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Mathematically Gifted Students' Experiences of Challenge with Cognitively Guided Instruction

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MATHEMATICALLY GIFTED STUDENTS' EXPERIENCES OF CHALLENGE
WITH
COGNITIVELY GUIDED INSTRUCTION

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ABSTRACT

This study examined to what extent and in what ways mathematically gifted students were challenged in two elementary classrooms taught by exemplary teachers using the principles of Cognitively Guided Instruction (CGI). The two case studies included a third grade class and a fourth/fifth split grade class, with five mathematically advanced participants from each grade. A top tier and a second tier of mathematically advanced students were identified, with the top tier of five students identified as mathematically gifted. Classroom observations of CGI math lessons, interviews with students and teachers, and analysis of students' problem-solving strategies were conducted over a five week period. A synthesis of literature from mathematics education and gifted education was used to craft an operational definition of what it means for a mathematically gifted student to be challenged, focusing on exploration of mathematical relationships, exposure to new mathematical ideas, and experience of Ascending Intellectual Demand on a continuum toward expertise. An alignment of CGI problem-solving strategy levels with levels of Ascending Intellectual Demand served as a conceptual framework for locating and describing the level of challenge experienced in their classrooms. This framework also was used to identify which elements were lacking and suggest what could provide further challenge. The findings revealed a classroom environment that was supportive of mathematical challenge with a variety of ways for extending students' thinking. However, although the students reported high levels of enjoying the lessons and worked with advanced topics, they reported low to mediocre levels of challenge, with the top tier group reporting less challenge than the second tier group. The self-reported challenge levels decreased as the students' grade level increased, with the fifth grade top tier student indicating the least perceived challenge. Analysis of the mixed results suggests that the challenge level of the assigned mathematical task should be elevated.

This dissertation is approved for recommendation
to the Graduate Council.

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DEDICATION

This dissertation is dedicated to the mathematically gifted students who participated in this study and to their equally-gifted teachers who understood the importance of challenge. It is also dedicated to mathematically gifted students who are still waiting to be challenged in their classrooms.

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CHAPTER 1: INTRODUCTION

Challenging mathematically gifted students in mainstream elementary classrooms can pose a challenge to teachers responsible for reaching a range of mathematical abilities. One type of learning environment that has demonstrated success in making mathematics accessible to a range of learners is a problem-solving oriented classroom that focuses on students' mathematical thinking, allowing students to solve problems using methods that make the most sense to them (Carey, Fennema, Carpenter, & Franke, 1995). The students use and discuss a variety of problem-solving strategies, engaging in mathematical reasoning in a "productive discourse" that is recommended for mathematically gifted students (Diezmann & Watters, 2002, p. 5). This study focused on mathematically gifted students in two classrooms that provided such a learning environment.

Cognitively Guided Instruction

Such a learning environment is one that is typically implemented by teachers who have participated in the research-based professional development program called Cognitively Guided Instruction (CGI). Although the program does not prescribe teaching methods, the teachers who have undergone this professional development (referred to as CGI teachers) deepen their knowledge of how children intuitively solve problems and are more likely to use a problem-solving approach to teaching mathematics to elicit mathematical thinking in their classrooms (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). Research has shown improved problem solving ability in classrooms taught by CGI teachers (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) and has additionally confirmed this benefit for the sub-populations of English language learners (Marshall, 2009), learning disabled students (Behrend, 1994), and lower ability math students (Empson, 2003). This study examined the experiences of students on the other end of the ability spectrum, namely mathematically gifted students, to find to what extent and in what ways they were challenged in a classroom using the principles of Cognitively Guided Instruction.

Mathematical Giftedness

There is a variety of nomenclature for referring to mathematically advanced students: mathematically advanced (Assouline & Lupkowski-Shoplik, 2005), mathematically gifted (Assouline & Lupkowski-Shoplik, 2005; Diezmann & Watters, 2000; Krutetskii, 1976; Ryser & Johnsen, 1998), mathematically talented (Stanley, Lupkowski, & Assouline, 1990), mathematically promising (Sheffield, 1999), and mathematically able (Krutetski, 1976). This study will identify two tiers of mathematically advanced students and refer to the top tier as "mathematically gifted". Although no one definition of mathematical giftedness has been embraced by the community of mathematics educators nor the gifted education community, Krutetskii's description (1976) has been referenced by authors from both fields such as Sheffield (1999), Leiken, Berman, and Koichu (2009), and Koshy, Ernest, and Casey (2009).

Based on his twelve year study of mathematical ability in schoolchildren in the Soviet Union, Krutetskii (1976) delineated a set of characteristics found in how mathematically-able children obtain, process and retain mathematical information as they engage in problem solving:

1. Obtaining mathematical information

   A. The ability for formalized perception of mathematical material, for grasping the formal structure of the problem.

2. Processing mathematical information

   A. The ability for logical thought in the sphere of quantitative and spatial relationships, number and letter symbols; the ability to think in mathematical symbols.
   B. The ability for rapid and broad generalization of mathematical objects, relations, and operations.

C. The ability to curtail the process of mathematical reasoning and the system of corresponding operations; the ability to think in curtailed structures.
D. Flexibility of mental processes in mathematical activity.
E. Striving for clarity, simplicity, economy, and rationality of solutions.
F. The ability for rapid and free reconstruction of the direction of a mental process, switching from a direct to a reverse train of thought (reversibility of the mental process in mathematical reasoning).

3. Retaining mathematical information

A. Mathematical memory (generalized memory for mathematical relationships, type characteristics, schemes of arguments and proofs, methods of problem-solving, and principles of approach) (p. 350).

These characteristics interrelate and form what he termed a "distinctive syndrome of mathematical giftedness, the mathematical cast of mind" (p. 351) that has a "distinctive tendency to perceive many phenomena through the prism of mathematical relationships" (p. 361). Although the number of mathematically gifted children depends on one's definition, Miller (1990) estimated that 2-3% of the population possess the characteristics described by Krutetskii.

Challenging the Mathematically Gifted Student

Mathematically gifted students thrive with a challenging, non-repetitive, mathematics curriculum that allows them to exercise and further develop their above average abilities by exploring the depth, complexity, and beauty of mathematics (Sheffield, 1999). Sheffield stressed the importance of a challenging curriculum in light of the potential contributions these students can make in our technological world. She pointed out that standardized test scores at the 99th percentile in mathematics do not necessarily indicate, however, that these students have been challenged nor inspired. In a study of mathematically gifted children, 5.7% said they never felt challenged, 47.1% felt rarely challenged, and 44.9% felt sometimes challenged (Assouline & Lupkowski-Shoplik, 2005, as cited in Cox, 2008).

Secondary students have access to advanced mathematics courses; however, there is little evidence that elementary gifted students are offered such challenges. Assouline and Lupkowski-Shoplik (2005) maintain that “the current educational landscape discourages mathematical talented students, as well as their educators and parents” (p. xvi). Services or special accommodations for mathematically gifted students tend to be sporadic, partly dependent on degree of interest and advocacy on the part of the teacher, principal, district administration, and parents.

There are numerous approaches to meeting the needs of the mathematically gifted. The National Mathematics Advisory Panel (2008) made the following recommendation: “Mathematically gifted students with sufficient motivation appear to be able to learn mathematics much faster than students proceeding through the curriculum at a normal pace, with no harm to their learning, and should be allowed to do so (p. 53).” Advancing more quickly through the curriculum can occur by grade acceleration, or within the classroom. A meta-analysis of research studies in gifted education reported that accelerating students capable of working above grade level is the most effective intervention, with long term benefits both academically and socially, but tends to be negatively viewed by educators and remains infrequently applied (Colangelo, Assouline, & Gross, 2004). Elementary grade acceleration for math class only, rather than skipping entire grades, is another option. Occasionally, schools will track students at grade level by forming a high ability math class. An approach considered more socially-equitable is that of clustering a small group of highest math ability students within one mainstream class to be taught by the grade level teacher who has the most expertise in teaching mathematics (Winebrenner, 2001). “Compacting” and “telescoping” refer to plans that can be implemented within the mainstream classroom, involving either increasing the speed at which the student goes through the curriculum, or reducing the curriculum to only the topics that have not yet been mastered by the student and adding enrichment or advanced topics in replacement (Renzulli, Smith, & Reis, 1982, p. 186).

Considering Cognitively Guided Instruction as an Option for Challenge

With the reality that most mathematically gifted students are in mainstream classes with their age peers who are not necessarily their intellectual peers (Assouline & Lupkowski-Shoplik, 2005) educators should consider how best to challenge them within that setting. The effects of Cognitively Guided Instruction on student achievement has been demonstrated in mainstream classrooms (Fennema et al., 1996). With its encouragement of mathematical thinking of students of all levels, a classroom taught by a CGI teacher (referred to as a CGI classroom) is an ideal choice for studying the potential for challenging mathematically gifted students.

Sheffield (1999) suggests that a classroom environment offering an “open-ended heuristic” (p. 46) for problem solving allows students with mathematical promise to learn at a faster pace and to explore mathematics with more breadth and depth. She cites Jensen’s (1980) model for gifted education as an example of such a heuristic approach. This model is illustrated with a five-point star featuring the points “relate”, “investigate”, “create”, “evaluate”, and “communicate”. In this model, the problem solving process begins with a problem posed by either the teacher or the student. The student may enter the process at any of these points, relating the problem to other known mathematical ideas, investigating strategies as possible solutions, creating new questions along the way, evaluating and verifying hypotheses, and communicating solutions to the class. In a study of teachers involved in four years of CGI professional development, Fennema et al. (1996) described the highest level of classroom instruction to be by the teachers whose instruction was guided by their knowledge of their students' thinking, or "cognitively guided" (p. 421). Their classrooms were devoted to problem solving, mathematical discourse, and providing opportunities for students to explore mathematics in the ways that made most sense to them, rather than by teacher-directed methods. The classrooms of these highest level teachers closely resemble Jensen's model of a student-centered classroom focused around problem-solving strategies and mathematical communication, further positioning a CGI classroom with exemplary teachers as a promising environment for studying mathematical gifted children.

CGI Professional Development Program

Cognitively Guided Instruction was established by Carpenter et al. (1989) as a professional development program for K-3 teachers, and has since expanded to include upper elementary grades. The research base behind CGI focused on how individual students solved additions and subtraction problems (Carpenter & Moser, 1984). In studying how children solved these problems without teacher direction, the researchers discovered that the children had an intuitive knowledge of mathematics that they were able to draw on to make sense of the problems, and that the sophistication of their strategies increased as they gained more experience in problem solving. Their research led them to create a taxonomy of problem types as well as a progression of problem-solving strategy levels that could aid teachers in understanding their students' thinking. CGI professional development was created to increase teachers' knowledge of how their students' think mathematically, with the idea that this knowledge will provide a framework for guiding the instruction in their classrooms. In this teacher development program, the teachers examine students' problem solving strategies to deepen their understanding of student thinking and learn to become facilitators of children's mathematical thinking. Fennema et al.'s study (1996) of CGI professional development found that the highest level of teacher-participants had created a student-centered classroom environment in which children had ample opportunities to investigate problems, communicate their questions and mathematical ideas to others, and justify and share their solutions.

The term, “Cognitively Guided Instruction,” has evolved into a somewhat colloquial term in which teachers refer to a “CGI teacher” as one who has participated in CGI professional development and a “CGI classroom” as one that employs the principles of CGI. This study focuses on two CGI classrooms each taught by a CGI teacher. The exemplary CGI teachers chosen for this study have created classroom environments and use an instructional approach similar to those of the highest level CGI teachers described by Fennema et al. (1996). The learning of mathematics curriculum was accomplished through problem-solving, featuring problems that would help develop mathematical concepts. Teachers encouraged students to invent their own strategies for solving problems and expected them to explain their thinking and justify their reasoning. There was an expectation that students would listen to other students’ strategies and develop mathematical communication skills. From frequent interaction with students, the teachers knew their students’ thinking well and actively sought ways to elicit deeper understanding and more sophisticated strategies from individual students, based on what they already knew. In this study, I refer to three distinct phases of the math lessons that I observed: the problem-posing Phase1, the problem-solving Phase 2, and the strategy-sharing discussion Phase 3.

Purpose and Significance of Study

This study examined the experiences of fifteen mathematically advanced students, five of whom were identified as mathematically gifted, in two CGI classrooms. In the interest of learning ways to address the problem of lack of challenge of mathematically gifted students in mainstream classrooms, the purpose of the study was to find out how Cognitively Guided Instruction can engage and challenge these students, and to what extent. By observing mathematically gifted students in CGI classrooms taught by exemplary teachers, analyzing their problem solving strategies, and interviewing both the students and their teachers, I hope to add to the literature on which strategies work best to challenge these students and where there may be room for improvement.

Research Question

To what extent and in what ways are mathematically gifted students challenged in CGI classrooms?

Theoretical Lens, Bias, and Scope of Study

This study was conducted within the theoretical lens of constructivism (Piaget, 1952, as cited in National Research Council, 2000), an understanding that persons construct meaning based on their experiences with the world around them. Mathematical meaning is constructed by individual students as they synthesize the classroom experience with their prior knowledge. With Cognitively Guided Instruction, teachers learn to tap into students' prior knowledge, come to understand it, then facilitate connections that lead to synthesis and learning.

The CGI literature offers a conceptual framework of problem solving strategy levels that was used to analyze student thinking (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011). The gifted education literature offers a conceptual framework for how a student moves from levels of novice to expert with Ascending Intellectual Demand (Tomlinson et al, 2002; Tomlinson et al, 2009). The levels of cognitive thought from Bloom's Revised Taxonomy (Anderson & Krathwohl, 2001) provided an overall framework for recognizing higher order thinking skills, as evidence of challenge was analyzed in this study.

The scope of this qualitative study has been limited to two classrooms in one geographic area with the main purpose of providing a description of experiences with Cognitively Guided Instruction as it relates to being challenged mathematically. Studying mathematically advanced students and their teachers in classrooms that use the principles of Cognitively Guided Instruction may shed light on how mathematically gifted students can be engaged, challenged, and working to their potential in a mainstream elementary math classroom. As an advocate for mathematically gifted children who wishes to see them reach their full potential, I recognize the possible bias involved in discussing an educational practice that may or may not be effective. Regardless, the description of the experiences of these students can lay the basis for recommendations of how to maximize the intellectual growth of mathematically gifted students in a CGI classroom.

CHAPTER 2: LITERATURE REVIEW

The background for this study of challenging mathematically gifted children in CGI classrooms emerged from a synthesis of the literature in Cognitively Guided Instruction (CGI) as well as gifted and mathematics education. The literature in Cognitively Guided Instruction includes its theoretical background and the influence that its professional development has had on teachers and students. The literature on giftedness includes characteristics of the domain specific area of mathematics, the issue of cognitive challenge, and what constitutes a challenging environment for mathematically gifted students. The synthesis of elements of this literature review was essential in developing an operational definition of mathematically challenge and a conceptual framework for viewing the challenge of elementary students.

In researching the literature, a Google or Google Scholar search was usually successful in locating, at a minimum, the correct citation and abstract, and often the entire document. Access to electronic journals and dissertations through the University of Arkansas library databases of Ebsco, Proquest, and ERIC, and National Council of Teachers of Mathematics membership provided much of this information. Books and journal articles were obtained that were not available electronically.

Cognitively Guided Instruction

Professional Development in Understanding Students' Thinking

The seminal work in Cognitively Guided Instruction described and studied effects of a teacher professional development program by the same name, which focused on deepening elementary teachers' knowledge of their students' mathematical thinking as they engage in mathematical problem solving (Carpenter et al., 1989). Having previously studied children's mathematical thinking and teachers' knowledge about their students' thinking, the researchers hypothesized that training teachers in CGI would increase student achievement, and designed an experimental study to test their hypothesis. They found that CGI classrooms were more oriented toward a problem solving context for teaching mathematics and the students scored higher in problem solving on standardized tests than the control group, but not significantly different on computation. Number facts were less likely to be explicitly taught in CGI classrooms, yet student recall of number facts was greater than that of the control group. CGI students reported significantly more confidence and mathematical understanding. The CGI teachers encouraged multiple strategies and listened to and better understood students' solution processes.

Taxonomy of Problem Types and Problem-solving Strategy Levels

Carpenter and Moser (1984) developed a framework for understanding how children think about and solve addition and subtraction problems. This framework marked a key step toward a professional development program designed to help teachers understand their students' mathematical thinking. Their longitudinal study followed how individual children's intuitive problem-solving strategies (without the direct instruction of teachers) developed from first through third grade. This analysis allowed them to categorize types of addition and subtraction problems as well as levels of strategies for solving them. The framework gave structure to the study of children's thinking as teachers in the CGI professional development analyzed student strategies and considered how to guide their students to more sophisticated strategies. The higher level, more sophisticated strategies were those that were based on an increased conceptual knowledge of mathematics. Subsequent literature characterized problem types and strategy levels for multiplication and division, and fractions. Description of these problem types and strategy levels follows. Fraction problem types and strategy levels are emphasized because both classrooms in this study were involved in fraction problem solving.

Addition and subtraction. For addition and subtraction, the basic categories of “join”, “separate”, “part-part-whole”, and “compare” lead to eleven problem types depending on the action within the word problem and which part of the information is unknown (Carpenter et al,, 1999, p. 7). For instance, the part-part-whole problem 2 + 3 = ?, has the whole unknown: Jen has 2 marbles and Jose has 3 marbles. How many do they have altogether? If we knew that they had 5 marbles altogether, and Jen had 2 of them, asking how many Jose has would change the direction of thinking to a “part unknown” problem, and would increase the challenge of the problem.

There is a progression of strategy levels in children’s solving of the various types of addition and subtraction problems, beginning with direct modeling in which students model the action and relationships of the problem by arranging physical objects or drawing a picture. When students no longer need to see the actual quantity represented, they transition to a more abstract, symbolic representation using counting strategies, such as counting on or counting down. Students’ experiences with different problem types encourage the use of a variety of counting strategies. Consequently, children’s strategy use becomes more flexible, allowing for invention of strategies based on number relationships. One example of using knowledge of relationships in solving a problem involves decomposing the numbers based on the concept of place value: 27 + 35 = 20 + 7 + 30 +5 = 50 + 12 = 62. In a later publication within CGI literature, the strategy of relating one numerical expression to another was termed “relational thinking” (Carpenter et al., 2003). Recognizing mathematical relationships, students may derive number facts from known facts throughout the strategy levels. Eventually, these strategies are replaced with applying knowledge of recalled number facts.

Multiplication and division. Multiplication, measurement division, and partitive division are problem types that are inter-related, and vary based on which piece of information is unknown (Carpenter et al., 1999). Consider this situation: There are 3 bags of candy with 10 pieces of candy in each bag which makes a total of 30 candies.

Multiplication Problem Type: Sue has 3 bags of candy with 10 pieces of candy in each bag. How many candies are there?

Measurement Division Problem Type: Sue has 30 pieces of candy. She puts 10 pieces of candy in each bag. How many bags will she need?

Partitive Division Problem Type: Sue has 30 pieces of candy. She puts the candy into 3 bags so that each bag contains the same amount. How pieces of candy will go in each bag?

Similar to the progression of strategy types for solving addition and subtraction problems, there are levels of strategies for solving multiplication and division problems. Students begin with direct modeling (using grouping, measurement, or partitive strategies). Following this concrete representation of quantities and relationships, they transition to the more efficient counting strategies for multiplication (such as repeated addition and skip counting) and deriving facts based on known facts, followed by recalling number facts. As students' conceptual knowledge of mathematics increases, they begin to see relationships within and between quantities and use relational thinking to solve problems.

Fractions. Empson and Levi (2011) categorized three problem types for fractions that help children understand fraction relationships for learning about fraction operations and equivalence: Equal -Sharing (Multiple Groups, Partitive Division), Multiple Groups – Multiplication, and Multiple Groups – Measurement Division. They describe levels of strategies for solving these types of fraction problems that show an increase in understanding of and use of relationships, leading to more efficient solutions.

Equal sharing. Children's knowledge of fractions stems from their intuitive sense of equal sharing or fair share, for instance sharing 7 sandwiches equally among 4 people (with an answer greater than one whole) or sharing 4 sandwiches equally among 7 people (with an answer less than one whole). Empson and Levi (2011) described strategies that children use to solve equal share problems, characterizing them based on how children coordinate the main two components of the problem: the people sharing and the things to be shared. If the student either does not distribute all that is to be shared or does not distribute equal shares, their solution shows "no coordination between sharers and shares" (p. 11). An example of "non-anticipatory coordination between sharers and shares" (p. 13) is where the student distributes whole pieces first, then cuts remaining pieces in half to distribute, then finally when a small amount remains, divides it into the number of pieces equal to the number of people. Students also may use "trial and error to coordinate" (p.14) trying a repertoire of different partitions until they find one that works for the amounts in the problem.

Children's strategies become more sophisticated when they become more "anticipatory" for coordinating the fractions with the number of sharers (Empson & Levi, 2011, p. 15). The first level of anticipatory coordination is "additive coordination – one item at a time". For example, in solving the problem of sharing 7 sandwiches among 8 people, the student divides up each item to be shared into the same number of pieces as there are people. Thus, each person gets $\frac{1}{8}$ from each of 7 sandwiches. When students realize that they do not need to divide up each item giving one little piece of each item to each person, they transition to "additive coordination – groups of items" (p.18). Students may group what is to be divided so that they can distribute bigger pieces first, for example: each person gets $\frac{1}{2}$ a sandwich, then $\frac{1}{4}$ of a sandwich, then $\frac{1}{8}$ of a sandwich. In "multiplicative coordination: fraction as quotient" (p. 19), students recognize that the two processes of partitioning into unit equal shares, then combining these fractional parts, are represented by a fraction interpreted as numerator divided by denominator. In one seamless mental strategy, a student can say that sharing 7 sandwiches 8 ways means each person gets 7/8 of sandwich.

Empson & Levi (2011) mention less common strategies for equal sharing such as relating the quantities in a problem to ratios. Eight people sharing 6 pizzas results in the same equal shares as half the amount of people sharing half the amount of pizzas (4 people sharing 3 pizzas and with repeated halving, 2 people sharing 1 ½ pizzas, etc.) Knowing factors and multiples helps extend this thinking into scaling down beyond halving, such as seeing 24 people sharing 8 pizzas the same as 6 people sharing 2 pizzas, having recognized the common factor of 4. Sometimes students use concrete modeling, placing cubes to stand for the number of people and cubes to stand for the number of things to be shared, then dividing up into equal groups, representing the idea of equal ratios. In the example 6 people sharing 4 pizzas, 6 cubes to 4 cubes could be separated into 3 cubes to 2 cubes. The student may then see that each of the 3 people would get 1/3 of each cube and since there are 2 cubes, each person would get 2/3.

Multiple groups: multiplication or measurement division. Seeing fractions as relational quantities between a unit and a subdivision of that unit into equal pieces, thus a unit fraction, and understanding the multiplicative relationship between the unit fraction and other fractions is an important step in understanding equivalence and operations on fractions. In "multiple group" problem types (Empson & Levi, 2011, p. 49), problems can involve multiplication or measurement division. In multiplication, the size of the groups (as a fraction, i.e. 2/3 of a group) and the number of groups are known, and the total amount in unknown. In measurement division, the size of the group (as a fraction) and the total amount are known. "Direct modeling and repeated addition" (p. 54) are the most basic strategies for solving these problem types in which the student represents each fractional quantity individually. In "grouping and combining strategies," (p. 57) students find more efficient ways to group the fractions, realizing that not all fractions need to be written or modeled. If they can add enough fractions to get to a whole number, they can combine groups of fractions (whose sums are whole numbers, such as 2/3 + 2/3 + 2/3 make 2 wholes) to get an answer more quickly. In a more sophisticated strategy, a student would see multiplicative relationships instead of additive (3 sets of 2/3 or 3 x 2/3 = 2).

Benchmarks for understanding equality as relational thinking. Carpenter et al. (2003) went beyond discussion of problem types and strategy levels to bring forth the importance of understanding equality. The development of how a student conceptualizes the use of the equal sign is important to the development of algebraic reasoning throughout the elementary curriculum. The authors proposed four benchmarks (not necessarily progressive stages) as students work toward the desired understanding of the equal sign as indicating a relationship between numerical expressions, rather than indicating simply a calculation. The authors termed this desired understanding as "relational thinking" (p. 27). For example, a student who uses relational thinking would look at both sides of the equal sign in the number sentence 7 + 5 = N + 4, and see that 5 is one more than 4 so N must be one more than 7, which is 8. Compare this to a student who would add 7 + 5 first to get 12, then think of what number is 4 less than 12. Relational thinking supports the thinking of problem solutions in terms of relationships, including the underlying properties of operations and equality, as well as the writing of number sentences to express the mathematical relationships. Students use higher level problem-solving strategies when they move beyond the more basic strategies of direct modeling and counting methods to routinely using a flexible choice of strategies and engaging in relational thinking (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, & Levi, 2003, Empson & Levi, 2011).

Relational thinking in strategy use. Students who think relationally tend to simplify calculations by using number relations, view the equal sign as signifying a relationship between two expressions, and generalize relations explicitly based on the underlying fundamental properties of arithmetic (Jacobs, Franke, Carpenter, Levi, & Battey, 2007, p. 260). Strategies that use "relational thinking" to relate one numerical expression to another rely upon fundamental properties of operations or equality (Empson & Levi, 2011, p. 78) as shown in a variation of the previous example: $7 + 5 = N + 4$ can be rewritten as $7 + (1 + 4) = N + 4$, which is equivalent to $(7 + 1) + 4 = N + 4$. This shows an implicit use of the associative property in the student's relational thinking. An example of relational thinking in grouping and combining strategies shows a student finding a total of 15 groups of 2/3 yards of fabric, by first combining 3 groups to get 2 yards, 6 groups to get 4 yards, then combining these as 9 groups to get 6 yards, thus using $(6 \times 2/3) + (3 \times 2/3) = (6 + 3) \times 2/3$, utilizing the distributive property. A more efficient, multiplicative strategy could be used by a student who has the relational understanding that 2/3 is a multiple of the unit fraction 1/3, to get $15 \times 2/3 = 30/3$. Underlying this thinking is the associative property: $15 \times 2/3 = 15 \times (2 \times 1/3) = (15 \times 2) \times 1/3 = 30 \times 1/3 = 30/3$.

Empson and Levi (2011) described relational thinking as integral in the highest levels of strategy use, whether for addition and subtraction, multiplication and division, or fraction problem solving:

> As students' strategies evolve, they incorporate Relational Thinking. Developing Relational Thinking enhances students' understanding of arithmetic and at the same time prepares students to understand algebra. Students who use Relational Thinking are:
>
> - using a relatively small set of fundamental properties of operations and equality and related principles to establish connections between quantities, operations on quantities, and equalities between quantities
> - developing the ability to look at a problem as a whole and to decide which relationships could be used to simplify the solution. (p. 230-231)

With mathematically gifted students' high ability to see relationships, having the opportunity to use relational thinking and express these relationships using properties of operations and equality is a promising approach to keeping them engaged.

Levels of justification in strategy use. With justification of one's solution an important part of the problem-solving process, Carpenter et al. (2003) classified three levels of justification: appeal to authority, justification by example, and generalizable arguments. In the first level, the student simply accepts something as true because he heard it from the teacher. When students do not go through the steps to convince themselves, there is no justification. In justification by example, giving multiple examples that indicate something is true can be convincing to a student and inductively lead to a conjecture, but students must realize that it does not constitute a proof. The final level of justification involves finding a provable generalization, or a counterexample that would disprove the conjecture.

CGI Professional Development's Influence on Teachers and Students

Detailing the categorization of problem types and solution strategies with relational thinking as a goal, the benchmarks for understanding equality, and the levels for justification of solutions within the CGI literature demonstrates a framework available for teachers to better understand their students' thinking as they participate in CGI professional development. A longitudinal study of teacher participants found improved student achievement directly related to the teachers' beliefs changing from a teacher-directed approach to math instruction to one that was student-centered with students engaged in problem-solving and mathematical communication. The increased achievement was attributed to the professional development as teachers gained understanding about the development of children's mathematical thinking and learned to encourage that thinking in their classrooms (Fennema et al., 1996). The achievement gains were in the areas of conceptual understanding and problem solving ability, with no change in computational ability. Carpenter, Franke, Jacobs, and Fennema (1998) found students in CGI classrooms who invented their own strategies for solving problems (as opposed to teacher-taught strategies or algorithms) were more flexible thinkers and able to transfer their problem solving techniques to new types of problems as well as to learning algorithms with understanding.

Empson (2003), in a thorough analysis of teacher-student interactions in a case study of two low-performing first grade students, maintained that student success in the CGI classroom "depends fundamentally on the teacher's role in making space and meaning for students' contributions to classroom discourse" (p.307). Empson described "participant frameworks" as a theoretical lens for understanding how teachers and students take on roles that support the key component to the participant framework. Its cognitive accessibility and potential for higher level mathematical thinking can vary depending on the participant framework's culture for individually and collectively discussing problem solutions.

Carpenter, Fennema, and Franke (1996) focused on what it means for a teacher to understand a student's mathematical thinking at the K-3 level. For a teacher to listen to a student's thinking, the teacher must know what to listen for by having a deep understanding of mathematics content and problem types and an astute awareness that there can be multiple paths for solving a math problem. The authors point out the importance of recognizing that children have an intuitive knowledge beyond what they have been taught about mathematics in school, which can allow them to make sense of a problem by direct modeling. The teacher who sees where the child's level of understanding lies can better assist in connecting this intuitive understanding to formal mathematical knowledge. Considering the importance of the teacher-student interaction, Franke et al., (2009) examined teacher questioning in three CGI classrooms, finding that although all teachers questioned students to explain their thinking, differences appeared in the extent and quality of follow up questioning after the initial question. Teachers who used follow up questioning elicited more knowledge about their students' thinking. The study described the nature of these follow up questions reporting that students' responses varied according to how closely the teachers' questions connected to the students' explanations. Teachers who pressed for more elaborate and explicit details were more able to guide students to completely-explained and correct solutions.

The presence of a chapter in a book on equity (Carey et al., 1995) cemented CGI as a viable approach to teaching mathematics to all children, regardless of ability, ethnicity, gender, or economic disadvantage. The authors explained how a CGI classroom immerses all students in the problem solving process and allows them to solve the problems in ways that make sense to them. They cited the example of the success of implementing CGI in the low-SES, predominantly African-American schools of Prince George's County, Maryland, compared to the former approach of giving low level math work accompanied by low expectations of the children. The CGI approach relates to equity in that all students' solutions are valued at their respective levels by their teachers. Additionally, the CGI classroom climate of sharing strategies helps develop a sense of respect among students for one another's work. This climate encourages students to take on an identity as a learner of math, rather than one who cannot do math. As students continue with this type of experience in problem solving, facilitated by the teacher who helps move students to more sophisticated and efficient strategies, their mathematical power grows. "Empowering children to make decisions about what is appropriate for them in terms of context and content of mathematics is a critical feature of equitable classrooms" (Carey et al., 1995, p. 122). Although this work of Carey et al. promoted CGI for all learners, its sentiment was geared toward the disadvantaged student who traditionally has been disenfranchised from the learning of mathematics. Disenfranchisement has not been perceived as an issue for the mathematically able students which may explain the lack of CGI research on the high ability subpopulation.

The potential of CGI to foster an equitable environment has continued to be confirmed. In Empson's (2003) study of two low-performing first graders in a CGI classroom, the students progressed from being disengaged as math learners to developing an identity as participants in classroom mathematical discourse. Empson attributed this positive identity to the nature of CGI that allows students to take control of their learning by choosing problem solving strategies appropriate for them. In an environment that respected all student strategies, these students increased their participation. With a respectful environment as well as the scaffolding by teachers, students' understanding of fractions and their facility with problem solving grew. Behrend (1994) found similar results in a study of math problem-solving processes of five second and third graders, identified as learning-disabled, as they experienced Cognitively Guided Instruction in a small group. When the focus was redirected from these students' deficiencies to what they could make sense of on their own, Behrend found that the learning disabled students increased their confidence as mathematical thinkers and could solve multi-step problems in a variety of ways, fairly consistently justifying their solutions. Marshall (2009) followed four English Language Learning elementary students in the U.S. from kindergarten through 2nd grade as they engaged in Cognitively Guided Instruction in bilingual classrooms, finding their ability to solve problems and explain their thinking developed, using both Spanish and English.

Carpenter, Ansell, Franke, Fennema, and Weisbeck (1993) found the majority of kindergarteners in their study could make sense of and solve a variety of multiplication and division problems without explicit instruction. The students were given time to explore the problems and invent their own strategies using direct modeling and were able to solve problems that would ordinarily have been introduced in later grades. Classroom problem-solving environments that offer opportunities to explore advanced topics are of interest to this study of challenging mathematically gifted students in mainstream classrooms.

Mathematical Giftedness

Characteristics of the Mathematically Gifted Child

Despite the lack of one definition of what it means to be mathematically gifted, several key authors in this field, including Sheffield (1999), Assouline & Lupkowski-Shoplik (2005), and Leiken, Berman, & Koichu (2009), have cited Krutetskii's (1976) research. Krutetskii characterized mathematically gifted children as possessing:

- numerical, symbolic, and spatial ability,
- a well-functioning mathematical memory,
- flexibility and economy of thought,
- ability to re-direct a mental process,
- ability to see quantitative and spatial relationships,
- ability to reason deductively, formalize and generalize mathematical ideas.

In looking at characteristics of mathematical giftedness, the intersection with creative thinking has been discussed (Leiken et al., 2009). Guilford (1950) described the creative mind as possessing the characteristics of fluency, flexibility, originality, and elaboration. Sheffield (2009) interpreted how these characteristics are manifested in mathematics problem-solving. She described fluency in problem solving as referring to the number of answers, strategies, or questions formulated, flexibility referring to the variety of answers, strategies, or questions, originality referring to the uniqueness of the solutions, strategies, or questions, and elaboration referring to the quality of expression of the mathematical thought. Sheffield included these four characteristics as criteria that could be used for assessing and assisting creative mathematical thought to its fullest potential, and adds three more to the list: depth of understanding as to the extent that math concepts are explored and developed, generalizations that result from noting and verifying patterns, and extensions that reveal themselves as related questions that are asked and explored.

Providing a Challenging Environment for Mathematically Gifted Students

Diezmann and Watters (2002) described types of academic tasks that benefit mathematically gifted students. The beneficial tasks tend to be challenging, introduce students to mathematical topics beyond what their age peers would typically learn, provide a rich learning experience with open-ended investigations, and connect with their interests which may be interdisciplinary. Henningsen and Stein (1997) stated, “Not only must the teacher select and appropriately set up worthwhile mathematical tasks, but the teacher must also proactively and consistently support students’ cognitive activity without reducing the complexity and cognitive demands of the task” (p. 546). Their study found that pressing the students for explanations, justifications, and meaning by teacher questioning, comments and feedback, as well as allotting an appropriate amount of time for solving problems, resulted in maintaining higher-level engagement.

The mathematics curriculum for the practitioner level of Ascending Intellectual Demand suggests open-ended, interest-based, and student-centered learning experiences (Hedrick & Flannagan, 2009). Students were found to experience increased engagement in learning when they felt more control of their learning environment and when their perceived challenge of the task and their own skills was high (Shernoff, Csikszentmihalyi, Schneider, & Shernoff, 2003). Diezmann and Watters (2002) described three main features of an effective learning environment in which mathematically gifted students can develop their skills for becoming autonomous learners. First, students should be able to investigate and learn through discourse. “Productive discourse incorporates evidence, logic, and argumentation and involves students in sharing ideas, building on each other’s ideas, and critiquing ideas” (p. 5). Second, teachers should respect students’ preferences for individual or group work for varying tasks, which tends to be working alone on easier tasks and in groups for more challenging tasks. Third, there should be opportunities provided for mathematically gifted students to work with one another so that they can have their ideas challenged by “like-minded peers” (p. 5) and have to re-examine or defend their thinking. Winebrenner and Brulles (2008) promote clustering the highest ability math students in one mainstream grade level class to ensure that they have an intellectual peer group with opportunities for working together and to increase the chances that the teacher will differentiate instruction for their needs.

Intersection between the Literature of CGI and Mathematical Giftedness

There are numerous comments in Sheffield’s book (1999) that call for mathematically promising students to have a classroom experience similar to those that support Cognitively Guided Instruction. Maker (as cited in Wheatley, 1999, p. 77) described productive classrooms that are learner-centered, emphasize independence over dependence, open to new ideas, focus on complexity, accepting of ideas rather than judgmental, and have flexible rather than rigid structure. He also noted that “the study of mathematics for promising students should be fast paced and problem centered, focusing on concepts rather than procedures” (p. 77). Krist (1999, p. 175) adds,

> Of crucial importance is the very act of answering the same question by using different techniques or working on problems that have more than one answer…Talented students seek to be taken seriously and want someone to listen carefully to their sometimes long, involved arguments. Bright youngsters seek opportunities for knowledge-based dialogue.

Hashimoto and Becker (1999, p. 102) propose that when leaving some aspect of a problem “open”, either the formulation of the problem, the process for solving, or the end product, opportunities arise for “bright students to exercise their creative abilities and devise insightful ways to deal with mathematical topics and problems”. They further stress the importance of discussing multiple solutions and connecting new ideas to prior knowledge. These descriptions from the literature on mathematical giftedness suggest practices for challenging the mathematically gifted. Their similarity to the principles and practices of CGI support the idea of investigating a CGI classroom for its potential for challenging mathematically gifted students.

Toward a Definition of Cognitive Challenge in Learning Elementary Mathematics

Higher Order Thinking Skills

The use of higher order thinking skills is an element in all the topics of this literature review, Cognitively Guided Instruction, mathematical giftedness, and challenge. Bloom’s Revised Taxonomy (Anderson & Krathwohl, 2001) provides a framework that was used in this study for referring to higher order thinking as it relates to evidence of challenge. The original taxonomy introduced a classification of six levels of cognition: Knowledge, Comprehension, Application, Analysis, Synthesis, and Evaluation (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956), referring to the higher order thinking skills as those at the higher levels of the taxonomy.

The revision of this taxonomy emphasized the cognitive processes dimension as it intersected with the knowledge dimension. The emphasis on process was evident in the re-naming of the six levels to: Remembering, Understanding, Applying, Analyzing, Evaluating, and Creating.

Mathematical Challenge

Determining and defining what it means for a child to experience mathematical challenge is an essential step in answering the research question “to what extent is a mathematically gifted student challenged in a CGI classroom?” Mathematical challenge was the topic of the International Commission for Mathematics Instruction annual study in 2006. Barbeau and Taylor (2005), co-chairs of the study, offered this description:

> The word 'challenge' denotes a relationship between a question or situation and an individual or a group…A challenge has to be calibrated so that the audience is initially puzzled by it but has the resources to see it through. The analysis of a challenging situation may not necessarily be difficult, but it must be interesting and engaging. (p. 126)

Explaining why challenge is important, they continued:

> We have some evidence that the process of bringing structure to a challenge situation can lead one to develop new, more powerful solution methods. One may or may not succeed in meeting a challenge, but the very process of grappling with its difficulties can result in fuller understanding. The presentation of mathematical challenges may provide the opportunity to experience independent discovery, through which one can acquire new insights and a sense of personal power. Thus, teaching through challenges can increase the level of the student's understanding of and engagement with mathematics. (p. 126)

However, Taylor (2009) commented that one of the realizations stemming from their investigation was that challenge is a difficult construct to measure. He further stated that mathematical challenge and its effect on the learning process has not been well-documented and has few experts. The need for research to better understand this component of mathematics education is suggested.

Taylor’s comments on the lack of research on mathematical challenge suggest that addressing the question of what constitutes “challenge” is a challenge in itself. Barbeau and Taylor’s (2005) description, however, of challenge as a relationship between a question and an individual such that the individual “is initially puzzled…but has the resources to see it through” (p. 126) relates well to Vygotsky’s zone of proximal development. This provides a good starting point for a deeper description of mathematical challenge.

Zone of Proximal Development

Vygotsky’s concept of a zone of proximal development (Vygotsky, 1978, as cited in National Research Council, 2000) describes the difference between what a child can learn independently and what a child can learn with help. Understanding this idea is an initial step toward determining when a student is challenged and how to continuously maintain the challenge:

> Vygotsky helped us understand an individual learns when a teacher presents tasks to the student at a level of difficulty somewhat beyond the learner’s capacity to complete the task independently. When a teacher presents tasks in the student’s ‘zone of proximal development’ and then scaffolds, coaches, or supports the student in successfully completing the tasks, the student’s independence zone ultimately expands. This causes the need for new tasks at a greater level of demand. For advanced learners in a subject area, the implication is that tasks will need to be more complex than would be appropriate for students who are less advanced in their capacities at that time. (Tomlinson et al., 2009, p. 11)

Ascending Intellectual Demand

The key to keeping a student in his or her zone of proximal development, and thus offering appropriate challenge, is to provide instruction that offers Ascending Intellectual Demand (AID), an “escalating match between learner and curriculum” (Tomlinson et al., 2009, p. 11). This involves elevating the challenge level of both the curriculum materials and the tasks assigned as students become more advanced in their knowledge, understanding, and skills. It requires a curriculum, such as the Parallel Curriculum Model (Tomlinson et al., 2002; Tomlinson et al., 2009) that is concept-based with tasks that demand complex thinking and provide opportunities for open inquiry. The continuum of Ascending Intellectual Demand describes a progression of knowledge, skills, attitudes, and habits of mind as a learner moves from novice to expert. There are four levels on this continuum toward expertise: novice, apprentice, practitioner, and expert, with specific characteristics of the mathematics learner associated with each level. The role of the teacher is to be attentive to students' needs and to plan learning experiences that help guide students along this path toward expertise. The purpose of scaffolding along the way is "to provide support for the learner so that he may master challenging content and skills" and move upward on the continuum toward expertise (Tomlinson et al., 2009, p. 237).

As a mathematics learner moves past the novice stage, we begin to see characteristics similar to those in Krutetskii's description of mathematically gifted students (1976), such as making connections and seeing relationships. These characteristics will play a central role in further defining what it means to be challenged mathematically in that a student is moving along the continuum of Ascending Intellectual Demand and, thus experiencing mathematical challenge, when these characteristics are observed. The progression beyond novice is described as follows. In reading through this continuum toward expertise, keep in mind that expertise is relative to the topic and to the individual. Thus a second grade student may be approaching expertise in inventing strategies for solving addition and subtraction problems, but may only be beginning to discover relationships involving fractions.

Learning Characteristics of the Apprentice in Mathematics:

- Connects the relationships among mathematical facts and skills through concepts
- Computes fluently and makes reasonable estimates

- Applies skills with confidence and develops greater understanding beyond number and operations
- Makes connections across mathematical ideas
- Understands the principles that frame a field (i.e. measurement, algebra, geometry, statistics)
- Develops skills and understanding through complex problem solving
- Sets goals that extend beyond computational accuracy

Learning Characteristics of the Practitioner in Mathematics:

- Uses the principles of mathematics to make connections among concepts across multiple fields within mathematics
- Makes appropriate selections about which tools and methods to use
- Understands patterns, relations, and functions
- Applies skills with automaticity
- Understands change in a variety of contexts
- Uses a variety of tools and methods with efficiency in the analysis of mathematical situations
- Appreciates the role of mathematics in other disciplines
- Formulates questions for research that can be addressed through one or more fields of mathematics

Characteristics of the Expert in Mathematics:

- Uses computation as merely a means to an end
- Questions existing mathematical principles
- Moves easily among the fields of mathematics through the use of macroconcepts
- Links mathematical principles to other fields through real-world problems
- Seeks the challenge of unresolved problems and the testing of existing theories
- Seeks flow through the manipulation of tools and methods in complex problem solving
- Views unanswered questions in other disciplines through the concepts of mathematics
- Uses reflection and practice as tools for self-improvement

(Hedrick & Flannagan, 2009, p. 262)

An Operational Definition of Mathematical Challenge

The characteristics of Ascending Intellectual Demand confirm and add detail to Krutetskii's (1976) description of mathematical giftedness. However, Barbeau and Taylor (2005) and Krutetskii remind us that these characteristics do not exist in isolation but are in relation to a problem that the student takes on. In Krutetskii's (1976) study of mathematical ability, he delineated levels of mathematical problems by associating them with the cognitive characteristics of the problem-solver which were revealed as the students solved increasingly more difficult problems:

> Experimental problems ought to fulfill their direct purpose: solving them should help to clarify the structure of abilities. In other words, as the problems are solved, those features of mental activity that are specific to mathematical activity should be manifested. (p. 91)

Drawing on Krutetskii's work, a challenging problem could be described as one that allows the characteristics of mathematical giftedness to be exhibited.

To define what it means for a student to be challenged mathematically, I begin by considering the elements common to the bodies of literature mentioned in this review. The existence of levels of cognitive thought, problem solver characteristics, and expertise suggest that we can look to the higher levels for where challenge lies. Bloom's Revised Taxonomy provided our framework for higher order thinking skills. Krutetskii's work (1976) associated certain characteristics with the highest level of mathematical problem solving ability. The literature on expertise provided the characteristics of progressing levels of Ascending Intellectual Demand for mathematics. Finally, the CGI literature laid out a progression of problem-solving strategy levels.

According to CGI literature, children reach the higher levels of problem-solving strategies when they move beyond the more basic strategies of direct modeling and counting methods to routinely using a flexible choice of strategies and engaging in relational thinking (Carpenter et al., 1999; Carpenter et al., 2003).

> Relational thinking entails a flexible approach to calculation in which expressions are transformed on the basis of at least implicit use of fundamental properties of number operations…Relational thinking represents a fundamental shift from an arithmetic focus (calculating answers) to an algebraic focus (examining relations). (Jacobs et al., 2007, p. 260)

More specifically, students who think relationally tend to simplify calculations by using number relations, view the equal sign as signifying a relationship between two expressions, and can generalize relations explicitly based on the underlying fundamental properties of arithmetic.

There is a common theme found in Krutetskii's description of mathematically gifted children, the characteristics of mathematics students as they experience Ascending Intellectual Demand on the continuum toward expertise, and the CGI focus on relational thinking as an elevated level of problem solving. They all describe higher levels of thinking as including the ability to think in terms of mathematical relationships. The continuum of Ascending Intellectual Demand further implies the element of being exposed to new ideas as students progress on a path toward expertise. Therefore, I propose the following operational definition of mathematical challenge: *Students are challenged mathematically when they engage in exploring, discovering, or utilizing mathematical relationships, are exposed to new mathematical ideas, and experience Ascending Intellectual Demand on a path toward expertise as mathematical thinkers.*

Adding to the Literature

This chapter has focused on the literature of Cognitively Guided Instruction, mathematical giftedness and challenge. Little has been done to bring together these fields of study. My study will contribute by examining the experiences of mathematically gifted students in CGI classrooms in relation to mathematical challenge. I have proposed a definition of challenge that begins to synthesize these various fields. Furthermore, I suggest a connection between the two major frameworks discussed in this chapter. The framework of Ascending Intellectual Demand is used by teachers as a curriculum planning guide to aid in facilitating students as they move along the continuum from novice to apprentice to practitioner to expert. The frameworks of problem-solving strategy levels guide teachers in understanding their students' thinking as it progresses from basic strategies toward more sophisticated strategies such as relational thinking. Students cycle through the levels of both frameworks as new topics are introduced and explored, but do not necessarily begin at the first level again. Although the AID framework for curriculum planning has a broader purpose than addressing specific problem-solving strategies, the commonalities between the two frameworks allow for alignment. I propose the following intersection between these two frameworks to help characterize the level of Ascending Intellectual Demand experienced by the mathematically gifted students in a CGI classroom as they engage in problem solving (see Figure 1). Chapter 5 will include further discussion of implications of this alignment of CGI strategy levels with AID for challenging mathematically gifted students.

CGI/AID Framework

Problem-Solving Strategy Levels from CGI Literature	Levels of Ascending Intellectual Demand, the Novice to Expert Continuum in Mathematics, from the Parallel Curriculum Model (Hedrick & Flannagan, 2009, p. 262)
Representing each item or group: Direct Modeling Repeated Addition for Multiplication Non-anticipatory Coordination Additive Coordination: Sharing One item at a Time	Novice: • Applies the skills of discrete mathematics, but lacks a conceptual understanding • Identifies the principles, but cannot apply them unless prompted • Computes, efficiently, but lacks fluency • Sees limited relationships among numbers and number systems • Identifies only the most basic patterns • Needs frequent feedback and assurance during problem solving • Sees the "right answer" as the goal

Figure 1. Alignment of CGI problem-solving strategy levels with levels of Ascending Intellectual Demand. Novice to Expert Continuum in Mathematics by Hedrick, K., & Flannagan, J. S. (2009). Ascending intellectual demand in the parallel curriculum model. In Tomlinson et al., (2009), *The parallel curriculum: A design to develop learner potential and challenge advance learners, 2nd ed.* (p. 262). Copyright 2009 by National Association of Gifted Children.

(CGI/AID Framework continued)

Problem-Solving Strategy Levels from CGI Literature	Levels of Ascending Intellectual Demand, the Novice to Expert Continuum in Mathematics, from the Parallel Curriculum Model (Hedrick & Flannagan, 2009, p. 262)
Counting Strategies, (i.e. counting on, repeated addition, skip counting) Additive Coordination: Sharing groups of items Grouping and Combining Strategies Ratio: Repeated Halving Factors Derived Facts Flexible Use of Strategies Relational Thinking	Apprentice: • Connects the relationships among mathematical facts and skills through concepts • Computes fluently and makes reasonable estimates • Applies skills with confidence and develops greater understanding beyond number and operations • Makes connections across mathematical ideas • Understands the principles that frame a field (i.e. measurement, algebra, geometry, statistics) • Develops skills and understanding through complex problem solving • Sets goals that extend beyond computational accuracy
Derived Facts Number Facts Computational Fluency Multiplicative Coordination Multiplicative Strategies Flexible Use of Strategies Relational Thinking Notation with Equations	Practitioner: • Uses the principles of mathematics to make connections among concepts across multiple fields within mathematics • Makes appropriate selections about which tools and methods to use • Understands patterns, relations, and functions • Applies skills with automaticity • Understands change in a variety of contexts • Uses a variety of tools and methods with efficiency in the analysis of mathematical situations • Appreciates the role of mathematics in other disciplines • Formulates questions for research that can be addressed through one or more fields of mathematics

Figure 1. Alignment of CGI problem-solving strategy levels with levels of Ascending Alignment of CGI problem-solving strategy levels with Ascending Intellectual Demand. Novice to Expert Continuum in Mathematics by Hedrick, K., & Flannagan, J. S. (2009). Ascending intellectual demand in the parallel curriculum model. In Tomlinson et al., (2009), *Parallel curriculum: A design to develop learner potential and challenge advance learners* (p. 262). Copyright 2009 by National Association of Gifted Children.

(CGI/AID Framework continued)

Problem-Solving Strategy Levels from CGI Literature	Levels of Ascending Intellectual Demand, the Novice to Expert Continuum in Mathematics, from the Parallel Curriculum Model (Hedrick & Flannagan, 2009, p. 262)
Computational Fluency Relational Thinking Notation with Equations	Expert: • Uses computation as merely a means to an end • Questions existing mathematical principles • Moves easily among the fields of mathematics through the use of macroconcepts (common concepts across disciplines or topics) • Links mathematical principles to other fields through real-world problems • Seeks the challenge of unresolved problems and the testing of existing theories • Seeks flow through the manipulation of tools and methods in complex problem solving • Views unanswered questions in other disciplines through the concepts of mathematics • Uses reflection and practice as tools for self-improvement

Figure 1. Alignment of CGI problem-solving strategy levels with levels of Ascending Alignment of CGI problem-solving strategy levels with Ascending Intellectual Demand. Novice to Expert Continuum in Mathematics by Hedrick, K., & Flannagan, J. S. (2009). Ascending intellectual demand in the parallel curriculum model. In Tomlinson et al., (2009), *Parallel curriculum: A design to develop learner potential and challenge advance learners* (p. 262). Copyright 2009 by National Association of Gifted Children.

Common Core Standards

In our current educational climate, it is relevant to place the preceding information and discussion in the context of the recommendations of the Common Core Standards for Mathematical Practice:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.

- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

These standards "describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years." (CCSSI, 2010, p. 8). It is interesting to note that both the Common Core practices and the idea of Ascending Intellectual Demand drew, in part, from the ideas of the National Research Council (2000) that suggested studying the characteristics of experts informs how people learn.

Summary of Literature Review

The bodies of literature reviewed in this chapter inform the central topic of challenging mathematically gifted students in CGI classrooms. The background of Cognitively Guided Instruction was described with its origins in the understanding of students' mathematical thinking, the taxonomy of problem types and strategy levels, and finally the development of the CGI professional development program. Research was presented that showed the positive influence that the professional development had on both teachers and students.

The literature on the characteristics of mathematical giftedness and challenging learning environments was discussed. Similarities were drawn between the learning environments recommended by the gifted education literature and the description of classrooms of teachers who had experienced CGI professional development. I then introduced literature on mathematical challenge, higher order thinking skills, zone of proximal development, and Ascending Intellectual Demand to support the operational definition of mathematical challenge for elementary students. The operational definition guided this study as I collected and analyzed data for evidence of challenge. The final focus was an alignment of two frameworks, CGI strategy levels and levels of Ascending Intellectual Demand, the CGI/AID Framework, used to further characterize the level of challenge in the CGI classrooms in this study.

CHAPTER 3: METHODS

This study investigated the experiences of mathematically gifted and advanced students in classrooms using the principles of Cognitively Guided Instruction (CGI) to find to what extent and in what ways they are engaged and challenged in mathematics learning. The literature in Chapter 2 presented a rationale for developing the following operational definition of mathematical challenge: students are challenged mathematically when they engage in exploring, discovering, or utilizing mathematical relationships, are exposed to new mathematical ideas, and experience Ascending Intellectual Demand (AID) on a path toward expertise as mathematical thinkers.

Using a multiple case study research design, two cases of CGI classrooms taught by exemplary teachers were studied. The research methods included observing the identified students and teachers during CGI math lessons, examining the students' problem-solving strategies, and interviewing the participants. Yin (2009) stated "the analytic benefits from having two (or more) cases may be substantial" (p. 61) and therefore suggests a minimum of two or three replications. Collecting data from two cases provides more compelling evidence for a more robust study (Herriot & Firestone, 1983,cited in Yin, 2009, p. 53).

Yin (2009) suggests a "replication, not sampling logic" (p. 54) for using multiple case studies such that carefully selected cases would predict similar results, allowing themes to emerge, and if similar results do not emerge, the cases would provide contrasting results for predictable reasons. In this study, the cases were selected based on the presence of an exemplary CGI teacher with the prediction that exemplary teachers would be most likely to provide challenge for gifted students. The data from three different grade levels of two cases (a third grade class, and a fourth/fifth split grade class) allowed for comparing and contrasting in a cross-case analysis which added to the overall picture of mathematically gifted students obtained from the within-case analysis.

This study used a parallel mixed design in which the quantitative and qualitative phases occurred approximately at the same time and addressed related elements of the same research question (Teddlie & Tashakkori, 2009). The sentiment of this study is more qualitative than quantitative but as quantitative data was collected at each classroom visit, information was gained that framed the collection of the qualitative data. For instance, measuring students' "finish time" for the problems revealed that some students finished their problems very quickly. This led to more attentive observation of what the teachers did to further engage these students once the students had finished the problems. Teddlie and Tashakkori (2009) warned that a parallel mixed design can be difficult to implement due to having to pay attention to both phases at the same time. However, with this study focusing on a small number of students and one teacher in one of two classrooms at a time, this design was more useful than burdensome.

Participants and Setting

The focus of this study was on mathematically gifted students situated in two CGI classrooms taught by exemplary teachers using a CGI approach to mathematics instruction. The purposive selection process of the two classrooms was based on finding exemplary CGI teachers who each had at least one mathematically gifted student in their classes. Although one aspect of the research question focused on the teachers and their strategies for engaging and challenging students, the students were the primary unit of analysis since it was the students' responses to the teachers' strategies that determined whether or not the strategies evoked a challenging situation.

Institutional Review Board approval was procured and all students, their parents, and the teachers in the participating classrooms were asked to sign informed consent (provided in

English and Spanish) to grant their permission to be included in the study. All but one of 24 third graders returned informed consent. Twenty of the 28 students of the fourth/fifth split grade class returned informed consent. To ensure confidentiality, the student names were given letter codes, which I used when taking field notes and transcribing interview and observation data. All records were kept secure and only used by the researcher. Pseudonyms were used for writing this report.

The CGI Teacher Participants

The two teachers selected for this study were chosen using the following criteria:

- An elementary school teacher who had participated in at least two years of CGI professional development and had been implementing the principles of CGI in their classrooms;
- An elementary school teacher recommended as an exemplary CGI teacher by a CGI professional development leader;
- An elementary school teacher whose classroom had at least one mathematically gifted student.

Although it was not a prerequisite for the teacher to have knowledge of mathematical giftedness, the teachers selected were able to identify their mathematically gifted students as well as describe characteristics that were similar to Krutetskii's description (1976).

Ms. B, the third grade teacher, began CGI professional development in her first year of teaching. Within her six years of teaching experience, she participated in three years of CGI professional development, the Arkansas CGI Leadership Institute, and has taught CGI Year 1 and apprenticed to teach CGI Year 2.

Ms. K, the teacher of a fourth/fifth grade split class, also has six years teaching experience. She had been first introduced to CGI through professional development offered by her school in her second and third years of teaching. For the next two years, she participated in CGI Years 1 and 2. In addition to this, she completed a Math Science Partnership workshop that focused on students' thinking about fractions and decimals.

Description of the Setting

Both case studies were conducted at the same elementary school within a district of 18,810 students, in an Arkansas city of 69,797 (U.S. Census Bureau, 2010). Although it was convenient to have both cases at the same elementary school, this factor was not considered when choosing the teachers for participation. This K-5 school has a student population of 529, with 97% eligible for the free/reduced lunch program, indicating low economic status, and 80% English Language Learners (ELL). The student body is made up of 76% Hispanic, 19% Caucasian, and 5% other ethnicities/races. The school is located in a city that is 65% Caucasian, 35% Hispanic, and 10% other ethnicities/races. The school district has supported CGI professional development for its teachers.

The Mathematically Gifted Student Participants

Once the exemplary teachers were identified and agreed to participate in this study, the teachers reviewed their list of students who had scored advanced on the previous year's standardized test (SAT 10 for the 3rd graders, and Arkansas Benchmark Exam for the 4th/5th graders). From the list of advanced students, the teachers indicated which students they most highly recommended as mathematically gifted as well as a second level of advanced students. These recommendations were based not only on test scores, but also from the teachers' observations of students' excellent problem-solving abilities for seven months over the course of the school year. They also recommended a few students as strong math students and good mathematical thinkers who had not scored advanced on the previous year's standardized exams, attributing the lower scores to language-related issues as English Language Learners. Of the 15 students selected for the study, 12 of the students were raised with Spanish as their first language and began learning English in kindergarten. One of the 15 participants was raised bilingual in Spanish and English. The remaining two students were raised in English-speaking homes. Only three of the 12 English Language Learners were considered fluent in English at the time of the previous year's standardized test and four students were considered fluent a year later at the time of this study (see English Language Development Assessment, ELDA, scores in Tables 1 and 2). Because this study was being conducted approximately one year after the previous year's exams had been given, and because the students' language skills continued to grow, the teachers felt that their knowledge of their students' mathematical ability was a more accurate judge of ability than the previous year's test score. The teachers attributed their confident knowledge of their students' abilities to the nature of Cognitive Guided Instruction which focuses on student thinking and mathematical communication. The class discussion phase of the CGI math lesson, in particular, was a fruitful time to hear the advanced mathematical thinking of their students. In interviews with the 15 student participants, all of them considered themselves to be fluent in English, even those with lower ELDA scores, and reported that they had no trouble understanding the English during the math discussions.

Test of Mathematical Abilities for Gifted Students (TOMAGS). To obtain an objective measure of mathematical giftedness, and because the school-administered standardized tests can have a ceiling effect in which high scores may not indicate the extent of the student's mathematical ability, the teachers administered the Test of Mathematical Abilities for Gifted Students (TOMAGS) (Ryser & Johnsen, 1998) to their classes. TOMAGS, a norm-referenced measure of mathematical reasoning and problem-solving ability associated with mathematical giftedness, has Cronbach's coefficient alpha of .92 for the Primary TOMAGS (grades K-3) and .88 for the Intermediate TOMAGS (grades 4-6), indicating little test error (Ryser & Johnsen, 1998). The sample used for norm-referencing contained a 12% Hispanic population, although there was no information about fluency in English within this subgroup. Testing for reliability within this subgroup revealed a coefficient alpha of .88 for the Primary TOMAGS (grades K-3) and .88 for the Intermediate TOMAGS (grades 4-6). For construct validity, TOMAGS has statistically significant correlations indicating concurrent validity with the Otis-Lennon School Ability Test (OLSAT), the quantitative battery of the Cognitive Abilities Test (CogAT), and the mathematics total of the Stanford Achievement Test (SAT). Both reliability and validity were tested for subgroup comparisons of Mexican American students versus non-Mexican American students, indicating little or no testing bias in this group.

Identifying mathematical gifted students among the advanced students. The TOMAGS test identified five students likely to be mathematically gifted, using a cut-off score of 95th percentile. Of these five students, there were three third graders, one fourth grader, and one fifth grader. The rationale for choosing this cut-off score is as follows. The authors of TOMAGS suggest that a student in the 98th percentile is very likely to be mathematically gifted, being two standard deviations above the mean. This is close to Miller's (1990) estimate that 2-3% of the population is mathematically gifted. TOMAGS further suggests that students in the 92nd to 97th percentile range may be gifted. Since TOMAGS has a standard error of measurement that suggests a student's true score could be about 5 percentile points lower, a cut-off score of 95th percentile would increase the chances that the students may be gifted. Further justifying the cut-off score, the Johns Hopkins Center for Talented Youth recommends above-level testing for students who score at the 95th percentile on a nationally-normed standardized test (Assouline & Lupkowski-Shoplik, 2005), to determine if they are capable of studying math at a higher grade level than their own. The rationale of identifying the five students who scored at or above the 95th percentile on the TOMAGS as mathematically gifted was later supported by the findings that this group of students consistently completed the daily class problems correctly, compared to a lesser rate of success among advanced students who scored below 95th percentile.

Top tier advanced (mathematically gifted) and second tier advanced students. The scope of this study was expanded beyond the five mathematically gifted students to include ten other advanced students. Classifying two tiers of advanced students, a "top tier" of mathematically gifted students and a "second tier" of advanced students, allowed for an interesting comparison showing differences between the two groups. The top tier advanced group ranked high on all three measures: TOMAGS, advanced on district standardized testing, and teacher recommendation (Table 1). The second tier advanced group of students ranked high on two of the three measures (Table 2), with the exception of one student who was included because he was highly recommended by the teacher but did not score advanced on the other two measures. Observation of his participation and work during pilot visits to the class and conversations with both his classroom teacher and ELL teacher indicated that he was likely mathematically gifted, but language and personal issues interfered with his success at test-taking.

Table 1

Top Tier Advanced Students based on Success on Three Measures

Pseudonym	Grade	TOMAGS Percentile	Teacher Recommendation (TR)	2010 Score, Level/ 2011 Score, Level	2010 ELDA/ 2011 ELDA
Dominic	3rd	97th	Highest TR	Adv/737 Adv	4*/4
Jasmin	3rd	95th	Highest TR	Adv/717 Adv	4*/4
Freddy	3rd	97th	Highest TR	Adv/641 Adv	4*/4
Andre	4th	97th	Highest TR	731 Adv/ 695 Adv	5/5
Geraldo	5th	99th	Highest TR	775 Adv/ 774 Adv	4/4

Note: Cut off score for advanced on Arkansas Benchmark: 3rd grade 640, 4th grade 640, 5th grade 698. 3rd graders took SAT 10 as 2nd graders in April 2010 and Arkansas Benchmark as 3rd graders in April 2011. 4th/5th graders took the Arkansas Benchmark as 3rd/4th graders in April 2010 and as 5th graders in April 2011.
ELDA: English Language Development Assessment Composite Score, standardized. *ELDA score for 2nd graders was based on teacher survey, not a standardized test. ELDA Scores Range from 1 to 5 (5 is full English proficiency). See Appendix A for more details.

Table 2
Second Tier Advanced Students based on Success on One or Two Measures

Pseudonym	Grade	TOMAGS Percentile	Teacher Recommendation (TR)	2010 Score, Level/ 2011 Score, Level	2010 ELDA/ 2011 ELDA
Maya	3rd	85th	High TR	Adv/674 Adv	NA
Julia	3rd	84th	TR	Adv/663 Adv	4*/4
Natalie	4th	90th	Highest TR	698 Adv/759 Adv	4/5
Alana	4th	90th	High TR	731 Ad/732 Adv	5/4
Anita	4th	70th	TR	681 Adv/682 Adv	3/3
Katerina	4th	35th	TR	745 Adv/646 Adv	4/5
Sylvia	5th	84th	High TR	694 Adv/727 Adv	NA
Allen	5th	73rd	TR	682 Adv/767 Adv	NA
Roberto	5th	73rd	Highest TR	615 Prof/633 Prof	2/3
Kara	5th	39th	TR	688 Adv/661 Prof	5/5

Note: Cut off score for advanced on Arkansas Benchmark: 3rd grade 640, 4th grade 640, 5th grade 698. 3rd graders took SAT 10 as 2nd graders in April 2010 and Arkansas Benchmark as 3rd graders in April 2011. 4th/5th graders took the Arkansas Benchmark as 3rd/4th graders in April 2010 and as 5th graders in April 2011.
ELDA: English Language Development Assessment Composite Score, standardized. *ELDA score for 2nd graders was based on teacher survey, not a standardized test. ELDA Scores Range from 1 to 5 (5 is full English proficiency). See Appendix A for more details.

Data Collection

In preparation for conducting this study, four pilot observations helped formulate the processes for collecting the data. The interview questions were piloted on five students. Observing the identified students and teachers during CGI math lessons, examining the students' problem-solving strategies, and interviewing the participants provided a rich description of the mathematically gifted students' experiences, their perception of being challenged, and the teacher strategies that enhanced this experience. In this description of methodology, three phases of a CGI math lesson are referred to: Phase 1, the problem-posing phase in which the teacher poses the problem(s), Phase 2, the problem-solving phase in which students are given ample time to solve the problem(s) in whatever way makes sense to them, and Phase 3, the strategy-sharing discussion phase, the culmination of the lesson in which teacher-selected student strategies are shared and discussed by the class.

Interviews

Pre- and post-study semi-structured interviews with students and teachers. The study began with a semi-structured interview of each student (Appendix B) focusing on attitudes toward math, math class, and the issue of being challenged. This interview also served as an icebreaker to help the student feel at ease. After the observations, when initial data analysis suggested the teachers served as mentors for the advanced students, students were further questioned about this possibility. The teachers were interviewed (Appendix C) about their knowledge of their mathematically gifted and advanced students and the issue of challenging them. Their responses informed the list of what to look for during class observations. Teachers were interviewed after the series of observations to further question them about ways in which they tried to extend the thinking of their advanced students (teacher extensions). Informal conversation with the teachers throughout the study allowed for more understanding of the teachers' goals and concerns. After a list of teacher extensions was completed from the data analysis, the teachers were asked if the list accurately reflected how they viewed their interactions with students. This member check indicated an accurate report.

Post-lesson student interviews. After each lesson observation, most of the advanced participants were interviewed. This structured interview (Appendix D) included a short survey of perceived challenge using a Likert scale of 1 to 5, with 5 being the highest.

Observation

The study included observations of the CGI math period 10 times per class. Ms. B's third grade CGI lessons lasted for about one hour, and one lesson was complete in one day. Ms. K's fourth/fifth grade class also spent at least an hour a day in a CGI math period, however several lessons spilled over into the next day due to the elaborate discussions, thus only seven distinct math lessons were observed throughout the 10 observations of Ms. K's class. I considered my role to be that of a "participant observer" which "combines participation in the lives of the people being studied with maintenance of a professional distance that allows adequate observation and recording of data" (Fetterman, 1998, pp. 34-35). I was primarily passive as I observed and usually did not participate in the lesson, but I did have some interaction with the students while casually walking around during the problem-solving process. Informal conversations with the teachers about the lessons took place before and after class. The teachers were aware that the focus of the study was challenging the mathematically gifted and advanced students. Many of our conversations centered on ideas for challenge, some of which were mine.

Recording the lesson observations involved a combination of field notes, audio recording, and video recording, looking for evidence of higher level thinking among the advanced students and what teachers did to extend their thinking. Field notes were taken on a coding/record sheet (Appendix E), using one coding sheet for Phase 1 and 2 (switching ink colors for each phase to keep track of the phase in which the evidence occurred) and a new copy of the coding sheet for Phase 3 (again switching ink colors). The coding/record sheet included a list of student and teacher actions that would indicate evidence of the operational definition of challenge. The items on the list were generated prior to beginning this study rather than coding all actions and looking for themes. This a priori approach was possible because I was familiar with the CGI literature and that of higher order thinking in relation to challenge, had observed CGI math lessons when piloting the study, and had these items confirmed by interviewing the teachers. Two items were added to the list after observing the teachers for this study. These items were coded during review of the videos and field notes after completion of the observations.

During observations, sometimes I simply checked off that I saw evidence relating to a certain category on the coding sheet, but often I wrote a few details about what I observed. When the evidence noted involved a specific student, I used the code letter specifying that student, rather than a checkmark. During Phase 2, the problem-solving phase in which students were working at their tables, I circulated the classroom for the first five minutes, noting on my clipboard coding/record sheet which students finished the problems within two minutes and then which students finished within five minutes (but not within 2 minutes). Per my request, Ms. K asked her students to record their finish times on their papers, but made it clear that this was not a race to see who finishes first. She explained to the class that the amount of time a student spends on a problem gives us information about the problem and the strategy chosen for solving the problem. I occasionally placed the portable audio recorder on a table at which the teacher was having a conversation with a student about his or her work, in order to capture evidence of the teacher's extending the student's thinking. To assist me in remembering the significance of certain teacher and student actions and interactions as I later reviewed the recordings, I jotted a few reminder notes. During the Phase 3 strategy-sharing phase, a video camera recorded the discussion of the selected strategies, focusing on the board at the front of the room with the students gathered in front of the board. Field notes were taken during this phase to highlight main points and to note comments from students that may have been inaudible with the recorder. During the observations, I recorded the beginning and ending time of each phase of the lessons.

Document Collection

Test scores for identification of advanced students. Arkansas State Mathematics Benchmark and SAT-10 scores were collected from the previous year for identifying students as advanced. The current year's scores became available at the end of the school year, thus were not used for identification, but are reported in Tables 1 and 2. The TOMAGS was administered and scored for identifying students as mathematically gifted.

Student work. Student work was collected to examine solution strategies for evidence of using higher level strategies and for thinking in terms of mathematical relationships. The student work also showed how many problems were completed, how many answers were correct, and if there were extra problems done by the participants that were not given to the rest of the class. Teacher feedback to problems was sometimes written on students' papers.

Data Analysis

A meta-inference process (Teddlie & Tashakkori, 2009, p. 12) was used to analyze the data in which quantitative and qualitative data were integrated with each type informing the other. Although there are two components to the research question "to what extent and in what ways are mathematically gifted students challenged in a CGI classroom?," it seemed natural to integrate the analysis of the data rather than separating the two components for analysis. One explanation for this natural integration might be that the challenge felt by the student was a function of the actions of the teacher.

Both within case and cross case analyses were used in this study. For within case analysis, third grade class results and fourth/fifth grade class results were reported separately. For cross case analysis, third grade and fourth/fifth grade results were compared. Another analysis that spanned across the cases was the comparison of the top tier advanced students of all grades to the second tier advanced students of all grades. In a combination of cross case and within case analysis, all three grades were compared for some data. Because the trends were similar for both cases, data from the two cases were also reported for the whole group of advanced participants.

All interviews and audio recordings of teacher/student interactions were transcribed. Only some of the video recordings were transcribed due to the extensive amount of time it took to type in mathematical conversation. Viewing and listening to the videos directly was favorable, rather than reading a video transcript, in that hearing voice intonations and seeing facial expressions added to the meaning gleaned from the conversations.

The first step of organizing the data was to create a chart (Appendix F) on which each lesson's assigned problem was recorded, along with the total time allotted for solving it. The chart displays information about each student's solutions for each problem set, including which problems were completed correctly and which were not. From this, the percentage correct of the problems completed by the top tier and the second tier groups was calculated. After examining a student's problem-solving strategies, the main strategies were noted on the chart for each problem (i.e. number facts, grouping and combining, relational thinking) and if students used relational thinking to help solve a problem based on their knowledge of a previous solution. Reviewing the coding/record sheets from observations, I noted on the chart whether the student finished within 2 or within 5 minutes (but not within 2 minutes). Since the students' were listed in order of my perception of their ability and included several non-advanced students at the bottom of the table for the sake of comparison, this table not only provided an important reference as I worked through other data, but it also gave a good visual of trends. The layout of the chart provided a visual for comparing top tier and second tier students and how the advanced students compared to students who were not advanced (i.e. use of certain strategies).

From the post-lesson student interview data, the average of students' ratings of perceived challenge for each problem, on a scale of 1 to 5, was calculated to get a sense of the intellectual demand according to the students. This data was calculated for top tier advanced and second tier advanced, and per grade level, to look for differences. Another suggestion of intellectual demand came from observing how quickly students finished the problems. Therefore, I created a table that showed which students finished the problems early, either within 2 minutes or within 5 minutes (but not within 2 minutes). Information about whether or not the teacher provided an extension of the student's thinking in Phase 1 or 2 (gathered from coding/record sheets, reviewing audio recordings of teacher-student interaction, and from examining students' work) was then overlaid on the same table to begin to see the picture of the teachers' efforts at challenging their students. The percentage of times the students received teacher extensions was calculated and related to if the students had finished early. I calculated the percentage of times the top tier, second tier, and each grade level finished early. To reveal much extra time these students had after finishing early, the average time allotted for the problem-solving phase for both classes was first calculated, as well as the average time for the third grade class and fourth/fifth grade class individually. Then, for those students who finished early, the percentage of the time spent solving the problems out of the total time allotted was calculated.

The next step of analysis was to review all coding/record sheets and audio recordings of Phase 1 and Phase 2 for frequency and examples, per class, of types of teacher extensions provided to challenge the students. This begins to address the component of the research question "in what ways do the teachers challenge their students?" For the analysis, I collapsed some of the existing categories on the coding/record sheet and organized them within Bloom's Revised Taxonomy to provide a stronger case for their connection with higher levels of thinking, which is necessary for Ascending Intellectual Demand. Both within case and cross case data were considered, but the cross case analysis of what both teachers frequently did to extend student thinking will hold more weight in the analysis. To supplement this presentation of ways in which teachers attempted to challenge their students, student interview comments were provided on one particular type of extension. Teacher interview data were examined to further understand the teachers' intent for using these extension strategies as well as for other ideas on how to challenge their mathematically gifted students.

Returning to the students' work for a second round of analysis, I re-examined the student work with the operational definition of challenge in mind to see if the students were thinking in terms of mathematical relationships. I studied the solutions of both the advanced and non-advanced students to describe them in terms of CGI strategy levels, with the premise that working at higher levels corresponds with Ascending Intellectual Demand. I compared the use of the higher level strategies between top tier and second tier students and compared each to the time that students took to solve the problems.

Analyzing the observational data (field notes and video recordings) for evidence of higher level thinking during the Phase 3 strategy-sharing discussion followed a similar format as in Phases 1 and 2. Viewing the Phase 3 videos, I coded for the categories (Appendix E) either with a checkmark, a direct quote with a code to mark who said it, or a brief account of the evidence. A few gaps were filled in by referring to the field notes. For analyzing this coded data, I collapsed some categories and rearranged them within the broader categories of Bloom's Revised Taxonomy. Then frequencies of types of higher level thinking were calculated and examples of these types were selected.

An important part of the analysis was to tie in students' perceptions of the lesson, from post-lesson interview data, with the frequency data on types of higher level thinking. For the top tier group and the second tier group, I calculated the average student ratings of how challenged they felt by the entire lesson experience (beyond the problems they solved) and how much they enjoyed the lesson. I compared the average ratings of perceived challenge of the entire lesson to the average ratings of perceived challenge of the problem solved, and selected student comments from interview data to show what they liked about the lesson.

The final analysis of data addressed if and how students were exposed to new ideas, a component of the operational definition of mathematical challenge. I reviewed the documents of problems assigned and the observational data, listing and categorizing the topics into fourth, fifth, and sixth grade standards. I counted, per grade level, how many lessons included above grade level topics. Student interview data was analyzed by grade level and for top tier students versus second tier students, to calculate the percentage of lessons in which students reported learning a new idea. Calculating the percentage of the new ideas that came in Phase 3 as opposed to Phase 2 added detail to the analysis. Another student interview question asked if students found any other student's strategy of interest, which suggested an interest in learning new ideas. I analyzed these responses for top tier and second tier, and by class, by calculating the percentage of lessons in which students found other students' strategies of interest.

Literature Used for the Methodology

In choosing to do a qualitative study, I considered how my findings can best contribute to the literature. Lester (2005) discussed the role of mathematics education research and pointed to the potential of a blended approach using both qualitative and quantitative methods to investigate questions. He considered the role of theory in education research and how the researcher's philosophical stance may affect the research, and suggests utilizing a conceptual framework for designing and conducting inquiry.

Yin (2009) provided guidance in how to perform a case study, utilizing recommended methods such as observation and interview to collect data. He promoted the case study as an appropriate methodology for answering the questions "how" and "why". In my case of mathematically gifted students in CGI classrooms, I focused on how they act and interact, how challenged they feel, and how the teacher interacts with them to provide intellectual stimulation. Yin refers to case studies as being explanatory, exploratory, and descriptive. I characterize my study as exploratory and descriptive.

Considering the CGI emphasis on children's strategies for solving math problems, examining student work was a key component of this study. Borko, Kuffner, and Arnold (2007) stated that classroom artifacts, such as student work, reflect actual instructional activities better than teachers' interpretations of those activities. In this study, analysis of the student work as well as student commentary relating to it was considered when looking for evidence of challenge.

Trustworthiness

The trustworthiness of a study is the degree to which the results of the study are convincing to an audience and "worth paying attention to" (Lincoln & Guba, 1985, p. 300 as cited in Teddlie & Tashakkori, 2009). It includes credibility and transferability of qualitative research (Teddlie & Tashakkori, 2009) that address issues similar to internal and external validity, respectively, of quantitative research.

This study used multiple methods of observation, interviews, and document analysis, with the prolonged engagement of multiple observations in two case studies to allow for themes to arise and to be confirmed. The data was analyzed both within case and cross case. Furthermore the data was analyzed with respect to meeting the criteria of an operational definition of challenge as well as how it aligned with a framework of CGI/AID levels. This attention to the triangulation of the multiple methods as well as the multiple analyses led to a "convergence of evidence," (Yin, 2009, p. 117) and increased the credibility of this study.

Other than conversations with my committee members regarding the methods and findings of this study, I had several "peer debriefing" discussions (Teddlie & Tashakorri, 2009, p. 210) through all phases of the study with another expert on Ascending Intellectual Demand, Dr. Sandra Kaplan. Dr. Kaplan is a professor of education at the University of Southern California, is well known for her work in gifted education, and is one of the authors of the Parallel Curriculum Model. In particular, Dr. Kaplan agreed that it was reasonable to adapt the AID framework in the way that I had chosen to align it with the CGI framework of problem-solving levels. Once the analysis of the data was complete and a list of ways in which teachers challenge their students had been made, I asked the teachers to review the list as a "member check" (Teddlie & Tashakkori, 2009, p. 213) to confirm if my interpretation of what they had done was reasonable, which they affirmed. These efforts of soliciting feedback from others further increased the credibility of this study.

A “thick description” of the data and data collection methods adds to the transferability of the interpretation and conclusions of the data (Teddlie & Tashakkori, 2009, p. 213). It increases the chances that the conclusions could be applied in similar settings. It also makes it possible for others to replicate the study. In this study, I was careful to document the data in an organized manner to allow for efficient retrieval for data analysis. The students’ work was filed per lesson per class in a labeled folder and strategy types were labeled on the original papers. The coding/record sheets and field notes for each lesson were kept in these folders. Transcriptions of any audio recording of student/teacher interaction during Phase 2 was printed out and kept in these “per lesson” folders. Interview responses were transcribed electronically and filed both per individual interview and per question (i.e. all the students responses to question #1, etc.). Observation videos were kept in two electronic folders, one for each class. When reviewing the recordings, I made one summary coding/record sheet for each class. When coding the data, which included writing direct quotes to use as examples, I notated from which lesson the data came. I also recorded the data such that it was clear from which phase of the lesson it originated. These efforts provided an audit trail that can be used to verify the methods and the results, thus increasing the transferability of this study.

CHAPTER 4: RESULTS

To address the research question "to what extent and in what ways are mathematically gifted students challenged in a CGI classroom?", data came from classroom observations of math lessons taught by two exemplary CGI teachers, analysis of student work of 15 of their mathematically advanced students, and interviews with these students and their teachers. The students were classified as either “top tier advanced” or “second tier advanced”. Third grade teacher, Ms. B, was observed teaching 10 lessons within a four week period. Fourth/fifth split grade teacher, Ms. A, was observed 10 times over a four week period, for a total of seven lessons, some of which took two days to complete. Each lesson had three phases: Phase 1 (problem-posing), Phase 2 (problem-solving), or Phase 3 (strategy-sharing and discussion).

In collecting data addressing the extent of mathematical challenge, the intent of data collection had to remain close to the operational definition of mathematical challenge. Reviewing that definition, mathematical challenge centers around the exploration, discovery, and utilization of mathematical relationships and involves Ascending Intellectual Demand. Such demand requires exposure to new ideas inviting the students to continue upon a path toward expertise. Thus the data collection from lesson observation and student work analysis focused on relationships and higher order thinking necessary for Ascending Intellectual Demand as well as exposure to new ideas. The data collection from student interviews focused more on their self-reports of the challenge level of the problems and the lesson as well as exposure to new ideas. Observation of how quickly students finished the problems further described the challenge level of the problems.

The report begins with evidence of the extent of challenge by reporting the students’ perspective of how challenged they felt by the problems assigned in Phase 2, then linking their self-report with data on the percentage of times they finished early, the percentage of time the early-finishing students worked on the problems out of the allotted problem-solving time, and their percentage of correct answers. The next section will introduce strategies that the teachers used to extend students' thinking during the problem-solving phase as well as frequency of these extensions, and provide student feedback from interview data on one strategy in particular. I then turn to the students' work to see if their problem-solving strategies showed evidence of relational thinking and other high level strategies. Transitioning to the Phase 3 strategy-sharing, I provide frequency and examples of higher level thinking, including focus on relationships, in this discussion phase. The students' ratings of perceived challenge and enjoyment of the overall lesson experience are then reported. The final piece of data regarding the extent of challenge involves to what degree the topics and ideas were new to these advanced students, beginning with the self-report by students of learning new topics as well as interest in other students' strategies. This is followed by a count of lesson topics that were above grade level standards. Throughout this report, teacher and student interview comments are provided to more fully reveal the results of this investigation.

Students' Perceptions of Challenge

In preparation for asking the students to rate how challenged they felt after each lesson, the students defined what the word "challenged" meant to them in relation to math class. The most common responses were similar to these two responses: "it means I really have to think" and "the problem is hard". Two other comments were "when I feel challenged, I might have to ask the teacher a question" and "it's more interesting". One student described challenge as "like being on an escalator" and said that she did not like being stuck on the escalator. The students did not mention the idea of "exploring relationships", but their references to "thinking harder" are suggestive of the higher order thinking and Ascending Intellectual Demand, the key elements of the operational definition of mathematical challenge applied throughout this study.

After each math lesson, I conducted a brief interview with the top tier advanced students and the majority of those in the second tier. One interview question targeted how challenged they felt solving the day's problems. An example of a typical problem used in both CGI classrooms is presented here to show how multiple problems are generated from one root problem. Following the root problem are four sets of "number choices" (as the teachers and students refer to them) to fill in the blanks, thus creating a total of four possible problems, usually increasing in difficulty:

> Angela is making _____ cookies. Each cookie will get _____ of a cup of frosting.
>
> How many cups of frosting are needed for all the cookies?
>
> (12, 1/3) (36, 1/3) (72, 1/3) (72, 2/3)

The students were familiar with this format and knew to place the number choices in the blanks, one at a time. For instance, the first problem was "Angela is making 12 cookies. Each cookie will get 1/3 of a cup of frosting. How many cups of frosting are needed for all the cookies?" A second problem was generated by replacing the "12" and the "1/3" with the next number choice, "36" and "1/3", and so on. To allow for differentiated instruction, the teachers allowed the students to begin with any set of number choices. However, the advanced students usually began with the first choice and worked through all number choices. When interviewed about their perceived challenge level of the day's problems, students gave a separate rating for each problem generated by each number choice. The number of ratings varies per student since not every student was interviewed each day and not all students completed the same number of problems. The averages of these ratings are reported in Table 3. The perceived challenge ratings provided by the students slightly declined with increased grade level. In addition, the perceived challenge ratings by the top tier advanced students were slightly lower than the ratings offered by the second tier advanced students.

Table 3

Averages of Students' "Perceived Challenge" Ratings of Individual Problems

Class		Top Tier Advanced		Second Tier Advanced
3rd graders	1.6	(n= 39 ratings, 3 students)	2.3	(n= 25 ratings, 2 students)
4th graders	1.5	(n= 23 ratings, 1 student)	1.7	(n= 48 ratings, 3 students)
5th graders	1	(n= 25 ratings, 1 student)	1.5	(n= 70 ratings, 4 students)
4th/5th graders	1.2	(n=48 ratings, 2 students)	1.6	(n=118 ratings, 7students)
Combined Results 3rd/4th/5th	1.4	(n= 87 ratings, 5 students)	1.7	(n=143 ratings, 9 students)

Note: The ratings were based on a Likert Scale of 1 to 5, with 1 "not challenging", and 5 "very challenging".

Challenge Suggested by Problem-Solving Times and Teachers' Extensions

Noting how quickly the advanced students finished the assigned problems in Phase 2, the problem-solving phase of the lesson, further suggests the degree of difficulty encountered and thus how challenged they were by the assigned problems. Observations focused on students who finished within 2 minutes and within 5 minutes (which does not include those who finished within 2 minutes). The finish time was established primarily by observation, along with students' self-report (from the fourth/fifth grade class) and confirmation during the post-lesson interview while discussing their work. The average time allotted for the problem-solving phase of the lesson was 22 minutes (more specifically, an average of 19 minutes for the third grade class and an average of 28 minutes for the fourth/fifth grade class). I also looked for evidence of teachers' extending the advanced students' thinking during Phase 1 and 2, regardless of finish times, and refer to these as teacher extensions. These findings are reported in Tables 4 and 5, displaying for each lesson which students finished in either 2 or 5 minutes and whether or not a teacher extension was offered. The types and frequency of extension strategies are in Table 6. The within 2 minute and within 5 minute finish times represent the time in which the initial set of problems was completed and does not include time spent on extensions prompted by the teacher.

Table 4

Top Tier Advanced Students who finished assigned Problems within 2 or 5 minutes, along with when Teachers provided Extensions

Lesson # Students	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
3rd graders										
Dominic	e	(2)e	(2)	(2)e	(2)e	(5)	e	(5)e	e	(2)e
Jasmin	e	e	(5)e	(5)e	e	e	e	(5)e	e	(5)e
Freddy	e	(5)	(2)	(5)e	e		e	(5)e	@	(2)e
Time allotted	23	22	11	15	23	20	30	5	25	15
4th graders										
Andre			(2)e		(5)e	e	e			
5th graders										
Geraldo	(2)e	(2)e	(2)e	(2)e	(2)e	(2)e	(2)e			
Time allotted	20	25	21	35	29	26	38			

Note: (2) denotes student finished within 2 minutes, (5) denotes student finished within 5 minutes, but not within 2 minutes, "e" denotes that teacher provided some kind of extension to the student during Phase 1 or Phase 2, @ denotes student was absent at the time of data collection.

For reporting percentages based on Table 4 and 5 data, each student's experience of an individual lesson is considered an "instance," and the percentage of times the students finished early are out of the total number of instances. The term "finished early" means that students finished the assigned problems within 2 minutes or 5 minutes while the rest of the class worked on the same problems for up to the entire time allotted to problem solving, which was on average 22 minutes. The three top tier students in the 3rd grade class finished early 55% of the time. The teacher then provided some type of extension to the students in 69% of these instances. In the other class, the top tier 4th grader finished early 29% of the time (or 2 of the 7 instances), with extensions provided both times. The top tier 5th grader finished early in 100% of the instances with extensions provided 100% of the time. Of the "finished early" instances, 60% of them involved students finishing the problem within 2 minutes.

Combining data for students of all grades, the top tier advanced students finished early in 58% of the 43 instances. The teachers provided extensions in 84% of these instances. Disregarding whether or not a student finished early, the teachers provided extensions to top tier students 58% of the time. Considering the amount of time allotted to the problem-solving phase of each lesson, the top tier students who finished early used 17% of the time allotted to complete their problems. Separating these data by class, the 3rd grade top tier students used 27% of the time and the 4th/5th grader top tier students used only 9% of the time.

Table 5 illustrates the frequency of the second tier advanced students finishing early. Fifth graders finished early more frequently than 3rd graders or 4th graders. For all grades combined, second tier advanced students finished early in 18% of the 71 instances. The teacher provided extensions to the students who finished early in 46% of these instances. One second tier student commented, "If I finish early and tell her they were easy she gives me another problem that's harder so I can get more advanced." Disregarding whether or not a student finished early, the teacher provided extensions to the second tier advanced students in 51% of the instances. Taking into account the time allotted to the problem-solving phase of each lesson, in the 13 out of 71 instances in which the second tier advanced students finished early, they used 16% of the allotted time to complete the problems.

Table 5

Second Tier Advanced Students who finished assigned Problems within 2 or 5 minutes, along with when Teachers provided Extensions

Lesson # Students	L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
3rd graders										
Maya	e	e	(5)	e	e	e	e	e	e	e
Julia								@	e	
Time allotted	23	22	11	15	23	20	30	5	25	15
4th graders										
Natalie	e	e		(5)	(5)					
Alana	e		e		e					
Anita	e			e	e	e				
Katerina	e		e	e						
5th graders										
Sylvia	(5)e	(5)e	e	(2)	(2)e	(5)e				
Allen	e	e		e	@	@				
Kara		e	e	e	(5)					
Roberto	e	(5)	(5)	@	(2)e	(5)e	@			
Time allotted	20	25	21	35	29	26	38			

Note: (2) denotes student finished within 2 minutes, (5) denotes student finished within 5 minutes, but not within 2 minutes, "e" denotes that teacher provided some kind of extension to the student during Phase 1 or Phase 2, @ denotes student was absent.

During interviews, the teachers were asked what they typically do to further challenge a student who has finished the problems early. Ms. K responded,

> I question students' strategies and try to push their thinking beyond what they have shown in their solutions. Sometimes I provide a different number choice to see if they are making a different type of mathematical connection. I also ask them to notate their thinking with a number sentence.

When introducing the problems for the day, she occasionally reminded students to try a different strategy for solving the problems if they finish early, but would make it clear the new strategy should be representative of the student's ability (i.e. if the student does not need to model the problem by drawing a picture to understand it, the time should not be spent on direct modeling). Ms. B's interview responses were similar but she elaborated on how she pushes for students' understanding of properties of operations as they notate their thinking by writing number sentences. When I asked Dominic what Ms. B does to challenge his thinking, he responded, "She might have done higher numbers for me, try harder problems, make me notate it down, not really, but it was a choice that would be harder to notate it down." At one observation, the need for extension came right as Ms. B had finished posing the problem when two of the top tier students quietly said that they already knew the answer. Without missing a beat, the teacher said "then I'd like you to work on the notation".

Another consideration that may help characterize the difficulty of the problems assigned is whether or not the students got the correct answer and if they finished the entire set of number choices. The top tier advanced students correctly completed 95% of the problems. The second tier advanced correctly completed 81% of the problems. These percentages were calculated out of all the problems assigned and any problem that a student got wrong or did not finish was counted incorrect.

Types of Extension Strategies used by Teachers to Challenge Students' Thinking during Problem-Solving

The teachers regularly used strategies to extend the thinking of their students beyond the scope of producing a correct solution to a specific problem. Tables 4 and 5 detailed the observed instances of the teachers' providing extensions for the advanced students during Phase 2. Many of these extensions were provided to other students, too, since the practice of extending a student's thinking is embedded in the CGI philosophy, but only extensions for the advanced participants were tracked for this study.

The types and examples of teacher extension strategies presented in Tables 6 and 7 came from 10 classroom observations per teacher, recordings of one-on-one interactions of teacher and student as well as teacher facilitation of small group challenges, analysis of students' work, and teacher interviews. These extensions ranged from simple verbal prompts to more complex strategies, such as providing a special challenge question intended for a small group of advanced students. However, 98% of the extensions were based on the students' working on the original root problem assigned to the whole class. Since the idea of "extending one's thinking" implies reaching higher levels of cognitive thought, the extension strategies in Table 7 are categorized based on Bloom's Revised Taxonomy.

Table 6

Frequency and Examples of Observed Teacher Extensions provided to Advanced Students in Phases 1 or 2

Extensions relating to Problems Assigned	3rd grade	4th/5th grade
Teacher assigns…		
Problems with differentiated number choices for students to choose at which level to begin (this extension was available to the entire class but only the advanced students typically solved using the harder number choices)	All lessons	All lessons
A harder number choice to begin with	5	-
A new number choice after student finishes all number choices from original problem, sometimes designed to elicit relational thinking based on the solutions to the previous number choices	10	12
Advanced students to work together on the problem	4	-
A challenge investigation, after students finish original problem, designed for a group to discuss and formulate a generalization	1	2
A challenge investigation, after student finishes original problem, designed for the individual to work on	-	1

Note: The frequency count is the number of observed instances of each extension throughout the 10 observations per classroom.

Grouping Practices

Two of these extension strategies involved the teacher forming a group of advanced students to work together on problems, rather than working individually or with other students at their assigned tables. Their regular table seating had a mix of student abilities, although there was always at least one good math student present in addition to the advanced student at the table. One grouping practice involved the selection of about four of the advanced students to work together on the daily problem, or a more challenging, yet related, problem. These students worked at a whiteboard easel in the corner of the room as well as on their individual clipboards. The teacher was observed initiating their discussion, then circulating the rest of the classroom helping other students, periodically returning to facilitate the advanced group's discussion using many of the higher level thinking prompts mentioned in Table 7.

Another grouping practice involved selecting the first few students who completed the problems of the day and giving them a challenge question to investigate based on an idea or question that had sprung from a previous day's discussion. Students worked in their group on questions that were designed to elicit a generalization while the rest of the class was at their seats finishing the daily problems. One group challenge investigation involved students making a generalization about the size of the product when multiplying by fractions versus whole numbers. Another group investigation related dividing by a fraction to multiplication due to increasing the number of groups. In all small group discussions, the students stayed occupied on the problems for the duration of the time allotted to Phase 2 problem solving.

I questioned students about their attitudes with respect to working in small groups with other advanced math students. All advanced students expressed positive attitudes toward engaging in math problem solving with peers whose mathematical ability was similar to theirs. For instance, Freddy said, "If there are harder problems, we can work together like when me and

Jasmin and Dominic did (referring to that day's problem they had worked on at the easel in the corner of the classroom). We can learn a lot from each other." Dominic added, "I'd feel glad to have other students that are smart with me [sic] and they'd encourage me." Alana confided, "Sometimes when I'm by myself I feel like I can't do it but when I'm with others and see that they are doing it, I tell myself I can, too." Sylvia said,

> If I was at the same table as those people, it would work out much better. I could talk to my group and they would actually understand what I was saying. I'll tell my [current] group what I was thinking, and they say, 'What? What do you mean by that? Show me,' and I'm thinking, 'Oh, great.'

And finally, Andre liked the idea of working together with other advanced math students because "the best students won't copy off you because they are already good at math."

Table 7

Frequency and Examples of Observed Teacher Extensions provided to Advanced Students in Phases 1 or 2, categorized within Bloom's Revised Taxonomy

Extensions relating to Interaction of Teacher with Students during Problem Solving

	3rd grade	4th/5th grade	Example
Applying: Teacher encourages student to use relationships between number choices to solve the problems (relational thinking).	7	9	Problem: 8 people each eating ¾ pizza, with the second number choice (16, ¾). "You found that 8 people will eat a total of 6 pizzas. If we change the problem to 16 people, do you need to start from scratch to solve it, or can you use something you found out in your first problem?"

(continued)

Table 7 (continued)

Frequency and Examples of Observed Teacher Extensions provided to Advanced Students in Phases 1 or 2, categorized within Bloom's Revised Taxonomy

Extensions relating to Interaction of Teacher with Students during Problem Solving

	3rd grade	4th/5th grade	Example
Analyzing:			
Teacher asks students to compare.	12	6	From a small group challenge that sprung from a class problem involving how many portions in 12 cups: "Geraldo wrote $12 \div \frac{1}{2} = 24$ and Kara has 12 x 2 =24. Are these the same?"
Teacher asks student to look for mistake or flaw in thinking, or utilizes a counterexample to clarify.	4	5	"How could dividing by different numbers give you the same answer? $12 \div \frac{1}{2} = 12 \div 2$?"
Teacher poses a 'what if' situation.	2	3	"What would happen if you divided by ¼ instead of ½ ?"
Teacher asks student if they agree or disagree with a statement.	3	3	"Maya said that you can split it into thirds. Do you agree?"
Evaluating:			
Teacher asks student to justify their reasoning.	19	11	"How did you know that 3/8 was half of ¾?
Teacher asks student if something is always true or to prove a conjecture	5	2	In response to Maya's conjecture that splitting a fraction in half gives you an even number denominator, "Will that always be true? Is there a way you can prove that?"

(continued)

Table 7 (continued)

Frequency and Examples of Observed Teacher Extensions provided to Advanced Students in Phases 1 or 2, categorized within Bloom's Revised Taxonomy

Extensions relating to Interaction of Teacher with Students during Problem Solving

	3rd grade	4th/5th grade	Example
Creating: Teacher facilitates the *creation* of new ideas.			
Teacher encourages students to try another strategy for solving the problem different from their first solution strategy.			This was an expectation of all students that both teachers had established at the beginning of the school year, and students were occasionally reminded.
Teacher encourages student to connect their thinking to underlying properties of arithmetic.	8	4	Teacher helped student see that when she multiplied 2 x 1 ½ , she was using the distributive property as her work indicated she had carried out the operations in this way (2 x 1) + (2 x ½) = 2 x (1 + ½).
Teacher encourages student to make connections, see relationships, and express them using mathematical notation.	29	28	"I want to challenge you to write a number sentence that explains the relationship that you found."
Teacher encourages student to make conjectures.	3	1	"What do you think you need to do whenever you add fractions with unlike denominators?"
Teacher encourages student to formulate a generalization.	5	9	"You've got $12 \div \frac{1}{2} = 12\text{x}2$, $12 \div \frac{1}{3} = 12\text{x}3$, $12 \div \frac{1}{4} = 12\text{x}4$. Will this always work? Tell me, is there a pattern here?"

Note: These frequencies are based on what was observed during class, or in reviewing recordings and student work. It is unlikely that all evidences of extension were captured since I was concentrating on multiple facets. Thus, it is likely that the actual frequencies are higher than reported here.

Evidence of Higher Level Strategies in Students' Problem-Solving

Tables 6 and 7 illustrated the efforts of the teachers to encourage higher level thinking, such as relational thinking, during the problem-solving phase of the lesson. To confirm that students were engaging in relational thinking and other higher level problem-solving strategies, I examined the students' work. This allowed me to investigate the issue of challenge from another direction by seeing if the teachers' extension efforts were reflected in the students' problem-solving strategies. I looked at their strategies in terms of the operational definition of challenge that included the key idea of thinking in terms of relationships, the more specific CGI definition of relational thinking that refers to relating numerical expressions, and the problem-solving strategy levels from the CGI literature. I noted when students expressed mathematical relationships using equations (number sentences as their teachers would say).

Every solution of every advanced student indicated the use of relational thinking, from the simple statement $1 + \frac{1}{2} = \frac{3}{2}$ to recognition of the more complex situation that $1\frac{1}{2}$ x 8 is equivalent to 3 x 4. The continuum of problem solving strategy levels described in the CGI literature was evident in both classrooms, with the less advanced students primarily using direct modeling and the more advanced students primarily expressing relationships with number sentences. The advanced students often had a flexible use of strategies for solving, including use of direct modeling for a portion of the problem or repeated addition, combining and grouping of numbers, and number facts.

The following are examples of advanced students' use of relational thinking. Solving a problem of "4 kids sharing 6 cakes", Dominic, a 3rd grader, knew that each kid would get at least 1 cake, and that there would be 2 cakes left to be shared 4 ways. Recognizing that 2 cakes divided 4 ways could be represented as $\frac{2}{4}$, Dominic was using multiplicative coordination, the highest strategy level for solving equal sharing problems (Empson & Levi, 2011). He related one expression to another by writing the total of 6 cakes as $4+1+1 = 4 + (\frac{2}{4} + \frac{2}{4}) + (\frac{2}{4} + \frac{2}{4})$, concluding that each kid would get $1 + \frac{2}{4}$ cakes.

In another equal sharing problem of 8 kids sharing 14 candy bars, a group of advanced students used direct modeling to help them divide up the remaining 6 candy bars, once each kid receives one bar. Four of the bars were split into 8 halves, and 2 of them into 8 fourths. With guidance from the teacher, they expressed the sharing of these 14 candy bars as 14 = 8 + 4 + 2 = (8x1) + (8 x ½) + (8 x ¼).

In a multiple groups problem, Dominic used the highest level strategy again, a multiplicative strategy, (Empson & Levi, 2011) as he solved the following :

Six kids, each with $2\frac{2}{3}$ cookies, how many total cookies?
He wrote $6 \times 2\frac{2}{3}$ then solved it by breaking it down in this way:

$6 \times 2 = 12$

$3 \times \frac{2}{3} = 2$

$3 \times \frac{2}{3} = 2$

$12 + 4 = 16$

Jasmin, a third grader, also used the following multiplicative strategy, but decomposed the 2/3 into 1/3's first:

$6 \times 2 = 12$

$3 \times \frac{1}{3} = 1$

$6 \times \frac{1}{3} = 2$

$3 \times \frac{1}{3} = 1$

$12 + 4 = 16$

Jasmin initially used a multiplicative strategy in solving the same root problem with the number choice of $2\frac{5}{6}$, by writing:

$$6 \text{ x } 2\frac{5}{6}$$

$$6 \text{ x } 2 = 12$$

$$6 \text{ x } \frac{5}{6}$$

Whereas Julia had expressed 6 x $\frac{5}{6}$ as 6 groups of $\frac{5}{6}$, Jasmin's relational thinking included a less obvious interpretation of the expression which led her to write the following relationships:

$$6 \text{ x } \frac{5}{6} = \frac{30}{6} = \frac{6}{6} + \frac{6}{6} + \frac{6}{6} + \frac{6}{6} + \frac{6}{6} = \frac{30}{6} = 5$$

Table 8 shows how the types of strategies of the $4^{th}/5^{th}$ grade advanced students primarily fall within the higher level strategies for a multiple groups problem compared to other students in the class. The students marked as "regular" students are the participants of the study who were not advanced participants. These students had an ability range from low ability to high average ability. The strategies of the top tier advanced 4^{th} grader and top tier advanced 5^{th} grader are in the two highest level categories.

Table 8

Frequency of Strategies for a Multiple Group Problem as defined by Empson and Levi (2011) with two additional intersecting Categories

Problem: 12 cookies each get 1/3 cup of frosting. How many total cups of frosting is needed? $(12, \frac{1}{3})$ $(36, \frac{1}{3})$ $(72, \frac{1}{3})$ $(72, \frac{2}{3})$

Strategy	Frequency	Example
Represents Each Group		
Direct Modeling or Repeated Addition	3 regular 4^{th} graders 1 advanced 4^{th} grader 3 regular 5^{th} graders	The student represented each fractional group, either drawing twelve $\frac{1}{3}$ cups of frosting (direct modeling), or writing $\frac{1}{3}$ repeatedly twelve times (repeated addition).

(continued)

Table 8 (continued)

Frequency of Strategies for a Multiple Group Problem as defined by Empson and Levi (2011) with two additional intersecting Categories

Problem: 12 cookies each get 1/3 cup of frosting. How many total cups of frosting is needed?
$(12, \frac{1}{3})$ $(36, \frac{1}{3})$ $(72, \frac{1}{3})$ $(72, \frac{2}{3})$

Strategy	Frequency	Example
(Intersecting Category): Repeated Addition & Grouping/Combining Strategies Flexible Use of Strategies	1 regular 4^{th} grader 3 advanced 4^{th} graders 2 regular 5^{th} graders 3 advanced 5^{th} graders	For the first number choice $(12, \frac{1}{3})$ the student represented each group using repeated addition to get 4 cups as an answer, then used more efficient grouping/combining strategies for solving with subsequent number choices, based on their knowledge of their first answer, i.e. 4+4+4 cups to frost 36 cookies.
Grouping/Combining Strategies	1 regular 5^{th} grader	Instead of representing each fractional group, the student combines fractional groups in an efficient way and counts these groups. *Ingrid knew that 3 thirds make 1 whole. She set up a table counting whole cups of frosting instead of fractional cup, listing cookies in multiples of 3 up to 72 cookies. The teacher extended her thinking by guiding her to change her counting strategy for subsequent number choices. The student did not see that 36 x 2 = 72, so continued to count in multiples of 12.*

Cookies	Cups of Frosting
3	1
6	2
9	3
12	4
24	8
36	12
48. . .	16. . .

(continued)

Table 8 (continued)

Frequency of Strategies for a Multiple Group Problem as defined by Empson and Levi (2011) with two additional intersecting Categories

Problem: 12 cookies each get 1/3 cup of frosting. How many total cups of frosting is needed?
$(12, \frac{1}{3})$ $(36, \frac{1}{3})$ $(72, \frac{1}{3})$ $(72, \frac{2}{3})$

Strategy	Frequency	Example
(Intersecting Category): Grouping/Combining Strategies & Multiplicative Strategies Flexible Use of Strategies Relational Thinking	1 advanced 4th grader (top tier advanced) 1 advanced 5th grader	Andre used a multiplicative strategy to relate the fractional group in the first number choice to a total (12 x 1/3 = 4 cups), then used efficient grouping/combining strategies to solve the other number choices: 12 + 12 = 24 cookies need 8 cups 24 + 12 = 36 cookies need 12 cups Using a multiplicative strategy again to relate the 2 number choices $(36, \frac{1}{3})$ and $(72, \frac{1}{3})$ by notating 36x2=72, he then state that 12 cups + 12 cups = 24 cups (to frost the 72 cookies) 24x2 = 48 cups (for 2/3 cup frosting)
Multiplicative Strategies Relational Thinking	1 advanced 5th grader (top tier advanced)	Geraldo related the fractional grouping to a total by multiplicative reasoning for all number choices: $(12, \frac{1}{3})$ 12 ÷ 3 = 4 cups $(36, \frac{1}{3})$ 4 x 3 = 12 cups $(72, \frac{1}{3})$ 12 x 2 = 24 cups $(72, \frac{2}{3})$ 24 x 2 = 48 cups

Note: n=20 students in Ms. K's 4th/5th grade class

Two top tier students finished the cookies and frosting problems within 2 minutes. One second tier student finished this problem within 5 minutes. All three of these students used multiplicative strategies and relational thinking as part of their solutions.

Evidence of Higher Level Thinking during Phase 3 Strategy Discussion

The final phase of the lesson, the Phase 3 strategy sharing and class discussion, began with the teachers purposefully selecting three or four students' solution strategies to be shared that would serve as a springboard to discuss mathematical relationships. Ms. B's students wrote their solutions on the board before discussion began, then explained their strategies when asked. Ms. A used a document camera to project the selected students' papers one at a time, asking other students to explain how the student solved the problem and then asking for clarification from the selected student when necessary. Ms. K commented, "When we share as a class, I can call on the advanced students when higher level concepts are brought up in discussions . . . and question them on what they know to push them to the next level of thinking." Both teachers spent considerable time asking students to compare strategies, which was a catalyst for much mathematical discussion. The length of Phase 3 for the 10 third grade lessons averaged 34 minutes per discussion. The length of Phase 3 for the seven fourth/fifth grade lessons averaged 50 minutes per class, with several discussions continuing over the course of two days as the discussion often evolved beyond the discussion of the original strategies. Reviewing the video recordings of the discussions, I coded for frequency of types of higher level thinking and categorized each within Bloom's Revised Taxonomy. Incidences of student use of higher level thinking and teachers encouraging students to use such thinking are presented in Table 9.

Table 9

Frequency and Examples of Higher Level Thinking (student or teacher-initiated) in Phase 3, Strategy Sharing and Discussion, categorized within Bloom's Revised Taxonomy

Category of Higher Level Thinking	3rd grade	4th/5th grade	Example
Analyzing:			
Comparing	21	15	*3rd grade Example:* Ms. B: "You're saying that these problems are similar (3 kids sharing 2 cakes and 6 kids sharing 4 cakes). They are different problems, yet why are we getting the same answer?" *4th/5th grade Example:* After confusion whether $6 \div \frac{1}{2}$ and $\frac{1}{2}$ x 6 meant the same thing, discussion ensued that resulted in students drawing pictures to model both expressions, as well as the related expressions 6 x $\frac{1}{2}$, 6 x 2, and $6 \div 2$, which then led to their misunderstandings being clarified.
Looking for mistakes or flaw in reasoning, or utilizing a counterexample to clarify	15	13	*3rd grade Example:* Ms. B: "You said earlier that $\frac{3}{2} = 3 \div 2$ is false. What made you change your mind?" Student remembered the answer on the board to the "2 kids share 3 cakes" problem was 3÷2= 1 ½ and reasoned that 1 ½ = $\frac{2}{2} + \frac{1}{2} = \frac{3}{2}$ therefore $\frac{3}{2} = 3 \div 2$ must be a true statement. *4th/5th grade Example:* After Andre explained how a grouping of "4 kids will eat 3 whole pizzas" helped him quickly solve "16 kids eating ¾ pizza each", Ms. K posed a counterexample of grouping 3 kids to make the point that not any number choice would work out as well as Andre's grouping of 4 kids.
Posing a "what if" situation.	5	4	*3rd grade Example:* Ms. B: "What if we changed the problem to $\frac{1}{4} + \frac{1}{8}$, can we use the same idea?" (doubling a denominator to get the common unit) *4th/5th grade Example:* Ms. K: "What if you had decided to split it into eighths. Would it have been easier or harder?"

(continued)

Table 9 (continued)

Frequency and Examples of Higher Level Thinking (student or teacher-initiated) in Phase 3, Strategy Sharing and Discussion, categorized within Bloom's Revised Taxonomy

Category of Higher Level Thinking	3rd grade	4th/5th grade	Example
(continued, Analyzing)			
Teacher asks students if they agree or disagree with a statement.	14	15	*3rd grade Example:* Ms. B: "He says that this 12 x ½ in his number sentence represents 12 groups of ½. Do you agree or disagree? *4th/5th grade Example:* Ms. K: "Will you say that again so we can make a good argument for or against it?"
Evaluating:			
Justifying reasoning.	34	25	*3rd grade Example:* Ms. B: "What is one half of a third?" Dominic: "$\frac{1}{6}$" Ms. B: Why do you say "$\frac{1}{6}$" ? Dominic: "Because it takes two sixths to make one third." *4th/5th grade Example:* Ms. K: "How can we verify that 12 x $\frac{1}{3}$ = 4 expresses the relationship in this problem? Where are the 12, $\frac{1}{3}$, and 4 in Eva's picture?"
Teacher asks students to consider if something is "always true" or how they could prove a conjecture.	10	7	*3rd grade Example:* Ms. B: "You think we can break down thirds into sixths? How can we prove that?" *4th/5th grade Example:* A student had conjectured that "when you multiply it's more and when you divide it's less." Ms. K responded, "Is that always true?" Geraldo offered the counterexample of 12 x ½ = 6 to disprove the conjecture.

(continued)

Table 9 (continued)

Frequency and Examples of Higher Level Thinking (student or teacher-initiated) in Phase 3, Strategy Sharing and Discussion, categorized within Bloom's Revised Taxonomy

Category of Higher Level Thinking	3rd grade	4th/5th grade	Example
Creating: Teacher facilitates the *creation* of new ideas.			
Connecting thought process in strategies to the underlying properties of arithmetic.	9	5	*3rd grade Example:* Jasmin's solution to "6 kids, $2\frac{2}{3}$ brownies each" included: $3 \times \frac{1}{3} = 1$, $6 \times \frac{1}{3} = 1$, and $3 \times \frac{1}{3} = 1$. Ms. B guided the class in seeing how her solution utilizes the distributive property: $6 \times \frac{2}{3} = (6 \times \frac{1}{3}) + (6 \times \frac{1}{3}) = (3 \times \frac{1}{3}) + (3 \times \frac{1}{3}) + (6 \times \frac{1}{3})$. *4th/5th grade Example:* Featuring two strategies for solving a problem, Ms. K guided the class to representing one strategy with the number sentence, $12 \times \frac{1}{3} = 4$, and the other strategy with the number sentence $4 \times (3 \times \frac{1}{3}) = 4$, then to seeing how the associative property verifies their equivalence.
Making connections, seeing relationships, and expressing them using mathematical notation.	58	46	*Examples of making connections:* *3rd grade:* For a "14 sharing 8" problem, the teacher guided students in making connections between the direct modeling solution and solution without pictures, only number sentences. The students matched the number sentences 8 x 1 = 8, 8 x ½ = 4, and 8 x ¼ = 2, with the pictures of 8 wholes, 8 groups of 1/2, and 8 groups of ¼. *4th/5th grade:* "We are trying to make a connection between Andre's and Kara's strategies by finding Andre's pattern in Kara's strategy."

Examples of seeing relationships:
3rd grade: Ms. B: "What is the relationship between ¼ and ½ ?"
Dominic: "You need two ¼'s to make a half."
Ms. B: "So if I need two ¼'s to make a half, then ¼ is <u>what</u> of ½ ? … Let's look at 2 and 4. 2 is <u>what</u> of 4?
Freddie: Half.
4th/5th grade: Students noticed that 1/3 x 12 = 4 and 4 ÷ 12 = 1/3 have an inverse relationship like "fact families" of whole numbers operations.

(continued)

Table 9 (continued)

Frequency and Examples of Higher Level Thinking (student or teacher-initiated) in Phase 3, Strategy Sharing and Discussion, categorized within Bloom's Revised Taxonomy

Category of Higher Level Thinking	3rd grade	4th/5th grade	Example
(continued, Creating: Teacher facilitates the *creation* of new ideas.)			
Examples of creating mathematical expressions to notate thinking: *3rd grade:* This notation represents a connection between the thinking behind two students' strategies and verifies equivalent forms of the same answer: $\frac{3}{4} + \frac{3}{4} = \frac{6}{4} = \frac{4}{4} + \frac{2}{4} = 1 + ½ = 1 ½$. *4th/5th grade:* This notation represents a connection between the thinking behind three students' strategies: 2 x 1 ½ = 1 ½ + 1 ½ = (½ + ½) + (1 + 1).			
Making conjectures	10	11	*3rd grade Example:* Ms. B: "Think of what kind of unit helped us add $\frac{1}{2} + \frac{1}{4}$. Can you make a conjecture of what kind of units would help us add $\frac{1}{2} + \frac{1}{3}$? 4th grade Example: A student conjectured, "Taking half of an odd whole number gives you a whole number and a half."
Formulating Generalizations	10	5	*3rd grade Example:* In response to a student who said that $\frac{3}{6}$ is the same as $\frac{1}{2}$ because 3 is half of 6, Ms. B asks the class, "Is it always true that the top of the fraction is half of the bottom when it's equal to $\frac{1}{2}$. *4th/5th grade Example:* The class had verified that dividing by ½ is the same as multiplying by 2, then explored other fractions (dividing by 1/3 is the same as multiplying by 3, dividing by ¼ is the same as multiplying by 4) and made the generalization that dividing by a unit fraction has the same result as multiplying by the number of parts the fraction is split into.

Students' Perceptions of the Challenge Level of the Overall Lesson

Having explored evidence of higher level thinking in all phases of the CGI lessons, the next report details the students' attitude regarding the entire lesson. In the daily interview, students' were asked to rate how challenged they felt considering the day's lesson as a whole, including the discussion and strategy sharing phase (Table 10) rather than just the challenge level of the problem alone (reported again for comparison in Table 11).

Table 10

Averages of Students' "Perceived Challenge" Ratings of the entire Lesson

Students	Top Tier Advanced	Second Tier Advanced
3rd graders	2.1 (n= 23 ratings, 3 students)	3.8 (n= 15 ratings, 2 students)
4th graders	2.4 (n= 7 ratings, 1 student)	2.8 (n= 15 ratings, 3 students)
5th graders	1.7 (n= 7 ratings, 1 student)	2.9 (n= 21 ratings, 4 students)
4th/5th graders	2 (n=14 ratings, 2 students)	2.8 (n= 36 ratings, 7students)
Combined Results 3rd/4th/5th	2.1 (n= 37 ratings, 5 students)	3.1 (n= 51 ratings, 9 students)

Note: The ratings were based on a Likert Scale of 1 to 5, with 1 "not challenging", and 5 "very challenging".

Table 11

Comparison of Students' "Perceived Challenge" Ratings of Individual Problems to their "Perceived Challenge" Ratings of the Entire Lesson

Students	Top Tier Advanced Individual Problem/Entire Lesson	Second Tier Advanced Individual Problem/Entire Lesson
3rd graders	1.6 /2.1 (.5 increase)	2.3 / 3.8 (1.5 increase)
4th graders	1.5 / 2.4 (.9 increase)	1.7 / 2.8 (1.1 increase)
5th graders	1.0 / 1.7 (.7 increase)	1.5 / 2.9 (1.4 increase)
4th/5th graders	1.2 / 2 (.8 increase)	1.6 / 2.8 (1.2 increase)
Combined Results 3rd/4th/5th	1.4 /2.1 (.7 increase)	1.7 / 3.1 (1.4 increase)

Note: The ratings were based on a Likert Scale of 1 to 5, with 1 "not challenging", and 5 "very challenging".

The perceived challenge levels increased when the students considered the class discussion as part of the criteria for rating, compared to the ratings of the perceived challenge level of the assigned problems alone (Tables 3 and 11). On average, without separation by tier nor class, this was a 1 point increase, on a 5 point scale. The perceived challenge level of the whole lesson was 1 point lower for the top tier students than the second tier students, and slightly lower for fifth graders than third graders.

Table 12 shows evidence for challenge based on Barbeau and Taylor's (2005) description of mathematical challenge:

> A challenge has to be calibrated so that the audience is initially puzzled by it but has the resources to see it through. The analysis of a challenging situation may not necessarily be difficult, but it must be interesting and engaging. (p. 126)

An example of how students were interested and engaged comes from the following quote. At the end of a lesson in which students were enthusiastically arguing if a certain conjecture was true or not, Ms. K said, "we're going to have to stop here for today to go to specials" immediately followed by a disappointed and somewhat-in-unison "ohhhhh!" from the students.

Table 12

Averages of Students' Ratings in Response to the Post-lesson Question: "On a scale of 1 to 5, where 5 is the most, how much did you enjoy today's lesson?"

Class	Top Tier Advanced		Second Tier Advanced	
3rd graders	4.96	(n= 23 ratings, 3 students)	4.73	(n= 15 ratings, 2 students)
4th graders	5.00	(n= 7 ratings, 1 student)	5.00	(n= 15 ratings, 3 students)
5th graders	5.00	(n= 7 ratings, 1 student)	4.86	(n= 21 ratings, 4 students)
4th/5th graders	5.00	(n=14 ratings, 2 students)	4.92	(n= 36 ratings, 7students)
Combined Results 3rd/4th/5th	4.97	(n= 37 ratings, 5 students)	4.86	(n= 51 ratings, 9 students)

Note: The ratings were based on a Likert Scale of 1 to 5, with 1 "I did not enjoy the lesson", and 5 "I enjoyed the lesson very much".

Details of how the students enjoyed the lesson came from the post-lesson interview question, "what was your favorite part of today's lesson?" Some student responses include:

- The discussion
- How we figured out that dividing by 3 is same as taking 1/3 of something

- How both mine and Jasmin’s strategies were very efficient
- Sharing strategies and learning from others’ mistakes
- I liked the math because it was tricky. I got a chance to think about it.
- Working in a small group
- My extra problem (from the top tier student who typically reported not being challenged)
- I liked solving the problem and giving more details
- My favorite part was when I couldn’t figure out . . . because it made me figure out the relationship between…

Evidence of Challenge by Exposure to New Ideas

The final criterion in the evaluation of the extent of challenge is that of exposure to new ideas as students move along a path toward expertise. I begin by suggesting that the advanced students’ interest in other students’ strategies indicates an interest in learning new ideas.

Table 13 suggests some evidence that advanced students found other students’ strategies particularly interesting during the discussion phase. From both observing these students paying attention during the discussion phase of the lesson and from listening to students explain during the daily interviews why they liked certain strategies, I got a sense that they were interested in other ways to solve problems beyond their own strategies. Alana commented, “I like seeing other strategies because I can see other people how they got their ideas like number sentences.” The interest in others’ strategies was stronger among the third graders; however, the third grade post-lesson interviews were conducted in the classroom in which the students could still see all the shared strategies on the board. When asked this question, the students typically turned to the board and looked at the strategies before answering. The fourth/fifth grade interviews were conducted outside the classroom. Also, the fourth/fifth grade teacher shared the strategies one at a time on the document camera so they were visible for a shorter amount of time.

Table 13

Percentage of Students' Interview Responses indicating they particularly liked another Student's Strategy shared during Discussion Time

Class	Top Tier Advanced	Second Tier Advanced	Both Tiers
3rd graders	83% (n=23)	80% (n=15)	82% (n=38)
4th/5th graders	36% (n=14)	50% (n=36)	46% (n=50)
Combined Results 3rd/4th/5th	54% (n=37)	76% (n=51)	61% (n=88)

Note: n = # of interview responses

The data in Table 14 came from two post-lesson interview questions. One question asked if students learned a new mathematical idea during the lesson and the other asked if they tried a new strategy. Noticing that students rarely reported trying a new strategy, I collapsed the results of these two questions into one category, deciding that trying a new strategy fits under the category of learning a new mathematical idea. The percentage of new mathematical ideas was less for the top tier advanced students than the second tier advanced students and less for the fifth graders compared to the third graders. Table 15 reports during which phase the new ideas arose.

Table 14

Percentage of Student Interview Responses indicating they learned a new Mathematical Idea

Class	Top Tier Advanced	Second Tier Advanced
3rd graders	57% (n=23)	80% (n=15)
4th graders	57% (n=7)	73% (n=15)
5th graders	29% (n=7)	67% (n=21)
4th/5th graders	43% (n=14)	69% (n=36)
Combined Results 3rd/4th/5th	51% (n=37)	76% (n=51)

Note: n = # of interview responses

Table 15

Percentage of (self-reported) New Ideas learned during the Problem-Solving Phase and the Discussion Phases

Class	Phase 2 Problem-Solving	Phase 3 Discussion/Strategy-Sharing
3rd graders	28%	72%
4th/5th graders	28%	72%
Combined Results 3rd/4th/5th	28%	72%

Note: Percentage is out of 88 interview responses

Finally, to provide further evidence for the extent to which these students were exposed to new mathematical ideas, Tables 16 and 17 identify the topics discussed in these lessons that were above grade level standards based on the K-8 Mathematics Curriculum Framework (Arkansas Department of Education, 2004). Even though Arkansas is transitioning to Common Core Mathematics Standards, the 2011 Arkansas benchmark test was based on the existing state framework.

Table 16

Mathematics Topics explored in the 3rd Grade Class that were Above 3rd Grade Level Standards

4th Grade Standard	5th Grade Standard	6th Grade Standard
Using a fraction to represent division of whole numbers	Fraction addition with like denominators	Mixed number addition
Associative property	Fraction addition with unlike denominators	Multiplication of whole number by mixed number
	Multiplication of fractions	
	Simplifying fractions	Division by a fraction or mixed number
	Converting mixed numbers to improper fractions	
	Distributive property	Proportional reasoning

Table 17

Mathematics Topics explored in the 4^{th}/5th Grade Class that were Above Grade Level for 4^{th} Graders (both columns) or above level for 5^{th} graders (second column)

5^{th} Grade Standard	6^{th} Grade Standard
Fraction addition with like denominators	Mixed number addition
Fraction addition with unlike denominators	Multiplication of whole number by mixed number
Multiplication of fractions	
Simplifying fractions	Division by a fraction or mixed number
Converting mixed numbers to improper fractions	
Distributive property	Proportional reasoning

I saw evidence of exposure to above grade level topics in each of the ten third grade lessons. In the fourth/fifth grade class, the fourth graders were exposed to above grade level topics (fifth grade or sixth grade) in all seven of the lessons. In that same class, the fifth graders were exposed to above grade level topics (sixth grade) five out of the seven lessons. In many cases, these topics were not necessarily an intended goal of the lesson. Ms. K alluded to how the discussion often evolved into talking about advanced topics when she said to her students, "Sometimes it depends on what you guys say and I think, oooohhh, we can go there! So sometimes what you guys say and discover changes my direction."

CHAPTER 5: DISCUSSION

This study sought to answer the central question "to what extent and in what ways are mathematically gifted students challenged in a CGI classroom?" I developed the following operational definition and used it as a basis for answering this question:

> A student is challenged mathematically when he or she engages in exploring, discovering, or utilizing mathematical relationships, is exposed to new mathematical ideas, and experiences Ascending Intellectual Demand on a path toward expertise as a mathematical thinker.

I begin with an analysis of the findings in relation to each of the three components of the operational definition of challenge: 1) mathematical relationships, 2) Ascending Intellectual Demand, and 3) exposure to new ideas. This is followed by a discussion of the strengths and weaknesses of the CGI classroom experience for mathematically gifted students. The CGI/AID Framework will then be examined to provide recommendations for what teachers can do to fill in the gaps to further challenge mathematically gifted students. Finally, limitations of this study and implications for future research will be discussed.

In this concluding chapter, I speak in general of the mathematically advanced students, referring to both the top tier advanced and the second tier advanced, because the trends were similar for both. However, the top tier advanced students experienced less challenge than the second tier advanced students, further suggesting that having used the 95th percentile cutoff score on the Test of Mathematical Abilities for Gifted Students identified a group that can reasonably be called "mathematically gifted." Although both groups were clearly advanced math students and benefited from the rich mathematical discourse in their CGI classrooms, it seems that any deficiency in challenge is more an issue for the top tier students and that the second tier advanced students are relatively well-situated. Thus, the conclusions of this study pertain particularly to the top tier advanced students, the mathematically gifted students. Similarly, perceived challenge as expressed by both groups of students declined as the grade levels increased. The following discussion can be read with both these trends in mind.

Addressing the Operational Definition of Mathematical Challenge

Component 1 for Mathematical Challenge: Focus on Mathematical Relationships

Learning of mathematics in these two CGI classrooms was focused on mathematical relationships. This criterion of the operational definition of mathematical challenge was in place. Teachers posed problems to elicit relational thinking and prompted students to express mathematical relationships during the problem-solving phase. The advanced students were exploring, discovering, and utilizing mathematical relationships, evident by their solution strategies and their participation in the discussions. This was most evident during the extensive discussion time, as the teachers encouraged students to find relationships within and between strategies, express a variety of mathematical ideas by writing equations, and connect other math topics to that of the problem they solved.

Component 2 for Mathematical Challenge: Ascending Intellectual Demand

With emphasis on mathematical relationships, these CGI classrooms were fertile grounds for challenge to take place. However, challenge is relative to the individual, and the relationships must be at a level permitting each individual to experience Ascending Intellectual Demand. For instance, Geraldo, the top tier fifth grader, reported to have understood the concept of common denominators since third grade and did not find the discussion of equivalent fractions challenging. However, when the class digressed one day to explore how multiplication of fractions is like division, he was very interested in exploring this further. The teacher picked up on his interest and gave him a related challenge problem the next day, after he had finished the regularly-assigned problem. The challenge problem was designed to discover the fraction division algorithm. Geraldo was temporarily stumped but, with a small amount of teacher's scaffolding, could see the relationship between dividing by a number and multiplying by its reciprocal. The teacher had placed him in his zone of proximal development, thus increasing the intellectual demand. Knowing when a gifted student needs intellectual challenge requires skill and attentiveness on the part of the teacher, recognizing even brief signs of student intrigue and making instant decisions to facilitate that student in a new direction. Reminding that Ascending Intellectual Demand is an "escalating match between learner and curriculum" (Tomlinson et al., 2009, p. 11) it guides teachers as they consider how to challenge their students.

Knowing the extent to which students are challenged is related to ways in which they were challenged. The next sections will focus on other significant ways that the CGI teachers pressed the intellectual demand for their students beyond what might be considered a more average classroom experience. Limitations of these ways are also discussed to further describe the extent of the challenge.

Problemization of simple problems. The students' self-reports of perceived challenge can approximate the degree of challenge they experienced as they solved the assigned problems. Their average perceived challenge level of the assigned problems was low (1.4 for top tier advanced students and 1.7 for second tier advanced students, on a scale of 1 to 5 with 1 meaning 'not challenged'). The top tier advanced students finished the problems within 5 minutes (out of an average of 22 minutes allotted to the class for problem solving) 57% of the time, which suggests the problems were relatively easy for them. The teachers did, however, frequently use strategies to extend their students' thinking during this problem solving phase.

The students perceived the challenge of the entire lesson as higher than their rating of the assigned problem alone (2.1 for the top tier students and 3.1 for the second tier students). Hiebert (as cited in Empson, 2003) pointed out that the cognitive level of a problem could be low but the problemization of it in the hands of a skilled teacher and an active group of participating students could make it a high cognitive level experience. Problemization of a math problem refers to the extensive elaboration of the problem that extends the mathematical thinking beyond a simple solution. In the cases of the two CGI classrooms in my study, at each observation I witnessed problemization as the teachers used the somewhat-simple word problem assigned that day as a springboard for a deeper mathematical discussion. Teachers led discussions in which students compared different strategies for solving the problem and looked for connections between them. They delved into more depth at the math concepts behind the solutions as different number choices for the same root problem were discussed. Teachers' resourcefulness for problemization was evident as they took advantage of opportunities for students to discover new related concepts and to express the numerous mathematical relationships that were discovered by using mathematical notation of equations. These rich discussions almost always led to exposure to above grade level topics and lasted an average of 42 minutes, further supporting the idea that although the individual problems may have been easy for some to solve, the problemization raised considerably the level of thinking.

Number choices for solving problems. The most common strategy of differentiated instruction in these CGI classrooms was the use of more challenging number choices for the same root problem. Although it is a practice that efficiently allows students in a mainstream classroom to self-differentiate, its limitations are apparent when the results show top tier students completing all number choices in a fraction of the time that other students take. It works for a while with a new topic, such as when the third grade students were transitioning from adding thirds and halves with a common unit of sixths, to being challenged to finding a common unit for thirds and fourths.

With mathematically gifted students possessing a high ability for processing mathematical information and keen mathematical memory (Krutetskii, 1976), these students catch on quickly to both procedures and concepts and retain the knowledge to apply at a subsequent time. Once the concept is mastered, a mathematically gifted student may see adding eighths and twelfths as no more difficult than adding halves and thirds. The teacher must be attentive to the moment when these students have conceptually mastered a topic and are ready for a greater challenge. This may be evidenced by a quick finish time with complete explanation of the process.

The selection of number choices for the root problems observed in this study were often designed to elicit relational thinking from one number choice to the next, encouraging students to look for relationships between sets of numbers. Problems that could be solved using relational thinking between number choices often led students to quick solutions particularly when the advanced students recognized the relationship between number sets meant there would be a relationship between answers. Although the third graders were just beginning to see such relationships, the fourth and fifth graders were getting accustomed to looking for them and getting good at applying them to produce quick answers to subsequent problems. However, the students did not always take advantage of the relationships between number choices to solve the problems, perhaps finding their chosen methods to be just as easy. However, these relationships between the number choices did offer another interesting way to solve problems that ordinarily may have been easy for some students, and serves as another way for the students to look at the problem once they have finished with their initial strategies. The teachers must recognize that when relational thinking in this manner becomes easy for these students, they should be prepared to challenge the student in new ways.

Emphasis on notation of mathematical relationships. Both teachers emphasized expressing mathematical relationships using equations and making connections to mathematical properties. This elevated the level of instruction beyond any specific math topic, promoting a deeper algebraic understanding of numbers. The prevalence of this notation in the students' work and in class discussion is a reflection of the teachers' efforts to push this higher level understanding.

Intellectual Peer Groups. Placing the advanced students together provided increased opportunities for discussion among these intellectual peers that prolonged their engagement with the problem. This also presented students with the opportunity to lead their own mathematical discussions, increasing autonomy, a kind of challenge in itself (Diezmann & Watters, 2002). In the group investigations, the students were challenged to develop generalizations beyond those expected of the other students.

Higher order thinking skills emphasized in teacher extensions. The teachers interacted with the advanced students on a regular basis with the intent of extending their thinking to higher levels. This attention facilitated students in seeing connections between ideas, notating their work, making conjectures, and justifying their solutions. Other types of higher order thinking were prevalent, too, as categorized in Tables 6 – 9, thus increasing the intellectual demand.

Component 3 for Mathematical Challenge: Exposure to New Ideas

Mathematical challenge requires that students are exposed to new mathematical ideas in order to move them along the path toward expertise. The evidence of exposure to new ideas is discussed next.

Above Grade Level Topics. Third and fourth graders were exposed to above grade level topics, based on the Arkansas Mathematics Framework (2004), during all the observed lessons. The fifth graders experienced above grade level topics in all but two of the lessons. The assigned problems were such that they could be solved with knowledge expected of students at grade level using either intuitive methods such as direct modeling or with more sophisticated methods and advanced knowledge. For instance, for the third grade problem "How many cookies are there altogether if there are 6 kids each with $2\frac{2}{3}$ cookies," most regular students modeled the problem by drawing $2\frac{2}{3}$ cookies for each of 6 kids, then adding. The advanced students manipulated number relationships such as 6 x 2 and 6 x $\frac{2}{3}$ in various ways, writing equations to express their thinking. Their multiplicative solutions used the sixth grade standard of multiplying a whole number by a mixed number. The teachers took advantage of the students' choice of strategies to connect their work to the distributive property, a fifth grade standard.

Embedded in the CGI philosophy is the idea that if students are allowed to make sense of mathematics in their own ways, even advanced concepts can be accessible to them. This touches on the idea of autonomy, in this case 'open process' for solving problems, as an important element of a challenging mathematical task (Diezmann & Watters, 2002). In this study, the advanced students had the freedom in their strategy choice to take full advantage of working with advanced concepts. Their teachers were skilled at seizing opportunities to extend students' thinking to advanced concepts when their readiness was apparent.

Students were also exposed to above grade level topics during the class discussion phase, when the sharing of and discussion of strategies often led to exploration of or introduction of advanced ideas. Third grade had 12 above grade level topics, fourth grade had 10 above level topics, and fifth grade had four above grade level topics. The number of above grade level topics decreased as the students' grade level increased, but only slightly from third to fourth grade. The nature of a split grade class for the fourth/fifth grade class may have caused the fourth grade exposure to new topics to be higher than expected and the fifth grade number to be lower than expected. Perhaps the presence of fifth graders raises the bar of instruction for the fourth graders, and the presence of fourth graders lowers the bar for the fifth graders. Schools should consider whether or not a mathematically gifted student would benefit from being placed in a split level class, and only place them strategically.

Despite the relatively high exposure of above grade level topics noted in these observations, students' self-reports on the challenge level of the problems were low, even for problems that offered above grade level opportunities. Students' self-reports on the challenge level of the entire lesson (that included the class discussion) were higher but still on the low side for the top tier students and middle of the range for the second tier students. This finding may be an indication that the advanced students, particularly the top tier students, had mastered some of these above grade level topics already. Students' self-reports did claim that they greatly enjoyed the math lessons, so perhaps the exposure to and discussion of the above grade level topics kept them interested.

A final consideration with implications for increasing the challenge of advanced students is that several topics that were above grade level at the time of this study will be introduced in earlier grades with the Common Core Standards (CCSSI, 2010). As grade levels realign math topics with the Common Core Standards, teachers may need to reconsider which topics will challenge mathematically gifted students.

Self-report of learning new mathematical ideas. It is interesting to note that although math topics that were above grade level standards were mentioned at every lesson for the third and fourth graders, the students' self-reports of learning new mathematical ideas presented conflicting data. The self-reports of the top tier students claimed they learned new ideas only 57% of the time, and the second tier self-reports claimed learning new ideas 73% and 80% of the time, respectively. The fifth graders were exposed to above grade level standards in five of the seven (71%) lessons. This was close to the percentage reported by second tier fifth graders, but the top tier student reported learning a new idea only 29% of the time. It may be that these above grade level topics were discussed earlier in the year and were no longer new to *these* students. However, recalling that 72% of these reported new ideas were encountered during the strategy-sharing discussion time further describes the opportunities for mathematical challenge that the discussions offered.

Interest in other students' strategies. I proposed earlier that a student demonstrating interest in another student's strategy is tangentially related to learning something new because it suggests the willingness of a person to learn. In the third grade class, 82% of the students reported particularly liking another student's strategy. In the fourth/fifth grade class, the percentage dropped to 46%. However, both results show strong student interest in others' work. The difference in the two percentages may suggest that the strategy-sharing format of the third grade class may be preferable to that of the fourth/fifth grade class. Ms. B had her selected third graders write their strategies on a long white board and all strategies stayed on the board during the discussion, along with what was added during the discussion as connections were made between strategies. I interviewed Ms. B's students in the classroom where they had access to the board when asked if they like any of the other strategies in particular. Ms. K collected the students' papers that she wanted to share. She then projected them on a screen, one at a time, using a document camera. She would overlap two strategies when talking about connections between them. When the discussion ended, the strategies were no longer visible. It could be beneficial to the students to have full view of all the strategies for the sake of comparison.

Discussion of the Extent of Challenge

Taylor (2009) stated that mathematical challenge is a hard construct to measure. Although I could count the frequency of teacher extensions and connect them to the use of higher order thinking skills to imply increased intellectual demand, I could not measure the degree to which these extensions increased the intellectual demand with each student, other than by the self-report. The teachers provided a classroom environment that emphasized mathematical relationships, focused on higher order thinking skills to increase the intellectual demand, and exposed to students to new mathematical ideas. Seemingly, the three components of the operational definition of mathematical challenge were present, and the students were on a promising path toward expertise. Yet the mathematically advanced students did not claim to feel challenged. Perhaps there is an inherent feeling within gifted students always to be reaching for something more, thus they honestly do not feel completely satisfied or challenged (S. N. Kaplan, personal communication, September 26, 2011). However, assuming that the students' perception of low challenge is accurate, there must be a deficit in some element of the mathematical environment these exemplary teachers provide.

A Weak Link in the Problems Assigned

Students' reports of low challenge and their ability to quickly finish problems suggest the assigned problem as a weak link, in spite of the masterful problemization that occurred in the lessons. The literature review emphasized the importance of elevating the challenge level of the task assigned to mathematically gifted students (Diezmann & Watters, 2000; Henningsen & Stein, 1997; Tomlinson, et al., 2009). Elevating the challenge level of the task in a CGI classroom beyond the selection of number choices may be a key to elevating the challenge to be experienced by advanced students.

Underestimating the Mathematical "Gift"

The examination of students' problem-solving strategies showed that the advanced students comfortably used mathematical notation to express relational thinking, whereas the regular students did not. This indicates that the advanced students responded to their teachers' diligent encouragement to work on notation and relational thinking. It is unlikely that third graders would ordinarily use such extensive notation in their solutions if their teacher had failed to encourage it. Yet, ability to formalize ideas such as expressing relationships using mathematical notation is one of Krutetskii's (1976) characteristics of mathematical giftedness. Thus, these students may have readily gravitated to the practice of notation when it was introduced by their teachers, and not considered it a challenge per se, even in 3rd grade. Perhaps educators underestimate how quickly mathematically gifted students catch on to concepts. What teachers consider challenging to mathematically gifted students may be right in the realm of their ability rather than challenging, or challenging for only a short while.

Even though educators understand the constructivist view that students have prior knowledge and that instruction should build on that prior knowledge, perhaps the depth of the mathematically gifted students' prior knowledge is underestimated in a mainstream classroom, as is the rate at which these students learn. Noting the trend that the students' perceived level of challenge decreased with the older students may suggest that these students are absorbing information at a "compound rate" rather than a "simple rate" and that educators' demands are not keeping up with the pace of the students' minds. The students in this study were exposed to above grade level topics and had teachers who pressed for higher level thinking, yet still rated a low to mediocre level of challenge; this may be more indicative of how much they are capable of rather than a deficiency of any instructional approach.

Perhaps building on the prior knowledge of students who have a great deal more prior knowledge than other students entails 1) teachers' learning more about the prior knowledge that mathematically gifted students have, and 2) providing more complex problems to more fully engage the prior knowledge. With the emphasis on mathematical communication and understanding students' thinking in a CGI classroom, CGI teachers already are quite knowledgeable of what their students know and how they think. They are, therefore, well-poised to look more deeply. Assigning tasks that are more complex may allow for deeper student knowledge to be revealed than would otherwise surface from a simpler task. With the principles of Cognitively Guided Instruction, teachers gain more understanding of their students' to further know how to guide their progress in the students' zone of proximal development.

Conclusions from a Somewhat Paradoxical Situation

One key to answering the central question regarding the extent to which mathematically gifted students are challenged in CGI classrooms lies in resolving what may first seem like a paradoxical situation presented by the self-reports. On one hand, the advanced students did not feel very challenged by the problems they were assigned. Their ratings of the overall lesson were higher than the ratings of the problems, but still only 2.7 for all advanced students, and 2.1 for the top tier advanced students, on the 5 point scale. Although Alana spoke of how she liked the strategy-sharing discussions, she also said, “but not when there is a lot of time spent on easy strategies.” Yet when students were asked to rate how much they *enjoyed* the day’s lesson, their responses were almost invariably a “5”. A possible paradox arises here, in that the students did not feel very challenged yet they really enjoyed the lessons. As Barbeau and Taylor (2005) stated, “. . . a challenging situation may not necessarily be difficult, but it must be interesting and engaging” (p. 126). However, a student could certainly enjoy a math lesson without feeling challenged by it. For instance, playing math baseball and speedily calculating easy problems to get to the next base, or dividing up real cookies to show fractions is pleasurable but easy. Perhaps there is no paradox at all. Consider, though, that these students were not playing math baseball nor dividing real cookies. They were engaging in mathematical discussion and writing advanced mathematical notation and found it very enjoyable. There could be a deeper significance here.

I suggest that the reason that students gave “enjoying the lesson” the highest rating was that they found a mentor in their teacher. Perhaps they found a kindred spirit in someone else who loves math as much as they do. They found someone who was interested in their mathematical thinking and was not only willing to discuss mathematical ideas but could keep up with their arguments and their questions. These teachers provided an environment in which exploring and talking about mathematics was enjoyable for these students. So where a lower level of challenge might have otherwise caused a lack of interest, the exemplary mentor teacher sustained students' interest. Mentorship is a recommended scaffolding strategy for students at the practitioner level of Ascending Intellectual Demand (Hedrick & Flannagan, 2009). With the idea of mentoring in mind, I returned to interview students again after analyzing the data for this study. When asked if they admired their teachers for their knowledge of math and their interest in helping them learn math, they responded favorably. Dominic said,

> I like math all the time and she made me like it more by showing me new strategies like letting me notate down my strategies or if I can use a better strategy. She asks me questions like 'how did you do this?' and 'how did you times this so fast?'

Natalie commented, "She also tells us what math she learns like at her workshops."

Anita said it was important that her teacher likes math because "if she likes math it means she can teach us more about math. She can help us with any questions and help us get advanced with equations." I followed up by asking Ms. K if she felt that her students see her as a mentor. She responded,

> I can definitely see the kids seeing us as a mentor because my kids really rely on me when they want to try a new strategy or push themselves to a more efficient or advanced strategy. They rely on me to say if it's more efficient or more reasonable. Sometimes they just look up to us wanting to know if it's advanced or not.

Participant Frameworks Revisited: The Mentor-Mentoree Relationship

Empson (2003) reported the importance of participant frameworks to the animation of low-achieving students in a classroom that focused on children's mathematical thinking. She found that, "the use of task-based participant frameworks to analyze student engagement and learning in mathematics classrooms can provide useful insights into the nature of success and failure in mathematics" (p. 337). In her study of low achieving students, the mathematical success of the students depended on their interaction with the teacher. "Under other circumstances – for example, working with a teacher who did not recognize the mathematical potential of informal strategies or attempting to solve problems that did not engage what they knew – one can easily imagine their failure" (p. 337-338). Although this statement was written with low-achieving students in mind, I propose that it has significant meaning for high-achieving students, too, and their interaction with a mentoring teacher. But in the case of mathematically gifted students, 'engaging what they know' when they know a lot requires a teacher with a solid understanding of mathematical content and pedagogical knowledge so as not to risk failure of *not* taking these students to the next level. The knowledgeable teacher would engage mathematically gifted students by inviting them into a participant framework in a mentor-mentoree relationship, devoted to challenging mathematical thought within the CGI classroom culture.

So why is this so compelling? Clasen and Clasen (2003) state, "the mentor serves as teacher, advisor, and role model, guiding the youth toward excellence and helping validate both the individual and the talent" (p. 265). Although the degree of challenge experienced by the students was not high in these two case studies, other elements of the CGI experience with an exemplary teacher kept students on a path toward expertise. The exemplary teachers encouraged mathematical discourse which allowed students to think and discuss like a practitioner in the discipline of mathematics, along the continuum of Ascending Intellectual Demand. With a teacher-mentor, the CGI experience puts mathematically gifted students in their element, beyond what a traditional math class would tend to offer.

The Research Question: To what Extent were the Mathematically Gifted Students Challenged?

In short, these students were in an environment that supported the components of the operational definition of mathematical challenge to a respectable degree: focus on mathematical relationships, exposure to new ideas, and attention to Ascending Intellectual Demand. The rich mathematical discourse, attention to notation of mathematical relationships, and the opportunity to have an exemplary teacher as a mentor makes a CGI classroom a promising environment for mathematically gifted students. The students' voices said, however, that although they enjoyed the CGI math lessons, they were not particularly challenging. This discussion has already suggested some elements of the lessons that need attention. The following continues that discussion with the goal of determining how teachers can improve on what already is a solid foundation for challenging mathematically gifted students in a mainstream classroom.

The CGI/AID Framework: Moving to the Next Level of Ascending Intellectual Demand

It is useful to turn to the framework for guidance to further challenge these advanced students. The CGI/AID Framework (Figure 3 in Chapter 2) aligned the levels of CGI problem-solving strategies with the levels of Ascending Intellectual Demand from novice to apprentice to practitioner to expert. This alignment allowed the opportunity to describe mathematically gifted students in a CGI classroom primarily as practitioners, one step away from the expert level, relative to elementary mathematics. Thinking as practitioners of mathematics, they were making connections, understanding relationships, using efficient strategies for problem solving, and formulating conjectures (the presence of the latter two characteristics were the main criteria for setting them apart from the previous level of apprentice).

The mathematically gifted students' high ability along with the opportunities of the CGI classroom allowed these students some entry to the expert level of the Ascending Intellectual Demand continuum. This was initially evidenced by their problem-solving strategies that exhibited computational fluency and relational thinking. What follows is a discussion of each of the eight characteristics of the expert level of Ascending Intellectual Demand (Hedrick & Flannagan in Tomlinson et al., 2009, p. 269) in relation to the students' experiences, or lack of, to help pinpoint areas for increasing the challenge level in CGI classrooms.

Expert Characteristic #1: Uses Computation as merely a Means to an End

Recall that expertise is relative to the grade level, so these students were fluent with respect to grade level standards of computation. They were also fluent in some above grade level computation. Students efficiently used computation as a tool for solving problems.

Expert Characteristic #2: Moves easily among the Fields of Mathematics through the Use of Macroconcepts

The Parallel Curriculum Model defines a “macroconcept” as a general idea or understanding that extends across disciplines or topics (Tomlinson et al., 2009, p. 127). Thinking in terms of relationships is a macroconcept in that it spans all areas of study in mathematics. Relational thinking, or relating numerical expressions with an underlying understanding of properties of operations, is also a macroconcept, giving an algebraic focus to a variety of math topics. The strong emphasis on exploring mathematical relationships, including relational thinking, and making connections between topics in a CGI classroom gives these students entry to the expert level of Ascending Intellectual Demand. Mathematically gifted students’ aptitude for understanding mathematical relationships and making connections between topics clearly matches them with this characteristic of expertise.

Expert Characteristic #3: Questions existing Mathematical Principles

The extensive discussion phase of the CGI lesson offered opportunities for questioning mathematical ideas, especially as new concepts were developed. Observation of the advanced students during these discussions revealed their interest and ability in questioning the principles of elementary mathematics, and making related conjectures.

Expert Characteristic #4: Seeks Flow through the Manipulation of Tools and Methods in Complex Problem Solving

Flow is a term that was coined to describe being deeply engrossed in an intrinsically enjoyable activity that has a stimulating amount of challenge commensurate with one's ability (Csikszentmihalyi, 1990). Considering how quickly many of the advanced students finished the assigned problems, the nature of the problem solving cannot likely be described as complex enough to have allowed for a sustained period of engagement in the problem itself. This further points to the assigned problem as needing to offer more complex challenge in order to move students beyond the level of practitioner. However, the engagement and enjoyment of the advanced students in the extensive discussion phase of the lesson was high, and that experience could likely be considered flow. Ideally, the assigned problems should be challenging and engaging enough such that the experience of solving the problem is as rewarding as discussing the solutions.

Expert Characteristic #5: Seeks the Challenge of Unresolved Problems and the Testing of Existing Theories

Relative to elementary mathematics, an unresolved problem could be interpreted as one that contains a mathematical idea new to the student but that does not have a solution that is readily apparent to the student without more engaged investigation. There were a few times when I observed advanced students who were not sure how to begin a problem. In most other cases, however, the solution seemed readily apparent as they were able to begin their strategies immediately. Providing more complex problems could develop the students' skills for planning an approach to a problem whose solution is not clear at first.

'Testing of existing theories' relative to elementary mathematics could be interpreted as the next logical step to 'questioning existing principles' (characteristic #3). Questions that lead to exploration and developing conjectures should lead to testing of the conjectures, or proof and justification. If a theory is interpreted as an explanation, students would investigate and test the explanation, looking for justification of the existing explanation. Justification was a common theme in the CGI classrooms but I would suggest it could be taken to a higher level for the mathematically gifted students, with more targeted teacher guidance.

Expert Characteristic #6: Links Mathematical Principles to other Fields through Real-World Problems

The assigned problems receive further scrutiny with this characteristic of Ascending Intellectual Demand, perhaps also revealing the most promising suggestion for change. Although sharing sandwiches and pizza are part of the real world for elementary students, they do not offer the type of interdisciplinary application of mathematics that a student would need to experience to move to the next level of expertise. Assigning problems with more authentic real-world applications could increase the challenge level and engage the students for the length of Phase 2.

Expert Characteristic #7: Views Unanswered Questions in other Disciplines through the Concepts of Mathematics

The CGI lessons observed did not include topics from other disciplines, although Ms. B referred to some interdisciplinary activities that her classes do throughout the year that utilize mathematics. At the elementary level, unanswered questions in other disciplines could reveal themselves to students either during other subject lessons or during their own research investigations. Individual students, or those in groups, who find a question of interest which could have a mathematical solution could be encouraged to investigate that question further. This idea would involve a longer term assignment that would extend beyond a typical class period. If students who typically finish problems early were involved in a long term research investigation, for instance, they could work on it after they finish the assigned problems. This type of extension suggested in the expert level of Ascending Intellectual Demand may suggest that a mathematically gifted student may need something beyond the realm of a typical CGI lesson, as enriching as it is, to reach toward expertise.

Expert Characteristic #8: Uses Reflection and Practice as Tools for Self-Improvement

Not surprisingly, the CGI emphasis on understanding students' thinking has both classroom teachers well-prepared for using reflection as a tool for students' self-improvement. In the interviews, students showed interest in and the ability to reflect upon their own problem-solving strategies, other students' strategies, and how they felt about the lesson of the day. The strategy-sharing discussion allowed students to reflect on their own solutions in comparison to others. The idea of reflection could be more formally put into place in a CGI classroom with teacher-student conferencing on a regular basis or by students' writing in journals.

The idea of using practice as a tool for self-improvement needs further exploration in a CGI classroom. With the CGI lessons having a minimum length of one hour including an extensive discussion period, the students were certainly experiencing ample practice at mathematical discourse. However, with only a few problems assigned, despite the fact that problemization occurred, the mathematically gifted students may want and need more practice on challenging mathematics problems. They may need problems beyond those the rest of the class is doing to advance them to the next level in mathematical thinking. It could be that the rich mathematical discourse in a CGI lesson is enough to sustain the interest and motivation of the mathematically gifted, but not to significantly advance it.

Summary of Recommendations

Increase the Challenge Level of the Problem

Several indicators in this study pointed to the assigned problem as the root of the students' perception of low challenge, particularly for top tier advanced and higher grade level students. These students need more significant differentiation, which may mean a shift from using number choices to differentiate to using different problems altogether. More challenging, complex problems may keep students engaged for the entire problem-solving time and increase the chances that they can experience flow in the act of problem-solving. The previous analysis of characteristics of the expert level suggests ways to accomplish this goal, such as real-world applications of mathematics that connect with other disciplines. Challenging mathematically advanced students with open-ended problems was a common suggestion in the literature review.

Provide Feedback

When students are given differentiated assignments, it is essential that students receive feedback to validate the worth of the assignment for the student and to monitor and guide the students' progress. The differentiated instruction for the two CGI classrooms in this study was providing number choices of increasing difficulty for the root problems. Although only the advanced students completed the problems using all the number choices, the teachers were skilled at bringing their solutions into the class discussion and making connections with the easier solutions. If challenging mathematically gifted students inevitably requires assigning them problems different from what the other students work on, it should be done in a way that does not sacrifice the rich connections that are made during the whole-class discussion that benefit all levels of students. The problems should be related to the other students' problems so that incorporating some aspect of them into the discussion is still possible and productive, while still offering feedback to the advanced students who worked on the problems. However, it is likely that there are elements of advanced problems that are not appropriate for the rest of the class. Teachers then should make a point of providing feedback during the problem-solving process or during individual conferences at other times of the day.

Small Group Challenges

The advanced students favored the idea of working with other advanced students in groups. Whether they were working on the daily problem or a special challenge problem, with the occasional facilitation of the teacher, they stayed engaged with mathematics for the length of the problem-solving time. Teachers should be attentive to ideas that come up during discussion time that particularly intrigue the advanced students, and pose a group challenge investigation of that topic the next day. These are opportunities to elevate the challenge level as students prove conjectures and make generalizations working with their peers. The group could be made up of top tier students with an ongoing rotation of second tier students into the group.

Vertical Alignment between Grade Levels

With mathematically gifted students coming into a classroom each year with advanced knowledge and abilities, it is important that the new teachers find out from the former teachers what advanced work the student has mastered to avoid unnecessary repetition in the upcoming year. Considering how the top tier 5th grader reported the lowest challenge, and that he would be in middle school the next year with the possibility of taking Algebra early, conversations with middle school math teachers may be important in deciding how to challenge a student in the last year of elementary school. CGI teachers also should not rule out the possibility that certain mathematically gifted students are capable of being accelerated to a higher grade for mathematics. Pre-assessment and above level testing (Stanley, Lupkowski, & Assouline, 1990) can aid in these situations.

Mentor-based Clustering for Top Tier Advanced Students

The benefits offered by CGI to mathematically gifted students in a CGI classroom have been clarified in this study, yet so have the shortcomings. I propose that the deficiency in challenge felt by the top tier advanced students in a CGI classroom could be more easily remedied by implementing mentor-based clustering. Mentor-based clustering would cluster a small group of top tier advanced students with the grade level CGI teacher who can best challenge them mathematically and serve as their mentor. Creating a critical mass of the highest ability students makes it easier and more likely that teachers will differentiate for their unique needs and capabilities (Winebrenner, 2001; Winebrenner & Brulles, 2008). This grouping practice could be implemented in a CGI classroom without compromising other principles close to the CGI philosophy.

Limitations

This "best practices" study focused on two math classrooms of exemplary CGI teachers. The results would not necessarily be repeated in another CGI classroom with teachers who did not have as much CGI experience, as much mathematical content knowledge to keep up with the advanced thinking of these students, nor as much enthusiasm and inclination to press for challenge as these teachers exhibited. These teachers were aware of their mathematically advanced students and had considered ways to challenge them even before I had entered their classrooms. They both admitted, to some degree however, that my presence in their classrooms with the intent of this study further pressed their awareness of challenging mathematically gifted students. I suggest that it is possible for a CGI teacher to be exemplary yet not be aware of the needs of mathematical gifted students, and results may be different if such a teacher were studied instead. Furthermore, this study took place in the last six weeks of the school year following the benchmark testing. Although the teachers said they covered topics that they normally would have covered, the relief of the pressure from standardized testing may have influenced what they chose to do. Also, having been in the classroom for a condensed period of time, approximately four weeks per teacher, rather than multiple visits over a longer period of time, prevented the gaining of a long term perspective.

Implications for Future Research

The teachers in this study were exemplary in their ability to understand their students' mathematical thinking and for their methods of extending their students' thinking to higher levels. The teachers' also had enough depth in their own mathematics knowledge to recognize the advanced thinking of their mathematically gifted students. Likewise, they were artful in orchestrating a productive mathematical conversation with both individual students and a class of students. This study sought out *exemplary* teachers to see what was possible for challenging mathematically gifted students, but it could be useful to explore the opposite. Studying randomly chosen CGI teachers on a broader scale could further describe the experience of mathematically gifted students in CGI classrooms. Surveying CGI teachers on which strategies they use to challenge their advanced students might suggest interventions that could increase the challenge levels for such learners. Specific interventions recommended by this study such as mentor-based clustering or increasing the complexity of the assigned problem, could be studied for their effectiveness in CGI classrooms.

Closing Summary

This study examined the issue of challenge for top tier and second tier mathematically-advanced students in two CGI classrooms taught by exemplary teachers. I developed an operational definition of what it means for an elementary student to be challenged in mathematics learning. Although there were many elements of the CGI classrooms that created a challenging environment, and the advanced students reported enjoying the lessons, they did not report feeling very challenged. I created a conceptual framework in which I aligned levels of Ascending Intellectual Demand with levels of CGI problem-solving strategies to help describe the level of challenge experienced by these students, and to suggest what could be done to increase the challenge. A central conclusion was that the complexity of the assigned problems should increase for the advanced students, particularly for the mathematically gifted students.

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Appendix A English Language Development Assessment (ELDA)

Explanation of Composite Proficiency Levels for grades K, and 1-2

Level 1 – Pre-functional indicates that the student who is limited English proficient:
• may understand some isolated spoken words, commands, and questions, but often requires non-verbal cues and frequent repetition
• may speak or repeat common phrases and words and can ask one- to two-word questions
• demonstrates an understanding of concepts of print (left to right, top to bottom) and can follow one-step directions depicted graphically
• achieves written communication only through drawing pictures; may be able to copy letters or words successfully; or may form letters from memory but is unable to transmit meaning

Level 2 – Beginning indicates that the student who is limited English proficient:
• understands short, simple oral statements on familiar topics; follows simple multi-step directions; requires frequent repetition and rephrasing
• predominantly uses formulaic speech patterns and memorized phrases; responds to questions with one- to twoword answers
• begins to identify the names of letters; begins to recognize the different functions of words; can follow multi-step directions depicted graphically
• achieves written communication through drawing pictures or dictating words; can revise or edit with teacher support; commits frequent mechanical errors

Level 3 – Intermediate indicates that the student who is limited English proficient:
• understands sentence-length statements and questions; understands main idea and some details from
conversations and simple oral texts; is beginning to develop key vocabulary, interpret meaning, and understand some idioms
• restructures learned language into original speech; has limited vocabulary and marked errors in speech; can use language to retell, describe, narrate, question, and instruct, but not fluently
• comprehends single words and simple text, as well as simple sentence structure and simple compounding; recognizes the different functions of words, and that words have multiple meanings
• participates in writing activities with teacher support; writes simple and compound sentences; is beginning to write with phrases; uses transition words; can edit, usually with teacher support; most writing is descriptive, expository, procedural, or narrative

Level 4 – Advanced indicates that the student who is limited English proficient:
• understands most school/social conversations; grasps main ideas and relevant details; comprehends most gradelevel vocabulary and idioms; is developing a wide range of academic vocabulary in the content areas
• restructures language to communicate orally; uses connective devices; responds in a mostly coherent, unified, and sequenced manner; has sufficient vocabulary to communicate in most situations; is fluent but may hesitate or make errors in spontaneous communicative situations

• reads familiar text with little support, but needs support to comprehend unfamiliar text; identifies all story elements; is beginning to read across text types and apply what they read to other activities
• participates in writing activities with minimal support; restructures known language in writing; writes mostly coherent, unified, and sequenced sentences; uses connective devices and a range of grammatical structures, with some errors; possesses a strong social vocabulary and a functional academic vocabulary; writes and edits all text types

Level 5 – Full English Proficiency indicates that the student who is limited English proficient:
• understands most grade-level speech, both social and academic; understands main ideas and relevant details at a level comparable to a native English speaker at the same grade level; has a broad range of vocabulary including idiomatic language
• responds orally in a coherent, unified, and sequenced manner; uses a variety of connective devices; understands and uses a range of simple and complex grammatical structures; has grammar and vocabulary comparable to a native English speaker at the same grade level and shows flexibility, creativity and spontaneity speaking in many contexts
• participates in reading activities at grade level comparable to their English speaking peers with little teacher support; reads across text types; has an increasing range of social and academic vocabulary; understands multiple word meanings
• participates in writing activities with no teacher support; edits complex sentence structures with some errors; utilizes precise social and academic vocabulary; understands the use of nuance and subtlety in writing for different audiences

Explanation of Composite Proficiency Levels for grades 3-12

Level 1 - Pre-functional indicates that the student who is limited English proficient is:
• Beginning to understand short utterances
• Beginning to use gestures and simple words to communicate
• Beginning to understand simple printed material
• Beginning to develop communicative writing skills

Level 2 - Beginning indicates that the student who is limited English proficient can:
• Understand simple statements, directions, and questions
• Use appropriate strategies to initiate and respond to simple conversation
• Understand the general message of basic reading passages
• Compose short informative passages on familiar topics

Level 3 - Intermediate indicates that the student who is limited English proficient can:
• Understand standard speech delivered in school and social settings
• Communicate orally with some hesitation
• Understand descriptive material within familiar contexts and some complex narratives
• Write simple texts and short reports

Level 4 - Advanced indicates that the student who is limited English proficient can:
• Identify the main ideas and relevant details of discussions or presentations on a wide range of topics
• Actively engage in most communicative situations familiar or unfamiliar
• Understand the context of most text in academic areas with support

• Write multi-paragraph essays, journal entries, personal/business letters, and creative texts in an organized fashion with some errors

Level 5 - Full English Proficiency indicates that the student who is limited English proficient can:

• Understand and identify the main ideas and relevant details of extended discussion or presentations on familiar and unfamiliar topics

• Produce fluent and accurate language

• Use reading strategies the same as their native English-speaking peers to derive meaning from a wide range of both social and academic texts

• Write fluently using language structures, technical vocabulary, and appropriate writing conventions with some circumlocutions

Arkansas Department of Education, (2010). *Guide to understanding scores on the English Language Development Assessment (ELDA)*. arkansased.org/testing/assessment/elda.html

Appendix B Semi-Structured Student Interview

Questions to be asked one student at a time, at the beginning of the study:

1. I understand that your school considers you as “advanced” in math. Do you feel like you are good at math?
2. Do you remember always being good at math?
3. Is math one of your favorite subjects? If so, what do you like about it?
4. What other subjects do you like a lot?
5. I know your class spends a lot of time discussing different strategies for solving math problems. Have you noticed if your ways of solving problems are very different from your classmates?
6. During a math lesson, do you think you spend more time than your classmates solving the assigned problems or less time?
7. Give me some examples of what it means to you to be challenged in math class? (or how do you feel when you are being challenged in math?)
8. Do you feel like you are being challenged in your math class? (may need explanation such as “do the problems take a fair amount of thought to do?”)
9. If you finish the assigned problems early (before the discussion time), what do you typically do?
10. Do you have any ideas as to what would make math class more challenging for you?
11. Would you like to be grouped with other students who are as good at math as you are?
12. Is there anything else you would like to tell me about “you and math”?

Relational Thinking Problems to do (for researcher to have a baseline sense for how the participants think):

Before you begin this problem, I'd like you to read it to me and think about it for a moment before you start to solve it.

Whole numbers: 78 + 63 = ____ + 64 Time: ______

25 + 37 + 75 = _____ Time: ______

Fractions: 5 1/3 + 2 1/3 = 2 + ____ Time: ______

2 ¾ + 1 ¼ = _____ + 3 ¼ Time: _______

Post-Study Interview, exploring the idea of mentoring (performed at the beginning of the next school year)

1. Tell me what were the kinds of things that your teacher last year did to make you like math class?
2. What did you like best about the way that your teacher last year taught math?
3. Can you think of things that your teacher did last year that made math more interesting for you?
4. Can you think of things your teacher did last year that made math more challenging for you last year?
5. Can you think of things your teacher did last year to make you learn math better?
6. Did your teacher inspire you and encourage you to be even better at math?
7. Was your teacher last year someone you felt you could look up to, admire her for her knowledge of math, and you could ask her any math question and she would be able to help you with it? (also how much, on a scale of 1 to 5)
8. How important is it to you that your teacher likes math and is really interested in helping you learn math? (on a scale of 1 to 5)

Appendix C Semi-structured Teacher Interview

Questions to be asked at the beginning of the study:

1. Tell me about your background with CGI (professional development, how many years you have tried it in the classroom, etc.) and how many years you've been teaching.
2. How would you describe the characteristics of your mathematical advanced students?
3. At the beginning of the school year, was there one characteristic in particular that stood out and helped you identify them as mathematically advanced?
4. Have you taught math using a non-CGI approach?
5. (If yes,) was it any easier to identify your mathematically talented students in a CGI classroom?
6. How do your mathematically advanced students differ from your other students?
7. (If not answered above,) Can you describe how their problem-solving strategies differ from your other students?
8. Are than any students who are not labeled "advanced" for the mathematics benchmark whom you have found to be advanced mathematical thinkers?
9. In what ways do you try to challenge your mathematically talented students during a CGI lesson?
10. Do you have any students who you worry are not being challenged?
11. How do you arrange the seating of your advanced students?

Questions to be asked near the end of the study:

1. In these past few weeks, do you feel that your awareness, perception, and understanding of your mathematically advanced students have changed, and if so, in what ways?
2. Which of your strategies for challenging them do you think works best?
3. In the previous interview, I asked you to describe strategies that you use to challenge your mathematically advanced students. Which of these strategies do you think are the most effective (or do you feel that each one has its own merit?)
4. Which strategies do you typically use when you see that a student has solved a problem very quickly and is done?
5. How do you think the CGI approach benefits the mathematically advanced students?
6. Do you think there are any drawbacks of the CGI approach to learning math for the mathematically advanced students? Please describe, if so.
7. Beyond the CGI lessons, do you offer any other mathematical experiences to your mathematically advanced students (enrichment activities, challenge assignments, projects, etc.)?
8. How do you think that the CGI approach is beneficial to ELL students?
9. Do you think that there are disadvantages to this approach for the ELL students?
10. Some teachers feel that when they work with the gifted students, they neglect the struggling students, and when they work with the struggling students, they neglect the gifted students.

    a) Do you find it is difficult to reach both the mathematically advanced students and the struggling students in the same class?

b) Does the CGI approach make it any easier to handle both ends of the ability spectrum or does it hinder it?

11. Has every student had a chance to share strategies this year? Some more than others?

12. If you could easily make any change happen,

    a) what would you like to see happen to help keep mathematically talented students challenged?

    b) What resources would be useful to make this happen?

13. So far I have observed CGI lessons after benchmark exams. I have observed how, despite the fact that students only work on a few problems in a lesson, there is so much math that is discussed in the sharing time. I have also observed how you notice when the class seems to be struggling with a certain concept, you go back and review it. However, I don't have a clear picture of how you "cover the standards" throughout the year. Are most of your lessons CGI lessons and you ensure that the "content" is within? Are there some lessons that are more traditional? Do the students ever do practice sheets that involve more repetition of certain skills and concepts?

14. One additional question with respect to my being in the classroom after benchmarks: Between the fact that it was after benchmark testing so you did not need to be concerned about covering certain topics in your lesson, and the approach that we took with this study that we were pressing to see to what extent we could challenge the advanced students, how typical or not were the math lessons that I observed? i.e. were they longer than usual, were those problems typical of what you would have covered after benchmarks anyway, etc.?

15. Finally, did you sense that by pressing more toward the advanced students that you were "losing" any of the lowest ability students more than on a typical day?

16. Anything else that you would like to tell me about CGI and mathematically advanced students?

Post-study question:

Do you think your mathematically advanced students look up to you as a mentor in their learning of mathematics?

Appendix D Post-lesson Student Interviews

Questions to be asked after each math lesson (with the student work of the day in front of student):

1. What was your favorite part of today's math lesson?
2. Was there anything you didn't like about today's lesson?
3. Did you learn a new math concept today (something you hadn't known before)? (can point to their work)
4. Did you come up with a new strategy for solving today's problem or a mathematical idea that you had not thought of or used before?
5. Was there any strategy that one of your classmates shared during discussion time that you found particularly interesting? (did you learn anything new from it?)
6. On a scale of 1 to 5, how challenging (easy or hard) did you find today's problems to be? (5 is hardest)
7. On a scale of 1 to 5, how challenged did you feel today when you solved today's problems? (5 is the most challenged)
8. On a scale of 1 to 5, how much did you enjoy today's lesson?

Appendix E Coding/Record Sheet for Classroom Observation

Date: __________________ Teacher ________________

Math topic/problem:

(Red ink) Phase 1, Problem Posing begins ______
(Blue ink) Phase 2, Problem Solving begins _________ Problem Solving Ends __________
(Green ink) Phase 3, Strategy Sharing ends ________

Record times of students' finishing problems if finished within 5 minutes.
If exact time is not known, "2" means under 2 minutes, and "5" means under 5 minutes

Code name (initials)	Done with First problem	**Done with All problems**

Each participant is assigned a letter(or initials). Each letter noted on the checklist indicates a coding of that action involving the student represented by that letter. A checkmark also indicates coding of an action when it did not seem significant to associate with a particular student. The different colors of ink indicate in which of the 3 phases the event occurred.

Comments/Questions that indicate Higher-level Thinking: Phase 2 *(times: _____ to ______)*

Student	Teacher
• Student compares/critiques strategies with another student.	• Teacher asks student to compare.
• Student detects other student's or teacher's mistake or flaw in thinking.	• Teacher asks student to look for mistake or flaw in thinking.
• Student expresses a conjecture of or realization of a mathematical relationship (may be an "aha" moment).	• Teacher asks student to look for a relationship or make a conjecture.
	• Teacher encourages student to express relationship with mathematical notation, including relating to number properties
• Student makes a comment that indicates a generalization (inductive reasoning, formalized).	• Teacher encourages student to make a generalization.
• Student asks a "what if" question or shows curiosity for more knowledge beyond the main topic.	• Teacher asks student a "what if" question or another question to extend student's thinking
Other categories (student or teacher) below:	• Teacher asks student to justify their reasoning.
	• Teacher gives problem to the class that has number choices built in for the same root problem • Teacher gives extra problems to the class in case they finish the main problem early
	• Teacher gives student another problem to do (not given to rest of class)
	• Teacher asks student for a different strategy that is representative of student's ability.
	• Teacher gives student feedback on student's work (on extra problems?)

Comments/Questions that indicate Higher-level Thinking: Phase 1 (red), Phase 3 (green)

(times: _____ to _______)

Student	Teacher
• Student compares/critiques strategies with another student.	• Teacher asks student to compare.
• Student detects other student's or teacher's mistake or flaw in thinking.	• Teacher asks student to look for mistake or flaw in thinking.
• Student expresses a conjecture of or realization of a mathematical relationship (may be an "aha" moment).	• Teacher asks student to look for a relationship or make a conjecture.
•	• Teacher encourages student to express relationship with mathematical notation, including relating to number properties
• Student makes a comment that indicates a generalization (inductive reasoning, formalized).	• Teacher encourages student to make a generalization.
• Student asks a "what if" question or shows curiosity for more knowledge beyond the main topic.	• Teacher asks student a "what if" question or another question to extend student's thinking
Other categories (student or teacher) below:	• Teacher asks student to justify their reasoning.
	• Teacher gives problem to the class that has number choices built in for the same root problem • Teacher gives extra problems to the class in case they finish the main problem early
	• Teacher gives student another problem to do (not given to rest of class)
	• Teacher asks student for a different strategy that is representative of student's ability.
	• Teacher gives student feedback on student's work (on extra problems?)

Other Observation Notes: Teacher: Date:

Appendix F Problem Information Sheet

Per Student Reporting of Correctness of Solution, Relational Thinking, Strategy Use, Finish Time, etc.						
Name (5th) *pseudonym*	Scores TOMAGS/ 4th gr. Benchmark/ 5th gr. Benchmark *(advanced 697+)*	Problem 1 *3 kids share 4 sandwiches. How much for each kid? (3,4) (3,8) (6,20) (12,40) (6,10) (10,25)*	Problem 2 *Maria ate ½ bag of candy. Lori ate ½ bag of candy. Total eaten? (½, ½) (½, 1/4)(1/2, 1/8) 2 ½ ,3 ¼) (4 ½, 3 1/3)*	Problem 3 *12 cupcakes, 1/3 c. frosting, How much total? (12,1/3) (36, 1/3) (72, 1/3) (72, 2/3)*	Problem 4 *8 kids, ¾ pizza each, How much total? (8, ¾) (16, ¾) (20, ¾) (40, ¾)*	Problem 5 *12 cups cat food, 1 ½ cups per day, How many days? (12, 1 ½) Plus group challenge*
Geraldo	99th/775/774 Adv/Adv Highest TR	C C C C C (2) Number facts (mental)	C C C C C (2) Number facts (Mental)	C C C C (2) Number sentences RT	C C C C + *C* (2) Number sentences RT	C (2) Number sentence
Sylvia	84th/694/725 Adv /Adv Highest TR	C C C C X (5) Number sentences RT	C C C C (5) Number sentences Equivalent fractions	C C C X Number facts RT	C C C C (2) Direct modeling Chunking RT	C (2) Number sentence
Allen	73rd/682/767 Adv /Adv TR	C C C C 0 Direct modeling RT	C C C 0 0 Number sentences Direct Modeling	C C 0 0 Direct modeling Number facts	C C C (c) Direct modeling Chunking	
Roberto	73rd /615/633 Prof/Prof HighestTR	C C C C 0 Number sentences RT	C C C C C (5) Mental math Direct modeling	C C C X (5) Number facts RT		C (2) Chunking
Kara	39th /688/661 Adv /Prof TR	C C C C 0 Number sentences Direct modeling	C C C C X Number facts Equivalent fractions	C C C C Table Number facts RT	C C 0 0 Direct Modeling Chunking	C (5) Chunking
Ingrid	37th/661/_@__ Adv/		C C C C (c) Direct modeling Equivalent fractions	C C C 0 Chunking Table Number sentences	C C X 0 Chunking Table RT	C Direct modeling Number sentence
Ruby	___/___/685 /Prof	C C C C C Number sentence Direct modeling	C C X C C Direct modeling	C C 0 0 Direct modeling Table Number facts	C C C 0 Direct modeling Chunking RT	C Direct modeling Table
Sharina	___/661/677 /Adv/Prof	C C 0 0 0 Direct modeling	C C C C C Direct modeling Equivalent fractions Number sentences	0 0 C 0 Number facts Chunking	0 0 C C Number sentence	C Direct modeling Chunking
Katey	___/___/562 /Bas	X X 0 0 0 Direct modeling	X X X 0 0 Direct modeling	X X 0 0 Direct modeling	X X X 0 Table	
Time allotted for Problem Solving		*20 minutes*	*25 minutes*	*21 minutes*	*35 minutes*	*29 minutes* *Includes small group challenge*

C: correct answer
C: correct answer gotten by using relational thinking based on previous answers
(c): problem not finished but was on the way to a correct answer
X: incorrect answer X: incorrect answer but was using relational thinking based on previous answers
X+: incorrect answer but strategy was sound and could have led to correct answer
0: problem not attempted
(2) solved in under 2 minutes (5) solved in under 5 minutes

Name (5th) *pseudonym*	Scores TOMAGS/ 4th gr. Benchmark/ 5th gr. Benchmark *(advanced 697+)*	Problem 6 *Lawnmower holds 3 gallons gas. Each lawn takes ½ gal. How many lawns? (1/2, 1/4, ¾, 3/8)*	Problem 7 *¼ + 1/8 cup syrup. Total? If there is 1 cup of syrup is it enough for 2 days? (1/4, 1/8) (1/4, 1/3) Plus group challenge*
Geraldo	99th/775/774 Adv/Adv Highest TR	C C C C +ext (2) Number sentences RT	C C (2) Number sentences Equivalent fractions
Sylvia	84th/694/725 Adv/Adv Highest TR	C C C C (5) Number sentences RT	C X Direct modeling
Allen	73rd/682/767 Adv/Adv TR		C X Direct modeling
Roberto	73rd/615/633 Prof/Prof Highest TR	C C C C (5) Counting Number sentence	
Kara	39th/688/661 Adv/Prof TR	C C C C Direct modeling	C 0 Direct modeling
Ingrid	___/661/___ /Prof/Prof	C (X) C C Chunking Direct modeling	X X Direct modeling
Ruby	___/___/685 /Prof	C C 0 0 Counting	X 0 Direct modeling
Sharina	___/661/677 /Prof/Prof	C C X 0 Chunking	C X Direct modeling
Katey	___/___/562 /Bas	X X X X	
Time allotted for Problem Solving		*26 minutes*	*38 minutes*

C: correct answer
C: correct answer gotten by using relational thinking based on previous answers
(c): problem not finished but was on the way to a correct answer
X: incorrect answer X: incorrect answer but was using relational thinking based on previous answers
X+: incorrect answer but strategy was sound and could have led to correct answer
0: problem not attempted
(2) solved in under 2 minutes (5) solved in under 5 minutes

Name (4^{th}) *pseudonym*	Scores TOMAGS/ 4^{th} gr. Benchmark/ 5^{th} gr. Benchmark *(advanced 640+)*	Problem 1 *3 kids share 4 sandwiches. How much for each kid?* *(3,4) (3,8) (6,20) (12,40) (6,10) (10,25)*	Problem 2 *Maria ate ½ bag of candy. Lori ate ½ bag of candy. Total eaten?* *(½,½)* *(½, 1/4)(1/2, 1/8) 2 ½ ,3 ¼) (4 ½, 3 1/3)*	Problem 3 *12 cupcakes, 1/3 c. frosting, How much total?* *(12, 1/3)(36, 1/3) (72,1/3) (72, 2/3)*	Problem 4 *8 kids, ¾ pizza each, How much total?* *(8, ¾) (16, ¾) (20, ¾) (40, ¾)*	Problem 5 *12 cups cat food, 1 ½ cups per day, How many days?* *(12, 1 ½)*
Andre	97^{th}/731/695 Adv/Adv Highest TR	C C C C C Direct modeling Number facts RT	C C C C C Direct modeling	C C C C (2) Number sentences RT	C C 0 0 Chunking RT	C (5) Guess and check Decomposing Number sentence
Natalie	90^{th}/698/759 Adv /Adv Highest TR	C C C C C Direct modeling Number facts RT	C C X C C Direct modeling Number sentences	C C 0 0 Number sentences RT	C C C C (5) Chunking RT	C (5) Chunking
Alana	90^{th}/731/732 Adv/Adv TR	C C C C C Number facts Number sentence RT	C C C C C Direct modeling Number sentences	0 0 0 (X) Direct modeling Chunking	C C C C Chunking Table	C + *C* (2) Chunking
Anita	70th/681/682 Adv/Adv TR	C C C C C Number facts Number sentences RT	C C C C C Direct modeling Number sentences	C C 0 0 Number sentence Chunking	C C C C + *C* Number sentences Grouping RT	C Chunking Number sentence
Katerina	12^{th}/745/646 Adv/Adv TR	C C C C C Direct modeling Number facts	C C C C C Direct modeling	C C C 0 Direct modeling	C C C C Direct modeling RT	X Direct modeling
Henry	/ /663 /Adv	C C X O O Direct modeling	C C C C 0 Direct modeling	C C 0 0 Direct modeling	X C X C Direct modeling	
Gloria	/ /588 /Prof		X X 0 0 0 Direct modeling	X 0 0 0 Direct modeling Number sentences	X 0 0 0 Direct modeling	C Direct modeling
Lala	/ /572 /Prof		C X X 0 0 Direct modeling	X X 0 0 Direct modeling	X 0 0 0 Direct modeling	
Jenni	/ /550 /Bas	C X O O O Direct modeling	X X 0 X 0 Direct modeling	X X X 0 Direct modeling Counting	X X 0 0 Direct modeling	
Time allotted for Problem Solving		*20 minutes*	*25 minutes*	*21 minutes*	*35 minutes*	*29 minutes*

C: correct answer
C: correct answer gotten by using relational thinking based on previous answers
(c): problem not finished but was on the way to a correct answer
X: incorrect answer X: incorrect answer but was using relational thinking based on previous answers
X+: incorrect answer but strategy was sound and could have led to correct answer
0: problem not attempted
(2) solved in under 2 minutes (5) solved in under 5 minutes

Name (4th) *pseudonym*	Scores TOMAGS/ 4th gr. Benchmark/ 5th gr. Benchmark *(advanced 640+)*	Problem 6 *Lawnmower holds 3 gallons gas. Each lawn takes ½ gal. How many lawns? (1/2,1/4, ¾, 3/8)*	Problem 7 *¼ + 1/8 cup syrup. Total? If there is 1 cup of syrup is it enough for 2 days? (1/4, 1/8) (1/4, 1/3)*
Andre	97th/731/695 Adv/Adv Highest TR	C C C C Number sentence Chunking Direct modeling	C X Direct modeling
Natalie	90th/698/759 Adv /Adv Highest TR	C C C <u>C</u> Chunking RT	C C Direct modeling
Alana	90th/731/732 Adv/Adv TR	C C C C Chunking Direct modeling	C X Direct modeling
Anita	70th/681/682 Adv/Adv TR	C C C <u>C</u> Chunking Direct modeling RT	C X Direct modeling
Katerina	12th/745/646 Adv/Adv TR	X X 0 0 Direct modeling	X X Direct modeling
Henry	/ /663 /Adv		C X Direct modeling
Gloria	/ /588 /Prof	C C 0 0 Direct modeling	C X Direct modeling
Lala	/ /572 /Prof	C C 0 0 Direct modeling	X X Direct modeling
Jenni	/ /550 /Bas	C C X X Chunking	X X Direct modeling
Time allotted for Problem Solving		*26 minutes*	*38 minutes*

C: correct answer
<u>C</u>: correct answer gotten by using relational thinking based on previous answers
(c): problem not finished but was on the way to a correct answer
X: incorrect answer
<u>X</u>: incorrect answer but was using relational thinking based on previous answers
X+: incorrect answer but strategy was sound and could have led to correct answer
0: problem not attempted
(2) solved in under 2 minutes (5) solved in under 5 minutes

Name (3rd) *pseudonym*	Scores TOMAGS/ 3rd gr.Benchmark *(advanced 586+)*	Problem 1 *8 kids share 14 candy bars. How much per kid?* *(8,14)* *Small group: (4,7)*	Problem 2 *4 kids share 7 sandwiches. How much per kid?* *(4,7) (12,8)*	Problem 3 *2 kids share 3 cakes. How much per kid?* *(2.3) (4,6)*	Problem 4 *6 kids, each with 2 2/3 cookies. How many in all?* *(6, 2 2/3)*	Problem 5 *6 kids, each with 2 ¼ brownies. How many in all?* *(6, 2 ¼) (6, 2 5/6)*
Dominic	97th/737 Adv Highest TR	C (c) Direct modeling Number facts Number sentences	C C (5) Number facts Number Sentences	C C (2) Decomposing Number facts	C (5) C Number facts Number sentences	C C (2) Number facts Number sentences
Jasmin	95th/717 Adv Highest TR	C (c) Direct modeling Number facts Number sentences	C X+ Number facts Number sentences Pictures	C C (5) C Number facts Number sentences	C (5) (c) Decomposition Number facts Number sentences	C C Decomposition Number facts Number sentences
Freddie	97th/641 Adv Highest TR	C (c) Direct modeling Number facts	C C (5) Number facts Number sentences Counting w/model	C C (2) Number facts Number sentences	C (5) Decomposing Counting w/model	C C Counting w/model
Maya	85th/674 Adv TR	C (c) Direct modeling Number facts Number sentences	C Number facts Number sentences Direct modeling	C C (5) Number facts Number sentences Count w/Model	C Counting w/model	C C Counting w/model Number sentences
Julia	84th/663 Adv TR	C Direct modeling Counting	C C Counting w/model	C C Number sentences Count w/Model	X+ Counting w/model	C C Counting w/model
Jaime	84th/692 Adv		X X Counting		X	X X Number sentences Direct modeling
Juan	84th/686 Adv		C Direct modeling	C C Direct modeling	X Counting w/model	C (c) Counting w/model
Gissela	23rd/486 Bas	X Direct modeling	X X Counting w/model	X X Direct modeling	X Counting w/model	X 0 Direct modeling Counting
Time allotted for Problem Solving		*23 minutes*	*22 minutes*	*11 minutes*	*15 minutes*	*23 minutes*

C: correct answer
C: correct answer gotten by using relational thinking based on previous answers
(c): problem not finished but was on the way to a correct answer
X: incorrect answer X: incorrect answer but was using relational thinking based on previous answers
X+: incorrect answer but strategy was sound and could have led to correct answer
0: problem not attempted
(2) solved in under 2 minutes (5) solved in under 5 minutes

Name (3rd) *pseudonym*	Scores TOMAGS/ 3rd gr.Benchmark *(advanced 586+)*	Problem 6 *6 cups dog food. ½ cup each day. How many days? (1/2)(1½)(1/4)(3/4)*	Problem 7 *Eating 1/3 pizza and ½ pizza. How much total? How much left? (1/3, ½) (1/3, ¼)* *Small group*	Problem 8 *Francisco drank ½ cup milk. Jay drank ¼ cup. How much total? (1/2, ¼) (1/4, 1/8) Small group (1/2,1/5)*	Problem 9 *Daniel has 2/3 Hershey bar, Fernando has 1/3. How much more? (2/3,1/3) (5/6,1/3) (7/12,1/2)(3/4,2/3)* *Small group*	Problem 10 *Jacquelyn ate 2/3 candy bar. Brother ate 1/12. How much total? (2/3, 1/12)*
Dominic	97th/737 Adv Highest TR	C C C <u>C</u> (5) Number sentences Number facts Counting RT	C C C C *(easel)* Direct modeling	C C (5)(easel) Equivalent fractions Direct modeling	C <u>C</u> C C *(easel)* Direct modeling Equivalent fractions Number sentences RT	C (2) Equivalent fractions Direct modeling Number sentences
Jasmin	95th/717 Adv Highest TR	C C C X+ Number sentences Number facts Chunking	C C C C *(easel)* Direct modeling	C C (5)(easel) Equivalent fractions Direct modeling	C <u>C</u> C C *(easel)* Direct modeling Equivalent fractions Number sentences	C (5) Equivalent fractions Direct modeling Number sentences
Freddie	97th/641 Adv Highest TR	C C X+ 0 Counting	C C C C *(easel)* Direct modeling	C C (5) (easel) Equivalent fractions Direct modeling	@	C (2) Equivalent fractions Direct modeling Number sentences
Maya	85th/674 Adv TR	C C C C Direct modeling Number sentences	C C C C *(easel)* Direct modeling	C C (5) (easel)	C C C X+ *(board)* Direct modeling Number Sentences	
Julia	84th/663 Adv TR	C C C X+ Direct modeling Number sentences	C C C C *(easel)* Direct modeling Number sentences	@	C C C X+ *(board)* Direct modeling Number Sentences	C *(easel)* Direct modeling Number sentence Equivalent fractions
Jaime	84th/692 Adv	C 0 0 0 Counting w/model				(c) Direct modeling Number sentence
Juan	84th/686 Adv	C 0 0 0 Counting w/model	X X 0 0 Direct modeling			
Gissela	23rd/486 Bas	C X 0 0 Counting w/model	X X X 0 Direct modeling			(c) Direct modeling Number sentence
Time allotted for Problem Solving		*20 minutes*	*30 minutes*	*5 then group*	*25 minutes*	*15 minutes*

C: correct answer
<u>C</u>: correct answer gotten by using relational thinking based on previous answers
(c): problem not finished but was on the way to a correct answer
X: incorrect answer <u>X</u>: incorrect answer but was using relational thinking based on previous answers
X+: incorrect answer but strategy was sound and could have led to correct answer
0: problem not attempted
(2) solved in under 2 minutes (5) solved in under 5 minutes

Appendix G Institutional Review Board (IRB) Approval

210 Administration • Fayetteville, Arkansas 72701 • (479) 575-2208 • (479) 575-3846 (FAX)
Email: irb@uark.edu

Research Compliance
Institutional Review Board

April 26, 2011

MEMORANDUM

TO:	Kim McComas Laura Kent
FROM:	Ro Windwalker IRB Coordinator
RE:	New Protocol Approval
IRB Protocol #:	11-04-614
Protocol Title:	*Mathematically Talented Students' Experiences with Cognitively Guided Instruction*
Review Type:	☐ EXEMPT ☒ EXPEDITED ☐ FULL IRB
Approved Project Period:	Start Date: 04/26/2011 Expiration Date: 04/25/2012

Your protocol has been approved by the IRB. Protocols are approved for a maximum period of one year. If you wish to continue the project past the approved project period (see above), you must submit a request, using the form *Continuing Review for IRB Approved Projects*, prior to the expiration date. This form is available from the IRB Coordinator or on the Compliance website (http://www.uark.edu/admin/rsspinfo/compliance/index.html). As a courtesy, you will be sent a reminder two months in advance of that date. However, failure to receive a reminder does not negate your obligation to make the request in sufficient time for review and approval. Federal regulations prohibit retroactive approval of continuation. Failure to receive approval to continue the project prior to the expiration date will result in Termination of the protocol approval. The IRB Coordinator can give you guidance on submission times.

This protocol has been approved for 114 participants. If you wish to make *any* modifications in the approved protocol, including enrolling more than this number, you must seek approval *prior to* implementing those changes. All modifications should be requested in writing (email is acceptable) and must provide sufficient detail to assess the impact of the change.

If you have questions or need any assistance from the IRB, please contact me at 210 Administration Building, 5-2208, or irb@uark.edu.