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When does a Bernoulli convolution admit a spectrum?

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Abstract

In this paper, we solve a long-standing problem on Bernoulli convolutions. In particular, we show that the Bernoulli convolution μ_ρ with contraction rate $\rho \in (0, 1)$ admits a spectrum if and only if ρ is the reciprocal of an even integer.

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1. Introduction

The *Bernoulli convolution* μ_ρ with contraction rate $\rho \in (0, 1)$ is the distribution of $\sum_{n=1}^{\infty} \pm \rho^n/2$, where the signs are chosen independently with probability 1/2. It is the infinite convolution product of $\frac{1}{2}(\delta_{-\rho^n/2} + \delta_{\rho^n/2})$ and its Fourier transform $\widehat{\mu}_\rho(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi t} d\mu_\rho(t)$ is given by

$$\widehat{\mu_\rho}(\xi) = \prod_{j=1}^{\infty} \cos(\pi \rho^j \xi). \tag{1.1}$$

Bernoulli convolutions have been studied since the 1930s and renewed in the 1980s. They have surprising connections with harmonic analysis, number theory, dynamical system, and fractal geometry; see the review paper [25]. In this paper, we consider spectra of Bernoulli convolutions. Here a discrete set Λ is said to be a *spectrum* of a probability measure μ if $\{\exp(-2\pi i \lambda \cdot) : \lambda \in \Lambda\}$ forms an orthogonal basis for $L^2(\mu)$.

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A spectral measure is a probability measure that admits a spectrum. It is a natural extension of a spectral set, which means that the Lebesgue measure restricted to the spectral set is a spectral measure. The spectral set has close connection to tiling as formulated in Fuglede's spectral set conjecture [9]: *A measurable set is a spectral set if and only if it tiles the whole Euclidean space by translation*. The spectral set conjecture has attracted a lot of attention in the past decade. It was proved to be false by Tao and others in dimension three or higher [19,20,22,31], but it is still open in dimension one and two. On the other hand, strong connection between spectral sets and tiling has been observed especially in low dimensions; see [1,12,13,17,21,23] and the survey [18].

The first singular, non-atomic, spectral measure was constructed by Jorgensen and Pedersen. It is the Bernoulli convolution μ_ρ whose contraction rate ρ is the reciprocal of an even integer [15,29]; see [2,3,5–7,14,28,30] and references therein for recent advances on singular spectral measures. A long-standing problem is whether the contraction rate ρ must be the reciprocal of an even integer if the Bernoulli convolution μ_ρ is a spectral measure. Significant progress on that problem has been recently made by Hu and Lau. They show that $\rho = (p/q)^{1/r}$ for some positive integers p, q, r with p, q being odd and even respectively, provided that the Bernoulli convolution μ_ρ is a spectral measure [11]. In this paper, we solve the above problem by proving the following results.

Theorem 1.1. *The Bernoulli convolution with rational contraction rate is not a spectral measure unless the contraction rate is the reciprocal of an even integer.*

Theorem 1.2. *The Bernoulli convolution with irrational contraction rate is not a spectral measure.*

2. Bernoulli convolutions with rational contraction rate

In this section, we prove Theorem 1.1. The crucial steps are the adaptive listing of an orthogonal set for the Bernoulli convolution μ_ρ (Propositions 2.4 and 2.8) and logarithmic decay for the Fourier transform $\widehat{\mu}_\rho$ (Proposition 2.5). Here a discrete set Λ is said to be an *orthogonal set* for a probability measure μ if $\{\exp(-2\pi i\lambda\cdot) : \lambda \in \Lambda\}$ is an orthogonal set of $L^2(\mu)$, while it is said to be a *maximal orthogonal set* if it is not a proper subset of another orthogonal set for that measure.

To prove Theorem 1.1, we first recall two results on Bernoulli convolutions in [11,15].

Theorem 2.1. *Let μ_ρ be the Bernoulli convolution with contraction rate $\rho \in (0, 1)$. Then the following statements hold.*

(i) *If $\rho = 1/q$ for some even integer q, then μ_ρ is a spectral measure.*
(ii) *If μ_ρ is a spectral measure, then $\rho = (p/q)^{1/r}$ for some positive integers p, q, r such that p is odd and q is even.*

By Theorem 2.1, the proof of Theorem 1.1 reduces to proving that the Bernoulli convolution μ_ρ is not a spectral measure under the assumption that $\rho = p/q$ for some positive coprime integers p, q satisfying the following condition:

$$q \text{ is even} \quad \text{and} \quad p \neq 1. \tag{2.1}$$

Given a probability measure μ and a discrete subset Λ, set

$$Q_{\mu,\Lambda}(t) := \sum_{\lambda\in\Lambda} |\widehat{\mu}(t-\lambda)|^2, \quad t \in \mathbb{R}. \tag{2.2}$$

To prove Theorem 1.1, we need the following characterization of spectra and orthogonal sets for a probability measure, which we will prove in Section 2.1.

Proposition 2.2. *Let μ be a probability measure with compact support and Λ be a discrete set on the real line. Then the following statements hold.*

(i) *Λ is a spectrum for the measure μ if and only if $Q_{\mu,\Lambda}(t) = 1$ for all $t \in \mathbb{R}$.*
(ii) *Λ is an orthogonal set for the measure μ if and only if $Q_{\mu,\Lambda}(t) \leq 1$ for all $t \in \mathbb{R}$.*

By Proposition 2.2, the proof of Theorem 1.1 reduces further to establishing the following proposition.

Proposition 2.3. *Let $\rho = p/q < 1$ for some coprime positive integers p, q satisfying (2.1). Then*

$$\inf_{t\in\mathbb{R}} Q_{\mu_\rho,\Lambda}(t) < 1 \tag{2.3}$$

for any maximal orthogonal set Λ.

Our first crucial step to establish (2.3) is the listing of a maximal orthogonal set for the Bernoulli convolution μ_ρ in the next proposition. The proof depends on a characterization on the listing of an orthogonal set for the Bernoulli convolution (Proposition 2.8); see Section 2.2 for the proofs on our adaptive labeling. We remark that different labeling with similar flavor is used in [5] to study spectra for the Bernoulli convolution whose contraction rate is the reciprocal of an even integer.

Proposition 2.4. *Let p, q be coprime integers satisfying (2.1), set $\rho = p/q$, and let Λ be a maximal orthogonal set for the Bernoulli convolution μ_ρ. Then Λ can be listed as a sequence $\{\lambda(n)\}_{n=0}^\infty$ satisfying*

$$|\lambda(n-2^l) - \lambda(0)| \leq |\lambda(n) - \lambda(0)| \quad \text{and} \quad \lambda(n) - \lambda(n-2^l) \in \rho^{-d_0} q^l (\mathbb{Z} + 1/2) \tag{2.4}$$

whenever $n \in [2^l, 2^{l+1})$ for some nonnegative integer l, where d_0 is the smallest positive integer such that $(\Lambda - \Lambda) \cap \rho^{-d_0}(\mathbb{Z} + 1/2) \neq \emptyset$.

We observe that the sequence $\{\lambda(n)\}_{n=0}^\infty$ in Proposition 2.4 satisfies the following size estimate:

$$\begin{aligned} |\lambda(n) - \lambda(0)| &\geq \frac{1}{2}(|\lambda(n) - \lambda(0)| + |\lambda(n-2^l) - \lambda(0)|) \\ &\geq \frac{1}{2}|\lambda(n) - \lambda(n-2^l)| \geq \frac{1}{4}\rho^{-d_0} q^l \geq \frac{1}{4q}\rho^{-d_0} n^{\ln q/\ln 2} \end{aligned} \tag{2.5}$$

for all positive integers n, where l is the unique nonnegative integer such that $n \in [2^l, 2^{l+1})$.

We also observe that the following identity holds for the sequence $\{\lambda(n)\}_{n=0}^\infty$ in Proposition 2.4:

$$\sum_{n=0}^{2^{l+1}-1} \prod_{j=d_0}^{l+d_0} \cos^2(\pi\rho^j (t - \lambda(n))) = 1 \quad \text{for all } t \in \mathbb{R}, \tag{2.6}$$

where l is an arbitrary nonnegative integer (see Section 2.3 for the proof).

Our second crucial step to establish Proposition 2.3, proved in Section 2.4, is to show that the Fourier transform of the Bernoulli convolution μ_ρ decays logarithmically.

Proposition 2.5. *Let p, q be coprime positive integers that satisfy $1 < p < q$, and set $\rho = p/q$. Then*

$$\sup_{\xi\in\mathbb{R}} |\widehat{\mu}_\rho(\xi)| \, (\ln(2+|\xi|))^{\alpha_0} < \infty, \tag{2.7}$$

where $\alpha_0 = \frac{\ln(\cos\frac{\pi}{2q})^{-1}}{\ln(\ln q/\ln p)} > 0$.

We remark that the decay of the Fourier transform $\widehat{\mu}_\rho$ at infinity is closely related to the fundamental problem of determining for which $\rho \in (1/2, 1)$ the Bernoulli convolution μ_ρ is absolutely continuous and for which $\rho \in (1/2, 1)$ it is singular [25]. For instance, Erdös proved that the Bernoulli convolution μ_ρ is singular when $\rho \in (1/2, 1)$ is the reciprocal of a Pisot number, by showing that the Fourier transform $\widehat{\mu}_\rho$ does not tend to zero at infinity [8]. On the other hand, Salem proved that $\widehat{\mu}_\rho(\xi)$ tends to zero as $\xi \to \infty$ if the contraction rate ρ is not the reciprocal of a Pisot number [26]. The interested reader is referred to [4,10,16,24,25,27] for further information on the decay of the Fourier transform $\widehat{\mu}_\rho$ at infinity and its various applications.

Now let us continue the proof of Theorem 1.1, particularly the verification of the inequality (2.3) in Proposition 2.3. We list the set Λ as the sequence $\{\lambda(n)\}_{n=0}^\infty$ in Proposition 2.4 that satisfies (2.4). Let $\alpha_0 > 0$ be as in Proposition 2.5, L be an integer larger than $1/(2\alpha_0)$, and set

$$A_l(t) = \sum_{n=0}^{2^{l^L}-1} |\widehat{\mu}_\rho(t-\lambda(n))|^2, \quad l \geq 1.$$

Recall from (1.1) that

$$\widehat{\mu}_\rho(\xi) = \left(\prod_{j=1}^{n} \cos(\pi\rho^j\xi)\right) \widehat{\mu}_\rho(\rho^n\xi) \quad \text{for all } n \geq 1. \tag{2.8}$$

Then there exists a positive constant C_0 such that

$$\begin{aligned}
A_{l+1}(t) &= A_l(t) + \sum_{n=2^{l^L}}^{2^{(l+1)^L}-1} \prod_{j=1}^{(l+1)^L+d_0-1} \cos^2\left(\pi\rho^j(t-\lambda(n))\right) \\
&\quad \times |\widehat{\mu}_\rho \rho^{(l+1)^L+d_0-1}(t-\lambda(n))|^2 \\
&\leq A_l(t) + C_0 \sum_{n=2^{l^L}}^{2^{(l+1)^L}-1} \prod_{j=1}^{(l+1)^L+d_0-1} \cos^2\left(\pi\rho^j(t-\lambda(n))\right) \\
&\quad \times \left(\ln(2+\rho^{(l+1)^L+d_0-1}|t-\lambda(n)|)\right)^{-2\alpha_0} \\
&\leq A_l(t) + C_0\left(\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)\right)^{-2\alpha_0} \\
&\quad \times \sum_{n=2^{l^L}}^{2^{(l+1)^L}-1} \prod_{j=1}^{(l+1)^L+d_0-1} \cos^2\left(\pi\rho^j(t-\lambda(n))\right) \\
&\leq A_l(t) + C_0\left(\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)\right)^{-2\alpha_0}
\end{aligned}$$

$$\times \left(\sum_{n=0}^{2^{(l+1)^L}-1} \prod_{j=1}^{(l+1)^L+d_0-1} \cos^2\left(\pi\rho^j(t-\lambda(n))\right) - A_l(t) \right)$$

$$\le A_l(t) + C_0 \left(\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)\right)^{-2\alpha_0}$$

$$\times \left(\sum_{n=0}^{2^{(l+1)^L}-1} \prod_{j=d_0}^{(l+1)^L+d_0-1} \cos^2\left(\pi\rho^j(t-\lambda(n))\right) - A_l(t) \right)$$

$$= A_l(t) + C_0 \left(\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)\right)^{-2\alpha_0}(1-A_l(t))$$

for all $t \in \lambda(0)+\frac{1}{8}[-\rho^{-d_0}q^{l^L-1}, \rho^{-d_0}q^{l^L-1}]$, where the first equality follows from (2.8), the first inequality holds by Proposition 2.5, the second inequality is true by (2.5), and the last equality is obtained from (2.6) with l replaced by $(l+1)^L-1$. Thus

$$1 - A_{l+1}(t) \ge (1-A_l(t)) \times \left(1 - C_0\left(\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)\right)^{-2\alpha_0}\right) \tag{2.9}$$

for all $t \in \lambda(0)+\frac{1}{8}[-\rho^{-d_0}q^{l^L-1}, \rho^{-d_0}q^{l^L-1}]$.

Note that

$$\lim_{l\to\infty} \frac{\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)}{l^L} = \ln p > 0.$$

This, together with the assumption that $2L\alpha_0 > 1$, implies the existence of a sufficiently large integer l_0 such that

$$\prod_{l=l_0}^{\infty} \left(1 - C_0\left(\ln(2+p^{(l+1)^L-1}q^{l^L-(l+1)^L}/8)\right)^{-2\alpha_0}\right) := r_0 > 0. \tag{2.10}$$

Applying (2.9) repeatedly and using (2.10), we obtain

$$1 - \sum_{n=0}^{\infty} |\widehat{\mu}_\rho(t-\lambda(n))|^2 = 1 - \lim_{l\to\infty} A_l(t) \ge r_0(1-A_{l_0}(t))$$

for all $t \in \lambda(0)+\frac{1}{8}[-\rho^{-d_0}q^{l_0^L-1}, \rho^{-d_0}q^{l_0^L-1}]$. We observe that

$$\sup\left\{1 - A_{l_0}(t) : \ t \in \lambda(0)+[-\rho^{-d_0}q^{l_0^L-1}/8, \rho^{-d_0}q^{l_0^L-1}/8]\right\}$$

$$\ge \sup\left\{ \sum_{n=2^{(l_0+1)^L}}^{\infty} |\widehat{\mu}_\rho(t-\lambda(n))| : \ t \in \lambda(0)+[-\rho^{-d_0}q^{l_0^L-1}/8, \rho^{-d_0}q^{l_0^L-1}/8] \right\}$$

$$> 0$$

by Proposition 2.2. Therefore

$$\inf_{t\in\mathbb{R}} \sum_{n=0}^{\infty} |\widehat{\mu}_\rho(t-\lambda(n))|^2 < 1.$$

This together with Proposition 2.2 proves Proposition 2.3, and hence completes the proof of Theorem 1.1.

We conclude this section with the proofs of Propositions 2.2, 2.4 and 2.5, and identity (2.6).

2.1. *Proof of Proposition 2.2*

(i) See [15, Lemma 3.3] for a proof.

(ii) ($\Longrightarrow$) Suppose that Λ is an orthogonal set of a probability measure μ. Then

$$\begin{aligned} Q_{\mu,\Lambda}(t) &= \sum_{\lambda\in\Lambda} \left|\langle \exp(-2\pi i\lambda\cdot), \exp(-2\pi i t\cdot)\rangle_{L^2(\mu)}\right|^2 \\ &\le \|\exp(-2\pi i t\cdot)\|^2_{L^2(\mu)} = 1 \quad \text{for all } t\in\mathbb{R}. \end{aligned}$$

($\Longleftarrow$) Suppose that $Q_{\mu,\Lambda}(t) \le 1$ for all $t\in\mathbb{R}$. Then for any given $\lambda'\in\Lambda$,

$$1 \ge Q_{\mu,\Lambda}(\lambda') = |\widehat{\mu}(0)|^2 + \sum_{\lambda\ne\lambda'} |\widehat{\mu}(\lambda'-\lambda)|^2 = 1 + \sum_{\lambda\ne\lambda'} |\widehat{\mu}(\lambda'-\lambda)|^2.$$

This implies that $\widehat{\mu}\left(\lambda'-\lambda\right) = 0$ whenever $\lambda\ne\lambda'$, and hence completes the proof.

2.2. *Proof of Proposition 2.4*

In this subsection, we prove the following generalization of Proposition 2.4.

Proposition 2.6. *Let $\rho = (p/q)^{1/r} < 1$ for some positive integers p, q, r with the property that p and q are coprime integers, q is even and r is the minimal integer with ρ^r being rational, and let Λ be a maximal orthogonal set for the Bernoulli convolution μ_ρ. Then the maximal orthogonal set Λ can be listed as a sequence $\{\lambda(n)\}_{n=0}^\infty$ satisfying (2.4).*

To prove Proposition 2.6, we need some technical lemmas. One may verify that a discrete set Λ is an orthogonal set of a probability measure μ if and only if

$$\Lambda - \Lambda \subset Z(\widehat{\mu}) \cup \{0\}. \tag{2.11}$$

Here $Z(f) = \{x : f(x) = 0\}$ is the set of all zeros of a continuous function f on the real line. By (1.1), the zero set $Z(\widehat{\mu_\rho})$ of the Fourier transform of the Bernoulli convolution μ_ρ contains all ρ^{-j}-dilated half-integers, $j \ge 1$; i.e.,

$$Z(\widehat{\mu_\rho}) = \bigcup_{j=1}^{\infty} \rho^{-j}(\mathbb{Z}+1/2). \tag{2.12}$$

Then we obtain from (2.11) and (2.12) that a discrete set Λ is an orthogonal set for the Bernoulli convolution μ_ρ if and only if

$$\Lambda - \Lambda \subset \left(\bigcup_{j=1}^{\infty} \rho^{-j}(\mathbb{Z}+1/2)\right) \cup \{0\}. \tag{2.13}$$

In the case that the contraction rate ρ has the form $(p/q)^{1/r}$ for some integers p, q, r, ρ satisfying the conditions in Proposition 2.6, we have the following result about the difference of an orthogonal set for the Bernoulli convolution μ_ρ, which will also be used in the proof of Theorem 1.2.

Lemma 2.7. *Let p, q, r, ρ be as in Proposition 2.6, and let Λ be an orthogonal set for the Bernoulli convolution μ_ρ. Then*

$$\Lambda - \Lambda \subset \left(\rho^{-d_0} \bigcup_{l=0}^{\infty} q^l(\mathbb{Z}+1/2)\right) \cup \{0\}, \tag{2.14}$$

where d_0 is the smallest positive integer such that $(\Lambda - \Lambda) \cap \rho^{-d_0}(\mathbb{Z}+1/2) \neq \emptyset$.

Proof. Let d_0 be the smallest positive integer such that $(\Lambda - \Lambda) \cap \rho^{-d_0}(\mathbb{Z}+1/2) \neq \emptyset$, and let $\lambda_0 \in \Lambda$ be so chosen that $\Lambda \cap (\lambda_0 + \rho^{-d_0}(\mathbb{Z}+1/2)) \neq \emptyset$. The existence of such a positive integer d_0 follows from (2.13). First we establish the following weaker result than the desired inclusion (2.14):

$$\Lambda \subset \lambda_0 + \rho^{-d_0}\left(\left(\bigcup_{l=0}^{\infty} q^l(\mathbb{Z}+1/2)\right) \cup \{0\}\right); \tag{2.15}$$

i.e., $\lambda \in \lambda_0 + \rho^{-d_0}\left((\cup_{l=0}^{\infty} q^l(\mathbb{Z}+1/2)) \cup \{0\}\right)$ for any $\lambda \in \Lambda$. Let $\lambda_1 \in \Lambda$ be so chosen that $\lambda_1 - \lambda_0 = \rho^{-d_0}(k_0 + 1/2)$ for some integer k_0. Notice that $\lambda \in \lambda_0 + \rho^{-d_0}\left((\cup_{l=0}^{\infty} q^l(\mathbb{Z}+1/2)) \cup \{0\}\right)$ when $\lambda \in \{\lambda_0, \lambda_1\}$. So we may assume that $\lambda \in \Lambda \setminus \{\lambda_0, \lambda_1\}$. Thus there exist integers j_i and k_i, $i = 1, 2$, by (2.13) and the definition of the integer d_0 such that $j_1 \geq d_0$, $j_2 \geq d_0$, $\lambda - \lambda_0 = \rho^{-j_1}(k_1 + 1/2)$ and $\lambda - \lambda_1 = \rho^{-j_2}(k_2 + 1/2)$. Therefore

$$\rho^{-j_1+d_0}(k_1 + 1/2) - \rho^{-j_2+d_0}(k_2 + 1/2) - (k_0 + 1/2) = 0. \tag{2.16}$$

As $pz^r - q = 0$ is the minimal polynomial in the domain of integers by the assumption on the integer r, we obtain from (2.16) that $j_1 - d_0 = l_1 r$ and $j_2 - d_0 = l_2 r$ for some nonnegative integers l_1 and l_2. Hence we can rewrite (2.16) as

$$(q/p)^{l_1}(k_1 + 1/2) - (q/p)^{l_2}(k_2 + 1/2) - (k_0 + 1/2) = 0.$$

Thus one and only one of two integers l_1 and l_2 must be zero, as q is even and p is odd by the assumption. This proves that $\lambda \in \lambda_0 + \rho^{-d_0} q^{l_1}(\mathbb{Z}+1/2)$ and hence (2.15) follows.

Now we prove the desired inclusion (2.14). Clearly it suffices to prove

$$\lambda_3 - \lambda_2 \in \rho^{-d_0} \bigcup_{l=0}^{\infty} q^l(\mathbb{Z}+1/2) \tag{2.17}$$

for any two given distinct elements $\lambda_2, \lambda_3 \in \Lambda$. If one of those two given distinct elements in Λ is λ_0, then (2.17) follows from (2.15). So we may assume that $\lambda_2 \neq \lambda_0$ and $\lambda_3 \neq \lambda_0$. Hence by (2.15) there exist nonnegative integers l_1, l_2 and integers k_1, k_2 such that $\lambda_2 - \lambda_0 = \rho^{-d_0} q^{l_1}(k_1 + 1/2)$ and $\lambda_3 - \lambda_0 = \rho^{-d_0} q^{l_2}(k_2 + 1/2)$. If $l_1 \neq l_2$, then $\lambda_3 - \lambda_2 \in \rho^{-d_0} q^{\min(l_1,l_2)}(\mathbb{Z}+1/2)$ as q is an even integer, and hence (2.17) holds. If $l_1 = l_2$, then $\lambda_3 - \lambda_2 = \rho^{-d_0} q^{l_1}(k_2 - k_1) = \rho^{-n}(k' + 1/2)$ for some integer k' and natural number $n \geq d_0$ by (2.13). Hence $n = d_0 + l_3 r$ and $(2k' + 1) \in p^{l_3}\mathbb{Z}$ for some nonnegative integer l_3, as p, q are coprime and q is an even integer. Thus $\lambda_3 - \lambda_2 = \rho^{-d_0} q^{l_3}(k' + 1/2)/p^l \in \rho^{-d_0} q^{l_3}(\mathbb{Z}+1/2)$ and hence (2.17) is established. □

To prove Proposition 2.6, we also need the following characterization on listing an orthogonal set for the Bernoulli convolution μ_ρ.

Proposition 2.8. *Let p, q, r, ρ be as in Proposition 2.6. Then the following statements are equivalent.*

(i) *A discrete set Λ is an orthogonal set for the Bernoulli convolution μ_ρ.*

(ii) *A discrete set Λ can be listed as an indexed set $\{\lambda(n)\}_{n\in N}$ such that the index set N, a subset of nonnegative integers, contains zero and has the binary-tree structure property starting from the origin,*

$$n-2^l \in N \quad \text{whenever } n \in N\cap[2^l, 2^{l+1}) \text{ for some nonnegative integer } l, \tag{2.18}$$

and that

$$\begin{aligned} &|\lambda(n-2^l)-\lambda(0)| \le |\lambda(n)-\lambda(0)| \quad \text{and} \\ &\lambda(n)-\lambda(n-2^l) \in \rho^{-d_0}q^l(\mathbb{Z}+1/2) \end{aligned} \tag{2.19}$$

for some positive integer d_0 whenever $n \in N\cap[2^l, 2^{l+1})$ for some nonnegative integer l.

Proof. (ii)$\Longrightarrow$(i) By (2.12), it suffices to prove that

$$\lambda(n)-\lambda(n') \in \rho^{-d_0}\bigcup_{l=0}^{\infty} q^l(\mathbb{Z}+1/2) \quad \text{for all } n, n' \in N. \tag{2.20}$$

Without loss of generality, we assume that $n < n'$. Write $n=\sum_{i=0}^m \epsilon_i 2^i$ and $n'=\sum_{i=0}^m \epsilon_i' 2^i$, where $\epsilon_i, \epsilon_i' \in \{0,1\}, 0\le i\le m$ and $\epsilon_m'=1$. Set $n''=0$ if $\epsilon_0 \ne \epsilon_0'$ and set $n''=\sum_{i=0}^{m'}\epsilon_i 2^i$ where $m'<m$ is the unique integer such that $\epsilon_i=\epsilon_i'$ for all $0\le i\le m'$ and $\epsilon_{m'+1}\ne\epsilon_{m'+1}'$. Then $0\le n''\le n<n'$ and $n''\in N$ from the binary-tree structure of the index set N. In the case that $n''\ne n$, we have that $n''<2^{\min(l,l')}$, $n-n''\in 2^l(2\mathbb{Z}+1)$ and $n'-n''\in 2^{l'}(2\mathbb{Z}+1)$ for two distinct nonnegative integers l and l'. Hence $\lambda(n)-\lambda(n'')\in\rho^{-d_0}q^l(\mathbb{Z}+1/2)$ and $\lambda(n')-\lambda(n'')\in\rho^{-d_0}q^{l'}(\mathbb{Z}+1/2)$ by (2.19). This implies that $\lambda(n)-\lambda(n')\in\rho^{-d_0}q^{\min(l,l')}(\mathbb{Z}+1/2)$ and hence (2.20) holds when $n''\ne n$. In the case that $n''=n$, there exists a nonnegative integer l such that $n<2^l$ and $n'-n\in 2^l(2\mathbb{Z}+1)$. Hence $\lambda(n')-\lambda(n)=\lambda(n')-\lambda(n'')\in\rho^{-d_0}q^l(\mathbb{Z}+1/2)$ by (2.19), and thus (2.20) follows when $n''=n$.

(i)$\Longrightarrow$(ii) Let d_0, λ_0 be as in Lemma 2.7. Set $N_0=\{0\}$ and define $\lambda(0)=\lambda_0$. Inductively for $l\ge 0$, let $\tilde N_l$ contain all integers $n\in[2^l,2^{l+1})$ such that $n-2^l\in N_l$ and $\Lambda\cap(\lambda(n-2^l)+\rho^{-d_0}q^l(\mathbb{Z}+1/2))\ne\emptyset$, set $N_{l+1}=N_l\cup\tilde N_l$, and for $n\in\tilde N_l$ let

$$\lambda(n):=\operatorname{argmin}\left\{|\lambda-\lambda(0)| : \lambda\in\Lambda\cap(\lambda(n-2^l)+\rho^{-d_0}q^l(\mathbb{Z}+1/2))\right\} \tag{2.21}$$

be an element in $\Lambda\cap(\lambda(n-2^l)+\rho^{-d_0}q^l(\mathbb{Z}+1/2))$ such that

$$|\lambda(n)-\lambda(0)|=\min\left\{|\lambda-\lambda(0)| : \lambda\in\Lambda\cap(\lambda(n-2^l)+\rho^{-d_0}q^l(\mathbb{Z}+1/2))\right\}.$$

Let $N=\cup_{l=0}^\infty N_l$. Then the index set N contains 0 as $0\in N_0$, and has the binary-tree structure (2.18) since $N\cap[2^l,2^{l+1})=\tilde N_l$ and $\tilde N_l-2^l\subset N_l$.

Next we show that the indexed set $\{\lambda(n)\}_{n\in N}$ satisfies (2.19). Let $n\in N\cap[2^l,2^{l+1})=\tilde N_l$ for some $l\in\mathbb{Z}_+$. Then the conclusion that $\lambda(n)\in\lambda(n-2^l)+\rho^{-d_0}q^l(\mathbb{Z}+1/2)$ follows from (2.21). Now we prove that $|\lambda(n-2^l)-\lambda(0)|\le|\lambda(n)-\lambda(0)|$. Clearly the above conclusion holds when $n-2^l=0$. So we may assume that $n-2^l\ne 0$. Thus $n-2^l\in\tilde N_{l'}$ for some $0\le l'<l$. This implies that $\lambda(n-2^l)-\lambda(n-2^l-2^{l'})\in\rho^{-d_0}q^{l'}(\mathbb{Z}+1/2)$. Hence $\lambda(n)-\lambda(n-2^l-2^{l'})=(\lambda(n)-\lambda(n-2^l))+(\lambda(n-2^l)-\lambda(n-2^l-2^{l'}))$ also belongs to $\rho^{-d_0}q^{l'}(\mathbb{Z}+1/2)$, which together with (2.21) implies that $|\lambda(n-2^l)-\lambda(0)|\le|\lambda(n)-\lambda(0)|$, and hence the indexed set $\{\lambda(n)\}_{n\in N}$ satisfies (2.19).

Finally we prove that $\{\lambda(n)\}_{n\in N}$ is obtained by indexing the orthogonal set Λ. Given two distinct integers n, n' in N, there exists a nonnegative integer l such that $\lambda(n)-\lambda(n')\in$

$\rho^{-d_0}q^l(\mathbb{Z}+1/2)$ (and hence $\lambda(n) \neq \lambda(n')$) by applying the same argument as the one used to prove (2.20). Therefore it suffices to show that given any $\lambda \in \Lambda$ there exists $n \in N$ such that $\lambda = \lambda(n)$. For $\lambda = \lambda_0$, we have that $\lambda = \lambda(0)$. So now we may assume that $\lambda \neq \lambda_0$. By Lemma 2.7, there exists an unique nonnegative integer l_1 such that $\lambda - \lambda(0) \in \rho^{-d_0}q^{l_1}(\mathbb{Z}+1/2)$. Then $|\lambda-\lambda(0)| \geq |\lambda(2^{l_1})-\lambda(0)|$ by (2.21). If $\lambda = \lambda(2^{l_1})$, then the proof is completed. Otherwise, $|\lambda-\lambda(0)| \geq |\lambda(2^{l_1})-\lambda(0)|$ and

$$\lambda - \lambda(2^{l_1}) \in \rho^{-d_0}q^{l_1}\mathbb{Z} \setminus \{0\}, \tag{2.22}$$

which also implies that

$$|\lambda-\lambda(0)| \geq \frac{1}{2}(|\lambda-\lambda(0)| + |\lambda(2^{l_1})-\lambda(0)|) \geq \frac{1}{2}|\lambda-\lambda(2^{l_1})| \geq \frac{1}{2}\rho^{-d_0}q^{l_1}.$$

Recall from Lemma 2.7 that $\lambda - \lambda(2^{l_1}) \in \rho^{-d_0}q^{l_2}(\mathbb{Z}+1/2)$ for some integer $l_2 \geq 0$. Therefore $l_2 \geq l_1 + 1$ by (2.22) and the assumption that q is an even integer. If $\lambda = \lambda(2^{l_1}+2^{l_2})$, then the conclusion follows, otherwise as in the above argument we have that $|\lambda-\lambda(0)| \geq |\lambda(2^{l_1}+2^{l_2})-\lambda(0)|$ and

$$\lambda - \lambda(2^{l_1}+2^{l_2}) \in \rho^{-d_0}q^{l_2}\mathbb{Z} \setminus \{0\} \quad \text{and} \quad |\lambda-\lambda(0)| \geq \frac{1}{2}\rho^{-d_0}q^{l_2}.$$

Applying the above argument iteratively, there exist integers $l_1 < \cdots < l_m$ with $m \leq \log_q(2\rho^{d_0}|\lambda-\lambda_0|) - l_1 + 1$ such that $\lambda = \lambda(2^{l_1}+2^{l_2}+\cdots+2^{l_m})$. $\square$

Now we are ready to prove Proposition 2.6.

Proof of Proposition 2.6. Given a maximal orthogonal set Λ for the Bernoulli convolution μ_ρ, let d_0 be the smallest positive integer such that $(\Lambda-\Lambda)\cap\rho^{-d_0}(\mathbb{Z}+1/2) \neq \emptyset$. By Proposition 2.8, Λ can be listed as an indexed set $\{\lambda(n)\}_{n\in N}$ which satisfies (2.18) and (2.19). Then the proof reduces to showing that the index set N contains all nonnegative integers. Suppose, on the contrary, that $N \neq \mathbb{Z}_+$. Then from the binary-tree structure property (2.18) for the set N established in Proposition 2.8 there exists an integer $n_1 \in [2^{l_1}, 2^{l_1+1})$ for some nonnegative integer l_1 such that $n_1 - 2^{l_1} \in N$ and $\Lambda \cap (\lambda(n_1-2^{l_1}) + \rho^{-d_0}q^{l_1}(\mathbb{Z}+1/2)) = \emptyset$. Define $\lambda(n_1) = \lambda(n_1-2^{l_1}) \pm \rho^{-d_0}q^{l_1}/2$, where the sign is so chosen that $|\lambda(n_1)-\lambda(0)| \geq |\lambda(n_1-2^{l_1})-\lambda(0)|$. One may verify that $\{\lambda(n)\}_{n\in N\cup\{n_1\}}$ satisfies (2.19). Then $\Lambda \cap \{\lambda(n_1)\}$ is an orthogonal set for the Bernoulli convolution μ_ρ by Proposition 2.8 and $\Lambda \cap \{\lambda(n_1)\}$ is a true supset of Λ. This is a contradiction. $\square$

2.3. *Proof of identity (2.6)*

In this subsection, we will prove the following slight generalization of identity (2.6).

Proposition 2.9. *Let p, q be coprime integers with q being even, set $\rho = p/q \in (0,1)$, and let the sequence $\{\gamma(n)\}_{n=0}^{\infty}$ satisfy $\gamma(n) - \gamma(n-2^l) \in q^l(\mathbb{Z}+1/2)$ whenever $n \in [2^l, 2^{l+1})$ for some nonnegative integer l. Then*

$$\sum_{n=0}^{2^{l+1}-1} \prod_{j=0}^{l} \cos^2(\pi\rho^j(t-\gamma(n))) = 1 \quad \text{for all } t \in \mathbb{R}, \tag{2.23}$$

for all nonnegative integers l.

Proof. For $l = 0$, we have

$$\sum_{n=0}^{2^{l+1}-1} \prod_{j=0}^{l} \cos^2 \left(\pi \rho^j (t - \gamma(n))\right) = \cos^2 (\pi(t - \gamma(0))) + \cos^2 (\pi(t - \gamma(1)))$$
$$= \cos^2 (\pi(t - \gamma(0))) + \sin^2 (\pi(t - \gamma(0)))$$
$$= 1 \quad \text{for all } t \in \mathbb{R}. \tag{2.24}$$

For $l \geq 1$, we obtain

$$\begin{aligned}\sum_{n=0}^{2^{l+1}-1} \prod_{j=0}^{l} \cos^2 \left(\pi \rho^j (t - \gamma(n))\right) &= \sum_{n=0}^{2^{l}-1} \prod_{j=0}^{l} \cos^2 \left(\pi \rho^j (t - \gamma(n))\right) \\ &\quad + \sum_{n=0}^{2^{l}-1} \prod_{j=0}^{l} \cos^2 \left(\pi \rho^j (t - \gamma(n+2^l))\right) \\ &= \sum_{n=0}^{2^{l}-1} \left(\prod_{j=0}^{l-1} \cos^2 \left(\pi \rho^j (t - \gamma(n))\right)\right) \cdot \cos^2 \left(\pi \rho^l (t - \gamma(n))\right) \\ &\quad + \sum_{n=0}^{2^{l}-1} \left(\prod_{j=0}^{l-1} \cos^2 \left(\pi \rho^j (t - \gamma(n))\right)\right) \cdot \sin^2 \left(\pi \rho^l (t - \gamma(n))\right) \\ &= \sum_{n=0}^{2^{l}-1} \prod_{j=0}^{l-1} \cos^2 \left(\pi \rho^j (t - \gamma(n))\right),\end{aligned} \tag{2.25}$$

where the second equality follows from the observation that $\rho^j(\gamma(n+2^l) - \gamma(n)) \in \mathbb{Z}, 0 \leq j \leq l-1$ and $\rho^l(\gamma(n+2^l) - \gamma(n)) \in \mathbb{Z} + 1/2$ for all $n \in [0, 2^l)$ by the assumptions on integers p, q, ρ and the sequence $\{\gamma(n)\}_{n=0}^{\infty}$. Then identity (2.23) follows from (2.24) and (2.25). $\square$

2.4. *Proof of Proposition 2.5*

To prove Proposition 2.5, we need a technical lemma.

Lemma 2.10. *Let p, q, ρ, μ_ρ be as in Proposition 2.5. Then for any $\xi \in \mathbb{R} \setminus (-1, 1)$ there exists $\xi' \in \mathbb{R}$ such that $\rho|\xi| \geq |\xi'| \geq \rho^2 |\xi|^{\frac{\ln p}{\ln q}}$ and*

$$\left|\widehat{\mu}_\rho(\xi)\right| \leq \cos \frac{\pi}{2q} \cdot \left|\widehat{\mu}_\rho(\xi')\right|. \tag{2.26}$$

Proof. For any real number x, we let $\langle x \rangle$ be the unique number in $(-\frac{1}{2}, \frac{1}{2}]$ satisfying $x - \langle x \rangle \in \mathbb{Z}$. We first consider the case that $\langle \rho|\xi| \rangle \notin (-\frac{1}{2q}, \frac{1}{2q})$. In this case,

$$|\widehat{\mu}_\rho(\xi)| = |\cos(\pi\rho\xi)| \cdot |\widehat{\mu}_\rho(\rho\xi)| \leq \cos \frac{\pi}{2q} \cdot |\widehat{\mu}_\rho(\rho\xi)|$$

by (2.8). Hence the desired conclusion (2.26) follows by letting $\xi' = \rho\xi$.

We then consider the case that $\langle \rho|\xi|\rangle \in (-\frac{1}{2q}, \frac{1}{2q})$. In this case, we write

$$\rho|\xi| = \langle \rho|\xi|\rangle + \sum_{l=0}^{m} \varepsilon_l q^l, \tag{2.27}$$

where $\varepsilon_l \in \{0, 1, \dots, q-1\}$ for all $0 \le l \le m \in \mathbb{Z}$ with $\varepsilon_m \neq 0$. Let m_0 be the smallest nonnegative integer such that $\varepsilon_{m_0} \neq 0$. Then

$$\langle \rho^{m_0+2}|\xi|\rangle = \left\langle \rho^{m_0+1}\langle \rho|\xi|\rangle + \varepsilon_{m_0} p^{m_0+1}/q \right\rangle \notin \left(-\frac{1}{2q}, \frac{1}{2q}\right) \tag{2.28}$$

as $\varepsilon_{m_0} p^{m_0+1}/q \notin (-1/q, 1/q) + \mathbb{Z}$, and

$$\rho|\xi| \ge -1/(2q) + \epsilon_{m_0} q^{m_0} \ge \rho q^{m_0} \tag{2.29}$$

by (2.27). Hence, in the case that $\langle \rho|\xi|\rangle \in (-\frac{1}{2q}, \frac{1}{2q})$, the desired conclusion (2.26) holds with $\xi' = \rho^{m_0+2}\xi$, because

$$|\xi'| \ge \rho^2 |\xi|^{1+\ln\rho/\ln q} = \rho^2 |\xi|^{\ln p/\ln q}$$

and

$$|\widehat{\mu_\rho}(\xi)| \le |\cos(\pi \rho^{m_0+2}\xi)| \cdot |\widehat{\mu_\rho}(\rho^{m_0+2}\xi)| \le \cos\frac{\pi}{2q} \cdot |\widehat{\mu_\rho}(\rho^{m_0+2}\xi)|$$

by (2.8), (2.28) and (2.29). □

Now we prove Proposition 2.5.

Proof of Proposition 2.5. For $\xi \in \mathbb{R} \setminus (-2, 2)$, we start from $\xi_0 = \xi$ and apply Lemma 2.10 iteratively to obtain $\xi_k, 0 \le k \le m$, until it cannot be applied. Then the sequence of numbers $\xi_k, 0 \le k \le m$, satisfies

$$\begin{cases} \xi_0 = \xi,\ |\xi_m| \le 1,\ \rho|\xi_k| \ge |\xi_{k+1}| \ge \rho^2 |\xi_k|^{\ln p/\ln q} & \text{for all } 0 \le k \le m-1, \text{ and} \\ |\widehat{\mu}_\rho(\xi_k)| \le \cos\dfrac{\pi}{2q} |\widehat{\mu}_\rho(\xi_{k+1})| & \text{for all } 0 \le k \le m-1. \end{cases}$$

This together with (1.1) implies that

$$\begin{aligned} 1 &\ge |\xi_m| \ge \rho^2 |\xi_{m-1}|^{\ln p/\ln q} \ge \rho^{2+2\ln p/\ln q} |\xi_{m-2}|^{(\ln p/\ln q)^2} \\ &\ge \cdots \ge \rho^{2(1+\ln p/\ln q+\cdots+(\ln p/\ln q)^{m-1})} |\xi_0|^{(\ln p/\ln q)^m} \\ &\ge \rho^{2\ln q/(\ln q-\ln p)} |\xi|^{(\ln p/\ln q)^m} = q^{-2} |\xi|^{(\ln p/\ln q)^m}, \end{aligned}$$

and

$$|\widehat{\mu}_\rho(\xi)| \le \cos\frac{\pi}{2q} \cdot |\widehat{\mu}_\rho(\xi_1)| \le \cdots \le \left(\cos\frac{\pi}{2q}\right)^m |\widehat{\mu}_\rho(\xi_m)| \le \left(\cos\frac{\pi}{2q}\right)^m.$$

Hence the desired estimate (2.7) follows. □

3. Bernoulli convolutions with irrational contraction rate

In this section, we prove Theorem 1.2. To prove it, we need a lemma for Bernoulli convolutions with different contraction rates.

Lemma 3.1. *Let p, q, r, ρ be as in Proposition 2.6, and μ_ρ be the Bernoulli convolution with contraction rate $\rho \in (0, 1)$. If Λ is an orthogonal set for the Bernoulli convolution μ_ρ with contraction rate ρ, then there exists a nonnegative integer $n_1 \le r - 1$ such that $\rho^{-n_1}\Lambda$ is an orthogonal set for the Bernoulli convolution μ_{ρ^r} with contraction rate ρ^r.*

Proof. By Lemma 2.7, there exists a positive integer d_0 such that

$$\Lambda - \Lambda \subset \rho^{-d_0}\left(\bigcup_{n=0}^{\infty} q^n(\mathbb{Z} + 1/2)\right) \cup \{0\}. \tag{3.1}$$

Take the unique integer n_1 between 0 and $r - 1$ such that $d_0 + n_1 = rl$ for some positive integer $l \ge 1$. This together with (2.12) and (3.1) implies that

$$\begin{aligned}\rho^{-n_1}\Lambda - \rho^{-n_1}\Lambda &\subset \rho^{-d_0-n_1}\left(\bigcup_{n=0}^{\infty} q^n(\mathbb{Z} + 1/2)\right) \cup \{0\} \\ &\subset \left(\bigcup_{m=l}^{\infty} (\rho^r)^{-m}(\mathbb{Z} + 1/2)\right) \cup \{0\} \subset Z(\widehat{\mu}_{\rho^r}) \cup \{0\}. \end{aligned} \tag{3.2}$$

Then the desired conclusion that $\rho^{-n_1}\Lambda$ is an orthogonal set for the Bernoulli convolution μ_{ρ^r} follows from (2.11) and (3.2). □

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 2.1 and Proposition 2.2, it suffices to prove

$$\inf_{t\in\mathbb{R}} Q_{\mu_\rho,\Lambda}(t) < 1 \tag{3.3}$$

for any orthogonal set Λ, where $\rho = (p/q)^{1/r}$ for some positive integers p, q and r such that p, q are coprime, q is even, and r is the minimal positive integer such that ρ^r is rational. Let n_1 be the positive integer in Lemma 3.1 such that $\rho^{-n_1}\Lambda$ is an orthogonal set for the Bernoulli convolution μ_{ρ^r} with contraction rate $\rho^r = p/q$. Then it follows from (1.1), Proposition 2.2 and Lemma 3.1 that

$$\begin{aligned}\sum_{\lambda\in\Lambda} |\widehat{\mu}_\rho(t-\lambda)|^2 &= \sum_{\lambda\in\Lambda}\prod_{n=1}^{\infty} \cos^2(\pi\rho^n(t-\lambda)) \\ &\le \sum_{\lambda\in\Lambda}\prod_{m=1}^{\infty} \cos^2(\pi\rho^{mr-n_1}(t-\lambda)) \\ &= \sum_{\lambda\in\Lambda} |\widehat{\mu}_{\rho^r}(\rho^{-n_1}t - \rho^{-n_1}\lambda)|^2 \le 1. \end{aligned} \tag{3.4}$$

Notice that $r \ge 2$ as ρ is irrational by the assumption. Hence (3.3) holds as the first inequality in (3.4) is strict for some $t \in \mathbb{R}$, for instance $t \notin \lambda_0 + \cup_{n=1}^{\infty} \rho^{-n}(\mathbb{Z} + 1/2)$ where $\lambda_0 \in \Lambda$. □

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