2015

Output Consensus Control for Heterogeneous Multi-Agent Systems

Abhishek Pandey
Louisiana State University and Agricultural and Mechanical College, apande3@tigers.lsu.edu

OUTPUT CONSENSUS CONTROL FOR HETEROGENEOUS
MULTI-AGENT SYSTEMS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Division of Electrical and Computer Engineering

by
Abhishek Pandey
B.E., Visvesvaraya Technological University, India, 2008
M.S., Louisiana State University, USA, 2012
August 2015

Acknowledgments

I would like to express my never ending gratitude to my advisor and committee chair Professor Guoxiang Gu, for providing guidance and expertise during all stages of this research project. His enthusiasm and support gave me the confidence to tackle problems that seemed overwhelming at the time. His suggestions helped me to overcome hurdles and kept me enthusiastic and made this work a wonderful learning experience.

I wish to express my gratitude to Professors Kemin Zhou, Shuangqing Wei, Shahab Mehraeen and Frank Tsai for being a part of my dissertation committee and providing constructive criticism and insight on this dissertation. I also take this opportunity to thank Professor James J. Spivey for his moral support throughout my studies. I would also like to extend my thanks to Dr. Luis D. Alvergue for the many project related meetings and discussions we had to help me through this research work.

Finally, I would like to thank my family for their care, unconditional love, and invaluable support throughout my entire life.

Table of Contents

List of Tables

List of Figures

Acronyms

ARE Algebraic Riccati Equation

ATC Air Traffic Control

ATM Air Traffic Management

FAA Federal Aviation Administration

FI Full Information

GM Gain Margin

LMI Linear Matrix Inequality

LQG Linear Quadratic Gaussian

LTR Loop Transfer Recovery

MAS Multi-agent System

MIMO Multi-Input/ Multi-Output

PR Positive Real

SISO Single-Input/ Single-Output

Abstract

We study distributed output feedback control of a heterogeneous multi-agent system (MAS), consisting of N different continuous-time linear dynamical systems. For achieving output consensus, a virtual reference model is assumed to generate the desired trajectory for which the MAS is required to track and synchronize. A full information (FI) protocol is assumed for consensus control. This protocol includes information exchange with the feed-forward signals. In this dissertation we study two different kinds of consensus problems. First, we study the consensus control over the topology involving time delays and prove that consensus is independent of delay lengths. Second, we study the consensus under communication constraints. In contrast to the existing work, the reference trajectory is transmitted to only one or a few agents and no local reference models are employed in the feedback controllers thereby eliminating synchronization of the local reference models. Both significantly lower the communication overhead. In addition, our study is focused on the case when the available output measurements contain only relative information from the neighboring agents and reference signal. Conditions are derived for the existence of distributed output feedback control protocols, and solutions are proposed to synthesize the stabilizing and consensus control protocol over a given connected digraph. It is shown that the $\mathcal{H}_\infty$ loop shaping and LQG/LTR techniques from robust control can be directly applied to design the consensus output feedback control protocol. The results in this dissertation complement the existing ones, and are illustrated by a numerical example.

The MAS approach developed in this dissertation is then applied to the development of autonomous aircraft traffic control system. The development of such systems have already started to replace the current clearance-based operations to trajectory based operations. Such systems will help to reduce human errors, increase efficiency, provide safe flight path, and improve the performance of the future flight.

Chapter 1
Introduction

1.1 Motivation

The consensus control is a research topic which has attracted great attention from many research communities, ever since the theoretical framework of the consensus problem for multi-agent systems (MASs) was proposed and analyzed by Olfati-Saber and Murray in [62]. It leads to the research field of consensus control.

The main objective of the consensus control is to develop algorithms for MASs such that the group of dynamic agents reaches an agreement regarding a certain quantity of interest by communicating information with neighboring agents and itself. The MASs differs from traditional control systems because it requires the convergence of control theory and communications. The challenges to MASs lie in the design of control systems that achieve robust cooperation, despite disconnections of some agents, inherent to most distributed environments. Had no notion of cooperative control evolved, each agent would be running separately, utilizing more resources and increasing the cost. It would not be able to utilize the availability of several agents in a distributed environment. It is the need for the cooperation which reveals many problems which otherwise would have been undiscovered.

Most of the existing consensus study is for homogeneous MASs. But in real world, most systems are heterogeneous in nature. In fact for practical systems, the agents coupled with each other have different dynamics because of various restrictions or depending on the common goal which they are trying to achieve together. For truly heterogeneous MASs, the state consensus may not be meaningful due to possible difference in their dynamics and state dimensions. Hence it makes more sense to consider output consensus. It should be pointed out that even if all the agent systems are made by the same manufacturer, the system dynamics may change due to aging and working environments. Therefore there is a need to study the more complex consensus problem of heterogeneous MASs for example;

heterogeneous MASs with delays, heterogeneous MASs under directed graphs/switching topologies/random networks, discrete heterogeneous MASs etc. Parameters like friction, changing masses, damping coefficients, material properties, and the like can not be ignored in real-life.

1.2 Applications of Consensus Control

Consensus control has received a lot of attention in the literature due to its numerous applications in various areas, e.g. unmanned aerial vehicles [12, 13, 82], mobile robots [84, 85], satellites [14, 16, 71], formation control [25, 43], distributed sensor networks [59, 60], flocking [57], automated highway systems [7, 69] and synchronization of complex networks [45, 68, 73], to name only a few.

1.3 Early Work in Consensus Control

The consensus problem is a fundamental research topic in the field of distributed computing [42]. The problem of cooperative control of networked MASs [25] is important because in real-life networked systems have limitations such as restricted network bandwidth, limited sensing capabilities of agents or packet loss during communication. This makes the area of cooperative control interesting, where the agents may have limited information about their environment and the state of the other agents while they should also adjust themselves to the changing environment according to their system dynamics.

Formation control is one of the important applications of cooperative MASs. Existing approaches to solve this type of problem are classified as leader-follower method and virtual-leader method. The leader-follower approach has a leader which defines a reference trajectory for others to follow. Although the method is simple to implement, it requires each follower to have information about its leader. This dependence on a leader during formation may be undesirable and can lead to a bottleneck situation. Additionally, this approach is known to have poor disturbance rejection properties. Such situations can be tackled by having decentralized control where each agent may look for information from its neighbors, thereby reducing the complexity of information exchange between agents.

Another approach is based on creation of a virtual-leader. A fictitious leader is created to replace the real leader and all other agents are considered as followers. This approach simplifies analysis and requires fewer sensors for control law implementation. Such an approach is also capable of overcoming the problems associated with disturbance rejection. Although the advantage is achieved at the expense of high communication and computation capabilities which are essential to identify the virtual leader and then to communicate its position in real-time to the other followers.

Early work in the field of cooperative MASs includes [35] where the agents' dynamics are modeled as a switched linear system, and [52, 62] where agents consist of a scalar integrator. In [41] the agents are modeled as double integrators. Also in [74] the authors investigate the motion of vehicles modeled as double integrators. The objective for them is to achieve a common velocity while avoiding collision between vehicles. The results for integrator chains more than two has been discussed in [79].

Some of the recent work concerned with homogeneous MASs [45, 48, 87] have state-space representation. They are more general and include integrator dynamics as a special case. The results in aforementioned papers and others solve the problems of designing distributed and local control protocols for state feedback and state estimation. As the problem can be decomposed into two parts, the solutions to cooperative control based on output feedback are also available.

The survey paper [61] by Olfati-Saber and Murray, and references therein, provide a good overview of system-theoretic framework for expression and analysis of consensus algorithms in both continuous-time and discrete-time of MASs along with results, applications and challenges in this area. The common feature between these approaches is the assumption about the communication topology which allows us to use a particular cooperative formation control methodology. Communication topologies in networked systems can be fixed. They can also be dynamic or be a switching network [62, 64] either due to node and link failures/creations, formation reconfigurations [58] or due to flocking [57, 63]. The

3

network with switching topology is interesting as the graph is changing i.e, a node is being removed or added which could affect the consensus between the agents. The information flow between graphs could be directed or undirected, with or without time-delays. The strategy to form a communication topology should satisfy stability and meet performance requirements, and should be robust to any changes in the communication topology. A directed graph (digraph) can be strongly connected or connected. It is called strongly connected if there is a path from each node in the graph to every other node [Figure 1.1(a)], whereas it is called connected if between any two nodes there is a path from one to another [Figure 1.1(b) - Node 6 is a connected node].
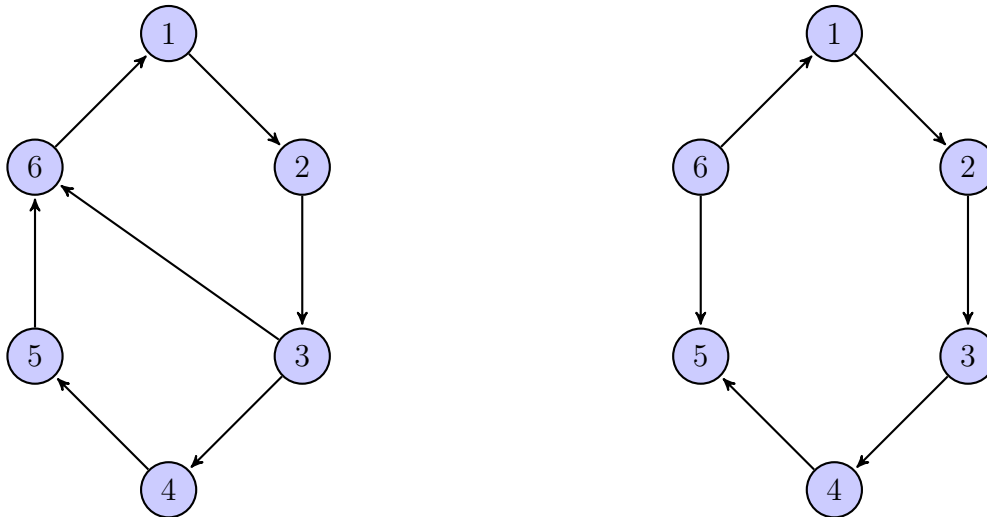
Figure 1.1: a) Left: Strongly Connected Graph. b) Right: Connected Graph.

1.4   An Overview on Consensus Control in Homogeneous MASs

In systems theory to achieve a desired behavior from a complicated system it is usually preferred to design an interconnection of simpler subsystems whose dynamics are similar to the complex system. Luc Moreau puts forward the stability properties for a class of linear time-varying systems in [52, 53, 54]. Moreau considers each individual system in the network to be a scalar integrator. He provided a condition for convergence to a consensus value for minimal connectivity of the graph which allowed the communication from one

system to another to be indirect, assuming that a system need not communicate directly with all other systems in the network at any given time. A linear time-varying system is described as

$$\dot{x}(t) = A(t)x(t). \tag{1.1}$$

We assume that $A(t)$ is piecewise continuous and bounded. In addition, $A(t)$ satisfies

$$A(t) = \begin{cases} \sum_{k=1}^{N} a_{ik}(t), & j = i \\ -a_{ij}(t), & j \neq i. \end{cases} \tag{1.2}$$

If there exists a $T > 0$ such that for all t

$$\int_{t}^{t+T} A(\tau)d\tau, \tag{1.3}$$

represents a connected graph, then the system equilibrium sets of the consensus states is uniformly exponentially stable. Each component of $x(t)$ in the MAS described by (1.1) represents an agent.

The communication between the set of interconnected systems is encoded through a time-varying weighted directed graph (digraph) specified by $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where $\mathcal{V} = \{v_i\}_{i=1}^{N}$ is the set of nodes and $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ is the set of edges or arcs, where an edge starting at node i and ending at node j is denoted by $(v_i, v_j) \in \mathcal{E}(t)$. The node index set is denoted by $\mathcal{N} = \{1, \cdots, N\}$. The neighborhood of node i at time t is denoted by the set $\mathcal{N}_i(t) = \{j \mid (v_j, v_i) \in \mathcal{E}(t)\}$. A path on the digraph is an ordered set of distinct nodes $\{v_{i_1}, \cdots, v_{i_K}\}$ such that $(v_{i_{j-1}}, v_{i_j}) \in \mathcal{E}(t)$. Let $\mathcal{A}(t) = [a_{ij}(t)] \in \mathbb{R}^{N \times N}$ be weighted adjacency matrix. The value of $a_{ij}(t) \geq 0$ represents the coupling strength of edge (v_j, v_i) at time t. Self edges are not allowed, i.e., $a_{ii}(t) = 0 \; \forall \; i \in \mathcal{N}$ for all t. Denote the degree matrix for $\mathcal{A}(t)$ by $\mathcal{D}(t) = \text{diag}\{\deg_1(t), \cdots, \deg_N(t)\}$ with $\deg_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(t)$ and the Laplacian matrix as $\mathcal{L}(t) = \mathcal{D}(t) - \mathcal{A}(t)$ which is equivalent to

5

$$l_{ij}(t) = \begin{cases} \sum_{k=1}^{N} a_{ik}(t), & j = i \\ -a_{ij}(t), & j \neq i. \end{cases} \tag{1.4}$$

If $v_i \to v_j \ \forall \ j \in \mathcal{N}$, then v_i is called a connected node of $\mathcal{G}(t)$. The digraph is called connected if there exists a connected node. The graph $\mathcal{G}(t)$ is uniformly connected if there exists a time horizon $T > 0$ and a node v_i such that $v_i \to v_j \ \forall \ j \in \mathcal{N}$ across $[t, t+T]$.

Notice that $A(t)$ as defined by Moreau may be interpreted as $-\mathcal{L}(t)$ due to the properties of the Laplacian matrix. As a result,

$$\dot{x}(t) = -\mathcal{L}(t)x(t). \tag{1.5}$$

Scardovi and Sepulchre [68] provide an extension to the work done by Moreau. Consider N agents exchanging information about their state vectors x_i, for $i = 1, \ldots, N$, according to a communication graph $\mathcal{G}(t)$. They describe the consensus protocol as

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij}(t)(x_j - x_i), \qquad i = 1, \ldots, N. \tag{1.6}$$

Using the definition of Laplacian matrix we can rewrite the above equation as

$$\dot{x}(t) = -\hat{\mathcal{L}}_n(t)x(t), \tag{1.7}$$

where $\hat{\mathcal{L}}_n(t) = \mathcal{L}(t) \otimes I_n$.

A more general MAS is the one in which each agent is a dynamic system. An instance is the N identical linear state-space models described by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \qquad y_i(t) = Cx_i(t), \tag{1.8}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector, $u_i(t) \in \mathbb{R}^m$ is the control input, and $y_i(t) \in \mathbb{R}^p$ is

the output vector for $k = 1, \ldots, N$. The authors in [68] consider a special case where B and C are $n \times n$ nonsingular matrices and all the eigenvalues of A are on the imaginary axis. Under the assumption that the communication graph $\mathcal{G}(t)$ is uniformly connected and the corresponding Laplacian matrix $\mathcal{L}(t)$ be piecewise continuous and bounded. Then the control law is given by

$$u_i = B^{-1}C^{-1} \sum_{j=1}^{N} a_{ij}(t)(y_j - y_i), \qquad i = 1, \ldots, N, \tag{1.9}$$

uniformly exponentially synchronizes all the solutions of linear systems to a solution of the system $\dot{x} = Ax$. This discussion may not be true as B and C are not invertible in general. Hence, we are unlikely to obtain an equation similar to (1.7).

The above assumption of square nonsingular matrices B and C is removed by considering the condition which only requires stabilizability of the pair (A, B), detectability of the pair (C, A) and by employing dynamic couplings. Then the control law is given by

$$\dot{\eta}_i = (A + BK)\eta_i + \sum_{j=1}^{N} a_{ij}(t)(\eta_j - \eta_i + \hat{x}_i - \hat{x}_j),$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + H(\hat{y}_i - y_i),$$

$$u_i = K\eta_i, \qquad i = 1, \ldots, N, \tag{1.10}$$

where K is the an arbitrary stabilizing feedback matrix, H is the observer matrix and $\hat{y}_k = C\hat{x}_k$, solves the synchronization problem. The result can be stated under the condition that the communication graph $\mathcal{G}(t)$ is uniformly connected and the Laplacian matrix $\mathcal{L}(t)$ is piecewise continuous and bounded. The eigenvalues of A are on the closed left half complex plane. If pairs (A, B) is stabilizable and (A, C) is detectable then we can choose K and H such that $A + BK$ and $A + HC$ are Hurwitz, then the solution of the linear system with dynamic couplings will uniformly exponentially synchronize to a solution of the system $\dot{x} = Ax$.

It would be of interest to identify other classes of systems beyond the simple integrators considered by Moreau. Consensus control for such systems would require us to design distributed state and output feedback controllers. Next we discuss the work done by Li, Duan, Chen and Huang in [45] in the field of consensus control. The authors present a unified way to achieve consensus in MAS and synchronization of complex networks. They propose distributed observer-type consensus protocol based on relative output measurements for the agents whose dynamics are extended to be in a general linear form (1.8). The static consensus protocol is given by the relative measurements of other agents with respect to agent i

$$\zeta_i = c \sum_{j=1}^{N} a_{ij}(t)(y_j - y_i), \qquad i = 1, \ldots, N. \tag{1.11}$$

A distributed observer type consensus protocol is proposed

$$\dot{v}_i = (A + BK)v_i + F[c \sum_{j=1}^{N} a_{ij}C(v_i - v_j) - \zeta_i], \qquad u_i = Kv_i, \tag{1.12}$$

where F and K are feedback gain matrices. This observer based protocol solves the consensus problem for a directed network of agents having a spanning tree if and only if $(A+BK)$ and $(A+c\lambda_i FC)$, for $i = 2, \ldots, N$, are Hurwitz. This allows the use of separation principle for a multi-agent setting and converts the consensus problem into stability problem for a set of matrices with the same dimension as a single agent.

Based on leader-follower approach another paper which discusses agent dynamics for the identical general linear form is by Zhang, Lewis and Das [87]. A leader node is used to generate the desired tracking trajectory. An optimal design for synchronization of cooperative systems is proposed including full state feedback control, observer design, and output feedback control.

8

1.5 An Overview on Consensus Control in Heterogeneous MASs

The recent development in the area of consensus problem has motivated the researchers to now think about the more difficult situation and extend it to the case of heterogeneous MASs. There are some results which are reported in the literature [27, 39, 47, 75, 78], dealing with the complex problem of heterogeneous MASs. It is important for us to know the requirement for consensusability for such agents. The problem could be to design a controller such that the output of the closed loop system asymptotically tracks a reference signal [26] or as a special case of output regulation [10], regardless of external disturbance and the initial state.

The well know internal model principle for the classical regulator problem for linear, time-invariant, finite dimensional systems is introduced by Francis and Wonham [26]. They embed an internal model of the disturbance and reference signals in the open loop system. The purpose of introducing this internal model is to supply closed loop transmission zeros which will cancel the unstable poles for the disturbance or reference signals.

Wieland and Allgöwer introduced the internal model principle to the area of consensus control [77]. They show that each agent with its controller requires an internal model of the consensus dynamics for it to have a solution to the consensus problem. They provide a necessary condition for existence of a solution to the consensus problem which applies to both output and state consensus over a constant communication graph. Later the authors extended their work in [78] to put forward a more generalized version and provide necessary and sufficient requirements for output synchronization in case of time-varying connected graphs based on the results provided by Moreau [52, 53, 54] and Scardovi and Sepulchre [68] by employing dynamic couplings to the system model. They solve the heterogeneous synchronization problem for N linear systems by finding a distributed control law, dependent on the relative information only over the uniformly connected communication graph $\mathcal{G}(t)$,

of which the outputs of the closed loop system asymptotically synchronize to a common trajectory.

Consider N heterogeneous agents with the dynamics of the ith agent described by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \qquad y_i(t) = C_i x_i(t), \tag{1.13}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state, $u_i(t) \in \mathbb{R}^m$ is the input, and $y_i(t) \in \mathbb{R}^p$ is the measurement output of the ith dynamic agent. It follows that $A_i \in \mathbb{R}^{n_i \times n_i}, B_i \in \mathbb{R}^{n_i \times m}$, and $C_i \in \mathbb{R}^{p \times n_i}$. Thus the ith agent admits transfer matrix $P_i(s) = C_i(sI_{n_i} - A_i)^{-1} B_i$ with I_n the $n \times n$ identity matrix. Note that the state dimension n_i can be different from each other. However, all agents have the same number of inputs and outputs. The global system of (1.13) is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{1.14}$$

where $A = \operatorname{diag}(A_1, \ldots, A_N), B = \operatorname{diag}(B_1, \ldots, B_N), C = \operatorname{diag}(C_1, \ldots, C_N)$, and

$$x(t) = \operatorname{vec}\{x_1(t), \ldots, x_N(t)\} := \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix},$$

$u(t) = \operatorname{vec}\{u_1(t), \ldots, u_N(t)\}, y(t) = \operatorname{vec}\{y_1(t), \ldots, y_N(t)\}$. For heterogeneous MASs, the consensus problem is concerned with the agents' outputs and requires that

$$\lim_{t \to \infty} [y_i(t) - y_j(t)] = 0, \qquad \forall \, i, j \in \mathcal{N}. \tag{1.15}$$

The necessary condition extended from the results of [77] to solve the heterogeneous synchronization problem is stated as follows. Consider N linear state space models coupled through dynamic controllers. Assume that the closed loop system has no asymptotically

stable equilibrium set on which outputs vanish. Then there exists a number $m \in \mathbb{N}$, matrices $S \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{q \times m}$, where the eigenvalues of S are on the closed right-half complex plane and (S, R) is observable, and matrices $\Pi_i \in \mathbb{R}^{n_i \times m}$ and $\Gamma_i \in \mathbb{R}^{p_i \times m}$ for $i = 1, \ldots, N$ satisfying

$$A_i \Pi_i + B_i \Gamma_i = \Pi_i S,$$

$$C_i \Pi_i = R, \tag{1.16}$$

for $i = 1, \ldots, N$, which is necessary for synchronizability of heterogeneous network.

The authors propose the following dynamic couplings to achieve synchronization of heterogeneous networks

$$\dot{\zeta}_i = S\zeta_i + \sum_{j=1}^{N} a_{ij}(t)(\zeta_j - \zeta_i),$$

$$\dot{\hat{x}}_i = A_i \hat{x}_i + B_i u_i + H_i(\hat{y}_i - y_i),$$

$$u_i = K_i(\hat{x}_i - \Pi_i \zeta_i) + \Gamma_i \zeta_i, \tag{1.17}$$

with controller states $\zeta_i \in \mathbb{R}^m$ and $\hat{x}_i \in \mathbb{R}^{n_i}$ for $i = 1, \ldots, N$. Comparing the controller equation, u_i of (1.17) with (1.10) we see that two new terms $\Pi_i \zeta_i$ and $\Gamma_i \zeta_i$ are added, these are needed for synchronization of heterogeneous networks.

The sufficient condition to achieve synchronization of heterogeneous networks is the main result of [78] and is stated as follows. Consider N heterogeneous linear state-space models with (A_i, B_i) stabilizable and (A_i, C_i) detectable for $i = 1, \ldots, N$. Let Laplacian matrix $\mathcal{L}(t)$ be piecewise continuous and bounded for a uniformly connected communication graph $\mathcal{G}(t)$. Gain matrices K_i and H_i can be chosen such that $A_i + B_i K_i$ and $A_i + H_i C_i$ are Hurwitz for $i = 1, \ldots N$, then a solution to heterogeneous problem exists which uniformly exponentially synchronizes if and only if there exists a number $m \in \mathbb{N}$, matrices $S \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{q \times m}$, $\Pi_i \in \mathbb{R}^{n_i \times m}$ and $\Gamma_i \in \mathbb{R}^{p_i \times m}$ for $i = 1, \ldots, N$ satisfies the necessary conditions and has eigenvalues of S on the imaginary axis.

11

Under the assumption that (A_i, B_i) stabilizable and (A_i, C_i) detectable for $i = 1, \ldots, N$ the proposed dynamic couplings achieve synchronization by assigning each individual system a reference generator in the form

$$\dot{\zeta}_i = S\zeta_i + \sum_{j=1}^{N} a_{ij}(t)(\zeta_j - \zeta_i). \tag{1.18}$$

All the individual systems then asymptotically track their reference generators to achieve synchronization.

Output synchronization for heterogeneous networks of non-introspective agents was proposed by Grip, Yang, Saberi and Stoorvogel in [27]. Most of the design methods for output synchronization of heterogeneous agents rely on self-knowledge, or in other words they may be required to know their state, their output or their own state/output relative to that of reference trajectory, which is different from the information transmitted via the network. As pointed out by the authors there may be situations when this self-knowledge is unavailable. The authors refer to agents which possess self-knowledge as introspective agents, while non-introspective agents are those which possess no self-knowledge except for what is received via the network. A multi-input/multi-output (MIMO) network of N non-introspective agents is described by linear state-space model. The only knowledge the agent receives is from a constant communication network (in this case a weighted digraph $\mathcal{G}$) is in the form of linear combination of its own output relative to that of the other agents

$$\dot{\zeta}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{1.19}$$

where $a_{ij} = 0$ and $a_{ii} = 0$. Using the definition of Laplacian matrix the above equation is equivalent to

$$\dot{\zeta}_i = \sum_{j=1}^{N} l_{ij} y_j. \tag{1.20}$$

Apart from the information which agents receives from output relative to other agents,

12

they are also assumed to exchange relative information about the internal estimates via the network and that is given by

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{N} l_{ij}\eta_j, \qquad (1.21)$$

where $\eta_j \in \mathbb{R}^p$ is produced internally by the controller for agent j.

Certain assumptions about the network topology and the agents are made. A directed spanning tree which is considered here is a directed tree that contains all the nodes of $\mathcal{G}$. The digraph $\mathcal{G}$ has a directed spanning tree with root agent $W \in \{1, \ldots, N\}$, such that for each $i \in \{1, \ldots, N\} \setminus W$, has the following properties

1) (A_i, B_i) is stabilizable

2) (A_i, C_i) is observable

3) (A_i, B_i, C_i, D_i) is right-invertible

4) (A_i, B_i, C_i, D_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of A_W.

Let $\bar{\mathcal{L}}_W = [l_{ij}]_{i,j \neq W}$ be defined from $\mathcal{L}$ by removing the row and column corresponding to the root agent W.

A 3-step design procedure of the decentralized controllers which can achieve output synchronization is provided in [27], we discuss it briefly here. The control output and the internal estimate of the root agent is set to 0. The goal is to set the dynamics of the synchronization error variable, $e_i := y_i - y_W$ to 0. The dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_W \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & A_W \end{bmatrix} \begin{bmatrix} x_i \\ x_W \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i,$$

$$e_i = \begin{bmatrix} C_i & -C_W \end{bmatrix} \begin{bmatrix} x_i \\ x_W \end{bmatrix} + D_i u_i. \qquad (1.22)$$

13

The system defined above in general is not stabilizable. To achieve the goal of making $e_i = 0$ a standard output regulation method is used with the only available information to agent i being ζ_i and $\hat{\zeta}_i$. First step reduces the dimension of the model to $\bar{x}_i$ by performing state transformation. This removes the redundant modes which have no effect on e_i so even though the original model may be unobservable, the reduced model is always observable.

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i,$$

$$e_i = \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i u_i. \tag{1.23}$$

Second step designs a state feedback controller as a function of $\bar{x}_i$ to regulate e_i to 0. Consider the following regulator equations with unknowns $\Pi_i \in \mathbb{R}_{n_i \times r_i}$ and $\Gamma_i \in \mathbb{R}_{m_i \times r_i}$, where $r_i = \eta_W - q_i$. The null space dimension of the observability matrix corresponding to the system is defined as q_i. Based on Π_i and Γ_i, find matrix $\bar{F}_i = \begin{bmatrix} F_i & \Gamma_i - F_i \Pi_i \end{bmatrix}$, where F_i is chosen such that $A_i + B_i F_i$ is Hurwitz. This controller cannot be directly implemented as $\bar{x}_i$ is not available to agent i. Third step, construct an observer that makes an estimate of $\bar{x}_i$ available to agent i. This observer is based on the information ζ_i and $\hat{\zeta}_i$ received via the network. A second state transformation is performed as the network in heterogeneous in nature, in order to obtain a dynamical model which is similar to other agents. In conclusion we can state that by implementing the observer estimates for each agent along with the state feedback controller output synchronization is achieved.

Kim, Shim and Sio studied output synchronization for uncertain linear MASs with single-input/single-output (SISO), minimum phase systems in [39]. The block diagram in Figure 1.2 shows how they achieve synchronization by embedding an identical generator in each agent, the output of which is tracked by the actual agent output.
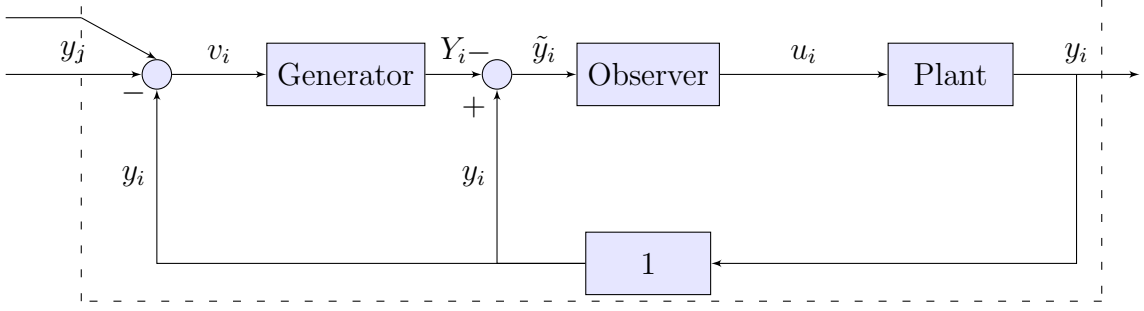
Figure 1.2: Block diagram of the proposed scheme in [39].

Consider a group of heterogeneous uncertain N agents given by

$$\dot{x}_i = A_i(\mu_i)x_i + B_i(\mu_i)u_i, \qquad y_i = C_i(\mu_i)x_i, \qquad i = 1, \ldots, N, \qquad (1.24)$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^i$ the control input, $y_i \in \mathbb{R}^i$ the output of the ith agent, and the uncertain vector μ_i ranges over a compact subset $\mathcal{M}_i$ of $\mathbb{R}_{n_i}$ for all $i = 1, \ldots, N$. A weighted, directed and fixed network topology is considered here.

The output feedback controller is written as

$$\dot{\zeta}_i = F_i\zeta_i + G_{1i}y_i - G_{2i}\sum_{j \in \mathcal{N}} l_{ij}y_j,$$

$$u_i = H_i\zeta_i + J_{1i}y_i - J_{2i}\sum_{j \in \mathcal{N}} l_{ij}y_j, \qquad i = 1, \ldots, N, \qquad (1.25)$$

where $\zeta_i \in \mathbb{R}^{p_i}$. Recall the consensus problem which implies that a certain signal $\phi(t)$ exists such that for all i

$$\lim_{t \to \infty} \{y_i(t) - \phi(t)\} = 0 \qquad (1.26)$$

Consider a group of auxiliary linear systems which are termed as generators and are of the form

$$\dot{w} = Sw, \qquad \phi = Rw, \qquad (1.27)$$

where $w \in \mathbb{R}^q$, $Re\{\lambda_j(S)\} = 0$ for $j = 1, \ldots, q$ and (S, R) is observable. The generator

15

produces signals which is an outcome of online consensus among the agents. The fact which motivates to embed the dynamics $\dot{w} = Sw$ into the controller is the presence of all eigenvalues of S for all $\mu_i \in \mathcal{M}_i$ and $i = 1, \ldots, N$ in the closed loop system matrix given by

$$\bar{A}_i(\mu_i) := \begin{bmatrix} A_i(\mu_i) + B_i(\mu_i)J_{1i}C_i(\mu_i) & B_i(\mu_i)H_i \\ G_{1i}C_i(\mu_i) & F_i \end{bmatrix}, \tag{1.28}$$

so that the solution of the closed loop system satisfies the consensus condition given by

$$\lim_{t \to \infty} \left\{ y_i(t) - Re^{St}w_o \right\} = 0. \tag{1.29}$$

Synchronization of heterogeneous agents with arbitrary linear dynamics given by (1.14) based on internal reference model requires the agents to have a common intersection so that they become synchronizable is proposed by Lunze in [47]. The agents are said to be synchronized if the following statements are satisfied

1) Agents have a common intersection, i.e. for specific initial states all outputs $y_i(t)$ follow a common trajectory $y_s(t)$.

2) For all initial states, the agents asymptotically approach the same trajectory $y_s(t)$. It is assumed that the communication between the agents is restricted to transfer of its outputs. If the agents are synchronized then agents generate a synchronous trajectory $y_s(t)$ without interactions such that the error vanishes. This synchronous trajectory is generated by an exosystem. The overall system has a plant model and an exosystem with a communication graph which is fixed, directed spanning tree.

1.6 Organization of the Dissertation

We introduce consensus control for the heterogeneous MAS in Chapter 1, consisting of N different continuous-time dynamical systems. The motivation for studying heterogeneous MASs is discussed in this chapter along with the numerous application areas. Before we go into the detail of our work we provide an overview of how this area has evolved both

in terms of homogeneous MASs and heterogeneous MASs. In Chapter 2 and Appendix A we provide the preliminaries required to proceed with our work which includes topics like internal model principle, graph theory etc. Our main results for heterogeneous MASs are provided in Chapters 3 to 5. Two different kinds of consensus problems are presented in this dissertation. In Chapter 3 we consider consensus control when time delays exist in the communication topology. In Chapter 4 we consider consensus under communication constraints where the reference trajectory is transmitted to only one or few agents. We also consider the problem of designing the feedback control law in order to achieve tracking for typical test signals in MAS environment. In Chapter 5 we consider the consensus control of aircrafts in an attempt to build autonomous aircraft traffic control systems. The dissertation is concluded in Chapter 6, which also lists some ideas about possible future research topics.

Chapter 2
Preliminaries

2.1 Internal Model Principle

As mentioned in the earlier section, Francis and Wonham [26] put forward the internal model principle for the classical regulator problem. Disturbance or reference signals have a known structure and is being generated by an exosystem or reference model. The aim of the controller is to provide disturbance rejection and reference tracking by embodying the model of disturbance or reference signal within itself. In this section, we give a brief overview of internal model principle method from [30].

Consider the state space model described by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^p$ is the measured output with $p = m$. The exosystem or reference model is described by

$$\dot{r}(t) = A_r r(t), \qquad y_r(t) = C_0 r(t), \tag{2.2}$$

where $r(t) \in \mathbb{R}^{n_r}$. The condition for tracking requires

$$\lim_{t \to \infty} e(t) = \lim_{t \to \infty} \left[y(t) - y_r(t) \right]$$
$$= \lim_{t \to \infty} \left[y(t) - C_0 r(t) \right]. \tag{2.3}$$

The static state feedback controller can be modeled as

$$u(t) = K_x x(t) + K_r r(t) \tag{2.4}$$

where $K_x \in \mathbb{R}^{m \times n}$ and $K_r \in \mathbb{R}^{m \times n_r}$ are constant matrices. The closed loop system can then be written as

$$\dot{x}_c(t) = A_c x_c(t) + B_c r(t), \qquad y_c(t) = C_c x_c(t) + D_c r(t), \tag{2.5}$$

where $A_c = (A + BK_x)$, $B_c = BK_r$, $C_c = (C + DK_x)$ and $D_c = DK_r$.

Lemma 1 *The linear output regulation problem can be solved by using the control law of the form* (2.4) *under the following assumptions*

   1) A_r *has no eigenvalues with negative real parts.*   *2)* (A, B) *is stabilizable.*

   3) Closed loop system in (2.5) *is Hurwitz.*

The closed loop system is said to have output regulation property if it follows

$$\lim_{t \to \infty} e(t) = \lim_{t \to \infty} [C_c x_c(t) + D_c r(t)] = 0.$$

Then there exists a unique matrix X_c that satisfies the following matrix equations

$$X_c A_r = A_c X_c + B_c, \qquad 0 = C_c X_c + D_c. \tag{2.6}$$

The following steps can be followed to synthesize a desired static state feedback controller.

   Step 1: Find a feedback gain K_x such that $(A + BK_x)$ is stable.

   Step 2: Solve for both X_c and K_r from the set of linear equations given by

$$X_c A_r = (A + BK_x)X_c + BK_r, \qquad 0 = (C + DK_x)X_c + DK_r. \tag{2.7}$$

This approach has a drawback wherein X_c and K_r depend on K_x. This dependency requires a recomputation of X_c and K_r each time K_x is redesigned. To overcome this issue a better

approach is obtained by making the following linear transformation

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} I_n & 0_{n \times m} \\ K_x & I_m \end{bmatrix} \begin{bmatrix} X_c \\ K_r \end{bmatrix}$$

in Equation (2.7). After the transformation we get the following set of linear matrix equations in unknown matrices X and U given by

$$X A_r = AX + BU, \qquad 0 = CX + DU. \tag{2.8}$$

Theorem 1 *Under the assumptions in Lemma 1, let the feedback gain K_x be computed such that $(A + BK_x)$ is exponentially stable. Then the linear output regulation problem is solvable by a static state feedback control of the form*

$$u = K_x x + K_r r$$

if and only if there exists two matrices X and U that satisfy (2.8), with the feedforward gain K_r given by

$$K_r = U - K_x x.$$

Theorem 2 *Equations in (2.8) admit a solution pair (X, U), if and only if*

$$\text{rank} \left\{ \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} \right\} = \# \text{ rows } \forall s = \lambda_i(A_r).$$

The regulator equations in (2.8) can be written as

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} X A_r - \begin{bmatrix} A & B \\ C & D \end{bmatrix} X = \begin{bmatrix} 0_{n \times n_r} \\ -C_0 \end{bmatrix}, \qquad X = \begin{bmatrix} X \\ U \end{bmatrix}.$$

20

Denote $\underline{x} = \text{vec}(X)$ that packs columns of X into a single vector column in order. Then the above equation is equivalent to $M\underline{x} = \underline{b}$ with

$$M = A_r' \otimes \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} - I_{n_r} \otimes \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \underline{b} = \text{vec}\left(\begin{bmatrix} 0_{n \times n_r} \\ -C_0 \end{bmatrix} \right).$$

Hence using the above synthesis procedure we can track a reference input by using a static state feedback controller of the form (2.4).

2.2 Positive Real Property

Positive real (PR) transfer function matrices have been studied extensively in network theory [4] and for stability analysis in control theory [22]. Consider a continuous linear time-invariant system described by (2.1). Let $T(s) = C(sI - A)^{-1}B$ be a square transfer function matrix of a complex variable $s = j\omega$. Then $T(s)$ is termed PR [4, 5] if the following conditions are satisfied

1) All the elements of $T(s)$ are analytic in $Re[s] > 0$.

2) $T(s)$ is real for real positive s.

3) $T^*(s) + T(s) \geq 0$ for $Re[s] > 0$ where superscript * denotes complex conjugate transpose.

2.3 Greshgorin Circle Theorem

There are many areas in engineering and physics, where eigenvalues and eigenvectors play important roles. In linear algebra eigenvalues are defined for a square matrix A. An eigenvalue for the matrix A is a scalar λ such that there is a non-zero vector x which satisfies the equation $Ax = \lambda x$. In linear algebra the eigenvalues are also roots of the characteristic polynomial $\det(A - \lambda I)$. Unfortunately, it is often difficult to find the eigenvalues of A as it requires solving a degree n polynomial equation. Another way of estimating the eigenvalues is to find the trace of the matrix, $tr(A) = \sum_{i=1}^{n} |a_{ii}|$. The trace of a matrix is the sum of the eigenvalues but it does not give us any range for the eigenvalues.

In order to bound the eigenvalues in the complex plane the Greshgorin circle theorem is used. The theorem can be stated as follows [32]. Let $A = [\,a_{ij}\,]$ be a $n \times n$ matrix, let $d_i = \sum_{i \neq j} |a_{ij}|$. Then the set $D_i = \{z \in \mathbb{C} : |z - A_{ii}| \leq d_i\}$ is called the ith Greshgorin disc of a matrix A. The eigenvalues of A are the union of Greshgorin discs

$$G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - A_{ii}| \leq d_i\}. \tag{2.9}$$

Furthermore, if the union of k of the n discs that comprise $G(A)$ forms a set $G_k(A)$ that is disjoint from the remaining $n - k$ discs, then $G_k(A)$ contains exactly k eigenvalues of A, counted according to their algebraic multiplicities.

2.4   Dominant and M-matrices

Denote $\mathbb{R}^N$ as the N-dimensional real space. Let $A = [\,a_{ij}\,]$ be a matrix with a_{ij} the (i, j)th entry. The real square matrix A is called row dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$, column dominant if $|a_{jj}| \geq \sum_{i \neq j} |a_{ij}|$, and doubly dominant if it is both row and column dominant. If the inequalities are strict then one calls such matrices strictly row or column or doubly dominant.

A certain class of matrices which is extensively studied for stability analysis in control theory is M-matrices [1, 18, 65, 66]. A square matrix M is called an M-matrix (resp. semi M-matrix), if all its off-diagonal elements are either negative or zero, and all its principal minors are positive (resp. nonnegative).

The following properties of M-matrices are useful [81]. Suppose that all the off-diagonal elements of the square matrix M are either negative or zero. Then the following are equivalent

1) M is an M-matrix;   2) $-M$ is Hurwitz;

3) The leading principal minors of M are all positive;

4) There exists a diagonal matrix $D = \text{diag}\,\{d_1, \ldots, d_N\} > 0$ such that MD (resp. DM) is strictly row (resp. column) dominant.

Chapter 3
Output Consensus Control with Time Delays

The design of distributed and local control protocols to achieve not only feedback stability but also output consensus in tracking reference trajectories remains a major challenge. In this chapter we develop a more accessible method for consensus control and derive a consensusability condition for heterogeneous MASs and also consider the issue of time delays over communication topology. It will be shown that similar results to the ones found in [45, 48, 87] for homogeneous MASs are available for heterogeneous MASs. The existing design methods, such as linear quadratic Gaussian (LQG) and loop transfer recovery (LTR) [4], and $\mathcal{H}_\infty$ loop shaping [51], developed for MIMO feedback control systems can be employed to synthesize consensus controllers for heterogeneous MASs. Since the controller gains are computed based on either $\mathcal{H}_\infty$ loop shaping or LQG/LTR methods, each controlled agent is robust to perturbations in the form of coprime factor uncertainties or gain/phase uncertainties, respectively.

3.1   Problem Formulation

We consider N heterogeneous agents with the dynamics of ith agent described by (1.13). In studying output consensus, tracking performance is often taken into account [87]. In particular, the N outputs of the MAS are required to track the output of some exosystem or reference model described by

$$\dot{x}_0(t) = A_0 x_0(t), \qquad y_0(t) = C_0 x_0(t), \tag{3.1}$$

with zero steady-state error. This is a virtual reference generator, and all eigenvalues of A_0 are restricted to lie on the imaginary axis. A real-time reference trajectory may not be actually from this exosystem, but consists of piece-wise step, ramp, sinusoidal signals, etc.

whose poles coincide with eigenvalues of A_0. Following [87], we call these agents controlled agents. The consensus control requires that

$$\lim_{t \to \infty} [y_i(t) - y_0(t)] = 0 \; \forall i \in \mathcal{N}. \tag{3.2}$$

Such a consensus problem has more control flavor, and deserves attention from the control community.

Assume that the realizations of N agents are all stabilizable and detectable. We will study under what condition for the feedback graph, there exist distributed stabilizing controllers and consensus control protocols such that the outputs of N agents satisfy (3.2). Moreover we will study how to synthesize the required distributed and local controllers in order to achieve output consensus, taking performance into account.

For consensus control of the MAS involving time delays under our study, two useful facts are stated next. The first is purely algebraic.

Fact 1 *If two square matrices $M_1 \in \mathbb{C}^{N \times N}$ and $M_2 \in \mathbb{C}^{N \times N}$ satisfy*

$$M_1 + M_1^* \geq 0, \quad M_2 + M_2^* > 0,$$

then $\det(I + M_1 M_2) \neq 0$.

The next fact is concerned with positive realness (PR) of the dynamic system under state feedback control [4].

Fact 2 *Suppose (A, B) is stabilizable for system*

$$\dot{x} = Ax + Bu,$$

and $F = B'X$ is the stabilizing feedback control gain, where $X > 0$ is the stabilizing solution

to the algebraic Riccati equation (ARE)

$$A'X + XA - XBB'X + Q = 0$$

with $Q \geq 0$, then the closed-loop transfer matrix

$$T_F(s) = F(sI - A + BF)^{-1}B$$

is PR, *i.e., $T_F(s) + T_F(s)^* \geq 0 \ \forall \ \mathrm{Re}[s] > 0$.*

An interesting algebraic property for the digraph is derived and referred to as fundamental lemma in [2] in studying the consensus control for heterogeneous MASs. Here we provide a more complete version with a simpler proof for this lemma.

- **A Fundamental Lemma**

Let $e_{iR} \in \mathbb{R}^N$ be a vector with 1 in the i_Rth entry and zeros elsewhere. We state the following lemma that is instrumental to the main results.

Lemma 2 *Suppose that $\mathcal{L}$ is the Laplacian matrix associated with the digraph $\mathcal{G}$. The following statements are equivalent:*

(i) *There exists an index $i_R \in \mathcal{N}$ such that*

$$\mathrm{rank}\left\{\mathcal{L} + e_{i_R}e'_{i_R}\right\} = N; \tag{3.3}$$

(ii) *There exist diagonal matrices $D > 0$ and $G \geq 0$ (with rank 1) such that*

$$\mathcal{M} + \mathcal{M}' > 0, \quad \mathcal{M} = D\mathcal{L} + G; \tag{3.4}$$

(iii) *The digraph $\mathcal{G}$ is connected.*

Proof: Let $\Rightarrow$ stand for "implies". We will show that (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in order to establish the equivalence of the three statements. For (iii) $\Rightarrow$ (i), assume that $\mathcal{G}$ is

connected. Then there exists a reachable node $v_{i_R} \in \mathcal{V}$ for some index $i_R \in \mathcal{N}$. We can construct an augmented graph $\overline{\mathcal{G}}$ by adding a node v_0, and adding an edge from v_{i_R} to v_0 with weight 1. The augmented graph is again connected with v_0 as the only reachable node. It follows that the Laplacian matrix associated with the augmented graph $\overline{\mathcal{G}}$ is given by

$$\overline{\mathcal{L}} = \begin{bmatrix} 0 & \cdots & 0 \\ -e_{i_R} & \mathcal{L} + e_{i_R} e'_{i_R} \end{bmatrix}.$$

Since the augmented graph is connected, the Laplacian matrix $\overline{\mathcal{L}}$ has only one zero eigenvalue, implying the rank condition (3.3), and thus (i) is true.

For (i) $\Rightarrow$ (ii), assume that the rank condition (3.3) is true. Then $\mathcal{L} + e_{iR} e'_{iR}$ is an M-matrix, because it is not only a semi M-matrix but also has all its eigenvalues on strict right half plane, in light of the Gershgorin circle theorem. Properties of M-matrices from Section 2.4 can then be applied to conclude the existence of a diagonal matrix D such that

$$\mathcal{M} = D(\mathcal{L} + e_{i_R} e'_{i_R}) = D\mathcal{L} + G, \tag{3.5}$$

is strictly column dominant where $G = De_{i_R} e'_{i_R}$ is diagonal and has rank 1. Since $\mathcal{M}$ is row dominant, although not strictly, $\mathcal{M} + \mathcal{M}'$ is both strictly row and column dominant, thereby concluding (ii).

For (ii) $\Rightarrow$ (iii), assume that (3.4) is true. Then $\mathcal{M}$ is an M-matrix, by the fact that all its eigenvalues lie on strict right half plane. Hence there holds

$$N = \text{rank}\{D^{-1}\mathcal{M}\} = \text{rank}\{\mathcal{L} + ge_{i_R} e'_{i_R}\}$$
$$\leq \text{rank}\{\mathcal{L}\} + 1,$$

by the rank inequality and rank$\{ge_{i_R}e'_{i_R}\} = 1$, where $D^{-1}G = ge_{i_R}e'_{i_R}$ for some scalar $g > 0$ and some index $i_R \in \mathcal{N}$. The above implies that rank$\{\mathcal{L}\} \geq N - 1$. It follows that the Laplacian matrix $\mathcal{L}$ has only one zero eigenvalue, concluding that the graph $\mathcal{G}$ is connected, and thus (iii) is true. The proof is now complete. □

Although Lemma 2 considers only the directed graph, the result holds for the undirected graph with a similar and much simpler proof. The requirement that a digraph is strongly connected has been reported in several papers to ensure consensusability [15, 35], whereas the condition (3.3) with the addition of the connected condition is new and crucial to our main results. In practice N is large, and hence condition (3.3) holds generically for some $i \in \mathcal{N}$.

Remark 1 (a) Lemma 2 is an improved version of the fundamental lemma in [2] with a much simpler proof. More important the rank condition (3.3) in Lemma 2 is true for an arbitrary node v_{i_R}, provided that it is a reachable node. In addition the Gershgorin circle theorem can be used to conclude that all eigenvalues of $\mathcal{L} + e_{i_R}e'_{i_R}$ lie on strict right half plane, and thus it is an M-matrix. As pointed out in [2], (ii) implies

$$\mathcal{M} + \mathcal{M}' = (D\mathcal{L} + G) + (D\mathcal{L} + G)' > 2\kappa I, \tag{3.6}$$

for some $\kappa > 0$; In fact $\kappa = 1$ can be taken with no loss of generality. Efficient algorithms for linear matrix inequality (LMI) can be used to search for D and G. In fact $G = ge_{i_R}e'_{i_R}$ with $g > 0$ for those i_R satisfying (3.3) can be taken. Hence computation of the required D and G in Lemma 2 is not an issue.

(b) For MIMO agents with m-input/p-output, a commonly adopted graph has the weighted adjacency matrix in form of $\mathcal{A} = \{a_{ij}I_q\}$ with $q = m$ or $q = p$. Thus

$$\begin{aligned} D &= \text{diag}(d_1I_q, \cdots, d_NI_q), \\ G &= \text{diag}(g_1I_q, \cdots, g_NI_q), \end{aligned} \tag{3.7}$$

with only one nonzero $g > 0$. In this case Lemma 2 holds true, and (3.3) is extended to

$$\text{rank}\left\{\mathcal{L} + (e_{i_R} \otimes I_q)(e'_{i_R} \otimes I_q)\right\} = Nq. \tag{3.8}$$

Basically the state-space system consists of q decoupled identical scalar systems to which the result of Lemma 2 is applicable.

3.2   Full Information Distributed Protocol

We consider a full information (FI) protocol, assuming that all $\{x_i(t)\}_{i=0}^N$ are available for consensus control. Let F_i be a stabilizing state feedback gain in the sense that $(A_i - B_i F_i)$ is a Hurwitz matrix and $r_i(t) = F_{0i} x_0(t)$ with F_{0i} the feed-forward gain. The distributed control protocol over the topology is given by

$$u_i(t) = g_i \left[r_i(t) - F_i x_i(t)\right] + d_i \sum_{j=1}^N a_{ij} \left[(r_i(t) - F_i x_i(t)) - (r_j(t) - F_j x_j(t))\right], \tag{3.9}$$

for $i, j \in \mathcal{N}$ and $1 \le i \le N$. Recall that $d_i > 0$ for all i and $g_i > 0$ for only one i with the rest zero. This protocol includes information exchange with the feed-forward signals $\{r_i(t)\}$. The collective control protocol (3.9) can now be written as

$$u(t) = (D\mathcal{L} + G)\left[r(t) - Fx(t)\right]. \tag{3.10}$$

Applying the Laplace transform to the above control input yields

$$U(s) = (D\mathcal{L} + G)\left[R(s) - FX(s)\right] \tag{3.11}$$

where $U(s)$, $R(s)$, and $X(s)$ are Laplace transform of $u(t)$, $r(t)$, and $x(t)$, respectively. Substituting the control protocol in (3.10) into the state equation (1.14) yields

$$\dot{x}(t) = (A - B\mathcal{M}F)x(t) + B\mathcal{M}r(t). \tag{3.12}$$

Now consider the collective output

$$y(t) = Cx(t), \quad C = \text{diag}(C_1, \cdots, C_N). \tag{3.13}$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and $R(s) = \mathcal{L}\{r(t)\}$ be the Laplace transforms of $y(t)$ and $r(t)$ respectively. The Laplace transform of (3.12) with the output equation (3.13) is given by

$$Y(s) = C(sI - A + B\mathcal{M}F)^{-1}B\mathcal{M}R(s). \tag{3.14}$$

3.3   FI Distributed Protocol with Time Delays

While consensus of heterogeneous MASs has been studied in [27, 78], the issue of time delays is not considered, which is the focus of this chapter. Recall the diagonal matrices $D = \text{diag}(d_1, \cdots, d_N)$ and $G = \text{diag}(g_1, \cdots, g_N)$ in (ii) of Lemma 2. We begin with the FI protocol, assuming that all $\{x_i(t)\}_{i=0}^{N}$ are available for consensus control over the topology involving time delays. Let F_i be a stabilizing state feedback gain in the sense that $(A_i - B_i F_i)$ is a Hurwitz matrix and $r_i(t) = F_{0i}x_0(t - \tau_i)$ with F_{0i} the feed-forward gain and $\tau_i \geq 0$ for $1 \leq i \leq N$. Denote

$$\varepsilon_i(t) = r_i(t) - F_i x_i(t), \quad \varepsilon_{ji}(t; \tau) = r_j(t) - F_j x_j(t - \tau_{ij}), \tag{3.15}$$

for $i, j \in \mathcal{N}$. The distributed control protocol over the topology involving time delays is proposed as

$$u_i(t) = g_i \varepsilon_i(t) - d_i \sum_{j=1}^{N} a_{ij}[\varepsilon_i(t) - \varepsilon_{ji}(t; \tau)], \tag{3.16}$$

for $1 \leq i \leq N$. Recall that $d_i > 0$ for all i and $g_i > 0$ for only one i with the rest zero. The above protocol is modified by adding delay parameters $\tau_i \geq 0$ and $\tau_{ij} \geq 0$, and by adding the information exchange for the feed-forward signals $\{r_i(t)\}$. In this initial study, $\tau_{ii} = 0$

for all $i \in \mathcal{N}$ is assumed. That is, there is no time delay for each agent's controller to receive its own agent's output measurement. We will prove an interesting result that the consensus goal as required in (3.2) can be achieved by using the delayed control protocol in (3.16) independent of the delay lengths.

Define delay operator q_τ^{-1} via $q_\tau^{-1} s(t) = s(t - \tau)$. The graph topology involving time delays introduces the Laplacian matrix $\mathcal{L}_q$ involving delay operators. Specifically the (i, j)th entry of $\mathcal{L}_q$, denoted as $\ell_{ij}^{(q)}$, is specified by

$$\ell_{ij}^{(q)} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & \text{if } j = i, \\ -a_{ij} q_{\tau_{ij}}^{-1}, & \text{if } j \neq i. \end{cases} \tag{3.17}$$

Let $F = \text{diag}(F_1, \cdots, F_N)$ and $r(t) = \text{vec}\{r_1(t), \cdots, r_N(t)\}$. The collective control protocol (3.16) can now be rewritten as

$$u(t) = (D\mathcal{L}_q + G) [r(t) - Fx(t)]. \tag{3.18}$$

Applying Laplace transform to the above control input yields

$$U(s) = (D\mathcal{L}(s) + G) [R(s) - FX(s)], \tag{3.19}$$

where $U(s)$, $R(s)$, and $X(s)$ are Laplace transform of $u(t)$, $r(t)$, and $x(t)$, respectively, and $\mathcal{L}(s)$ is the Laplacian in s-domain with $\ell_{ij}(s)$ as its (i, j)th element given by

$$\ell_{ij}(s) = \begin{cases} \sum_{k=1}^{N} a_{ik}, & \text{if } j = i, \\ -a_{ij} e^{-s\tau_{ij}}, & \text{if } j \neq i. \end{cases} \tag{3.20}$$

For convenience define

$$\mathcal{M}_q := D\mathcal{L}_q + G, \quad \mathcal{M}(s) := D\mathcal{L}(s) + G. \tag{3.21}$$

Upon substituting the control protocol in (3.18) into the state equation in (1.14) yields

$$\dot{x}(t) = (A - B\mathcal{M}_q F)x(t) + B\mathcal{M}_q r(t). \tag{3.22}$$

Distributed stabilization is aimed at synthesizing the collective state feedback gain $F = \text{diag}(F_1, \ldots, F_N)$ such that the characteristic polynomial

$$\lambda(s) := \det[sI - A + B\mathcal{M}(s)F] \neq 0 \ \forall \ \text{Re}[s] \geq 0. \tag{3.23}$$

The following result is instrumental to the synthesis of the stabilizing state feedback gains $\{F_i\}$ later.

Lemma 3 *Let $\mathcal{L}(s)$ be the Laplacian matrix associated with the delayed feedback graph with its (i, j)th element defined in (3.20), and $\mathcal{M}(s) = D\mathcal{L}(s) + G$. The following statements are equivalent:*

(a) *There exists diagonal matrices $D > 0$ and $G \geq 0$ (with rank 1) such that*

$$\mathcal{M}(0) + \mathcal{M}'(0) > 0;$$

(b) *There exist diagonal matrices $D > 0$ and $G \geq 0$ with rank 1 such that*

$$\mathcal{M}(s) + \mathcal{M}^*(s) > 0 \ \forall \ \text{Re}[s] \geq 0;$$

(c) *The corresponding graph is connected.*

Proof: Since $\mathcal{M} := \mathcal{M}(0)$ and $\mathcal{L} := \mathcal{L}(0)$ correspond to the delay-free case, the equivalence of (a) and (c) is true in light of Lemma 2. We need only to prove the equivalence of (a) and (b) in order to to establish the equivalence of the three statements. Now suppose (a) is true. Then $\mathcal{M} = \mathcal{M}(0)$ is an M-matrix, and it is strictly column dominant in addition to be row dominant. Let μ_{ij} be the (i, j)th element of $\mathcal{M}$. The fact that $\mathcal{M} + \mathcal{M}'$ being

31

both strictly row and column dominant implies that

$$2\mu_{ii} > \sum_{j=1,j\neq i}^{N} (|\mu_{ij}| + |\mu_{ji}|) \ \forall \ i \in \mathcal{N}.$$

By an abuse of notation, denote $\mu_{ij}(s)$ as the (i,j)th element of $\mathcal{M}(s)$. Then there holds $|\mu_{ij}(s)| \leq |\mu_{ij}| \ \forall \ \text{Re}[s] \geq 0$ and for all $j \neq i$ in light of the off-diagonal elements of $\mathcal{L}$ and $\mathcal{M}$. It follows that

$$2\mu_{ii} > \sum_{j=1,j\neq i}^{N} (|\mu_{ij}(s)| + |\mu_{ji}(s)|) \ \forall \ i \in \mathcal{N}$$

and for all $\text{Re}[s] \geq 0$, thereby proving that (b) holds true. If (b) is true, then (a) holds as well by simply taking $s = 0$ that concludes the proof. $\square$

Lemma 3 shows that $\mathcal{M}(s)$ is strictly PR regardless of the delay lengths. In addition there exists $\kappa > 0$ such that

$$\mathcal{M}(s) = D\mathcal{L}(s) + G(s) = \kappa[Z(s) + I], \tag{3.24}$$

where $Z(s) + Z(s)^* > 0$ for all $\text{Re}[s] \geq 0$, i.e., $Z(s)$ is strictly PR as well. Furthermore $\kappa = 1$ can be taken without loss of generality. Hence we can obtain the following main result in this section.

Theorem 3 *Suppose that (A_i, B_i) is stabilizable $\forall i \in \mathcal{N}$ for the MAS described in (1.13), and the feedback digraph $\mathcal{G}$ is connected. Then there exist distributed stabilizing state feedback control protocols in the form of (3.16) for the underlying MAS over the delayed feedback topology.*

Proof: The closed-loop stability of the heterogeneous MAS is hinged on the inequality (3.23). By (3.24),

$$Z(s) = \kappa^{-1}\mathcal{M}(s) - I,$$

is strictly PR as well for some $\kappa > 0$. We set $\kappa = 1$ that has no loss of generality. As a

32

result the characteristic equation $\lambda(s)$ in (3.23) can be written as

$$\lambda(s) = \det[sI - A + BF + BZ(s)F]. \tag{3.25}$$

Hence the inequality (3.23) is equivalent to

$$\det[I + (sI - A + BF)^{-1}BZ(s)F] \neq 0 \ \forall \ \text{Re}[s] \geq 0$$

that is in turn equivalent to

$$\det[I + F(sI - A + BF)^{-1}BZ(s)] \neq 0 \ \forall \ \text{Re}[s] \geq 0. \tag{3.26}$$

The stabilizability of (A_i, B_i) for all $i \in \mathcal{N}$ implies the existence of F_i such that

$$T_{F_i}(s) = F_i(sI - A_i + B_iF_i)^{-1}B_i,$$

is PR in light of Fact 2. It follows that

$$T_F(s) = F(sI - A + BF)^{-1}B = \text{diag}[T_{F_1}(s), \ldots, T_{F_N}(s)], \tag{3.27}$$

is PR as well. Consequently the inequality (3.26) can be made true by those state feedback gains rendering $T_F(s)$ PR, in light of Fact 1 by setting $M_1 = T_F(s)$ and $M_2 = Z(s)$ at each s with $\text{Re}[s] \geq 0$. Therefore there indeed exist state feedback gains $\{F_i\}$ that stabilize the heterogeneous MAS asymptotically. $\square$

If the states of the MAS are not available for feedback, then a distributed observer can be designed to estimate the states of the agent, and used in the control protocol (3.18). We consider neighborhood observers [87], assuming that only relative outputs are available from and to the neighboring agents. Let $\hat{x}_i(t)$ and $\hat{y}_i(t) = C_i\hat{x}_i(t)$ be the estimated state

and output of the ith agent, respectively, and denote

$$e_{x_i}(t) = x_i(t) - \hat{x}_i(t), \quad e_{y_i}(t) = y_i(t) - \hat{y}_i(t), \quad (3.28)$$

for $0 \leq i \leq N$. The state of the reference model $x_0(t)$ may require estimation as well at the ith agent based on the received noisy feed-forward signal $F_{0i}x_0(t)$. The observer for the heterogeneous MAS over the feedback topology involving time delays is given by

$$\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + B_i u_i(t) + g_i L_i \left[e_{y_i}(t) - e_{y_0}(t - \tau_i) \right] \quad (3.29)$$

$$+ d_i L_i \sum_{j=1}^{N} a_{ij} \left[e_{y_i}(t) - e_{y_j}(t - \tau_{ij}) \right] \ \forall \ i \in \mathcal{N}$$

where L_i is a stabilizing state estimation gain, i.e., $(A_i - L_i C_i)$ is a Hurwitz matrix for each i. Taking difference between the state equation of (1.13) and (3.29) for $1 \leq i \leq N$ leads to

$$\dot{e}_{x_i}(t) = A_i e_{x_i}(t) - g_i L_i \left[C_i e_{x_i}(t) - C_0 e_{x_0}(t - \tau_i) \right] - d_i L_i \sum_{j=1}^{N} a_{ij} [C_i e_{x_i}(t) - C_j e_{x_j}(t - \tau_{ij})].$$

Denote $\underline{e}_{y_0}(t) = \text{vec}\{e_{y_0}(t - \tau_1), \cdots, e_{y_0}(t - \tau_N)\}$. The collective error dynamics can be written as

$$\dot{e}_x(t) = [A - L\mathcal{M}_q C] \, e_x(t) + GL\underline{e}_{y_0}(t), \quad L = \text{diag}(L_1, \ldots, L_N).$$

It follows that the above and (3.18) with $x(t)$ replacing by $\hat{x}(t)$ lead to the following overall MAS system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}_x(t) \end{bmatrix} = \begin{bmatrix} A - B\mathcal{M}_q F & B\mathcal{M}_q F \\ 0 & A - L\mathcal{M}_q C \end{bmatrix} \begin{bmatrix} x(t) \\ e_x(t) \end{bmatrix} + \begin{bmatrix} B\mathcal{M}_q r(t) \\ GL\underline{e}_{y_0}(t) \end{bmatrix}. \quad (3.30)$$

The separation principle holds true for neighborhood observers as shown in the above

34

collective dynamics. We have the following result for output feedback for which the proof is omitted.

Theorem 4 *Suppose that (A_i, B_i, C_i) is both stabilizable and detectable for all $i \in \mathcal{N}$, and the feedback graph is connected. Then there exist distributed output feedback stabilizing controllers for the underlying heterogeneous* MAS *over the delayed feedback topology.*

3.4 Consensus Tracking of Reference Inputs

We consider the problem of designing a FI control protocol in which both states of the plant model and reference model are available without time delays in the communication topology. We focus on the SISO agents, but our consensus results are applicable to MIMO systems. The next lemma is useful.

Lemma 4 *For each distinct eigenvalue of A_0, denoted as s_κ with $\mathrm{Re}[s_\kappa] \geq 0$ and multiplicity $\mu_\kappa \geq 1$, assume that s_κ is also a pole of $P_i(s) = C_i(sI - A_i)^{-1}B_i$ with the same multiplicity μ_κ for $1 \leq i \leq N$. Let both $\det[sI - A + B\mathcal{M}F]$ and $\det[sI - A + L\mathcal{M}C]$ be Hurwitz. Denote*

$$T_{\mathcal{M}F}(s) = C[sI - A + B\mathcal{M}F]^{-1}B\mathcal{M},$$

and $\Delta(s) = T_{\mathcal{M}F}(s) - T_F(s)$ with

$$T_F(s) = F(sI - A + BF)^{-1}B. \tag{3.31}$$

Then there holds

$$\lim_{s \to s_\kappa} \frac{\Delta(s)}{(s - s_\kappa)^{\mu_\kappa - 1}} = 0. \tag{3.32}$$

Proof: Denote $P(s) = C(sI - A)^{-1}B$ and $P_F(s) = F(sI - A)^{-1}B$. The hypotheses imply

$$\Delta(s) = C[sI - A + B\mathcal{M}F]^{-1}B\mathcal{M} - C(sI - A + BF)^{-1}B$$

$$= P(s)[I + \mathcal{M}P_F(s)]^{-1}\mathcal{M} - P(s)[I + P_F(s)]^{-1}$$

$$= P(s)P_F(s)^{-1}[I + \mathcal{M}^{-1}P_F(s)^{-1}]^{-1} - P(s)P_F(s)^{-1}[I + P_F(s)^{-1}]^{-1}$$

$$= P(s)P_F(s)^{-1}\left\{[I + \mathcal{M}^{-1}P_F(s)^{-1}]^{-1} - [I + P_F(s)^{-1}]^{-1}\right\}.$$

Since $P(s)$ and $P_F(s)$ are diagonal transfer matrices and each of their diagonal entry has s_κ as pole with multiplicity μ_κ, $P_F(s)^{-1} \to 0$, $[(s - s_\kappa)^{\mu_\kappa - 1}P_F(s)]^{-1} \to 0$, and $P(s)P_F(s)^{-1}$ approaches a finite diagonal matrix as $s \to s_\kappa$. Moreover

$$[I + P_F(s)^{-1}]^{-1} = I - P_F(s)^{-1} + o(\{P_F(s)^{-1}\}^2,$$

$$[I + \mathcal{M}^{-1}P_F(s)^{-1}]^{-1} = I - \mathcal{M}^{-1}P_F(s)^{-1} + o(\{P_F(s)^{-1}\}^2,$$

with $o\left([P_F(s)^{-1}]^2\right)$ indicating that each of its terms approaches zero in the order of $[P_F(s)^{-1}]^2$ as $s \to s_\kappa$. Consequently there holds

$$\Delta(s) \to P(s)P_F(s)^{-1}\left\{I - \mathcal{M}^{-1}\right\}P_F(s)^{-1} + o([P_F(s)^{-1}]^2),$$

as $s \to s_\kappa$. Substituting the above into the left hand side of (3.32) yields

$$\lim_{s \to s_\kappa} \frac{\Delta(s)}{(s - s_\kappa)^{\mu_\kappa - 1}} = \frac{P(s)P_F(s)^{-1}\left\{I - \mathcal{M}^{-1}\right\}P_F(s)^{-1} + o([P_F(s)^{-1}]^2)}{(s - s_\kappa)^{\mu_\kappa - 1}} = 0,$$

that concludes the proof. □

Lemma 4 indicates that $T_{\mathcal{M}F}(s)R(s) - T_F(s)R(s) = \Delta(s)R(s)$ has no pole at s_κ. Otherwise its partial fraction in computing the term with pole at s_κ would contradict the limit in (3.32). Since s_κ is an arbitrary eigenvalue of A_0, no eigenvalue of A_0 is a pole of $\Delta(s)R(s)$. This is ensured by taking all eigenvalues of A_0 to be poles of $P_i(s)$ for all

36

$i \in \mathcal{N}$. That is, each eigenvalue of A_0 needs to be eigenvalue of $\{A_i\}_{i=1}^N$. This has no loss of generality, as weighting functions $W_i(s)$ can be used to augment the agent dynamics so that $P_{ai}(s) = P_i(s)W_i(s)$ satisfies the hypothesis of Lemma 4 for all $i \in \mathcal{N}$. Such a technique is used often in engineering practice. Hence s_κ cannot be transmission zero of $P(s)$ and $T_F(s)$, and $\{F_{0i}\}$ can thus be designed by solving

$$
\begin{aligned}
(A_i - B_i F_i)\Pi + B_i F_{0i} &= \Pi_i A_0, \\
(C_i - D_i F_i)\Pi + C_{0i} &= 0,
\end{aligned}
\tag{3.33}
$$

for each i where $D_i = 0$ is assumed. In light of the internal model principle [30, 40], each output of $T_F(s)$ tracks $y_0(t)$ with zero steady state error. More importantly the following result is true.

Theorem 5 *Under the same hypotheses as those of* Lemma 4, *there exist solutions* $\{F_{0i}\}$ *to* (3.33), *and the tracking performance* (3.2) *is satisfied with feed-forward signals* $r_i(t) = F_{0i}x_0(t) \; \forall \; i \in \mathcal{N}$.

Proof: It is straightforward to see that the tracking error in the s-domain is given by

$$
E(s) = T_{\mathcal{M}F}(s)R(s) - \underline{Y}_0(s) = \Delta(s)R(s) + [T_F(s)R(s) - \underline{Y}_0(s)],
$$

where $\underline{Y}_0(s)$ and $R(s)$ are the Laplace transform of $\underline{y}_0(t) = \text{vec}\{C_0 x_0(t), \cdots, C_0 x_0(t)\}$, and $r(t) = \text{vec}\{r_1(t), \cdots, r_N(t)\}$, respectively. The existence of $F_0 = \text{diag}(F_{01}, \cdots, F_{0N})$ to achieve the zero tracking error for $E(s) := T_F(s)R(s) - \underline{Y}_0(s)$ is well known by the form of $T_F(s)$ in (3.31). On the other hand $\Delta(s)$ is stable and $\Delta(s)R(s)$ has no pole at each distinct eigenvalue of A_0 by Lemma 4, implying that the tracking error indeed approaches zero based on the final value theorem. $\qquad \square$

3.5 Output Consensus with Time Delays

For heterogeneous MASs, the consensus problem is concerned with the agents' outputs and requires that (3.2) be satisfied. We consider the problem of designing a FI control protocol in order to achieve the tracking performance in (3.2) that is modified to

$$\lim_{t \to \infty} \|y_i(t) - y_0(t - \tau_i)\| \ \forall \ i \in \mathcal{N}, \tag{3.34}$$

due to the existence of time delays $\{\tau_i\}$. Indeed if the overall closed-loop MAS is internally stable, and $A_0 - L_0 C_0$ is a Hurwitz matrix, then $e_x(t) \to 0$ and $e_{y_0}(t) \to 0$ as $t \to \infty$. As a result the estimated output $\hat{y}(t)$ under output feedback control approaches $y(t)$ under FI control. For this reason it is adequate to consider FI control protocol. We focus on the SISO agents, but our consensus results are applicable to MIMO systems. The next lemma is useful.

Lemma 5 *For each distinct eigenvalue of A_0, denoted as s_κ with $\mathrm{Re}[s_\kappa] \geq 0$ and multiplicity $\mu_\kappa \geq 1$, assume that s_κ is also a pole of $P_i(s) = C_i(sI - A_i)^{-1} B_i$ with the same multiplicity μ_κ for $1 \leq i \leq N$. Let both $\det[sI - A + B\mathcal{M}(s)F]$ and $\det[sI - A + L\mathcal{M}(s)C]$ be Hurwitz. Denote*

$$T_{\mathcal{M}F}(s) = C[sI - A + B\mathcal{M}(s)F]^{-1} B\mathcal{M}(s),$$

and $\Delta(s) = T_{\mathcal{M}F}(s) - T_F(s)$ with $T_F(s)$ the same as in (3.27). Then there holds

$$\lim_{s \to s_\kappa} \frac{\Delta(s)}{(s - s_\kappa)^{\mu_\kappa - 1}} = 0. \tag{3.35}$$

Proof: The proof is similar to Lemma 4 and is omitted here. $\qquad\square$

Theorem 6 *Under the same hypotheses as those of* Lemma 5, *there exist solutions* $\{F_{0i}\}$ *to* (3.33), *and the tracking performance* (3.34) *is satisfied with feed-forward signals* $r_i(t) = F_{0i}x_0(t - \tau_i) \; \forall \; i \in \mathcal{N}$.

Proof: The proof is similar to Theorem 5 and is omitted here. □

It is worth pointing out that the tracking performance as required in (3.34) does not imply that in (3.2) unless the reference signals are step functions, or all delays $\{\tau_i\}$ (from the reference model to the N agents) are the same. Hence there is an incentive to estimate τ_i at each local agent i in order to adjust the transmission delay from the reference model locally, if the track performance in (3.2) is required.

3.6   Simulation Setup and Results

Following [44], consider a system of N point masses moving in one spatial dimension. Dynamics are governed by

$$\dot{x}_i = \frac{1}{m_i}u_i, \qquad y_i = x_i,$$

for $i = 1, 2, \ldots, \ell$ and

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad y_i = C_i x_i,$$

for $i = \ell + 1, \ldots, N$, where

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & -f_{d_i} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \frac{1}{m_i} \end{bmatrix},$$

and $C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The first set of dynamics represents agents whose velocity is directly controlled, while the second set represents agents that experience drag forces and whose acceleration is directly controlled. The output signal corresponds to the position of the point mass. Figure 3.1 shows the interconnection graph for a network of 4 agents. Agents

39

1 and 2 consist of scalar dynamics, while 3 and 4 of second order dynamics. We assume that the communication graph has $\tau_{ij} = 1s$ if $a_{ij} = 1$ and $\tau_{ij} = 0$ if $a_{ij} = 0$. On the other hand $\tau_{ii} = 0$ for all $i \in \mathcal{N}$ as we have mentioned earlier that there is no time delay for each agent's controller to receive its own agent's output measurement.
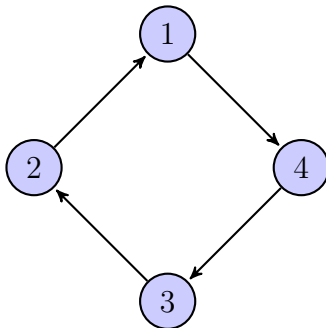
Figure 3.1: Graph for $N = 4$ point masses.

The parameters of the agent dynamic models are specified by $\{m_i\}_{i=1}^{4} = \{0.5, 2, 2.5, 3\}$, and $\{f_{d_i}\}_{i=1}^{4} = \{0, 0, 0.5, 0.9\}$. Our goal is output consensus with the position as the controlled output. Figure 3.2 shows each agent's position for the closed loop MAS. It needs to be pointed out that the feedback graph is strongly connected, and that $g_1 = 0.5$ with the rest $g_i = 0$. In addition the rank condition (3.3) is satisfied by taking $D = \operatorname{diag}(0.1608, 0.4348, 0.5683, 0.7168)$, and $\kappa = 0.1$. For simplicity, the control protocol in (3.16) is used with the state feedback gain designed using the LQR control for each agent. If the output feedback is required, then the tracking error will involve estimation errors.

The tracking of the ramp reference input is shown in Figure 3.3 over the same feedback graph in Figure 3.1. In order to satisfy the assumption that each eigenvalue of A_0 is also an eigenvalue of $\{A_i\}_{i=1}^{N}$, a weighting functions $W_i(s) = \frac{1}{s}$ is used to augment the agent dynamics so that $P_{ai}(s) = P_i(s)W_i(s)$ for all $i \in \mathcal{N}$. It can be seen that the tracking is achieved for the ramp input in the presence of time delays.

Figure 3.2:   Position of each agent under step reference input

Figure 3.3: Position of each agent under ramp reference input.

Chapter 4
Output Consensus Control with Communication Constraints

In this chapter we study output consensus of heterogeneous MASs with communication constraints. A major distinction of our work compared to other investigations is that we provide a solution to the consensus problem for the case when not all agents have access to the reference t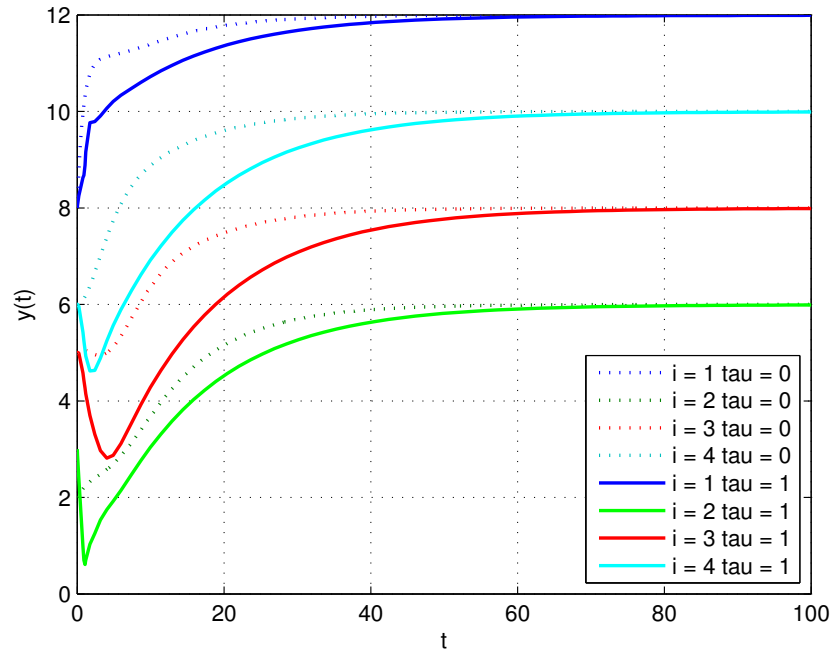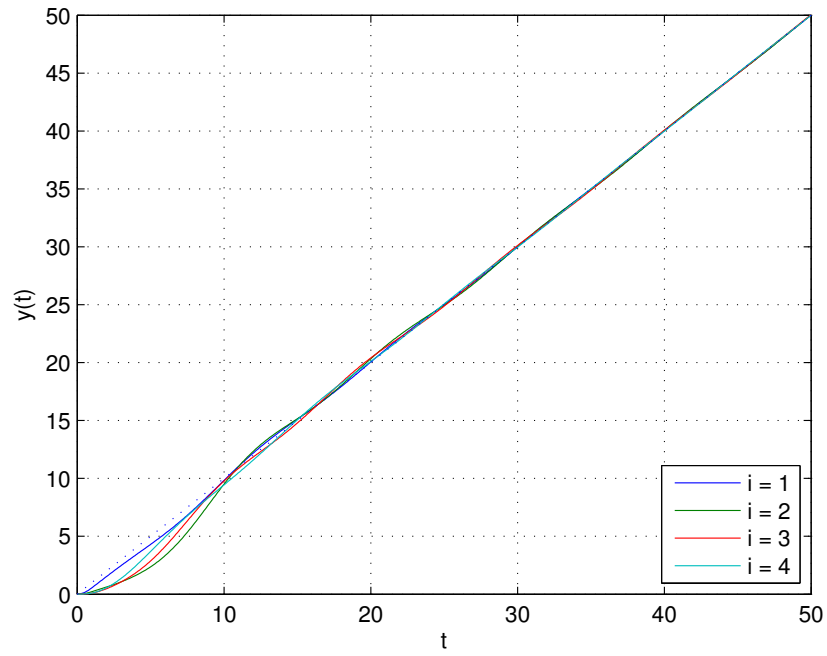rajectory. In fact, if the communication graph is connected (or contains a spanning tree) and the Laplacian matrix of the graph satisfies a rank condition, then it is sufficient for one agent to have access to the reference input for the heterogeneous MAS to achieve consensus. Moreover our proposed design method does not require duplication of the reference model in each of the local controllers thereby eliminating synchronization of the local reference models commonly adopted in the existing work. Both features lower significantly the communication overhead between agents by avoiding the need to communicate the reference trajectory to all agents and by removing additional synchronization between local reference models. Furthermore we focus on non-introspective agents as in [27] and use only relative information for both state feedback and state estimation.

We consider N heterogeneous agents with the dynamics of ith agent described by (1.13). The consensus problem is concerned with agents' output and should satisfy (1.15). The N outputs of the MAS are required to track the output of the reference model described by (3.1) with zero steady-state error. To reduce the communication overhead, the reference signal is often transmitted to only one or a few of the N agents. The realizations of N agents are all stabilizable and detectable. We will study under what condition for the feedback graph, there exist distributed stabilizing controllers and consensus control protocols such that the outputs of N agents satisfy not only (1.15) but also $y_i(t) \to y_0(t) = r(t) \,\forall\, i$ asymptotically. We will study how to synthesize the required distributed and local controllers in order to achieve output consensus, taking performance into account.

The following two facts are useful for our work.

Fact 3 If two square matrices M_1 and M_2 satisfy

$$M_1 + M_1' \geq 0, \quad M_2 + M_2' > 0,$$

with $'$ transpose, then $\det(I + M_1 M_2) \neq 0$. Note that

$$M_1 = (I + R_1)^{-1}(I - R_1), \quad M_2 = (I + R_2)^{-1}(I - R_2),$$

for some (R_1, R_2) satisfying $\bar{\sigma}(R_1) \leq 1$ and $\bar{\sigma}(R_2) < 1$.

Fact 4 Let $X_\mathrm{a} \geq 0$ be the stabilizing solution to the following algebraic Riccati equation (ARE)

$$A_\mathrm{a}' X_\mathrm{a} + X_\mathrm{a} A_\mathrm{a} - X_\mathrm{a} B_\mathrm{a} R_\mathrm{a}^{-1} B_\mathrm{a}' X_\mathrm{a} + Q_\mathrm{a} = 0, \tag{4.1}$$

where $Q_\mathrm{a} \geq 0$ and $R_\mathrm{a} > 0$. Then with $F_\mathrm{a} = R_\mathrm{a}^{-1} B_\mathrm{a}' X_\mathrm{a}$, $(A_\mathrm{a} - B_\mathrm{a} F_\mathrm{a})$ is Hurwitz, and the transfer matrix

$$T_{F_\mathrm{a}}(s) = R_\mathrm{a} F_\mathrm{a}(sI - A_\mathrm{a} + B_\mathrm{a} F_\mathrm{a})^{-1} B_\mathrm{a}, \tag{4.2}$$

is PR [4] (page 106). That is, $T_{F_\mathrm{a}}(s)^* + T_{F_\mathrm{a}}(s) \geq 0 \ \forall \ \mathrm{Re}[s] \geq 0$ with superscript * denoting conjugate transpose. Specifically, let $A_{F_\mathrm{a}} = A_\mathrm{a} - B_\mathrm{a} F_\mathrm{a}$ and $s = \frac{1}{2}\sigma + j\omega$, $\sigma \geq 0$. The ARE (4.1) can be written as

$$(sI - A_{F_\mathrm{a}})^* X_\mathrm{a} + X_\mathrm{a}(sI - A_{F_\mathrm{a}}) = Q_\mathrm{a} + F_\mathrm{a}' R_\mathrm{a} F_\mathrm{a} + \sigma X_\mathrm{a} \geq 0.$$

Multiplying the above equation by $B_\mathrm{a}'(sI - A_{F_\mathrm{a}})^{*-1}$ from left, by $(sI - A_{F_\mathrm{a}})^{-1} B_\mathrm{a}$ from

43

right, and using the relation $R_a F_a = B_a' X_a$ leads to $T_{F_a}(s)^* + T_{F_a}(s) \geq 0 \ \forall \ \mathrm{Re}[s] \geq 0$, concluding the PR property.

Finally we recall a normal form for a $p \times p$ plant model $P(s) = C(sI_n - A)^{-1}B$ satisfying $\det(CB) \neq 0$ and $n > p$. Without loss of generality, its realization can be assumed to be of the normal form [34]

$$
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C = \begin{bmatrix} 0 & I_p \end{bmatrix},
$$

where $A_{22} \in \mathbb{R}^{p \times p}$. It can be easily shown that transmission zeros of $P(s)$ are eigenvalues of A_{11}, thus termed as zero dynamics.

4.1 Distributed Stabilization

This section is focused on the distributed control protocol over the connected graph $\mathcal{G}$, represented by its Laplacian matrix $\mathcal{L}$. Distributed stabilization will be studied and stabilizability condition will be derived for both the case of state feedback and output feedback.

4.1.1 State Feedback

Consider the control protocol for the ith agent given by (recall $r(t) = C_0 x_0(t)$ and that only one $g_i \neq 0$)

$$
u_i(t) = g_i(r - F_i x_i) - d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j), \tag{4.3}
$$

with $d_i > 0$, $g_i \geq 0$, and F_i the state feedback gain for the state vector of agent i where $1 \leq i \leq N$. Basically the control signal for the ith agent consists of relative information, i.e., its error signals with respect to the neighboring agents, plus its tracking error with respect to the reference signal. Such a control protocol is different from (3.9) where the agents had FI. We propose (4.3) in order to minimize the communication overhead, only one

of $\{g_i\}_{i=1}^n$ being nonzero is required, that is why condition (3.3) in Lemma 2 becomes useful. However, there is a trade off between stability/consensus robustness and communication overhead. If more than one $g_i \neq 0$, then the single point of failure scenario is avoided, if the rank condition (3.3) is enforced for each i such that $g_i \neq 0$. This way the closed loop system is more robust in the presence of broken communication links.

Substituting (4.3) into (1.13) yields

$$\dot{x}_i = A_i x_i - B_i d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j) - B_i g_i (F_i x_i - r).$$

Let D and G be in (3.7) with $q = m$, and $\mathcal{L}$ be the corresponding Laplacian matrix as in (b) of Remark 1. By denoting $x(t)$ as the collective state, i.e., the stacked vector of $\{x_i(t)\}_{i=1}^{N}$, the collective closed loop dynamics are now described by

$$\dot{x} = [A - B(D\mathcal{L} + G)F] x + BG [1_N \otimes r], \tag{4.4}$$

where $A = \text{diag}(A_1, \cdots, A_N)$, $B = \text{diag}(B_1, \cdots, B_N)$, and $F = \text{diag}(F_1, \cdots, F_N)$. Recall $\mathcal{M}$ in (3.5). Since $\mathcal{L}(1_N \otimes r) = 0 \; \forall \; r \in \mathbb{R}^m$, (4.4) is equivalent to

$$\dot{x} = [A - B\mathcal{M}F] x + B\mathcal{M} [1_N \otimes r]. \tag{4.5}$$

The following result is concerned with stabilization for the underlying MAS under state feedback.

Theorem 7 *Suppose that (A_i, B_i) is stabilizable $\forall \; i \in \mathcal{N}$. There exists a stabilizing state feedback control protocol (4.3) for the underlying MAS over the directed graph $\mathcal{G}$, if $\mathcal{G}$ is connected, and the condition (3.8) holds for $q = m$.*

Proof: If $\mathcal{G}$ is connected and (3.8) holds for $q = m$, then Lemma 2 and Remark 1 imply the existence of a diagonal $G \geq 0$ with only one of $\{g_i\}_{i=1}^{N}$ nonzero and a diagonal $D > 0$

45

such that the inequality (3.4) holds. In fact (3.6) holds for some $\kappa > 0$. Thus $Z + Z' > 0$ by taking $Z = (D\mathcal{L} + G)/\kappa - I$, i.e., $(D\mathcal{L} + G) = \kappa(Z + I)$. Feedback stability of the underlying MAS requires that

$$\det\left[sI - A + B(D\mathcal{L} + G)F\right] \neq 0 \quad \forall \, \mathrm{Re}\{s\} \geq 0. \tag{4.6}$$

Recall that $\kappa = 1$ can be taken with no loss of generality. Substituting $(D\mathcal{L} + G) = (Z + I)$ into the above inequality yields

$$\det\left(sI - A + BF + BZF\right) \neq 0 \quad \forall \, \mathrm{Re}\{s\} \geq 0.$$

Simple manipulation shows the equivalence of the above inequality to

$$\det\left[I + T_F(s)Z\right] \neq 0 \, \forall \, \mathrm{Re}\{s\} \geq 0 \tag{4.7}$$

where $T_F(s) = F(sI - A + BF)^{-1}B$. Stabilizability of (A_i, B_i) assures the existence of a stabilizing state feedback control gain F_i such that for each $i \in \mathcal{N}$,

$$T_{F_i}(s) = F_i(sI - A_i + B_iF_i)^{-1}B_i, \tag{4.8}$$

is PR. Recall Fact 4 in Section 3.1 with $R_{\mathrm{a}} = I > 0$. As a result,

$$T_F(s) = \mathrm{diag}\left\{T_{F_1}(s), \cdots, T_{F_N}(s)\right\},$$

is PR as well. That is,

$$T_F(s) + T_F^*(s) \geq 0 \, \forall \, \mathrm{Re}\{s\} \geq 0.$$

It follows that the inequality (4.7) holds by $(Z + Z') > 0$ and in light of Fact 3. $\qquad \square$

Theorem 7 provides a sufficient condition for stabilizability under the distributed state feedback control. This sufficient condition becomes necessary for two special cases as shown next.

Corollary 1 *Consider state feedback control for the MAS over the directed graph $\mathcal{G}$. If feedback stability holds for the MAS consisting of either* (i) *homogeneous multi-input unstable agents or* (ii) *heterogeneous single input unstable agents with $\{A_i\}_{i=1}^{N}$ having a common unstable eigenvalue, then the directed graph $\mathcal{G}$ is connected and the rank condition in either* (3.3) *or* (3.8) *holds.*

Proof: For case (i), the homogeneous hypothesis implies

$$F(sI - A)^{-1}B = I_N \otimes F_\mathrm{a}(sI - A_\mathrm{a})^{-1}B_\mathrm{a},$$

where $(A_i, B_i, F_i) = (A_\mathrm{a}, B_\mathrm{a}, F_\mathrm{a}) \forall\ i \in \mathcal{N}$. Using the same procedure as in [45, 62], the feedback stability condition in (4.6) can be shown to be equivalent to

$$\det[I + F_\mathrm{a}(sI - A_\mathrm{a})^{-1}B_\mathrm{a}\lambda_i(\mathcal{M})] \neq 0 \ \forall \ \mathrm{Re}[s] \geq 0.$$

Since A_a has unstable eigenvalues by the hypothesis, the above inequality implies $\lambda_i(\mathcal{M}) \neq 0$ for all i, concluding that the graph $\mathcal{G}$ is connected and the rank condition (3.8) for $\mathcal{M} = (D\mathcal{L} + G)$ holds for some diagonal $D > 0$ and $G \geq 0$ with rank of G equal to 1. For case (ii), feedback stability implies stability of $(A - B\mathcal{M}F)$ for some F. Thus

$$\mathrm{rank}\left\{\begin{bmatrix} sI_n - A & B\mathcal{M} \end{bmatrix}\right\} = n \ \forall \ \mathrm{Re}[s] \geq 0$$

where $n = n_1 + \cdots + n_N$. Recall A_i has dimension $n_i \times n_i$. For single input agents, $B\mathcal{M}$ has N columns. Taking s to be the common unstable eigenvalue of $\{A_i\}_{i=1}^{N}$ implies

that $(sI_n - A)$ has rank $(n - N)$, and thus $B\mathcal{M}$ has rank N, leading to the conclusion of nonsingular $\mathcal{M} = (D\mathcal{L} + G)$ again that concludes the proof. $\qquad\square$

We now consider a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ where the edge set $\mathcal{E}(t)$ is time-varying. This results in a time-varying adjacency matrix $\mathcal{A}(t)$ and Laplacian $\mathcal{L}(t)$. A concept of uniformly connected graph has been proposed in [52, 70, 78]. Because its formal definition is rather lengthy and abstract, we adopt the following slightly modified notion.

Definition 1 A time-varying graph $\mathcal{G}(t)$ with Laplacian $\mathcal{L}(t)$ is uniformly connected with time interval $h > 0$, if

$$\overline{\mathcal{L}}_h(t) := \frac{1}{h} \int_t^{t+h} \mathcal{L}(\tau) \ d\tau, \tag{4.9}$$

is a Laplacian matrix corresponding to some connected graph at all time t.

The above notion is essentially the same as that in [52, 68, 78] and is motivated by practical considerations. Roughly speaking the time-varying graph should be more often connected, in some uniform sense, in the time interval $h > 0$ and at all time t. The next result extends Theorem 7 to the case of time-varying graphs.

Corollary 2 Let (A_i, B_i) be stabilizable $\forall \ i \in \mathcal{N}$. There exists a stabilizing state feedback control protocol

$$u_i(t) = g_i(r - F_i x_i) - d_i \sum_{j=1}^{N} a_{ij}(t)(F_i x_i - F_j x_j) \tag{4.10}$$

for the underlying MAS over the directed graph $\mathcal{G}(t)$, if $\mathcal{G}(t)$ is uniformly connected with sufficiently small time interval $h > 0$, and

$$\text{rank} \left\{ \overline{\mathcal{L}}_h(t) + e_{i_R(t)} e'_{i_R(t)} \right\} = N; \tag{4.11}$$

for at least one index $i_R(t) \in \mathcal{N} \ \forall \ t.$

Proof: Since $\overline{\mathcal{L}}_h(t)$ corresponds to a connected graph and condition (4.11) holds at all t, there exists diagonal $D(t) > 0$ and $G(t) \geq 0$ in the form of (3.7) with only one $g_i(t) > 0$ such that $\overline{\mathcal{M}}(t) = D(t)\overline{\mathcal{L}}_h(t) + G(t)$ satisfies

$$\overline{Z}(t) + \overline{Z}(t)' > 0, \quad \overline{Z}(t) = \overline{\mathcal{M}}(t) - I.$$

Note that (4.4), with $r(t) = 0$, is equivalent to the closed loop system in Figure 4.1, where $T_F(s)$ is block diagonal with the ith block $T_{F_i}(s)$ defined in (4.8) and $Z(t) = \mathcal{M}(t) - I$ with $\mathcal{M}(t) = D(t)\mathcal{L}(t) + G(t)$.

Let $G(s) = [I + T_F(s)]^{-1} [I - T_F(s)]$ and $R(t) = [I + Z(t)]^{-1} [I - Z(t)]$. Then the closed loop in Figure 4.1 is equivalent to the one in Figure 4.2 with $d_+(t) = z(t) + w(t)$ and $d_-(t) = z(t) - w(t)$.
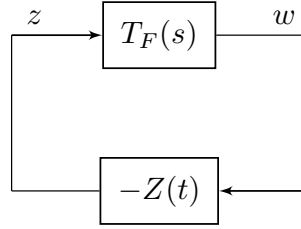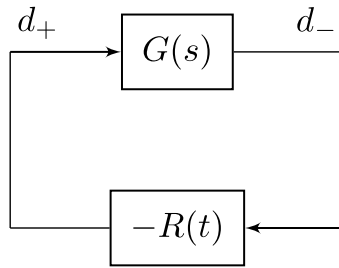

Figure 4.1: Closed loop system

Figure 4.2: Equivalent closed loop system

Since $T_F(s)$ is PR, $G(s)$ is bounded real [4], i.e.,

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\mathrm{Re}[s]>0} \overline{\sigma}[G(s)] \leq 1. \tag{4.12}$$

49

It follows that for each pair of input/output signals, there holds $\|d_-\|_2 \leq \|d_+\|_2$ with $\|\cdot\|_2$ the $\mathcal{L}_2$ norm defined via

$$\|d\|_2^2 = \int_{-\infty}^{\infty} d(t)'d(t) \; dt.$$

On the other hand $d_+(t) = -R(t)d_-(t)$ by Figure 4.2 and

$$
\begin{aligned}
< w, z > \; &:= \int_{-\infty}^{\infty} w(t)'z(t) \; dt \\
&= -\frac{1}{2} \int_{-\infty}^{\infty} w(t)' \left[Z(t) + Z(t)'\right] w(t) \; dt,
\end{aligned}
$$

by Figure 4.1. Denote $f_k(t) = f(kh + t)$. Then

$$< w, z > = -\frac{1}{2} \sum_{k=-\infty}^{\infty} \int_0^h w_k(t)'[Z_k(t) + Z_k(t)']w_k(t) \; dt.$$

Now choosing $D(t)$ and $G(t)$ piecewise constant over each time interval $[kh, \; (k+1)h)$ leads to

$$\sum_{k=-\infty}^{\infty} \int_0^h \left[Z_k(t) + Z_k(t)'\right] dt = \sum_{k=-\infty}^{\infty} h \left[\overline{Z}_k(0) + \overline{Z}_k(0)'\right].$$

Note that $\overline{Z}_k(0) + \overline{Z}_k(0)' > 0$ for all k. Since $T_F(s)$ is PR, its output $w(t)$ is a smooth function, which is approximately constant over $[kh, \; (k+1)h)$ for each k by the hypothesis that h is adequately small. Consequently $< w, z >$ is strictly negative. It follows that

$$
\begin{aligned}
\|d_+\|_2^2 &= \|z\|_2^2 + \|w\|_2^2 + 2 < w, z > \\
&< \|z\|_2^2 + \|w\|_2^2 - 2 < w, z > \; = \|d_-\|_2^2.
\end{aligned}
$$

That is, $\mathcal{L}_2$ induced norm for the block with gain $-R(t)$ in Figure 4.2 is strictly less than 1. By the small gain theorem, the closed loop system in Figure 4.2 is stable, implying the stability of the closed loop system in Figure 4.1. $\qquad\square$

Corollary 2 is weaker than the results in [78] due to the stronger condition on the adequately small time interval h. However this condition does not sacrifice much in practice

by noting that h consists of h_{off} and h_{on}, the time intervals over which the graph losses its connectivity, and keeps its connectivity, respectively. We would expect that in practice a graph will be more often connected, so as long as h_{off} is adequately small, h can be taken adequately small too. It is important to observe that large h_{off} can cause divergence of the output signal inducing system breakdown if the agents are unstable.

4.1.2 Output Feedback

When the states of the MAS are not available for feedback, a distributed observer can be designed to estimate the state of each agent, which can then be used for feedback control. See [45, 87] for homogeneous MASs. We will modify some of the distributed observers in [87] for design of distributed output feedback controllers in the case of heterogeneous MASs.

Our first distributed observer has the local form

$$
\begin{aligned}
\dot{\hat{x}}_i &= A_i\hat{x}_i + B_iu_i - L_i(\hat{y}_i - y_i) \\
&= (A_i - L_iC_i)\hat{x}_i + L_iC_ix_i + B_iu_i,
\end{aligned}
\tag{4.13}
$$

for each i. Notice that there is no communication graph for the output estimation part. Since $\dot{x}_i = A_ix_i + B_iu_i$, taking the difference of the two leads to

$$
\dot{e}_{x_i} = (A_i - L_iC_i)e_{x_i}, \quad e_{x_i} = x_i - \hat{x}_i,
$$

where $i \in \mathcal{N}$. By packing $\{e_{x_i}\}_{i=1}^N$ into a single vector e_x, we obtain $\dot{e}_x = (A - LC)e_x$ with $L = \text{diag}(L_1, \cdots, L_N)$.

Using the estimated states $\{\hat{x}_i\}$ for the control input in (4.3) leads to

$$
u_i(t) = g_i(r - F_i\hat{x}_i) - d_i \sum_{j=1}^N a_{ij}(F_i\hat{x}_i - F_j\hat{x}_j).
\tag{4.14}
$$

Recall $\mathcal{M} = D\mathcal{L} + G$. Substituting the above into $\dot{x} = Ax + Bu$ for $1 \le i \le N$ yields

$$\dot{x} = [A - B\mathcal{M}F] x + B\mathcal{M}Fe_x + B\mathcal{M} (1_N \otimes r).$$ (4.15)

Thus the overall MAS satisfies the state space equation

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \end{bmatrix} = \begin{bmatrix} A - B\mathcal{M}F & B\mathcal{M}F \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} B\mathcal{M} \\ 0 \end{bmatrix} (1_N \otimes r).$$ (4.16)

It follows that the internal stability holds, if and only if $(A - B\mathcal{M}F)$ and $(A - LC)$ are both Hurwitz.

A drawback for the local form in (4.13) and also for the results reported in [39, 78] lies in the use of $\{y_i(t)\}$. For applications to vehicle formation, $y_i(t)$ often represents the absolute position of the ith agent and requires a GPS signal which may not be available to all agents for feedback except the one with $g_i \ne 0$. Instead the relative positions are available from and to the neighboring vehicles, which motivates the second observer, termed neighborhood observer [87]. We propose the following modified observer for heterogeneous MASs

$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_iu_i + g_i[L_i(\hat{y}_i - y_i) - L_0(\hat{y}_0 - y_0)]$$ (4.17)

$$+ d_iL_i\sum_{j=1}^{N} a_{ij}[(\hat{y}_i - \hat{y}_j) - (y_i - y_j)] \ \forall \ i \in \mathcal{N}$$

where $\hat{y}_i = C_i\hat{x}_i$ for $0 \le i \le N$. Notice that in this observer there is a communication graph for the output estimation part. The above observer employs the error signals with respect to neighbors of the ith agent plus the reference signal, and thus can be more preferred in some applications, such as vehicle formation.

In fact the state of the reference model $x_0(t)$ also requires estimation at the ith agent whenever $g_i \ne 0$ due to possible corrupting noise in the received reference signal $r(t) =$

$C_0 x_0(t)$. Taking difference between $\dot{x}_i = A_i x_i + B_i u_i$ and (4.17) leads to

$$\dot{e}_{x_i} = A_i e_{x_i} - d_i L_i \sum_{j=1}^{N} a_{ij}(C_i e_{x_i} - C_j e_{x_j}) - g_i L_i C_i e_{x_i} + g_i L_0 C_0 e_{x_0},$$

which results in the collective error dynamics

$$\dot{e}_x = [A - L (D\mathcal{L} + G) C] e_x + (I_N \otimes L_0)\mathcal{M} (1_N \otimes C_0 e_{x_0}),$$

with $\mathcal{M} = (D\mathcal{L} + G)$ and $e_{x_0}(t) = x_0(t) - \hat{x}_0(t)$. Denote $\underline{L}_0 = I_N \otimes L_0$. In connection with (4.15), the overall MAS has the state space equation

$$\begin{bmatrix} \dot{x} \\ \dot{e}_x \end{bmatrix} = \begin{bmatrix} A - B\mathcal{M}F & B\mathcal{M}F \\ 0 & A - L\mathcal{M}C \end{bmatrix} \begin{bmatrix} x \\ e_x \end{bmatrix} + \begin{bmatrix} B\mathcal{M} (1_N \otimes C_0 \hat{x}_0) \\ \underline{L}_0 \mathcal{M} (1_N \otimes C_0 e_{x_0}) \end{bmatrix}. \qquad (4.18)$$

For both local and neighborhood observers, the separation principle for stabilization holds true as manifested in the collective dynamics (4.16) and (4.18). Hence we have the following result for which we omit the proof.

Theorem 8 *Suppose that (A_i, B_i, C_i) is both stabilizable and detectable for all $i \in \mathcal{N}$. Then there exist distributed output feedback stabilizing controllers for the underlying heterogeneous* MAS, *if the feedback graph is connected and (3.8) holds for both $q = p$ and $q = m$. For the* MAS *over the time-varying graph, the distributed output stabilization requires that the graph be uniformly connected with adequately small $h > 0$ and (4.11) holds for both $q = p$ and $q = m$.*

In light of Corollary 1, the sufficient condition in Theorem 8 for distributed output stabilizability condition can be made necessary for SISO heterogeneous MASs when all agents share some common unstable poles. In synthesizing the distributed output protocol controllers, the state estimation gain L_i is required not only to be stabilizing but also satisfy

the PR property for the resulting

$$T_{L_i}(s) = C_i(sI - A_i + L_iC_i)^{-1}L_i \ \ \forall \ i \in \mathcal{N} \tag{4.19}$$

which is dual to the state feedback case. Consequently $e_x(t) \to 0$ as $t \to \infty$. Hence the observer-based control results in the same closed-loop transfer matrix in steady-state. We also point out that Theorem 8 assumes the same communication graph at both input and output. In practice they can be different from each other which can give more design freedom. In case that the graphs differ, then both need to be connected and satisfy the rank condition (3.8) or (4.11). Moreover each agent may have different measurement output from the consensus output. For simplicity, our paper considers only the case when the measurement output is the same as the consensus output; however our design method can be easily adapted to fit to the case when they differ. For this case, the current matrix, C_i, in the state estimator needs to be replaced by a different C_i corresponding to the measurement output.

Remark 2 Many known output feedback controllers are observer based and satisfy the required PR property, including the controllers designed using $\mathcal{H}_\infty$ loop shaping and LQG/LTR methods. Indeed for $\mathcal{H}_\infty$ loop shaping based on a right coprime factorization (a dual result is presented in [51], page 69-72), $F_i = B_i'X_i$ and $L_i = Y_{i\infty}C_i'$ with $X_i \geq 0$ the stabilizing solution to the control ARE

$$A_i'X_i + X_iA_i - X_iB_iB_i'X_i + C_i'C_i = 0, \tag{4.20}$$

and $Y_{i\infty} \geq 0$ the stabilizing solution to the filtering ARE

$$A_iY_{i\infty} + Y_{i\infty}A_i' - Y_{i\infty}C_i'C_iY_{i\infty} + B_iB_i' + \Delta_i = 0, \tag{4.21}$$

where $\Delta_i = (\gamma_i^2 - 1)(I + Y_{i\infty}X_i)B_iB_i'(I + X_iY_{i\infty})$. Since $\gamma_i > \gamma_{iopt} = \sqrt{1 + \lambda_{\max}(X_iY_i)} \geq 1$,

$\Delta_i \geq 0$ is true. In light of Fact 4, both $T_{F_i}(s)$ defined in (4.8) and $T_{L_i}(s)$ defined in (4.19) are PR.

For LQG/LTR, $F_i = B_i' X_i$ with $X_i \geq 0$ the stabilizing solution to the ARE (4.20), and $L_i = Y_i C_i'$ with $Y_i \geq 0$ the stabilizing solution to the ARE

$$A_i Y_i + Y_i A_i' - Y_i C_i' C_i Y_i + q_i^2 \tilde{Q}_i = 0,$$

for some design parameter $q_i > 0$ sufficiently large. The matrix $\tilde{Q}_i = B_i B_i'$ if $P_i(s) = C_i(sI - A_i)^{-1} B_i$ is minimum phase. Otherwise $\tilde{Q}_i = B_{im} B_{im}'$ where $P_i(s) = P_{im}(s) B_{ia}(s)$ with $P_{im}(s) = C_i(sI - A_i)^{-1} B_{im}$ being the minimum phase part of $P_i(s)$ and $B_{ia}(s)$ satisfying $B_{ia}(-s)' B_{ia}(s) = I$ and containing all unstable zeros of $P_i(s)$ [86]. Hence both $T_{F_i}(s)$ defined in (4.8) and $T_{L_i}(s)$ defined in (4.19) are again PR in light of Fact 4. $\square$

The next result is concerned with the existence of static output stabilizing control law for heterogeneous MASs.

Corollary 3 *Suppose that (A_i, B_i, C_i) is both stabilizable and detectable satisfying $\det(C_i B_i) \neq 0$ and (A_i, B_i, C_i) is strictly minimum phase for all $i \in \mathcal{N}$. Then there exist distributed static output stabilizing controllers for the underlying heterogeneous MAS, if the feedback graph is connected and (3.8) holds for $q = m$.*

Proof: It is shown in [28] that under the hypotheses for (A_i, B_i, C_i), there exists K_i such that

$$u_i(t) = K_i y_i(t) = K_i C_i x_i(t),$$

is an LQR control law for the system $\dot{x}_i = A_i x_i + B_i u_i$ with $R = I$. Since this is true for each $i \in \mathcal{N}$, the result for distributed state feedback control can be applied. Specifically $T_F(s)$ in the proof of Theorem 7 with $F_i = K_i C_i$ for each i can be made PR. The corollary is thus true. $\square$.

4.1.3 Robust Analysis

It is well known that both $\mathcal{H}_\infty$ loop shaping and the LQG/LTR methods provide robustness for the designed feedback control system. For instance a feedback controller designed using LQG/LTR has good gain margin (GM), while one designed using $\mathcal{H}_\infty$ loop shaping is optimally robust against model uncertainty described by normalized coprime factors. However the distributed controllers with distributed observer may obscure the robustness of these two different controllers. This subsection presents a preliminary analysis.

Consider first the LQG/LTR design method. Suppose that the ith plant is described by

$$P_i(s)(I + \Delta_i) = C_i(sI - A_i)^{-1}B_i(I + \Delta_i),$$

where Δ_i is diagonal with its kth diagonal element in the range of $(\alpha_{i,k}, \ \beta_{i,k})$ satisfying

$$0 < \alpha_{i,k} < 1 < \beta_{i,k} < \infty.$$

The overall gain margin for the MAS can be defined as

$$\text{GM} = 20 \log_{10}\left(\prod_{i=1}^{N}\prod_{k=1}^{m}\frac{\beta_{i,k}}{\alpha_{i,k}}\right)\text{dB}$$

that is the largest possible subject to feedback stability for all Δ_i in the interval. For convenience, denote $\Delta = \text{diag}(\Delta_1, \cdots, \Delta_N)$. We'll focus on the control protocol based on neighborhood observers in form of (4.17). Since the reference input does not change the gain margin, $r(t) = 0$ is taken for (4.14). It results in

$$\dot{x} = Ax - B(I + \Delta)\mathcal{M}F\hat{x}, \quad y = Cx,$$
$$\dot{\hat{x}} = (A - B\mathcal{M}F - L\mathcal{M}C)\hat{x} - L\mathcal{M}y.$$

The feedback system described above is depicted in the following block diagram Figure 4.3.
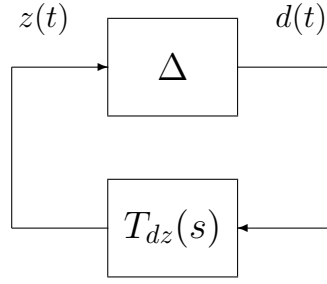
Figure 4.3: Gain margin analysis

Denote $P(s) = \text{diag}[P_1(s), \cdots, P_N(s)]$ and

$$\tilde{K}(s) = \tilde{F}(sI - A + B\tilde{F} + \tilde{L}C)^{-1}\tilde{L}, \tag{4.22}$$

with $\tilde{F} = \mathcal{M}F$ and $\tilde{L} = L\mathcal{M}$. The transfer matrix $T_{dz}(s)$ can be obtained as

$$T_{dz}(s) = -\tilde{K}(s)[I - P(s)\tilde{K}(s)]^{-1}P(s).$$

Although $P(s)$ is diagonal, $\tilde{K}(s)$ is not due to the presence of $\mathcal{M} = D\mathcal{L} + G$ in $\tilde{F}$ and $\tilde{L}$. It can be verified that

$$T_{dz}(s) = \tilde{F}(sI - A + B\tilde{F})^{-1}\tilde{L}C(sI - A + \tilde{L}C)^{-1}B.$$

Hence in general the gain margin calculation is a μ-analysis problem that can be difficult, especially for the case when N is large.

In the case of $\mathcal{H}_\infty$ loop shaping, Δ_i is a stable transfer matrix, representing the modeling uncertainty of the system, different from the previous real diagonal Δ. Moreover the expression of the equivalent $T_{dz}(s)$ is more complex. Let $P_i(s) = N_i(s)M_i(s)^{-1}$ be the normalized right coprime factorization of $P_i(s)$. The true unknown model for the ith agent is assumed to be

$$[N_i(s) + \Delta_{iN}(s)][M_i(s) + \Delta_{iM}(s)]^{-1}$$

and $\Delta_i(s) = [\ \Delta_{iN}(s)'\ \Delta_{iM}(s)'\ ]'$ satisfying $\|\Delta_i\|_{\mathcal{H}_\infty} \leq \delta_i$ [51] with $\|\cdot\|_{\mathcal{H}_\infty}$ the $\mathcal{H}_\infty$-norm defined in (4.12). Let $\tilde{K}(s)$ be as in (4.22). An equivalent $\tilde{T}_{dz}(s)$ in the sense of the same $\mathcal{H}_\infty$-norm as that for $T_{dz}(s)$ is given by

$$\tilde{T}_{dz}(s) = \begin{bmatrix} P(s) \\ I \end{bmatrix} \Big(I - \tilde{K}(s)P(s) \Big)^{-1} \begin{bmatrix} \tilde{K}(s) & -I \end{bmatrix}.$$

Since the derivation is similar, it is skipped. The stability margin can be obtained via computing

$$\delta_{\max} = \Big(\|\tilde{T}_{dz}\|_{\mathcal{H}_\infty} \Big)^{-1}$$

implying that feedback stability holds as long as $\delta_i < \delta_{\max}$ for all $i \in \mathcal{N}$. Again, the computation of $\delta_{\max}$ can be very demanding.

Before concluding this section, we would like to point out that while both $\mathcal{H}_\infty$ loop shaping and LQG/LTR methods produce robust controllers, the robustness measures such as GM and $\delta_{\max}$ of the feedback MAS can be very different from the case in absence of $\mathcal{M} = D\mathcal{L} + G$.

4.2 Output Consensus

Several results exist regarding the condition for heterogeneous MASs to achieve output consensus, including [27, 39, 78]. Our results differ from the existing work due to the absence of a local reference model at each agent and in the explicit conditions for the consensusability in terms of the connected graph and the rank condition in Lemma 2. Hence synchronization of the local reference models can be avoided and consensus control can be achieved directly using local and distributed feedback control protocols. More importantly the existing well-developed design methods such as $\mathcal{H}_\infty$ loop shaping [51] and LQG/LTR [4] can be used to synthesize the output consensus control law.

Prior to study of output consensus, we introduce a known result from [30].

Lemma 6 *Let the plant model be described by*

$$\dot{x}_\mathrm{a}(t) = A_\mathrm{a}x(t) + B_\mathrm{a}u_\mathrm{a}(t), \quad y_\mathrm{a}(t) = C_\mathrm{a}x_\mathrm{a}(t),$$

where $A_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a} \times n_\mathrm{a}}$, $B_\mathrm{a} \in \mathbb{R}^{n_\mathrm{a} \times m_\mathrm{a}}$, and $C_\mathrm{a} \in \mathbb{R}^{p_\mathrm{a} \times n_\mathrm{a}}$, and the reference model be described in (3.1) with $A_0 \in \mathbb{R}^{n_0 \times n_0}$, $C_0 = I_p$, and $p = p_\mathrm{a}$. Assume that $(A_\mathrm{a}, B_\mathrm{a})$ is stabilizable and consider the control law $u_\mathrm{a}(t) = -F_\mathrm{a}x_\mathrm{a}(t) + F_{0\mathrm{a}}r(t)$. Then for each stabilizing state feedback gain $F_\mathrm{a} \in \mathbb{R}^{m_\mathrm{a} \times n_0}$, there exists a reference feed-forward gain $F_{0\mathrm{a}} \in \mathbb{R}^{m_\mathrm{a} \times p}$ such that

$$\lim_{t \to \infty} [y_\mathrm{a}(t) - r(t)] = 0, \tag{4.23}$$

i.e., the output of the plant model tracks the reference input with zero steady-state error, if and only if

$$\mathrm{rank}\left\{ \begin{bmatrix} \lambda I - A_\mathrm{a} & B_\mathrm{a} \\ C_\mathrm{a} & 0 \end{bmatrix} \right\} = n + p, \tag{4.24}$$

at $\lambda = \lambda_\ell(A_0)$ for $\ell = 1, \cdots, n_0$.

The above lemma implicitly assumes $p \leq m$. That is, the plant model $P_\mathrm{a}(s) = C_\mathrm{a}(sI - A_\mathrm{a})^{-1}B_\mathrm{a}$ is a wide or square transfer matrix. Thus the feed-forward gain $F_{0\mathrm{a}}$ is a tall or square matrix. It follows that the closed-loop transfer matrix from the reference input $r(t)$ to output $y_\mathrm{a}(t)$ is square and given by

$$T_\mathrm{a}(s) = C_\mathrm{a}(sI - A_\mathrm{a} + B_\mathrm{a}F_\mathrm{a})^{-1}B_\mathrm{a}F_{0\mathrm{a}}. \tag{4.25}$$

Computation of $F_{0\mathrm{a}}$, given a stabilizing F_a, requires first computing the solution $(W_\mathrm{a}, U_\mathrm{a})$

to the equation

$$
\begin{bmatrix} I_{n_a} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_a \\ U_a \end{bmatrix} A_0 - \begin{bmatrix} A_a & B_a \\ C_a & 0 \end{bmatrix} \begin{bmatrix} W_a \\ U_a \end{bmatrix} = \begin{bmatrix} 0 \\ C_0 \end{bmatrix}, \tag{4.26}
$$

and then setting $F_{0a} = U_a - F_a W_a$ [30] (page 7-9). Generally F_{0a} has full rank, and in fact, the full rank condition can be assured if the synthesis of the stabilizing state feedback F_a admits design degrees of freedom.

It is important to observe that the tracking condition in (4.23) does not require that the plant model include the modes $\{\lambda_\ell(A_0)\}$ due to the existence of the feed-forward gain F_{0a}. In practice, though, the inclusion of the modes $\{\lambda_\ell(A_0)\}$ in the plant dynamics help to improve performance of both tracking and disturbance rejection. For this reason we assume the following:

Assumption 1 Each distinct eigenvalue of $\{\lambda_\ell(A_0)\}$ is a pole of $P_i(s)$ and satisfies

$$
\mathrm{rank}\left\{ \lim_{s \to \lambda_\ell(A_0)} [s - \lambda_\ell(A_0)] P_i(s) \right\} = \mathrm{full} \ \forall \ i.
$$

If Assumption 1 does not hold, then dynamic weighting functions $\{W_i(s)\}$ (having poles at the missing modes of $\{\lambda_\ell(A_i)\}$) can be employed so that the weighted plant $P_{W_i}(s) = P_i(s)W_i(s)$ satisfies Assumption 1 $\forall \ i$. In fact adding weighted dynamics such as integrators and lead/lag compensators to obtain a desired frequency shape has been a standard procedure in LQG/LTR and $\mathcal{H}_\infty$ loop shaping design methods [4, 51]. Controller design can then proceed for $P_{W_i}(s)$ and implementation of the controller needs to take $W_i(s)$ as part of the controller. Assumption 1 then results in no loss of generality, since it can always be made true. The next result provides the output consensusability condition for heterogeneous MASs in the case $p = m$.

Theorem 9 *Consider the heterogeneous* MAS *with equal number of inputs and outputs, and agent model* $P_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i$ *having stabilizable and detectable realization for all* i. *Let the reference model be described in* (3.1) *with* $C_0 = I_p$. *Under* Assumption 1, *the given* MAS *over the feedback graph* $\mathcal{G}$ *is output consensusable, if* $\mathcal{G}$ *is connected, the condition* (3.8) *holds for* $q = p = m$, *and* (4.24) *is true for all* $a = i \in \mathcal{N}$.

Proof: The connected graph $\mathcal{G}$ and rank condition (3.8) imply the existence of observer-based controllers with feedback gains $\{F_i\}$ and $\{L_i\}$ which achieve distributed local stabilization. In fact stabilizing state feedback and state estimation gains can be synthesized such that $\{T_{F_i}(s)\}$ in (4.8) and $\{T_{L_i}(s)\}$ in (4.19) are not only stable but also PR using either LQG/LTR or $\mathcal{H}_\infty$ loop shaping design method. It follows from condition (4.24) with $a = i$ and Lemma 6 that the feed forward gain $F_{0i} \in \mathbb{R}^{m \times m}$ exists such that the fictitious closed-loop system with transfer matrix

$$T_{C_i}(s) = C_i(sI - A_i + B_iF_i)^{-1}B_iF_{0i},$$

achieves tracking with zero steady-state error. Note that F_{0i} can be made nonsingular by choosing a suitable stabilizing state feedback gain due to the many design degrees of freedom in both LQG/LTR and $\mathcal{H}_\infty$ loop shaping design methods. For output consensus control, the control input in (4.14) is modified by setting $u_i(t) = F_{0i}\widehat{u}_i(t)$ and

$$\widehat{u}_i(t) = \widehat{G}_i(r - \widehat{F}_i\hat{x}_i) - \widehat{D}_i \sum_{j=1}^{N} a_{ij}(\widehat{F}_i\hat{x}_i - \widehat{F}_j\hat{x}_j), \tag{4.27}$$

where $\{\hat{x}_i(t)\}$ are estimated states based on either local observers (4.13) or neighborhood observers (4.17), and

$$\widehat{G}_i = R_{0i}^{-1}g_i, \quad \widehat{D}_i = R_{0i}^{-1}d_i, \quad R_{0i} = F'_{0i}F_{0i}. \tag{4.28}$$

The above leads to replacement of B_i by $\widehat{B}_i = B_iF_{0i}$ and F_i by $\widehat{F}_i = F_{0i}^{-1}F_i$. Thus $B_iF_i =$

$\widehat{B}_i \widehat{F}_i$, leading to

$$\widehat{T}_{C_i}(s) = C_i(sI - A_i + \widehat{B}_i \widehat{F}_i)^{-1} \widehat{B}_i, \tag{4.29}$$

for each $i \in \mathcal{N}$. The control input $u_i(t) = F_{0i} \widehat{u}_i(t)$ with $\widehat{u}_i(t)$ in (4.27) shows that the collective state equation in (4.15) is now replaced by

$$\dot{x} = [A - \widehat{B}\widehat{\mathcal{M}}\widehat{F}]x + \widehat{B}\widehat{\mathcal{M}}\widehat{F}e_x + \widehat{B}\widehat{\mathcal{M}}(1_N \otimes r), \tag{4.30}$$

where $\widehat{\mathcal{M}} = \widehat{D}\mathcal{L} + \widehat{G}$ and

$$\widehat{D} = \text{diag}(\widehat{D}_1, \cdots, \widehat{D}_N), \qquad \widehat{G} = \text{diag}(\widehat{G}_1, \cdots, \widehat{G}_N),$$
$$\widehat{B} = \text{diag}(\widehat{B}_1, \cdots, \widehat{B}_N), \qquad \widehat{F} = \text{diag}(\widehat{F}_1, \cdots, \widehat{F}_N).$$

Recall (4.28). Denoting $R_0 = \text{diag}(R_{01}, \cdots, R_{0N})$ yields

$$\widehat{\mathcal{M}} = R_0^{-1}\mathcal{M}, \quad \mathcal{M} = D\mathcal{L} + G.$$

The connectedness of $\mathcal{G}$ and the rank condition (3.8) imply that $\mathcal{M} + \mathcal{M}' > 0$ for some diagonal $D > 0$ and $G \geq 0$ in (3.7) with only one nonzero $g_i > 0$. Let $F_0 = \text{diag}(F_{01}, \cdots, F_{0N})$. Then $R_0\widehat{\mathcal{M}} + \widehat{\mathcal{M}}'R_0 = \mathcal{M} + \mathcal{M}' > 0$ that is equivalent to (recall that $\kappa = 1$ can be taken)

$$F_0\widehat{\mathcal{M}}F_0^{-1} + (F_0\widehat{\mathcal{M}}F_0^{-1})' > I \iff \tilde{\mathcal{M}} + \tilde{\mathcal{M}}' > I \tag{4.31}$$

with $\tilde{\mathcal{M}} = F_0\widehat{\mathcal{M}}F_0^{-1}$. It is claimed that $(A - \widehat{B}\widehat{\mathcal{M}}\widehat{F})$ is Hurwitz, provided that $(A_i - B_iF_i)$ is for all $i \in \mathcal{N}$. Indeed

$$\det(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F}) = \det(sI - A + B\tilde{\mathcal{M}}F),$$

where $\tilde{\mathcal{M}} = F_0\widehat{\mathcal{M}}F_0^{-1}$ by $\widehat{B} = BF_0$ and $\widehat{F} = F_0^{-1}F$. Hence a similar argument in the proof

of Theorem 7 can be used to conclude the claim.

Now consider the collective output

$$y(t) = Cx(t), \quad C = \mathrm{diag}(C_1, \cdots, C_N). \tag{4.32}$$

Let $Y(s) = \mathcal{L}\{y(t)\}$, $R(s) = \mathcal{L}\{r(t)\}$, and $E_x(s) = \mathcal{L}\{e_x(t)\}$ be the Laplace transforms of $y(t)$, $r(t)$, and $e_x(t)$, respectively. The Laplace transform of (4.30) with the output equation (4.32) is given by

$$\begin{aligned}
Y(s) = {} & C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}x(0) \\
& + C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\widehat{F}E_x(s) \\
& + C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}(1_N \otimes R(s)),
\end{aligned} \tag{4.33}$$

taking the modification in (4.27) into account. Because $(A - \widehat{B}\widehat{\mathcal{M}}\widehat{F})$ is Hurwitz, the term associated with $x(0)$ approaches zero as $t \to \infty$. In addition the estimation error $e_x(t)$ also approaches zero as $t \to \infty$ regardless of local or neighborhood observers being used. Hence the steady-state response of $y(t)$ is determined by

$$\widehat{T}_C(s) = C\left(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F}\right)^{-1}\widehat{B}\widehat{\mathcal{M}}, \tag{4.34}$$

that is the transfer matrix from $1_N \otimes r(t)$ to $y(t)$. Recall that $P(s) = C(sI - A)^{-1}B = \mathrm{diag}(P_1, \cdots, P_N)$. Denote

$$\widehat{P}(s) = P(s)F_0 = C(sI - A)^{-1}\widehat{B}.$$

In light of the hypotheses that $p = m$ and Assumption 1, there holds

$$\lim_{s \to \lambda_\ell(A_0)} P(s)^{-1} = \lim_{s \to \lambda_\ell(A_0)} \widehat{P}(s)^{-1} = 0.$$

Hence for a simple eigenvalue $\lambda_\ell(A_0) = s_\ell$, there holds

$$
\begin{aligned}
\lim_{s \to s_\ell} \widehat{T}_C(s) &= \lim_{s \to s_\ell} C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} \\
&= \lim_{s \to s_\ell} \left[\widehat{\mathcal{M}}\widehat{F}(sI - A)^{-1}\widehat{B}\widehat{P}(s)^{-1} \right]^{-1} \widehat{\mathcal{M}} \\
&= \lim_{s \to s_\ell} \left[\widehat{F}(sI - A)^{-1}\widehat{B}\widehat{P}(s)^{-1} \right]^{-1} \\
&= \lim_{s \to s_\ell} C(sI - A + \widehat{B}\widehat{F})^{-1}\widehat{B} \\
&= \lim_{s \to s_\ell} [T_C(s)],
\end{aligned}
$$

with $T_C(s) = \text{diag}\{T_{C_1}(s), \cdots, T_{C_N}(s)\}$ and $T_{C_i}(s)$ in (4.29). Since $T_{C_i}(s)$ achieves tracking with zero steady-state error for each i, the output consensus for the heterogeneous MAS is also achieved in the case of simple eigenvalue $\lambda_\ell(A_0) = s_\ell$. In the case of multiple eigenvalue $\lambda_\ell(A_0) = 0$ with multiplicity $\mu > 1$, perfect tracking requires that

$$
\lim_{s \to 0} \frac{\widehat{T}_C(s) - \widehat{T}_C(0)}{s^{\mu - 1}} = 0.
$$

Since the limit of $\widehat{T}_C(s)$ is the same as the limit of $T_C(s)$ at $\lambda_\ell(A_0)$, the above equality is true, implying the output consensus at $\lambda_\ell(A_0) = 0$. The proof for nonzero repeated eigenvalues is similar so it is skipped. □

Theorem 9 does not consider the issue of $p \neq m$, i.e., the number of inputs does not equal to the number of outputs for each agent. A simple way to bypass the issue of $p < m$ is to append additional $(m - p)$ linearly independent rows to C_i for all i before synthesizing the output consensus. The augmented agents are square and thus Theorem 9 can be applied to synthesize the output consensus controllers. Note that the added $(m-p)$ outputs are fictitious for which the tracking performance can be ignored in design of the output consensus control protocol. The issue of $p > m$ is more complex, because it is not possible for p outputs to track m reference inputs with zero steady-state error in general. A convenient way is to consider consensus control for the first m outputs. However instead

of deleting the last $(p - m)$ rows of C_i, we can consider appending $(p - m)$ zero columns to B_i, and then apply the design procedure from the proof of Theorem 9. Although the invertibility of the augmented $P_i(s)$ does not hold, the first $m \times m$ block is invertible, and thus tracking of the first m outputs to m reference inputs can be assured.

In practice, we have a virtual reference model that does not exist physically. What is available to the MAS is the reference input $r(t)$ that is piecewise step, ramp, sinusoidal, etc, i.e., $r(t) = C_0(t)x_0(t)$ with $C_0(t)$ piecewise constant in time to pick up the step or ramp or sinusoidal signal generated by $\dot{x}(t) = A_0 x(t)$ with different initial condition. For this reason, $n_0 > m$ and $C_0(t) \neq I$ in general. However for each fixed reference input, it can be generated by a reference model with much smaller dimension. Thus $C_0 = I$ assumed in Theorem 9 has no loss of generality.

Finally it needs to be reminded that the tracking performance is influenced by the eigenvalues of the Laplacian matrix $\mathcal{L}$. How to take $\mathcal{L}$ into consideration for design of high performance feedback controllers remains a challenging issue. For some ideas in this direction, see [3, 15, 76], where the issue of how the topology of the graph influences the dynamics of the agents is discussed. While time-varying graphs are not considered in Theorem 9, the output consensusability condition can be easily extended to the uniformly connected graph with adequately small $h > 0$ and the rank condition in (4.11).

4.3 Simulation Setup and Results

Consider the simulation setup as described in Section 3.6 without time delays in the communication graph. We want to asymptotically regulate the position such that the final positions are 10, 4, 6, and 8 for agent 1, 2, 3, and 4, respectively. In order to achieve this, we add an offset value ψ_i to each agent as shown below

$$u_i(t) = g_i(r - F_i x_i) - d_i \sum_{j=1}^{N} a_{ij}(F_i x_i - F_j x_j) + \phi_i,$$

$$\phi_i = g_i \psi_i - d_i \sum_{j=1}^{N} a_{ij}(\psi_i - \psi_j).$$

Note that ϕ_i is a fixed value for each i which can be computed and stored locally beforehand. In this manner, the closed loop collective dynamics are given by

$$\dot{x} = [A - B\mathcal{M}F]\,x + B\mathcal{M}\,[\psi + 1_N \otimes r]\,.$$

where ψ is a vector with its ith element containing the offset value ψ_i. Figure 4.4 shows each agent's position in closed loop using the state feedback control signal (4.3) and the local observer based control signal (4.14) with offset term.
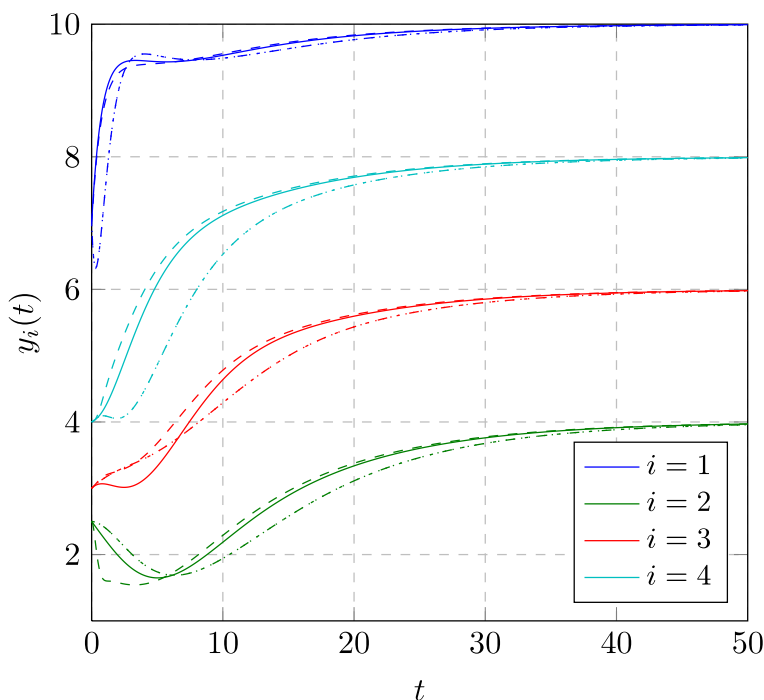

Figure 4.4: Evolution of the output signals under state feedback (solid), local observer-based feedback with LQG (dotted), local observer-based feedback with LTR (dashed), and $\mathcal{H}_\infty$ loop shaping (dash-dot). Signals are communicated through the graph in Figure 3.1

It is important to note that for all cases, the network graph is connected and that $g_i \neq 0$ only for $i = 1$, i.e., only agent 1 has direct access to $r(t)$. In addition, the rank condition (3.3) is satisfied with $i = 1$. With respect to (3.6), $\kappa = 0.1$, $g_1 = 0.5$, and $D = \mathrm{diag}\,(0.1608, 0.4348, 0.5683, 0.7168)$. Notice that the 4th agent is the best one at following the trajectory of the 1st agent while the 2nd agent is the worst one. This may be

explained by the graph interconnection since the 2nd agent is the last in the communication chain.

4.4 Consensus Tracking

We consider the problem of designing a feedback control law in order to achieve tracking of reference inputs in control systems focusing on unit step and ramp functions. The tracking is a complicated problem for MASs. In addition to design of each stabilizing state feedback F_i we also need to design a feed-forward gain F_{0i} such that steady-state error

$$e_{ss} = \lim_{s \to \infty} e(t) = \lim_{s \to \infty} [y(t) - r(t)] = 0. \tag{4.35}$$

The final-value theorem can be employed to obtain the condition for output consensus given by

$$\lim_{s \to 0} s\left[Y(s) - 1_N \otimes R(s)\right] = 0. \tag{4.36}$$

We consider two different methods to achieve tracking of unit step and ramp functions.

4.4.1 Offset Method

In this method, we assume that $P(s)$ has exactly one pole at the origin. The steady-state response of $y(t)$ is determined by (4.34). The following derivation shows that

$$\begin{aligned}
\widehat{T}_C(s) &= C(sI - A)^{-1}\widehat{B}\left[I + \widehat{\mathcal{M}}\widehat{F}(sI - A)^{-1}\widehat{B}\right]^{-1}\widehat{\mathcal{M}} \\
&= \left[P(s)^{-1} + \widehat{\mathcal{M}}P_F(s)P(s)^{-1}\right]^{-1}\widehat{\mathcal{M}},
\end{aligned}$$

67

in light of the assumption that $P_i(s)$ has a pole at the origin where $P(s) = C(sI - A)^{-1}B$ and $P_F(s) = F(sI - A)^{-1}B$. We also note that

$$
\begin{aligned}
T_C(s) &= C(sI - A + \widehat{B}\widehat{F})^{-1}\widehat{B} = C(sI - A)^{-1}\widehat{B}\left[I + \widehat{F}(sI - A)^{-1}\widehat{B}\right]^{-1} \\
&= \left[P(s)^{-1} + P_F(s)P(s)^{-1}\right]^{-1}.
\end{aligned}
$$

By the assumption that $P_i(s)$ has a pole at the origin,

$$
\lim_{s \to 0} \widehat{T}_C(s) = \lim_{s \to 0} P(s)P_F(s)^{-1} = \lim_{s \to 0} T_C(s). \tag{4.37}
$$

We begin with tracking of step input.

- **Tracking a Step Input**

  Output consensus for step input requires

$$
\lim_{s \to 0} s\left[Y(s) - 1_N \otimes \frac{1}{s}\right] = 0. \tag{4.38}
$$

Consider the LHS of (4.38) where $Y(s)$ is given by (4.33) we get

$$
\begin{aligned}
\lim_{s \to 0} s\left[Y(s) - 1_N \otimes \frac{1}{s}\right] &= \lim_{s \to 0} s\left[C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\left(1_N \otimes \frac{1}{s}\right) - 1_N \otimes \frac{1}{s}\right] \\
&= \lim_{s \to 0}\left[C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - I\right]1_N \\
&= \lim_{s \to 0}\left[C(sI - A + \widehat{B}\widehat{F})^{-1}\widehat{B} - I\right]1_N,
\end{aligned}
$$

in light of (4.37). The following result is thus true.

Lemma 7 *Under the assumption that $P_i(s)$ has a pole at the origin for each i, tracking of step input with zero steady-state error for the MAS is achievable, if and only if (A_i, B_i) is stabilizable for all i.*

Proof: Since $s = 0$ is a pole of $P_i(s)$, it cannot be a zero for $P_i(s)$. That is,

$$\text{rank}\left\{\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix}\right\} = \text{rank}\left\{\begin{bmatrix} A_i - B_i F_i & B_i \\ C_i & 0 \end{bmatrix}\right\} = n_i + 1,$$

for each i, and any state feedback gain F_i. If (A_i, B_i) is stabilizable for all i, then stabilizing F_i exists for each i, and thus the above rank condition implies that

$$\lim_{s \to 0} C_i(sI - A_i + B_i F_i)^{-1} B_i \neq 0 \ \forall \ i.$$

The above shows that F_{0i} exists such that

$$\lim_{s \to 0} C_i(sI - A_i + B_i F_i)^{-1} B_i F_{0i} = 1,$$

by setting that $F_{0i} = [C_i(-A_i + B_i F_i)^{-1} B_i]^{-1}$. It follows that

$$\lim_{s \to 0} \left[C(sI - A + \widehat{B}\widehat{F})^{-1}\widehat{B} - I \right] 1_N = 0,$$

and thus the zero steady-state error is achieved for the step input. Conversely the zero steady-state error implies that $\widehat{T}_C(s)$ and $T_C(s)$ are all stable implying that $(A_i - B_i F_i)$ is a stability matrix, and this (A_i, B_i) is stabilizable for all i. $\qquad\square$

- **Tracking a Ramp Input**

Here we consider the output consensus condition for tracking a ramp with an offset term ϕ_i added. For output consensus control, the control input in (4.27) is modified and written as

$$\widehat{u}_i(t) = \widehat{G}_i(r - \widehat{F}_i \widehat{x}_i) - \widehat{D}_i \sum_{j=1}^{N} a_{ij}(\widehat{F}_i \widehat{x}_i - \widehat{F}_j \widehat{x}_j) + \phi_i, \tag{4.39}$$

$$\phi_i = \widehat{G}_i \psi_i - \widehat{D}_i \sum_{j=1}^{N} a_{ij}(\psi_i - \psi_j). \tag{4.40}$$

69

Subsequently the collective state equation in (4.30) can now be replaced by

$$\dot{x} = [A - \widehat{B}\widehat{\mathcal{M}}\widehat{F}]x + \widehat{B}\widehat{\mathcal{M}}\widehat{F}e_x + \widehat{B}\widehat{\mathcal{M}}(\psi + 1_N \otimes r). \tag{4.41}$$

Note that ϕ_i is a fixed value for each i which can be computed and stored locally beforehand. As the estimation error $e_x(t)$ approaches zero, the closed loop collective dynamics can be written as

$$\dot{x} = \left[A - \widehat{B}\widehat{\mathcal{M}}\widehat{F}\right]x + \widehat{B}\widehat{\mathcal{M}}\left[\psi + 1_N \otimes r\right], \tag{4.42}$$

where ψ is a vector with its ith element containing the offset value ψ_i.

Output consensus for ramp input with offset method requires

$$\lim_{s \to 0} s \left[Y(s) - 1_N \otimes \frac{1}{s^2}\right] = 0. \tag{4.43}$$

Consider the collective output (4.32). The Laplace transform of (4.41) with the output equation (4.32) is given by

$$
\begin{aligned}
Y(s) &= C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}x(0) \\
&\quad + C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\widehat{F}E_x(s) \\
&\quad + C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\left[\frac{\psi}{s} + 1_N \otimes R(s)\right],
\end{aligned}
\tag{4.44}
$$

taking the modification in (4.39) into account. Because $(A - \widehat{B}\widehat{\mathcal{M}}\widehat{F})$ is Hurwitz, the same arguments as earlier concludes that steady-state output $y_{ss}(t)$ reduces to

$$Y_{ss}(s) = C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\left[\frac{\psi}{s} + 1_N \otimes R(s)\right]. \tag{4.45}$$

The steady-state response of $y(t)$ is again determined by $\widehat{T}_C(s)$ in (4.34). Consider the LHS

of (4.43) where $Y(s)$ is given by (4.45) we get

$$\lim_{s \to 0} s \left[Y(s) - 1_N \otimes \frac{1}{s^2} \right] = \lim_{s \to 0} s \left[C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} \left(\frac{\psi}{s} + 1_N \otimes R(s) \right) - 1_N \otimes R(s) \right]$$

$$= \lim_{s \to 0} s \left[\left(C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - I \right) 1_N \otimes R(s) + \frac{C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\psi}{s} \right]$$

$$= \lim_{s \to 0} s \left[\frac{\left(C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - I \right) 1_N}{s^2} + \frac{C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\psi}{s} \right]$$

$$= \lim_{s \to 0} s \left[\frac{\left(C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - I \right) 1_N}{s^2} \right] + \lim_{s \to 0} s \left[\frac{C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\psi}{s} \right]$$

$$= \lim_{s \to 0} \left[\frac{\left(C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - I \right) 1_N}{s} \right] + \psi.$$

Recall that $C(-A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} = I$ in the earlier case of step input where we design $F_{0i} = (C_i(-A_i + B_iF_i)^{-1}B_i)^{-1}$ such that $\lim_{s \to 0} \widehat{T}_C(s) = I$. Define

$$\Omega = \lim_{s \to 0} \frac{\left[C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - I \right] 1_N}{s}.$$

The limit is of the form $\frac{0}{0}$ and the L'Hospital's rule can be applied. Using the result obtained from tracking of step input where F_i is designed such that $(A_i + B_iF_i)$ is Hurwitz for all i, applying L'Hospital's rule yields

$$\Omega = \lim_{s \to 0} -C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-2}\widehat{B}\widehat{\mathcal{M}}1_N$$

$$= -C(-A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-2}\widehat{B}\widehat{\mathcal{M}}1_N.$$

The consensus condition reduces to

$$-C(-A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-2}\widehat{B}\widehat{\mathcal{M}}1_N + \psi = 0. \tag{4.46}$$

Using (4.46) we can calculate ψ that is given by

$$\psi = C(-A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-2}\widehat{B}\widehat{\mathcal{M}}1_N. \tag{4.47}$$

Figure 4.5 shows each agent's position as they track the ramp input with an offset term (4.39).
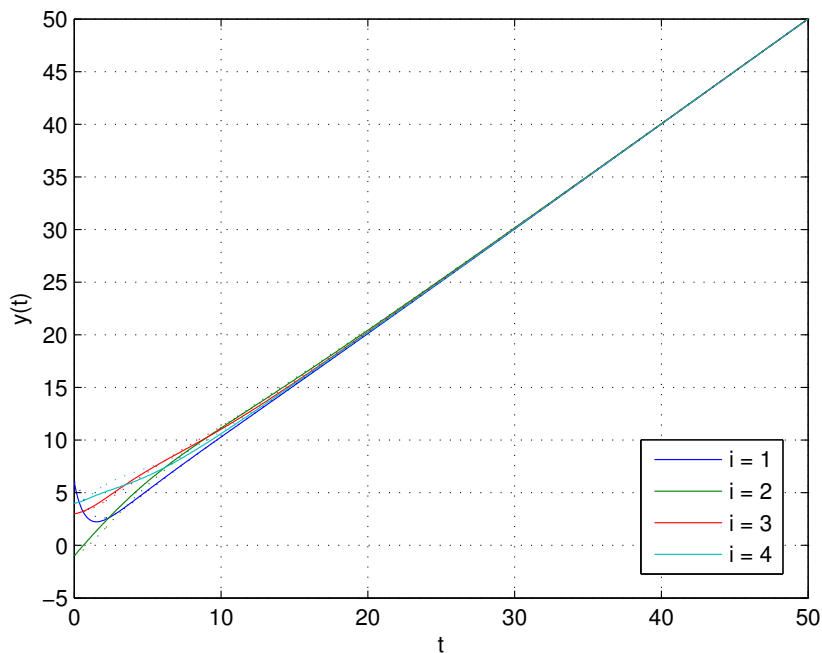

Figure 4.5: Evolution of the output signals tracking a ramp function. Signals are communicated through the graph in Figure 3.1.

Although consensus is achieved the method proposed is not a local and distributed approach to MASs, because of the use of $\widehat{\mathcal{M}}$ matrix in computing the offset vector ψ. To overcome this weakness we propose another method in the following section.

4.4.2 Tracking a Ramp Input - Local and Distributed Approach

This approach assumes that $s = 0$ is a double pole of $P_i(s)$ for all i. We thus have a similar lemma to Lemma 7.

Lemma 8 *Under the assumption that $s = 0$ is a double pole of $P_i(s)$ for all i, tracking of the ramp input with zero steady-state error for the MAS is achievable, if and only if*

(A_i, B_i) *is stabilizable for all* i.

Instead of proving this lemma, we will provide derivation of the consensus result assuming the stabilizability of (A_i, B_i) for all i. Consider the following MAS output under ramp reference plus the offset

$$Y(s) = \widehat{T}_C(s)R(s) = C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\left[\frac{\psi}{s} + 1_N \otimes \frac{1}{s^2}\right].$$

The error of the system is defined by

$$
\begin{aligned}
E(s) &= Y(s) - R(s) \\
&= Y(s) - 1_N \otimes \frac{1}{s^2} \\
&= C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\left[\frac{\psi}{s} + 1_N \otimes \frac{1}{s^2}\right] - \left(1_N \otimes \frac{1}{s^2}\right) \\
&= \frac{[C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}1_N - 1_N]}{s^2} + \frac{C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\psi}{s}. \quad (4.48)
\end{aligned}
$$

In light of the final value theorem, the steady-state error is given by

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}1_N - 1_N}{s} + \lim_{s\to 0} C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\psi$$

Next we consider the following error function in s-domain:

$$s\Delta(s) = C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - C(sI - A + BF)^{-1}\widehat{B}.$$

By noting that $\widehat{B}_i = B_i F_{0i}$ and $\widehat{F}_i = F_{0i}^{-1}F_i$ for each i, there holds

$$
\begin{aligned}
s\Delta(s) &= P(s)F_0[I + \widehat{\mathcal{M}}F_0^{-1}P_F(s)F_0]^{-1}\widehat{\mathcal{M}} - P(s)[I + P_F(s)]^{-1}F_0 \\
&= P(s)[I + F_0\widehat{\mathcal{M}}F_0^{-1}P_F(s)]^{-1}F_0\widehat{\mathcal{M}} - P(s)[I + P_F(s)]^{-1}F_0.
\end{aligned}
$$

Denote $\mathcal{M}_0 = F_0\widehat{\mathcal{M}}F_0^{-1}$. Then

$$s\Delta(s) = P(s)[I + \mathcal{M}_0 P_F(s)]^{-1}\mathcal{M}_0 F_0 - P(s)[I + P_F(s)]^{-1}F_0$$

$$= P(s)P_F(s)^{-1}[I + \mathcal{M}_0^{-1}P_F(s)^{-1}]^{-1}F_0 - P(s)P_F(s)^{-1}[I + P_F(s)^{-1}]^{-1}F_0$$

$$= P(s)P_F(s)^{-1}\left\{[I + \mathcal{M}_0^{-1}P_F(s)^{-1}]^{-1} - [I + P_F(s)^{-1}]^{-1}\right\}F_0.$$

We assume that $P(s)^{-1} \to 0$ and $[sP(s)]^{-1} \to 0$ as $s \to 0$ which is equivalent to $P(s)$ having double pole at the origin. Hence there holds

$$s\Delta(s) = P(s)P_F(s)^{-1}\left\{[I - \mathcal{M}_0^{-1}P_F(s)^{-1}] - [I - P_F(s)^{-1}]\right\}F_0 + o([P_F(s)^{-1}]^2)$$

$$= P(s)P_F(s)^{-1}\left\{I - \mathcal{M}_0^{-1}\right\}P_F(s)^{-1}F_0 + o([P_F(s)^{-1}]^2),$$

where $o\left([P_F(s)^{-1}]^2\right)$ indicates that the term approaches zero in the order of $[P_F(s)^{-1}]^2$, i.e., $o(s^2)$, as $s \to 0$. This is ensured by the double zero eigenvalue of the A matrix. It follows that

$$\lim_{s\to 0}\Delta(s) = P(s)P_F(s)^{-1}\left\{I - \mathcal{M}_0^{-1}\right\}[sP_F(s)]^{-1}F_0 + o([P_F(s)^{-1}]) = 0.$$

The above shows that so long as the plant has a double pole at the origin, there holds

$$\lim_{s\to 0}\frac{C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}} - C(sI - A + BF)^{-1}\widehat{B}}{s} = 0. \qquad (4.49)$$

The equality (4.49) yields the steady state error

$$\lim_{s\to 0}sE(s) = \lim_{s\to 0}\frac{C(sI - A + BF)^{-1}\widehat{B}1_N - 1_N}{s} + \lim_{s\to 0}C(sI - A + \widehat{B}\widehat{\mathcal{M}}\widehat{F})^{-1}\widehat{B}\widehat{\mathcal{M}}\psi$$

$$= \lim_{s\to 0}\frac{C(sI - A + BF)^{-1}BF_0 1_N - 1_N}{s} + \lim_{s\to 0}C(sI - A + BF)^{-1}BF_0\psi, \quad (4.50)$$

74

by $\widehat{B} = BF_0$. Hence we can design F_0 such that

$$\lim_{s \to 0} C_i(sI - A_i + B_iF_i)^{-1}B_iF_{0i} = 1 \ \forall \ i$$

or

$$\lim_{s \to 0} C(sI - A + BF)^{-1}BF_0 = I \Rightarrow F_0 = \left(C(-A + BF)^{-1}B\right)^{-1}$$

that is ensured by the nonzero DC gain for each $P_i(s)$. Substituting F_0 in (4.50) the tracking condition becomes

$$\lim_{s \to 0} \frac{C(sI - A + BF)^{-1}BF_01_N - 1_N}{s} + \psi = 0$$

$$\psi = \lim_{s \to 0} \frac{1_N - C(sI - A + BF)^{-1}BF_01_N}{s}. \qquad (4.51)$$

The limit is of the form $\frac{0}{0}$, applying L'Hospital's rule to calculate the limit, we get

$$\psi = C(-A + BF)^{-2}BF_01_N$$

or

$$\gamma_i = \lim_{s \to 0} \frac{C_i(sI - A_i + B_iF_i)^{-1}B_iF_{0i} - 1}{s}$$

for $i = 1, 2, \cdots, N$. Then the MAS achieves the zero steady-state error to unit ramp reference, if $\psi_i = -\gamma_i$ for $i = 1, 2, \cdots, N$.

The proposed method is local and distributed in nature as the design of each agent is decoupled and does not depend on $\widehat{\mathcal{M}}$. This is crucial as the MAS has the advantage of being scalable. That is, the consensus is achievable even if some agents are removed while new ones are added in. In such situations, $\widehat{\mathcal{M}}$ changes all the time, and thus the local and distributed results for consensus are preferred.

4.4.3 Tracking a Sinusoid Input

The reference model which is required for tracking a sinusoid input is described as

$$\dot{x}_0(t) = A_0 x_0(t) \tag{4.52}$$

where

$$A_0 = \begin{bmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{bmatrix}.$$

Taking Laplace transform of (4.52)

$$s X_0(s) - X_0(0) = A_0 X_0(s). \tag{4.53}$$

Rearranging (4.53) as

$$(sI - A_0) X_0(s) = X_0(0). \tag{4.54}$$

The initial conditions for the states can be chosen to be

$$X_0(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Substituting the initial conditions in (4.54)

$$X_0(s) = (sI - A_0)^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} s & -\omega_0 \\ \omega_0 & s \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} \alpha s + \beta \omega_0 \\ \beta s - \alpha \omega_0 \end{bmatrix}. \tag{4.55}$$

Consider $\alpha = 0$ and $\beta = 1$, we can then write (4.55) as

$$X_0(s) = \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} \omega_0 \\ s \end{bmatrix}. \tag{4.56}$$

Taking inverse Laplace transform of (4.56) we get

$$x_0(t) = \begin{bmatrix} \sin \omega_0 t \\ \cos \omega_0 t \end{bmatrix}.$$

To track the reference signal we need to design the feedback $\underline{F_0} = \begin{bmatrix} a & b \end{bmatrix}$. The reference signal can be written as

$$r(t) = \underline{F_0} x_0(t) = a \sin \omega_0 t + b \cos \omega_0 t. \tag{4.57}$$

Taking Laplace transform of (4.57) we get

$$R(s) = \begin{bmatrix} \dfrac{a\omega_0}{s^2 + \omega_0^2} + \dfrac{bs}{s^2 + \omega_0^2} \end{bmatrix}. \tag{4.58}$$

Lemma 9 *Under the assumption that $s = \pm j\omega$ is a pair of complex conjugate poles on the imaginary axis of $P_i(s)$ for all i, tracking of sinusoid input with zero steady state error for the MAS is achievable, if and only if (A_i, B_i) is stabilizable for all i.*

For tracking of a sinusoid input in a MAS environment we consider the feedforward gain, $F_{0i} = I$. By using the design procedure as discussed in the earlier section $R_{0i} = I$, which in turn yields $\widehat{\mathcal{M}} = \mathcal{M}$. Here it is required that we calculate $\underline{F_0} = \begin{bmatrix} a_i & b_i \end{bmatrix}$ for each agent beforehand. To calculate a_i and b_i we follow the method explained below. The output equation can be written as

$$\begin{aligned}
Y(s) = \overline{T}_C(s)R(s) &= C(sI - A + B\mathcal{M}F)^{-1}B\mathcal{M}\begin{bmatrix} \dfrac{a_i\omega_0}{s^2 + \omega_0^2} + \dfrac{b_i s}{s^2 + \omega_0^2} \end{bmatrix} \\
&= C(sI - A)^{-1}B\left[I + \mathcal{M}F(sI - A)^{-1}B\right]^{-1}\mathcal{M}\begin{bmatrix} \dfrac{a_i\omega_0}{s^2 + \omega_0^2} + \dfrac{b_i s}{s^2 + \omega_0^2} \end{bmatrix}.
\end{aligned}$$

Define

$$\overline{K} = \lim_{s \to j\omega_0} C(sI - A)^{-1}B\left[I + \mathcal{M}F(sI - A)^{-1}B\right]^{-1}\mathcal{M}$$

$$= \lim_{s \to j\omega_0} P(s)\left[I + \mathcal{M}P_F(s)\right]^{-1}\mathcal{M}.$$

We assume that $P(j\omega_0)^{-1} \to 0$ which is equivalent to $P_0(s)$ having double complex pole at the origin. Hence we define $P(s) = P_1(s)P_0(s)$ and $P_F(s) = P_{F1}(s)P_0(s)$. Further we can rewrite

$$\overline{K} = \lim_{s \to j\omega_0} P_1(s)P_0(s)\left[I + \mathcal{M}P_{F1}(s)P_0(s)\right]^{-1}\mathcal{M}$$

$$= \lim_{s \to j\omega_0} P_1(s)\left[P_0(s)^{-1} + \mathcal{M}P_{F1}(s)P_0(s)P_0(s)^{-1}\right]^{-1}\mathcal{M}$$

$$= \lim_{s \to j\omega_0} P_1(s)\left[P_0(s)^{-1} + P_{F1}(s)^{-1}\right]$$

$$= \lim_{s \to j\omega_0} P(s)\left[I + P_F(s)\right]^{-1} = \lim_{s \to j\omega_0} C(sI - A)^{-1}B\left[I + F(sI - A)^{-1}B\right]^{-1}$$

$$= \lim_{s \to j\omega_0} C(sI - A + BF)^{-1}B = \lim_{s \to j\omega_0} T_F(s).$$

From above we can conclude that the design of each agent is decoupled and does not depend on $\mathcal{M}$. Then the output equation can be written as

$$Y_i(s) = C_i(sI - A_i + B_iF_i)^{-1}B_i\left[\frac{a_i\omega_0}{s^2 + \omega_0^2} + \frac{b_is}{s^2 + \omega_0^2}\right]$$

$$= \frac{K_i}{(s - j\omega_0)} + \frac{K_i^*}{(s + j\omega_0)}.$$

Steady state solution is given by $Y_{ss}(t) = 2|K_i|\cos(\omega_0 t + \angle K_i)$. Calculating the partial fractions and taking the limits we can write

$$
\begin{aligned}
2K_i &= \lim_{s \to j\omega_0} 2T_{F_i}(s) \left[\frac{a_i\omega_0}{(s+j\omega_0)} + \frac{b_i j\omega_0}{(s+j\omega_0)} \right] \\
&= 2T_{F_i}(j\omega_0) \left[\frac{a_i\omega_0}{2j\omega_0} + \frac{b_i j\omega_0}{2j\omega_0} \right] \\
&= T_{F_i}(j\omega_0) \left[\frac{a_i + jb_i}{j} \right].
\end{aligned}
$$

For tracking $\cos\omega_0 t$, we set $\angle K_i = 0$

$$
\angle K_i = \angle T_{F_i}(j\omega_0) + \angle [a_i + jb_i] - 90 = 0.
$$

The angle can be calculated to be equal to

$$
\angle [a_i + jb_i] = 90 - \angle T_{F_i}(j\omega_0) = \frac{\pi}{2} - \angle T_{F_i}(j\omega_0), \tag{4.59}
$$

and the magnitude is equal to

$$
|K_i| = |T_{F_i}(j\omega_0)| \sqrt{a_i^2 + b_i^2} = 1 \Rightarrow \sqrt{a_i^2 + b_i^2} = \frac{1}{|T_{F_i}(j\omega_0)|}. \tag{4.60}
$$

Equations (4.59) and (4.60) implies $Y_{ss}(t) = \cos\omega_0 t$.

Hence we can track a sinusoid input by locally generating the sinusoid and cosine functions for each agent. As these functions are generated locally for each agent we will have to provide corresponding reference signals to each of the agents to achieve consensus. Therefore all the agents must receive the reference signal in order to achieve consensus unlike the cases of step and ramp input discussed earlier in this chapter where only one agent was required to receive the reference signal. For simplicity, we have considered $N = 2$ agents whose dynamics are described by second set of agents in Section 3.6. The dynamics of the agents do not have a pair of complex conjugate poles on the imaginary axis, hence

the plant model needs to be augumented with a weighting function equal to $W_i(s) = \frac{1}{s^2+\omega_0^2}$. We assume that $\omega_0 = \pi$, and the graph is strongly connected. The rank condition (3.3) is satisfied by taking $D = \text{diag}(-0.0019, 0.4981)$, $\kappa = 0.1$ and $g_1 = 0.5$ with the rest $g_i = 0$. Figure 4.6 shows each agent's position as they track a sinusoid input.
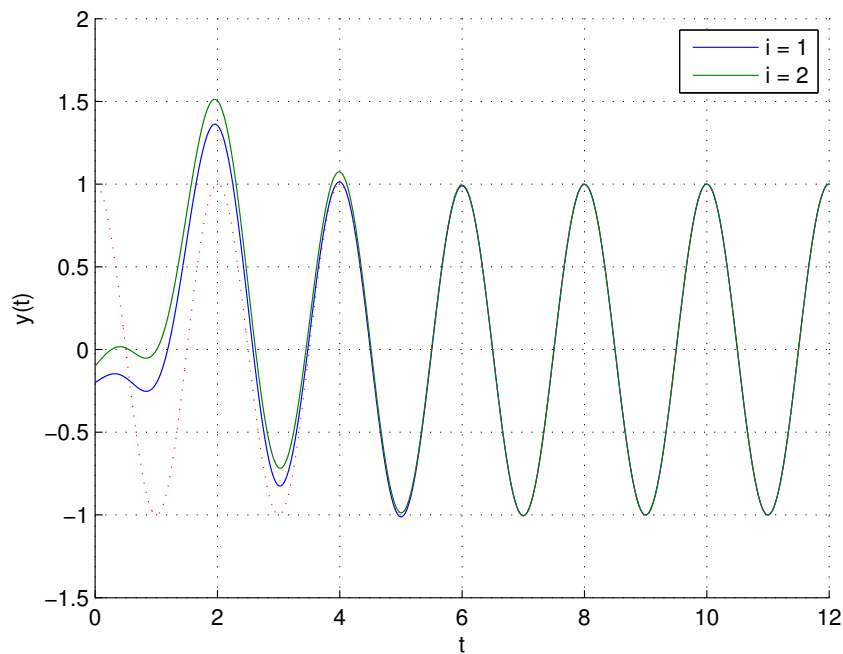

Figure 4.6:   Evolution of the output signals tracking a sinusoid function for $N = 2$.

Chapter 5
Application: Aircraft Traffic Control

5.1   Introduction

Today's air traffic capacity has doubled compared to the last decade and faces many challenges of managing an ever-growing amount of air traffic. Hence it has become a need to update the design of airspace continuously to meet the demands and provide the best, safest and the shortest routes for the increasing number of flights. Currently air traffic control (ATC) provides this service with the help of ground-based controllers who direct the aircraft safely based on traffic separation rules. The rules help the ATC operator to direct the aircraft on the ground and in controlled airspace, and also provide advisory services when it is in the uncontrolled airspace by maintaining a minimum amount of no-fly space around it at all times. This is known as Air Traffic Management (ATM). The aviation industry today has to find a transformational ATM solution which will help to do away with outdated infrastructure and operating techniques. In an attempt to optimize and enhance the efficiency in the aviation industry, advances have been made based on trajectory-based operations for replacing the current clearance-based operations in many parts of the airspace thereby reducing the human interaction in the operations. This shift in technique gives rise to a number of issues for distributed coordination in future ATM systems, as safe separation between aircrafts can only be achieved by coordinated management of the aircraft.

The flight dynamics of an aircraft are nonlinear in nature and can be described by a set of simultaneous second-order differential equations. The mathematical model of an aircraft provides us a way to simulate the aircraft conditions on a computer, which otherwise, would involve high costs if a real aircraft had to be built and tested upon. The detailed derivation of the mathematical model can be found in any aircraft or flight control design textbook [9, 23, 19, 56].

In mathematics, a nonlinear system is the one that does not satisfy the superposition principle. It refers to a set of nonlinear equations used to describe a physical system that cannot be written as a linear combination of the inputs. The theory of nonlinear systems have evolved with time and today is used to describe a great variety of scientific and engineering phenomena. For linear systems, the design of controller for output feedback is a two step process comprising of state feedback and state estimation. Separation principle can be applied for controller design and the two steps can be carried out independently for assigning the closed-loop eigenvalues. The design methods based on separation principle are fundamental and widely used as many techniques are developed based on this principle for the design of linear systems. For the general nonlinear systems, unlike the linear systems, the problem of output feedback control is much more difficult and less understood.

The study of nonlinear problem is important because most of the physical systems are inherently nonlinear in nature. The complexity of nonlinear systems poses a major challenge to the control community and requires design procedures which could meet control objectives for the desired specifications. These real-world control problems can be solved by using the nonlinear design tools presented in [36]. These useful methods include Lie algebra and differential geometry [34], sliding mode control, a Lyapunov redesign - which uses a Lypunov function of a nominal system to design a control component which is robust, backstepping - a recursive process [38] and passivity-based control. Most of these design tools require state feedback although a situation may arise when the control designer chooses not to measure the state variables due to technical or economical reasons. This may create a need to extend design techniques to output feedback. One of the methods used in practice by researchers is the introduction of high-gain observers for a class of nonlinear systems using separation principle [24, 37, 49, 50, 72]. Also certain control problems have an abstract mathematical model which make it difficult to find a property that could correspond to physical energy. These physical systems could be an electrical network or a mechanical machine. The notion of stored energy, passivity and dissipativity are widely

appreciated in the study of such physical systems as these provide a useful tool for the analysis of system behavior [11, 31, 80].

Despite the success of the nonlinear control design methods, the linearization methods are still widely used due to its simplicity and the availability of the many linear design tools. The process involves design by linearizing the system about the desired equilibrium point and then design a stabilizing linear feedback control for the linearized model but this can only guarantee asymptotic stability locally. Various other linearization techniques used are discussed in [36]. Techniques such as gain-scheduling can be used to extend the region of validity of linearization.

5.2 Linearized Aircraft Model

The stability of an aircraft is studied under two different categories i.e. static and dynamic stability. Static stability can be defined as the initial tendency of the aircraft to return to its equilibrium state after a disturbance while dynamic stability is related to actual time history of the systems state as $t \to \infty$. It is important to mention here that static stability does not always imply dynamic stability. However, dynamically stable systems are statically stable. Our focus will be on dynamic stability as it affects the actual motion of the aircraft when a control input is provided or a disturbance is injected. An aircraft is considered as a rigid body with movable surfaces like rudder, elevator and ailerons designed to control it. Usually in the study of flight control systems the equations of motion for an aircraft are derived as a rigid body with three components of translation and three components of rotation, which means that it has six degrees of freedom. For the sake of convenience, reference axis fixed with respect to the earth is considered.

The mathematical basis for the analysis of an aircraft quickened after the first manned flight was made possible by Wright brothers. This caused development in the area of aeronautics to suddenly gain momentum in the early 1900. Many scientists and mathematicians got involved in studying the stability and control problems faced by these early flights. The works of G. H. Bryan (1911) and Frederick Lanchester (1908) are recognized to have laid

the foundations for the subject. Lanchester conducted experiments with hand-launched gliders and was able to identify and describe certain mathematical dynamic characteristics of an aircraft. On the other hand Bryan was the first to develop the general equations of motion for the dynamic stability analysis which are still in use with some modifications. These equations of motion of an aircraft are the foundation on which the entire framework of flight dynamics is constructed. He also recognized that the equations of motion could be separated into symmetric longitudinal motion and an unsymmetric lateral motion. Bryan's mathematical theory was complex and lacked information about the various stability derivatives. In order to determine these stability derivatives experimental studies were carried out on scaled aircraft models inside the wind-tunnel by L. Bairstow and B. M. Jones at the National Physical Laboratory in England. They also showed that under certain assumptions the equations of motion can have two independent solutions, i.e., one longitudinal and one lateral. Around the same time Jerome Hunsaker at Massachusetts Institute of Technology conducted more wind-tunnel studies on scaled models of several flying aircrafts and added valuable information about them and their dynamic stability.

The rigid body equations of motion can be derived by applying Newton's second law. To find the solution for these equations of motion perturbation theory is applied. According to linear theory it is possible to write states as a sum of nominal value and a perturbation. We consider that the aircraft motion consists of small deviations from the equilibrium flight conditions. Perturbation theory is used to linearize the equations and the results thus obtained are decoupled state-space models for longitudinal and lateral motions which have sufficient accuracy for practical engineering purposes. The assumptions made are reasonable provided the aircraft is not undergoing a large amplitude or very rapid maneuver. The kinematic and dynamic equations for an aircraft are summarized in the Table 5.1 where u, v and w are the components of velocity along the x, y and z axes respectively. The angular velocities are denoted by p, q and r and the Euler's angles are denoted by ψ, θ and ϕ. The mass moments of inertia of the body about the x, y and z axes are represented

Table 5.1: Kinematic and dynamic equations for an aircraft.

Force Equations	$m\left(\dot{u} + qw - rv\right) = X - mg\sin\theta$ $m\left(\dot{v} + ru - pw\right) = Y + mg\cos\theta\sin\phi$ $m\left(\dot{w} + pv - qu\right) = Z + mg\cos\theta\cos\phi$
Moment Equations	$L = I_{xx}\dot{p} + qr(I_{zz} - I_{yy})$ $M = I_{yy}\dot{q} + pr(I_{xx} - I_{zz})$ $N = I_{zz}\dot{r} + pq(I_{yy} - I_{xx})$
Attitude Dynamics	$\dot{\theta} = q\cos\phi - r\sin\phi$ $\dot{\phi} = p + \tan\theta(q\sin\phi + r\cos\phi)$ $\dot{\psi} = \sec\theta(q\sin\phi + r\cos\phi)$

by the terms I_{xx}, I_{yy} and I_{zz} respectively. To linearize each of the kinematic and dynamic equations we need to replace the variables by a nominal value plus a perturbation

$$u = u_0 + \Delta u \qquad\qquad v = v_0 + \Delta v \qquad\qquad w = w_0 + \Delta w$$

$$p = p_0 + \Delta p \qquad\qquad q = q_0 + \Delta q \qquad\qquad r = r_0 + \Delta r$$

$$X = X_0 + \Delta X \qquad\qquad Y = Y_0 + \Delta Y \qquad\qquad Z = Z_0 + \Delta Z$$

$$M = M_0 + \Delta M \qquad\qquad N = N_0 + \Delta N \qquad\qquad L = L_0 + \Delta L$$

$$\delta_e = \delta_e + \Delta\delta_{e_0} \qquad\qquad \delta_t = \delta_t + \Delta\delta_{t_0}.$$

For convenience the equilibrium flight condition is assumed to be symmetric and with no angular velocity. This implies that $v_0 = p_0 = q_0 = r_0 = \phi_0 = \psi_0 = 0$. Also the x-axis is set along the direction of aircraft's velocity vector and hence $w_0 = 0$. The aircraft is flying with the speed $u = u_0$ and θ_0 is the reference angle of climb. In our discussion we restrict ourselves only to finding the longitudinal equations. The X-force, Z-force and pitching moment form the longitudinal equations whereas the Y-force, yawing and rolling moment form the lateral equations. The perturbed linearized longitudinal equations of motion are as follows

$$\left(\frac{d}{dt} - X_u\right)\Delta u - X_w\Delta_w + (g\cos\theta_0)\Delta\theta = X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t$$

$$-Z_u \Delta u + \left[(1 - Z_{\dot{w}})\frac{d}{dt} - Z_w\right]\Delta w - \left[(u_0 + Z_q)\frac{d}{dt} - g\sin\theta_0\right]\Delta\theta = Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t$$

$$-M_u \Delta u - \left(M_{\dot{w}}\frac{d}{dt} + M_w\right)\Delta w + \left(\frac{d^2}{dt^2} - M_q\frac{d}{dt}\right)\Delta\theta = M_{\delta_e}\Delta\delta_e + M_{\delta_t}\Delta\delta_t.$$

$$(5.1)$$

- **Longitudinal Aircraft Dynamics**

The perturbed linearized longitudinal equations are simple, ordinary linear differential equations with constant coefficients. These equations can be written in state-space form and represented mathematically as

$$\dot{x} = Ax + B\eta \qquad (5.2)$$

where x is a state vector and η is the control vector defined as follows:

$$x = \begin{bmatrix} \Delta u & \Delta w & \Delta q & \Delta\theta \end{bmatrix}^T \qquad \text{and} \qquad \eta = \begin{bmatrix} \Delta\delta_e \\ \Delta\delta_t \end{bmatrix}.$$

The control or actuator inputs δ_e and δ_t are the change in elevator angle and thrust, respectively. The coefficients of system matrix A are the aerodynamic stability derivatives and the coefficient of input matrix B are the control derivatives of the aircraft. The linearized longitudinal set of equations in (5.1) can be rewritten as

$$\Delta\dot{u} = X_u \Delta u + X_w \Delta w + X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t - (g\cos\theta_0)\Delta\theta$$

$$\Delta\dot{w} = Z_u \Delta u + Z_w \Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta q + Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t + u_0\Delta q - (g\sin\theta_0)\Delta\theta$$

$$\Delta\dot{q} = M_q\Delta q + M_{\dot{w}}\Delta\dot{w} + M_u\Delta u + M_w\Delta w + M_{\delta_e}\Delta\delta_e + M_{\delta_t}\Delta\delta_t$$

$$\Delta\dot{\theta} = \Delta q \qquad (5.3)$$

where $X_{(\cdot)} = \frac{1}{m}\frac{\partial X}{\partial\delta(\cdot)}$, $Z_{(\cdot)} = \frac{1}{m}\frac{\partial Z}{\partial\delta(\cdot)}$ and $M_{(\cdot)} = \frac{1}{I_{yy}}\frac{\partial M}{\partial\delta(\cdot)}$. Putting the equations in state-space form we obtain

$$
A = \begin{bmatrix}
X_u & X_w & 0 & -g\cos\theta_0 \\
Z_u & Z_w & u_0 & -g\sin\theta_0 \\
M_u + M_{\dot{w}}Z_u & M_w + M_{\dot{w}}Z_w & M_q + M_{\dot{w}}u_0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
X_{\delta_e} & X_{\delta_t} \\
Z_{\delta_e} & Z_{\delta_t} \\
M_{\delta_e} + M_{\dot{w}}Z_{\delta_e} & M_{\delta_t} + M_{\dot{w}}Z_{\delta_t} \\
0 & 0
\end{bmatrix}. \tag{5.4}
$$

In our discussion we consider the perturbation equations of longitudinal motion for the Boeing 747 transport aircraft in level flight at an altitude of $40,000$ ft and velocity of 774 ft/sec (Mach number $= 0.8$) [29]. For convenience the effect of wind is ignored. The state-space model is given by

$$
\begin{bmatrix}
\Delta\dot{u} \\
\Delta\dot{w} \\
\Delta\dot{q} \\
\Delta\dot{\theta}
\end{bmatrix}
=
\begin{bmatrix}
-0.003 & 0.039 & 0 & -0.322 \\
-0.065 & -0.319 & 7.74 & 0 \\
0.020 & -0.101 & -0.429 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta u \\
\Delta w \\
\Delta q \\
\Delta\theta
\end{bmatrix}
+
\begin{bmatrix}
0.01 & 1 \\
-0.18 & -0.04 \\
-1.16 & 0.598 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta\delta_e \\
\Delta\delta_t
\end{bmatrix}
$$

where the units are ft, sec and crad. The outputs of interest are aircraft speed Δu and climb rate $\Delta\dot{h}$. The output equation can be written as

$$
\begin{bmatrix}
\Delta u \\
\Delta\dot{h}
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 7.74
\end{bmatrix}
x.
$$

The above state-space model of an aircraft is used in our work for all simulations.

5.3 MAS Approach for Aircraft Traffic Control

We propose to use the above aircraft model to provide a solution based on the MAS approach for local and distributed coordination using the concept of free flight. Free flight is being developed to replace the current ATMs as it transfers the responsibility to the aircraft pilots who will now have the ability to change the trajectory in mid-flight independently. This allows the airspace to be reserved dynamically and automatically in a distributed manner using computer communication to ensure the required safety separation between aircrafts. Each aircraft will coordinate with the neighboring aircraft in achieving free flight as much as possible. Although conflicts may arise in determining the aircraft trajectories which can in turn lead to more new conflicts with the other aircrafts in the space and hence simple changes of flight path may not help and this will increase the chances of potential accidents and also increase flight delays and fuel consumption. To overcome these issues the MAS approach is proposed for ATM using a two-level architecture [20]. In the first level trajectory predictions are made based on the information available about the density of the given airspace. The second level ATM assumes full information about the airspace guarantees trajectory clearance and assures separation in order to avoid any collision. Our objective is to develop design tools that can apply the known results from consensus and develop practical design tools which will help to improve the performance of future flight.

The aircraft model is a MIMO system. Hence we need to satisfy the rank condition in (3.8). The FI distributed protocol in (3.9) is used for the simulation. We can achieve consensus because each aircraft will have access to its own reference along with the information from its neighbors. This will also make the closed loop system more robust in case of communication failure or broken links.

5.4 Simulation Results

We consider two aircrafts for our simulation which are heading towards each other and may collide. The Federal Aviation Administration (FAA) regulations require a vertical separation of 1000 ft between the two aircrafts to maintain safe separation. Each aircraft

acts as an agent in the airspace. The simulation model is set up as shown in the Figure 5.1 where the collective dynamics of the MAS are described by (3.12) and (3.13). Note that we have only two aircrafts for our simulation and hence the graph is strongly connected. Next we establish a rule for the flight path taken by the aircrafts. As soon as the two aircrafts enter in a zone of minimum horizontal separation the developed consensus protocol takes over and change the course of the flight. The aircrafts are cruising at a height of 40000 ft. To change the course of flight we ascend the height of one of the aircrafts by 500 ft while the height of the other aircraft descends by 500 ft. Such a rule will help avoid head-on collision and maintain the FAA regulation of 1000 ft vertical separation. Once they have crossed each other we can bring the aircrafts to their initial height. A step input is given to the collective dynamics of the two aircrafts which causes a change in the flight path. The simulated flight paths in two different views are shown in Figure 5.2 and Figure 5.3.
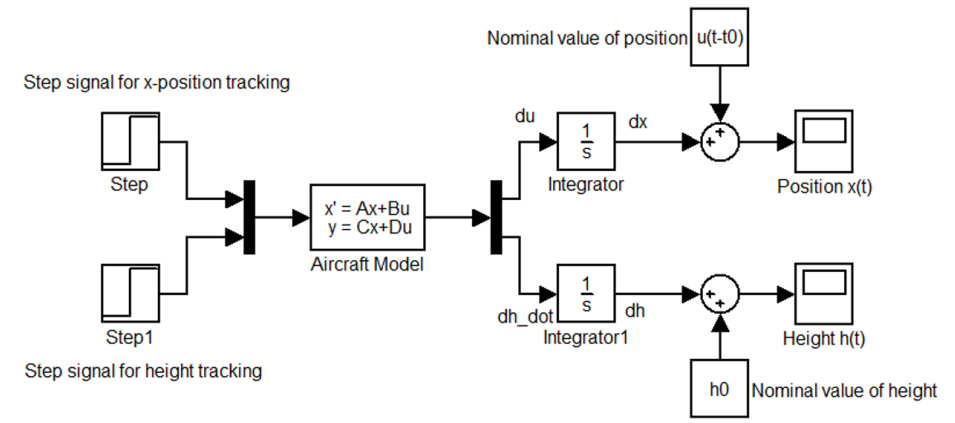

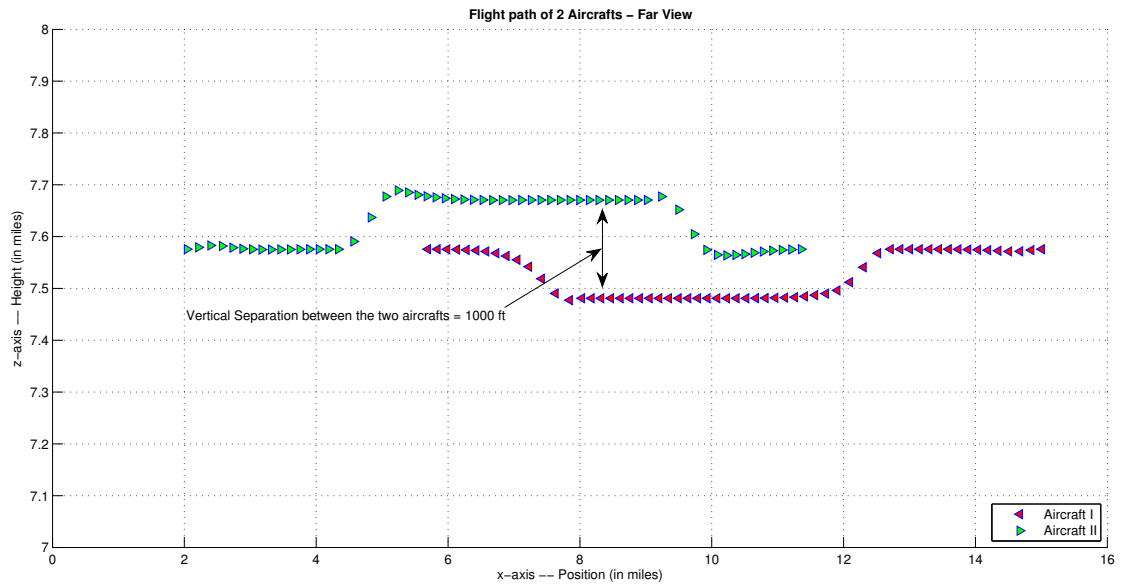Figure 5.1: Block diagram of the simulation model.

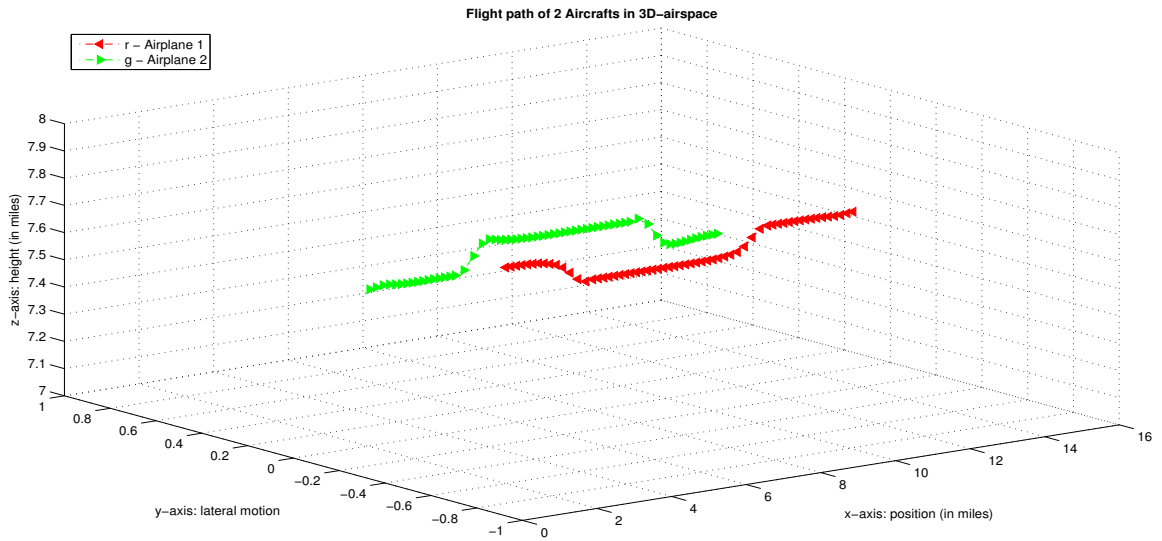Figure 5.2: Flight path of 2 aircrafts: Far View.

Figure 5.3: Simulation of flight phases in 3D-airspace.

Chapter 6
Conclusion and Future Work

We have studied output consensus control for heterogeneous MASs with FI protocol as it makes more sense to have all the states available for consensus control and provide each of them with reference signals. Such an arrangement can provide a more reliable control between communicating agents. We also introduce time delays in the communication topology and show that consensus is independent of the delay lengths. We have shown that under some mild rank condition involving a connected digraph, there exist distributed stabilizing controllers and consensus control protocols for heterogeneous MASs for both systems with or without time delays. The consensus control under communication constraints is also studied here. In order to achieve consensus to a reference trajectory, it is sufficient for one agent to have access to the reference signal, which lowers the communication overhead for the MAS. In addition it is not necessary to duplicate the reference model in each of the N local and distributed feedback controllers, thereby eliminating synchronization of the local reference models commonly required in the existing work for consensus control. Thus the communication cost can be lowered further. Although our work has focused on a fixed topology of feedback graph, we have also studied the switching topology and we have presented a similar result to that in [78]. The controller synthesis is based on $\mathcal{H}_\infty$ loop shaping and LQG/LTR methods, and therefore can accommodate performance and robustness requirements. We have also studied consensus tracking for various reference inputs. Our results show that each agent dynamics need to have modes of the reference model as internal modes in order to achieve the tracking performance and output consensus.

This dissertation also includes application to aircraft traffic control. The air traffic is expected to double in the next decade and this would require advanced ATMs. Today aircraft's are heavily dependent on the ground based air traffic controllers which require humans to operate. The human intervention could become an issue when traffic increases

and there is a need to cooperate between many aircraft's. The next generation ATMs will be trajectory-based and will replace today's clearance based methods. These ATMs would perform majority of operations automatically though humans would still be responsible for handing noncritical operations. This shift in technology would give rise to a number of issues in distributed coordination in ATMs, as separation assurance can be achieved only by coordinated management of aircraft. In order to address the issues of coordination, we proposed an MAS approach. The goal is to allow cooperation between aircraft's to achieve as much free flight as possible subject to safety constraints. The local and distributed coordination methods which were developed for MASs are used to achieve minimum separation between aircrafts.

In the following we would like to provide our view for the future work on MAS.

Time-varying Graphs:

We discussed about consensus of MASs over time-varying graphs in our work. Our result though is under the assumption that graph is more often connected. Such an assumption is too strong. There can be more work done to improve upon it and provide a weaker condition as in [53, 67, 68, 78].

Non-cooperative Consensus Control for Aircrafts:

To provide conflict detection and resolution we would like to extend our results to non-cooperative consensus control of aircrafts. As we have considered FI protocol and provided reference signal to each agent we need to modify our current protocol such that it can provide an offset based on the output measurements. Such a modification can allow many aircrafts to fly in the airspace in any direction with required safety separation between them and keeping the distance to destination minimum.

Consensus Control for Discrete-time Heterogeneous MASs:

The distributed output feedback of a heterogeneous MASs, consisting of N different continuous-time linear dynamical systems which satisfies the positive real condition was studied in this dissertation. There is a need to study and provide an approach for output feedback of discrete-time heterogeneous systems. Our method based on positive real condition is difficult to be extended to discrete-time MASs. This is because a strictly proper transfer function cannot be positive real. Although there has been significant work in the past for studying the positive real condition for discrete-time systems [33, 83] but none of them have considered a strictly proper transfer function. It will be interesting to derive results similar to the case of continuous-time systems from the dissertation where the reference input $r(t)$ which is to be tracked is piecewise step, ramp, sinusoidal etc and the MASs are connected with an underlying graph could have either fixed or switching topology. The discrete-time dynamic agents can be networked over the communication topology represented by an undirected graph or directed graph. The aim would be to achieve consensus which can accommodate performance and robustness requirements.

Consensus under Communication Constraints:

The communication topology considered in our work is under the assumption that there are no packet drops. But in reality due to network congestion, fading and faulty network hardware or drivers there will be packet losses, considering that the MASs are network centric and digital data are often employed for transmitting and receiving signals. It will be interesting to extend our discussion on consensus under communication constraints in future by addressing the issues due quantization error and packet drops.

Nonlinear Multi-agent Systems:

Most of the physical systems are nonlinear in nature and hence there is a need to study nonlinear MASs. Nonlinear systems are complex and require design procedures to meet

the control objectives. To study nonlinear MASs it will be advantageous to identify a class of nonlinear systems to which the separation principle can be applied. In past researchers have achieved separation by using various techniques such as the use of high-gain observers, bilinear approach, backstepping methods, etc. The known approaches to achieve synchronization behavior for nonlinear MASs are based on the concepts of dissipativity and passivity. The future work should relax these assumptions and meet the specified performance objectives by developing more practical design methods for consensus control of nonlinear systems.

MASs involving Uncertainty:

The discussion on robust analysis in Section 4.1.3 showed that the distributed controllers with distributed observers may obscure the robustness for the controllers designed by both $\mathcal{H}_\infty$ loop shaping and the LQG/LTR methods. The stability margin given by

$$\delta_{\max} = \left(\|\tilde{T}_{dz}\|_{\mathcal{H}_\infty} \right)^{-1}$$

can be arbitrarily small. Our calculations indicate that for the example with $N = 4$ considered in this dissertation there is a 90% reduction in robustness. The concern here is that for a very large N this can be diminishing to zero. Robust consensus control is a challenge and should be studied in the future.

References

[1] M. Araki and B. Kondo, "Stability and transient behavior of composite nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 17, no. 4, pp. 537-541, 1972.

[2] Alvergue, L.; Pandey, A.; Gu, G.; Chen, X., "Output consensus control for heterogeneous multi-agent systems," *Proc. IEEE Conf. Decision and Control* , pp.1502-1507, Italy, 2013.

[3] W. Abbas and M. Egerstedt, "Distribution of agents in heterogeneous multiagent systems," *Proc. IEEE Conf. Decision and Control*, pp. 976-981, Orlando, FL, 2011.

[4] B. D. O. Anderson and J. B. Moore, *Optimal Control – Linear Quadratic Methods*, Prentice-Hall, Englewoods Cliffs, NJ, 1990.

[5] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis – A Modern System Theory Approach*, Prentice-Hall, Englewoods Cliffs, NJ, 1973.

[6] B. Bollobas, *Modern Graph Theory*, Springer, 1998.

[7] J. G. Bender, "An overview of systems studies of automated highway systems," *IEEE Trans. Veh. Technol.*, vol. 40, pp. 82-99, 1991.

[8] J. A. Bondy and U. S. R. Murty, *Graph Theory*, vol. 244, New York: Springer-Verlag, 2008.

[9] A. E. Bryson, *Control of Spacecraft and Aircraft* Princeton, N.J: Princeton University Press, 1994.

[10] C. I. Byrnes, F. Delli Priscoli, and A. Isidori, "Output regulation of uncertain nonlinear systems," *Systems and Control: Foundations and Applications*, Birkhöuser, 1997.

[11] C. I. Byrnes, A. Isidori, and J. C. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 36, no. 11, pp. 1228-1240, 1991.

[12] R. W. Beard, T. W. McLain, M. A. Goodrich, and E. P. Anderson, "Coordinated target assignment and intercept for unmanned air vehicles," *IEEE Trans. Robot. Autom.*, vol. 18, no. 6, pp. 911-922, 2002.

[13] L. E. Buzogany, M. Pachter, and J. J. d'Azzo, "Automated control of aircraft in formation flight," *Proc. AIAA Conf. Guidance, Navigation, and Control*, pp. 1349-1370, 1993.

[14] R. Burns, C. A. McLaughlin, J. Leitner, M. Martin, "TechSat21: Formation design, control and simulation," *Proc. IEEE Aerospace Conf.*, pp. 19-25, 2000.

[15] S. Di Cairano, A. Pasini, A. Bemporad, and R. M. Murray "Convergence properties of dynamic agents consensus networks with broken links," *Proc. American Control Conference*, pp. 1362-1367, Seattle, WA, 2008.

[16] J. R. Carpenter, "Decentralized control of satellite formations," *Int. J. Robust Nonlinear Control*, vol. 12, no. 2/3, pp. 141-161, 2002.

[17] F. R. K. Chung, "Spectral Graph Theory, ser. Regional Conference Series in Mathematics," *Conference Board of the Mathematical Sciences*, vol. 92, 1997.

[18] P. A. Cook, "On the stability of interconnected systems," *Int. J. Control*, vol. 20, pp. 407-415, 1974.

[19] M. V. Cook, *Flight Dynamic Principles*, Butterworth-Heinemann, Oxford, 2007.

[20] P. DellOlmo and G. Lulli, "A New Hierarchical Architecture for Air Traffic Management: Optimization of Airway Capacity in a Free Flight Scenario," *European Journal of Operational Research*, vol. 144, pp. 179193, 2003.

[21] R. Diestel, *Graph Theory, ser. Grad. Texts in Math.*, New York: Springer-Verlag, vol. 173, 1997.

[22] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, New York: Academic Press, 1975.

[23] B. Etkin and L.D. Reid, *Dynamics of Flight: Stability and Control*, Wiley New York, 1996.

[24] F. Esfandiari and H. K. Khalil, "Output feedback stabilization of fully linearizable systems," *Int. J. of Contr.*, vol. 56, no. 5, pp. 1007-1037, 1992.

[25] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Automat. Contr.*, vol. 49, no. 9, pp. 1465-1476, 2004.

[26] B. A. Francis and W. M. Wonham, "The internal model principle of control theory," *Automatica*, Volume 12, Issue 5, Pages 457-465, 1976.

[27] H. Grip, T. Yang, A. Saberi, and A. Stoorvogel, "Output synchronization for heterogeneous networks of non-introspective agents," *Automatica*, vol. 48, pp. 2444-2453, 2012.

[28] G. Gu, "On the existence of linear optimal control with output feedback," *SIAM J. Contr. and Optimiz.*, vol. 28, pp. 711-719, 1990.

[29] R. K. Heffley and W. F. Jewell, "Aircraft Handling Qualities," *Systems Technology Inc. Tech. Report 1004-1*, Hawthrone, California, 1972.

[30] J. Huang, *Nonlinear Output Regulation: Theory and Applications*, SIAM Publisher, 2004.

[31] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Trans. Automat. Contr.*, vol. 21, no. 5, pp. 708-711, 1976.

[32] A.R. Horn and R.C. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[33] L. Hitz and B. D. O. Anderson, "Discrete positive-real functions and their application to system stability," *Proc. IEE*, vol. 116, no. 1, pp. 153-155, 1969.

[34] A. Isidori, *Nonlinear Control Systems*, Springer, 1995.

[35] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automat. Contr.*, vol. 48, no. 6, pp. 988-1001, 2003.

[36] H. K. Khalil, *Nonlinear Systems*, Prentice Hall PTR, 2002.

[37] H. K. Khalil and F. Esfandiari, "Semiglobal stabilization of a class of nonlinear systems using output feedback," *IEEE Trans. Automat. Contr.*, vol. 38, no. 9, pp. 1412-1415, 1993.

[38] M. Krstic, I. Kanellalopoulos and P. Kokotovió, *Nonlinear and Adaptive Control Design*, Wiley, 1995.

[39] H. Kim, H. Shim, and J. H. Seo, "Output consensus of heterogeneous uncertain linear multi-agent systems," *IEEE Trans. Automat. Contr.*, vol. 56, no. 1, pp. 200-206, 2011.

[40] H.W. Knobloch, A. Isidori, and D. Flockerzi, Dietrich, *Topics in Control Theory* (Chapter 1: The problem of output regulation), Birkhäuser, 1993.

[41] J. R. T Lawton, R. W. Beard, and B. J. Young, "A decentralized approach to formation maneuvers," *IEEE Trans. Robotics and Automation*, vol. 19, no. 6, pp. 933-941, 2003.

[42] N. A. Lynch, *Distributed Algorithms*, Morgan Kaufmann, California, 1997.

[43] G. Lafferriere, A. Williams, J. Caughman, and J. Veerman, "Decentralized control of vehicle formations," *Syst. & Contr. Lett.*, vol. 54, no. 9, pp. 899-910, 2005.

[44] D. Lee and M. W. Spong, "Stable flocking of multiple inertial agents on balanced graphs," *IEEE Trans. Automat. Contr.*, vol. 52, no. 8, pp. 1469-1475, 2007.

[45] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of multi-agent systems and synchronization of complex networks: A unified viewpoint," *IEEE Trans. on Circuits Syst. I*, vol. 57, no. 1, pp. 213-224, 2010.

[46] Z. Lin, B. Francis, and M. Maggiore, "Necessary and suficient graphical conditions for formation control of unicycles," Dept. ECE, Univ. Toronto, Toronto, ON, Canada.

[47] J. Lunze, "Synchronization of heterogeneous agents," *IEEE Trans. Automat. Contr.*, vol. 57, no. 11, pp. 2885-2890, 2012.

[48] C. Ma and J. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems," *IEEE Trans. Automat.Contr.*, vol. 55, no. 5, pp. 1263-1268, 2010.

[49] M. Maggiore and K. M. Passino, "A separation principle for a class of non-UCO systems," *IEEE Trans. Automat. Contr.*, vol. 48, no. 7, pp. 1122-1133, 2003.

[50] N. A. Mahmoud and H. K. Khalil, "Asymptotic regulation of minimum phase nonlinear systems using output feedback," *IEEE Trans. Automat. Contr.*, vol. 41, no. 10, pp. 1402-1412, 1996.

[51] D.C. McFarlane and K. Glover, *Robust Controller Design using Normalized Coprime Factor Plant Descriptions*, Springer Verlag, Lecture Notes in Control and Information Sciences, vol. 138, 1989.

[52] L. Moreau, "Stability of continuous-time distributed consensus algorithms," *Proc. IEEE Conf. Decision and Control*, pp. 3998-4003, Nassau, 2004.

[53] L. Moreau, "Stability of continuous-time distributed consensus algorithms," *arXiv:math/0409010 [math.OC]*, 2004.

[54] L. Moreau, "Stability of multiagent systems with time-dependent communication links," *IEEE Trans. Automat. Contr.*, vol. 50, no. 2, pp. 169-182, 2005.

[55] R. Merris, "A survey of graph Laplacians," *Linear Multilinear Alg.*, vol. 39, pp. 19-31, 1995.

[56] R. C. Nelson, *Flight Stability and Automatic Control*, McGraw-Hill, New York, 1989.

[57] R. Olifati-Saber, "Flocking for multi-agent dynamic agents: Algorithms and theory," *IEEE Trans. Automat. Contr.*, vol.51, no.3, pp. 401-420, 2006.

[58] R. Olfati-Saber and R. M. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems," *Proc. IEEE*, vol. 3, pp. 2965-2971, 2002.

[59] R. Olfati-Saber and J. S. Shamma, "Consensus filters for sensor networks and distributed sensor fusion," *Proc. IEEE*, pp. 6698-6703, Seville, Spain, 2005.

[60] R. Olfati-Saber and P. Jalalkamali, "Coupled distributed estimation and control for mobile sensor networks," *IEEE Trans. Automat. Contr.*, vol. 57, no. 10, pp. 2609-2614, 2012.

[61] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215-233, 2007.

[62] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automat.Contr.*, vol. 49, no. 9, pp. 1520-1533, 2004.

[63] R. Olfati-Saber and R. Murray, "Flocking with obstacle avoidance: cooperation with limited communication in mobile networks," *Proc. IEEE*, vol. 2, pp. 2022-2028, 2003.

[64] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," IEEE Trans. Automat.Contr., vol. 50, no. 5, pp. 655-661, 2005.

[65] H.H. Rosenbrock, "Lyapunov functions with applications to some nonlinear physical systems," *Automatica*, vol. 1, pp. 31-53, 1963.

[66] H.H. Rosenbrock, "A Lyapunov function for some naturally-occurring linear homogeneous time-dependent equations," *Automatica*, vol. 1, pp. 97-109, 1963.

[67] Y. Su and J. Huang, "Stability of a Class of Linear Switching Systems with Applications to Two Consensus Problems," *IEEE Trans. Automat. Control*, vol.57, no.6, pp.1420-1430, 2012.

[68] L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," *Automatica*, Vol. 45, Issue 11, pp. 2557-2562, 2009.

[69] D. Swaroop and J. K. Hedrick,"Constant spacing strategies for platooning in automated highway systems,"*ASME J. Dyna. Syst., Measure., Control*, vol. 121, pp. 462-470, 1999.

[70] S. Sternberg, *Lie Algebras*, Orange Grove Books, 2009.

[71] H. Schaub, S. R. Vadali, J.L. Junkins, and K. T. Alfriend, "Spacecraft formation flying control using mean orbital elements," *J. Astronaut. Sci.*, vol. 48, no. 1, pp. 69-87, 2000.

[72] A. Teel and L. Parly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Syst. Control Lett.*, vol. 22, pp. 313-325, 1994.

[73] S. E. Tuna, "Synchronizing linear systems via partial-state coupling," *Automatica*, vol. 44, no. 8, pp. 2179-2184, 2008.

[74] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Trans. Automat.Contr.*, vol. 52, no. 5, pp. 863-868, 2007.

[75] Y. Tian and Y. Zhang, "High-order consensus of heterogeneous multi-agent systems with unknown communication delays," *Automatica*, vol. 48, pp. 1205-1212, 2012.

[76] S. Tonetti and R. M. Murray, "Limits on the network sensitivity function for homogeneous multi-agent systems on a graph," *Proc. American Control Conference*, pp. 3217-3222, Baltimore, MD, 2010.

[77] P. Wieland and F. Allgöwer, "An internal model principle for consensus in heterogeneous linear multi-agent systems," *1st IFAC Workshop on Estimation and Control of Networked Systems*, vol. 1, part 1, Artigianelli, Italy, 2009.

[78] P. Wieland, R. Sepulchre, and F. Allgöwer, "An internal model principle is necessary and sufficient for linear output synchronization," *Automatica*, vol. 47, pp. 1068-1074, 2011.

[79] R. Wei, K. Moore, and C. Yang-Quan, "High-order consensus algorithms in cooperative vehicle systems," *Proc. IEEE*, pp.457-462, 2006.

[80] J. C. Willems, "Dissipative dynamical systems Part I: General Theory; Part II: Linear systems with quadratic supply rates," *Arch. Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321-393, 1972.

[81] J. C. Willems, "Lyapunov functions for diagonally dominant systems," *Automatica*, vol. 12, pp. 519-523, 1976.

[82] J. D. Wolfe, D. F. Chichka, and J. L. Speyer, "Decentralized controllers for unmanned aerial vehicle formation fight," *Proc. AIAA Conf. Guidance Navigation, and Control*, AIAA Paper 96-3833, 1996.

[83] C. Xiao and D. J. Hill, "Generalizations and new proof of the discrete-time positive real lemma and bounded real lemma," *IEEE Transactions on Circuits Syst. I*, vol. 46, no. 6, pp. 740-743, 1999.

[84] H. Yamaguchi, "A cooperative hunting behavior by mobile robot troops," *Int. J. Robot Res.*, vol. 18, no. 9, pp. 931-940, 1999.

[85] H. Yamaguchi, T. Arai, and G. Beni, "A distributed control scheme for multiple robotic vehicles to make group formations," *Robot. Auton. Syst.*, vol. 36, no. 4, pp. 125-147, 2001.

[86] Z. Zhang and J.S. Freudenberg, "Loop transfer recovery for nonminimum phase plants," *IEEE Trans. Automat. Contr.*, AC-35:547-553, May 1990.

[87] H. Zhang, F. L. Lewis, and A. Das, "Optimal design for synchronization of cooperative systems: State feedback, observer and output feedback," *IEEE Trans. Automat. Contr.*, vol. 56, no. 9, pp. 1948-1952, 2011.

Appendix A
Algebraic Graph Theory

Consider the graph $\mathcal{G}$ description in Section 1.4. The graph can be defined as time-varying or time-invariant (fixed). In this section we provide the terminologies used in graphs and selected properties of graph theory. There are many references available to study graph theory, we use the following for our discussion [6, 8, 21, 46], along with some online resources cited later in the section.

A.1 Terminologies

A. Undirected Graph: An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a finite set of $\mathcal{V}$ nodes and a set $\mathcal{E}$ of unordered pairs (v_i, v_j) where $v_i, v_j \in \mathcal{V} : v_i \neq v_j$. By definition, mathematical sets are unordered. This means the set $\{a, b\}$ is the same as the set $\{b, a\}$, and so the edges have no direction. See Figure $A.1$.

B. Directed Graph or Digraph: Contrary to a undirected graph, a graph in which the edges have a direction is called a directed graph. The set of edges $\mathcal{E}$ is a set of ordered pairs of elements $\mathcal{V}$; we write an ordered pair as (u, v) which is different from (v, u). See Figure $A.1$.

Figure A.1:   a) Left: Undirected Graph: $\mathcal{V} = \{1, 2, 3, 4\}$, $\mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 4)\}$. b) Right: Directed Graph: $\mathcal{V} = \{1, 2, 3, 4\}$, $\mathcal{E} = \{(1, 3), (2, 1), (1, 4), (2, 4)\}$.

C. In-degree and Out-degree of Directed Graphs: In a directed graph, number of edges directed into a vertex is called the in-degree of the vertex, and the number of edges directed out is called the out-degree.

D. Incident: If v_1 and v_2 are vertices and (v_1, v_2) where $(v_1 \neq v_2)$ is an arc then this arc is

said to be incident on v_1 and v_2.

E. Adjacent: The vertices of the graph v_1 and v_2 are said to be adjacent if they are joined by an edge.

F. Weighted Graph: A weighted graph $\mathcal{G}$ is one where each arc $(v_i, v_j) \in \mathcal{E}$ has associated with its weight w_{ij}.

G. Subgraph: A graph $\mathcal{G}' = (\mathcal{V}, \mathcal{E})$ is a subgraph of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $\mathcal{G}'$ is a graph, $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{E}' \subset \mathcal{E}$.

H. Walk: A walk is like a path except that there is no restriction on the number of times a node can be visited. A walk of length r in a digraph is a sequence of nodes $\{v_0, v_1, \ldots, v_r\}$, where a node may appear more than once. A path is a kind of walk with no repeated nodes. A walk with no repeated edges (but not necessarily all the vertices) is called a tour/trail. The terms directed walk and directed path have the expected meanings. Also if $v_0 = v_r$, then the walk is closed. See Figure $A.2$.
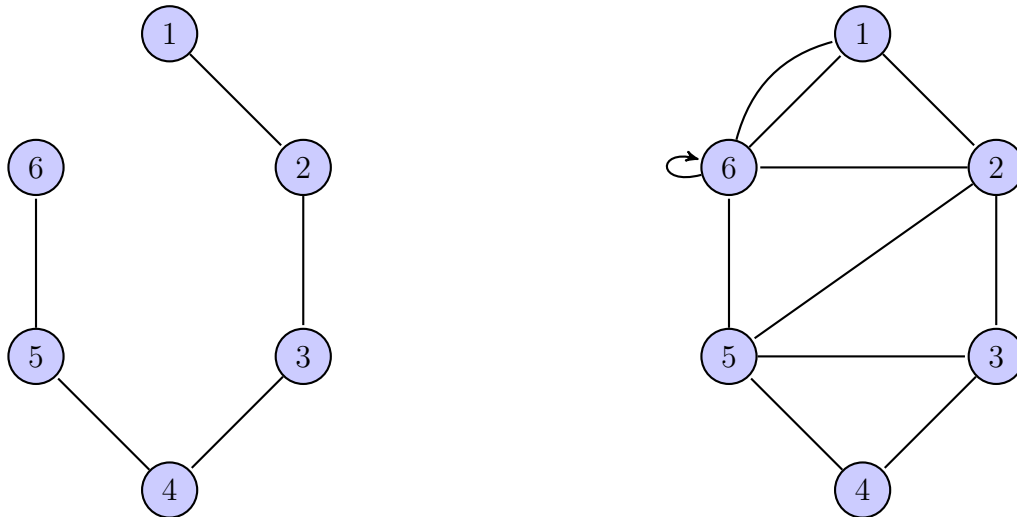

Figure A.2: a) Left: Example of Walk of length $r = 6$ in an graph. $1 \to 2 \to 3 \to 4 \to 5 \to 6$. b) Right: Example of Trail: Walk of $2 \to 6 \to 6 \to 5 \to 3 \to 4 \to 5$. Since the vertices $\{6, 5\}$ both occur twice. c) Right: Example of Path: Walk of $2 \to 3 \to 4 \to 5 \to 6$.

I. Connected Graph: Connected graph as defined earlier exists if $v_i \to v_j \ \forall \ j \in \mathcal{N}$, then v_i is called a connected node in $\mathcal{G}$. The digraph $\mathcal{G}$ is called connected if there exists a connected node in $\mathcal{G}$. Refer Figure 1.1. For an undirected graph, a connected graph is defined as a

graph where for any two nodes v_i and v_j we can find a walk which begins at v_i and ends at v_j.

J. Strongly Connected Graph: The digraph is called strongly connected if $v_i \rightarrow v_j$ and $v_j \rightarrow v_i \ \forall \ i, j \in \mathcal{N}$. A digraph with at least two nodes is strongly connected if and only if each node is globally reachable. It is weakly connected if there exists an undirected path between any two distinct nodes of $\mathcal{G}$. See Figure 1.1.

K. Bipartite Graphs: A graph $\mathcal{G}$ is bipartite if the vertex set of $\mathcal{G}$ can be partitioned into at most 2 independent sets.

L. Reachable Node: If there is a path in $\mathcal{G}$ from node v_i to node v_j, then v_j is said to be reachable from v_i, denoted as $v_i \rightarrow v_j$, else v_j is not reachable from v_i, denoted as $v_i \nrightarrow v_j$.

M. Globally Reachable Node: If a node v_i is reachable from every other node in the digraph then it is called globally reachable node. See Figure $A.3$.
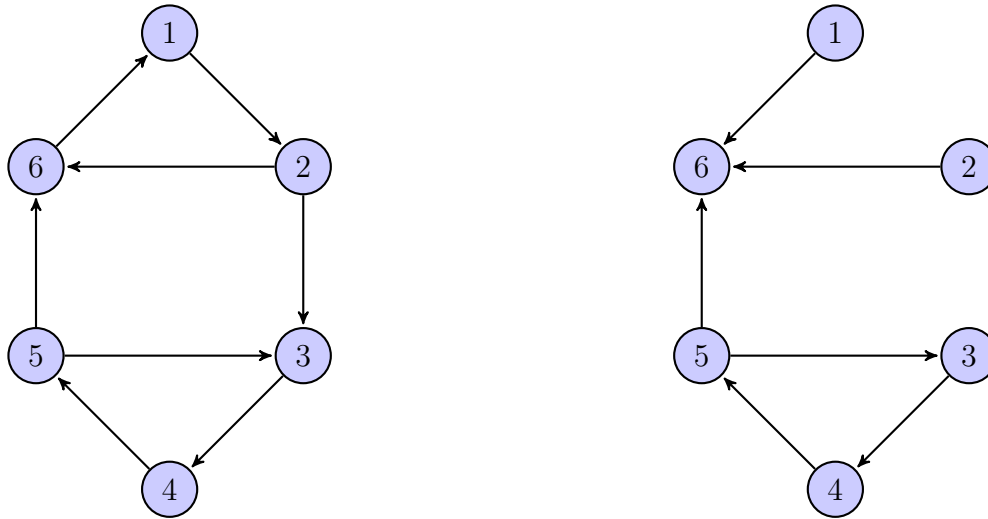

Figure A.3: Examples of Globally Reachable Node Sets. a) Left: $\{1, 2, 6\}$. b) Right: $\{6\}$.

N. Trees: A graph which does not contain a cycle is called acyclic, or a forest. A connected acyclic graph is called a tree. The edges of the tree are called branches. A graph is a tree if and only if there is exactly one path between every pair of its vertices. If removal of anyone of the edges from the graph disconnects it then such a graph is called minimally connected. A graph is a tree if and only if it is minimally connected. A directed tree is a

digraph where except the root node every other node has exactly one parent.

O. Spanning Tree: A spanning tree of a connected graph is a subtree which includes all the vertices of the graph. Alternatively, a spanning tree for a directed graph can be defined as a directed tree formed by graph edges that connect all the vertices of the graph. See Figure A.4. Every connected graph has at least one spanning tree.
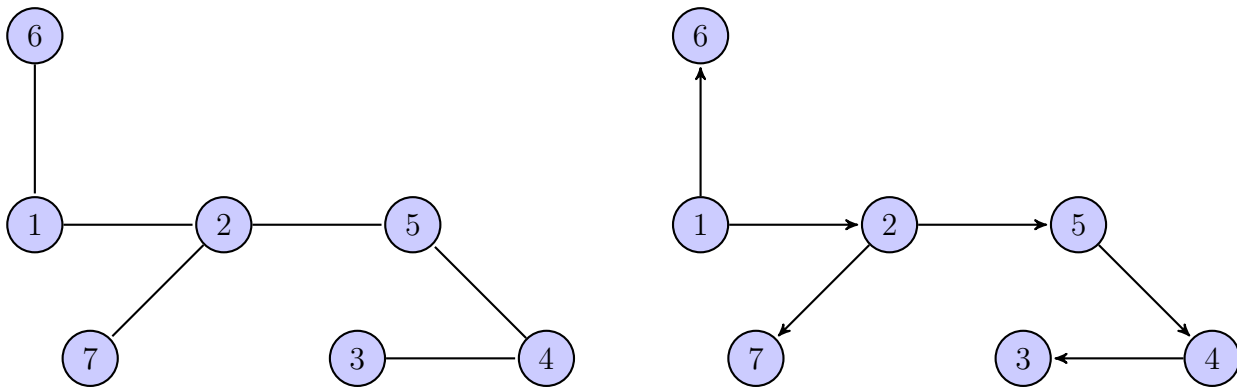

Figure A.4: a) Left: Example of Spanning Tree for Undirected graph. b) Example of Spanning Tree for digraph which is equivalent to the case that there exists a node having a directed path to all other nodes. Node 1 has a directed path to all other nodes.

A.2 Matrices Associated with Graphs

A stochastic matrix is used to describe a Markov chain. Therefore they are also called as Markov matrices. All entries are nonnegative real numbers. There are several types of stochastic matrices. A nonnegative square matrix consisting of real numbers is called row stochastic matrix, if all row sums are equal to one. A nonnegative square matrix consisting of real numbers is called column stochastic matrix, if all column sums are equal to one. A doubly stochastic matrix is a square matrix of nonnegative real numbers with each row and column summing to one.

Laplacian matrix $\mathcal{L}$ was earlier defined in Section 1.4. The properties of Laplacian matrices can be found in [17, 55, 58]. Some of them are discussed here.

Example 1 *Consider the digraph in Figure A.3a as an example with weights associated with each edge equal to 1. Then we can write the adjacency matrix $\mathcal{A}$, degree matrix $\mathcal{D}$ and*

the Laplacian matrix $\mathcal{L}$ as follows

$$
\mathcal{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\qquad
\mathcal{D} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

$$
\mathcal{L} = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 0 & 2 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\tag{A.1}
$$

The eigenvalues of $\mathcal{L}$ are $2 \pm i, 1, 1, 0, 2$.

It is clear that $\mathcal{L}1_N = 0$ and thus it has at least one zero eigenvalue. It is also known that $\mathrm{Re}\{\lambda_i(\mathcal{L})\} \geq 0 \; \forall \; i$. In fact the only eigenvalues of the Laplacian matrix on the imaginary axis are zero in light of the Gershgorin circle theorem. In addition zero is a simple eigenvalue of $\mathcal{L}$, if $\mathcal{G}$ is a connected digraph. Also the Laplacian matrix is a semi M-matrix.

Perron-Frobenius Theorem:

A non-negative matrix square A is called primitive if there is a k such that all the entries of A^k are positive. It is called irreducible if for any i, j there is a $k = k(i,j)$ such that $(A^k)_{ij} > 0$. Let A be a $n \times n$ matrix which has all its entries nonnegative ($A \geq 0$) and is irreducible, that is, the digraph of matrix A is strongly connected. Then the following statements hold true for A [70]

1) $\rho(A) > 0$;

2) A has a positive eigenvector $x > 0$ corresponding to $\rho(A)$;

3) $\rho(A)$ is a simple eigenvalue of A.

Lemma 10 *If $\mathcal{G}$ is strongly connected, then the degree matrix, $\mathcal{D}$, is invertible.*

Proof: If $\mathcal{G}$ is strongly connected, then v_i must have at least one edge ending at v_i, therefore $\deg_i > 0 \ \forall \ i$. Recall that $\mathcal{D} = \text{diag}\{\deg_1, \cdots, \deg_N\}$, therefore $\mathcal{D}$ is invertible.

Lemma 11 *If A is nonnegative and row stochastic, then $\rho(A) = \max\limits_{1 \leq i \leq n} |\lambda_i(A)| = 1$.*

Proof: Consider $Ax = \lambda x$ with A nonnegative and row stochastic. Notice that each component of Ax is of the form

$$(Ax)_i = a_{i,1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$$

with $\sum\limits_{j=1}^n a_{ij} = 1 \ \forall \ i$ and $a_{ij} \geq 0 \ \forall \ i, j$. Set $x_{\max} = \max\limits_{1 \leq i \leq n} |x_i|$, then, for all i,

$$|(Ax)_i)| \leq |a_{i1}x_1| + |a_{i2}x_2| + \cdots + |a_{in}x_n|$$

$$\leq a_{i1}x_{\max} + a_{i2}x_{\max} + \cdots + a_{in}x_{\max}$$

$$= x_{\max}$$

since A is row stochastic. Now suppose that $|\lambda| > 1$ exists, then for some i

$$|\lambda x_i| = |\lambda| x_{\max} > x_{\max}.$$

Since

$$Ax = \lambda x \Rightarrow |(Ax)_i| = |\lambda x_i| \ \forall \ i,$$

we have the contradiction that $|(Ax)_i| \leq x_{\max}$ and $|\lambda x_i| > x_{\max}$ for some i. Therefore $|\lambda| \leq 1$. Now we just have to show that $\lambda = 1$ is and eigenvalue of A to conclude that

106

$\rho(A) = 1$. Indeed

$$A1_n = 1_n$$

so $x = 1_n$, $\lambda = 1$ is an eigenpair. Therefore $\rho(A) = 1$.

Lemma 12 *If $\mathcal{G}$ is strongly connected, then 0 is a simple eigenvalue of $\mathcal{L}$.*

Proof: By Lemma 10 we know that $\mathcal{D}^{-1}$ exists. Define $\tilde{\mathcal{L}} = \mathcal{D}^{-1}\mathcal{L} = I - \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} = \mathcal{D}^{-1}\mathcal{A}$. Since $\tilde{a}_{ij} = \frac{1}{\deg_i}a_{ij}$, $\tilde{\mathcal{A}}$ has the same zero and non-zero entries as $\mathcal{A}$ and so $\mathcal{G}\left(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}\right)$ is also strongly connected. Furthermore, $\tilde{\mathcal{A}}$ is row stochastic since

$$\sum_{j=1}^{N} \tilde{a}_{ij} = \sum_{j=1}^{N} \frac{a_{ij}}{\deg_i} = \sum_{j=1}^{N} \frac{a_{ij}}{\sum\limits_{j=1}^{N} a_{ij}} = 1 \; \forall \; i.$$

By Lemma 11, $\rho\left(\tilde{\mathcal{A}}\right) = 1$ and by the Perron-Frobenius theorem, $\rho\left(\tilde{\mathcal{A}}\right)$ is a simple eigenvalue. Since $\tilde{\mathcal{L}} = I - \tilde{\mathcal{A}}$, 0 is a simple eigenvalue of $\tilde{\mathcal{L}}$ and also of $\mathcal{L}$ since $\mathcal{L} = \mathcal{D}\tilde{\mathcal{L}}$.

Theorem 10 *$\mathcal{G}$ is connected if and only if 0 is a simple eigenvalue of $\mathcal{L}$.*

Proof for sufficiency: We use the contrapositive argument to show that if $\mathcal{G}$ is not connected then 0 is not a simple eigenvalue of $\mathcal{L}$. Since $\mathcal{G}$ is not connected, we can renumber the nodes to obtain the following form for $\mathcal{L}$,

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 & 0 \\ 0 & \mathcal{L}_{22} & 0 \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}.$$

By the properties of Laplacians, 0 is an eigenvalue of both $\mathcal{L}_{11}$ and $\mathcal{L}_{22}$, therefore 0 is not a simple eigenvalue of $\mathcal{L}$.

Proof for necessity: Assume $\mathcal{G}$ is connected. Set $\mathcal{V}'$ as the set containing all connected nodes. Since $\mathcal{G}$ is connected, $\mathcal{V}'$ contains either all N nodes or r, $1 \leq r < N$, nodes. If

$\mathcal{V}' = \mathcal{V}$, then $\mathcal{G}$ is strongly connected and by property 4, 0 is a simple eigenvalue of $\mathcal{L}$.

If $\mathcal{V}'$ contains r nodes, then $u \to v$ and $v \not\to u$ for $u \in \mathcal{V}'$, $v \in \mathcal{V}\backslash\mathcal{V}'$. If necessary, renumber the nodes of $\mathcal{G}$ so that $\mathcal{V}' = \{1, 2, \ldots, r\}$. Therefore $\mathcal{D}$, $\mathcal{A}$, and $\mathcal{L}$ have the block partition form

$$\mathcal{D} = \begin{bmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_3 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 \\ \mathcal{A}_2 & \mathcal{A}_3 \end{bmatrix}, \quad \text{and } \mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & 0 \\ \mathcal{L}_3 & \mathcal{L}_3 \end{bmatrix},$$

with the $(1,1)$ block of each matrix of size $r \times r$. If $r = 1$, then $\mathcal{L}_1 = \mathcal{D}_1 = \mathcal{A}_1 = 0$. In this case define

$$\mathcal{D}_{s1} = \begin{cases} 1 & \text{if } r = 1 \\ \mathcal{D}_1 & \text{if } 1 < r < n \end{cases}, \quad \mathcal{A}_{s1} = \begin{cases} 1 & \text{if } r = 1 \\ \mathcal{A}_1 & \text{if } 1 < r < n \end{cases},$$

and $\mathcal{L} = \begin{bmatrix} \mathcal{D}_{s1} & 0 \\ 0 & \mathcal{D}_3 \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{s1} & 0 \\ \mathcal{A}_2 & \mathcal{A}_3 \end{bmatrix}$. Notice that $\mathcal{L}$ has not changed since for $r = 1$, $\mathcal{L}_1 = 0$ still holds. Since $\mathcal{L}_1$ corresponds to a strongly connected set of nodes, by Lemma 12, 0 is a simple eigenvalue of $\mathcal{L}_1$. Now we just need to show that 0 is not an eigenvalue of $\mathcal{L}_3$ to conclude that 0 is a simple eigenvalue of $\mathcal{L}$. Recall that $\mathcal{A}_{s1}$ is strongly connected therefore $\mathcal{D}_{s1}$ is invertible. Likewise, $u \to v$ for $u \in \mathcal{V}'$, $v \in \mathcal{V}\backslash\mathcal{V}'$ so $\mathcal{D}_3$ is invertible. Therefore $\mathcal{D}_s = \begin{bmatrix} \mathcal{D}_{s1} & 0 \\ 0 & \mathcal{D}_3 \end{bmatrix}$ is invertible. Consider

$$\tilde{\mathcal{A}} = D_s^{-1} \begin{bmatrix} \mathcal{A}_{s1} & 0 \\ \mathcal{A}_2 & \mathcal{A}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\mathcal{A}}_{s1} & 0 \\ \tilde{\mathcal{A}}_2 & \tilde{\mathcal{A}}_3 \end{bmatrix}$$

and $\tilde{\mathcal{L}} = D_s^{-1}\mathcal{L} = I - \tilde{\mathcal{A}}$. Notice that each row of $\tilde{\mathcal{A}}_2$ must have at least one positive entry. Since $\tilde{\mathcal{A}}$ is row stochastic, the maximum row sum of $\tilde{\mathcal{A}}_3$ must be less than one. This implies $\rho\left(\tilde{\mathcal{A}}_3\right) < 1$. Therefore, $\tilde{\mathcal{L}}_3 = I - \tilde{\mathcal{A}}_3$ is invertible and so is $\mathcal{L}_3$. Therefore 0 is not an eigenvalue of $\mathcal{L}_3$ and so 0 is a simple eigenvalue of $\mathcal{L}$.

Vita

Abhishek Pandey was born in March, 1987, in Dhanbad, India. He obtained his Bachelor's degree in Electrical and Electronics Engineering in 2008 at Visvesvaraya Technological University, India. In 2010 he enrolled at Louisiana State University to pursue graduate studies in the Division of Electrical and Computer Engineering with focus in control systems. He was awarded the degree of Master of Science in Electrical Engineering by Louisiana State University in August 2012. He is currently a candidate for the degree of Doctor of Philosophy in Electrical Engineering to be awarded by Louisiana State University in August 2015.