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Three Essays On Differential Games And Resource Economics

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THREE ESSAYS ON DIFFERENTIAL GAMES AND RESOURCE ECONOMICS

by

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ABSTRACT

This dissertation consists of three chapters on the topic of differential games and resource economics. The first chapter extends the envelope theorem to the class of discounted infinite horizon differential games that posses locally differentiable Nash equilibria. The theorems cover both the open-loop and feedback information structures, and are applied to a simple analytically solvable linear-quadratic game. The results show that the conventional interpretation of the costate variable as the shadow value of the state variable along the equilibrium path is only valid for feedback Nash equilibria, but not for open-loop Nash equilibria. The specific linear-quadratic structure provides some extra insights on the theorem. For example, the costate variable is shown to uniformly overestimate the shadow value of the state variable in the open-loop case, but the growth rate of the costate variable are the same as the shadow value under open-loop and feedback information structures.

Chapter two investigates the qualitative properties of symmetric open-loop Nash equilibria for a ubiquitous class of discounted infinite horizon differential games. The results show that the specific functional forms and the value of parameters used in the game are crucial in determining the local asymptotic stability of steady state, the steady state comparative statics, and the local comparative dynamics. Several sufficient conditions are provided to identify a local saddle point type of steady state. An important steady state policy implication from the model is that functional forms and parameter values are not only quantitatively important to differentiate policy tools, but they are also qualitatively important.

Chapter three shifts the interests to the lottery mechanism for rationing public resources. It characterizes the optimal pricing strategies of lotteries for a welfare-maximization agency. The optimal prices are shown to be positive for a wide range of individual private value distributions, suggesting that the sub-optimal pricing may result in a significant efficiency loss and that the earlier studies under zero-pricing may need to be re-examined. In addition, I identify the revenue and welfare equivalency propositions across lottery institutions. Finally, the numerical simulations strongly support the findings.

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INTRODUCTION

This dissertation is concerned with qualitative characterizations of Nash equilibria in differential games and optimal pricing of lotteries for public resources rationing. Differential games are a group of strategic problems related to the modeling and analysis of conflict in the context of a dynamic system. With a few exceptions, the difficulty of deriving analytically equilibria for a general differential game is notoriously known. Thus, as Clemhout and Wan (1994) pointed out, "… the theory of differential game is effective for what it does best, in characterizing qualitatively the time-profile of equilibrium plays."

A qualitative characterization of a model has its origin in Samuelson's (1947) seminal work, in which he derived the comparative statics properties of a general class of unconstrained static optimization problem. The theory was generalized to constrained static optimization problems by Silberberg (1974) and Partovi and Caputo (2006), to static game theory by Caputo (1996), and to optimal control theory by Caputo (1990, 1997, and 2003). However, the investigation and application of the qualitative properties in differential games usually took place in the context of tightly specified models, in most case, the linear-quadratic differential games (e.g. Fershtman and Kamien 1987). One purpose of this dissertation, therefore, is to provide a systematic approach to uncovering the qualitative properties of symmetric open-loop Nash equilibria in a generic class of differential games.

Secondly, this dissertation further extends the envelope theorem to a class of infinite horizon differential games, and applies the results to a simple analytically solvable linear-quadratic game. The envelope theorem, which describes how the value of an optimization problem changes as the parameters of the problem change, has been used in many economics and management science problems since it was first introduced into economic theory by Hotelling

(1931). A rigorous proof of the envelope theorem for Nash equilibria in general differential games is not available until Caputo (2007), in which he formally derived the envelope results for a class of finite horizon differential games. An "unexpected" envelope result in Caputo's finding is that the conventional interpretation of the costate vector as the shadow value of the state along the equilibria path is only valid for feedback Nash equilibria, but not for open-loop Nash equilibria. As expected, the similar envelope results arise in infinite horizon games, compare to its finite counterparts, under some reasonable transversality conditions, and the special linear-quadratic structure of the game further convinces the "unexpected" envelope result and provides some extra insight on the theorem.

After deriving the envelope theorems and the qualitative properties of Nash equilibria for differential games, the focus is shift to characterize the optimal pricing strategy of lotteries. The lotteries are commonly used to ration public resources when shortages exist at established prices. Examples include hunting privileges and recreational uses of state and federal lands throughout the U.S. The literature to date has largely focused upon the merits and shortcomings of lotteries relative to other non-market rationing mechanisms, such as waiting line-auctions, with prices generally being naively fixed at zero. However, empirical observations reveal that lottery prices tend to be positive yet appear in a number of forms. Thus, it is naturally to ask how the lottery prices are optimally determined. This dissertation provides a general framework to optimally characterize the lottery prices for an expected consumer-surplus maximization agency. In the meantime, the numerical simulation results indicate that the optimal lottery prices tend to be positive when the individual private values are relatively homogeneous in the lower tail of the distribution, which implies that the earlier studies of lotteries under zero pricing may need to be reconsidered.

This dissertation is organized as follow. Chapter 1 derives the envelope theorem for locally differential Nash equilibria in discounted infinite horizon differential games, and applied it to a simple linear-quadratic game. Chapter 2 derives the local asymptotical stability of steady state, the steady state comparative statics, and the local comparative dynamics of symmetric open-loop Nash equilibria in a ubiquitous class of discounted infinite horizon differential games. Finally, chapter 3 characterizes the optimal pricing strategies of lotteries, and performs numerical simulations to support the findings.

CHAPTER ONE: THE ENVELOPE THEOREM FOR LOCALLY DIFFERERNTIALBE NASH EQUILIBRIA OF DISCOUNTED INFINITE HORIZON DIFFERENTIAL GAMES

1.1. Introduction

The envelope theorem, which describes how the value of an optimization problem changes as the parameters of the problem change, has been widely applied in economics since it was first introduced into economic theory by Hotelling (1931). For instance, it may be used to prove Hotelling's Lemma, Shephard's lemma, and Roy's identity in microeconomics. It also allows for easier derivation of comparative statics results in generalized economic models (Silberberg and Suen 2001).

Although the envelope theorem achieved great success in single player optimization problems, it remains under-developed in a strategic setting with multiple players. In the past several decades, researchers devoted most of their efforts at deriving the equilibrium outcomes for games and determining how the equilibrium changes as the parameters change in tightly specified games. The envelope results, however, received little attention in the game theory literature. To my knowledge, only two theoretical papers exist in this field. Caputo (1996) derived the envelope theorem for static games with locally differentiable Nash equilibria, and Caputo (2007) further extended the envelope theorem to the class of finite horizon differential games. For the sake of completeness, it is natural to derive the envelope theorem for the class of discounted infinite horizon games.

Given the similarities and differences between finite and infinite horizon problem in optimal control theory, it would not be surprising to obtain similar envelope results in infinite horizon games under some reasonable transversality conditions. In general, the choice of the horizon length should be made based on the appropriateness of the assumption for the economic question under consideration, as well as the qualitative implications it implies, and its consistency with observed behavior. However, most interesting economics problems are indeed defined over an infinite time horizon, such as in macroeconomic policy, interboundary pollution control, and natural resources extraction, simply because most economic problems cannot reasonably predetermine a finite ending date. Thus, examining the envelope results to the class of discounted infinite horizon differential games is a natural extension. That is the first task of this chapter.

Among all the envelope expressions, the change in the value function with respect to the state variable, which is typically referred to as the shadow value in economics, has received the most attention. The shadow value usually possesses a fruitful economics meaning. For example, it can be used to interpret an optimal tax in environmental economics (Farzin 1996) and in the common-pool resource problems (Arnason 1990). It also can be viewed as the *in situ* value in natural resources economics (Hotelling 1931, Farzin 1992, and Krautkraemer 1998). This envelope expression has been proved to be equal to the value of the co-state variables that define Hamiltonian of the control problems (Léonard 1987). Thus, people can conveniently treat the costate variable as the shadow value in the analysis. Since this envelope results is so important, it usually has been naively extended to the strategic setting with multiple players without any rigorous proof (Mohr 1988, Negri 1989, Arnason 1990, Slade 1995, Dockner et al. 2000, among others). However, this is not always the case in a game setting. Caputo (2007) showed that the costate vector is indeed equal to the shadow value of the state vector for *feedback* Nash equilibria, but that this relationship does not hold for *open-loop* Nash equilibria. In this case, any statement about shadow values that has been declared based on the costate vectors may be misleading.

The breakdown between the shadow value and the costate vectors calls for further investigation of this envelope expression. First of all, it is helpful to convince people that the costate vector in general is not the shadow value of the state vector in a game setting by providing a supportive example. Second, it may be interesting to examine the magnitude of the quantitative difference and the dynamic evolution over time between the shadow values and the costate vectors, at least for some special classes of games. These two factors may tell us how misleading the results are if one treats the costate variable as the shadow value. Thus, examining this vital relationship for discounted infinite horizon differential game is the second task of the chapter.

Although the envelope theorem has been developed for some generic classes of games, its application is still rare. One possible reason for that was stated in Reinganum (1982), where she asserted that "it is difficult to obtain results from general differential games". Fortunately, there are a few classes of differential games that are analytically or qualitatively solvable (Dockner et al. 2000). Thus, the natural step is to apply the generic envelope theorem established here to those special classes of differential games. Moreover, some extra results may emerge once we impose some functional forms on the games. This may help deepen our understanding for these models. This is the third task for the current research.

In this chapter, I further extend the envelope theorem for the class of locally differentiable Nash equilibria of discounted infinite horizon differential games. Two different information structures and their corresponding Nash equilibria are considered. Specifically, they are (i) an *open-loop* information structure and its associated *open-loop* Nash equilibria, and (ii) a *feedback* information structure and its corresponding *feedback* Nash equilibria. I apply the general envelope results to a linear-quadratic differential game, the workhorse model in this field.

The simple linear-quadratic model allows us to analytically derive a Nash equilibrium, the associated costate vectors, and all the envelope expressions. Thus, one can examine the envelope results for the linear-quadratic model, and compare its costate vectors and the shadow values.

The results show that similar envelope results as in Caputo (2007) emerge for discounted infinite horizon differential games under some reasonable assumptions. That is, the costate variable is indeed equal to the shadow value for feedback Nash equilibria, but not so for open-loop Nash equilibria. Moreover, both the open-loop and feedback Nash equilibria are derived for our linear-quadratic model, and hence explicit envelope results are obtained, thereby confirming the theorem. The linear-quadratic game shows that the costate variable is uniformly less than the shadow value in the open-loop case, but their dynamic patterns are the same. The results suggest that for open-loop Nash equilibria, at least for the linear quadratic model, treating costate variable as shadow value raises some issues quantitatively. The magnitude of the quantitative difference is based on the parameters, which requires further empirical investigation.

1.2. Technical Preliminaries

Consider the ubiquitous class of discounted infinite horizon differential games consisting of a finite number of P players, indexed by $p = 1, 2, \cdots, P$. Denote player-specific variables, functions, and parameters by upper indices. The state of the differential game at each instant $t \in [0, \infty)$ is given by the vector $\mathbf{x}(t) \in \mathbb{R}^N$. The initial value of the state vector, denoted by $\mathbf{x}(0)$, is fixed at the value $\mathbf{x}_0 \in \mathbb{R}^N$. At each instant $t \in [0, \infty)$, each player $p \in \{1, 2, \ldots, P\}$ chooses a control variable $\mathbf{u}^p(t) \in \mathbb{R}^{M^p}$ from her set of feasible controls $\mathbf{U}^p(\mathbf{x}(t), \mathbf{u}^{-p}(t), t) \subseteq \mathbb{R}^{M^p}$ so as to maximize his payoff function. Define the vector $\mathbf{u}^{-p}(t)$ consisting of all other players' controls at time t, i.e.,

$$\mathbf{u}^{-p}(t) \stackrel{\text{def}}{=} (\mathbf{u}^{1}(t), \mathbf{u}^{2}(t), \ldots, \mathbf{u}^{p-1}(t), \mathbf{u}^{p+1}(t), \ldots, \mathbf{u}^{P}(t)) .$$

The state of the differential game evolves according to the system of ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}^{1}(t), \ldots, \mathbf{u}^{P}(t); \boldsymbol{\alpha}), \quad \mathbf{x}(0) = \mathbf{x}_0 ,$$

where $\boldsymbol{\alpha} \in \mathbb{R}^{A}$ is a vector of time independent parameters that enters the transition functions $\mathbf{g}(\cdot)$ and the instantaneous payoff function $f^{p}(\cdot)$ of each player $p \in \{1, 2, \ldots, P\}$. For simplicity, we assume that the integrand of every player contains an exponential discount factor but the game is otherwise autonomous. That is, $f^{p}(\cdot)$ and $\mathbf{g}(\cdot)$ do not explicitly depend on the time t. The discount factors are assumed to be the same among players, and is denoted by r. Let $\mathbf{z}(t)$ and $\mathbf{v}(t)$ represent $\mathbf{x}(t)$ and $\mathbf{u}(t)$ along the optimal respectively. Putting the above information together, the generic class of infinite horizon differential games under consideration is given by

$$W^{p}(\boldsymbol{\beta}) = \max_{\mathbf{u}^{p}(\cdot)} \int_{0}^{\infty} e^{-rt} f^{p}(\mathbf{x}(t), \mathbf{u}^{p}(t), \mathbf{u}^{-p}(t); \boldsymbol{\alpha}) dt \tag{1.1}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}^{p}(t), \mathbf{u}^{-p}(t); \boldsymbol{\alpha}), \quad \mathbf{x}(0) = \mathbf{x}_0 ,$$

where $p = 1, 2, \cdots, P$ and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (r, \mathbf{x}_0, \boldsymbol{\alpha}) \in \mathbb{R}^{A+N+1}$ is the vector of exogenous parameters of the game.

1.3. The Envelope Theorem for Open-Loop Nash Equilibria

In this section, I derive the envelope theorem for open-loop Nash equilibria. The following assumptions are imposed on the game that defined in Eq. (1.1) for all players.

(A1) $f^{p}(\cdot)$ and $\mathbf{g}(\cdot)$ are $C^{(1)}$ on their domains, and $\| f(\cdot) \| \le C$ and $\| f_{\boldsymbol{\alpha}}(\cdot) \| \le C$, where C is a positive constant and $\| \cdot \|$ is Euclidean norm in corresponding spaces.

(A2) The information structure is *open-loop*.

(A3) The open-loop Nash equilibria exists for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta)$, and is denoted by the P-tuple of vector $\mathbf{v}(t;\boldsymbol{\beta}) = (\mathbf{v}^1(t;\boldsymbol{\beta}), \mathbf{v}^2(t;\boldsymbol{\beta}), \ldots, \mathbf{v}^P(t;\boldsymbol{\beta}))$, where $\mathbf{z}(t;\boldsymbol{\beta}) \in \mathbb{R}^N$ is the associated state trajectory, $\boldsymbol{\lambda}^p(t;\boldsymbol{\beta}) \in \mathbb{R}^N$ is the corresponding current value costate trajectory of player p, $p = 1, 2, \ldots, P$.

(A4) The solution vector $(\mathbf{z}(\cdot), \mathbf{v}(\cdot), \boldsymbol{\lambda}(\cdot)) \in C^{(1)}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta)$.

(A5) $\lim_{t\to\infty} e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^{\dagger} \dfrac{\partial \mathbf{z}}{\partial \boldsymbol{\beta}}(t;\boldsymbol{\beta}) = \mathbf{0}_{N\times(A+N+1)}$.

The assumption (A1) guarantees that the improper integral $W^p(\cdot)$ converges for all admissible pairs $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$, and belongs to the class of $C^{(1)}$ functions. It also allows the interchange between the integral and differentiation operators below. Thus it is required for the differential characterization of the envelope theorem. Assumption (A2) specifies the information that available to players at any time of the game. Assumption (A3) is a result of assumption (A2), and it explicitly assumes the existence of the Nash equlibria. Alternatively, one can impose assumptions on the primitives that are sufficient to imply the existence of Nash equilibria. However, this approach implies that the envelope results so obtained are not intrinsic to the model but are in fact conditioned on those sufficient conditions. Thus, it reduces the generality and applicability of the results. It is important to observe that the open-loop information requires that the corresponding open-loop Nash equilibria $\mathbf{v}(t;\boldsymbol{\beta})$, only depends on time and the exogenous parameters. Finally, to keep the theorem as general as possible, I do not require the optimal solutions converge to a steady state.

Assumption (A4) is crucial to the analysis, since there is no guarantee that the optimal solution is differentiable from assumption (A1). This assumption precludes some important classes of games, such as those that admit non-smooth solutions, e.g., the bang-bang solution. Thus it may be stronger than needed to obtain the envelope results herein. The assumption can be relaxed to allow a zero measure set of non-smoothness points for the solutions, and hence cover a broader class of games, but such a generalization does not affect the envelope theorem and the linear-quadratic games below. It is important, therefore, to remember that the envelope theorem derived here is also valid for larger classes of games.

Assumption (A5) is unique to the infinite horizon assumption. It restricts the optimal values of the state and costate vectors in the limit. The assumption plays essentially the same role as the transversality condition on the costate vector in finite horizon games, and hence acts as a transversality condition for the infinite horizon games. Notice that the assumption only requires convergence along the equilibrium path, and thus is not very restrictive in economics. The typical steady state assumption in analyzing infinite horizon control problems, along with a regularity condition, is sufficient to yield it.[1]

By the definition of open-loop Nash equilibria, the optimal time-path of the pth player's control vector, i.e., $\mathbf{v}^p(t;\boldsymbol{\beta})$, is the solution to the optimal control problem

$$W^p(\boldsymbol{\beta}) = \max_{\mathbf{u}^p(\cdot)} \int_0^\infty e^{-rt} f^p(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{v}^{-p}(t;\boldsymbol{\beta});\boldsymbol{\alpha})dt \tag{1.2}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}^p(t), \mathbf{v}^{-p}(t;\boldsymbol{\beta});\boldsymbol{\alpha}), \quad \mathbf{x}(0) = \mathbf{x}_0 .$$

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[1] The regularity condition required here is that $\partial\mathbf{z}(\cdot)/\partial\boldsymbol{\beta}$ is continuous on the closure of $B(\boldsymbol{\beta}^\circ, \delta)$, it is bounded. The steady state assumption implies that $\lim_{t\to\infty} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}) = \boldsymbol{\lambda}^p_{ss}(\boldsymbol{\beta})$, and hence it follows that $\lim_{t\to\infty} e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \, \partial\mathbf{z}(t;\boldsymbol{\beta})/\partial\boldsymbol{\beta} = \mathbf{0}_{N\times(A+N+1)}$ Since $\mathbf{z}(\cdot)$ is a $C^{(1)}$ function on $B(\boldsymbol{\beta}^\circ, \delta)$ by assumption (A.4), the smoothness of $\mathbf{z}(\cdot)$ at the boundary is sufficient to reach the regularity condition. Intuitionly, it means that there is no sudden jump for state variable when some parameters reach boundaries. Most control problems in economics meet this regularity requirement.

The current value Hamiltonian function $H^p(\cdot)$ corresponding to optimal control problem (1.2) is defined as

$$H^p(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} f^p(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t;\boldsymbol{\beta});\boldsymbol{\alpha}) + \boldsymbol{\lambda}^{p\dagger}\mathbf{g}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t;\boldsymbol{\beta});\boldsymbol{\alpha}) . \tag{1.3}$$

From Theorem 6.11 of Basar and Olsder (1995) and Theorem 14.3 of Caputo (2005), the necessary conditions obeyed by open-loop Nash equilibria are

$$\frac{\partial H^p}{\partial \mathbf{u}^p}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha}) = \mathbf{0}^{\dagger}_{M^p}, \tag{1.4}$$

$$\dot{\boldsymbol{\lambda}}^p = r\boldsymbol{\lambda}^p - \frac{\partial H^p}{\partial \mathbf{x}}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha}) , \tag{1.5}$$

$$\dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t),\mathbf{u}^p(t),\mathbf{v}^{-p}(t;\boldsymbol{\beta});\boldsymbol{\alpha}), \quad \mathbf{x}(0) = \mathbf{x}_0 , \tag{1.6}$$

for all $p = 1, 2, ..., P$. With these conditions, the central results of this section are given below.

Theorem 1.1 (Open-Loop Nash Equilibria). *If the differential game* (1.2) *satisfies assumptions (A1)-(A5), then for all* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta)$ *and* $p = 1, 2, ...P$, $W^p(\cdot) \in C^{(1)}$ *and*

$$\begin{aligned}\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) &= \int_0^\infty e^{-rt}[H^p_{\boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})]dt \\ &\quad + \int_0^\infty e^{-rt}[\sum_{j=1,\, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})]dt ,\end{aligned} \tag{1.7}$$

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \boldsymbol{\lambda}^p(0;\boldsymbol{\beta}) + \int_0^\infty e^{-rt}[\sum_{j=1,\, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})]dt , \tag{1.8}$$

$$\begin{aligned}\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) &= \int_0^\infty -te^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})dt \\ &\quad + \int_0^\infty e^{-rt}[\sum_{j=1,\, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta})]dt .\end{aligned} \tag{1.9}$$

Proof. The optimal value function $W^p(\cdot)$ defined in Eq. (1.2) can be equivalently defined as

$$W^p(\boldsymbol{\beta}) \overset{\text{def}}{=} \int_0^\infty e^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) dt \,. \tag{1.10}$$

By Weierstrass M-test and the Theorem A. 14.1 in Caputo (2005), Assumption (A1) implies that $W^p(\cdot)$ converges for all admissible pairs, $W^p(\cdot) \in C^{(1)}$, and

$$\frac{\partial}{\partial \boldsymbol{\beta}} \int_0^\infty e^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) dt = \int_0^\infty \frac{\partial}{\partial \boldsymbol{\beta}} [e^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha})] dt \,, \tag{1.11}$$

for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$ and $p \in \{1, 2, ...P\}$. To prove Eq. (1.7), differentiate Eq. (1.10) with respect to $\boldsymbol{\alpha}$ to get

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^\infty e^{-rt} [\frac{\partial f^p}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})$$
$$+ \sum_{j=1}^P \frac{\partial f^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}} + \frac{\partial f^p}{\partial \boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})] dt \,. \tag{1.12}$$

Next, differentiate the identity $\mathbf{g}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_N$ with respect to $\boldsymbol{\alpha}$ to get

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) + \sum_{j=1}^P \frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})$$
$$+ \frac{\partial \mathbf{g}}{\partial \boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) = \mathbf{0}_{N \times A} \,. \tag{1.13}$$

Premultiply Eq. (1.13) by the row vector $e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over the time horizon, add the result to Eq. (1.12), and then use the Eq. (1.3) to simplify the notation. These procedures yield the expression

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^\infty e^{-rt} [H^p_{\boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) + \sum_{j=1}^P \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})$$
$$+ \frac{\partial H^p}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) - \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta})] dt \tag{1.14}$$

Now notice the fact that

$$\frac{d}{dt}[e^{-rt}\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta})^{\dagger}\frac{\partial\mathbf{z}}{\partial\boldsymbol{\alpha}}(t;\boldsymbol{\beta})]=e^{-rt}\dot{\boldsymbol{\lambda}}^{p}(t;\boldsymbol{\beta})^{\dagger}\frac{\partial\mathbf{z}}{\partial\boldsymbol{\alpha}}(t;\boldsymbol{\beta})$$

$$+e^{-rt}\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta})^{\dagger}\frac{\partial\dot{\mathbf{z}}}{\partial\boldsymbol{\alpha}}(t;\boldsymbol{\beta})-re^{-rt}\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta})^{\dagger}\frac{\partial\mathbf{z}}{\partial\boldsymbol{\alpha}}(t;\boldsymbol{\beta}). \quad (1.15)$$

Use the necessary conditions given in Eqs. (1.4) and (1.5) in identity form, along with the Eq. (1.15), to reduce Eq. (1.14) to

$$\frac{\partial W^{p}}{\partial\boldsymbol{\alpha}}(\boldsymbol{\beta})=\int_{0}^{\infty}e^{-rt}[H_{\boldsymbol{\alpha}}^{p}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta});\boldsymbol{\alpha})$$

$$+\sum_{j=1,\,j\neq p}^{P}\frac{\partial H^{p}}{\partial\mathbf{u}^{j}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial\mathbf{v}^{j}}{\partial\boldsymbol{\alpha}}(t;\boldsymbol{\beta})]dt$$

$$-\lim_{t\to\infty}e^{-rt}\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta})^{\dagger}\frac{\partial\mathbf{z}}{\partial\boldsymbol{\alpha}}(t;\boldsymbol{\beta})+\boldsymbol{\lambda}^{p}(0;\boldsymbol{\beta})^{\dagger}\frac{\partial\mathbf{z}}{\partial\boldsymbol{\alpha}}(0;\boldsymbol{\beta}). \quad (1.16)$$

Since $\mathbf{z}(0;\boldsymbol{\beta})\equiv\mathbf{x}_0$, it follows that $\partial\mathbf{z}(0;\boldsymbol{\beta})/\partial\boldsymbol{\alpha}=\mathbf{0}_{N\times A}$ Finally, by assumption (A.5), Eq. (1.16) reduces to Eq. (1.7).

The procedures to prove Eqs. (1.8) and (1.9) are similar. For Eq. (1.8), differentiate Eq. (1.10) with respect to $\mathbf{x}_0$ using Leibniz's rule to get

$$\frac{\partial W^{p}(\cdot)}{\partial\mathbf{x}_0}(\boldsymbol{\beta})=\int_{0}^{\infty}[\frac{\partial f^{p}}{\partial\mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial\mathbf{z}}{\partial\mathbf{x}_0}(t;\boldsymbol{\beta})$$

$$+\sum_{j=1}^{P}\frac{\partial f^{p}}{\partial\mathbf{u}^{j}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial\mathbf{v}^{j}}{\partial\mathbf{x}_0}(t;\boldsymbol{\beta})]dt. \quad (1.17)$$

Differentiate the identity $\mathbf{g}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})-\dot{\mathbf{z}}(t;\boldsymbol{\beta})\equiv\mathbf{0}$ with respect to $\mathbf{x}_0$ to get

$$\frac{\partial\mathbf{g}}{\partial\mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial\mathbf{z}}{\partial\mathbf{x}_0}(t;\boldsymbol{\beta})+\sum_{j=1}^{P}\frac{\partial\mathbf{g}}{\partial\mathbf{u}^{j}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial\mathbf{v}^{j}}{\partial\mathbf{x}_0}(t;\boldsymbol{\beta})-\frac{\partial\dot{\mathbf{z}}(t;\boldsymbol{\beta})}{\partial\mathbf{x}_0}=\mathbf{0}_{N\times N}. \quad (1.18)$$

Premultiply Eq. (1.18) by the row vector $e^{-rt}\boldsymbol{\lambda}^{p}(t;\boldsymbol{\beta})^{\dagger}$, integrate the resulting inner product over the time horizon, add the result to Eq. (1.17), and simplify it using Eqs. (1.3), (1.4) and (1.5). These steps yield

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \int_0^\infty e^{-rt}\{-\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) + r\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})$$

$$+\sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})\}dt. \quad (1.19)$$

Similar to Eq. (1.15), we have

$$\frac{d}{dt}(e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})) = e^{-rt}\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})$$

$$+e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - re^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}). \quad (1.20)$$

Notice that $\mathbf{z}(0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$, hence it follows that $\partial\mathbf{z}(0;\boldsymbol{\beta})/\partial\mathbf{x}_0 = \mathbf{I}_N$. This fact, combined with Eq. (1.20) and assumption (A5), simplifies Eq. (1.19) to yield Eq. (1.8).

Lastly, For Eq. (1.9), differentiate Eq. (1.10) with respect to r to get

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^\infty e^{-rt}[-tf^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) + \frac{\partial f^p}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})$$

$$+\sum_{j=1}^{P} \frac{\partial f^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta})]dt. \quad (1.21)$$

Differentiate the identity $\mathbf{g}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_N$ with respect to r to get

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}) + \sum_{j=1}^{N} \frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta}) - \frac{\partial \dot{\mathbf{z}}}{\partial r}(t;\boldsymbol{\beta}) = \mathbf{0}_{N\times N}. \quad (1.22)$$

Again, premulitply Eq. (1.22) by the row vector $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over the time horizon, add the result to Eq. (1.21), and simplify it using Eqs. (1.3), (1.4) and (1.5) and

$$\frac{d}{dt}(e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})) = e^{-rt}\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})$$

$$+e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial r}(t;\boldsymbol{\beta}) - re^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}). \quad (1.23)$$

yields

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^\infty -te^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})dt + \boldsymbol{\lambda}^p(0,\boldsymbol{\beta})\frac{\partial \mathbf{z}}{\partial r}(0,\boldsymbol{\beta})$$

$$+\int_0^\infty [e^{-rt} \sum_{j=1, j\neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial r}(t;\boldsymbol{\beta})]dt \,. \quad (1.24)$$

Since $\mathbf{z}(0;\boldsymbol{\beta}) = \mathbf{x}_0$, it follows that $\partial \mathbf{z}(0;\boldsymbol{\beta})/\partial r \equiv \mathbf{0}_N$. Thus, Eq. (1.24) reduces to Eq. (10).

Until now, the envelope theorem has been established for open-loop Nash equilibria of the discounted infinite horizon class of games defined in Eq. (1.1). The result shows that the envelope theorem for infinite horizon differential games consists of two distinct effects when the exogenous parameters change. Specifically, it consists of a *direct* or *explicit* effect, the first term on the right-hand side of Eqs. (1.7), (1.8), and (1.9). This direct effect of the parameter perturbation on the value function of player p is identical to its corresponding counterpart in an optimal control problem (Caputo 2005, Th.14.10). The second effect is a *strategic effect* comprised of the second term on the right-side of Eqs. (1.7), (1.8), and (1.9). This reflects the effects of a perturbation in the parameters on player p through the other $P-1$ players' response to the perturbation. This effect is identically zero in optimal control theory.

The envelope result in Eq. (1.8) indicates that the shadow value of the initial state is not equal to the costate variable evaluated at the initial time. It implies that interpreting the costate vector as the shadow value in an open-loop game is incorrect. Moreover, given the difficulty of solving general differential games, Equation (1.8) suggests that it may be difficult to know the true shadow value for the initial state in the game setting, and also its dynamic pattern in general.[2]

Without putting any further structure on the game, we cannot compare the costate vector and the shadow value. On the other hand, if we can sign the second part of Eq. (1.8) and even be able to show, at least for some special classes of games, the equivalence of the dynamic pattern of the costate vector and the shadow value, then it is safe to conclude that the qualitative propositions will not be severely affected if we still interpret the costate vector as the shadow value for convenience. Section 1.5 considers a simple analytically solvable linear-quadratic differential game, which provides an opportunity to investigate all these possibilities. Another issue is the quantitative difference between the costate vector and shadow value, where the magnitude depends on the parameters of the model. An empirical exercise or numerical simulation may be needed to examine the quantitative difference, but this is not the goal of this chapter. The next section will examine the analogous envelope theorem for the feedback information structures.

1.4. The Envelope Theorem for Feedback Nash Equilibria

In this section, a similar result to Theorem 1.1 for differential game (1.1) under the assumption that its information structure is a *feedback* pattern is established. I retain assumption (A1), but assumptions (A2), through (A5) are replaced with their feedback counterparts.

(B2) The information structures is *feedback.*

[2] Economists usually have more interests in the evolution of the shadow value (or the costate varibale), e.g., the in-situ value of the stock in the Hotelling setting (Hotelling 1931, Salant et al. 1976) or the pollution tax over time (Farzin 1996 and Arnason 1990),

(B3) The Nash equilibria exists for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta)$, and is denoted by the P-tuple of vector $\mathbf{v}(t,z(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}) = (\mathbf{v}^1(t,z(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\mathbf{v}^2(t,z(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),...,\mathbf{v}^P(t,z(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}))$, where $z(t;\boldsymbol{\beta}) \in \mathbb{R}^N$ is the associated state trajectory, $\boldsymbol{\lambda}^p(t;\boldsymbol{\beta}) \in \mathbb{R}^N$ is the corresponding current value costate trajectory of player p, $p = 1,2,...,P$.

(B4) The solution vector $(\mathbf{z}(\cdot),\mathbf{v}(\cdot),\boldsymbol{\lambda}(\cdot)) \in C^{(1)}$ for all t and $\boldsymbol{\beta}$.

(B5) $\lim_{t\to\infty} e^{-rt}\boldsymbol{\lambda}(t;\boldsymbol{\beta})^{\dagger} \dfrac{\partial \mathbf{z}}{\partial \boldsymbol{\beta}}(t,z(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}) = \mathbf{0}_{N\times(A+N+1)}$.

Similar to the open-loop case, assumption (B2) describes the information that is available to all the players at time t. Given this, assumption (B3) assumes the existence of the Nash equilibrium and its corresponding optimal state and costate trajectories, while assumption (B4) assumes that the solution is locally differentiable with respect to the parameters. The most striking difference here is that the feedback Nash equilibria $\mathbf{v}(t;\mathbf{z}(t,\boldsymbol{\beta});r,\boldsymbol{\alpha})$, depends explicitly on the current state variables and the parameters $\boldsymbol{\alpha}$ and r, but not on the initial state $\mathbf{x}_0$ and initial time, in contrast to open-loop Nash equilibria. Assumption (B5), again, analogous to the open-loop case, is essentially a transversality condition.

By the definition of feedback Nash equilibria, the optimal time-path of the pth player's control vector, i.e., $\mathbf{v}^p(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})$, is a solution to the optimal control problem

$$W^p(\boldsymbol{\beta}) = \max_{\mathbf{u}^p(\cdot)} \int_0^{\infty} e^{-rt} f^p(\mathbf{x}(t),\mathbf{u}^p(t),\mathbf{v}^{-p}(t,z(t,\boldsymbol{\beta});r,\boldsymbol{\alpha}))dt \tag{1.25}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t),\mathbf{u}^p(t),\mathbf{v}^{-p}(t,\mathbf{z}(t,\boldsymbol{\beta});r,\boldsymbol{\alpha}), \quad \mathbf{x}(0) = \mathbf{x}_0 .$$

The current value Hamiltonian for player p is defined as

$$H^p(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p};r,\boldsymbol{\alpha}) \overset{\text{def}}{=} f^p(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p};\boldsymbol{\alpha}) + \boldsymbol{\lambda}^{p\dagger}\mathbf{g}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p};\boldsymbol{\alpha}). \tag{1.26}$$

By Theorem 2.2 of Mehlmann (1988), the necessary conditions that a feedback Nash equilibrium must satisfy are given by

$$H^p_{\mathbf{u}^p}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha})=\mathbf{0}_{M^p}\,, \tag{1.27}$$

$$\dot{\boldsymbol{\lambda}}^p=r\boldsymbol{\lambda}^p-\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha})$$

$$-\sum_{j=1,\,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{x},\mathbf{u}^p,\mathbf{v}^{-p}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p;\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})\,, \tag{1.28}$$

$$\dot{\mathbf{x}}(t)=\mathbf{g}(\mathbf{x}(t),\mathbf{u}^p(t),\mathbf{v}^{-p}(t,\mathbf{z}(t,\boldsymbol{\beta});r,\boldsymbol{\alpha}),\quad \mathbf{x}(0)=\mathbf{x}_0\,. \tag{1.29}$$

With these conditions, similar results for feedback Nash equilibria are stated below:

Theorem 1.2 (Feedback Nash Equilibria). *If the differential game* (1.25) *satisfies assumptions (B1)-(B5), then for all* $\boldsymbol{\beta}\in B(\boldsymbol{\beta}^0;\delta)$ *and* $p=1,2,...,P$, $W^p(\cdot)\in C^{(1)}$ *and*

$$\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta})=\int_0^\infty e^{-rt}[H^p_{\boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})]dt$$

$$+\int_0^\infty e^{-rt}\sum_{j=1,\,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})dt\,, \tag{1.30}$$

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta})=\boldsymbol{\lambda}^p(0;\boldsymbol{\beta})\,, \tag{1.31}$$

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta})=\int_0^\infty -te^{-rt}f^p(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})dt$$

$$+\int_0^\infty e^{-rt}\sum_{j=1,\,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial r}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})dt\,. \tag{1.32}$$

Proof. By assumption (B3), the optimal value function $W^p(\cdot)$ defined in Eq. (1.25) can be equivalently defined as

$$W^p(\boldsymbol{\beta})\overset{\text{def}}{=}\int_0^\infty e^{-rt}f^p(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}^p(t,\mathbf{z}(t,\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})dt\,. \tag{1.33}$$

It then follows from Eq. (1.33) and assumption (A1) that $W^p(\cdot) \in C^{(1)}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ, \delta)$, and

$$\frac{\partial}{\partial \boldsymbol{\beta}} \int_0^\infty e^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}^p(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) dt = \int_0^\infty \frac{\partial}{\partial \boldsymbol{\beta}} [e^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}^p(t, \mathbf{z}(t,\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha})] dt \quad (1.34)$$

To establish Eq. (1.30), differentiate Eq. (1.33) with respect to $\boldsymbol{\alpha}$ to get

$$\begin{aligned}\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_0^\infty e^{-rt} \{ & \frac{\partial f^p}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \sum_{j=1}^{P} \frac{\partial f^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) [\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha})] + \frac{\partial f^p}{\partial \boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) \} dt \, . \end{aligned} \quad (1.35)$$

Next, differentiate the identity form of Eq. (1.29) with respect to $\boldsymbol{\alpha}$ to arrive at

$$\begin{aligned} & \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), v(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \sum_{j=1}^{P} \frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), v(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) [\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \\ & + \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha})] + \frac{\partial \mathbf{g}}{\partial \boldsymbol{\alpha}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}); \boldsymbol{\alpha}) - \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_{N \times A} \, . \end{aligned} \quad (1.36)$$

Since $\mathbf{z}(0;\boldsymbol{\beta}) = \mathbf{x}_0$, it follows that

$$\partial \mathbf{z}(0;\boldsymbol{\beta}) / \partial \boldsymbol{\alpha} = \mathbf{0}_{N \times A} \, . \quad (1.37)$$

Premultiply Eq. (1.36) by the row vector $e^{-rt} \boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over the time horizon, add the result to Eq. (1.35), and then use Eqs. (1.26), (1.27), and (1.28) to simplify the resulting expression to get

$$\begin{aligned}\frac{\partial W^p}{\partial \boldsymbol{\alpha}}(\boldsymbol{\beta}) = & \int_0^\infty e^{-rt} [H_{\boldsymbol{\alpha}}^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha})] dt \\ & + \int_0^\infty e^{-rt} \sum_{j=1, j \neq p}^{P} \frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}), \boldsymbol{\lambda}^p(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{v}^j}{\partial \boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}); r, \boldsymbol{\alpha}) dt \end{aligned}$$

$$+\int_0^\infty re^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) - e^{-rt}\dot{\boldsymbol{\lambda}}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \mathbf{z}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) - e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger \frac{\partial \dot{\mathbf{z}}}{\partial \boldsymbol{\alpha}}(t;\boldsymbol{\beta}) . \tag{1.38}$$

Further simplifying Eq. (1.38) using Eqs. (1.15) and (1.37) and assumption (B5) gives Eq (1.30).

To prove Eq. (1.31), first differentiate Eq. (1.33) with respect to $\mathbf{x}_0$ to get

$$\frac{\partial W^p}{\partial \mathbf{x}_0}(\boldsymbol{\beta}) = \int_0^\infty e^{-rt}[\frac{\partial f^p}{\partial \mathbf{x}_0}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}))\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})$$

$$+\sum_{j=1}^P \frac{\partial f^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}))\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})]dt . \tag{1.39}$$

Next, differentiate the identity form of Eq. (1.29) with respect to $\mathbf{x}_0$ to derive the expression

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) + [\sum_{j=1}^P \frac{\partial g}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), v(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})$$

$$\times \frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta}) - \frac{\partial \dot{\mathbf{z}}}{\partial \mathbf{x}_0}(t;\boldsymbol{\beta})] = \mathbf{0}_{N\times N} . \tag{1.40}$$

Premultiply Eq. (1.40) by the row vector $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over the time horizon, add the result to Eq. (1.39), and then use Eqs. (1.20), (1.26), (1.27), (1.28), and (1.37), and assumption (B5), to simplify to get Eq.(1.31).

To prove Eq. (1.32), differentiate Eq. (1.33) with respect to r to get

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta}) = \int_0^\infty -re^{-rt} f^p(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})dt$$

$$+\int_0^\infty e^{-rt}\{\frac{\partial f^p}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})$$

$$+\sum_{j=1}^P \frac{\partial f^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})$$

$$+\frac{\partial \mathbf{v}}{\partial r}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})] + \frac{\partial f^p}{\partial r}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})\}dt . \tag{1.41}$$

Differentiate the identity form of Eq. (1.29) with respect to r to get

$$\frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})+\sum_{j=1}^{P}\{\frac{\partial \mathbf{g}}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})$$

$$\times[\frac{\partial \mathbf{v}^j}{\partial \mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta})+\frac{\partial \mathbf{v}}{\partial r}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})]\}-\frac{\partial \dot{\mathbf{z}}}{\partial r}(t;\boldsymbol{\beta})=\mathbf{0}_N. \tag{1.42}$$

Premultiply Eq. (1.42) by the row vector $e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger$, integrate the resulting inner product over the time horizon, add the result to Eq. (1.41), and then use Eqs. (1.23), (1.26), (1.27), and (1.28) to simplify the expression to arrive at

$$\frac{\partial W^p}{\partial r}(\boldsymbol{\beta})=\int_0^\infty -re^{-rt}f^p(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha});\boldsymbol{\alpha})dt$$

$$+\int_0^\infty e^{-rt}\sum_{j=1,\,j\neq p}^{P}\frac{\partial H^p}{\partial \mathbf{u}^j}(\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha}),\boldsymbol{\lambda}^p(t;\boldsymbol{\beta});\boldsymbol{\alpha})\frac{\partial \mathbf{v}^j}{\partial r}(t,\mathbf{z}(t;\boldsymbol{\beta});r,\boldsymbol{\alpha})dt$$

$$+\boldsymbol{\lambda}^p(0;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(0;\boldsymbol{\beta})-\lim_{t\to\infty}e^{-rt}\boldsymbol{\lambda}^p(t;\boldsymbol{\beta})^\dagger\frac{\partial \mathbf{z}}{\partial r}(t;\boldsymbol{\beta}). \tag{1.43}$$

since $\mathbf{z}(0;\boldsymbol{\beta})\equiv\mathbf{x}_0\Rightarrow\partial\mathbf{z}(0;\boldsymbol{\beta})/\partial r=\mathbf{0}_N$, and assumption (B5), Eq. (1.43) reduces to Eq. (1.32).

Theorem 1.2 establishes the envelope theorem for feedback Nash equilibria of discounted infinite horizon differential games. Equations (1.30) and (1.32) characterize the effects of a perturbation in $\boldsymbol{\alpha}$ and r on the optimal value function, where they have a similar structure to their open-loop counterparts given in Eqs. (1.7) and (1.9) The first terms on the right-hand side of Eqs. (1.30) and (1.32) represent the *direct* or *explicit* effect that results from the explicit appearance of the parameters $\boldsymbol{\alpha}$ and r, and the second terms on the right-hand side of Eqs. (1.30) and (1.32) again denote the *strategic* effect that results from the other $P-1$ players response to the perturbation of the parameters. Only the direct effect is present in optimal control theory, as the strategic effect is degenerately absent in optimal control theory. The main distinctive feature in the feedback information structure lies in Eq. (1.31). Contrary to the open-loop case, it implies that the costate vector is equal to the shadow value of the initial state, the familiar envelope result in the control literature.

The above two theorems characterize the envelope theorem for open-loop and feedback Nash equilibria of differential games. In what follows, I will apply these general envelope results to a simple linear-quadratic game. Compared to the general theorem, the linear-quadratic structure allows the explicit derivation of Nash equilirbia and hence the envelope expressions, It also allow one to calculate and sign the strategic term of Eq. (1.8), and thus a comparison of the dynamic pattern for the costate vector and shadow value for the open-loop case. This exercise can be viewed as a supportive example and an application of the general envelope theorems.

1.5. A Linear-Quadratic Differential Game

In this section, I explicitly derive the envelope results for a simple linear-quadratic differential game. The linear-quadratic structure is one of the few for which an analytical solution can be derived for open-loop and feedback Nash equilibria. Such a game is characterized by a linear system of state equations and quadratic objective functionals. The class of linear-quadratic differential games is the workhorse model in this field and has been widely applied in macroeconomics (Miller and Salmon 1985; Cohen and Michel 1988; Dockner and Neck 2008), international pollution control (Dockner and Long 1993; Rubio and Ulph 2006; Diamantoudi and Sartzetakis 2006) and natural resources extracting (Groot et al. 2003).

Consider a simple linear-quadratic model with two players, one state variable and one control variable for each player. Following Dockner et al. (2000, p.171-72), the game is given by the pair of optimal control problems:

$$W^1(\boldsymbol{\beta}) = \max_{u^1(t)} -\frac{1}{2}\int_0^\infty e^{-rt}\{k_1 x(t)^2 + k_2[u^1(t)]^2\}dt\,, \tag{1.44}$$

$$W^2(\boldsymbol{\beta}) = \max_{u^2(t)} -\frac{1}{2}\int_0^\infty e^{-rt}\{l_1 x(t)^2 + l_2[u^2(t)]^2\}dt\,, \tag{1.45}$$

$$\text{s.t.} \quad \dot{x}(t) = ax(t) + bu^1(t) + cu^2(t), \quad x(0) = x_0\,,$$

where $k_1, k_2, l_1, l_2, a, b, c$, and r are constant scalars, and x_0 is the given initial value of the state variable. Thus we have $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (k_1, k_2, l_1, l_2, a, b, c)$. The above game can be generalized to include more than two players, more than one state variable, or more than one control variable for each player. It is also possible to allow for nonstationary games, in which $k_1, k_2, l_1, l_2, a, b,$ and c are time functions. It is also can be generalized to include more general quadratic objective functions. However, there is no fundamental difference among different quadratic objective functions. Without losing generality, I assume that the parameters k_1, k_2, l_1, and l_2 are positive.

Notice that I am attempting to make the model as general as possible, and thus do not wish to impose specific economic content to further constrain the variables at this moment.[3] However, it is not difficult to provide some economic interpretation for the model with the proper signs of the parameters. For instance, one can view this as an international pollution control problem as in Dockner and Long (1993), if we assume $a \le 0$ as the natural decay rate, and all other parameters are positive. Thus $x(t)$ is the nonnegative stock of pollution at time t, $u^p(t)$ is the abatement activity for country p, and the goal of each country p is to minimize the present discounted cost due to the pollution damage and the abatement activity.

In what follows, I explicitly derive both the open-loop and feedback Nash equilibria for the above linear-quadratic game, and directly calculate the envelope results. This procedure can

[3] Most Economic problems usually require the nonnegativity of the state variable and choice variables.

be considered as a supportive example of the general envelope theorems that have been derived in the previous two sections. I will start with the open-loop case.

Let $(v^1(\cdot), v^2(\cdot))$ be an open-loop Nash equilibrium, and $z(\cdot)$ be its corresponding state variable trajectory. The current-value Hamiltonians of players 1 and 2 are defined as

$$H^1(x,u^1,v^2,\lambda^1;\boldsymbol{\alpha}) \overset{\text{def}}{=} -\frac{1}{2}[k_1x^2 + k_2(u^1)^2] + \lambda^1(ax + bu^1 + cv^2)\,, \tag{1.46}$$

$$H^2(x,u^2,v^1,\lambda^2;\boldsymbol{\alpha}) \overset{\text{def}}{=} -\frac{1}{2}[l_1x^2 + l_2(u^2)^2] + \lambda^2(ax + bv^1 + cu^2)\,. \tag{1.47}$$

From Eqs. (1.4) and (1.5), the necessary conditions are

$$u^1(t) = (b/k_2)\lambda^1(t)\,, \tag{1.48}$$

$$u^2(t) = (c/l_2)\lambda^2(t)\,, \tag{1.49}$$

$$\dot{\lambda}^1 = k_1x(t) + (r-a)\lambda^1(t)\,, \tag{1.50}$$

$$\dot{\lambda}^2 = l_1x(t) + (r-a)\lambda^2(t)\,. \tag{1.51}$$

In order to solve the necessary conditions, substitute Eqs. (1.48) and (1.49) into the state equation, and combine it with the Eqs. (1.50) and (1.51). This results in the canonical system

$$\begin{aligned} \dot{x}(t) &= ax(t) + (b^2/k_2)\lambda^1(t) + (c^2/l_2)\lambda^2(t) \\ \dot{\lambda}^1 &= k_1x(t) + (r-a)\lambda^1(t) \\ \dot{\lambda}^2 &= l_1x(t) + (r-a)\lambda^2(t). \end{aligned} \tag{1.52}$$

In vector notation the canonical system (1.52) can be written as

$$\dot{\mathbf{y}}(t) = \mathbf{A}\cdot\mathbf{y}(t)\,, \tag{1.53}$$

where $y(t) = (x(t),\ \lambda^1(t),\ \lambda^2(t))'$, and

$$\mathbf{A} \overset{\text{def}}{=} \begin{pmatrix} a & b^2/k_2 & c^2/l_2 \\ k_1 & r-a & 0 \\ l_1 & 0 & r-a \end{pmatrix}.$$

The eigenvalues s_1, s_2 and s_3 of $\mathbf{A}$ can be derived from the characteristic function $\det(\varsigma\mathbf{I}-\mathbf{A})=0$, where $\mathbf{I}$ is the identity matrix. They are

$$\begin{aligned} s_1 &= r/2-\sqrt{r^2/4-a(r-a)+M}\,, \\ s_2 &= r/2+\sqrt{r^2/4-a(r-a)+M}\,, \\ s_3 &= r-a\,, \end{aligned} \tag{1.54}$$

where $M \overset{\text{def}}{=} c^2(l_1/l_2)+b^2(k_1/k_2)>0$ as k_1, k_2, l_1, and l_2 are all positive. For simplicity, we only consider the case that $s_1<0$, $s_2>0$, and $s_3>0$. This occurs whenever the determinant of the system matrix $\mathbf{A}$ is negative.[4] I further focus on the optimal solutions that converge to the steady state. Given the initial state condition $x(0)=x_0$, the open-loop Nash equilibrium is given by

$$\begin{aligned} z(t;\boldsymbol{\beta}) &= x_0 e^{s_1 t}, \\ v^1(t;\boldsymbol{\beta}) &= x_0 \frac{b}{k_2}\left(\frac{w_{21}}{w_{11}}\right)e^{s_1 t}, \\ v^2(t;\boldsymbol{\beta}) &= x_0 \frac{c}{l_2}\left(\frac{w_{31}}{w_{11}}\right)e^{s_1 t}, \\ \lambda^1(t;\boldsymbol{\beta}) &= x_0 \frac{w_{21}}{w_{11}} e^{s_1 t}, \\ \lambda^2(t;\boldsymbol{\beta}) &= x_0 \frac{w_{31}}{w_{11}} e^{s_1 t}, \end{aligned} \tag{1.55}$$

where $\mathbf{w}_i \overset{\text{def}}{=} (w_{1i}, w_{2i}, w_{3i})^{\dagger} \in \mathbb{R}^3$ is the eigenvector of $\mathbf{A}$ corresponding to s_i. Thus we have

$$\begin{pmatrix} a-s_1 & b^2/k_2 & c^2/l_2 \\ k_1 & r-a-s_1 & 0 \\ l_1 & 0 & r-a-s_1 \end{pmatrix}\begin{pmatrix} w_{11} \\ w_{21} \\ w_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

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[4] There are five possible cases according to the signs of the eigenvalues. The other four are (i) $s_1>0$, $s_2>0$, and , $s_3>0$ (ii) $s_1<0$, $s_2>0$, and $s_3=0$ (iii) $s_1<0$, $s_2>0$, and $s_3<0$, and (iv) $s_1=0$, $s_2>0$, and $s_3>0$. It is assumed that the state equation has a global asymptotic stable steady state when the system is uncontrolled, hence $a<0$. Moreover, because $r^2/4-a(r-a)>0$ and $M>0$ by assumption, no complex solutions can occur in any case.

$$\frac{w_{21}}{w_{11}} = -\frac{k_1}{r-a-s_1} < 0, \tag{1.56}$$

$$\frac{w_{31}}{w_{11}} = -\frac{l_1}{r-a-s_1} < 0, \tag{1.57}$$

$$(a-s_1)w_{11} + (b^2/k_2)w_{21} + (c^2/l_2)w_{31} = 0. \tag{1.58}$$

Inspection of Eqs. (1.55) shows that assumption (A.5) is satisfied. Given the above explicit open-loop Nash equilibrium, it is straightforward to calculate the optimal value function for each player over the entire planning horizon. Namely,

$$W^1(\boldsymbol{\beta}) = -\frac{x_0^2}{2(r-2s_1)}[k_1 + (\frac{b^2}{k_2})(\frac{w_{21}}{w_{11}})^2] < 0, \tag{1.59}$$

$$W^2(\boldsymbol{\beta}) = -\frac{x_0^2}{2(r-2s_1)}[l_1 + (\frac{b^2}{l_2})(\frac{w_{31}}{w_{11}})^2] < 0. \tag{1.60}$$

Armed with above, one can investigate the envelope results in Theorem 1. For player 1, the shadow value of the initial state is

$$\frac{\partial W^1}{\partial x_0}(\boldsymbol{\beta}) = -\frac{x_0}{r-2s_1}[k_1 + \frac{b^2}{k_2}(\frac{w_{21}}{w_{11}})^2] < 0. \tag{1.61}$$

Similarly, the shadow value of the initial state for player 2 is

$$\frac{\partial W^2}{\partial x_0}(\boldsymbol{\beta}) = -\frac{x_0}{r-2s_1}[l_1 + \frac{c^2}{l_2}(\frac{w_{31}}{w_{11}})^2] < 0. \tag{1.62}$$

On the other hand, the value of the initial costate variable $\lambda^1(0;\boldsymbol{\beta})$ is equal to $x_0(w_{21}/w_{11})$, which is also negative, but not equal to the shadow value of the initial state $\partial W^1(\boldsymbol{\beta})/\partial \mathbf{x}_0$. Therefore, in the open-loop information structure, the initial costate value is not the shadow value of the initial state.

This model also provides an opportunity to explicitly show the envelope results. For player 1, the right-hand side of Eq. (1.8) is

$$\lambda^1(0;\boldsymbol{\beta}) + \int_0^\infty e^{-rt}[\frac{\partial H^1}{\partial u^2}(z(t;\boldsymbol{\beta}), v^1(t;\boldsymbol{\beta}), v^2(t;\boldsymbol{\beta}), \lambda^1(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial v^2}{\partial x_0}(t;\boldsymbol{\beta})]dt$$

$$= x_0 \frac{w_{21}}{w_{11}} + \frac{x_0}{r - 2s_1} \frac{c^2}{l_2} \frac{w_{21}w_{31}}{w_{11}^2}. \qquad (1.63)$$

Substituting Eqs. (1.56), (1.57), and (1.58) in Eq. (1.63), It follows that

$$\lambda^1(0;\boldsymbol{\beta}) + \int_0^\infty e^{-rt} \frac{\partial H^1}{\partial u^2}(z(t;\boldsymbol{\beta}), v^1(t;\boldsymbol{\beta}), v^2(t;\boldsymbol{\beta}), \lambda^1(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial v^2}{\partial x_0}(t;\boldsymbol{\beta})dt$$

$$= \frac{x_0}{r - 2s_1} \frac{w_{21}}{w_{11}}[(r - 2s_1) + \frac{c^2}{l_2}(\frac{w_{31}}{w_{11}})]$$

$$= -\frac{x_0}{r - 2s_1} \frac{k_1}{r - a - s_1}[r - a - s_1 + a - s_1 + \frac{c^2}{l_2} \frac{w_{31}}{w_{11}}]$$

$$= -\frac{x_0}{r - 2s_1} \frac{k_1}{r - a - s_1}[r - a - s_1 - \frac{b^2}{k_2} \frac{w_{21}}{w_{11}}]$$

$$= -\frac{x_0}{r - 2s_1}[k_1 + \frac{b^2}{k_2}(\frac{w_{21}}{w_{11}})^2]$$

$$= \frac{\partial W^1}{\partial x_0}(\boldsymbol{\beta}) < 0. \qquad (1.64)$$

Similar results can be derived for player 2. Equation (1.64) confirms one of the envelope results for open-loop Nash equilibria of an infinite planning horizon differential game. For other envelope results, namely Eqs. (1.7) and (1.9), the derivations are similar.

The linear quadratic structure yields an important outcome that cannot be obtained in general. Specifically, Eqs. (1.63) and (1.64) demonstrate that the costate variable $\lambda^p(0;\boldsymbol{\beta})$ is less than the shadow value of the initial state $\partial W^p(\boldsymbol{\beta}) / \partial x_0$, seeing as the strategic effect is positive under the stated assumptions on the parameters k_2 and l_2. Hence the costate variable overestimates the increase in minimum costs due to an increase in the initial state.

Second the growth rates of the costate variable ($\dot{\lambda}/\lambda$) and shadow values are the same. For convenience, denote $\psi(0;\boldsymbol{\beta})$ as the true shadow value of the state at time *t*, i.e., $\psi^p(0;\boldsymbol{\beta}) \overset{\text{def}}{=} \partial W^p / \partial x_0$. Consider the truncated game with initial time s, in which case the initial state variable is $x(s) = x_s = x_0 e^{s_1 s}$. Since the open-loop Nash equilibria are time consistence, the true shadow value of the state at time s, for player 1, is

$$\psi^1(s;\boldsymbol{\beta}) = \frac{\partial W^1}{\partial x_s}(\boldsymbol{\beta})\Big|_{x_s = x_0 e^{s_1 s}} = \frac{-x_0 e^{s_1 s}}{r - 2s_1}[k_1 + \frac{b^2}{k_2}(\frac{w_{21}}{w_{11}})^2] < 0 . \tag{1.65}$$

Thus, for player 1, it follows that

$$\frac{\dot{\psi}^1(s;\boldsymbol{\beta})}{\psi^1(s;\boldsymbol{\beta})} = s_1 = \frac{\dot{\lambda}^1(s;\boldsymbol{\beta})}{\lambda^1(s;\boldsymbol{\beta})} . \tag{1.66}$$

Again, similar results can be obtained for player 2. Equation (1.66) shows that the growth rates of the shadow value and costate variable are the same in the class of linear-quadratic differential games under consideration. Thus, this result demonstrates that the main problem with treating the costate variable as the shadow value is the magnitude of their quantitative difference in the class of games under consideration. This difference depends on the parameters and is solely an empirical question, hence empirical or simulation exercises may be worth to carry out. It also suggests that a statistical calibration may be sufficient to recover the true shadow value of the state from the costate variable.

To get some flavor of the magnitudes involved in the comparison of the costate variable and the shadow value, I specify the following values for the parameters of the model: $k_1 = l_1 = 1$, $k_2 = l_2 = 0.75$, $a = -0.25$, $b = c = 1$, $r = 0.05$ and $x_0 = 25$. Figure 1 graphs the dynamics of the costate variable and the shadow value for player 1. The numerical results show that the costate variable is about 25% lower than the shadow value of the state at the initial time, but it eventually converges to the shadow value in the linear-quadratic differential game.

These results have an important policy implication. For instance, in the resource extracting models, if the policy maker somehow wants to protect the in situ stocks by regulating the extraction rates of the firms, he would like to know the in situ value of the stock. However, he will overestimate the in situ value of the stock if he interprets the costate variable as the shadow value in the conventional way. Therefore, the policy maker will over-regulate the extraction rates and hence reduce the welfare of the economy. It is important to remember that the above results do not hold in general but are limited to the linear-quadratic class of games under consideration.

Now I turn to the derivation of a feedback Nash equilibrium of the game considered in Eqs. (1.44) and (1.45). The HJB equations for this game are

$$rW^1(x;\boldsymbol{\alpha},r)=\max_{u^1}\{-\frac{1}{2}[k_1x^2+k_2(u^1)^2]+\frac{\partial W^1}{\partial x}(x;\boldsymbol{\alpha},r)[ax+bu^1+cv^2(t,\mathbf{x};r,\boldsymbol{\alpha})]\}\,, \quad (1.67)$$

$$rW^2(x;\boldsymbol{\alpha},r)=\max_{u^2}\{-\frac{1}{2}[k_1x^2+k_2(u^2)^2]+\frac{\partial W^2}{\partial x}(x;\boldsymbol{\alpha},r)[ax+bv^1(t,\mathbf{x};r,\boldsymbol{\alpha})+cu^2]\}\,. \quad (1.68)$$

To solve above equations, I propose a general functional form for $W^p(\cdot)$, and then determine a set of parameter values for it so that the proposed function satisfies Eqs. (1.67), and (1.68). Noticed the fact that the model under consideration is linear-quadratic, I guess the optimal value functions are of the form

$$W^p(x;\boldsymbol{\alpha},r)=(1/2)\phi^p x^2, \quad p=1,2\,, \quad (1.69)$$

where ϕ^p, $p=1,2$, are unknown parameters of the optimal value functions which are to be determined. Substituting Eq. (1.69)into the HJB equations, the first-order necessary conditions are

$$u^1 = (b/k_2)\phi^1 x\,, \tag{1.70}$$

$$u^2 = (c/l_2)\phi^2 x\,. \tag{1.71}$$

From Eqs. (1.70) and (1.71), a feedback Nash equilibrium is $v^1(x;r,\boldsymbol{\alpha}) = (b/k_2)\phi^1 x$ and $v^2(x;r,\boldsymbol{\alpha}) = (c/k_2)\phi^2 x$. Substituting the Eqs. (1.70) and (1.71) back into the HJB equations, I get a standard system of algebraic Riccati equations, namely,

$$k_1 + (r-2a)\phi^1 - (b^2/k_2)(\phi^1)^2 - 2(c^2/l_2)(\phi^1\phi^2) = 0\,, \tag{1.72}$$

$$l_1 + (r-2a)\phi^2 - (c^2/l_2)(\phi^2)^2 - 2(b^2/k_2)(\phi^1\phi^2) = 0\,. \tag{1.73}$$

If we retain the assumptions that the parameters k_1, k_2 , l_1 , and l_2 are positive and the uncontrolled system is stable, i.e. $a \le 0$, then the above algebraic Riccati equations admit a unique negative solution, i.e., $\phi^1 < 0$ and $\phi^2 < 0$ (Engwerda 2000, Th.1). Hence the state equation becomes

$$\dot{x}(t) = ax(t) + (b^2/k_2)\phi^1 x + (c^2/l_2)\phi^2 x, \quad x(0) = x_0\,. \tag{1.74}$$

According to Theorem 2.4 of Boyce and Diprima (1977), there exist a unique absolutely continuous solution $z(\cdot)$ for the initial value problem (1.74). Moreover, the quadratic optimal value functions $W^i(\cdot)$ are bounded on any bounded interval, and x is bounded between $[0, x_0]$ since $a + (b^2/k_2)\phi^1 + (c^2/l_2)\phi^2 < 0$. Therefore the proposed strategies $v^1(x;r,\boldsymbol{\alpha})$ and $v^2(x;r,\boldsymbol{\alpha})$ constitute a feedback Nash equilibrium (Dockner et al. 2000, Th 4.4). The equilibrium state trajectory is $z(t;\boldsymbol{\beta}) = x_0 e^{\eta t}$, where

$$\eta \overset{\text{def}}{=} a + (b^2/k_2)\phi^1 + (c^2/l_2)\phi^2\,. \tag{1.75}$$

It follows immediately that the open-loop representation of the control path generated by the feedback Nash equilibrium is given by

$$\tilde{v}^1(t;\boldsymbol{\beta}) = x_0(b/k_2)\phi^1 e^{\eta t}, \tag{1.76}$$

$$\tilde{v}^2(t;\boldsymbol{\beta}) = x_0(c/l_2)\phi^2 e^{\eta t}. \tag{1.77}$$

The resulting optimal value functions of the two players are

$$W^1(\boldsymbol{\beta}) = -\frac{x_0^2}{2(r-2\eta)}[k_1 + (\frac{b^2}{k_2})(\phi^1)^2] < 0, \tag{1.78}$$

$$W^2(\boldsymbol{\beta}) = -\frac{x_0^2}{2(r-2\eta)}[l_1 + (\frac{c^2}{l_2})(\phi^2)^2] < 0. \tag{1.79}$$

The shadow value of the initial state x_0 for player 1 is

$$\frac{\partial W^1}{\partial x_0}(\boldsymbol{\beta}) = -\frac{x_0}{r-2\eta}[k_1 + (\frac{b^2}{k_2})(\phi^1)^2] < 0. \tag{1.80}$$

Next I show one of the envelope results for feedback Nash equilibrium, namely, Eq. (1.31) in Theorem 1.2. In contrast to the open-loop case, the costate variable $\lambda^p(\cdot)$ is not explicitly specified in the HJB equations for the feedback Nash equilibrium. Thus I consider the open-loop representations of the current value Hamiltonian for player 1 generated by the feedback Nash equilibrium strategies. Without loss any generality, I only consider for player 1. Similar results can be derived for player 2. The current value Hamiltonian of player 1 is defined as

$$H^1(x,u^1,v^2,\lambda^1;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} -\frac{1}{2}[k_1x^2 + k_2(u^1)^2] + \lambda^1[ax + bu^1 + cv^2(x;r,\boldsymbol{\alpha})]. \tag{1.81}$$

One of the first order conditions, Eq. (1.27), now is

$$\frac{\partial H^1}{\partial u^1}(x,u^1,v^2,\lambda^1;\boldsymbol{\alpha}) = 0 \Rightarrow \lambda^1 = \frac{k_2}{b}u^1. \tag{1.82}$$

By substituting from Eq. (1.76), the costate variable becomes

$$\lambda^1(t;\boldsymbol{\beta}) = x_0\phi^1 e^{\eta t}, \tag{1.83}$$

hence $\lambda^1(0;\boldsymbol{\beta}) = x_0\phi^1$. In order to show

$$\frac{\partial W^1(\boldsymbol{\beta})}{\partial x_0} = \lambda^1(0;\boldsymbol{\beta}), \tag{1.84}$$

rearrange Eq. (1.72), to yield

$$\begin{aligned} k_1 + \frac{b^2}{k_2}(\phi^1)^2 &= -r\phi^1 + 2a\phi^1 + 2\frac{b^2}{k_2}(\phi^1)^2 + 2\frac{c^2}{l_2}\phi^1\phi^2 \\ &= -r\phi^1 + 2\phi^1[a + \frac{b^2}{k_2}\phi^1 + \frac{c^2}{l_2}\phi^2]. \end{aligned} \tag{1.85}$$

Multiply Eq. (1.85) by x_0, simplify it by Eq. (1.75), and rearrange the equation to get

$$-\frac{x_0}{r-2\eta}[k_1 + \frac{b^2}{k_2}(\phi^1)^2] = x_0\phi^1. \tag{1.86}$$

Equation (1.86) is identical to the Eq. (1.84), and hence confirms one of the envelope results for feedback Nash equilibria of an infinite planning horizon differential game. It shows that the costate variable $\lambda^1(\cdot)$ reflects the shadow value of the state at the initial time in the feedback Nash equilibrium, the conventional interpretation of the costate variable.

1.6. Conclusion

This chapter extended the envelope theorem so as to apply to the class of discounted infinite horizon differential games under proper assumptions. The presence of the strategic effects in the envelope expressions require that the optimal solutions of the games must be known in order to compute envelope results and sign the first-order partial derivative of the optimal value function. This fact greatly limits the application of the envelope results in differential games relative to optimal control problems. An important implication that directly follows from the envelope results is that only in the feedback case, does the costate vector represents the shadow value of the state vector at the initial time. This conventional interpretation of the costate is not valid in the open-loop information structure.

Finally, the general envelope theorem has been applied to a simple linear-quadratic model. By actually computing the solution of the games, I established the envelope results for both open-loop and feedback Nash equilibria. The linear-quadratic structure of the game yields extra outcomes that cannot be obtained in the general setting. Specifically, the costate variable uniformly overestimates the shadow value of the state in the considered model, but their growth rates are equal. These facts suggest that a statistical calibration may be sufficient to recover the archetypal economic interpretation of the costate variable.

CHAPTER TWO: A QUALITATIVE CHARACTERIZATION OF SYMMETRIC OPEN-LOOP NASH EQUILIBRIA IN DISCOUNTED INFINITE HORIZON DIFFERENTIAL GAMES

2.1. Introduction

Many economics and management science problems are characterized by multiple interdependent decision makers and the results enduring consequences of their decisions. Differential game theory conceptualizes problems of this kind by assuming that time evolves continuously and the system dynamics can be described by a set of ordinary differential equations. Examples of differential games in economics and management science can be found in Case (1979), Mehlmann (1988), Clemhout and Wan (1994), and Dockner et al (2000), among others. The Proceedings and the Associated Annals of the International Symposia on Dynamic Games and Applications (for example, Haurie et al., 2006) held every other year document the development of both the theory and application of differential games over the past 20 years.

In this chapter I examine the symmetric open-loop Nash equilibria (OLNE) of a common class of differential games for its particular qualitative structures that leads to sharp predictions regarding the local asymptotic stability of its steady state, the steady state comparative statics, and the local comparative dynamics. The class of differential games under consideration is the P-players, discounted infinite time horizon variety, with time entering explicitly only through the exponential discount factor, a single state variable, a single control variable for each player, a time independent vector of parameters influencing the system, a given initial state, and a symmetric OLNE that converges to a simple steady state. This class of differential games covers a large body of applied models in economics and management science. For example, the common property resources models of Ploude and Yeung (1989), Negri (1989), Arnason (1990), and Caputo and Leuck (2003), the international pollution control models of Long (1992), Dockner and Long (1993), List and Mason (2001), and Kossioris et al (2008), and the industrial organization models of Clemhout et al. (1971), Spence (1979), Fershtman and Kamien (1987), Reynolds (1987), Cellini and Lambertini (2004), and Lambertini (2010).

The idea of systematical qualitative characterization of an optimization model has its origins in Samuelson's (1947) seminal work, in which he derived the comparative statics properties of a general class of unconstrained static optimization problems. This theory was generalized to constrained static optimization problems by Silberberg (1974) and Partovi and Caputo (2006), to static game theory by Caputo (1996), and to the optimal control theory by Caputo (1990 and 2003). In addition, Caputo (1997) examined a class of optimal control models for its particular qualitative properties, namely, the local asymptotic stability of the steady state, the steady state comparative statics, and the local comparative dynamics. A striking common feature among these papers is that the resulting qualitative structures are generic in the sense that only the underlying mathematical structure of the problems responsible for the qualitative properties of the model, while the specific functional forms (e.g. the Cobb-Douglas production function in cost minimization problem) are independent of determining these qualitative results.

The differential game literature is not unfamiliar with qualitative results. In fact, a qualitative characterization of an equilibrium is usually the best one can do, due to the difficulty of deriving analytically the equilibria for all but most simple functional forms. As Clemhout and Wan (1994) pointed out, "… the theory of differential games is effective for what it does best, in characterizing qualitatively the time-profile of equilibrium plays."

Brock (1977) was the first to investigate the local stability properties of OLNE. Fershtman and Muller (1984) derived a conditional global asymptotic stability result for OLNE of a capital accumulation game. Reynolds (1987) showed the conditional global asymptotic stability of the OLNE for a linear quadratic capacity investment game. Fershtman and Kamien (1987) studied the local asymptotic stability properties of the symmetric steady state price, as well as some of its comparative statics, for a dynamic duopolistic competition model with sticky prices. Negri (1989) examined the local asymptotic stability property of the symmetric steady state and its comparative statics for a common property aquifer problem. Dockner and Takahashi (1990) studied the local asymptotic stability properties of a class of capital accumulation games, which essentially was the linear-quadratic class. Most of these works concluded that the steady state equilibria are characterized by a general saddle-point property. Caputo and Lueck (2003) showed that, in addition to a saddle point, there is a possibility that the steady state is locally asymptotically stable in a common-property exploitation problem. They also provide a sufficient condition to identify a local saddle point type of steady state, and a necessary condition for a locally asymptotically stable steady state.[5] Despite the qualitative results in the literature, it has focused on special classes of differential game, in most cases, the linear-quadratic class. Therefore, the existing literature does not give generic results. Moreover, little work has been done regarding the local comparative dynamic properties of differential games.

This chapter is the first to take a systematic approach to uncovering the qualitative properties of symmetric OLNE of a widely applied class of differential games. This special class of games allows the dynamic system to be reduced to a pair of differential equations, and hence the standard phase-diagram method can be applied to study the equilibria of the model. Moreover, it is well known that the steady state properties of higher dimensional ordinary differential equations are not as well-developed compared to their counterpart in the plane. If and when more progress is made in the qualitative theory of ordinary differential equations, more

[5] Unfortunately, their sufficient condition requires one to compute the steady state OLNE, which significantly reduces its applicability.

findings regarding the qualitative structure of equilibria in differential games are to be expected. Meanwhile, the qualitative results of symmetric OLNE in this chapter can serve as a benchmark model for future studies.

The major contribution of this chapter is the demonstration that the qualitative properties of symmetric OLNE for the class of differential games under consideration depend critically on the functional forms that are specified in the game. In fact, it is possible that the qualitative results are completely opposite for any given game by using different functional forms with the same underlying mathematical structure. This result is contrary to a common feature in previous literatures of charactering the qualitative results for static optimization problems, static game theory, and optimal control theory, and hence justifies the importance of this chapter. Our result then raises a concern about generalizing any qualitative results, which are derived from games with specific functional forms, e.g., the linear-quadratic class of differential games. It also has an important policy implication. For a certain policy goal, the agency may receive opposite policy recommendations simply because different researchers use different functional forms in simulating the same underlying model. Moreover, this chapter also provides some simple sufficient conditions to identify a local saddle point type of steady state for the generic class of games under consideration.

2.2. The Problem and Assumptions

The class of P-players discounted infinite horizon differential games under consideration is given by

$$W^p(x_0, \boldsymbol{\theta}) \overset{\text{def}}{=} \max_{u^p(\cdot)} \int_0^\infty e^{-rt} f^p(x(t), u^p(t), \mathbf{u}^{-p}(t); \boldsymbol{\alpha}) dt \tag{2.1}$$

$$\text{s.t.} \quad \dot{x}(t) = g(x(t), u^p(t), \mathbf{u}^{-p}(t); \boldsymbol{\beta}), \qquad x(0) = x_0,$$

where P is the total number of players indexed by $p=1,2,...,P$, $r\in\mathbb{R}_+$ is the discount rate, and $\boldsymbol{\theta}\overset{\text{def}}{=}(r,\boldsymbol{\alpha},\boldsymbol{\beta})$ is the vector of time-independent parameters of the game. The state of the game at each instant $t\in[0,\infty)$ is given by $x(t)\in X\subset\mathbb{R}$, and the initial value of the state is fixed at the value $x_0\in X$. At each instant $t\in[0,\infty)$, player p chooses a control variable from her set of admissible controls, i.e., $u^p(t)\in U^p\subset\mathbb{R}$, to maximize her discounted payoff functional, where vector $\mathbf{u}^{-p}(t)$ consists of all other players' controls at time *t*, i.e.,

$$\mathbf{u}^{-p}(t)\overset{\text{def}}{=}(u^1(t),u^2(t),\ldots,u^{p-1}(t),u^{p+1}(t),\ldots,u^P(t)).$$

The following assumptions are imposed on the game for all players.

(C1) $f^p(\cdot):X\times\mathbf{U}\times A\to\mathbb{R}$, $f^p(\cdot)\in C^{(2)}$ on its domain $(x,\mathbf{u};\boldsymbol{\alpha})\in X\times\mathbf{U}\times A$ for any $p\in\{1,2,...,P\}$, where $\mathbf{u}$ and $\mathbf{U}$ denote the vector of all players' control variables and the population feasible control set, respectively, and $X,\mathbf{U}$ and $A\subset\mathbb{R}^{K_1}$ are convex and compact.

(C2) $g(\cdot):X\times\mathbf{U}\times B\to\mathbb{R}$, $g(\cdot)\in C^{(2)}$ on the domain $(x,\mathbf{u};\boldsymbol{\beta})\in X\times\mathbf{U}\times B$, $g_{u^p}(\cdot)\neq 0$ along any OLNE, and $B\subset\mathbb{R}^{K_2}$ is convex and compact.

(C3) Players are symmetric and the information structure of the game is an open-loop pattern and thus consists of the set $(P,\boldsymbol{\theta},x_0)$ at time $t\in[0,\infty)$ for each player.

(C4) A symmetric ONLE exists for all $P\in\mathbb{Z}_+$ and $(x_0,\boldsymbol{\theta})\in\text{int}(X\times\mathbb{R}_+\times A\times B)$. The symmetric OLNE is denoted by $v(t;x_0,\boldsymbol{\theta})=v^p(t;x_0,\boldsymbol{\theta})$, $p=1,2,...,P$, and its corresponding state and current value costate trajectories are $z(t;x_0,\boldsymbol{\theta})$, and

$\lambda(t;x_0,\boldsymbol{\theta})=\lambda^p(t;x_0,\boldsymbol{\theta}),\ p=1,2,...,P$. Furthermore, $(z(t;x_0,\boldsymbol{\theta}),v(t;x_0,\boldsymbol{\theta}))\in \text{int}(X\times U)$, where $U^p=U$, $p=1,2,...,P$.

(C5) The symmetric OLNE functions $(z(\cdot),v(\cdot),\lambda(\cdot))\in C^{(1)}$ on their domains.

(C6) $H^p_{u^p u^p}(x(t),\mathbf{u}(t),\lambda^p(t);\boldsymbol{\alpha},\boldsymbol{\beta})\neq 0$ and $\sum_{j=1}^{P} H^p_{u^p u^j}(x(t),\mathbf{u}(t),\lambda^p(t);\boldsymbol{\alpha},\boldsymbol{\beta})<0$, $p\in\{1,2,...,P\}$, along the symmetric OLNE, where $H^p(\cdot)$ is the current value Hamiltonian defined in (2.6) and $\lambda^p(t)$ is the current value costate variable for player *p*.

(C7) There exists an interior, symmetric, and simple steady state OLNE to game (2.1) for any $P\in\mathbb{Z}_+$ and $(x_0,\boldsymbol{\theta})\in \text{int}(X\times\mathbb{R}_+\times A\times B)$, denoted by $(x^*(\boldsymbol{\theta}),u^*(\boldsymbol{\theta}),\lambda^*(\boldsymbol{\theta}))$, which is the solution to the pair of steady state equations (2.13) and (2.14).

(C8) The symmetric OLNE converges to the simple steady state for any $\boldsymbol{\theta}\in \text{int}(\mathbb{R}_+\times A\times B)$, i.e., $\lim_{t\to\infty}(z(t;x_0,\boldsymbol{\theta}),v(t;x_0,\boldsymbol{\theta}),\lambda(t;x_0,\boldsymbol{\theta}))=(x^*(\boldsymbol{\theta}),u^*(\boldsymbol{\theta}),\lambda^*(\boldsymbol{\theta}))$.

The differentiability properties imposed on $f^p(\cdot)$ and $g(\cdot)$ in assumptions (C1) and (C2) are common to virtually all applied papers in economics and management science that use differential game theory. The convex and compact domains and the $C^{(2)}$ assumption on the functions $f^p(\cdot)$ and $g(\cdot)$ imply that they and their first- and second-order partial derivative are bounded. This fact, along with the exponential discount factor, implies that the improper integral $W^p(\cdot)$ converges for all admissible pairs $(x,\mathbf{u})$, so differential game (2.1) is well defined. Finally $g_{u^p}(\cdot)\neq 0$ along the symmetric OLNE implies that each player's control variable affects the evolution of the state variable in equilibrium.

Assumption (C3) specifies the symmetric characteristic among players and the information that is available to them at any instant of the game. The symmetric property is crucial throughout the chapter, as it allows the analysis focus on symmetric OLNE, in which case the dynamic system of game (2.1) can be reduced to the pair of ordinary differential equations in (x,u), thus implying that the standard phase-diagram method can be applied. Additionally, the symmetric property impose several extra requirements on the functions $f^p(\cdot)$ and $g(\cdot)$ along the symmetric OLNE for any $p,\, q \in \{1,2,...,P\}$. Specifically,

(C3.1) $f_x^{p\bullet}(\cdot) = f_x^{q\bullet}(\cdot)$, $f_{u^p}^{p\bullet}(\cdot) = f_{u^q}^{q\bullet}(\cdot)$, $f_{xx}^{p\bullet}(\cdot) = f_{xx}^{q\bullet}(\cdot)$, $f_{u^p u^p}^{p\bullet}(\cdot) = f_{u^q u^q}^{q\bullet}(\cdot)$, $f_{xu^p}^{p\bullet}(\cdot) = f_{xu^q}^{q\bullet}(\cdot)$,

$$\sum_{j=1,\, j\neq p}^{P} f_{u^j}^{p\bullet}(\cdot) = \sum_{j=1,\, j\neq q}^{P} f_{u^j}^{q\bullet}(\cdot),\ \sum_{j=1,\, j\neq p}^{P} f_{u^p u^j}^{p\bullet}(\cdot) = \sum_{j=1,\, j\neq q}^{P} f_{u^q u^j}^{q\bullet}(\cdot),\ \text{and}\ \sum_{j=1,\, j\neq p}^{P} f_{xu^j}^{p\bullet}(\cdot) = \sum_{j=1,\, j\neq q}^{P} f_{xu^j}^{q\bullet}(\cdot),$$

(C3.2) $g_{u^p}^{\bullet}(\cdot) = g_{u^q}^{\bullet}(\cdot)$, $g_{u^p u^p}^{\bullet}(\cdot) = g_{u^q u^q}^{\bullet}(\cdot)$, and $g_{u^p x}^{\bullet}(\cdot) = g_{u^q x}^{\bullet}(\cdot)$,

where "$\bullet$" denotes that the functions are evaluated along a symmetric OLNE. Assumption (C3.1) specifies some structural requirements on the players' instantaneous payoff functions $f^p(\cdot)$ in order to comply with the assumption of symmetric players. The following interpretations for (C3.1) and (C3.2) hold only along the symmetric OLNE. The first equality asserts that the state variable has the same marginal effect on every player's instantaneous payoff function. The second says that the marginal effect of a player's own control variable on her instantaneous payoff function is the same across players. The third equality says that the own second-order effects of the state variable on $f^p(\cdot)$ are the same across all players. The fourth equality shows that the second-order effects of a player's own control on her instantaneous payoff function are the same across players. The fifth equality implies that the second-order cross effects between the state variable and a player's own control on any player's instantaneous payoff function are the same. The last three equality shows that other players' controls have the same effects on one player's instantaneous payoff function, and on the marginal impact a player's own control and the state variable have on a player's instantaneous payoff function.

Assumption (C3.2) specifies similar internal structure requirements on the state dynamic function $g(\cdot)$, and as such, need not be interpreted. Assumptions (C3.1) and (C3.2) are useful in simplifying the derivations of the qualitative results in the next sections. It is worthwhile to point out that (C3.1) and (C3.2) are only valid along the symmetric OLNE paths. In fact, they are usually not valid off the symmetric OLNE paths, as the examples in sections 2.7 and 2.8 will illustrate. These symmetric assumptions are similar to Mehlmann (1988, p.148), where he presented them in the form of the players' current value Hamiltonian.

Assumption (C4) is a result of the assumption (C3), and it explicitly assumes the existence of a symmetric OLNE. Alternatively, one can impose assumptions on the primitives that are sufficient to imply the existence of an OLNE, e.g. Dockner et al (2000, Theorem 4.2). However, this approach implies the qualitative results so obtained are not intrinsic to the model but are in fact conditioned on those sufficient conditions. Thus, such an approach reduces the generality and the applicability of the results. Furthermore, the restriction that the parameters lie in the interior of their convex and compact sets rules out the mathematical complications when differentiating the equilibria with respect to a parameter at the boundary of its set. This assumption also insures that the equilibrium paths are not at the boundary of the admissible region. In practice, it may imply that the non-negativity restrictions on state and control variables are not binding along the symmetric OLNE.

Assumption (C5) is another crucial assumption since there is no guarantee that the equilibria are differentiable from assumptions (C1) and (C2). It allows the use of differential calculus in computing the desired qualitative results. Assumption (C6) implies that the second-order sufficient condition for maximizing each player's current value Hamiltonian holds along the symmetric OLNE. It also allows the necessary conditions to be reduced to a pair of differential equation in (x,u). Notice that $\sum_{j=1}^{P} H^p_{u^p u^j}(\cdot) < 0$ is a strong, but reasonable, assumption for symmetric players, and asserts that the second-order own effect of a player's control dominates the second-order cross effects from the other players' controls along the symmetric OLNE. If sign($H^p_{u^p u^p}$)=sign($H^p_{u^p u^j}$), $p, j \in \{1, 2, ..., P\}$, or players' control variables are additively separable in their current value Hamiltonian, i.e., $H^p_{u^p u^j} = 0 \;\; j = 1, 2, ..., P$ and $j \neq p$, then along with the fact that $H^p_{u^p u^p}(\cdot) \neq 0$ and the necessary condition (2.3), it follows that $\sum_{j=1}^{P} H^p_{u^p u^j}(\cdot) < 0, \;\; p \in \{1, 2, ..., P\}$.

Assumption (C7) asserts the existence of a symmetric and simple steady state equilibrium to game (2.1). A simple steady state is one in which the determinant of the Jacobian matrix of the pair of differential equations defining the symmetric OLNE is nonzero at the steady state.

Finally, assumption (C8) asserts that the symmetric OLNE and its corresponding state variable converge to their steady state values as $t \to \infty$. Therefore, it implies that the steady state of game (2.1) is hyperbolic. This implies that the linearization method of ordinary differential equations can be applied to study the local stability properties of the steady state, and the phase diagram of the original dynamic system and its linearization at the steady state are qualitatively equivalent in a neighborhood of the steady state. It will be shown that assumptions (C7) and (C8) imply that the steady state is either a local saddle point or locally asymptotically stable.

2.3. The Necessary Conditions

According to Basar and Olsder (1995, Theorem 6.11), the necessary conditions for OLNE of differential game (2.1), for each player $p \in \{1, 2, ..., P\}$, are

$$H^p_{u^p}(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = f^p_{u^p}(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\alpha}) + \lambda^p g_{u^p}(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\beta}) = 0, \tag{2.2}$$

$$H^p_{u^p u^p}(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\alpha}, \boldsymbol{\beta}) = f^p_{u^p u^p}(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\alpha}) + \lambda^p g_{u^p u^p}(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\beta}) \leq 0, \tag{2.3}$$

$$\dot{\lambda}^p = r\lambda^p - H^p_x(x, u^p, \mathbf{u}^{-p}, \lambda^p; \boldsymbol{\alpha}, \boldsymbol{\beta}) = (r - g_x)\lambda^p - f^p_x, \tag{2.4}$$

$$\dot{x} = g(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\beta}), \tag{2.5}$$

where $H^p(\cdot)$ is the current value Hamiltonian for player *p*, which is defined as

$$H^p(x, u^p, \mathbf{u}^{-p}, \lambda^p; \boldsymbol{\alpha}, \boldsymbol{\beta}) \overset{\text{def}}{=} f^p(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\alpha}) + \lambda^p g(x, u^p, \mathbf{u}^{-p}; \boldsymbol{\beta}). \tag{2.6}$$

By assumption (C3), I consider symmetric OLNE throughout this chapter. Thus, the symmetric OLNE paths and their corresponding current-value costate and state paths for each player

$$v^1(t; x_0, \boldsymbol{\theta}) = v^2(t; x_0, \boldsymbol{\theta}) = \cdots = v^P(t; x_0, \boldsymbol{\theta}) = v(t; x_0, \boldsymbol{\theta}), \tag{2.7}$$

$$\lambda^1(t; x_0, \boldsymbol{\theta}) = \lambda^2(t; x_0, \boldsymbol{\theta}) = \cdots = \lambda^P(t; x_0, \boldsymbol{\theta}) = \lambda(t; x_0, \boldsymbol{\theta}), \tag{2.8}$$

$$z(t; x_0, \boldsymbol{\theta}). \tag{2.9}$$

In order to deduce the qualitative properties of the symmetric OLNE, it is useful to reduce the necessary conditions (2.2), (2.4) and (2.5) to a pair of ordinary differential equations in (x, u). To do so, consider the symmetric OLNE and differentiate Eq. (2.2) with respect to *t* to derive

$$\sum_{j=1}^{P} f^p_{u^p u^j}\dot{u} + f^p_{u^p x}\dot{x} + \lambda[\sum_{j=1}^{P} g_{u^p u^j}\dot{u} + g_{u^p x}\dot{x}] + \dot{\lambda} g_{u^p} = 0. \tag{2.10}$$

Since $g_{u^p}(x,u^p,\mathbf{u}^{-p};\boldsymbol{\beta})\neq 0$ along the symmetric OLNE by assumption (C2), one can solve Eq. (2.2) to get $\lambda=-f^p_{u^p}\big/g_{u^p}$, $p\in\{1,2,...,P\}$, and substitute it along with Eqs. (2.4) and (2.5) into (2.10) to get

$$\dot{u}=h(x,\mathbf{u};\boldsymbol{\theta})\overset{\text{def}}{=}\frac{f^p_{u^p}(r-g_x)+g_{u^p}f^p_x+g[f^p_{u^p}(g_{u^p})^{-1}g_{u^px}-f^p_{u^px}]}{\sum_{j=1}^{P}[f^p_{u^pu^j}-f^p_{u^p}(g_{u^p})^{-1}g_{u^pu^j}]}, \tag{2.11}$$

$$\dot{x}(t)=g(x,\mathbf{u};\boldsymbol{\beta}). \tag{2.12}$$

According to assumption (C6), the denominator of (2.11) is negative, and hence Eq. (2.11) is well defined. Equations (2.11) and (2.12) are the pair of ordinary differential equations for game (2.1) that form the basis of the analysis that follows. Notice that one could also reduce the necessary conditions (2.2), (2.4) and (2.5) to a pair of ordinary differential equation in (x,λ). To do so, one could apply the implicit function theorem to solve u locally as a function of $(x,\lambda;\boldsymbol{\alpha},\boldsymbol{\beta})$, i.e., $u=\hat{u}(x,\lambda;\boldsymbol{\alpha},\boldsymbol{\beta})$ from Eq. (2.2) by assumption (C6), and substitute it into (2.4) and (2.5) to get the desired result. However, the information in the (x,λ) dynamic system is the same as that in (x,u), hence pursuing one approach is sufficient.

2.4. Stability and the Steady State

The symmetric steady state of game (2.1) is defined as the solution to $\dot{x}=\dot{u}=0$, in which case Eqs. (2.11) and (2.12) reduce to

$$f^p_{u^p}(x,\mathbf{u};\boldsymbol{\alpha})[r-g_x(x,\mathbf{u};\boldsymbol{\beta})]+g_{u^p}(x,\mathbf{u};\boldsymbol{\beta})f^p_x(x,\mathbf{u};\boldsymbol{\alpha})=0, \tag{2.13}$$

$$g(x,\mathbf{u};\boldsymbol{\beta})=0. \tag{2.14}$$

Use the facts that $\sum_{j=1}^{P}g^*_{u^j}=Pg^*_{u^p}$ and $\sum_{j=1}^{P}g^*_{xu^j}=Pg^*_{xu^p}$ from assumption (C3.2), the steady state Jacobian determinant of system (2.13) and (2.14) may be written as

$$\left|J_s\right| = g_x^*[(r-g_x^*)(\sum_{j=1}^{P} f_{u^p u^j}^{p*}) + f_x^{p*}\sum_{j=1}^{P} g_{u^p u^j}^*] - Pg_{u^p x}^*[g_{u^p}^* f_x^{p*} + f_{u^p}^{p*} g_x^*] - P(g_{u^p}^*)^2 f_{xx}^{p*}$$

$$+g_{u^p}^*[Pf_{u^p}^{p*} g_{xx}^* - P(r-g_x^*) f_{u^p x}^{p*} + (\sum_{j=1}^{P} f_{xu^j}^{p*}) g_x^*]. \tag{2.15}$$

Equation (2.15) is non-zero when evaluated at the symmetric steady state by assumptions (C6), (C7), and Eq. (2.26), where $g_x^* = g_x(x^*(\boldsymbol{\theta}), \mathbf{u}^*(\boldsymbol{\theta}); \boldsymbol{\beta})$ and so on signify the evaluation of the derivatives at the symmetric steady state OLNE. By (C7) the symmetric steady state OLNE solution exits, and the implicit function theorem implies that

$$u = u^*(\boldsymbol{\theta}), \tag{2.16}$$

$$x = x^*(\boldsymbol{\theta}), \tag{2.17}$$

$$\lambda = \lambda^*(\theta) = \frac{-f_{u^p}^p(x^*(\boldsymbol{\theta}), \mathbf{u}^*(\boldsymbol{\theta}); \boldsymbol{\alpha})}{g_{u^p}(x^*(\boldsymbol{\theta}), \mathbf{u}^*(\boldsymbol{\theta}); \boldsymbol{\beta})}, \quad p \in \{1, 2, ..., P\}. \tag{2.18}$$

Given the above, one can obtain the local stability properties of the steady state by taking a linear approximation of Eqs. (2.11) and (2.12), i.e.,

$$\begin{pmatrix} \Delta\dot{u} \\ \Delta\dot{x} \end{pmatrix} = \begin{pmatrix} h_u^* & h_x^* \\ g_u^* & g_x^* \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta x \end{pmatrix}, \tag{2.19}$$

where

$$h_u^* = \frac{(r-g_x^*)(\sum_{j=1}^{P} f_{u^p u^j}^{p*}) + f_x^*(\sum_{j=1}^{P} g_{u^p u^j}^*) + g_{u^p}(\sum_{j=1}^{P} f_{xu^j}^{p*}) - Pg_{u^p}^* f_{xu^p}^{p*}}{\sum_{j=1}^{P}[f_{u^p u^j}^{p*} - f_{u^p}^{p*}(g_{u^p}^*)^{-1} g_{u^p u^j}^*]}, \tag{2.20}$$

$$h_x^* = \frac{(r-2g_x^*) f_{u^p x}^{p*} - f_{u^p}^{p*} g_{xx}^* + g_{u^p}^* f_{xx}^{p*} + g_{u^p x}^*[f_{u^p}^{p*}(g_{u^p}^*)^{-1} g_x^* + f_x^{p*}]}{\sum_{j=1}^{P}[f_{u^p u^j}^{p*} - f_{u^p}^{p*}(g_{u^p}^*)^{-1} g_{u^p u^j}^*]}, \tag{2.21}$$

$$g_u^* = \sum_{p=1}^{P} g_{u^p}^* = Pg_{u^p}^*, \tag{2.22}$$

$\Delta u \overset{\text{def}}{=} u - u^*(\boldsymbol{\theta})$ and $\Delta x \overset{\text{def}}{=} x - x^*(\boldsymbol{\theta})$, and thus $\Delta \dot{u} \equiv \dot{u}$ and $\Delta \dot{x} \equiv \dot{x}$. Equation (2.20) is simplified by using the same identities from assumption (C3.2) that were used in deriving Eq. (2.15).

The local stability of the simple steady state is analyzed by finding the eigenvalues of the coefficient matrix of (2.19), the dynamic Jacobian J_d, where $|J_d| \neq 0$ by assumption (C7). The eigenvaules δ_1 and δ_2 are found by solving the characteristic equation $|J_d - \delta \mathbf{I}| = 0$. Notice that Eq. (2.13) implies that

$$f_x^{p*} = -f_{u^p}^{p*}(g_{u^p}^*)^{-1}(r - g_x^*), \tag{2.23}$$

and the fifth equation in (C3.1) implies that

$$P f_{xu^p}^{p*} = \sum_{j=1}^{P} f_{xu^j}^{j*} . \tag{2.24}$$

Armed with Eqs. (2.23) and (2.24), algebraic manipulation shows that the eigenvalues δ_1 and δ_2 satisfy

$$\delta_1 + \delta_2 = \text{tr} J_d = h_u^* + g_x^* = r + \frac{g_{u^p}^*\left(\sum_{j=1, j\neq p}^{P} f_{xu^j}^{p*} - \sum_{j=1, j\neq p}^{P} f_{xu^j}^{j*}\right)}{\sum_{j=1}^{P}[\, f_{u^p u^j}^{p*} - f_{u^p}^{p*}(g_{u^p}^*)^{-1} g_{u^p u^j}^*\,]}, \tag{2.25}$$

$$\delta_1 \delta_2 = |J_d| = h_u^* g_x^* - g_u^* h_x^* = |J_s| \{\sum_{j=1}^{P}[\, f_{u^p u^j}^{p*} - f_{u^p}^{p*}(g_{u^p}^*)^{-1} g_{u^p u^j}^*\,]\}^{-1} \neq 0. \tag{2.26}$$

First note that when $P = 1$, equations (2.25) and (2.26) reduce to their counterparts in optimal control theory, as seen in Eqs. (10a) and (10b) of Caputo (1997). Furthermore, in this case, the simple steady state is a local saddle point (Caputo 1997, stability lemma). For $P > 1$, assumption (C8) assures that the symmetric OLNE converge to a symmetric simple steady state when $t \to \infty$, thus the theory of ordinary differential equations implies that the steady state can take five possible forms: *a local saddle point*, *a local asymptotically stable (LAS) proper node*, *a LAS* *spiral, a LAS improper node, and a LAS star node.* In what follows, I discuss the properties of different types of steady states and specify some sufficient conditions to identify them. I will also identify several special classes of games that possess the saddle point type of steady state.

A saddle point steady state is the most frequently encountered case in the literature, and hence warrants an extensive discussion. A saddle point occurs when both eigenvalues are real, but with different signs. Given assumption (C8), a sufficient condition for a local saddle point steady state is that $\mathrm{tr}(J_d) > 0$, which is equivalent to the sum of the eigenvalues being positive, i.e., $\delta_1 + \delta_2 > 0$. The logic behind this is that if the sum of the eigenvalues is positive, then at least one real eigenvalue, or the real part of the complex conjugates, are positive. But assumption (C8) indicates that the symmetric OLNE must converge to the simple and symmetric steady state, which rules out the possibility that both eigenvalues are real and positive, or are complex conjugates with positive real parts. As a result, the only possibility left is that one eigenvalue is positive and the other is negative, say $\delta_1 > 0 > \delta_2$, so that the steady state is a local saddle point.

This sufficient condition can help to identify a special class of differential games that possess local saddle point steady states. The special class of games referred to here is the one in which the instantaneous payoff function for each player *p* is additively separable in the state variable and all control variables, i.e., $f^p_{xu^j} = 0,\ j = 1, 2, ..., P$. In this case, the numerator in Eq. (2.25) vanishes, and hence the sum of the eigenvalues equals the discount rate, i.e., $\delta_1 + \delta_2 = r > 0$, thus satisfying the sufficient condition for a local saddle point steady state given assumption (C8). The following proposition summarizes this result.

Proposition 2.1. *Under assumptions (C1)-(C8), if the instantaneous payoff function of every player is additively separable in state variable and all control variables, i.e.,* $f^p_{xu^j} = 0,\ p \in \{1, 2, ..., P\}$ *and* $j = 1, 2, ..., P$*, then the symmetric steady state OLNE of game (2.1) is a local saddle point with real eigenvalues* $\delta_1 > r > 0$*,* $\delta_2 < 0$*,* $\delta_1 > |\delta_2|$ *and* $|J_d| < 0$.

This proposition covers a broad class of differential games, including the popular linear-quadratic differential games specified in Engwerda (2005), the investment games of Spence (1979) and Reynolds (1987), and other applications in economics and management science as documented by Dockner et al. (2000). Thus Proposition 2.1 implies that the steady states are local saddle points in these models. In practice it is relatively easy to inspect the separability of a player's instantaneous payoff function. Proposition 2.1 thus provides an easy way to identify if a steady state is a local saddle point for the class of differential games under consideration. A common-pool resources extraction game in section 2.7 will demonstrate this advantage.

Necessary and sufficient condition for a local saddle point steady state is that the determinant of the dynamic Jacobian matrix is negative, as $|J_d| = \delta_1\delta_2$. This condition implies that one eigenvalue is positive and the other is negative, and hence that the steady state is a local saddle point. Since it is relative difficulty to compute the sign of the $|J_d|$ by Eqs (2.15) and (2.26), this necessary and sufficient condition is only applicable to a rather restricted class of differential games with simple functional forms that permit vast simplification of $|J_d|$. This is illustrated by the dynamic duopolistic competition model of section 2.8.

In addition to a local saddle point, four LAS steady states might occur in game (2.1). They occur when both eigenvalues are real and negative, or are complex conjugates with negative real parts. The exact type of the LAS steady state can be identified by examining the sign of $\{[\mathrm{tr}(J_d)]^2 - 4|J_d|\}$ and the eigenvectors of J_d. However, this is rather difficult given the complexity of Eqs. (2.15) and (2.26). Nonetheless, it can be shown that the resulting local stability properties and the steady state comparative statics are qualitatively identical in these four cases by Proposition 2.2 and Theorem 2.1. Since all I am interested in is the qualitative structure of the symmetric OLNE, I can combine them together and only consider the one case of a LAS steady state.

In all four cases, it follows that $\delta_1 + \delta_2 < 0$ and $|J_d| = \delta_1\delta_2 > 0$. Notice that $\delta_1 + \delta_2 < 0$ is only a necessary condition for a steady state of game (2.1) to be locally asymptotically stable since a local saddle point is also possible when the sum of eigenvalues is negative.[6] Thus, this cannot be used to identify differential games that possess locally asymptotically stable steady states, as Proposition 2.1 did in the local saddle point case. Given assumption (C8), a sufficient condition for a LAS stable steady state is that $|J_d| = \delta_1\delta_2 > 0$. This follows because it implies that the eigenvalues either are real and have the same sign, or are complex conjugates. But assumption (C8) rules out the possibility that both eigenvalues are real and positive, or are complex conjugates with positive real parts. Thus the real parts of the eigenvalues must be negative, or if real they must both be negative, and hence the steady state is locally asymptotically stable. Given the complexity of the determinant of Jacobian matrix $|J_d|$ from Eqs. (2.15) and (2.26), it is difficult, if not impossible, to identify general classes of games that posses a locally asymptotically stable steady state. However, Caputo and Lueck (2003) indicated the

[6] In this case, two eigenvalues satisfy $\delta_1 > 0 > \delta_2$ and $\delta_1 < |\delta_2|$.

possibility of a LAS steady state in a common-pool resources problem, and gave a sufficient condition for it to occur. The following proposition concludes the discussion.

Proposition 2.2. *Under assumptions (C1)-(C8), the steady state of differential game (2.1) is a local saddle point (locally asymptotically stable), if and only if the determinant of the dynamic matrix* $|J_d| = \delta_1\delta_2$ *is negative (positive).*

As the above discussion made clear, the local stability property of the symmetric steady state OLNE is not, in general, independent of the functional forms that are specified in the game. Also, it will be shown that the comparative statics of the steady state and the local comparative dynamic results depend crucially on the local stability property of the steady state. Therefore, the specific functional forms in the games are important in determining the qualitative structure of the equilibria. It thus follows that it is dangerous to generalize from qualitative results derived in games with special structures, e.g., a linear-quadratic differential game, without further investigation of the impact the said functional forms have on the qualitative conclusions. This finding represents the main contribution of this chapter.

Because the steady state has been shown to be either a local saddle point or locally asymptotically stable, and thus can be reached from $x_0 \neq x^*(\boldsymbol{\theta})$, attention is now shifted to the steady state comparative statics of game (2.1).

2.5. Steady State Comparative Statics

The local stability of the steady state will provide useful information in deriving the steady state comparative statics. First, Proposition 2.2 shows that the determinant of the dynamic

Jacobian is negative if and only if the steady state is a local saddle point, and positive if and only if the steady state is locally asymptotically stable. Next, assumption (C6) asserts that $\sum_{j=1}^{P} H^{p}_{u^p u^j} = \sum_{j=1}^{P} [f^{p*}_{u^p u^j} - f^{p*}_{u^p} (g^{*}_{u^p})^{-1} g^{*}_{u^p u^j}] < 0$ along a symmetric OLNE. These two facts indicate that the determinant of the steady state Jacobian is positive ($|J_s| > 0$) if and only if the steady state is a local saddle point, and negative ($|J_s| < 0$) if and only if it is locally asymptotically stable. This further implies that a steady state comparative static analysis of game (2.1) can be carried out via the implicit function theorem. The following proposition summarizes this discussion.

Proposition 2.3. *Under assumptions (C1)-(C8), the steady state Jacobian determinant in (2.15) is positive, i.e.,* $|J_s| > 0$*, if and only if the steady state is a local saddle point, while it is negative i.e.,* $|J_s| < 0$*, if and only if the steady state is locally asymptotically stable.*

This proposition suggests that the sign of the determinant of steady state Jacobian $|J_s|$ is opposite under different types of local stability of a steady state. Moreover, it will be shown that $|J_s|$ appears in the denominator of all the steady state comparative statics expressions, and thus one could expect that the steady state comparative statics results may change sign as the nature of the steady state changes. Remember that the type of the steady state depends on the functional forms and the value of parameters that are specified in a game, therefore the steady state comparative statics results also depend on them as well.

To find the steady state comparative statics for game (2.1), substitute the symmetric steady state OLNE $(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}))$ into Eqs. (2.13) and (2.14) to create local identities in the parameters $\boldsymbol{\theta}$:

$$f^p_{u^p}(x^*(\boldsymbol{\theta}),\mathbf{u}^*(\boldsymbol{\theta});\boldsymbol{\alpha})[r-g_x(x^*(\boldsymbol{\theta}),\mathbf{u}^*(\boldsymbol{\theta});\boldsymbol{\beta})]+g_{u^p}(x^*(\boldsymbol{\theta}),\mathbf{u}^*(\boldsymbol{\theta});\boldsymbol{\beta})\,f^p_x(x^*(\boldsymbol{\theta}),\mathbf{u}^*(\boldsymbol{\theta});\boldsymbol{\alpha})\equiv 0\,, \tag{2.27}$$

$$g(x^*(\boldsymbol{\theta}),\mathbf{u}^*(\boldsymbol{\theta});\boldsymbol{\beta})\equiv 0\,. \tag{2.28}$$

Differentiating Eqs. (2.27) and (2.28) with respect to the parameter of interest via the chain rule and solving the resulting linear system with Cramer's rule yields the steady state comparative statics. For instance, by differentiating Eqs. (2.27) and (2.28) with respect to the parameter α_{k_1}, and employing Eq. (2.22) to simplify, one arrives at

$$\begin{pmatrix} \sum_{j=1}^{P}[\,f^{p*}_{u^p u^j}(r-g^*_x)-f^{p*}_{u^p}g^*_{xu^j} + f^{p*}_x g^*_{u^p u^j}+g^*_{u^p}f^{p*}_{xu^j}] & f^{p*}_{u^p x}(r-g^*_x)-f^{p*}_{u^p}g^*_{xx}+g^*_{u^p x}f^{p*}_x+f^{p*}_{xx}g^*_{u^p} \\ Pg^*_{u^P} & g^*_x \end{pmatrix}\begin{pmatrix}\dfrac{\partial u^*}{\partial\alpha_{k_1}}\\ \dfrac{\partial x^*}{\partial\alpha_{k_1}}\end{pmatrix}=\begin{pmatrix}-f^{p*}_{u^p\alpha_{k_1}}(r-g^*_x)-f^{p*}_{x\alpha_{k_1}}g^*_{u^p}\\ 0\end{pmatrix}.$$

The steady state comparative states results are summarized in following theorem.

Theorem 2.1. *Under assumptions (C1)-(C8), and for* $k_1=1,2,\ldots,K_1$ *and* $k_2=1,2,\ldots,K_2$ *, the steady state comparative statics of differential game (2.1) are given by*

$$\frac{\partial u^*(\boldsymbol{\theta})}{\partial\alpha_{k_1}}=\frac{-f^{p*}_{u^p\alpha_{k_1}}(r-g^*_x)g^*_x-f^{p*}_{x\alpha_{k_1}}g^*_{u^p}g^*_x}{|J_s|}, \tag{2.29}$$

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial\alpha_{k_1}}=\frac{P[\,f^{p*}_{u^p\alpha_{k_1}}(r-g^*_x)g^*_{u^p}+f^{p*}_{x\alpha_{k_1}}(g^*_{u^p})^2]}{|J_s|}, \tag{2.30}$$

$$\frac{\partial u^*(\boldsymbol{\theta})}{\partial\beta_{k_2}}=\frac{g^*_{\beta_{k_2}}[\,f^{p*}_{u^p x}(r-g^*_x)-f^{p*}_{u^p}g^*_{xx}+g^*_{u^p x}f^{p*}_x+f^{p*}_{xx}g^*_{u^p}]+g^*_x[g^*_{x\beta_{k_2}}f^{p*}_{u^p}-f^{p*}_x g^*_{u^p\beta_{k_2}}]}{|J_s|}, \tag{2.31}$$

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial \beta_{k_2}} = \frac{-g^*_{\beta_{k_2}}\{\sum_{j=1}^{P}[f^{p*}_{u^p u^j}(r - g^*_x) - f^{p*}_{u^p} g^*_{xu^j} + f^{p*}_x g^*_{u^p u^j} + g^*_{u^p} f^{p*}_{xu^j}]\} - P g^*_{u^p}(g^*_{x\beta_{k_2}} f^{p*}_{u^p} - g^*_{u^p \beta_{k_2}} f^{p*}_x)}{|J_s|}, \quad (2.32)$$

$$\frac{\partial u^*(\theta)}{\partial r} = \frac{-f^{p*}_{u^p} g^*_x}{|J_s|}, \quad (2.33)$$

$$\frac{\partial x^*(\theta)}{\partial r} = \frac{P f^{p*}_{u^p} g^*_{u^p}}{|J_s|}. \quad (2.34)$$

By assumptions (C3.1) and (C3.2), the steady state comparative statics results (2.29)-(2.34) are identical for any $p \in \{1, 2, ..., P\}$. If $P = 1$, Theorem 2.1 reduces to the steady state comparative statics in optimal control theory found in Caputo (1997, Theorem 1). Thus it is not surprising that no signs are implied for any of the symmetric steady state comparative statics at this level of generality. Also note that the steady state comparative statics for the current value costate variable can be derived from identity (2.18) and Theorem 2.1. Furthermore, since the determinant of the steady state Jacobian $|J_s|$ enters the denominator of the steady state comparative static expressions, Proposition 2.2 implies that the steady state comparative statics may have the opposite sign for different types of local stability. Such a situation cannot occur in optimal control problems under the assumptions adopted here. Moreover, because different functional forms in a game are responsible for different types of steady states, an important policy implication follows, Namely, polices aiming for correcting the value of steady state variables may be completely opposite simply by using different functional forms to simulate the same model.

Note in passing, that the theorem does not cover the steady state comparative statics with respect to the total number of players since P is an integer and thus the differential method cannot be applied. This does not mean that the qualitative results with respect to total number of players are not important. Indeed, equations (2.25) and (2.26) imply that the nature of the steady state in a game could depend on the total number of players, and thus Proposition 2.2 and Theorem 2.1 suggests that P may be important to the steady state comparative statics. However, a more advanced technique may be needed to derive the qualitative results for this particular parameter. The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.1. *Under assumptions (C1)-(C8), if the symmetric steady state OLNE of game* (2.1) *is a local saddle point, then the steady state comparative statics for game (2.1) can be simplified as follows.*

(a) If a perturbation in the discount rate r occurs in the steady state, then

$$\operatorname{sign}\left[\frac{\partial u^*(\boldsymbol{\theta})}{\partial r}\right] = -\operatorname{sign}\left[f_{u^p}^{p*} g_x^*\right], \tag{2.35}$$

$$\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial r}\right] = \operatorname{sign}\left[f_{u^p}^{p*} g_{u^p}^*\right]. \tag{2.36}$$

(b) If $f_{u^p \alpha_{k_1}}^{p*} = 0$, $k_1 = 1, 2, ..., K < K_1$, then

$$\operatorname{sign}\left[\frac{\partial u^*(\boldsymbol{\theta})}{\partial \alpha_{k_1}}\right] = -\operatorname{sign}\left[f_{x\alpha_{k_1}}^{p*} g_{u^p}^* g_x^*\right], \tag{2.37}$$

$$\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \alpha_{k_1}}\right] = \operatorname{sign}\left[f_{x\alpha_{k_1}}^{p*}\right]. \tag{2.38}$$

(c) If $f_{x\alpha_{k_1}}^{p*} = 0$, $k_1 = 1, 2, ..., K < K_1$, then

$$\operatorname{sign}\left[\frac{\partial u^*(\boldsymbol{\theta})}{\partial \alpha_{k_1}}\right] = -\operatorname{sign}\left[f_{u^p \alpha_{k_1}}^{p*} (r - g_x^*) g_x^*\right], \tag{2.39}$$

$$\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \alpha_{k_1}}\right]=\operatorname{sign}\left[f^{p*}_{u^p\alpha_{k_1}}(r-g^*_x)g^*_{u^p}\right]. \tag{2.40}$$

If the steady state is locally asymptotically stable, then expressions (2.35)-(2.40) have the opposite sign.

Corollary 2.1 shows that the steady state comparative statics results are structurally similar for a local saddle point steady state and a locally asymptotically stable steady state. Thus, it is sufficient to interpret the results for one type of steady state. In what follows, I provide an economic interpretation of expressions (2.35)-(2.40) for the case in which the steady state of game (2.1) is a local saddle point.

First, equation (2.35) asserts that the effect of an increase in the discount rate on the symmetric steady state value of the control variable is qualitatively the opposite sign of the product of the marginal impact that any player's own strategy has on their instantaneous payoff function and the marginal impact of the state variable on the state equation, evaluated at the symmetric steady state. In economic and management science problems, if the stock is a good, say capital or resource reserves, then typically $f^{p*}_{u^p}g^*_x > 0$, so that $\partial u^*(\boldsymbol{\theta})/\partial r < 0$, while if the stock is bad, say pollution or toxic waste, then often $f^{p*}_{u^p}g^*_x < 0$ so that $\partial u^*(\boldsymbol{\theta})/\partial r > 0$. In other words, if the stock is a good, then an increase in the discount rate usually decreases the level of the control in a symmetric steady state, while if the stock is a bad, then an increase in the discount rate increases the level of the control in a symmetric steady state.

Equation (2.36) says that the effect of an increase in the discount rate on the symmetric steady state value of the state variable is qualitatively the same as the product of the marginal impact that any player's own control has on his instantaneous payoff function and the marginal impact of his controls on the state equation, evaluated at the symmetric steady state. If the stock is good, then typically $f_{u^p}^{p*} g_{u^p}^* < 0$, so that an increase in the discount rate decreases the stock in the steady state. Similarly, if the stock is a bad, then often $f_{u^p}^{p*} g_{u^p}^* > 0$, so that an increase in the discount rate increases the stock in the steady state. These two results suggest that the symmetric steady state strategy and state variables usually have the same qualitative response to an increase in the discount rate.

Part (b) describes the situation when an exogenous parameter α_{k_1} is additively separable from a player's control variable in that player's instantaneous payoff function at the steady state. Equation (2.38) says that the impact of an increase in α_{k_1} on the steady state value of state variable is qualitatively the same as the effect an increase in α_{k_1} has on the marginal impact the state variable has on a player's instantaneous payoff function, evaluated at the symmetric steady state. This means that all one has to do to compute the sign of $\partial x^* / \partial \alpha_{k_1}$ is to find the sign of $f_{x\alpha_{k_1}}^{p*}$ for any $p \in \{1, 2, ..., P\}$. This represents a vast simplification over using Theorem 1 in computing the steady state comparatives statics. The corresponding expression (2.37) for the symmetric equilibrium strategy is slightly more complicated due to the presence of the product $g_{u^p}^* g_x^*$, though, in general, it is also easy to compute.

Part (c) is similar to part (b), though in this case α_{k_1} is additively separable from the state variable in a player's instantaneous payoff function. In this instance the sign of the steady state comparatives statics depends not only on the sign of the partials $g_{u^p}^*$, g_x^*, and $f_{u^p\alpha_{k_1}}^{p*}$, but also on the relative magnitude of r and g_x^*. Thus, except for the information about the magnitude of g_x^*, this part of corollary is no more difficult to use than part (b).

Notice that in Theorem 1, if $g_x^* = 0$, then the steady state comparative statics expressions (2.29) and (2.33) vanish. Thus, under this situation, the steady state value of the symmetric OLNE is independent of the discount factor and any parameter that appears in the payoff functions of game (2.1). The following corollary summarizes this result.

Corollary 2.2 *Under assumptions (C1)-(C8), if* $g_x^* = 0$ *for differential game (2.1), then* $\partial u^* / \partial r = 0$ *and* $\partial u^* / \partial \alpha_{k_1} = 0$, $k_1 = 1, 2, \ldots, K_1$.

It is important to note that this corollary is true regardless of the type of steady state. Also, just because the steady state value of a symmetric OLNE is independent of $(r, \boldsymbol{\alpha})$, this does not mean that the trajectory corresponding to the symmetric OLNE is independent of these same parameters. This will be carefully discussed in section 2.6.

Up to this point, our discussion has focused on the steady state comparative statics with respect to the discount rate and the parameters that appear in a player's instantaneous payoff function. Equations (2.31) and (2.32) show that a perturbation in $\boldsymbol{\beta}$ is more complicated compared to one in $(r, \boldsymbol{\alpha})$. Hence, there are no "nice" steady state comparative statics results for $\boldsymbol{\beta}$ in Corollary 1. This scenario also exists in the optimal control theory, and for the same reason. It is mainly because r and $\boldsymbol{\alpha}$ appear explicitly in only one of the two steady state necessary conditions (equations (2.27) and (2.28)), while $\boldsymbol{\beta}$ appears in both of them. To keep the chapter tight, I will not explain this logic in detail. The interested readers can consult the explanation in optimal control theory in Caputo (1997, p.204). These observations indicate that when a parameter enters the state equation explicitly, few, if any, refutable steady state comparative statics will be forthcoming.

It is worthwhile to mention that the steady state comparative statics of Theorem 2.1 have been simplified by allowing a parameter to enter only a player's instantaneous payoff function $f^p(\cdot)$ or the state equation function $g(\cdot)$. It is expected that the steady state comparative statics will be quite complicated if a parameter appears in both of these functions. In this case, there are no user-friendly steady state comparative statics results like those in Corollaries 2.1 and 2.2. Fortunately, most problems in the economics and management science satisfy this separability condition.

Also worth mentioning is the fact that the sufficient conditions for parts (b) and (c) of Corollary 2.1, and the sufficient conditions for Corollary 2.2, are local conditions. Thus they need only hold in a neighborhood of the steady state for one to use these corollaries. However, if $f^p_{u^p\alpha_{k_1}} = 0$, $f^p_{x\alpha_{k_1}} = 0$, or $g_x = 0$ hold globally, then the corollaries can also be used. Naturally, the global requirements on the functions are much stronger, but in practice they are easier to verify since this can be done by inspection of the functional forms of the game under consideration.

Finally, although $|J_s|$ has the opposite sign in a local saddle point steady state versus a locally asymptotically stable steady state, the steady state comparative statics are not necessarily the opposite sign in the different cases. This is because a different type of steady state usually corresponds to a different level of a symmetric OLNE, and therefore the numerators of the steady state comparative statics expressions may also have different signs in different types of steady state.

2.6. Local Comparative Dynamics

In this section, I consider the local comparative dynamic properties of game (2.1). First note that if the steady state of the game is locally asymptotically stable, then any trajectory of the symmetric OLNE eventually converges to the steady state. This implies that the local comparative dynamics can take on any values in this case. Therefore, refutable local comparative dynamics can only exist, if they do at all, in the case of a local saddle point steady state. Throughout this section, I therefore assume that the steady state of the game is a local saddle point.

The local comparative dynamics of game (2.1) are determined by solving the linearized pair of ordinary differential equations (2.19). The eigenvalues and eigenvectors of the dynamic Jacobian matrix J_d are needed to solve (2.19). The eigenvectors (Q_1^i, Q_2^i), $i=1,2$, corresponding to the eigenvalues δ_i, $i=1,2$, are defined by

$$\begin{pmatrix} h_u^* - \delta_i & h_x^* \\ g_u^* & g_x^* - \delta_i \end{pmatrix}\begin{pmatrix} Q_1^i \\ Q_2^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad i=1,2, \tag{2.41}$$

where h_u^*, h_x^*, and g_u^* are defined in (2.20)-(2.22). Without loss of generality, one can take $Q_2^i = 1$, $i=1,2$, as the normalization, thus the eigenvectors are

$$(Q_1^i, Q_2^i) = \left(\frac{-h_x^*}{h_u^* - \delta_i}, 1\right) = \left(\frac{\delta_i - g_x^*}{g_u^*}, 1\right), \quad i=1,2, \tag{2.42}$$

Given the eigenvalues and eigenvectors of J_d, the general solution to the pair of ordinary differential equations (2.41) is given by

$$\begin{pmatrix} v(t;x_0,\boldsymbol{\theta}) \\ z(t;x_0,\boldsymbol{\theta}) \end{pmatrix} = \begin{pmatrix} u^*(\boldsymbol{\theta}) \\ x^*(\boldsymbol{\theta}) \end{pmatrix} + c_1\begin{pmatrix} Q_1^1 \\ 1 \end{pmatrix} e^{\delta_1 t} + c_2\begin{pmatrix} Q_1^2 \\ 1 \end{pmatrix} e^{\delta_2 t}, \tag{2.43}$$

where c_1 and c_2 are arbitrary constants of integration to be determined from the initial condition $x(0) = x_0$ and the convergence requirement on the state variable along the symmetric OLNE from assumption (A8), i.e., $\lim_{t\to\infty} z(t; x_0, \boldsymbol{\theta}) = x^*(\boldsymbol{\theta})$. Substituting this latter requirement into Eq. (2.43) yields

$$\lim_{t\to\infty} z(t; x_0, \boldsymbol{\theta}) = x^*(\boldsymbol{\theta}) + c_1 \lim_{t\to\infty} e^{\delta_1 t} + c_2 \lim_{t\to\infty} e^{\delta_2 t} = x^*(\boldsymbol{\theta}) . \tag{2.44}$$

Since the steady state is a saddle point, it implies that $\delta_1 > 0$ and $\delta_2 < 0$. Thus the term $c_2 \lim_{t\to\infty} e^{\delta_2 t}$ goes to zero for any c_2, while the term $c_1 \lim_{t\to\infty} e^{\delta_1 t}$ goes to $\pm\infty$ if $c_1 \neq 0$, which violates assumption (C8). This implies that $c_1 = 0$ for assumption (C8) to be met. Moreover, c_2 can be found by using the initial condition, i.e.,

$$z(0; x_0, \boldsymbol{\theta}) = x^*(\boldsymbol{\theta}) + c_2 = x_0 \quad \Rightarrow \quad c_2 = x_0 - x^*(\boldsymbol{\theta}), \tag{2.45}$$

Therefore, the specific solution that satisfies the initial condition and convergence requirement is

$$\begin{pmatrix} v(t; x_0, \boldsymbol{\theta}) \\ z(t; x_0, \boldsymbol{\theta}) \end{pmatrix} = \begin{pmatrix} u^*(\boldsymbol{\theta}) \\ x^*(\boldsymbol{\theta}) \end{pmatrix} + (x_0 - x^*(\boldsymbol{\theta})) \begin{pmatrix} Q_1^2 \\ 1 \end{pmatrix} e^{\delta_2 t}, \tag{2.46}$$

along with its time derivative

$$\begin{pmatrix} \dot{v}(t; x_0, \boldsymbol{\theta}) \\ \dot{z}(t; x_0, \boldsymbol{\theta}) \end{pmatrix} = \delta_2 (x_0 - x^*(\boldsymbol{\theta})) \begin{pmatrix} Q_1^2 \\ 1 \end{pmatrix} e^{\delta_2 t} . \tag{2.47}$$

The local comparative dynamics of game (2.1) are derived by differentiating Eqs. (2.46) and (2.47) with respect to the parameters $(x_0, \boldsymbol{\theta})$ and evaluating the resulting expression at the initial state $x_0 = x^*(\boldsymbol{\theta})$. The following theorem summarizes these results.

Theorem 2.2. *Under assumptions (C1)-(C8), if the symmetric steady state of differential game (2.1) is a local saddle point, then the effects of perturbations in the parameters* $(x_0, \boldsymbol{\theta})$ *on the time path of the symmetric OLNE in a neighborhood of a steady state are*

$$\left.\frac{\partial v(t;x_0,\boldsymbol{\theta})}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})} = Q_1^2 e^{\delta_2 t}, \tag{2.48}$$

$$\left.\frac{\partial z(t;x_0,\boldsymbol{\theta})}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})} = e^{\delta_2 t} \geq 0, \tag{2.49}$$

$$\left.\frac{\partial \dot{v}(t;x_0,\boldsymbol{\theta})}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})} = \delta_2 Q_1^2 e^{\delta_2 t}, \tag{2.50}$$

$$\left.\frac{\partial \dot{z}(t;x_0,\boldsymbol{\theta})}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})} = \delta_2 e^{\delta_2 t} \leq 0, \tag{2.51}$$

$$\left.\frac{\partial v(t;x_0,\boldsymbol{\theta})}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})} = \frac{\partial u^*(\boldsymbol{\theta})}{\partial \theta_k} - Q_1^2 e^{\delta_2 t}\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}, \quad k = 1, 2, \ldots, K_1 + K_2 + 1, \tag{2.52}$$

$$\left.\frac{\partial z(t;x_0,\boldsymbol{\theta})}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})} = \frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}(1 - e^{\delta_2 t}), \quad k = 1, 2, \ldots, K_1 + K_2 + 1, \tag{2.53}$$

$$\left.\frac{\partial \dot{v}(t;x_0,\boldsymbol{\theta})}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})} = -\delta_2 Q_1^2 e^{\delta_2 t}\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}, \quad k = 1, 2, \ldots, K_1 + K_2 + 1, \tag{2.54}$$

$$\left.\frac{\partial \dot{z}(t;x_0,\boldsymbol{\theta})}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})} = -\delta_2 e^{\delta_2 t}\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}, \quad k = 1, 2, \ldots, K_1 + K_2 + 1. \tag{2.55}$$

Notice that Theorem 2.2 is only valid if the symmetric steady state of the game under consideration is a local saddle point. It does not cover the local comparative dynamics with respect to the parameter P, for the same reason as was stated before. In passing, observe that the impact effects of parameter perturbations follow from Theorem 2.2 by evaluating the derivatives at $t = 0$. Since $\delta_2 < 0$, the ensuing corollary follows immediately from Theorem 2.2.

Corollary 2.3. *Under assumptions (C1)-(C8), the following local comparative dynamics results hold* $\forall t \in [0,\infty)$ *in differential game (2.1),*

$$\operatorname{sign}\left[\left.\frac{\partial v(t;\boldsymbol{\theta},x_0)}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})}\right]=\operatorname{sign}[Q_1^2], \tag{2.56}$$

$$\operatorname{sign}\left[\left.\frac{\partial \dot{v}(t;\boldsymbol{\theta},x_0)}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})}\right]=-\operatorname{sign}[Q_1^2], \tag{2.57}$$

$$\operatorname{sign}\left[\left.\frac{\partial z(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})}\right]=\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}\right],\quad k=1,2,\dots,K_1+K_2+1, \tag{2.58}$$

$$\operatorname{sign}\left[\left.\frac{\partial \dot{v}(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})}\right]=\operatorname{sign}\left[Q_1^2\,\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}\right],\quad k=1,2,\dots,K_1+K_2+1, \tag{2.59}$$

$$\operatorname{sign}\left[\left.\frac{\partial \dot{z}(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})}\right]=\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}\right],\quad k=1,2,\dots,K_1+K_2+1. \tag{2.60}$$

Theorem 2.2 and Corollary 2.3 lay out the local comparative dynamics for a symmetric saddle point steady state in differential game (2.1). Although the steady state is independent of the initial value of the state variable x_0, equations (2.48)-(2.51) show that the time path of the symmetric OLNE and the state variable are not, in general, independent of x_0. In fact, those local comparative dynamics depend on the sign of the eigenvector.

Second, equation (2.52) shows that one must know the steady state comparative statics for the state and control variables before any refutable local comparative dynamics can be derived for the time-path of the control. That is, players must know how their destination is affected by a parameter change before determining the best path to reach it. Moreover, it also shows that even if the steady state value of the control is independent of some parameter, i.e., if

$\partial u^* / \partial \theta_k = 0$, this does not necessarily mean that the time path of the equilibrium strategy is independent of this parameter, since equation (2.52) is not equal to zero as long as the $\partial x^* / \partial \theta_k \neq 0$ and $Q_1^2 \neq 0$. The intuition is that as long as the terminal position of the state variable changes, then players' strategies must change, at least temporarily, for them to drive the state to its new destination.

Third, equations (2.53)-(2.55) show that it is in general easier to determine the local comparative dynamics for the time derivative of the control, the state variable, and its time derivative, since these local comparative dynamic results only depend on the eigenvector Q^2 and the steady state comparative statics for state variable, as the steady state comparative statics of control do not play a role. Finally, it is important to point out that Theorem 2.2 and Corollary 2.3 are only valid in the case that the symmetric steady state of differential game (2.1) is a local saddle point.

The local comparative dynamics derived here can in many cases lead to a sound economic explanation of the steady state comparative statics, which may appear counter-intuitive if one considers the latter alone. The next section provides an example to demonstrate this point. Moreover, local comparative dynamics often yield qualitative results which show that the steady state policy implications can differ quite substantially from their short-run counterparts.

Up to this point, we derived the local asymptotic stability properties of the steady state, the steady state comparative statics, and the local comparative dynamics for differential game (1). Before we jump to the applications, it is worthwhile to point that the steady state comparative statics and local comparative dynamics share the similar mathematic structures with their counterparts in the optimal control theory as in Caputo (1997) if the steady state is a local saddle point. In this case, therefore, the extension to a differential game with symmetric OLNE would not significantly change the qualitative properties of the original optimal control problem.

In the rest of this chapter, two classical differential games are investigated in order to show how the propositions, theorems, and corollaries can be used to derive the qualitative properties of differential games admitting symmetric OLNE.

2.7. The Common-Pool Resources Extraction Games

Beginning with Gordon (1954) and Scott (1955), the common-pool resource problem attracted a lot of attention by economists. A classical common-pool resources extraction model is given by

$$W^p(\boldsymbol{\theta}) = \max_{c^p(\cdot)} \int_0^\infty e^{-rt}[U^p(c^p(t))]dt \qquad (2.61)$$

$$\text{s.t.} \quad \dot{x}(t) = F(x(t)) - \sum_{p=1}^{P} c^p(t), \quad x(0) = x_0 ,$$

where $c^p(t),\ p = 1,2,...,P$, is the resource extraction rate for player p at time $t \in [0,\infty)$, $x(t)$ is the stock of the resource at time $t \in [0,\infty)$, with the initial reserve x_0, $U^p(\cdot)$ is the instantaneous utility function for player p, with $U^p_{c^p} > 0$, $F(\cdot)$ is the inverse-U shaped natural growth function which satisfies $F'(x) > 0$, and $r > 0$ is the discount rate. It is assumed that assumptions (C1)-(C8) hold for this model.

Notice that players' control variables are additively separable with respect to the state variable x in this model, i.e., $f^p_{xu^j} = 0 \quad j = 1,2,...,P$, thus Proposition 2.1 implies that the steady state of the symmetric OLNE of game (2.61) is a local saddle point, and that the determinant of the dynamic Jacobian matrix is negative ($|J_d| < 0$). Therefore, the steady state comparative statics of the game follow from Theorem 2.1.

Suppose that the steady state of OLNE lies in the region that $F'(\cdot) > 0$, by equations (2.33) and (2.34) of Theorem 2.1, the steady state comparative statics with respect to the discount rate are

$$\frac{\partial c^*(\boldsymbol{\theta})}{\partial r} = -\frac{U_{c^p}^{p*} F_x^*}{|J_s|} \le 0, \tag{2.62}$$

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial r} = -\frac{P U_{c^p}^{p*}}{|J_s|} < 0, \tag{2.63}$$

since $|J_s| > 0$ by Proposition 2.3. Notice that the steady state comparative statics (2.62) and (2.63) can also be reached using part (a) of Corollary 2.1. They show that an increase in the discount rate reduces the symmetric steady state extraction rate and the stock of the resource, which is counterintuitive at first glance. For instance, how can the stock fall if the extraction rate has also fallen? This question can only be answered after the local comparative dynamics are investigated.

The local comparative dynamics with respect to r follows from Eq. (2.52) of Theorem 2.2,

$$\left.\frac{\partial c(t;\boldsymbol{\theta},x_0)}{\partial r}\right|_{\substack{x=x^*(\boldsymbol{\theta})\\ t=0}} = \frac{\partial c^*(\boldsymbol{\theta})}{\partial r} - Q_1^2 \frac{\partial x^*(\boldsymbol{\theta})}{\partial r}, \tag{2.64}$$

where $\delta_2 < 0$, $g_u^* = P g_{u^p}^* < 0$, $g_x^* > 0$, and (2.42) imply that $Q_1^2 = (\delta_2 - g_x^*)/g_u^* > 0$. Substituting Eqs. (2.62) and (2.63) into (2.64) yields

$$\left.\frac{\partial c(t;\boldsymbol{\theta},x_0)}{\partial r}\right|_{\substack{x=x^*(\boldsymbol{\theta})\\ t=0}} = -\frac{\delta_2 U_{c^p}^{p*}}{|J_s|} > 0. \tag{2.65}$$

Thus the moment r rises each player also increases her extraction rate above its previous steady state value and it remains above this value before declining to the new steady state value as Eq. (2.54), i.e., $\partial \dot{c}(t;\boldsymbol{\theta},x_0)/\partial r\big|_{x_0=x^*(\boldsymbol{\theta})} = -\delta_2 Q_1^2 e^{\delta_2 t}\,\partial x^*(\boldsymbol{\theta})/\partial r < 0$ indicates.

The above analysis clearly shows that examination of the local comparative dynamics can result in a more complete economic understanding of the effect that changes in a parameter have on players' behavior and the state variables at the steady state in a differential game. However, due to the difficulty of deriving explicitly the symmetric OLNE for general differential games, little local comparative dynamics work has been done in the literature. The current chapter thus gives an applicable method for examining the local comparative dynamics in differential games.

Moreover, if the steady state lies in the region that $F'(\cdot) < 0$, then the steady state comparative statics (2.62) and (2.63) becomes

$$\frac{\partial c^*(\boldsymbol{\theta})}{\partial r} = -\frac{U_{c^p}^{p*} F_x^*}{|J_s|} > 0, \tag{2.66}$$

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial r} = -\frac{P U_{c^p}^{p*}}{|J_s|} < 0. \tag{2.67}$$

In this case, an increase in the discount rate increases the symmetric steady state extraction rate and therefore reduces the stock of the resource. However, the local comparative dynamics (2.64) cannot be signed in general.

This model shows the advantage of Proposition 2.1 in identifying the local saddle point steady state of a differential game. The next section considers the well-known dynamic duopolistic competition model with sticky prices to illustrate the usefulness of Proposition 2.2 in identifying the nature of the steady state when the functional forms in the game do not satisfy Proposition 2.1.

2.8. Dynamic Duopolistic Competition with Sticky Prices

Now consider the well-known dynamic model of duopolistic competition with sticky prices of Fershtman and Kamien (1987):

$$W^q(p_0, \boldsymbol{\theta}) = \max_{u^q(\cdot)} \int_0^\infty e^{-rt}[p(t)u^q(t) - cu^q(t) - (1/2)(u^q(t))^2]dt \tag{2.68}$$

$$\text{s.t.} \quad \dot{p}(t) = s[a - (u^1(t) + u^2(t)) - p(t))], \quad p(0) = p_0,$$

where $u^q(t) \geq 0$, $q = 1, 2$, is the qth firm's output rate, $p(t)$ is the sticky price at time $t \in [0, \infty)$, with initial price p_0, $r > 0$ is the discount rate, $s > 0$ denotes the speed in which the price converges to its level on the demand function, and $\boldsymbol{\theta} \stackrel{\text{def}}{=} (c, a, s)$ are the exogenous time-independent parameters. It is easily to verify that assumptions (A1) (A2) (A3) and (A6) hold for this model. Moreover the results in Fershtman and Kamien (1987) show that the rest of assumptions hold too. Meanwhile, this simple model can illustrate that the symmetric assumptions (C3.1) and (C3.2) are not generally valid off the symmetric OLNE, e.g., the marginal effect of qth player's own control on her instantaneous payoff function is $p - c - u^q$. Therefore, these two marginal effects are different as long as they are off the symmetric OLNE path, i.e., $u^1 \neq u^2$.

Notice that the player's own strategy is not additively separable with respect to the state variable p in this model, i.e., $f^1_{pu^1} = f^2_{pu^2} \neq 0$, thus Proposition 2.1 cannot be applied to identify the local saddle point nature of the steady state of the game. Also the sum of eigenvalues from Eq. (2.25) is $r - s$, which is not positive in general. Fortunately, the structure of this game is simple enough to directly compute $|J_d|$, in which case Eq. (2.26) yields

$$|J_d| = -s(3r + 4s) < 0. \tag{2.69}$$

Since $|J_d| < 0$, Proposition 2.2 implies that the steady state of game (2.68) is a local saddle point, and thus Proposition 2.3 implies that $|J_s| > 0$. Therefore the steady state comparative statics of the game follow from Theorem 2.1, and are given by

$$\frac{\partial u^*}{\partial c} = \frac{-(r+s)s}{|J_s|} < 0, \tag{2.70}$$

$$\frac{\partial p^*}{\partial c} = \frac{2s(s+r)}{|J_s|} > 0, \tag{2.71}$$

$$\frac{\partial p^*}{\partial s} = \frac{2s(-f_{u^q}^{q*} + f_p^{q*})}{|J_s|} > 0, \tag{2.72}$$

$$\frac{\partial u^*}{\partial s} = \frac{-s(-f_{u^q}^{q*} + f_p^{q*})}{|J_s|} < 0, \tag{2.73}$$

$$\frac{\partial u^*}{\partial r} = \frac{sf_{u^q}^{q*}}{|J_s|} < 0, \tag{2.74}$$

$$\frac{\partial p^*}{\partial r} = \frac{-2sf_{u^q}^{q*}}{|J_s|} < 0, \tag{2.75}$$

where the signs of expressions (2.72)-(2.75) are determined by the facts $f_{u^q}^{q*} = -g_{u^q}^* f_p^{q*} \big/ (r - g_p^*) = su^* \big/ (r+s) > 0$ and $s(f_{u^q}^{q*} - f_p^{q*}) = -rf_{u^q}^q < 0$ from the steady state equation (2.13). Equations (2.72) and (2.75) recover the steady state comparative statics results in Fershtman and Kamien (1987, Proposition 1). They show that the steady state equilibrium price declines as the speed of adjustment decreases or the interest rate increases. Their results are obtained by explicitly deriving the symmetric steady state OLNE. This chapter provides an alternative method to get the same stability properties and steady state comparative statics. Instead of deriving the explicit Nash equilibria, which only possible for a few special classes of games, our approach instead simply requires one to compute the determinant of the steady state Jacobian matrix.

In the specified game, both approaches are quite easy to apply. However, if one slightly changes the quadratic cost function $C(u^q) = cu^q - (1/2)(u^q)^2$ to a more general one, for example, $C(u^q) = c(u^q)^\gamma, \gamma > 0$, then the game under consideration is no longer a linear-quadratic differential game, and hence may not be explicitly solvable. Therefore it is difficult, if not impossible, to examine the stability properties, steady state comparative statics, and local comparative dynamics by explicitly solving the game. On the other hand, our approach is still applicable since the steady state Jacobian determinant $|J_s|$ has not been affected by changing the quadratic cost function to a more general form. The steady state of the game thus remains a local saddle point regardless of which cost function is employed.

2.9. Conclusion

This chapter has provided a qualitative characterization of symmetric OLNE for a ubiquitous class of discounted infinite horizon differential games. The local asymptotic stability of a steady state, the steady state comparative statics, and the local comparative dynamics of the equilibria were all investigated. The results indicate that these qualitative characterizations of Nash equilibria are not independent of the functional forms and/or the value of the parameters that are specified in the game. An important policy implication that follows from this fact is that the policy recommendations may be opposite for the same stated goal simply because researchers use different functional forms to investigate the game. In addition, this chapter has provided several simple sufficient conditions to identify classes of differential games in which their steady state is a local saddle point. Finally, the propositions, theorems, and corollaries were applied to two classical differential games to demonstrate their advantages and conveniences.

The limitations of this method for studying the qualitative properties of differential games are important to spell out. For differential games with (i) more than one state variable, or (ii) each player having more than one control variable, or (iii) asymmetric players, but which otherwise meet the assumptions of this chapter, the method given here can be applied in principle, but the qualitative results will be more complicated. If the differential game has a finite horizon, the methods applied here cannot generally be used, for the steady state is not a focus in such games. Furthermore, if the game is non-autonomous, the method given here is also not applicable.

Finally, this chapter focuses on the symmetric OLNE, but it is as least as important to investigate the qualitative properties of symmetric feedback Nash equilibria. Our next step is to derive the qualitative properties of such equilibria for the class of differential games (2.1).

CHAPTER THREE: OPTIMAL PRICING OF LOTTERIES FOR RATIONING PUBLIC RESOURCES

3.1. Introduction

Lotteries are commonly used to ration public resources when shortages exist at established prices. Examples include hunting privileges and recreational uses of state and federal lands throughout the U.S. The economic study of lotteries to date has largely focused upon the merits and shortcomings of lotteries relative to other non-market rationing mechanisms, such as waiting line auctions (queues), with prices generally being naively fixed at zero. However, as empirical observation reveals that lottery prices tend to be positive yet appear in a number of forms (e.g., user-pay and all-pay lotteries; two-part tariff lotteries), several questions arise regarding how to optimally price public lotteries and determine lottery formats, given competing objectives of equity and efficiency in distributing under-priced resources.

This chapter is a first attempt to characterize the optimal prices of public lotteries. Treating the agency as an expected surplus maximizer, the optimal price of lotteries is characterized in settings where i) the units to be rationed are fixed in quantity; ii) the probability of being drawn is uniform over entrants; and iii) the awarded units are non-transferable. In addition, revenue and welfare equivalence are established across lottery mechanisms. Numerical simulations performed across a broad class of private value distributions and fixed quantities of the resource indicate that optimal lottery prices will tend to be positive when the individual private values are relatively homogenous in the lower tail of the distribution.

Three results directly follow. First, the framework can be used to better design and evaluate lotteries and their welfare implications. Second, the equivalency propositions suggest that the agency may not need to worry about the lottery format if revenue or efficiency is the main concern. Third, some previous comparisons that have been made between lotteries under zero prices or an arbitrary price and other rationing mechanisms may need to be reconsidered.

3.2. Literature Review

Allocating resources through lotteries has primarily been viewed as *fair* because each participant has the same chance of being drawn. Aubert (1959) and Goodwin (1992) argued that lotteries have commonly been chosen as the rationing mechanism because they represent a "fair" or "just" means of allocating the goods. Tobin (1970) argued that the fairness of lotteries is consistent with the distributive objective of "specific egalitarianism".[7] Elster (1992) argued that by allocating specific goods and services through a lottery, a greater degree of equality is attained relative to that of a competitive market.

In addition to issues of fairness and equality, the literature has focused on the merits and shortcoming of lotteries compared to alternative rationing mechanisms. Kahneman et al. (1986) found that people prefer queues to lotteries to auctions to allocate concert tickets, while Glass and More (1992) found that lotteries were preferred by hunters over market and queuing systems. Taylor et al. (2003) compared the relative efficiency between lotteries and waiting-line auctions, and showed that lotteries dominate waiting line auctions for a wide range of private value distributions. Koh et al. (2006) also investigated the allocative efficiency of lotteries and waiting-line auctions by simulating more private value distributions. Evans et al. (2009) examined the efficiency of hybrid allocation mechanisms that combine an auction with a lottery and showed that the presence of a lottery does not comprise the efficiency of the auction.

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[7]Specific egalitarianism in his paper is the view that certain specific scarce commodities should be distributed less unequally than the ability to pay for them.

A common assumption in these literatures is that the prices of entry and awarded units of the good are zero, despite the fact that empirical observation reveals that agencies usually charge positive prices in administering lotteries. Few studies consider the role of prices of lotteries, instead simply taking them as given (Wijkander 1988; Boyce 1994). A rigorous examination of the lottery pricing strategy and its associated incentive properties has not yet been undertaken. As Mumy and Hanke (1975, p.712) note, "..if the pricing policy that governs the sale of output is not explicitly taken into account in the analysis, serious errors will be encountered." Thus, examining the optimal pricing strategy is a necessary first step before one can correctly estimate the benefits or costs associated with lotteries and compare them to alternative rationing mechanisms.

Furthermore, those studies that have considered prices have focused on the user-pay lottery, although the price structure can take a variety of forms, with each possessing a unique incentive for participants.[8] Scrogin (2005) documented some evidence about the effects on entry and revenue due to the different price arrangements. While price has been given little consideration in the literature, just what are the optimal lottery price and format remain open questions. In this chapter, we try to answer these questions by presenting a general framework to determine the optimal price for lotteries and to evaluate the performance of different lottery institutions.

3.3. The Optimal Price of Public Lotteries

To begin, suppose that the agency has Q^* units of a homogenous good to be rationed among N risk neutral individuals through a lottery, where $Q^* < N$. Each individual may receive

[8] An exception is Boyce (1994), who considered the all-pay lottery instead. The definitions of the user-pay and all-pay lotteries will be given in next section.

no more than one unit of the good, and awarded units are nontransferable. The N individuals have private values for units of the good that are independently drawn from the interval $[0, \bar{v}]$ according to the continuous distribution function F with associated density function f. Let $v(k)$ define the private value of the kth individual, where $v(0)$ is assumed finite, $v(N) = 0$, and $v'(k) < 0$. An individual's decision to enter the lottery is based on the expected utility theorem, and she will choose to enter if and only if the expected payoff is nonnegative. Let $r : N \to \{0,1\}$ denote the realization of the lottery, where 0 and 1 represent the individual not being drawn and being drawn, respectively.

This chapter deals with a class of lotteries in which the allocation rule is uniform and payments are anonymous. That is, each entrant has the same chance of being awarded a unit, and the payments only depend on the realization of the lottery and not on any personal information about the participants. This class of lotteries is referred to as *regular* lotteries, and its rigorous definition is:

Definition 3.1. *A lottery is called regular if it satisfies the following properties: for any* $i, j \in Q$, $\pi(i) = \pi(j) = Q^* / Q$, *and* $\mu(i) = \mu(j)$, *if* $r(i) = r(j)$, *where* $\pi(\cdot)$ *is the prespecified probability rule of being drawn and* $\mu(\cdot)$ *is the prespecified payment rule.*

From the above definition, it is easy to show that the expected payments for individuals in all regular lotteries are the same. Thus, the mechanism has the desirable property of not requiring the agency to have information about individual preferences and endowments in order determine the resource allocation. As discriminatory lottery mechanisms can be constructed, the regular lottery provides a benchmark from which to evaluate these alternative mechanisms.

Regular lotteries have been considered rather exclusively in the literature and are the focus in the proceeding analysis. From definition 1 a lemma follows directly:

Lemma 3.1. *For any regular lottery, if an individual with private valuation v enters the lottery, then all individuals with private values greater than v also enter the lottery.*

Proof. Consider a regular lottery. If an individual *i* with private value $v(i)$ enters the lottery, then the expected payoff for individual *i* is $\pi(i)v(i) - E(\mu(i)) \geq 0$. For any individual *j* with value $v(j) > v(i)$, it follows that $\pi(j)v(j) - E(\mu(j)) > \pi(i)v(i) - E(\mu(i)) \geq 0$. Therefore, she will also enter the lottery.

From lemma 3.1, equilibrium entry occurs such that the expected value of a unit to the final (lowest valued) entrant is equal to the expected price of the lottery. Defining $\hat{v}$ as the private value of the final entrant, the entry condition is written $\pi\hat{v} - E(\mu) = 0$. It follows that the total number of entrant is:

$$Q(N, \pi, \mu) = Q(N, \hat{v}) = N[1 - F(\hat{v})], \tag{3.1}$$

where Q is the number of individuals in the population whose private valuations are at least as large as $\hat{v}$. Since N is fixed, we do not write N explicitly in $Q(\cdot)$ hereafter. Once the number of entrants is known, the expected revenue (R) and the expected consumer surplus (CS) can be defined as $R \stackrel{\text{def}}{=} Q(\pi, \mu)E(\mu)$ and $CS \stackrel{\text{def}}{=} \int_0^{Q(\pi,\mu)} \pi(k)v(k)dk - Q(\pi, \mu)E(\mu)$, respectively. Because a change in price μ will affect the number of entrants, the expected revenue and the expected consumer surplus generated by the remaining entrants will be affected. Obviously, the agency with different objectives (revenue or welfare maximization) may have different pricing strategies.

It is arguable that the agency may prefer to maximize the expected consumer surplus by using a lottery to ration public resources (Mumy and Hanke, 1975). Thus, the agency's optimization problem is to pick the μ that maximizes *CS*. That is,

$$\max_{\mu} CS(\pi,\mu) = \int_0^{Q(\pi,\mu)} \pi(k)v(k)dk - Q(\pi,\mu)E(\mu)\ . \tag{3.2}$$

The first term represents the aggregate expected benefits, and the second term is the aggregate expenditures by consumers. Figure 2 depicts a lottery system with $v(k)$ specified as a convex function. The curve cab is the entry function associated with the expected payment. (μ, π) and (μ', π') are two lotteries, where their corresponding expected consumer surplus are denoted by the shaded area *C* and *D* respectively, thus the agency is trying to find the optimal μ^* to maximize that shaded area. The managing agency maximizes (3.2) by determining the optimal price μ^*. Substituting entry equilibrium $\pi\hat{v} - E(\mu) = 0$ and $\pi(\cdot) = Q^*/Q$ into (3.2), the agency's problem may also be written as

$$\max_{\hat{v}} CS(\hat{v}) = \int_0^{Q(\hat{v})} \frac{Q^*}{Q(\hat{v})} v(k)dk - Q^*\hat{v}\ . \tag{3.3}$$

Instead of finding the optimal price directly, Eq. (3.3) allows us to maximize expected consumer surplus by finding the optimal private value of last entrant $\hat{v}^*$.[9] Rearranging Eq. (3.3) to yield

$$\max_{\hat{v}} CS(\hat{v}) = Q^*[\frac{1}{Q(\hat{v})} \int_0^{Q(\hat{v})} v(k)dk - \hat{v}]\ . \tag{3.4}$$

The first set of terms in the brackets is the average individual value over all the entrants. The second term is the private value of the final (lowest-valued) entrant. Thus, the difference between the average valuation and the last entrant's valuation, along with the total number of units to be

[9]One reason for doing this is because without knowing the format of lotteries we may not be able to differentiate $E(\mu)$ with respect to μ in general. Thus, the first order approach cannot be applied to problem (3.2).

rationed, determines the expected consumer surplus. Note that given Q^*, the first set of terms suggests that the more elastic the demand in the lower value population the greater the optimal value of the last entrant, and hence the greater the optimal price. Because more elastic demand in the lower value population corresponds to more homogeneous valuations among lower value consumers, the agency then may significantly increase the average individual valuation by slightly increasing the price to screen out a portion of lower value consumers. The second part may reflect the equity concern in the design of lottery. It suggests that the optimal individual value of the last entrant should be set as low as possible, and so for the price. Thus if the demand for the good is relative elastic in the lower value population, the surplus maximizing price needs to balance these two effects.

The first-order condition for an interior maximum of CS can be found by differentiating Eq. (3.3) with respect to $\hat{v}$ using Leibniz's rule and is given by:

$$Q^*[\frac{Q'(\hat{v})}{Q(\hat{v})}\hat{v} - \int_0^{Q(\hat{v})} \frac{Q'(\hat{v})}{Q^2(\hat{v})} v(k)dk - 1] = 0 . \tag{3.5}$$

Rearranging (3.5) yields:

$$\hat{v}^* = \frac{1}{Q(\hat{v}^*)} \int_0^{Q(\hat{v})} v(k)dk + \frac{Q(\hat{v}^*)}{Q'(\hat{v}^*)} . \tag{3.6}$$

Notice that Q^* does not appear explicitly in Eq. (3.6). This indicates that the last entrant's value at the optimum is solely determined by the distribution of individual private values. Thus, changing Q^* will require changing μ accordingly to maintain constant $\hat{v}^*$. If the agency has information about the individual private value distribution, she can derive the optimal $\hat{v}^*$ from (3.6), and the optimal expected price can be subsequently determined by entry equilibrium

$$E(\mu^*) = \pi\hat{v}^* = (Q^*/Q)\hat{v}^* . \tag{3.7}$$

Although Q^* is independent of $\hat{v}^*$, equation (3.7) indicates that the expected consumer surplus maximizing price $E(\mu^*)$ depends on Q^*. Thus, the individual private value distribution and the available units of goods jointly determine $E(\mu^*)$.

The optimal lottery price μ^* will depend upon the pricing arrangement chosen by the agency. If the lottery format is known, then the unique optimal (uniform) price may be determined. To illustrate, consider two widely used uniform-price lotteries: the user-pay (UP) lottery and all-pay (AP) lottery. The UP lottery is characterized by a tariff P that is incurred solely by the Q^* entrants who are drawn, while the AP lottery is characterized by a non-refundable tariff T that is paid by all entrants. These are regular lotteries if the probability of being drawn is equally likely among the entrants. In the UP lottery, the expected payment for an entrant is πP, while it is T in the AP lottery. Thus from Eq. (3.7), the optimal prices are given by $P^* = \hat{v}^*$ and $T^* = \pi\hat{v}^*$, respectively. This implies that P^* does not depend on Q^*, whereas T^* does. Thus, the effects of changes in the available units of resources on T^* may be examined by differentiating $T^* = \pi\hat{v}^*$ with respect to Q^* yields

$$\frac{\partial T^*}{\partial Q^*} = \frac{\hat{v}^*}{Q} \geq 0 . \qquad (3.8)$$

Equation (3.8) indicates that optimal prices of AP lotteries increase as the available quantity of the good increases. While this may appear counterintuitive at first glance, there is a straightforward explanation. Namely, when the available quantity of the good increases, more individuals are willing to enter the lottery in AP lottery, and the agency may be able to increase the price to screen the lower value participants to maintain efficiency. In the extreme, the lottery price may be equal to the market-clearing price when Q^* is sufficiently large, and zero when Q^*

is sufficiently small. In the former case, the lottery reduces to the competitive market. Since Q^* is the only exogenous parameter in the model, equation (3.8) is a prediction that can be empirically tested.

Thus far we have proposed a general framework to optimally price lotteries and characterize the optimal conditions. However, the optimal condition in Eq. (3.6) only considers the interior maximum prices of the model, and corner solutions may exist for some individual value distributions, in which case the optimal prices of lotteries are zero. The following example shows that the optimal prices are always zero if individual values are uniformly distributed. To begin, suppose that the individual value distribution function is denoted by

$$v(k) = a - bk\,. \tag{3.9}$$

Thus, the entry equilibrium implies that the entry function $Q(\hat{v})$ is

$$Q(\hat{v}) = (a - \hat{v})/b\,. \tag{3.10}$$

Substitute Eqs. (3.9) and (3.10) into the agency's optimization problem (3.3) to arrive at

$$\max_{\hat{v}} Q^*(a - \hat{v})/2\,. \tag{3.11}$$

Equation (3.11) clearly indicates that the optimal individual value of the last entrant is zero, i.e., $\hat{v}^*$ =0. It then follows that the optimal prices of lotteries for uniformly distributed individual value functions are always zero.

The above example shows that one must identify the existence of a positive optimal price for lotteries before one can conclude that zero-pricing is a sub-optimal strategy. To that end we show that the optimal price of a lottery is indeed positive for the power-function individual private values, that is,

$$v(k) = ak^{\gamma}\,, \tag{3.12}$$

where $a > 0$, $\gamma < 0$, and $\gamma \neq -1$. In this case, the entry function $Q(\hat{v})$ is characterized by

$$Q(\hat{v}) = (\hat{v}/a)^{\frac{1}{\gamma}} . \tag{3.13}$$

Again, substitute Eqs. (3.12) and (3.13) into the agency's optimization problem (3.3) to get

$$\max_{\hat{v}} - \frac{\gamma Q^* \hat{v}}{1+\gamma} - \frac{aQ^*}{1+\gamma}(\frac{\hat{v}}{a})^{-1/\gamma} . \tag{3.14}$$

Therefore, the first-order necessary condition is

$$-\frac{\gamma}{1+\gamma} + \frac{1}{\gamma(1+\gamma)}(\frac{\hat{v}^*}{a})^{-1-1/\gamma} = 0 . \tag{3.15}$$

Equation (3.15) may be rewritten to arrive at the optimal value of the last entrant

$$\hat{v}^* = a\gamma^{-2\gamma/(1+\gamma)} > 0 . \tag{3.16}$$

Thus the optimal prices of AP and UP lotteries are, respectively

$$P^* = \hat{v}^* = a(\gamma^{-2})^{\frac{\gamma}{1+\gamma}} > 0 , \tag{3.17}$$

$$T^* = \pi \hat{v}^* = Q^* a(\gamma^2)^{\frac{1-\gamma}{1+\gamma}} > 0 . \tag{3.18}$$

Equations (3.17) and (3.18) clearly indicate that the optimal prices of AP and UP lotteries are indeed positive if the individual value distribution is a power function.

The results suggest that the previous studies on examining the efficiency of lotteries or the relative efficiency comparison with other non-market mechanisms under zero-prices are misleading and need to be re-considered. An interesting point here is that the power distribution has been used to demonstrate the relative efficient of the zero-pricing lotteries to a queue in Taylor et al. (2003), while the uniform distribution was an example used to illustrate the optimal public investment for a lottery with a positive price in Mumy and Hanke (1975). Our model then shows that the conclusions in Taylor et al. can be made even stronger under optimal pricing, while the illustrative example in Mumy and Hanke may not be an appropriate one.

This section characterized a framework for determining the optimal price of regular lotteries. As we see from the examples of UP and AP lotteries, the agency cannot be able to determine the optimal lottery price until she chooses the lottery format. Thus, a question of what is the "best" lottery format naturally follows. In the next section, we will discuss this issue in some depth.

3.4. The Optimal Format of Lotteries

The previous section discussed how one can determine the optimal price for a lottery, and explicitly derived the optimal prices for user-pay and all-pay lotteries. However, different lottery formats will affect entry and the associated probability of being drawn, the net benefits realized by entrants, and the revenues collected by the managing agency. In order to better design and evaluate lottery mechanisms, the relative performance of alternative lottery formats needs to be addressed. This section characterizes the revenue and welfare properties of different regular lotteries and establishes the revenue and welfare equivalency propositions. Together, these two sections characterize the optimal lottery pricing strategy of regular lotteries. First of all, we have the following proposition.

Proposition 3.1. *Given Q^* and $v(k)$, the agency's expected revenues are equivalent in any two regular lotteries, if and only if the induced numbers of entrants are the same in the two lotteries.*

Proof. Suppose that any two regular lotteries (π, μ) and (π', μ') induce the same number of entrants, i.e., $Q = Q'$. Thus, we have $\pi = \pi' = Q^*/Q$. Equilibrium entry is characterized by the expected value of entry being equal to the final entrant in each lottery:

$$\pi v(k) - E(\mu) = 0\,, \tag{3.19}$$

$$\pi v(l) - E(\mu') = 0\,. \tag{3.20}$$

From lemma 3.1, these entrants' private values $\hat{v}$ must be equal in the two lotteries, i.e., $v(k) = v(l) = \hat{v}$. From Eqs. (3.19) and (3.20), it follows that $E(\mu) = E(\mu')$ and expected revenues (R) are therefore equal to

$$R = Q \times E(\mu) = Q' \times E(\mu')\,. \tag{3.21}$$

Thus, if regular lotteries induce the same number of entrants for a given Q^*, then the expected revenues are equivalent. Conversely, consider a regular lottery that generates revenues *R*. It follows that

$$R = Q \times E(\mu)\,. \tag{3.22}$$

For the final entrant, we have $E(\mu) = \pi \cdot \hat{v}$. Therefore

$$E(\mu) = R/Q = \pi\hat{v}\,. \tag{3.23}$$

This indicates that the expected payment is equal to the expected benefits derived by the last entrant. Substituting $\pi = Q^* / Q$ into Eq. (3.23) yields

$$\hat{v} = R/Q^*\,. \tag{3.24}$$

Thus, given the target revenue level *R*, the private values of the last entrants are unique, so the total number of entrants $Q = N[1 - F(\hat{v})]$ is therefore also unique. It follows that if two regular lotteries are revenue equivalent, then the lotteries induce the same number of entrants.

The expected consumer surplus of regular lotteries is calculated by Eq. (3.3). It implies that given the same number of entrants in two regular lotteries, the expected consumer surplus attained by entrants are equivalent. This is formalized in the proposition 3.2.

Proposition 3.2: *Expected consumer surplus is equivalent in any two regular lotteries if the induced numbers of entrants is the same.*

This proposition specifies a sufficient condition for the surplus equivalency. However, the necessary part of the proposition may not be valid as in the case of revenue equivalency. This is because if the optimal prices of lotteries are not equal to zero or the market clearing price, then the continuous expected consumer surplus function guarantees the existence of at least one expected consumer surplus level that corresponds to multiple last entrant valuation in the neighborhood of the optimum.[10] This idea can be better seen in the Figure 3.4 in the next section. We will discuss this issue in some depth later on.

Propositions 3.1 and 3.2 reveal that all the regular lotteries are revenue and welfare equivalent. Any revenue or welfare that can be generated by one lottery format can be generated by another lottery format by inducing the same number of entrants. However, different lottery formats may have significant differences with respect to Pareto implications or the wealth distribution. For example, unsuccessful participants in AP lottery will suffer losses, and hence are worse off, while no one gets hurt in UP lottery. In practice, the agency may prefer one lottery format over another if she takes other criteria into account, such as the wealth distribution.[11]

Moreover, a special lottery format that has been widely applied in the practice is the two-part tariff lottery, it requires a relative small admission fee for each participant, and only those drawn are required to pay the use fee. It can be shown that, unlike two-part tariffs in normal market (Feldstein 1972; Auerbach and Pellechio 1978), the above two equivalencies prevent the

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[10] Notice that we do not limit ourselves to the optimal level of consumer surplus in this proposition.

[11] Experimental evidence has shown that people do have preferences over the wealth distribution. See Kemp and Bolle (1999) and Rutström (2000).

agency from determining the unique optimal pricing strategy of two-part tariff lotteries by merely considering revenue or welfare, because different combinations of tariffs may induce the same aggregate entry and thus the same revenue and consumer surplus. One possible solution to this issue is set the non-refundable entry fee to balance the administration costs of the lottery, then determine the access fee by our model.

These two sections provide a general theoretical framework for determining the optimal price for regular lotteries. To more fully evaluate the model, we next apply this framework to derive the optimal prices for several simulated private value distributions.

3.5. Numerical Simulations

The previous sections characterized the surplus maximizing prices of lotteries and the equivalence of revenue and welfare. However, these results do not provide an indication of the optimal pricing strategy for different distributions of individual private values. For example, how concentrated do the lower valuation population or how relatively homogeneous do lower consumers' valuations need to be in order for the optimal price to be positive? And how large is the optimal expected consumer surplus relative to the pricing extremes of competitive pricing or zero pricing? And, how different will the optimal prices be between different lottery formats, such as the UP lottery and AP lottery? And, under what circumstances does the necessary part of proposition 2 fail? To address these issues, numerical simulations are performed across a broad class of distributions describing the population of individual private values.

Following Taylor et al. (2003) and Scrogin (2009), the population of individual private values is assumed to be characterized by the beta density function, which is given by

$$f(v;\alpha,\beta)=\frac{1}{B(\alpha,\beta)}v^{\alpha-1}(1-v)^{\beta-1},\ \alpha,\beta>0,\ v\in[0,1], \tag{3.25}$$

where $B(\alpha,\beta)=\int_0^1 v^{1-\alpha}(1-v)^{\beta-1}dt$ and α and β are shape parameters. The density is quite flexible in that it can generate distributions with widely different shapes. The density is symmetric if $\alpha=\beta$, with uniformity resulting when $\alpha=\beta=1$, and the density is left skewed if $\alpha<\beta$, and right skewed if $\alpha>\beta$.

In the simulations, we set $\beta=1$, and vary α across seven values ranging from 0.2 to 1. Four of these are used in Taylor et al. (2003), while the other three allow a broader examination of the effects the distribution of private values on optimal price and hence the efficiency of lotteries. In each case, 100 random samples of 10,000 values were drawn from Eq. (3.25). For a given sample, $v(\cdot)$ is approximated by arranging the 10,000 sample points in descending order. Table 1 reports the descriptive statistics calculated from the 100×7 random samples. We normalized the private value between $[0,1]$. With uniformly distribution private values (α=1), only five percent of the population has individual private value below 0.05, but the fraction increases sharply as the density skews left from uniformity, e.g., more than half of the population values lower than 0.05 when α=0.2.

With this information, we can conjecture that the lower the α is, the more elastic the entry at lower prices, and hence the higher optimal price suggested by Eq. (3.4). Moreover, the smaller value of α may more accurately describe the private value distribution for those goods rationed by lottery by state and federal agencies, such as hunting licenses throughout the US and Canada, since it is possible that only a small portion of people have high valuation for hunting, while a majority of people have very low valuations.

Assuming $Q^*=500$ and given the sample size of 10,000, this means that five percent of population will receive units of goods. In order to fully investigate the effect of price on expected consumer surplus, we evaluated the expected consumer surplus for each price, ranging from zero to market clearing price according to Eq. (3.4) in each case. Our results are summarized in Tables 2-4 and Figures 3-5.

In Table 2, the first two columns are the optimal expected prices for UP lottery and AP lottery, and the last three columns are the corresponding fraction of the population that enters, expected revenue, and expected consumer surplus for each private value distribution, respectively. From the table, the optimal prices are positive in six of the seven cases (e.g., all $\alpha<1$), and the effects on entry are significant when α is small (i.e., $\alpha\leq0.5$). Only about the third of the population will enter the lottery, compared to full participation under zero pricing. It suggests that there may be a sizable welfare loss if the agency simply sets the price equal to zero, at least for small αs. The second column in Table 4 reports the relative welfare between optimal pricing and zero pricing. We can see that the expected consumer surplus under optimal pricing could be two times as large as the expected consumer surplus under zero pricing when $\alpha=0.2$.

Moreover, the optimal price increases as the individual private valuation skews to the left (as α decreases), and the corresponding number of entrants decreases at the same time. These observations confirm our conjecture that the more dense the distribution in the lower tail (lower α), the higher optimal price. Thus, if the argument about the distribution of individual private values for hunting permits is reasonable, our model offers an explanation why federal and state agencies establishes positive tariffs in their lottery systems, and provides a framework for determining these tariffs.

In order to check the comparative statics of the available resources Q^*, we calculate the optimal AP lottery prices for different Q^*. Specifically, we increase Q^* to 1000 and 1500. Table

3 presents the optimal prices of AP lotteries for different Q^*. It clearly indicates that the optimal prices of AP lotteries will increase, other things equal, as Q^* increases.

In Table 4, we report some relative comparison results. The first column reports the relative expected consumer surplus between lotteries with optimal pricing and zero pricing. From this column, we can see that the expected consumer surplus under optimal pricing dominates zero pricing lotteries and this domination decreases when α increases as expected, and eventually converges to one. Thus, the agency may simply set the price equal to zero for large α.[12] The seven scenarios are depicted graphically in Figure 3. The figure clearly shows that the optimally priced lottery dominates zero pricing when the private value distribution is left skewed, and the difference becomes negligible as α increases. Thus, our results may provide another possible explanation for why agencies use lotteries to ration some types of goods to the public. The second column reports the relative optimal prices between UP and AP lottery. It shows that optimal prices in UP lotteries is much higher than in AP lotteries for some individual private value distributions, and hence indicates that the lottery format play an important in determining the optimal price of lotteries. Figure 4 depicts the optimal prices of UP and AP lotteries for all α's.

Regarding the revenue and welfare equivalency propositions, the first two columns in Table 2 denote the optimal price for the UP lottery and the AP lottery across the different individual private value distributions, where both prices generate the same number of entrants in the third column, hence the same expected revenue and expected consumer surplus, which is reported in the last two columns in Table 2. Thus, it provides evidence that two special lottery institutions, UP lottery and AP lottery, are equivalent in revenue and efficiency at the optimum.

[12] This claim can be made by the "payoff-dominance" effect. That is, the agency may choose the "easiest" strategy if the foregone benefits is negligible. See Harrison (1989).

The last two columns in table suggest that the expected consumer surplus increases, while the expected revenue decreases as the individual value function moves toward uniformity (as α increase).

For the necessary part of proposition 3.2, Figure 5 provides counter examples to illustrate its invalidity.[13] For example, consider the distribution function with parameters $\alpha = 0.33$ and $\beta = 1$, and the target expected consumer surplus is 150, two possible different numbers of entrants could generate this target expected consumer surplus. Specifically, they are 27.84% of the population and 75.67% of the population, with each number of entrants corresponds to its own price. Therefore, the necessary part of proposition 3.2 is not valid as its counterpart in proposition 3.1.

In summary, our numerical results clearly demonstrated optimal positive pricing over different individual value distributions, investigated the revenue and welfare equivalency propositions, and provided counter-examples to show the invalidity of the necessary part of proposition 3.2. It should be noted also that similar numerical exercises to all of the above were carried out for various right skewed beta densities, and, not unexpectedly, the optimal price is zero because of the low concentration of the population and inelastic demand associated with the lower tail of the distribution. That is, price plays no rule in balancing efficiency and equity.

3.6. Conclusion

This chapter developed a framework for optimally pricing lotteries for rationing public resources. An important aspect of this analysis leads to an operational pricing rule in terms of different individual private value distributions. As the outcomes of the theoretical analysis and

[13] The data for Figure 4 all come from the first draw in each sample.

numerical simulations indicate, the optimal lottery price will be positive for left skewed individual private value distributions and that all regular lotteries are equivalent in revenue and welfare at optimum.

Although this model is quite general, it has left some extensions for future research. First, how should the agency determine the optimal prices for non-regular lotteries, such as sequential lotteries and lotteries requiring different payments from different groups (e.g. residents vs. non-residents)? Since these lottery formats are common at the state and federal levels, examining their optimal pricing strategies may have great policy implications.

Second, how would a non-expected utility population affect the optimal pricing strategy and the corresponding empirical implications of lotteries? Abundant experimental evidence has shown that individual behavior may not be consistent with the expected utility theorem (e.g., Harrison and Rutström 2009). It may be informative to characterize the optimal prices of lotteries based on the prospect theorem or rank dependent utility theorem. Finally, empirical and experimental investigations of optimal pricing strategies and the revenue and welfare equivalency propositions may be a fruitful avenue for future research.

Table 1. Beta Distributed Individual Private Values: Summary Statistics on 100 Random Samples

	Mean Private Values	Standard Deviation	Fraction of Population with lower valuation ($v < 0.05$)	Min	Max
α=.20	0.17 (0.0025)	0.25 (0.0022)	0.55 (0.0051)	9.82e-19 (3.89e-18)	1.00 (0.0004)
α=.25	0.20 (0.0027)	0.27 (0.0020)	0.47 (0.0047)	2.40e-15 (8.62e-15)	1.00 (0.0004)
α=.33	0.25 (0.0029)	0.28 (0.0018)	0.37 (0.0043)	6.17e-12 (1.90e-11)	1.00 (0.0003)
α=.50	0.33 (0.0030)	0.30 (0.0015)	0.22 (0.0040)	1.96e-08 (4.52e-08)	1.00 (0.0002)
α=.67	0.40 (0.0030)	0.30 (0.0013)	0.14 (0.0032)	1.25e-06 (2.21e-06)	1.00 (0.0001)
α=.75	0.43 (0.0029)	0.30 (0.0013)	0.11 (0.0030)	2.08e-06 (8.04e-06)	1.00 (0.0001)
α=1	0.50 (0.0028)	0.29 (0.0013)	0.05 (0.0019)	0.91e-04 (0.0001)	1.00 (0.0001)

Note: $\beta = 1$, $Q^* = 500$, $N = 10,000$. The reported results are means (standard deviations) of the descriptive statistics calculated for each random sample.

Table 2. Optimal Price and Expected Consumer Surplus

	Optimal Price for UP lottery	Optimal price for AP lottery*	Fraction of the population enter the lottery	Expected Revenue	Expected CS
α=.20	0.13 (0.0150)	0.02 (0.0032)	0.34 (0.0172)	63.50 (7.4999)	162.63 (1.8575)
α=.25	0.12 (0.0140)	0.014 (0.0023)	0.41 (0.0187)	58.61 (7.0020)	166.30 (1.6531)
α=.333	0.10 (0.0135)	0.009 (0.0017)	0.53 (0.0227)	50.53 (6.7624)	172.73 (1.6735)
α=.50	0.06 (0.0096)	0.004 (0.0008)	0.75 (0.0199)	31.35 (4.7957)	187.82 (1.6757)
α=.667	0.03 (0.0067)	0.002 (0.0004)	0.91 (0.0158)	14.38 (3.3273)	206.08 (1.5618)
α=.75	0.01 (0.0049)	0.001 (0.0003)	0.96 (0.0113)	7.01 (2.4637)	216.49 (1.4473)
α=1	0.00 (0.0002)	0.000 (0.0000)	1.00 (0.0003)	0.08 (0.0927)	249.99 (1.3861)

Note: $\beta = 1,\ Q^* = 500,\ N = 10,000$. The numbers in parentheses are standard deviations.
* This column also represents the expected optimal price for a general regular lottery.

Table 3. Optimal prices of All-Pay Lotteries for different Q^*

	$Q^*=500$	$Q^*=1000$	$Q^*=1500$
α=.20	0.02 (0.0032)	0.038 (0.0064)	0.057 (0.0095)
α=.25	0.014 (0.0023)	0.029 (0.0047)	0.0043 (0.0070)
α=.333	0.009 (0.0017)	0.019 (0.0034)	0.0286 (0.0051)
α=.50	0.004 (0.0008)	0.008 (0.0015)	0.0126 (0.0023)
α=.667	0.002 (0.0004)	0.003 (0.0008)	0.0048 (0.0012)
α=.75	0.001 (0.0003)	0.001 (0.0005)	0.002 (0.0008)
α=1	----	----	----

Note: $\beta=1$, $Q^*=500$, $N=10,000$. The numbers in parentheses are standard deviations.

Table 4. Comparison of Relative Expected Consumer Surplus and Lottery Prices

	Expected *CS* in lottery with optimal price relative to zero price	Optimal price in UP lottery Relative to AP lottery (P^*/T^*)
α=.20	1.95 (0.0257)	6.76 (0.3123)
α=.25	1.66 (0.0180)	8.44 (0.4472)
α=.333	1.38 (0.0108)	10.64 (0.4113)
α=.50	1.13 (0.0057)	14.94 (0.3734)
α=.667	1.03 (0.0029)	18.26 (0.3066)
α=.75	1.01 (0.0017)	----
α=1	1.00 (0.0001)	----

Note: $\beta = 1,\ Q^* = 500,\ N = 10,000$. The numbers in parentheses are standard deviations.

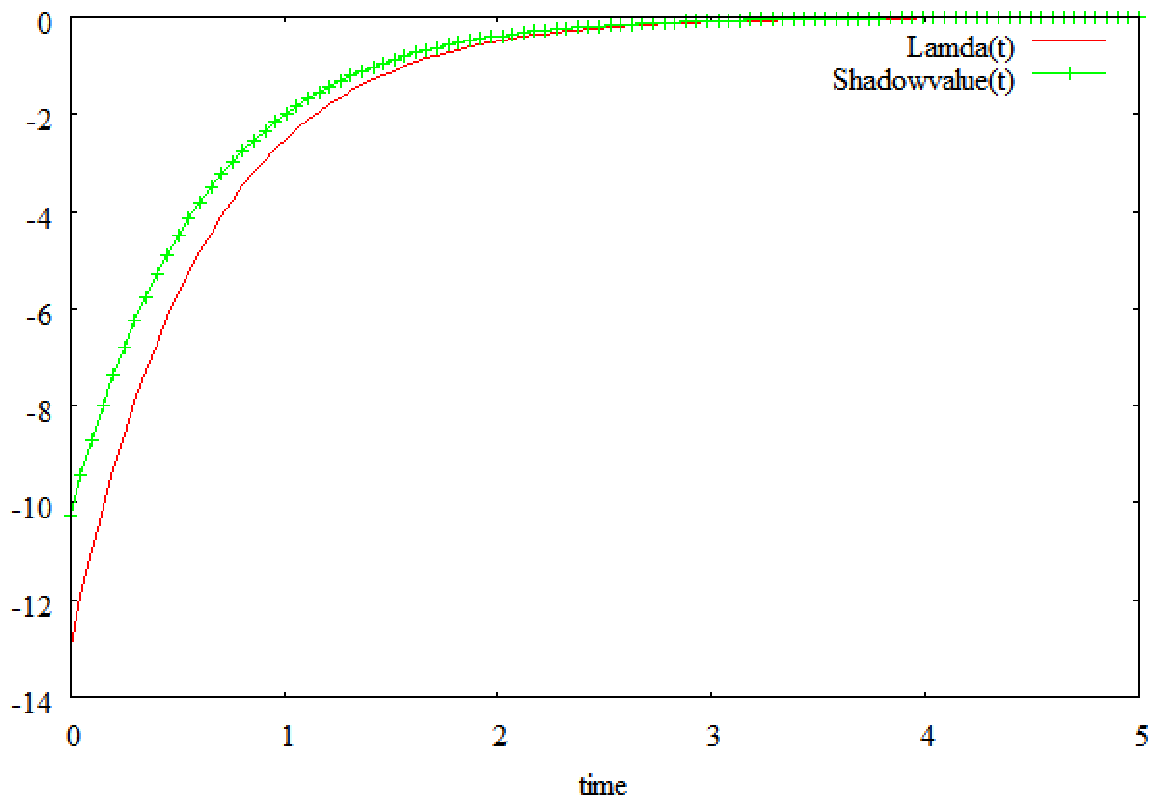

Figure 1. Costate variables (λ) and the shadow value of the state variable in the linear quadratic game.

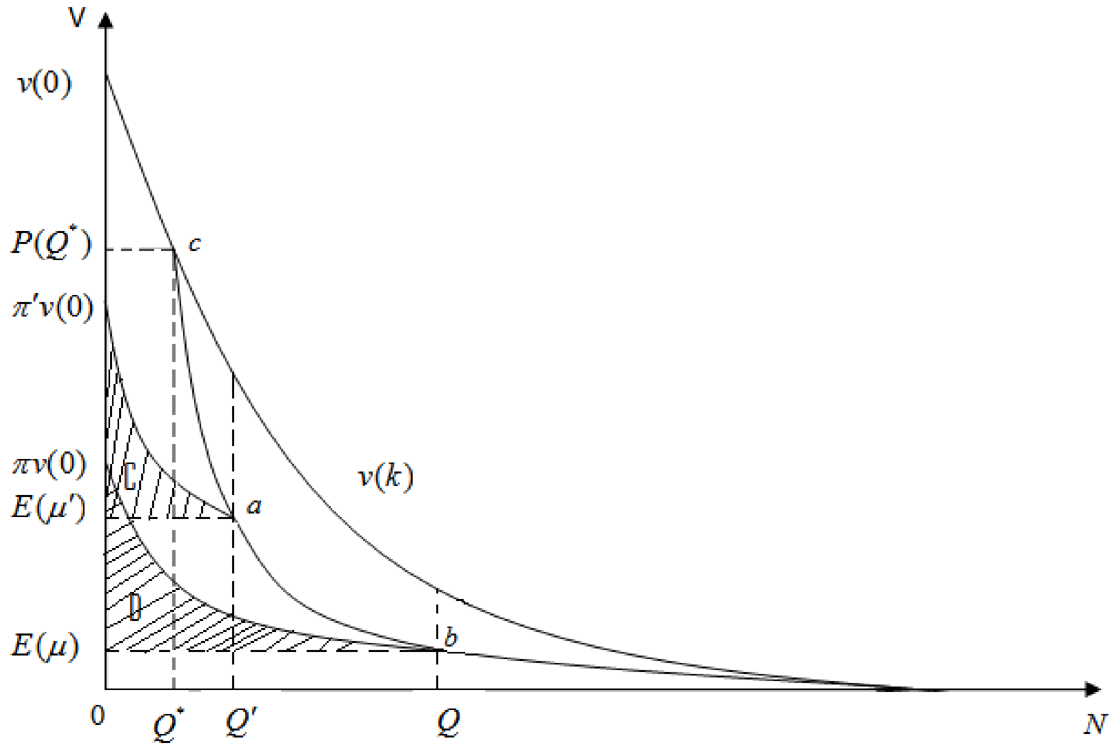

Figure 2. A graphical representation of a lottery system

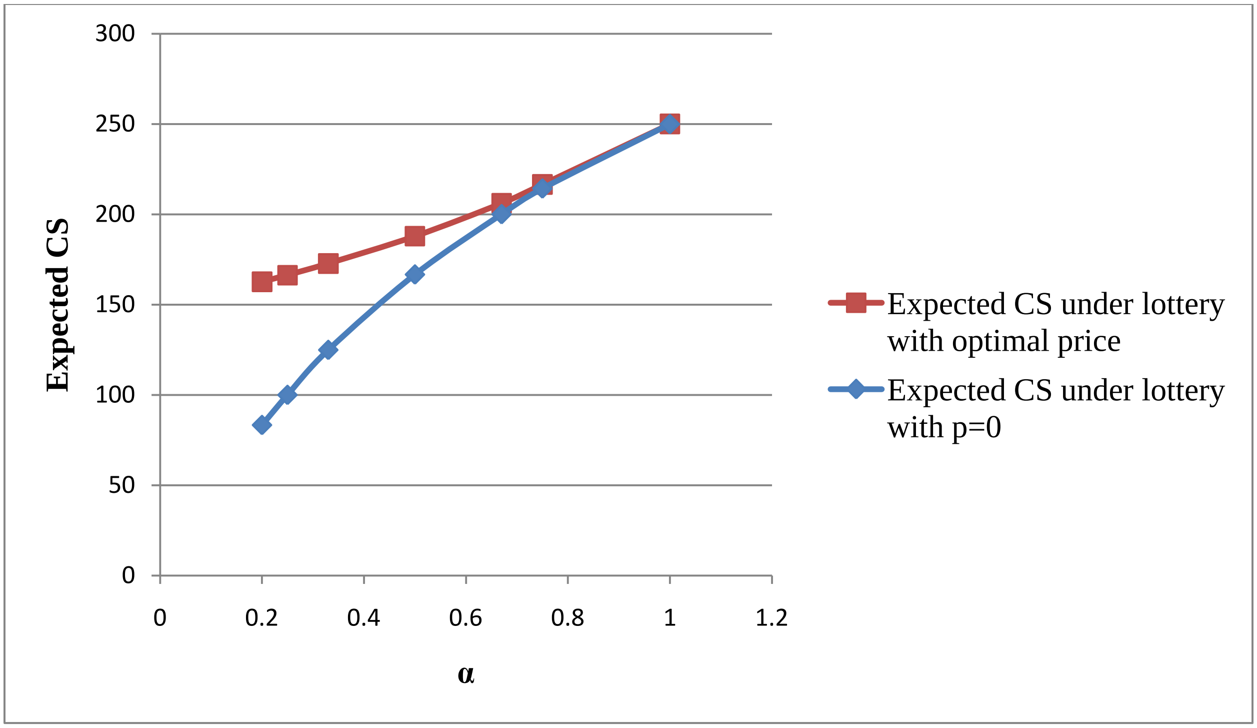

Figure 3. Expected CS comparison between lotteries under optimal price and zero price (β=1, Q^* =500, N=10,000)

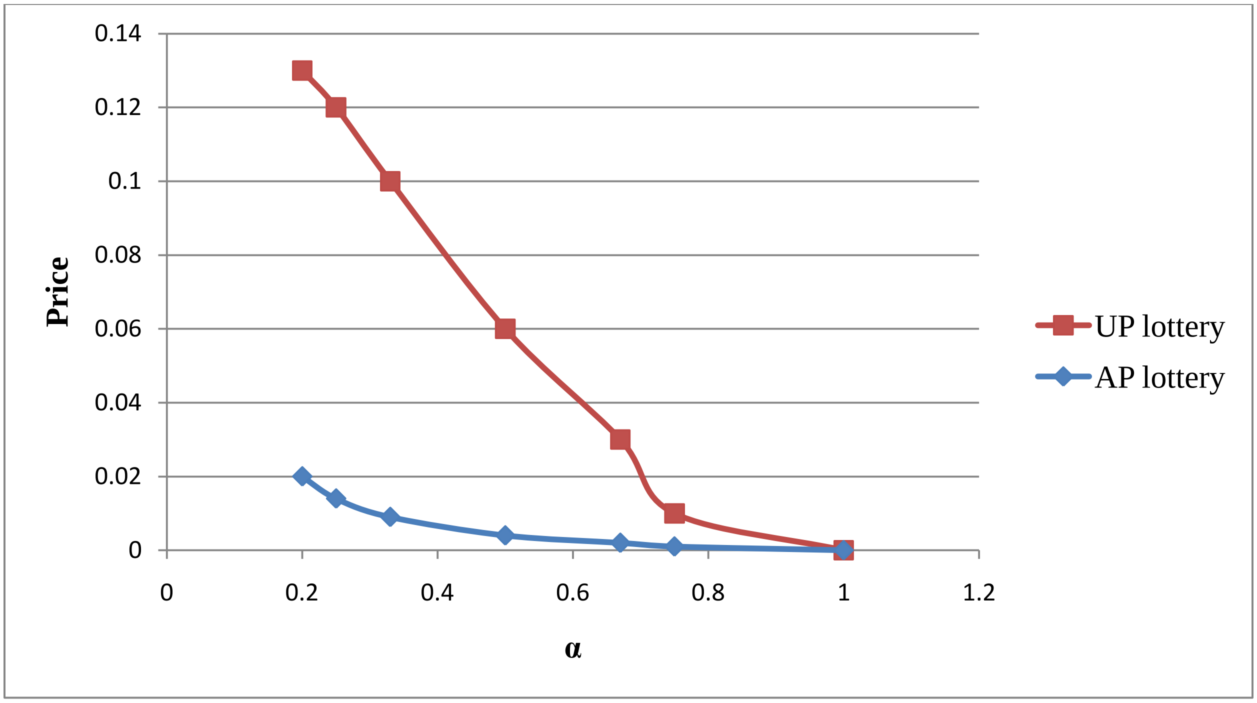

Figure 4. Optimal price for different format of lottery across different left skewed beta distributions (β=1, Q^*=500, N=10,000)

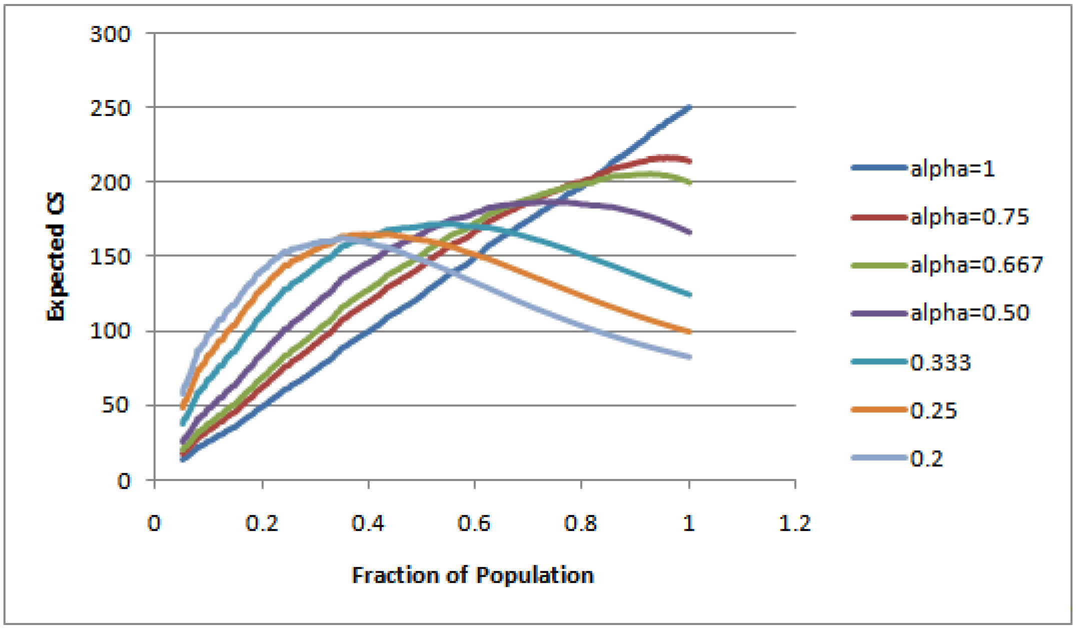

Figure 5. Expected consumer surplus for the entrants of fraction of the population (β=1, $Q^* = 500$, N=10,000)

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