow movers (class B skus) are located to the right of the fast movers. For class A skus, A_1 denotes the part of class A skus located at floor level and A_2 denotes the part of class A skus located at mezzanine level. For class B skus, B_1 denotes the part of class B skus located at floor level and B_2 denotes the part of class B skus located at mezzanine level. The number of class A or class B skus located at floor level equals the number of class A or class B skus located at mezzanine level.

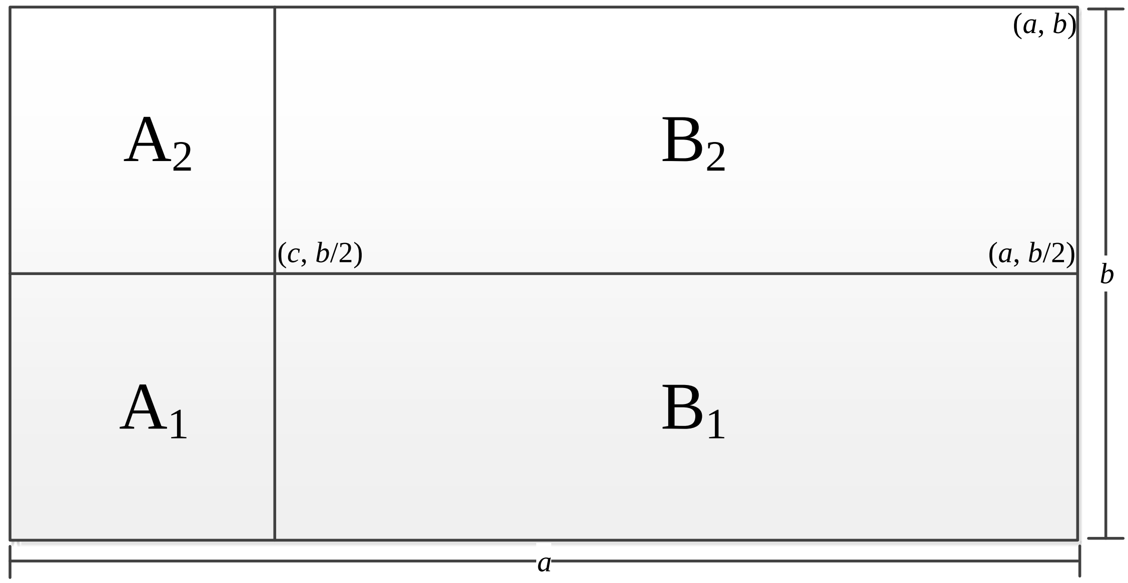

Figure 3: Layout 2 with each class divided equally between floor level and mezzanine level

The first and second moments for vertical travel time for Layout 2 in Figure 3 can be obtained from the results for Layout 1 in Figure 2. The moments for vertical travel time for floor-level class A_1 or class B_1 skus in Figure 3 are the same as the moments for vertical travel time for class A_1 or class B_1 in Figure 2. The moments for vertical travel time for mezzanine-level class A_2 or class B_2 skus in Figure 3 are the same as the moments for vertical travel time for class B_2 in Figure 2.

For storage and retrieval operations with dedicated storage or random storage, the first two moments for horizontal travel time for travel path AA in Figure 3 are the same as the first two moments for horizontal travel time for travel path A_1A_1 in Figure 2. The first two moments for horizontal travel time for travel path BB in Figure 3 are the same as the first two moments for horizontal travel time for travel path B_1B_1 in Figure 2. The first two moments for horizontal travel time for travel path AB in Figure 3 are the same as the first two moments for horizontal travel time for travel path AB_1 in Figure 2. The first two moments for horizontal travel time for travel path BA in Figure 3 are the same as the moments for travel path B_1A in Figure 2.

4.4.3. Travel time for Layout 3

For Layout 3 in Figure 4, fast movers (class A skus) are located at the left end of the S/R aisle; a portion of fast movers are located at floor level pick positions, with the balance located at mezzanine level pick positions. Slow movers (class B skus) are located to the right of fast movers at both floor and mezzanine levels. For class A skus, A_1 denotes the portion of class A skus located at floor level and A_2 denotes the part of class A skus located at mezzanine level. For class B skus, B_1 denotes the portion of class B skus located at floor level and B_2 denotes the portion of class B skus located at mezzanine level. The number of class A skus located at floor level is greater than the number located at mezzanine level. Likewise, the number of class B skus located at floor level is less than the number located at mezzanine level.

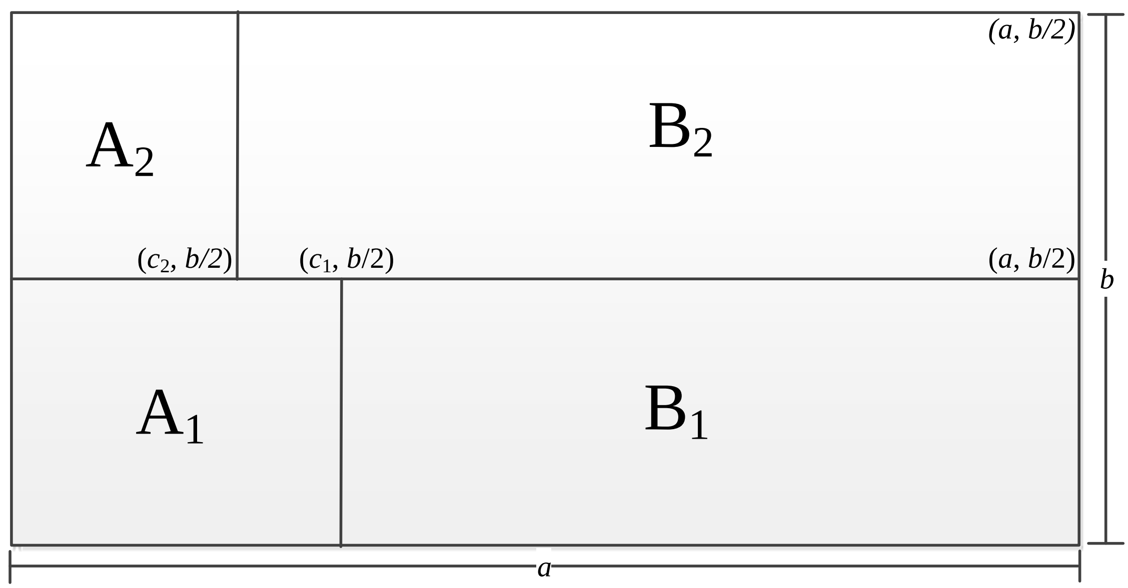

Figure 4: Layout 3 with each class divided unequally between floor level and mezzanine level

4.4.3.1. Vertical travel time for Layout 3

The first two moments for vertical travel time for floor-level class A_1 or class B_1 skus for Layout 3 in Figure 4 are the same as the first two moments for vertical travel time for class A_1 or class B_1 skus for Layout 1 in Figure 2. The first two moments for vertical travel time for mezzanine-level for class A_2 or class B_2 skus in Figure 4 are the same as the first two moments for vertical travel time for class B_2 for Layout 1 in Figure 2.

4.4.3.2. Storage operation horizontal travel time for Layout 3

The first two moments for storage operation horizontal travel time of each travel path can be obtained by combining the first two moments for storage operation horizontal travel time of the travel paths in Layout 1 in Figure 2.

First two moments for horizontal travel time for path A_1A_1 or path A_2A_2:

For the travel path A1A1, the first two moments for horizontal travel time are

$$E[T_h] = c_1 \tag{4.29}$$

$$E\left[{T_h}^2\right] = 7{c_1}^2/6 \tag{4.30}$$

For the travel path A_2A_2, the first two moments for horizontal travel time are obtained by replacing c_1 with c_2 in the formulas for the first two moments for horizontal travel time for travel path A_1A_1.

First two moments for horizontal travel time for path B_1B_1 or path B_2B_2:

For the travel path B_1B_1, the first two moments for horizontal travel time are

$$E[T_h] = a + c_1 \tag{4.31}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}\left(7a^2 + 10ac_1 + 7{c_1}^2\right) \tag{4.32}$$

For the travel path B_2B_2, the horizontal travel time first two moments are obtained by replacing c_1 with c_2 in the formulas for the horizontal travel time first two moments for travel path B_1B_1.

First two moments for horizontal travel time for path A_1B_1 or path B_1A_1:

$$E[T_h] = a/2 + c_1 \tag{4.33}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + 5ac_1 + 7{c_1}^2) \tag{4.34}$$

First two moments for horizontal travel time for path A_2B_2 or path B_2A_2:

$$E[T_h] = a/2 + c_2 \tag{4.35}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + 5ac_2 + 7{c_2}^2) \tag{4.36}$$

First two moments for horizontal travel time for path A_1A_2 or path A_2A_1:

$$E[T_h] = c_1/2 + c_2/2 \tag{4.37}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2{c_1}^2 + 3c_1c_2 + 2{c_2}^2) \tag{4.38}$$

First two moments for horizontal travel time for path A_1B_2 or path B_2A_1:

$$E[T_h] = \frac{1}{2}(a + c_1 + c_2) \tag{4.39}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + 3ac_1 + 2{c_1}^2 + 2ac_2 + 3c_1c_2 + 2{c_2}^2) \tag{4.40}$$

First two moments for horizontal travel time for path A_2B_1 or path B_1A_2:

$$E[T_h] = \frac{1}{2}(a + c_1 + c_2) \tag{4.41}$$

$$E\left[{T_h}^2\right] = \frac{2a^3+2{c_1}^3+3a^2c_2+3{c_1}^2c_2+2a{c_2}^2+2c_1{c_2}^2-14{c_2}^3}{6a-6c_1} \tag{4.42}$$

First two moments for horizontal travel time for path B_1B_2 or path B_2B_1:

$$E[T_h] = \frac{1}{2}(2a + c_1 + c_2) \tag{4.43}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(7a^2 + 2{c_1}^2 + 3c_1c_2 + 2{c_2}^2 + 5ac_1 + 5ac_2) \tag{4.44}$$

4.4.3.3. Retrieval operation horizontal travel time with dedicated storage for Layout 3

The first two moments for retrieval operation horizontal travel time of each travel path with dedicated storage can be obtained by combining the first two moments for retrieval operation horizontal travel time of the travel paths with dedicated storage in Layout 1 in Figure 2.

First two moments for horizontal travel time for path A_1A_1, path A_2A_2, path B_1B_1 or path B_2B_2:

For the travel path A1A1, the first two moments for horizontal travel time are

$$E[T_h] = c_1/3 \tag{4.45}$$

$$E\left[{T_h}^2\right] = {c_1}^2/6 \tag{4.46}$$

For the travel path A_2A_2, the first two moments for horizontal travel time are obtained by replacing c_1 with c_2 in the formulas for the first two moments for horizontal travel time for travel path A_1A_1. For the travel path B_1B_1, the first two moments for horizontal travel time are obtained by replacing c_1 with $a - c_1$ in the formulas for the first two moments for horizontal travel time for travel path A_1A_1. For the travel path B_2B_2, the first two moments for horizontal travel time are obtained by replacing c_1 with $a - c_2$ in the formulas for the first two moments for horizontal travel time for travel path A_1A_1.

First two moments for horizontal travel time for path A_1B_1, path B_1A_1, path A_2B_2 or path B_2A_2:

For the travel path A_1B_1 or the travel path B_1A_1, the first two moments for horizontal travel time are

$$E[T_h] = a/2 \tag{4.47}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}\left(2a^2 - ac_1 + {c_1}^2\right) \tag{4.48}$$

For the travel path A_2B_2 or the travel path B_2A_2, the first two moments for horizontal travel time are obtained by replacing c_1 with c_2 in the formulas for the first two moments for horizontal travel time for travel path A_1B_1.

First two moments for horizontal travel time for path A_1A_2 or path A_2A_1:

$$E[T_h] = \frac{c_1}{2} - \frac{c_2}{2} + \frac{{c_2}^2}{3c_1}(\tag{4.49}$$

$$E\left[{T_h}^2\right] = \frac{{c_1}^2}{3} - \frac{c_1 c_2}{2} + \frac{{c_2}^2}{3} \tag{4.50}$$

First two moments for horizontal travel time for path B_1B_2 or path B_2B_1:

$$E[T_h] = \frac{c_1 - c_2}{2} + \frac{(a-c_1)^2}{3(a-c_2)} \tag{4.51}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(a^2 + 2{c_1}^2 - 3c_1c_2 + 2{c_2}^2 - ac_1 - ac_2) \tag{4.52}$$

First two moments for horizontal travel time for path A_1B_2 or path B_2A_1:

$$E[T_h] = \frac{3a^2c_1 - 3a{c_1}^2 + 2{c_1}^3 - 3{c_1}^2c_2 + 3c_1{c_2}^2 - 2{c_2}^3}{6ac_1 - 6c_1c_2} \tag{4.53}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 - 3ac_1 + 2{c_1}^2 + 2ac_2 - 3c_1c_2 + 2{c_2}^2) \tag{4.54}$$

First two moments for horizontal travel time for path A_2B_1 or path B_1A_2:

$$E[T_h] = \frac{1}{2}(a + c_1 - c_2) \tag{4.55}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + 2ac_1 + 2{c_1}^2 - 3ac_2 - 3c_1c_2 + 2{c_2}^2) \tag{4.56}$$

4.4.3.4. Retrieval operation horizontal travel time with random storage for Layout 3

The first two moments for retrieval operation horizontal travel time of each travel path with random storage can be obtained by combining the first two moments for retrieval operation horizontal travel time of the travel paths with random storage in Layout 1 in Figure 2.

First two moments for horizontal travel time for path A_1A_1, path A_2A2, path B_1B_1 or path B_2B_2:

For the travel path A_1A_1, the first two moments for horizontal travel time are

$$E[T_h] = \frac{2c_1}{3} \tag{4.57}$$

$$E\left[{T_h}^2\right] = \frac{17{c_1}^2}{30} \tag{4.58}$$

For the travel path A_2A_2, the first two moments for horizontal travel time are obtained by replacing c_2 with c_1 in the formulas for the first two moments for horizontal travel time for the travel path A_1A_1. For the travel path B_1B_1, the first two moments for horizontal travel time are obtained by replacing $a - c_1$ with c_1 in the formulas for the first two moments for horizontal travel time for the travel path A_1A_1. For the travel path B_2B_2, the first two moments for horizontal travel time are obtained by replacing $a - c_2$ with c_1 in the formulas for the first two moments for horizontal travel time for the travel path A_1A_1.

First two moments for horizontal travel time for path A_1B_1 or path A_2B_2:

For the travel path A_1B_1, the first two moments for horizontal travel time are

$$E[T_h] = \frac{1}{6}(5a - 2c_1) \tag{4.59}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(5a^2 - 5ac_1 + 2{c_1}^2) \tag{4.60}$$

For the travel path A_2B_2, the first two moments for horizontal travel time are obtained by replacing c_2 with c_1 in the formulas for the first two moments for horizontal travel time for the travel path A_1B_1.

First two moments for horizontal travel time for path B_1A_1 or path B_2A_2:

For the travel path B_1A_1, the first two moments for horizontal travel time are

$$E[T_h] = \frac{1}{6}(3a + 2c_1) \tag{4.61}$$

$$E\left[{T_h}^2\right] = \frac{1}{6}(2a^2 + ac_1 + 2{c_1}^2) \tag{4.62}$$

For the travel path B_2A_2, the first two moments for horizontal travel time are obtained by replacing c_2 with c_1 in the formulas for first two moments for horizontal travel time for the travel path B_1A_1.

First two moments for horizontal travel time for path A_1A_2:

$$E[T_h] = \frac{1}{6}\left(3c_1 - c_2 + \frac{2{c_2}^2}{c_1}\right) \tag{4.63}$$

$$E\left[{T_h}^2\right] = \frac{1}{30}\left(10{c_1}^2 - 5c_1c_2 + 5{c_2}^2 + \frac{7{c_2}^3}{c_1}\right) \tag{4.64}$$

First two moments for horizontal travel time for path A_2A_1:

$$E[T_h] = \frac{1}{6}\left(5c_1 - 3c_2 + \frac{2{c_2}^2}{c_1}\right) \tag{4.65}$$

$$E\left[{T_h}^2\right] = \frac{1}{30{c_1}^2}(25{c_1}^4 - 25{c_1}^3c_2 + 20{c_1}^2{c_2}^2 - 5c_1{c_2}^3 + 2{c_2}^4) \tag{4.66}$$

First two moments for horizontal travel time for path B_1B_2:

$$E[T_h] = \frac{1}{6}\left(4(a - c_2) - (c_1 - c_2) + \frac{2(c_1 - c_2)^2}{a - c_2}\right) \tag{4.67}$$

$$E\left[{T_h}^2\right] = \frac{1}{30(\mathrm{a} - c_2)^2}(17a^4 - 8a^3c_1 + 17a^2{c_1}^2 - 3a{c_1}^3 + 2{c_1}^4 - 60a^3c_2 - 10a^2c_1c_2 -$$

$$25a{c_1}^2c_2 - 5{c_1}^3c_2 + 95a^2{c_2}^2 + 35ac_1{c_2}^2 + 20{c_1}^2{c_2}^2 - 75a{c_2}^3 - 25c_1{c_2}^3 + 25{c_2}^4) \tag{4.68}$$

First two moments for horizontal travel time for path B_2B_1:

$$E[T_h] = \frac{1}{6}\left[-a + c_1 + \frac{2(a - c_1)^2}{a - c_2} + 3(a - c_2)\right] \tag{4.69}$$

$$E\left[{T_h}^2\right] = \frac{1}{30}\left[5(a - c_1)^2 + \frac{7(a - c_1)^3}{a - c_2} - 5(a - c_1)(a - c_2) + 10(a - c_2)^2\right] \tag{4.70}$$

First two moments for horizontal travel time for path A_1B_2 or path B_2A_1:

For the travel path A_1B_2, the first two moments for horizontal travel time are

$$E[T_h] = \frac{5a^2c_1 - 7a{c_1}^2 + 6{c_1}^3 - 7{c_1}^2c_2 + 5c_1{c_2}^2 - 2{c_1}^3}{6ac_1 - 6c_1c_2} \tag{4.71}$$

$$E\left[{T_h}^2\right] = \frac{25a^3c_1 - 50a^2{c_1}^2 + 35a{c_1}^3 + 7{c_1}^4 - 43{c_1}^3c_2 + 52{c_1}^2{c_2}^2 - 33c_1{c_2}^3 + 7{c_2}^4}{30ac_1 - 30c_1c_2} \tag{4.72}$$

For the travel path B$_2$A$_1$, the first two moments for horizontal travel time are obtained by replacing c_1 with $a - c_2$ and c_2 with $a - c_1$ in the formulas for the first two moments for horizontal travel time for the travel path A$_1$B$_2$.

First two moments for horizontal travel time for path A$_2$B$_1$ or path B$_1$A$_2$:

For the travel path A$_2$B$_1$, the first two moments for horizontal travel time are

$$E[T_h] = \frac{1}{6}(5a + c_1 - 3c_2) \tag{4.73}$$

$$E\left[T_h{}^2\right] = \frac{1}{6}(5a^2 + c_1{}^2 - 5ac_2 - c_1c_2 + 2c_2{}^2) \tag{4.74}$$

For the travel path B$_1$A$_2$, the first two moments for horizontal travel time are obtained by replacing c_1 with $a - c_2$ and c_2 with $a - c_1$ in the formulas for the first two moments for horizontal travel time for the travel path A$_2$B$_1$.

4.4.4. Overall expected value and second moment of travel time

$$E[T_h] = E[T_h] + E[T_v] \tag{4.75}$$

$$E\left[T_h{}^2\right] = E\left[T_h{}^2\right] + E\left[T_v{}^2\right] + 2E[T_h]E[T_v] \tag{4.76}$$

$E\left[T_{ij}\right]$ and $E\left[T_{ij}{}^2\right]$ are the expected value and second moment of NALT travel time for travel path ij. $E[T_h]$ and $E\left[T_h{}^2\right]$ are first two moments of horizontal NALT travel time for the travel path ij. $E[T_v]$ and $E\left[T_v{}^2\right]$ are the first two moments of vertical NALT travel time for the travel path ij.

$$E[T_s] = \sum_i \sum_j p_{ij} E[T_{ij}] \tag{4.77}$$

$$E\left[T_s{}^2\right] = \sum_i \sum_j p_{ij} E[T_{ij}{}^2] \tag{4.78}$$

$$E[S_s] = E[T_s] + T_{PD} \tag{4.79}$$

$$E\left[S_s{}^2\right] = E\left[T_s{}^2\right] + T_{PD}{}^2 + 2T_{PD}E[T_s] \tag{4.80}$$

where p_{ij} is the probability travel path *ij* is taken in performing a storage operation, $E[T_{ij}]$ and $E[{T_{ij}}^2]$ are the first two moments of the travel time for travel path *ij*.

$$E[T_r] = \sum_i \sum_j p_{ij} E[T_{ij}] \tag{4.81}$$

$$E\left[{T_r}^2\right] = \sum_i \sum_j p_{ij} E[{T_i}^2] \tag{4.82}$$

$$E[S_r] = E[T_r] + T_{PD} \tag{4.83}$$

$$E\left[{S_r}^2\right] = E\left[{T_r}^2\right] + {T_{PD}}^2 + 2T_{PD}E[T_r] \tag{4.84}$$

where p_{ij} is the probability travel path *ij* is taken in performing a retrieval operation; $E[T_{ij}]$ and $E[{T_{ij}}^2]$ are the first two moments of the travel time for travel path *ij*.

$$E[S] = \frac{\lambda_r}{\lambda_r+\lambda_s} E[S_r] + \frac{\lambda_s}{\lambda_r+\lambda_s} E[S_s] \tag{4.85}$$

$$E[S] = \frac{\lambda_r}{\lambda_r+\lambda_s} E\left[{S_r}^2\right] + \frac{\lambda_s}{\lambda_r+\lambda_s} E\left[{S_s}^2\right] \tag{4.86}$$

4.5. Non-preemptive priority and non-priority queueing models

In the MIAPP-AS/RS system, travel of the NALT originates at a pick position and ends after replenishing another pick position or a reserve-storage location. Based on results obtained by Liu, et al (2019), for the number of pick positions we consider, we model the MIAPP-NALT system with a finite number of pick positions as an M/G/1/∞ queue.

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4.5.1. Non-preemptive priority queueing

The priority queue we consider, with two types of customers and non-preemptive priority service time discipline, is denoted M/G/1/NPRP/∞. From results in Jaiswal (1968), and Cox and Smith (1961), we obtain the following queueing steady state performance measures:

$$L_r = \frac{{\lambda_r}^2 E({S_r}^2) + \lambda_r \lambda_s E({S_s}^2)}{2(1-\rho_r)} + \rho_r, \tag{4.87}$$

$$L_s = \frac{\lambda_r \lambda_s E({S_r}^2) + {\lambda_s}^2 E({S_s}^2)}{2(1-\rho_r)(1-\rho_r-\rho_s)} + \rho_s, \tag{4.88}$$

$$L = L_r + L_s. \tag{4.89}$$

where λ_r is the arrival rate for retrieval requests and λ_s is the arrival rate for storage requests; the overall arrival rate for retrieval request or storage request is given by $\lambda = \lambda_r + \lambda_s$; μ_r is the service rate for retrieval requests and μ_s is the service rate for storage requests; ρ_r is the traffic intensity for retrieval requests and ρ_s is the traffic intensity for storage requests; $\mathrm{E}({S_r}^2)$ is the second moment of the service time pdf for retrieval requests and $\mathrm{E}({S_s}^2)$ is the second moment of the service time pdf for storage requests.

4.5.2. Non-priority queueing

The steady state expected number of customers in the system for the M/G/1/FCFS/∞ queue is given by the Pollaczek-Khinchine formula (Gross et al. 2008):

$$L = \rho + \frac{\lambda^2 Var[S] + \rho^2}{2(1-\rho)}, \tag{4.90}$$

$$L_r = \frac{\lambda_r}{\lambda_r + \lambda_s} \frac{\lambda^2 Var[S] + \rho^2}{2(1-\rho)} + \rho_r, \tag{4.91}$$

$$L_s = \frac{\lambda_s}{\lambda_r + \lambda_s} \frac{\lambda^2 Var[S] + \rho^2}{2(1-\rho)} + \rho_s. \tag{4.92}$$

where λ_r is the arrival rate for retrieval requests and λ_s is the arrival rate for storage requests; the overall arrival rate is given by $\lambda = \lambda_r + \lambda_s$

$$\mu = \frac{\lambda_r+\lambda_s}{\rho_r+\rho_s}, \tag{4.93}$$

$$E[S] = \frac{\lambda_r}{\lambda_r+\lambda_s}E[S_r] + \frac{\lambda_s}{\lambda_r+\lambda_s}E[S_s], \tag{4.94}$$

$$E[S^2] = \frac{\lambda_r}{\lambda_r+\lambda_s}E\left[{S_r}^2\right] + \frac{\lambda_s}{\lambda_r+\lambda_s}E\left[{S_s}^2\right], \tag{4.95}$$

$$Var[S] = E[S^2] - (E[S])^2. \tag{4.96}$$

4.6. Computational results and insights

In this section, we obtain computational results for an MIAPP-NALT operation and present insights gained. First, we compare results with non-preemptive priority and non-priority queues. Next, we compare results from dedicated storage and random storage policies. Third, results with skewnesses for the sku classes are analyzed. Fourth, results obtained for three different layouts of the two sku classes are compared. This section concludes with insights and practical considerations.

Consider an MIAPP-NALT operation with 400 skus in each S/R aisle, with each SKU assigned to one pick position. Six potential scenarios are considered: 1) dedicated storage policy for all SKUs and all SKUs belonging to one class; 2) dedicated storage policy for two classes of SKUs with class 1 (fast movers) located at floor level; 3) dedicated storage policy for two classes of SKUs with class 1 SKUs located at floor and mezzanine levels with fast movers located closest to the I/O; 4) scenario 1, but with a random storage; 5) scenario 2, but with a random storage policy; 6) scenario 3, but with a random storage policy.

With 400 pick positions (*N*) in each S/R aisle and 100 pick positions on each side of the floor level and mezzanine level, we assume the horizontal distance between the centerlines of two adjacent pick positions is 4 feet and the vertical distance between two adjacent reserve-storage levels is 5 feet. With 100 pick positions equally spaced along the S/R aisle, its length (A) is 4×100 or 400 feet; with 8 vertical pallet positions in the rack, the height of the rack (B) is 5×8 or 40 feet.. Let the horizontal travel speed (V_h) of the NALT be 200 feet per minute and its vertical travel speed (V_v) be 50 feet per minute; the time to travel a horizontal distance of A is 400/200 and the vertical travel time for a distance of B is 40/50.

Table 1. Service times used in queueing models.

queueing parameters		Layout 1					Layout 2	Layout 3
		dedicated				random	dedicated	dedicated
		10(90)	20(90)	30(70)	50(50)	50(50)	30(70)	30(70)
retrieval operation	expected value	0.721	0.819	1.138	1.467	2.133	1.360	1.244
	second moment	0.802	0.914	1.665	2.587	5.253	2.309	2.011
	variance	0.282	0.243	0.370	0.436	0.702	0.460	0.464
	service rate	1.386	1.221	0.879	0.682	0.469	0.735	0.804
storage operation	expected value	1.022	1.400	2.114	2.800	2.800	2.000	1.986
	second moment	1.419	2.359	5.211	8.720	8.720	4.773	4.709
	variance	0.374	0.399	0.741	0.880	0.880	0.773	0.764
	service rate	0.978	0.714	0.473	0.357	0.357	0.500	0.503

To apply the queueing model in Section 5, the first two moments and the variance of NALT service time must be calculated. Service times in Table 1 were obtained from formulas provided in Section 4.

4.6.1. Results for priority and non-priority queues without customer classes

With dedicated storage for the layout with 400 horizontal pick positions and 8 vertical pallet positions, consider a situation in which each sku can be assigned to any pick position. Retrieval requests have non-preemptive priority over storage requests; there are two classes of skus (fast movers and slow movers). Values for the expected number of retrieval and storage requests waiting or being served (L) are provided in Table 2 for the priority queue and the non-priority queue. For the latter, the arrival rate used to obtain values for L is $\lambda_s + \lambda_r$.

Table 2. Results obtained for the non-preemptive priority queue and non-priority queue.

arrival rates	traffic intensity	priority queue			non-priority queue		
λ_r=λ_s	$\rho_r + \rho_s$	L_r	L_s	$L_{priority}$	L_r	L_s	$L_{non\text{-}priority}$
0.130	0.633	0.384	0.822	1.206	0.553	0.726	1.279
0.135	0.657	0.407	0.909	1.315	0.612	0.792	1.403
0.140	0.681	0.430	1.008	1.438	0.679	0.866	1.545
0.145	0.706	0.455	1.124	1.579	0.758	0.951	1.709
0.150	0.730	0.480	1.261	1.741	0.850	1.050	1.901
0.155	0.754	0.506	1.426	1.933	0.961	1.167	2.128
0.160	0.779	0.533	1.628	2.162	1.095	1.308	2.403
0.165	0.803	0.561	1.881	2.443	1.262	1.482	2.744
0.170	0.827	0.590	2.207	2.798	1.476	1.703	3.178
0.175	0.852	0.621	2.641	3.262	1.759	1.993	3.752
0.180	0.876	0.652	3.249	3.900	2.153	2.393	4.546
0.185	0.900	0.684	4.156	4.840	2.739	2.985	5.724
0.190	0.925	0.717	5.655	6.372	3.701	3.955	7.656
0.195	0.949	0.752	8.593	9.345	5.581	5.841	11.422
0.200	0.973	0.788	16.911	17.699	10.888	11.155	22.043
0.205	0.998	0.825	198.956	199.781	126.857	127.131	253.988

As demonstrated by the results in Table 2, assigning a higher priority to replenishment requests reduces the overall number of requests for storage or replenishment waiting or being fulfilled (L). Likewise, the greater the traffic intensity, the greater the difference in the values of L_r and L_s with and without priority. As expected, providing priority service to replenishment requests provides smaller values of L does FCFS service.

4.6.2. Queueing results for dedicated and random storages

Here, we compare dedicated and random storage policies when all SKUs belong to one class and priority is given to replenishment requests. Based on service rates for dedicated and random storage given in Table 1, we obtain values for the number of replenishment requests waiting or being fulfilled and the number of storage requests waiting or being fulfilled. As expected, dedicated storage outperforms random storage. As traffic intensity increases, the benefit of using dedicated storage increases. However, the results in Table 3 are based on the same amount of space for dedicated and random storage; it is well known random storage requires less space than dedicated storage. Based on the work of Liu, et al (2019), we can expect a significant reduction in the space required for random storage must occur in order for it to yield the same value for $L_{\text{¸}}$ as obtained with dedicated storage.

Table 3. Results obtained for the non-preemptive priority queue with dedicated storage and random storage with Layout 1.

arrival rates	dedicated storage				random storage			
λ_r=λ_s	$\rho_r + \rho_s$	L_r	L_s	$L_{priority}$	$\rho_r + \rho_s$	L_r	L_s	$L_{priority}$
0.105	0.511	0.281	0.520	0.800	0.581	0.382	0.628	1.010
0.110	0.535	0.300	0.568	0.868	0.609	0.409	0.702	1.111
0.115	0.560	0.320	0.621	0.941	0.636	0.437	0.789	1.226
0.120	0.584	0.340	0.681	1.021	0.664	0.466	0.890	1.356
0.125	0.608	0.362	0.747	1.109	0.692	0.496	1.011	1.507
0.130	0.633	0.384	0.822	1.206	0.719	0.528	1.157	1.684
0.135	0.657	0.407	0.909	1.315	0.747	0.561	1.336	1.897
0.140	0.681	0.430	1.008	1.438	0.775	0.595	1.563	2.158
0.145	0.706	0.455	1.124	1.579	0.802	0.631	1.856	2.487
0.150	0.730	0.480	1.261	1.741	0.830	0.668	2.248	2.917
0.155	0.754	0.506	1.426	1.933	0.858	0.707	2.799	3.507
0.160	0.779	0.533	1.628	2.162	0.885	0.748	3.624	4.372
0.165	0.803	0.561	1.881	2.443	0.913	0.791	4.985	5.776
0.170	0.827	0.590	2.207	2.798	0.941	0.835	7.635	8.471
0.175	0.852	0.621	2.641	3.262	0.968	0.882	14.955	15.837
0.180	0.876	0.652	3.249	3.900	0.996	0.931	123.884	124.815

4.6.3. Impact on queueing performance of skewness for class A skus

To assess the impact of the ratio of the fraction of skus that are class A to the fraction of throughput represented by class A (skewness) on queueing performance, we consider dedicated storage results for Layout 1. Suppose all SKUs are separated into two classes: fast movers (class A) and slow movers (class B). If X percent of skus are class A and class A skus account for Y percent of overall throughput, we designate it as X(Y) skewness. We consider four skewnesses: 10(90); 20(90); 30(70); and 50(50). A skewness of 50(50) means all SKUs belong to the same class.

Table 4. Results obtained for Layout 1 with different relative sizes and throughputs for class A SKUs.

arrival rates	10(90)		20(90)		30(70)		50(50)	
$\lambda_r = \lambda_s$	$\rho_r + \rho_s$	$L_{priority}$	$\rho_r + \rho_s$	$L_{priority}$	$\rho_r + \rho_s$	$L_{priority}$	$\rho_r + \rho_s$	$L_{priority}$
0.125	0.293	0.367	0.352	0.463	0.482	0.733	0.608	1.109
0.130	0.305	0.387	0.366	0.488	0.501	0.782	0.633	1.206
0.135	0.316	0.406	0.381	0.515	0.520	0.834	0.657	1.315
0.140	0.328	0.426	0.395	0.542	0.539	0.890	0.681	1.438
0.145	0.340	0.447	0.409	0.570	0.559	0.949	0.706	1.579
0.150	0.352	0.468	0.423	0.600	0.578	1.013	0.730	1.741
0.155	0.363	0.490	0.437	0.630	0.597	1.082	0.754	1.933
0.160	0.375	0.512	0.451	0.662	0.616	1.156	0.779	2.162
0.165	0.387	0.535	0.465	0.694	0.636	1.238	0.803	2.443
0.170	0.398	0.559	0.479	0.729	0.655	1.326	0.827	2.798
0.175	0.410	0.583	0.493	0.764	0.674	1.424	0.852	3.262
0.180	0.422	0.609	0.507	0.802	0.693	1.532	0.876	3.900
0.185	0.434	0.635	0.522	0.841	0.713	1.654	0.900	4.840
0.190	0.445	0.662	0.536	0.882	0.732	1.790	0.925	6.372
0.195	0.457	0.690	0.550	0.925	0.751	1.946	0.949	9.345
0.200	0.469	0.718	0.564	0.970	0.770	2.125	0.973	17.699
0.205	0.480	0.748	0.578	1.018	0.790	2.334	0.998	199.781

From the results in Table 4, the more skewed the AB curve is, the more preferred is the two-class-based storage policy. Further, the greater the traffic intensity, the greater the benefit of separating skus into two classes.

4.6.4. Comparison of Layout 1, Layout 2, and Layout 3

Table 5 provides non-premptive priority results for a dedicated storage policy with a 30(70) skewness for the two classes of skus and the three layouts. For Layout 1, all class A skus are located at floor level pick positions; for Layout 2, one-half of class A skus are located at floor level pick positions; and, for Layout 3, 80% of class A skus are located at floor level pick positions.

Table 5. Results obtained for the non-preemptive priority queue with Layout 1, Layout 2 and Layout 3.

arrival rates	Layout 1		Layout 2		Layout 3	
$\lambda_r = \lambda_s$	$\rho_r + \rho_s$	$L_{priority}$	$\rho_r + \rho_s$	$L_{priority}$	$\rho_r + \rho_s$	$L_{priority}$
0.170	0.655	1.326	0.673	1.432	0.651	1.321
0.175	0.674	1.424	0.693	1.546	0.670	1.418
0.180	0.693	1.532	0.713	1.674	0.689	1.526
0.185	0.713	1.654	0.733	1.819	0.709	1.647
0.190	0.732	1.790	0.752	1.985	0.728	1.783
0.195	0.751	1.946	0.772	2.178	0.747	1.937
0.200	0.770	2.125	0.792	2.405	0.766	2.115
0.205	0.790	2.334	0.812	2.678	0.785	2.322
0.210	0.809	2.583	0.832	3.011	0.804	2.567
0.215	0.828	2.884	0.851	3.429	0.824	2.862
0.220	0.847	3.257	0.871	3.971	0.843	3.225
0.225	0.867	3.733	0.891	4.705	0.862	3.686
0.230	0.886	4.365	0.911	5.758	0.881	4.289
0.235	0.905	5.247	0.931	7.404	0.900	5.118
0.240	0.925	6.570	0.950	10.350	0.919	6.333
0.245	0.944	8.788	0.970	17.190	0.938	8.294
0.250	0.963	13.300	0.990	51.051	0.958	12.011

With an objective of minimizing the value of L, as shown in Table 5, Layout 3 outperforms the other layouts. As traffic intensity increases, the benefit of using Layout 3 versus Layout 1 or Layout 2 increases. On a relative basis, Layout 1 outperforms Layout 2 and the results obtained with Layout 1 and Layout 3 are quite similar, regardless of the traffic intensity. Of course, the results obtained for Layout 3 are greatly dependent on the proportion located at floor level. Further, the degree to which Layout 3 outperforms Layout 1 and Layout 2 is greatly dependent on class A skewness.

4.6.5. Insights and practical considerations

From Sections 6.1, 6.2 , 6.3 and 6.3, the following insights and practical considerations are shared:

a. Priority queue performance will provide a smaller value of number of operation requests waiting and being filled in the MIAPP-NALT system in comparison to the non-priority queue. The difference between the value of L_r and L_s increases as traffic intensity increases.

b. From a queueing perspective, a dedicated storage policy outperforms a random storage policy. The difference in the values of L, using dedicated storage versus random storage, increases as traffic intensity increases.

c. With equal arrival rates for retrieval and storage requests, the preference for a class-based storage policy increases as the fraction of throughput represented by fast movers increases.

d. For three different class-based storage layouts, the relative ranking of the queueing performances of the three layouts is highly dependent on traffic intensity and the skewness of class A skus. Depending on the skewness, each layout can yield lower values for L than the other two. Therefore, one should exercise caution in claiming one layout is superior to the other layouts.

4.7. Summary, conclusions and recommendations

To model the MIAPP-NALT retrieval operation as an M/G/1 queueing system, travel time pdfs were derived for retrieval and storage operations for dedicated storage, random storage and class-based storage. From retrieval and storage travel time pdfs, the expected value, the second moment and the variance for service time were calculated after combining pick up and deposit times (T_{PD}). MIAPP-NALT retrieval and storage operations are first analyzed with priority and non-priority queueing models and the results were compared. For the priority queueing model and non-priority queueing model in Section 5, the expected value, the second moment and the variance of service time were calculated using formulas in Section 4. Using a steady state queueing performance measure, priority and non-priority queues were compared, dedicated and random storage were compared, the skewness of the AB curve was analyzed, and three different layouts were compared.

Based on the research performed, the following conclusions are drawn:

- For priority and non-priority queues, the priority queue outperforms the non-priority queue in terms of the number of retrieval or storage requests waiting to be or being filled. Another benefit of a priority queue compared to a non-priority queue is the number of retrieval requests waiting to be or being filled is relatively small, especially when overall traffic intensity is high.
- For dedicated and random storage, dedicated storage can greatly reduce the value of the number of retrieval or storage requests waiting to be or being filled in comparison to random storage. Because travel time with random storage is greater than for dedicated storage, dedicated storage outperforms random storage; however, when traffic intensity is low, the relative benefit of using dedicated storage diminishes significantly (the reverse is true when traffic intensity is high).
- Regarding the skewness of the AB curve, the greater the skewness the greater the preference for a two-class based layout. Likewise, the greater the traffic intensity, the greater the benefit for a two-class based layout.
- From the analysis in Section 6.4, Layout 3 is the best allocation for a 30(70) skewness and for the traffic intensity underlying the calculations. The queueing results for Layout 1 and Layout 3 are very similar; both are better than Layout 2, especially when the traffic intensity is very high.

For future research, several extensions can be considered for the MIAPP-NALT:

- Include more than one mezzanine level with pick positions.
- Develop guidelines (as a function of traffic intensity and skewness) for the fraction of a class of skus located at each level of pick positions.

- Increase the number of sku classes from two to three and analyze the impact of skewness of ABC curves on queueing performance.
- Expand storage policies (dedicated, random, and class-based) to include a quasi-random storage policy. With a quasi-random policy, instead of a pallet being stored at any point within a region (random storage), it is stored in the open reserve-storage location closest to the pick position for which it is destined.
- Relax the assumption of the aisle-captive narrow aisle lift truck to allow the vehicle to perform storages and retrievals in multiple aisles.
- Model the MIAPP-NALT as a queueing network by combining the activities of the NALT and the activities of the order pickers.

References

Bozer, Y.A. and Cho, M. (2005) Throughput performance of automated storage/retrieval systems under stochastic demand, *IIE Transactions*, **37**(4), 367-378.

Bozer, Y.A. and White, J. A. (1984) Travel–time models for automated storage/retrieval systems, *IIE Transactions*, **16**(4), 329-338.

Bozer, Y.A. and White, J.A. (1990) Design and performance models for end-of-aisle order picking systems, *Management Science*, **36**(7), 852-866.

Cai, X., Heragu, S.S. and Liu, Y. (2013) *Handbook of stochastic models and analysis of manufacturing system operations*, Springer, 29-71.

Cai, X., Heragu, S.S. and Liu, Y. (2014) Modeling and evaluating the AVS/RS with tier-to-tier vehicles using a semi-open queueing network, *IIE Transactions*, **46**(9), 905-927.

Drury, J. (1988). Towards more efficient order picking, IMM monograph No. 1. Report, The Institute of Materials Management, Cranfield, U.K.

Francis, R.L., McGinnis, L.F., Jr. and White, J.A. (1992) *Facility Layout and Location: an Analytical Approach*, 2nd edition, Upper Saddle River (NJ): Prentice-Hall., 189-198, 258-280.

Fukunari, M. and Malmborg, C.J. (2009) A network queuing approach for evaluation of performance measures in autonomous vehicle storage and retrieval systems, *European Journal of Operational Research*, **193**(1), 152-167.

Gross, D., Shortle, J.F., Thompson, J.M. and Harris, C.M. (2008) *Fundamentals of Queueing Theory*, 4th ed., Hoboken (NJ): John Wiley & Sons, Inc.

Gue, K.R. (2006) Very high density storage systems, *IIE Transactions*, **38**(1), 79-90.

Gue, K.R. and Kim, B.S. (2007) Puzzle-based storage systems, *Naval Research Logistics*, **54**(8). 556-567.

Heragu, S.S, Cai X., Krishnamurthy, A. and Malmborg, C.J. (2011) Analytical models for analysis of automated warehouse material handling systems, *International Journal of Production Research*, **49**(22), 6833-6861.

Hur, S., Lee, Y.H., Lim, S.Y. and Lee, M.H. (2004) A performance estimation model for AS/RS by M/G/1 queuing system, *Computers & Industrial Engineering*, **46**(2), 233-241.

Hur, S. and Nam, J. (2006) Performance analysis of automatic storage/retrieval systems by stochastic modelling, *International Journal of Production Research*, **44**(8), 1613-1626.

Kuo, P., Krishnamurthy, A. and Malmborg, C.J. (2007) Design models for unit load storage and retrieval systems using autonomous vehicle technology and resource conserving storage and dwell point policies, *Applied Mathematical Modelling*, **31**(10), 2332-2346.

Larson, R.C. and Odoni, A.R. (1981) *Urban Operations Research*, Englewood Cliffs (NJ): Prentice-Hall, 138-139.

Lee, H.F. (1997) Performance analysis for automated storage/retrieval systems, *IIE Transactions*, **29**(1), 15-28.

Liu, J., Liao, H. and White, J.A. (2019) Queueing analysis of the replenishment of multiple in-the-aisle pick positons, working paper.

Liu, J., Liao, H. and White, J.A. (2019) Stochastic analysis of an automated storage and retrieval system with multiple in-the-aisle pick positions, working paper.

Malmborg, C.J. (2002) Conceptualizing tools for autonomous vehicle storage and retrieval systems, *International Journal of Production Research*, **40**(8), 1807-1822.

Malmborg, C.J. (2003) Interleaving dynamics in autonomous vehicle storage and retrieval systems, *International Journal of Production Research*, **41**(5), 1057-1069.

Miyagawa, M., (2010) Distributions of rectilinear deviation distance to visit a facility, *European Journal of Operational Research*, **205**(1), 106-112.

Park, B.C., Frazelle, E.H. and White, J.A. (1999) Buffer sizing models for end-of-aisle order picking systems, *IIE Transactions*, **31**(1), 31-38.

Ramtin, F. and Pazour, J.A. (2014) Analytical models for an automated storage and retrieval system with multiple in-the-aisle pick positions, *IIE Transactions*, **46**(9), 968-986.

Ramtin, F. and Pazour, J.A. (2015) Production allocation problem for an AS/RS with multiple in-the-aisle pick positions, *IIE Transactions*, **47**(1), 1-18.

Roy D., Krishnamurthy, A., Heragu, S. and Malmborg, C.J. (2015) Queuing models to analyze dwell-point and cross-aisle location in autonomous vehicle-based warehouse systems, *European Journal of Operational Research*, **242**(1), 72-87.

Saaty, T.L. (1961) *Elements of Queueing Theory with Applications*, New York (NY): McGraw-Hill Book Company, 323-329.

Sztrik, J. (2016) *Basic Queueing Theory: Foundations of System Performance Modeling*, Globe Edit, 160. (URL: https://www.amazon.com/Basic-Queueing-Theory-Foundations-Performance /dp/3639734718/ref=sr_1_1?s=books&ie=UTF8&qid=1513713761&sr=1-1&Keywords=Basic+Queueing+Theory%3A+Foundations+of+System+Performance+Modelin g)

Takács, L. (1960) *Stochastic Processes, Problems and Solutions*, Methuen & Co Ltd (NY): John Wiley & Sons, Inc., 73-74

Uludag, O. (2014) *GridPick: A High Density Puzzle Based Order Picking System with Decentralized control,* Ph.D. thesis, Auburn University.

Vlasiou, M., Adan, I.J.B.F. and Wessels, J. (2004) A Lindley-type equation arising from a carousel problem, *Journal of Applied Probability*, **41**(4), 1171-1181.

Zhang, L., Krishnamurthy, A, Malmborg, C.J. and Heragu, S. (2009) Variance-based approximation of transaction waiting times in autonomous vehicle storage and retrieval systems, *European Journal of Industrial Engineering*, **3**(2), 146-169.

Appendix

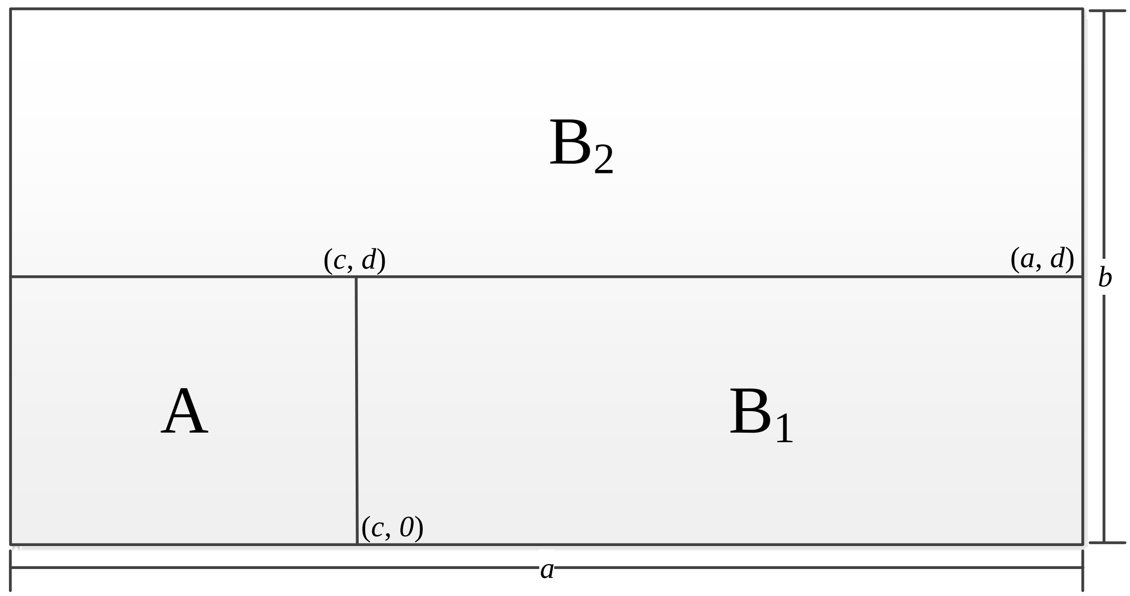

Figure A1. Layout for a two-class product allocation

Appendix A: Vertical travel time probability density function for floor-level storage location

For the class A or B_1 skus in Figure A1, the fork of the narrow aisle lift truck (NALT) will be lifted from the floor-level location to a random storage location of class A or B_1, pick up or deposit a pallet there, and, then the fork will be lowered back to the ground floor-level location. The expression for the NALT vertical travel time is:

$$T = 2y$$

where $x_1, x_2 \in$ unif$[0, d]$.

$$f(y) = \frac{1}{d}$$

With $0 \leq t \leq 2d$, the probability for T is

$$P_r(0 \leq T \leq t) = P_r(2y \leq t) = P_r(y \leq t/2) = \int_0^{t/2} \frac{1}{d} dy = \frac{t}{2d}$$

and take the derivative of $P_r(0 \leq T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(0 \leq T \leq t) = \frac{1}{2d}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \frac{1}{2d} \qquad t \in [0, 2d]. \tag{A1}$$

From the pdf, the expected value and second moment are

$$E[T] = d, \tag{A2}$$

$$E[T^2] = 4d^2/3. \tag{A3}$$

Appendix B: Vertical travel time pdf for mezzanine-level storage location

For the class B_2 skus in Figure A1, the fork of the narrow aisle lift truck (NALT) will be lifted from the floor-level location to a random storage location of class B_2, pick up or deposit a pallet there, and, then the fork will be lowered back to the ground floor-level location. The expression for the NALT vertical travel time is:

$$T = 2y$$

where $y \in$ unif$[d, b]$.

$$f(y) = \frac{1}{b - d}$$

With $2d \leq t \leq 2b$, the probability for T is

$$P_r(2d \leq T \leq t) = P_r(2d \leq 2y \leq t) = P_r(d \leq y \leq t/2,) = \int_d^{t/2} \frac{1}{b-d} dy = \frac{t-2d}{2(b-d)}$$

and take the derivative of $P_r(2d \leq T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(2d \leq T \leq t) = \frac{1}{2(b-d)}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \frac{1}{2(b-d)} \qquad t \in [2d, 2b]. \tag{B1}$$

From the pdf, the expected value and second moment are

$$E[T] = b + d, \tag{B2}$$

$$E[T^2] = \frac{4}{3}(b^2 + bd + d^2). \tag{B3}$$

Appendix C: Storage operation horizontal travel time pdf for travel path AA or B_2B_2

In the figure A1, the horizontal travel time pdf for travel path B_2B_2 can be obtained by replacing *c* with *a* in the horizontal travel time pdf of travel path AA. For the travel path AA, the NALT will travel from a random floor-level location x_i in class A to the input and output (I/O) location, and, then, the NALT will travel to another random floor-level location x_j in class A. The expression for the NALT horizontal travel time is:

$$T = x_i + x_j$$

where $x_i, x_j \in \text{unif}[0, c]$.

$$f(x_i, x_j) = f(x_i) f(x_i) = \frac{1}{c^2}$$

With $0 \le t \le c$, the probability for T is

$$P_r(0 \le T \le t) = \int_0^t \int_0^{-x_i+t} \frac{1}{c^2} dx_j dx_i = \frac{t^2}{2c^2}$$

and take the derivative of $P_r(0 \le T \le t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(0 \le T \le t) = \frac{t}{c^2}.$$

With $c < t \le 2c$, the probability for T is

$$P_r(T \le t) = \int_0^{t-c} \int_0^c \frac{1}{c^2} dx_j dx_i + \int_{t-c}^c \int_0^{-x_i+t} \frac{1}{c^2} dx_j dx_i = \frac{2t}{c} - \frac{t^2}{2c^2} - 1$$

and take the derivative of $P_r(T \le t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \le t) = \frac{2c-t}{c^2}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \begin{cases} \frac{t}{c^2} & t \in [0, c] \\ \frac{2c-t}{c^2} & t \in (c, 2c] \end{cases}. \tag{C1}$$

From the pdf, the expected value and second moment are

$$E[T] = c, \tag{C2}$$

$$E[T^2] = 7c^2/6. \tag{C3}$$

Appendix D: Horizontal travel time pdf for storage operation travel path B_1B_1

For the travel path B_1B_1 in Figure A1, the NALT will travel from a random floor-level location x_i in Class B_1 to the input and output (I/O), and, then, the NALT will travel to another random floor-level location x_j in class B_1. The expression for the NALT horizontal travel time is:

$$T = x_i + x_j$$

where $x_i, x_j \in \text{unif}[c, a]$.

$$f(x_i, x_j) = f(x_i)f(x_i) = \frac{1}{(a-c)^2}$$

With $2c \leq t \leq a + c$, the probability for T is

$$P_r(2c \leq T \leq t) = \int_c^{t-c} \int_c^{-x_i+t} \frac{1}{(a-c)^2} dx_j dx_i = \frac{(t-2c)^2}{2(a-c)^2}$$

and take the derivative of $P_r(0 \leq T \leq c)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(2c \leq T \leq c) = \frac{t-2c}{(a-c)^2}.$$

With $a + c < t \leq 2a$, the probability for T is

$$P_r(T \leq t) = \int_c^{t-a} \int_c^{a} \frac{1}{c^2} dx_j dx_i + \int_{t-a}^{a} \int_c^{-x_i+t} \frac{1}{c^2} dx_j dx_i = \frac{(t-2c)(4a-2c-t)}{2(a-c)^2}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{2a-t}{(a-c)^2}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \begin{cases} \frac{t-2c}{(a-c)^2} & t \in [2c, a+c] \\ \frac{2a-t}{(a-c)^2} & t \in (a+c, 2a] \end{cases}. \tag{D1}$$

From the pdf, the expected value and second moment are

$$E[T] = a - c, \tag{D2}$$

$$E[T^2] = \frac{1}{6}(7a^2 + 10ac + 7c^2). \tag{D3}$$

Appendix E: Horizontal travel time pdf for storage operation travel path AB$_1$ or B$_1$A

The horizontal travel time pdf of the travel path for AB$_1$ and B$_1$A are identical, therefore, we only need to derive the horizontal travel time pdf for the travel path AB$_1$. For the travel path AB$_1$ in Figure A1, the NALT will travel from a random floor-level location x_i in class A to the input and output (I/O) location, and, then, the NALT will travel to another random floor-level location x_j in class B$_1$. The expression for the NALT horizontal travel time is:

$$T = x_i + x_j$$

where $x_i \in \text{unif}[0, c]$ and $x_j \in \text{unif}[c, a]$.

$$f(x_i, x_j) = f(x_i) f(x_i) = \frac{1}{(a-c)c}$$

Condition 1: $c \le a/2$

With $c \le t \le 2c$, the probability for T is

$$P_r(c \le T \le t) = \int_0^{t-c} \int_c^{-x_i+t} \frac{1}{(a-c)c} dx_j dx_i = \frac{(t-c)^2}{2(a-c)c}$$

and take the derivative of $P_r(c \le T \le t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(c \le T \le t) = \frac{t-c}{(a-c)c}.$$

With $2c < t \le a$, the probability for T is

$$P_r(T \le t) = \int_0^c \int_c^{-x_i+t} \frac{1}{(a-c)c} dx_j dx_i = \frac{2t-3c}{2(a-c)}$$

and take the derivative of $P_r(T \le t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \le t) = \frac{1}{a-c}.$$

With $a < t \le a + c$, the probability for T is

$$P_r(T \le t) = \int_0^{t-a} \int_c^{a} \frac{1}{(a-c)c} dx_j dx_i + \int_{t-a}^{c} \int_c^{-x_i+t} \frac{1}{(a-c)c} dx_j dx_i$$

$$= \frac{2at + 2ct - t^2 - a^2 - 3c^2}{2(a-c)c}$$

and take the derivative of $P_r(T \le t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \le t) = \frac{a+c-t}{(a-c)c}.$$

Condition 2: $c > a/2$

With $c \le t \le a$, the pdf is the same as $c \le t \le 2c$ in Condition 1, therefore, the pdf is

$$f(t) = \frac{t-c}{(a-c)c}$$

With $a < t \le 2c$, the probability for T is

$$P_r(T \le t) = \int_0^{t-a} \int_c^{a} \frac{1}{(a-c)c} dx_j dx_i + \int_{t-a}^{t-c} \int_c^{-x_i+t} \frac{1}{(a-c)c} dx_j dx_i = \frac{2t-a-c}{2c}$$

and take the derivative of $P_r(T \le t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \le t) = \frac{1}{c}.$$

With $2c < t \le a + c$, the pdf is the same as $a < t \le a + c$ in Condition 1, therefore, the pdf is

$$f(t) = \frac{a+c-t}{(a-c)c}$$

In summary, from the derivations above, the pdf is

Condition 1: $0 \le c \le a/2$

$$f(t) = \begin{cases} \frac{t-c}{(a-c)c} & t \in [c, 2c] \\ \frac{1}{a-c} & t \in (2c, a]. \\ \frac{a+c-t}{(a-c)c} & t \in [a, a+c] \end{cases} \tag{E1}$$

Condition 2: $a/2 < c \le a$

$$f(t) = \begin{cases} \frac{t-c}{(a-c)c} & t \in [c, a] \\ \frac{1}{c} & t \in (a, 2c]. \\ \frac{a+c-t}{(a-c)c} & t \in [2c, a+c] \end{cases} \tag{E2}$$

Although the pdf are different, the expected value and second moment are

$$E[T] = a/2 + c, \tag{E3}$$

$$E[T^2] = \frac{1}{6}(2a^2 + 5ac + 7c^2). \tag{E4}$$

Appendix F: Horizontal travel time pdf for storage operation travel path AB_2 or B_2A

The horizontal travel time pdf of the travel path AB_2 and B_2A are identical, we only need to derive the pdf for horizontal travel path AB_2. For the travel path AB_2 in Figure A1, the NALT will travel from a random floor-level location x_i in class A to the input and output (I/O) location, and, then, the NALT will travel to another random floor-level location x_j in class B_2. The expression for the NALT horizontal travel time is:

$$T = x_i + x_j$$

where $x_i \in \text{unif}[0, \text{c}]$ and $x_j \in \text{unif}[0, \text{a}]$.

$$f(x_i, x_j) = f(x_i)f(x_i) = \frac{1}{ac}$$

With $0 \leq t \leq c$, the probability for T is

$$P_r(0 \leq T \leq t) = \int_0^t \int_0^{-x_i+t} \frac{1}{ac} dx_j dx_i = \frac{t^2}{2ac}$$

and take the derivative of $P_r(0 \leq T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(0 \leq T \leq t) = \frac{t}{ac}.$$

With $c < t \leq a$, the probability for T is

$$P_r(T \leq t) = \int_0^c \int_0^{-x_i+t} \frac{1}{ac} dx_j dx_i = \frac{2t - c}{2a}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{1}{a}.$$

With $a < t \leq a + c$, the probability for T is

$$P_r(T \leq t) = \int_0^{t-a} \int_0^a \frac{1}{ac} dx_j dx_i + \int_{t-a}^c \int_0^{-x_i+t} \frac{1}{ac} dx_j dx_i = \frac{2at - a^2 - (t-c)^2}{2ac}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{a+c-t}{ac}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \begin{cases} \frac{t}{ac} & t \in [0, c] \\ \frac{1}{a} & t \in (c, a]. \\ \frac{a+c-t}{ac} & t \in [a, a+c] \end{cases} \tag{F1}$$

From the pdf, the expected value and second moment are

$$E[T] = a/2 + c/2, \tag{F2}$$

$$E[T^2] = \frac{1}{6}(2a^2 + 3ac + 2c^2). \tag{F3}$$

Appendix G: Horizontal travel time pdf for storage operation travel path B_1B_2 or B_2B_1

The horizontal travel time pdf of the travel paths B_1B_2 and B_2B_1 are identical, we only need to derive the pdf for travel path B_1B_2. For the travel path B_1B_2 in Figure A1, the NALT will travel from a random floor-level location x_i in class B_1 to the input and output (I/O) location, and, then, the NALT will travel to another random floor-level location x_j in class B_2. The expression for the NALT horizontal travel time is:

$$T = x_i + x_j$$

where $x_i \in \text{unif}[c, a]$ and $x_j \in \text{unif}[0, a]$.

$$f(x_i, x_j) = f(x_i)f(x_i) = \frac{1}{a(a-c)}$$

With $c \leq t \leq a$, the probability for T is

$$P_r(c \leq T \leq t) = \int_c^t \int_0^{-x_i+t} \frac{1}{a(a-c)} dx_j dx_i = \frac{(t-c)^2}{2a(a-c)}$$

and take the derivative of $P_r(c \leq T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(c \leq T \leq t) = \frac{t-c}{a(a-c)}.$$

With $a < t \leq a + c$, the probability for T is

$$P_r(T \leq t) = \int_c^a \int_0^{-x_i+t} \frac{1}{a(a-c)} dx_j dx_i = \frac{2t-a-c}{2a}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{1}{a}.$$

With $a + c < t \leq 2a$, the probability for T is

$$P_r(T \leq t) = \int_0^{t-a} \int_0^a \frac{1}{a(a-c)} dx_j dx_i + \int_{t-a}^a \int_0^{-x_i+t} \frac{1}{a(a-c)} dx_j dx_i = \frac{2a^2+4at-t^2}{2a(a-c)}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \le t) = \frac{2a-t}{a(a-c)}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \begin{cases} \frac{t-c}{a(a-c)} & t \in [c, a] \\ \frac{1}{a} & t \in (a, a+c]. \\ \frac{2a-t}{a(a-c)} & t \in [a+c, 2a] \end{cases} \tag{G1}$$

From the pdf, the expected value and second moment are

$$E[T] = a + c/2, \tag{G2}$$

$$E[T^2] = \frac{1}{6}(7a^2 + 5ac + 2c^2). \tag{G3}$$

Appendix H: Horizontal travel time pdf for travel path AA, B_1B_1 or B_2B_2 with dedicated retrieval operation

The horizontal travel time pdf of the travel path B_1B_1 can be obtained by replacing *c* with *a*-*c* in the horizontal travel time pdf of travel path AA, therefore, we only need to derive the horizontal travel time pdf for travel path AA. For the horizontal travel path AA in Figure A1, the NALT will travel from a random floor-level location x_i in class A to another random floor-level location x_j in class A. The expression for the NALT horizontal travel time is:

$$T = |x_i - x_j| = \begin{cases} x_j - x_i & x_j \ge x_i & \text{Case 1} \\ x_i - x_j & x_j < x_i & \text{Case 2} \end{cases}$$

where $x_1, x_2 \in \text{unif}[0, \text{c}]$.

$$f(x_i, x_j) = f(x_i) f(x_i) = \frac{1}{c^2}$$

For the two cases, by symmetry, Case 1 and Case 2 are identical. Therefore, we only need to consider Case 1.

Case 1: With $x_j \geq x_i$, then $T = x_j - x_i$ and $0 \leq t \leq c$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 1}) = P_r\left(x_j - x_i \leq t\right) = P_r\left(x_j \leq x_i + t, x_j \geq x_i\right) = \int_0^{c-t} \int_{x_i}^{x_i+t} \frac{1}{c^2} dx_j dx_i + \int_{c-t}^{c} \int_{x_i}^{c} \frac{1}{c^2} dx_j dx_i = \frac{(2\text{c}-t)\text{t}}{2c^2}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 1}) = \frac{c-t}{c^2}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \frac{2(c-t)}{c^2} \qquad t \in [0, c]. \tag{H1}$$

From the pdf, the expected value and second moment are

$$E[T] = c/3, \tag{H2}$$

$$E[T^2] = c^2/6. \tag{H3}$$

Appendix I: Horizontal travel time pdf for travel path AB_1 or B_1A with dedicated retrieval operation

The pdf of the horizontal travel path AB_1 and B_1A are identical, we only need to derive the pdf for horizontal travel path AB_1. For the horizontal travel path AB_1 in Figure A1, the NALT will travel from a random floor-level location x_i in class A to another random floor-level location x_j in class B_1. The expression for the NALT horizontal travel time is:

$$T = x_j - x_i$$

where $x_i \in \text{unif}[0, \text{c}]$ and $x_j \in \text{unif}[\text{c}, \text{a}]$.

$$f\left(x_i, x_j\right) = f(x_i) f(x_i) = \frac{1}{(a-c)c}$$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t) = P_r(x_j - x_i \leq t) = P_r(x_j \leq x_i + t) = \int_{c-t}^{c} \int_{c}^{x_i+t} \frac{1}{(a-c)c} dx_j dx_i = \frac{t^2}{2(a-c)c}$$

and take the derivative of $P_r(0 \leq T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(0 \leq T \leq t) = \frac{t}{(a-c)c}.$$

With $c < t \leq a - c$, the probability for T is

$$P_r(T \leq t) = P_r(x_j - x_i \leq t) = P_r(x_j \leq x_i + t) = \int_{0}^{c} \int_{c}^{x_i+t} \frac{1}{(a-c)c} dx_j dx_i = \frac{2t-c}{2(a-c)}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{1}{(a-c)}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t) = P_r(x_j - x_i \leq t) = P_r(x_j \leq x_i + t) = \int_{0}^{a-t} \int_{c}^{x_i+t} \frac{1}{(a-c)c} dx_j dx_i +$$

$$\int_{a-t}^{c} \int_{c}^{a} \frac{1}{(a-c)c} dx_j dx_i = \frac{2a(c+t)-a^2-2c^2-t^2}{2(a-c)c}$$

and take the derivative of $P_r(T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{a-t}{(a-c)c}.$$

Condition 2: with $c \leq a/2$

With $0 \leq t \leq a - c$, the pdf is the same as $0 \leq t \leq c$ in Condition 1, therefore, the pdf is

$$f(t) = \frac{t}{(a-c)c}$$

With $a - c < t \leq c$, the probability for T is

$$P_r(T \leq t) = \int_{c-t}^{a-t} \int_{c}^{x_i+t} \frac{1}{(a-c)c} dx_j dx_i + \int_{a-t}^{c} \int_{c}^{a} \frac{1}{(a-c)c} dx_j dx_i = \frac{2t-a+c}{2c}.$$

and take the derivative of $P_r(a < T \leq t)$ with respect to T, the pdf is

$$f(t) = \frac{d}{dt} P_r(T \leq t) = \frac{1}{c}.$$

With $c < t \leq a$, the pdf is the same as $a - c < t \leq a$ in Condition 1, therefore, the pdf is

$$f(t) = \frac{a-t}{(a-c)c}$$

In summary, from the derivations above, the pdf is

Condition 1: $0 \leq c \leq a/2$

$$f(t) = \begin{cases} \frac{t}{(a-c)c} & t \in [0, c] \\ \frac{1}{(a-c)} & t \in (c, a-c]. \\ \frac{a-t}{(a-c)c} & t \in (a-c, a] \end{cases} \tag{I1}$$

Condition 2: $a/2 < c \leq a$

$$f(t) = \begin{cases} \frac{t}{(a-c)c} & t \in [0, a-c] \\ \frac{1}{c} & t \in (a-c, c]. \\ \frac{a-t}{(a-c)c} & t \in (c, a] \end{cases} \tag{I2}$$

Although the pdf are different, the expected value and second moment are the same for different pdfs and they are

$$E[T] = a/2, \tag{I3}$$

$$E[T^2] = \frac{1}{6}(2a^2 - ac + c^2). \tag{I4}$$

Appendix J: Horizontal travel time pdf for travel path AB_2, B_2A, B_1B_2 or B_2B_1 with dedicated retrieval operation

The pdf of the horizontal travel path AB_2 and B_2A are identical, the pdf of the horizontal travel path B_1B_2 and B_2B_1 are identical. The pdf of the horizontal travel path B_1B_2 can be obtained by replacing c with *a*-*c*, therefore, we only need to derive the pdf for horizontal travel path AB_2. For the horizontal travel path AB_2 in Figure A1, the NALT will travel from a random floor-level location x_i in class A to another random floor-level location x_j in class B_2. The expression for the NALT horizontal travel time is:

$$T = |x_i - x_j| = \begin{cases} x_i - x_j & x_i \geq x_j \quad \text{Case 1} \\ x_j - x_i & x_i < x_j \quad \text{Case 2} \end{cases}$$

where $x_i \in \text{unif}[0, c]$ and $x_j \in \text{unif}[0, a]$.

$$f(x_i, x_j) = f(x_i)f(x_i) = \frac{1}{ac}$$

Case 1: with $T = x_i - x_j$ and $x_i \geq x_j$

With $0 \leq t \leq c$, the probability for T is

$P_r(0 \leq T \leq t|\text{Case 1}) = P_r(x_i - x_j \leq t|\text{Case 1}) = P_r(x_i \leq x_j + t|\text{Case 1}) =$
$\int_0^{c-t} \int_{x_j}^{x_j+t} \frac{1}{ac} dx_i dx_j + \int_{c-t}^{c} \int_{x_j}^{c} \frac{1}{ac} dx_i dx_j = \frac{(2c-t)t}{2ac}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(0 \leq T \leq t|\text{Case 1}) = \frac{c-t}{ac}.$$

Case 2: with $T = x_j - x_i$ and $x_i < x_j$

With $0 \leq t \leq a - c$, the probability for T is

$P_r(T \leq t|\text{Case 2}) = P_r(x_j - x_i \leq t|\text{Case 2}) = P_r(x_j \leq x_i + t|\text{Case 2}) = \int_0^c \int_{x_i}^{x_i+t} \frac{1}{ac} dx_j dx_i =$
$\frac{t}{a}$

and take the derivative of $P_r(0 \leq T \leq t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 2}) = \frac{1}{a}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case 2}) = P_r\big(x_j - x_i \leq t|\text{Case 2}\big) = P_r\big(x_j \leq x_i + t|\text{Case 2}\big) =$$

$$\int_0^{a-t}\int_{x_i}^{x_i+t} \frac{1}{ac} dx_j dx_i + \int_{a-t}^{c}\int_{x_i}^{a} \frac{1}{ac} dx_j dx_i = \frac{2a(c+t)-a^2-c^2-t^2}{2ac}$$

and take the derivative of $P_r(T \leq t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt} P_r(T \leq t|\text{Case 2}) = \frac{a-t}{ac}.$$

In summary, from the derivations above, the pdf is

Condition 1: $0 \leq c \leq a/2$

$$f(t) = \begin{cases} \frac{2c-t}{ac} & t \in [0, c] \\ \frac{1}{a} & t \in (c, a-c]. \\ \frac{a-t}{ac} & t \in (a-c, a] \end{cases} \tag{J1}$$

Condition 2: $a/2 < c \leq a$

$$f(t) = \begin{cases} \frac{2c-t}{ac} & t \in [0, a-c] \\ \frac{a+c-2t}{ac} & t \in (a-c, c]. \\ \frac{a-t}{ac} & t \in (c, a] \end{cases} \tag{J2}$$

Although the pdf are different, the expected value and second moment are the same for different pdfs and they are

$$E[T] = \frac{a}{2} - \frac{c}{2} + \frac{c^2}{3a}. \tag{J3}$$

$$E[T^2] = \frac{a^2}{3} - \frac{ac}{2} + \frac{c^2}{3}. \tag{J4}$$

Appendix K: Horizontal travel time pdf for travel path AA B_1B_1 or B_2B_2 with random storage retrieval operation

The pdf of the horizontal travel path B_1B_1 can be obtained by replacing *c* with *a-c* in the pdf of the horizontal travel path AA, the pdf of the horizontal travel path B_2B_2 can be obtained by replacing *c* with a in the pdf of the horizontal travel path AA, therefore, we only need to derive the pdf for horizontal travel path AA. For the horizontal travel path AA in Figure A_1, the NALT will travel from a random floor-level location x_i in class A to another different random floor-level location x in class A, and, then, the NALT will travel to a different random floor-level location x_j in class A. The expression for the NALT horizontal travel time is:

$$T = |x - x_i| + |x - x_j| = \begin{cases} x_i + x_j - 2x & x_i \geq x, x_j \geq x \quad \text{Case 1} \\ x_i - x_j & x_j < x \leq x_i \quad \text{Case 2} \\ x_j - x_i & x_i < x \leq x_j \quad \text{Case 3} \\ 2x - x_i - x_j & x_i < x, x_j < x \quad \text{Case 4} \end{cases}$$

where x, x_i and $x_j \in \text{unif}[0, c]$.

$$f(x, x_i, x_j) = f(x)f(x_i)f(x_i) = \frac{1}{c^3}$$

Case 1: with $T = x_i + x_j - 2x$ and $x_i \geq x, x_j \geq x$

Case 1A: with $x_i \geq x_j$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r(x_i + x_j - 2x \leq t|\text{Case 1}) = \int_0^{t/2}\int_{x_j}^{-x_j+t}\int_0^{x_j}\frac{1}{c^3}dxdx_idx_j + \int_{t/2}^{t}\int_{-x_i+t}^{x_i}\int_{(x_i+x_j-t)/2}^{x_j}\frac{1}{c^3}dxdx_jdx_i + \int_t^c\int_{x_i-t}^{x_i}\int_{(x_i+x_j-t)/2}^{x_j}\frac{1}{c^3}dxdx_jdx_i = \frac{(2c-t)t^2}{8c^3}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(4c-3t)t}{8c^3}.$$

With $c < t \leq 2c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_0^{t/2}\int_0^{x_i}\int_0^{x_j}\frac{1}{c^3}dxdx_jdx_i +$$

$$\int_{t/2}^{c}\int_0^{-x_i+t}\int_0^{x_j}\frac{1}{c^3}dxdx_jdx_i + \int_{t/2}^{c}\int_{-x_i+t}^{x_i}\int_{(x_i+x_j-t)/2}^{x_j}\frac{1}{c^3}dxdx_jdx_i = \frac{t^3-4c^3+12c^2t-6ct^2}{24c^3}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(t-2c)^2}{8c^3}.$$

Case 1B: with $x_i < x_j$

With $0 \leq t \leq c$, the pdf for T is the same as $0 \leq t \leq c$ in Case 1A and therefore, the pdf is

$$f(t) = \frac{(4c - 3t)t}{8c^3}$$

With $c < t \leq 2c$, the pdf for T is the same as $c \leq t \leq 2c$ in Case 1A and therefore, the pdf is

$$f(t) = \frac{(t - 2c)^2}{8c^3}$$

Case 2: with $T = x_i - x_j$ and $x_j < x \leq x_i$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 2}) = P_r\left(x_i - x_j \leq t|\text{Case 2}\right) = P_r\left(x_i \leq x_j + t|\text{Case 2}\right) =$$

$$\int_0^{c-t}\int_{x_j}^{x_j+t}\int_{x_j}^{x_i}\frac{1}{c^3}dxdx_idx_j + \int_{c-t}^{c}\int_{x_j}^{c}\int_{x_j}^{x_i}\frac{1}{c^3}dxdx_idx_j = \frac{(3c-2t)t^2}{6c^3}$$

and take the derivative of $P_r(T \leq t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt}P_r(T \leq t|\text{Case 2}) = \frac{(c-t)t}{c^3}.$$

In summary, from the derivations above, the pdf is

$$f(t) = \begin{cases} \frac{(8c-7t)t}{2c^3} & t \in [0, c] \\ \frac{(t-2c)^2}{2c^3} & t \in (c, 2a] \end{cases}. \tag{K1}$$

From the derivations above, the expected value and second moment are

$$E[T] = \frac{2c}{3}, \tag{K2}$$

$$E[T^2] = \frac{17c^2}{30}. \tag{K3}$$

Appendix L: Horizontal travel time pdf for travel path AB$_1$ or B$_1$A with random storage retrieval operation

The pdf of the horizontal travel path B$_1$A can be obtained by replacing *c* with *a-c* in the pdf of the horizontal travel path AB$_1$, therefore, we only need to derive the pdf for horizontal travel path AB$_1$. For the horizontal travel path AB$_1$ in Figure A1, the NALT will travel from a random floor-level location x_i in class A to another different random floor-level location x in class B$_1$, and, then, the NALT will travel to a different random floor-level location x_j in class B$_1$. The expression for the NALT horizontal travel time is:

$$T = |x - x_i| + |x - x_j| = \begin{cases} 2x - x_i - x_j & x \geq x_j \quad \text{Case 1} \\ x_j - x_i & x < x_j \quad \text{Case 2} \end{cases}$$

where $x_i \in \text{unif}[0, c]$ and $x, x_j \in \text{unif}[\text{c}, a]$.

$$f(x, x_i, x_j) = f(x)f(x_i)f(x_i) = \frac{1}{(a-c)^2 c}$$

Case 1: with $T = 2x - x_i - x_j$ and $x \geq x_j$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r(2x - x_i - x_j \leq t|\text{Case 1}) = \int_{c-t}^{c}\int_{c}^{x_i+t}\int_{x_j}^{(x_i+x_j+t)/2} \frac{1}{(a-c)^2 c} dx dx_j dx_i = \frac{t^3}{12(a-c)^2 c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{t^2}{4(a-c)^2 c}.$$

With $c < t \leq a - c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(2x - x_i - x_j \leq t|\text{Case 1}\right) = \int_0^c \int_c^{x_i+t} \int_{x_j}^{(x_i+x_j+t)/2} \frac{1}{(a-c)^2 c} dx dx_j dx_i = \frac{c^2-3ct+3t^2}{12(a-c)^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{6t-3c}{12(a-c)^2}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(2x - x_i - x_j \leq t|\text{Case 1}\right) = \int_0^{a-t}\int_c^{x_i+t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{a-t}^{c}\int_c^{-x_i+2a-t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{a-t}^{c}\int_{-x_i+2a-t}^{a}\int_{x_j}^{a}\frac{1}{(a-c)^2c}dxdx_jdx_i = \frac{2a^3-c^3-9c^2t-3ct^2-2t^3-6a^2(c+t)+6a(c+t)^2}{12(a-c)^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{4a(c+t)-2a^2-3c^2-2ct-2t^2}{4(a-c)^2c}.$$

With $a < t \leq 2(a - c)$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(2x - x_i - x_j \leq t|\text{Case 1}\right) = \int_0^{c}\int_c^{-x_i+2a-t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_0^{c}\int_{-x_i+2a-t}^{a}\int_{x_j}^{a}\frac{1}{(a-c)^2c}dxdx_jdx_i = \frac{6a(c+2t)-6a^2-c^2-9ct-3t^2}{12(a-c)^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{4a-3c-2t}{4(a-c)^2}.$$

With $2(a - c) < t \leq 2a - c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(2x - x_i - x_j \leq t|\text{Case 1}\right) = \int_0^{2a-c-t}\int_c^{-x_i+2a-t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_0^{2a-c-t}\int_{-x_i+2a-t}^{a}\int_{x_j}^{a}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{2a-c-t}^{c}\int_c^{a}\int_{x_j}^{a}\frac{1}{(a-c)^2c}dxdx_jdx_i = \frac{(a-c)^3+(a-t)^3}{6(a-c)^2c} - \frac{(c-t)(t-2a+c)^2}{4(a-c)^2c} + \frac{t-2a+2c}{2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(t-2a+c)^2}{4(a-c)^2c}.$$

Condition 2: with $a/2 < c \leq 2a/3$

With $0 \leq t \leq a-c$, the pdf is the same as $0 \leq t \leq c$ in condition 1, the pdf is

$$f(t|\text{Case }1) = \frac{d}{dt}P_r(T \leq t|\text{Case }1) = \frac{t^2}{4(a-c)^2c}.$$

With $a-c < t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case }1) = P_r\big(2x - x_i - x_j \leq t|\text{Case }1\big) = \int_{c-t}^{a-t}\int_{c}^{x_i+t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{a-t}^{c}\int_{c}^{-x_i+2a-t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{a-t}^{c}\int_{-x_i+2a-t}^{a}\int_{x_j}^{a}\frac{1}{(a-c)^2c}dxdx_jdx_i = \frac{2a^3-2c^3-6c^2t-6ct^2-t^3-6a^2(c+t)+6a(c+t)^2}{12(a-c)^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case }1)$ with respect to T, the pdf is

$$f(t|\text{Case }1) = \frac{d}{dt}P_r(T \leq t|\text{Case }1) = \frac{4a(c+t)-2a^2-2c^2-4ct-t^2}{4(a-c)^2c}.$$

With $c < t \leq 2(a-c)$, the pdf is the same as $a-c < t \leq a$ in condition 1, the pdf is

$$f(t|\text{Case }1) = \frac{d}{dt}P_r(T \leq t|\text{Case }1) = \frac{4a(c+t)-2a^2-3c^2-2ct-2t^2}{4(a-c)^2c}.$$

With $2(a-c) < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case }1) = P_r\big(2x - x_i - x_j \leq t|\text{Case }1\big) = \int_{0}^{a-t}\int_{c}^{x_i+t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{a-t}^{2a-c-t}\int_{c}^{-x_i+2a-t}\int_{x_j}^{(x_i+x_j+t)/2}\frac{1}{(a-c)^2c}dxdx_jdx_i + \int_{c}^{a}\int_{-x_j+2a-t}^{c}\int_{x_j}^{a}\frac{1}{(a-c)^2c}dxdx_idx_j = \frac{7c^3-6a^3+3c^2t+3ct^2-t^3+6a^2(3c+t)-6ac(3c+2t)}{12(a-c)^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case }1)$ with respect to T, the pdf is

$$f(t|\text{Case }1) = \frac{d}{dt}P_r(T \leq t|\text{Case }1) = \frac{2a^2-4ac+c^2+2ct-t^2}{4(a-c)^2c}.$$

With $a < t \leq 2a - c$, the pdf is the same as $2(a-c) < t \leq 2a - c$ in condition 1, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{(t - 2a + c)^2}{4(a-c)^2 c}.$$

Condition 3: with $2a/3 < c \leq a$

With $0 \leq t \leq a - c$, the pdf is the same as $0 \leq t \leq a - c$in condition 2, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{t^2}{4(a-c)^2 c}.$$

With $a - c < t \leq 2(a-c)$, the pdf is the same as $a - c < t \leq c$ in condition 2, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{4a(c+t) - 2a^2 - 2c^2 - 4ct - t^2}{4(a-c)^2 c}.$$

With $2(a-c) < t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 1}\big) =$$

$$\int_c^a \int_{x_j - t}^{-x_j+2a-t} \int_{x_j}^{(x_i+x_j+t)/2} \frac{1}{(a-c)^2 c} dx dx_i dx_j + \int_c^a \int_{-x_j+2a-t}^{c} \int_{x_j}^{a} \frac{1}{(a-c)^2 c} dx dx_i dx_j = \frac{t-a+c}{2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{1}{2c}.$$

With $c < t \leq a$, the pdf is the same as $2(a-c) < t \leq a$ in condition 2, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{2a^2 - 4ac + c^2 + 2ct - t^2}{4(a-c)^2 c}.$$

With $a < t \leq 2a - c$, the pdf is the same as $a < t \leq 2a - c$ in condition 2, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{(t - 2a + c)^2}{4(a-c)^2 c}.$$

Case 2: with $T = x_j - x_i$ and $x < x_j$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$P_r(T \leq t|\text{Case } 2) = P_r\left(x_j - x_i \leq t|\text{Case } 2\right) = P_r\left(x_j \leq x_i + t|\text{Case } 2\right) =$

$\int_{c-t}^{c}\int_{c}^{x_i+t}\int_{c}^{x_j}\frac{1}{(a-c)^2 c}dxdx_jdx_i = \frac{t^3}{6(a-c)^2 c}$

and take the derivative of $P_r(T \leq t|\text{Case } 2)$ with respect to T, the pdf is

$$f(t|\text{Case } 2) = \frac{d}{dt}P_r(T \leq t|\text{Case } 2) = \frac{t^2}{2(a-c)^2 c}.$$

With $c < t \leq a - c$, the probability for T is

$P_r(T \leq t|\text{Case } 2) = P_r\left(x_j - x_i \leq t|\text{Case } 2\right) = P_r\left(x_j \leq x_i + t|\text{Case } 2\right) =$

$\int_{0}^{c}\int_{c}^{x_i+t}\int_{c}^{x_j}\frac{1}{(a-c)^2 c}dxdx_jdx_i = \frac{c^2-3ct+3t^2}{6(a-c)^2}$

and take the derivative of $P_r(T \leq t|\text{Case } 2)$ with respect to T, the pdf is

$$f(t|\text{Case } 2) = \frac{d}{dt}P_r(T \leq t|\text{Case } 2) = \frac{6t-3c}{6(a-c)^2}.$$

With $a - c < t \leq a$, the probability for T is

$P_r(T \leq t|\text{Case } 2) = P_r\left(x_j - x_i \leq t|\text{Case } 2\right) = P_r\left(x_j \leq x_i + t|\text{Case } 2\right) =$

$\int_{0}^{a-t}\int_{c}^{x_i+t}\int_{c}^{x_j}\frac{1}{(a-c)^2 c}dxdx_jdx_i + \int_{a-t}^{c}\int_{c}^{a}\int_{c}^{x_j}\frac{1}{(a-c)^2 c}dxdx_jdx_i = \frac{(a-c)^3+(c-t)^3}{6(a-c)^2 c} + \frac{t-a+c}{2c}$

and take the derivative of $P_r(T \leq t|\text{Case } 2)$ with respect to T, the pdf is

$$f(t|\text{Case } 2) = \frac{d}{dt}P_r(T \leq t|\text{Case } 2) = \frac{(a-t)(a+t-2c)}{2(a-c)^2 c}.$$

Condition 2: with $c > a/2$

With $0 \le t \le a-c$, the pdf is the same as $0 \le t \le c$ in condition 1, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt} P_r(T \le t|\text{Case 2}) = \frac{t^2}{2(a-c)^2 c}.$$

With $a-c < t \le c$, the probability for T is

$P_r(T \le t|\text{Case 2}) = P_r\left(x_j - x_i \le t|\text{Case 2}\right) = P_r\left(x_j \le x_i + t|\text{Case 2}\right) =$

$\int_{c-t}^{a-t} \int_{c}^{x_i+t} \int_{c}^{x_j} \frac{1}{(a-c)^2 c} dx dx_j dx_i + \int_{a-t}^{c} \int_{c}^{a} \int_{c}^{x_j} \frac{1}{(a-c)^2 c} dx dx_j dx_i = \frac{3t-2a+2c}{6c}$

and take the derivative of $P_r(T \le t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt} P_r(T \le t|\text{Case 2}) = \frac{1}{2c}.$$

With $c < t \le a$, the pdf is the same as $a-c < t \le a$ in condition 1, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt} P_r(T \le t|\text{Case 2}) = \frac{(a-t)(a+t-2c)}{2(a-c)^2 c}.$$

In summary, from the derivations above, the pdf is

Condition 1: $0 \le c \le a/2$

$$f(t) = \begin{cases} \frac{3t^2}{4(a-c)^2 c} & t \in [0, c] \\ \frac{3(2t-c)}{4(a-c)^2} & t \in (c, a-c] \\ \frac{-3c^2+2ct+4(a-t)t}{4(a-c)^2 c} & t \in (a-c, a]. \\ \frac{4a-3c-2t}{4(a-c)^2} & t \in (a, 2(a-c)] \\ \frac{(t-2a+c)^2}{4(a-c)^2 c} & t \in (2(a-c), 2a-c] \end{cases} \qquad \text{(L1)}$$

Condition 2: $a/2 < c \leq 2a/3$

$$f(t) = \begin{cases} \frac{3t^2}{4(a-c)^2c} & t \in [0, a-c] \\ \frac{(4a-4c-t)t}{4(a-c)^2c} & t \in (a-c, c] \\ \frac{-3c^2+2ct+4(a-t)t}{4(a-c)^2c} & t \in (c, 2(a-c)]. \\ \frac{4a^2-8ac+c^2+6ct-3t^2}{4(a-c)^2c} & t \in (2(a-c), a] \\ \frac{(t-2a+c)^2}{4(a-c)^2c} & t \in (a, 2a-c] \end{cases} \tag{L2}$$

Condition 3: $2a/3 < c \leq a$

$$f(t) = \begin{cases} \frac{3t^2}{4(a-c)^2c} & t \in [0, a-c] \\ \frac{(4a-4c-t)t}{4(a-c)^2c} & t \in (a-c, 2(a-c)] \\ \frac{1}{c} & t \in (2(a-c), c]. \\ \frac{4a^2-8ac+c^2+6ct-3t^2}{4(a-c)^2c} & t \in (c, a] \\ \frac{(t-2a+c)^2}{4(a-c)^2c} & t \in (a, 2a-c] \end{cases} \tag{L3}$$

Although the pdf are different, the expected value and variance are the same for different pdfs and they are

$$E[T] = \frac{1}{6}(5a - 2c), \tag{L4}$$

$$E[T^2] = \frac{1}{6}(5a^2 - 5ac + 2c^2). \tag{L5}$$

Appendix M: Horizontal travel time pdf for travel path AB_2 or B_1B_2 with random storage retrieval operation

The pdf of the horizontal travel path B_1B_2 can be obtained by replacing c with *a-c* in the pdf of the horizontal travel path AB_2, therefore, we only need to derive the pdf for horizontal travel path AB_2. For the horizontal travel path AB_2 in Figure A1, the NALT will travel from a random floor-level location x_i in class A to another different random floor-level location x in class B_2, and, then, the NALT will travel to a different random floor-level location x_j in class B_2. The expression for the NALT horizontal travel time is:

$$T = |x - x_i| + |x - x_j| = \begin{cases} x_i + x_j - 2x & x_i \geq x, x_j \geq x \quad \text{Case 1} \\ x_i - x_j & x_j < x \leq x_i \quad \text{Case 2} \\ x_j - x_i & x_i < x \leq x_j \quad \text{Case 3} \\ 2x - x_i - x_j & x_i < x, x_j < x \quad \text{Case 4} \end{cases}$$

where $x_i \in \text{unif}[0, c]$ and $x, x_j \in \text{unif}[0, a]$.

$$f(x, x_i, x_j) = f(x)f(x_i)f(x_i) = \frac{1}{a^2c}$$

Case 1: with $T = x_i + x_j - 2x$ and $x_i \geq x, x_j \geq x$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r(x_i + x_j - 2x \leq t|\text{Case 1}) = \int_0^{c-t}\int_x^{x+t}\int_x^{2x-x_i+t}\frac{1}{a^2c}dx_jdx_idx + \int_{c-t}^{c}\int_x^{c}\int_x^{2x-x_i+t}\frac{1}{a^2c}dx_jdx_idx = \frac{(3c-t)t^2}{6a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(2c-t)t}{2a^2c}.$$

With $c < t \leq a - c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r(x_i + x_j - 2x \leq t|\text{Case 1}) = \int_0^{c}\int_x^{c}\int_x^{2x-x_i+t}\frac{1}{a^2c}dx_jdx_idx = \frac{c(3t-c)}{6a^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{c}{2a^2}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r(x_i + x_j - 2x \leq t|\text{Case 1}) = \int_0^{a-t}\int_0^{x_i}\int_x^{2x-x_i+t}\frac{1}{a^2c}dx_jdxdx_i + \int_{a-t}^{c}\int_0^{(x_i+a-t)/2}\int_x^{2x-x_i+t}\frac{1}{a^2c}dx_jdxdx_i + \int_{a-t}^{c}\int_{(x_i+a-t)/2}^{x_i}\int_x^{a}\frac{1}{a^2c}dx_jdxdx_i = \frac{a^3-3c^3+3c^2t-3ct^2-t^3-3a^2(c+t)+3a(c+t)^2}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{c^2-a^2-2ct-t^2+2a(c+t)}{4a^2c}.$$

With $a < t \leq a + c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_{t-a}^{c}\int_{0}^{(x_i+a-t)/2}\int_{x}^{2x-x_i+t}\frac{1}{a^2c}dx_jdxdx_i + \int_{0}^{t-a}\int_{0}^{x_i}\int_{x}^{a}\frac{1}{a^2c}dx_jdxdx_i + \int_{t-a}^{c}\int_{(x_i+a-t)/2}^{x_i}\int_{x}^{a}\frac{1}{a^2c}dx_jdxdx_i = \frac{3a\left(c^2+2ct-t^2\right)-a^3-3c^3-3a^2(c-t)+3c^2t-3ct^2+t^3}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(a+c-t)^2}{4a^2c}.$$

Condition 2: with $c > a/2$

With $0 \leq t \leq a - c$, the pdf is the same as $0 \leq t \leq c$ in condition 1, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(2c-t)t}{2a^2c}.$$

With $a - c < t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_{0}^{c-t}\int_{x}^{x+t}\int_{x}^{2x-x_i+t}\frac{1}{a^2c}dx_jdx_idx + \int_{c-t}^{a-t}\int_{x}^{c}\int_{x}^{2x-x_i+t}\frac{1}{a^2c}dx_jdx_idx + \int_{a-t}^{(a+c-t)/2}\int_{2x+t-a}^{c}\int_{x}^{2x-x_i+t}\frac{1}{a^2c}dx_jdx_idx + \int_{a-t}^{c}\int_{(x_i+a-t)/2}^{x_i}\int_{x}^{a}\frac{1}{a^2c}dx_jdxdx_i = \frac{a^3-c^3-3c^2t+3ct^2-3t^3-3a^2(c+t)+3a(c+t)^2}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{2a(c+t)-a^2-c^2+2ct-3t^2}{4a^2c}.$$

With $c < t \leq a$, the pdf is the same as $a - c < t \leq a$ in condition 1, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{c^2-a^2-2ct-t^2+2a(c+t)}{4a^2c}.$$

With $a < t \leq a + c$, the pdf is the same as $a < t \leq a + c$ in condition 1, the pdf is

$$f(t|\text{Case } 1) = \frac{d}{dt} P_r(T \leq t|\text{Case } 1) = \frac{(a + c - t)^2}{4a^2c}.$$

Case 2: with $T = x_i - x_j$ and $x_j < x \leq x_i$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case } 2) = P_r\left(x_i - x_j \leq t|\text{Case } 2\right) = \int_0^{c-t} \int_{x_j}^{x_j+t} \int_{x_j}^{x_i} \frac{1}{a^2c} dx dx_i dx_j +$$

$$\int_{c-t}^{c} \int_{x_j}^{c} \int_{x_j}^{x_i} \frac{1}{a^2c} dx dx_i dx_j = \frac{(3c-2t)t^2}{6a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case } 2)$ with respect to T, the pdf is

$$f(t|\text{Case } 2) = \frac{d}{dt} P_r(T \leq t|\text{Case } 2) = \frac{(c-t)t}{a^2c}.$$

Case 3: with $T = x_j - x_i$ and $x_i < x \leq x_j$

With $0 \leq t \leq a - c$, the probability for T is

$$P_r(T \leq t|\text{Case } 3) = P_r\left(x_j - x_i \leq t|\text{Case } 3\right) = \int_0^{c} \int_{x_i}^{x_i+t} \int_{x_i}^{x_j} \frac{1}{a^2c} dx dx_j dx_i = \frac{t^2}{2a^2}$$

and take the derivative of $P_r(T \leq t|\text{Case } 3)$ with respect to T, the pdf is

$$f(t|\text{Case } 3) = \frac{d}{dt} P_r(T \leq t|\text{Case } 3) = \frac{t}{a^2}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case } 3) = P_r\left(x_j - x_i \leq t|\text{Case } 3\right) = \int_0^{a-t} \int_{x_i}^{x_i+t} \int_{x_i}^{x_j} \frac{1}{a^2c} dx dx_j dx_i +$$

$$\int_{a-t}^{c} \int_{x_i}^{a} \int_{x_i}^{x_j} \frac{1}{a^2c} dx dx_j dx_i = \frac{3a^2c - a^3 - 3ac^2 + c^3 + 3at^2 - 2t^3}{6a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case } 3)$ with respect to T, the pdf is

$$f(t|\text{Case } 3) = \frac{d}{dt} P_r(T \leq t|\text{Case } 3) = \frac{(a-t)t}{a^2c}.$$

Case 4: with $T = 2x - x_i - x_j$ and $x_i \geq x, x_j \geq x$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 4}\big) = \int_0^t \int_{x_i}^{(x_i+t)/2} \int_0^x \frac{1}{a^2c} dx_j dx dx_i +$$

$$\int_0^t \int_{(x_i+t)/2}^{x_i+t} \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i + \int_t^c \int_{x_i}^{x_i+t} \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i = \frac{(6c-t)t^2}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 1}) = \frac{(4c-t)t}{4a^2c}.$$

With $c < t \leq a - c$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 4}\big) = \int_0^c \int_{x_i}^{(x_i+t)/2} \int_0^x \frac{1}{a^2c} dx_j dx dx_i +$$

$$\int_0^c \int_{(x_i+t)/2}^{x_i+t} \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i = \frac{3t^2+3ct-c^2}{12a^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4}) = \frac{c+2t}{4a^2}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 4}\big) = \int_0^c \int_{x_i}^{(x_i+t)/2} \int_0^x \frac{1}{a^2c} dx_j dx dx_i +$$

$$\int_0^{a-t} \int_{(x_i+t)/2}^{x_i+t} \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i + \int_{a-t}^c \int_{(x_i+t)/2}^{a} \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i =$$

$$\frac{6a(c+t)^2+2a^3-3c^3-3c^2t-3ct^2-2t^3-6a^2(c+t)}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4}) = \frac{4a(c+t)-2a^2-c^2-2ct-2t^2}{4a^2c}.$$

With $a < t \leq 2a - c$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 4}\big) = \int_0^c \int_{x_i}^{(x_i+t)/2} \int_0^x \frac{1}{a^2c} dx_j dx dx_i +$$

$$\int_0^c \int_{(x_i+t)/2}^a \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i = \frac{2a(c+2t)-2a^2-c^2-ct-t^2}{4a^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4}) = \frac{4a-c-2t}{4a^2}.$$

With $2a - c < t \leq 2a$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 4}\big) = \int_0^{2a-t} \int_{x_i}^{(x_i+t)/2} \int_0^x \frac{1}{a^2c} dx_j dx dx_i +$$

$$\int_{2a-t}^c \int_{x_i}^a \int_0^x \frac{1}{a^2c} dx_j dx dx_i + \int_0^{2a-t} \int_{(x_i+t)/2}^a \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx dx_i = \frac{t^3+6a^2(c+2t)-8a^3-2c^3-6at^2}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4}) = \frac{(t-2a)^2}{4a^2c}.$$

Condition 2: with $c > a/2$

With $0 \leq t \leq a - c$, the pdf is the same as $0 \leq t \leq c$ in condition 1, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4}) = \frac{(4c-t)t}{4a^2c}.$$

With $a - c < t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\big(2x - x_i - x_j \leq t|\text{Case 4}\big) = \int_0^t \int_{x_i}^{(x_i+t)/2} \int_0^x \frac{1}{a^2c} dx_j dx dx_i +$$

$$\int_{t/2}^t \int_0^{2x-t} \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx_i dx + \int_t^c \int_{x-t}^x \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx_i dx +$$

$$\int_c^a \int_{x-t}^x \int_{2x-x_i-t}^x \frac{1}{a^2c} dx_j dx_i dx = \frac{2a^3-2c^3-6c^2t-3t^3-6a^2(c+t)+6a(c+t)^2}{12a^2c}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt} P_r(T \leq t|\text{Case 4}) = \frac{4a(c+t)-2a^2-2c^2-3t^2}{4a^2c}.$$

With $c < t \leq a$, the pdf is the same as $a - c < t \leq a$ in condition 1, the pdf is

$$f(t|\text{Case } 4) = \frac{d}{dt} P_r(T \leq t|\text{Case } 4) = \frac{4a(c+t) - 2a^2 - c^2 - 2ct - 2t^2}{4a^2 c}.$$

With $a < t \leq 2a - c$, the pdf is the same as $a < t \leq 2a - c$ in condition 1, the pdf is

$$f(t|\text{Case } 4) = \frac{d}{dt} P_r(T \leq t|\text{Case } 4) = \frac{2a(c+2t) - 2a^2 - c^2 - ct - t^2}{4a^2}.$$

With $2a - c < t \leq 2a$, the pdf is the same as $2a - c < t \leq 2a$ in condition 1, the pdf is

$$f(t|\text{Case } 4) = \frac{d}{dt} P_r(T \leq t|\text{Case } 4) = \frac{(t-2a)^2}{4a^2 c}.$$

In summary, from the derivations above, the pdf is

Condition 1: $0 \leq c \leq a/2$

$$f(t) = \begin{cases} \frac{(16c-7t)t}{4a^2c} & t \in [0, c] \\ \frac{3(c+2t)}{4a^2} & t \in (c, a-c] \\ \frac{6ac-3a^2+10at-4ct-7t^2}{4a^2c} & t \in (a-c, a] \\ \frac{a^2+6ac-2at-4ct+t^2}{4a^2c} & t \in (a, a+c] \\ \frac{4a-c-2t}{4a^2} & t \in (a+c, 2a-c] \\ \frac{(t-2a)^2}{4a^2c} & t \in (2a-c, 2a] \end{cases}. \qquad \text{(M1)}$$

Condition 2: $a/2 < c \leq a$

$$f(t) = \begin{cases} \frac{(16c-7t)t}{4a^2c} & t \in [0, a-c] \\ \frac{6ac-3a^2-3c^2+10at+6ct-14t^2}{4a^2c} & t \in (a-c, c] \\ \frac{6ac-3a^2+10at-4ct-7t^2}{4a^2c} & t \in (c, a] \\ \frac{a^2+6ac-2at-4ct+t^2}{4a^2c} & t \in (a, 2a-c] \\ \frac{5a^2+2ac+c^2-6at-2ct+2t^2}{4a^2c} & t \in (2a-c, a+c] \\ \frac{(t-2a)^2}{4a^2c} & t \in (a+c, 2a] \end{cases}. \qquad \text{(M2)}$$

Although the pdf are different, the expected value and variance are the same for different pdfs and they are

$$E[T] = \frac{5a}{6} - \frac{c}{2} + \frac{c^2}{3a}, \tag{M3}$$

$$E[T^2] = \frac{5a^2}{6} - \frac{5ac}{6} + \frac{2c^2}{3} - \frac{c^3}{6a} + \frac{c^4}{15a^2}. \tag{M4}$$

Appendix N: Horizontal travel time pdf for travel path B_2A or B_2B_1 with random storage retrieval operation

The pdf of the horizontal travel path B_2B_1 can be obtained by replacing c with a-c in the pdf of the horizontal travel path B_2A, therefore, we only need to derive the pdf for horizontal travel path B_2A. For the horizontal travel path B_2A in Figure A1, the NALT will travel from a random floor-level location x_i in class B_2 to another different random floor-level location x in class A, and, then, the NALT will travel to a different random floor-level location x_j in class A. The expression for the NALT horizontal travel time is:

$$T = |x - x_i| + |x - x_j| = \begin{cases} x_i + x_j - 2x & x_i \geq x, x_j \geq x \quad \text{Case 1} \\ x_i - x_j & x_j < x \leq x_i \quad \text{Case 2} \\ x_j - x_i & x_i < x \leq x_j \quad \text{Case 3} \\ 2x - x_i - x_j & x_i < x, x_j < x \quad \text{Case 4} \end{cases}$$

where $x_i \in \text{unif}[0, a]$ and $x, x_j \in \text{unif}[0, c]$.

$$f(x, x_i, x_j) = f(x)f(x_i)f(x_i) = \frac{1}{ac^2}$$

Case 1: with $T = x_i + x_j - 2x$ and $x_i \geq x, x_j \geq x$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_0^{c-t} \int_x^{x+t} \int_x^{2x-x_j+t} \frac{1}{ac^2} dx_i dx_j dx +$

$\int_{c-t}^{c} \int_x^{c} \int_x^{2x-x_j+t} \frac{1}{ac^2} dx_i dx_j dx = \frac{(3c-t)t^2}{6ac^2}$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(2c-t)t}{2ac^2}.$$

With $c < t \leq a - c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_0^c \int_x^c \int_x^{2x-x_j+t} \frac{1}{ac^2} dx_i dx_j dx = \frac{3t-c}{6a}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{1}{2a}.$$

With $a - c < t \leq a$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_0^{a-t} \int_0^{x_j} \int_x^{2x-x_j+t} \frac{1}{ac^2} dx_i dx dx_j +$$

$$\int_{a-t}^c \int_0^{(x_j+a-t)/2} \int_x^{2x-x_j+t} \frac{1}{ac^2} dx_i dx dx_j + \int_{a-t}^c \int_{(x_j+a-t)/2}^{x_j} \int_x^a \frac{1}{ac^2} dx_i dx dx_j =$$

$$\frac{a^3-3c^3+3c^2t-3ct^2-t^3-3a^2(c+t)+3a(c+t)^2}{12ac^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{2a(c+t)-a^2+c^2-2ct-t^2}{4ac^2}.$$

With $a < t \leq a + c$, the probability for T is

$$P_r(T \leq t|\text{Case 1}) = P_r\left(x_i + x_j - 2x \leq t|\text{Case 1}\right) = \int_{t-a}^c \int_0^{(x_j+a-t)/2} \int_x^{2x-x_j+t} \frac{1}{ac^2} dx_i dx dx_j +$$

$$\int_0^{t-a} \int_0^{x_j} \int_x^a \frac{1}{ac^2} dx_i dx dx_j + \int_{t-a}^c \int_{(x_j+a-t)/2}^{x_j} \int_x^a \frac{1}{ac^2} dx_i dx dx_j =$$

$$\frac{3a\left(c^2+2ct-t^2\right)-a^3-3c^3-3a^2(c-t)+3c^2t-3ct^2+t^3}{12ac^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \leq t|\text{Case 1}) = \frac{(a+c-t)^2}{4ac^2}.$$

Condition 2: with $c > a/2$

With $0 \le t \le a - c$, the pdf is the same as $0 \le t \le c$ in condition 1, the pdf is

$$f(t|\text{Case 1}) = \frac{(2c-t)t}{2ac^2}.$$

With $a - c < t \le c$, the probability for T is

$$P_r(T \le t|\text{Case 1}) = P_r\big(x_i + x_j - 2x \le t|\text{Case 1}\big) = \int_0^{c-t}\int_x^{x+t}\int_x^{2x-x_j+t}\frac{1}{ac^2}dx_i dx_j dx + \int_{c-t}^{a-t}\int_x^{c}\int_x^{2x-x_j+t}\frac{1}{ac^2}dx_i dx_j dx + \int_{a-t}^{(a+c-t)/2}\int_{2x+t-a}^{c}\int_x^{2x-x_j+t}\frac{1}{ac^2}dx_i dx_j dx + \int_{a-t}^{c}\int_{(x_j+a-t)/2}^{x_j}\int_x^{a}\frac{1}{ac^2}dx_i dx dx_j = \frac{a^3-c^3-3c^2t+3ct^2-3t^3-3a^2(c+t)+3a(c+t)^2}{12ac^2}$$

and take the derivative of $P_r(T \le t|\text{Case 1})$ with respect to T, the pdf is

$$f(t|\text{Case 1}) = \frac{d}{dt}P_r(T \le t|\text{Case 1}) = \frac{2a(c+t)-a^2-c^2+2ct-3t^2}{4ac^2}.$$

With $c < t \le a$, the pdf is the same as $a - c < t \le a$ in condition 1, the pdf is

$$f(t|\text{Case 1}) = \frac{2a(c+t)-a^2+c^2-2ct-t^2}{4ac^2}.$$

With $a < t \le a + c$, the pdf is the same as $a < t \le a + c$ in condition 1, the pdf is

$$f(t|\text{Case 1}) = \frac{(a+c-t)^2}{4ac^2}$$

Case 2: with $T = x_i - x_j$ and $x_j < x \le x_i$

Condition 1: with $c \le a/2$

With $0 \le t \le c$, the probability for T is

$$P_r(T \le t|\text{Case 2}) = P_r\big(x_i - x_j \le t|\text{Case 2}\big) = \int_0^{c-t}\int_{x_j}^{x_j+t}\int_{x_j}^{x_i}\frac{1}{ac^2}dx dx_i dx_j + \int_{c-t}^{c}\int_{x_j}^{c}\int_{x_j}^{x_i}\frac{1}{ac^2}dx dx_i dx_j + \int_{c-t}^{c}\int_{c}^{x_j+t}\int_{x_j}^{c}\frac{1}{ac^2}dx dx_i dx_j = \frac{(3c-t)t^2}{6ac^2}$$

and take the derivative of $P_r(T \le t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt}P_r(T \le t|\text{Case 2}) = \frac{(2c-t)t}{2ac^2}.$$

With $c < t \le a - c$, the probability for T is

$$P_r(T \le t|\text{Case 2}) = P_r\left(x_i - x_j \le t|\text{Case 2}\right) = \int_0^c \int_{x_j}^c \int_{x_j}^{x_i} \frac{1}{ac^2}\,dx dx_i dx_j +$$

$$\int_0^c \int_c^{x_j+t} \int_{x_j}^c \frac{1}{ac^2}\,dx dx_i dx_j = \frac{3t-c}{6a}$$

and take the derivative of $P_r(T \le t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt}P_r(T \le t|\text{Case 2}) = \frac{1}{2a}.$$

With $a - c < t \le a$, the probability for T is

$$P_r(T \le t|\text{Case 2}) = P_r\left(x_i - x_j \le t|\text{Case 2}\right) = \int_0^c \int_{x_j}^c \int_{x_j}^{x_i} \frac{1}{ac^2}\,dx dx_i dx_j +$$

$$\int_0^{a-t} \int_c^{x_j+t} \int_{x_j}^c \frac{1}{ac^2}\,dx dx_i dx_j + \int_{a-t}^c \int_c^a \int_{x_j}^c \frac{1}{ac^2}\,dx dx_i dx_j = \frac{a^3-2c^3-3ct^2-t^3-3a^2(c+t)+3a(c+t)^2}{6ac^2}$$

and take the derivative of $P_r(T \le t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt}P_r(T \le t|\text{Case 2}) = \frac{(a-t)(t-a+2c)}{2ac^2}.$$

Condition 2: with $c > a/2$

With $0 \le t \le a - c$, the pdf is the same as $0 \le t \le c$ in condition 1, the pdf is

$$f(t|\text{Case 2}) = \frac{(2c-t)t}{2ac^2}$$

With $a - c < t \le c$, the probability for T is

$$P_r(T \le t|\text{Case 2}) = P_r\left(x_i - x_j \le t|\text{Case 2}\right) = \int_0^{c-t} \int_{x_j}^{x_j+t} \int_{x_j}^{x_i} \frac{1}{ac^2}\,dx dx_i dx_j +$$

$$\int_{c-t}^c \int_{x_j}^c \int_{x_j}^{x_i} \frac{1}{ac^2}\,dx dx_i dx_j + \int_{c-t}^{a-t} \int_c^{x_j+t} \int_{x_j}^c \frac{1}{ac^2}\,dx dx_i dx_j + \int_{a-t}^c \int_c^a \int_{x_j}^c \frac{1}{ac^2}\,dx dx_i dx_j =$$

$$\frac{a^3-c^3-3c^2t-2t^3-3a^2(c+t)+3a(c+t)^2}{6ac^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 2})$ with respect to T, the pdf is

$$f(t|\text{Case 2}) = \frac{d}{dt}P_r(T \leq t|\text{Case 2}) = \frac{2a(c+t)-a^2-c^2-2t^2}{2ac^2}.$$

With $c < t \leq a$, the pdf is the same as $a - c < t \leq a$ in condition 1, the pdf is

$$f(t|\text{Case 2}) = \frac{(a-t)(t-a+2c)}{2ac^2}$$

Case 3: with $T = x_j - x_i$ and $x_i < \ x \leq x_j$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 3}) = P_r\left(x_j - x_i \leq t|\text{Case 3}\right) = \int_0^{c-t}\int_{x_i}^{x_i+t}\int_{x_i}^{x_j}\frac{1}{ac^2}dx dx_j dx_i +$$

$$\int_{c-t}^{c}\int_{x_i}^{c}\int_{x_i}^{x_j}\frac{1}{ac^2}dx dx_j dx_i = \frac{(3c-2t)t^2}{6ac^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 3})$ with respect to T, the pdf is

$$f(t|\text{Case 3}) = \frac{d}{dt}P_r(T \leq t|\text{Case 3}) = \frac{(c-t)t}{ac^2}.$$

Case 4: with $T = 2x - x_i - x_j$ and $x_i \geq x, x_j \geq x$

Condition 1: with $c \leq a/2$

With $0 \leq t \leq c$, the probability for T is

$$P_r(T \leq t|\text{Case 4}) = P_r\left(2x - x_i - x_j \leq t|\text{Case 4}\right) = \int_0^{t}\int_{x_j}^{(x_j+t)/2}\int_0^{x}\frac{1}{ac^2}dx_i dx dx_j +$$

$$\int_{t/2}^{t}\int_0^{2x-t}\int_{2x-x_j-t}^{x}\frac{1}{ac^2}dx_i dx_j dx + \int_t^c\int_{x-t}^{x}\int_{2x-x_j-t}^{x}\frac{1}{ac^2}dx_i dx_j dx = \frac{(2c-t)t^2}{4ac^2}$$

and take the derivative of $P_r(T \leq t|\text{Case 4})$ with respect to T, the pdf is

$$f(t|\text{Case 4}) = \frac{d}{dt}P_r(T \leq t|\text{Case 4}) = \frac{(4c-3t)t}{4ac^2}.$$

With $c < t \leq 2c$, the probability for T is

$$P_r(T \leq t|\text{Case } 4) = P_r\left(2x - x_i - x_j \leq t|\text{Case } 4\right)$$

$$= \int_0^{t/2}\int_0^{x}\int_0^{x}\frac{1}{ac^2}dx_i dx_j dx + \int_{t/2}^{c}\int_{2x-t}^{x}\int_0^{x}\frac{1}{ac^2}dx_i dx_j dx$$

$$+ \int_{t/2}^{c}\int_0^{2x-t}\int_{2x-x_j-t}^{x}\frac{1}{ac^2}dx_i dx_j dx = \frac{12c^2t - 6ct^2 + t^3 - 4c^3}{12ac^2}$$

and take the derivative of $P_r(T \leq t|\text{Case } 4)$ with respect to T, the pdf is

$$f(t|\text{Case } 4) = \frac{d}{dt}P_r(T \leq t|\text{Case } 4) = \frac{(t-2c)^2}{4ac^2}.$$

Condition 1: $0 \leq c \leq a/3$

$$f(t) = \begin{cases} \frac{(16c-11t)t}{4ac^2} & t \in [0, c] \\ \frac{4c^2+(t-2c)^2}{4ac^2} & t \in (c, 2c] \\ \frac{1}{a} & t \in (2c, a-c]. \\ \frac{c^2-3a^2-6ct-3t^2+6a(c+t)}{4ac^2} & t \in (a-c, a] \\ \frac{(a+c-t)^2}{4ac^2} & t \in (a, a+c] \end{cases} \qquad \text{(N1)}$$

Condition 2: $a/3 < c \leq a/2$

$$f(t) = \begin{cases} \frac{(16c-11t)t}{4ac^2} & t \in [0, c] \\ \frac{4c^2+(t-2c)^2}{4ac^2} & t \in (c, a-c] \\ \frac{5c^2-3a^2-10ct-2t^2+6a(c+t)}{4ac^2} & t \in (a-c, 2c]. \\ \frac{c^2-3a^2-6ct-3t^2+6a(c+t)}{4ac^2} & t \in (2c, a] \\ \frac{(a+c-t)^2}{4ac^2} & t \in (a, a+c] \end{cases} \qquad \text{(N2)}$$

Condition 3: $a/2 < c \leq a$

$$f(t) = \begin{cases} \frac{(16c-11t)t}{4ac^2} & t \in [0, a-c] \\ \frac{10ct-3a^2-3c^2-14t^2+6a(c+t)}{4ac^2} & t \in (a-c, c] \\ \frac{5c^2-3a^2-10ct-2t^2+6a(c+t)}{4ac^2} & t \in (c, a] \\ \frac{a^2+2ac+5c^2-2at-6ct+2t^2}{4ac^2} & t \in (a, 2c] \\ \frac{(a+c-t)^2}{4ac^2} & t \in (2c, a+c] \end{cases} \tag{N3}$$

The expected value and variance are the same for different pdfs and they are

$$E[T] = \frac{a}{2} - \frac{c}{6} + \frac{c^2}{3a}, \tag{N4}$$

$$E[T^2] = \frac{a^2}{3} - \frac{ac}{6} + \frac{c^2}{6} + \frac{7c^3}{30a}. \tag{N5}$$

Certification of Student Work

UNIVERSITY OF ARKANSAS

College of Engineering
Department of Industrial Engineering

MEMORANDUM

TO: University of Arkansas Graduate School

FROM: Haitao Liao, Professor, John and Mary Lib White Endowed Systems Integration Chair
John A. White, Chancellor Emeritus and Distinguished Professor (retired)

DATE: July 25, 2019

SUBJECT: Certification of Student Effort and Contribution

As co-directors of Mr. Jingming Liu's doctoral dissertation, we certify Mr. Liu contributed more than 51 percent of the work included in the chapter entitled, "A Working Paper on "Class-based storage and retrieval in an MIAPP-NALT system in stochastic conditions" contained in the doctoral dissertation entitled, "Probabilistic Models for Order-Picking Operations with Multiple in-the-Aisle Pick Positions".

Chapter 5

Summary, Conclusions and Future Research

5.1. Summary

Multiple in-the-aisle pick positions (MIAPP) order-picking systems were analyzed under stochastic conditions, considering both dedicated and random storage, when pick positions were replenished using either a narrow-aisle lift truck (NALT) or an automated storage and retrieval (AS/R) machine. All chapters included a case study, insights gained, and conclusions drawn. The following sub-sections provide summaries of the chapters.

5.1.1. Summary of Chapter 2

In modeling the MIAPP-NALT operation as a queueing system, pdfs were developed for NALT travel time for four scenarios: FD, FR, ID and IR. For each scenario, formulas for the expected value and variance for NALT travel time were derived. For finite population scenarios, the Laplace transform for service time were obtained. The MIAPP-NALT system was modeled as a finite population queue (M/G/1/N) and infinite population queue (M/G/1/∞); then, the MIAPP-NALT system was approximated using an M/G/1/∞ queueing model. For the M/G/1/N queue, Takács' formulas were used to determine the steady-state performance measures of the MIAPP-NALT operation using Laplace transforms of the service time pdfs. Pollaczek-Khinchine formulas were used to determine the steady-state performance measures for the M/G/1/∞ model by using expected values and variances for service times obtained.

5.1.2. Summary of Chapter 3

To model the MIAPP-AS/RS retrieval operation as a queueing system, travel time probability density functions (pdfs) were derived for four travel paths for both dedicated storage and random storage. For each path, pdfs were derived. The MIAPP-AS/RS retrieval operation was first modeled as a finite population queue (M/G/1/N), then, the pdfs for an infinite population queue (M/G/1/∞) were used to approximate the finite population queue. For the finite population queue, the steady state performance measure was obtained using Tak$\acute{a}$cs's results by applying the Laplace transform of the pdfs for the four travel paths. For the infinite population queue, the steady state performance measures can be obtained using Pollaczek-Khinchine formulas, based on expected values and variances for service times obtained for the four paths.

5.1.3. Summary of Chapter 4

To model the MIAPP-NALT retrieval operation as an M/G/1 queueing system, travel time pdfs were derived for retrieval and storage operations for dedicated storage, random storage and class-based storage. From retrieval and storage travel time pdfs, the expected value, the second moment and the variance for service time were calculated after combining pick up and deposit times (T_{PD}). MIAPP-NALT retrieval and storage operations were analyzed with priority and non-priority queueing models and the results were compared. For the priority queueing model and non-priority queueing model, the expected value, the second moment and the variance of service time were calculated. Using a steady state queueing performance measure, priority and non-priority queues were compared, dedicated and random storage were compared, the skewness of the AB curve was analyzed, and three different layouts were compared.

5.2. Conclusions

Each chapter included conclusions drawn from the research performed and results obtained from the case studies considered. In the following sub-sections, the conclusions from each chapter are provided.

5.2.1. Conclusions from Chapter 2

Based on the research results obtained, we drew the following conclusions: the M/G/1/N queueing model can be used to model the performance of an MIAPP-NALT operation when times to deplete pallets at pick positions are neither exponentially distributed nor identically distributed; expected values obtained from the pdfs we derived can be used to design an MIAPP-NALT system, but modeling the system as an M/G/1 queue provides greater design insights than relying solely on expected values for demands for replenishment and service times; depending on the traffic intensity and the number of pick positions in an S/R aisle, a significant penalty in order-picking productivity can result when using random storage versus dedicated storage unless random storage significantly reduces the amount of storage space required; for the parameter values we considered, an infinite population queueing approximation can be used to obtain reasonably accurate results when traffic intensity is less than approximately 80 percent and approximately 150 pick positions are assigned to an S/R aisle; and the smaller the population size, the smaller the traffic intensity must be for a reasonable infinite population approximation.

5.2.2. Conclusions from Chapter 3

In comparison with expected-value analysis used to determine the utilization of the S/R machine, the queueing results in Section 6 can be used to determine the values of the performance measures, as well as the utilization of the AS/R machine. Additionally, Takács's results can be used to obtain the probability mass function for the number of empty deposit positions. Further, by establishing an aspiration level for the expected number of pick positions waiting to be replenished, the models in Section 6 can be used to determine if the design meets the established aspiration level.

A large number of cases must be considered in developing the pdf for the finite population; doing so can be quire cumbersome. To simplify calculations for the finite population queueing model, the Laplace transform of the pdf for service time based on an infinite population can be used with Takács's results for the finite population queue. To justify economically the use of an MIAPP-AS/RS, a large number of pick positions is generally required; therefore, using the Laplace transform of the pdf for service time for an infinite population, the results obtained from Taka ́cs's results will closely approximate those that would be obtained using the Laplace transform of the pdf for service time for a finite population.

Comparing dedicated storage to random storage, when the same amount of reserve-storage space is used for dedicated and random storage, we observed dedicated storage yields significantly smaller values for the performance measures than random storage. The exception is when traffic intensity is quite low. (It is unlikely an MIAPP-AS/RS design will be used for very low traffic intensity applications.) The results in Section 6 can be used to establish an upper bound on the relative reserve-storage requirements for random storage to yield the same replenishment performance as dedicated storage.

For future research, several extensions can be considered for the MIAPP-AS/RS. One potential extension is to use a quasi-random storage policy by storing replenishment loads in the open storage position closest to the pick position destined for the replenishment load. Another extension is to assign SKUs to pick position locations based on a class-based or turnover-based storage policy. A third extension is to model both the AS/R machine and the order-picking workers in the MIAPP-AS/RS as a queueing network. A fourth research extension is to consider the storage and retrieval operations in the MIAPP-AS/RS and analyze interleaving of storage and retrieval operations.

5.2.3. Conclusions from Chapter 4

- For priority and non-priority queues, the priority queue outperforms the non-priority queue in terms of the number of retrieval or storage requests waiting to be or being filled. Another benefit of a priority queue compared to a non-priority queue is the number of retrieval requests waiting to be or being filled is relatively small, especially when overall traffic intensity is high.
- For dedicated and random storage, dedicated storage can greatly reduce the value of the number of retrieval or storage requests waiting to be or being filled in comparison to random storage. Because travel time with random storage is greater than for dedicated storage, dedicated storage outperforms random storage; however, when traffic intensity is low, the relative benefit of using dedicated storage diminishes significantly (the reverse is true when traffic intensity is high).
- Regarding the skewness of the AB curve, the greater the skewness the greater the preference for a two-class based layout. Likewise, the greater the traffic intensity, the greater the benefit for a two-class based layout.
- From the analysis in Section 6.4, Layout 3 is the best allocation for a 30(70) skewness and for the traffic intensity underlying the calculations. The queueing results for Layout 1 and Layout 3 are very similar; both are better than Layout 2, especially when the traffic intensity is very high.

5.3. Future Research

Each chapter included recommendations for future research, based on the research performed and documented in the chapters. In the following sub-sections, we repeat the recommendations provided, recognizing there will be duplications due to the commonalities among the chapters.

5.3.1. Future Research for Chapter 2

For future research, a variety of extensions to the MIAPP-NALT system can be considered; including, but not limited to, using the NALT to replenish double-deep pallet rack or flow racks in single or multiple aisles. As mentioned, a closest-open-location variation of the random storage policy merits further attention. In addition, replenishing pick positions based on a prioritization of pick positions or class-based storage in an MIAPP-NALT system can be explored. Yet another logical extension to the MIAPP-NALT system is incorporating mezzanines in the models developed; doing so introduces computational, not theoretical challenges – we chose to limit our research to floor-level order picking to shorten the paper and simplify modeling notation. An interesting extension of our queueing model is to incorporate people performing case picking in the model, resulting in a queueing network; the designer will not only decide the length of the S/R aisle, but also the number of people assigned to the pick positions in a picking aisle. Finally, interleaving storage and retrieval operations performed by the NALT could prove to be a fertile area for further research.

5.3.2. Future Research for Chapter 3

For future research, several extensions can be considered for the MIAPP-AS/RS. One potential extension is to use a quasi-random storage policy by storing replenishment loads in the open storage position closest to the pick position destined for the replenishment load. Another extension is to assign SKUs to pick position locations based on a class-based or turnover-based storage policy. A third extension is to model both the AS/R machine and the order-picking workers in the MIAPP-AS/RS as a queueing network. A fourth research extension is to consider the storage and retrieval operations in the MIAPP-AS/RS and analyze interleaving of storage and retrieval operations.

5.3.3. Future Research for Chapter 4

For future research, several extensions can be considered for the MIAPP-NALT:

- Include more than one mezzanine level with pick positions.
- Develop guidelines (as a function of traffic intensity and skewness) for the fraction of a class of skus located at each level of pick positions.
- Increase the number of sku classes from two to three and analyze the impact of skewness of ABC curves on queueing performance.
- Expand storage policies (dedicated, random, and class-based) to include a quasi-random storage policy. With a quasi-random policy, instead of a pallet being stored at any point within a region (random storage), it is stored in the open reserve-storage location closest to the pick position for which it is destined.
- Relax the assumption of the aisle-captive narrow aisle lift truck to allow the vehicle to perform storages and retrievals in multiple aisles.

- Model the MIAPP-NALT as a queueing network by combining the activities of the NALT and the activities of the order pickers.