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On a combinatorial problem of Erdős, Kleitman and Lemke

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Abstract

In this paper, we study a combinatorial problem originating in the following conjecture of Erdős and Lemke: given any sequence of n divisors of n, repetitions being allowed, there exists a subsequence the elements of which are summing to n. This conjecture was proved by Kleitman and Lemke, who then extended the original question to a problem on a zero-sum invariant in the framework of finite Abelian groups. Building among others on earlier works by Alon and Dubiner and by the author, our main theorem gives a new upper bound for this invariant in the general case, and provides its right order of magnitude.

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1. Introduction

Let G be a finite Abelian group, written additively. If G is cyclic of order n, it will be denoted by C_n. In the general case, we can decompose G as a direct product of cyclic groups $C_{n_1} \oplus \cdots \oplus C_{n_r}$ such that $1 < n_1 | \cdots | n_r \in \mathbb{N}$, where r and n_r are respectively called the rank and exponent of G. Usually, the exponent of G is simply denoted by $\exp(G)$. The order of an element g of G will be written $\mathrm{ord}(g)$ and for every divisor d of $\exp(G)$, we denote by G_d the

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subgroup of G consisting of all elements of order dividing d:

$$G_d = \{x \in G \mid dx = 0\}.$$

In this paper, any finite sequence $S = (g_1, \dots, g_\ell)$ of ℓ elements of G will be called a *sequence* over G of *length* $|S| = \ell$. We will also denote by $\sigma(S)$ the sum of all elements contained in S, which will be referred to as a *zero-sum sequence* whenever $\sigma(S) = 0$.

Given a sequence S over G, we denote by S_d the subsequence of S consisting of all elements of order d contained in S, and by $\mathsf{k}(S)$ the *cross number* of $S = (g_1, \dots, g_\ell)$, which is defined as follows:

$$\mathsf{k}(S) = \sum_{i=1}^{\ell} \frac{1}{\mathrm{ord}(g_i)}.$$

By $\mathsf{t}(G)$ we denote the smallest integer $t \in \mathbb{N}^*$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence $S' \mid S$ with $\mathsf{k}(S') \leq 1$. Such a subsequence will be called a *tiny zero-sum subsequence*.

The investigations on $\mathsf{t}(G)$ originate in the following conjecture, addressed by Erdős and Lemke in the late eighties (see [15, Introduction]). Is it true that out of n divisors of n, repetitions being allowed, one can always find a certain number of them that sum up to n? Motivated by this conjecture, Kleitman and Lemke [15, Theorem 1] proved the following stronger result.

Theorem 1.1 (*Kleitman and Lemke [15]*)**.** *For any given integers $a_1, \dots, a_n$ there is a non-empty subset $I \subseteq [\![1, n]\!]$ such that*

$$n \mid \sum_{i \in I} a_i \quad \text{and} \quad \sum_{i \in I} \gcd(a_i, n) \leq n.$$

Meanwhile, Lagarias and Saks framed a graph-theoretic approach, called graph pebbling, thanks to which the problem of Erdős and Lemke could be reduced to the study of a combinatorial game, played with pebbles on the vertices of a simple graph. In this context, Chung [2, Theorem 6] found a new elegant proof of Theorem 1.1, and, under one extra assumption made on the prime factors of n, Denley [3, Theorem 1] could sharpen this theorem. Graph pebbling led to a rapidly growing literature as well as many open problems. The interested reader is referred to the surveys [13,14] which contain many references on the subject.

In addition, let us underline that, with our notation, Theorem 1.1 simply asserts that $\mathsf{t}(C_n) \leq n$, and this upper bound is easily seen to be optimal. In the general case, Kleitman and Lemke [15, Section 3] conjectured that $\mathsf{t}(G) \leq |G|$ holds for every finite Abelian group G. This conjecture was proved, using tools from zero-sum theory, by Geroldinger [8]. An alternative proof was then found by Elledge and Hurlbert [5, Theorem 2] using graph pebbling.

In this paper, we prove that in the general case of finite Abelian groups, the currently known upper bound on $\mathsf{t}(G)$ can be improved significantly. More precisely, our main theorem shows that, for finite Abelian groups of fixed rank, $\mathsf{t}(G)$ grows linearly in the exponent of G, which gives the correct order of magnitude. To do so, we prove that $\mathsf{t}(G)$ can be bounded above using a result of Alon and Dubiner [1] on the following classical invariant in zero-sum theory.

By $\eta(G)$ we denote the smallest integer $t \in \mathbb{N}^*$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence $S' \mid S$ with $|S'| \leq \exp(G)$. Such a subsequence is called a *short zero-sum subsequence*.

Since $\mathsf{k}(T) \leq 1$ implies $|T| \leq \exp(G)$, one can notice that $\eta(G) \leq \mathsf{t}(G)$ always holds. Let us also mention that $\eta(G)$ is one out of many invariants studied because of their arithmetical applications. In this respect, the interested reader is referred to [10,6,9] for comprehensive surveys on non-unique factorization theory. For instance, it is known [10, Theorem 5.8.3] that for all integers $m, n \in \mathbb{N}^*$ such that $m \mid n$, one has

$$\eta(C_m \oplus C_n) = 2m + n - 2. \tag{1}$$

In particular, the equality $\mathsf{t}(C_n) = \eta(C_n) = n$ holds. Moreover, and even if little is known on the exact value of $\eta(G)$ for finite Abelian groups of rank $r \geq 3$, its behavior is better understood than it is for $\mathsf{t}(G)$, and the following key result, due to Alon and Dubiner [1, Theorem 1.1], will be extensively used throughout this article.

Theorem 1.2 (*Alon and Dubiner [1]*)**.** *For every $r \in \mathbb{N}^*$ there exists a constant $c_r > 0$ such that for every $n \in \mathbb{N}^*$, one has*

$$\eta\left(C_n^r\right) \leq c_r(n-1) + 1.$$

From now on, we will identify c_r with its smallest possible value in Theorem 1.2. On the one hand, a natural construction shows that $c_r \geq (2^r - 1)$. Indeed, if $(e_1, \ldots, e_r)$ is a basis of C_n^r with $\mathrm{ord}(e_i) = n$ for all $i \in [\![1, r]\!]$, it is easily checked that the sequence S consisting of $n-1$ copies of $\sum_{i \in I} e_i$ for each non-empty subset $I \subseteq [\![1, n]\!]$ contains no short zero-sum subsequence, so that

$$(2^r - 1)(n-1) + 1 \leq \eta(C_n^r) \leq \mathsf{t}(C_n^r). \tag{2}$$

In particular, one has $\mathsf{t}(C_2^r) = \eta(C_2^r) = 2^r$. For the time being, the exact value of $\mathsf{t}(G)$ is known for cyclic groups and elementary 2-groups only.

On the other hand, the method used in [1] yields $c_r \leq (cr \ln r)^r$, where $c > 0$ is an absolute constant. Also, it readily follows from (1) that it is possible to choose $c_1 = 1$ and $c_2 = 3$, with equality in Theorem 1.2, and it is conjectured in [1] that there actually is an absolute constant $d > 0$ such that $c_r \leq d^r$ for all $r \geq 1$. For a complete account on $\eta(G)$, see [4,7] and the references contained therein.

2. New results and plan of the paper

Let $\mathcal{P} = \{p_1 = 2 < p_2 = 3 < \cdots\}$ be the set of prime numbers. Given a positive integer n, let $\mathcal{D}_n$ be the set of its positive divisors. By $P^-(n)$ and $P^+(n)$, we denote the smallest and greatest prime elements of $\mathcal{D}_n$ respectively, with the convention $P^-(1) = P^+(1) = 1$. Finally, the p-adic valuation of n will be denoted by $\nu_p(n)$.

In this paper, we prove that in the general case of finite Abelian groups, the currently known upper bound on $\mathsf{t}(G)$ can be improved significantly.

Our starting point will be to prove it first in the case of finite Abelian p-groups. For this purpose, a classical variant of $\mathsf{t}(G)$, introduced by Geroldinger and Schneider in [11], will be studied in Section 3. Even though a single by-product of this investigation (see Corollary 3.6) will effectively be used in subsequent sections, we include this study in full, since it may be of interest in view of arithmetical applications.

Then, in Section 4, we prove the main theorem of this paper. This theorem provides a new upper bound for $\mathsf{t}(G)$, which depends on the rank and exponent of G only. It is proved thanks to an appropriate partition of the divisor lattice of $\exp(G)$. In addition, an interesting special case is the one of finite Abelian groups of rank two, where this theorem can be applied specifying $c_2 = 3$.

Theorem 2.1. *Let G be a finite Abelian group of rank r and exponent n. Then*

$$\mathsf{t}(G) \le c_r \sum_{d|n} \left(\frac{d}{P^+(d)^{\nu_{P^+(d)}(d)}} - 1 \right) + c_r\,(n-1) + 1.$$

In Section 5, we then obtain, as a corollary of Theorem 2.1, the following Alon and Dubiner type upper bound for $\mathsf{t}(G)$. In particular, it is easily deduced from (2) that this upper bound has the right order of magnitude.

Theorem 2.2. *For every $r \in \mathbb{N}^*$ there exists a constant $d_r > 0$ such that, for every finite Abelian group G of rank r and exponent n, one has*

$$\mathsf{t}(G) \le d_r(n-1) + 1.$$

The qualitative upper bound of Theorem 2.2 is voluntarily given in a form allowing to stress the connection between d_r and c_r. In this regard, a key argument in our proof of Theorem 2.2 actually comes from a simple, albeit somewhat surprising, property of the following arithmetic function

$$f(n) = \sum_{d|n} \frac{d}{P^+(d)}.$$

Indeed, one can show that $f(n) \le n$ always holds (see Proposition 5.2). Consequently, and even if no particular effort has been made to optimize the constant in Theorem 2.2, it turns out that d_r can be chosen to satisfy $1 \le d_r/c_r \le 2$, thus being at most twice as large as the best possible constant we could hope for.

Finally, in Section 6, we propose and discuss two open problems on $\mathsf{t}(G)$.

3. On a variant of $\mathsf{t}(G)$

Let G be a finite Abelian group, and $d',\ d \in \mathbb{N}^*$ be two integers such that $d'|d|\exp(G)$. This section is devoted to the following variant of $\mathsf{t}(G)$, which was first introduced by Geroldinger and Schneider [11, Section 3] (see also [10, Section 5.7]):

$$\rho(G) = \max\{\mathsf{k}(S) \mid S \text{ contains no tiny zero-sum subsequence}\}.$$

Using a compression argument, we will obtain a new upper bound for $\rho(G)$, which has the correct order of magnitude and applies to any finite Abelian group G. For this purpose, we consider the following invariant, which was introduced in [12].

By $\eta_{(d',d)}(G)$ we denote the smallest integer $t \in \mathbb{N}^*$ such that every sequence S over G_d of length $|S| \ge t$ contains a non-empty subsequence $S' \mid S$ of length $|S'| \le d'$ and with sum in $G_{d/d'}$.

The numbers $\eta_{(d',d)}(G)$ and $\eta(G)$ are closely related to each other. First, note that the two definitions coincide when $d' = d = \exp(G)$. In addition, and as shown in [12, Proposition 3.1], there exists a subgroup $G_{\upsilon(d',d)} \subseteq G$ such that $\eta_{(d',d)}(G) = \eta(G_{\upsilon(d',d)})$. In order to define this particular subgroup $G_{\upsilon(d',d)}$ properly, we introduce the following notation. Given the decomposition of G as a product of cyclic groups

$$G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}, \quad \text{with } 1 < n_1 | \cdots | n_r \in \mathbb{N},$$

we set, for all $i \in [\![1, r]\!]$,

$$A_i = \gcd(d', n_i), \qquad B_i = \frac{\operatorname{lcm}(d, n_i)}{\operatorname{lcm}(d', n_i)}, \qquad \upsilon_i(d', d) = \frac{A_i}{\gcd(A_i, B_i)}.$$

Therefore, whenever d divides n_i, we have $\upsilon_i(d', d) = \gcd(d', n_i) = d'$, and in particular $\upsilon_r(d', d) = d'$. We can now state our result on $\eta_{(d',d)}(G)$.

Proposition 3.1. *Let $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$, with $1 < n_1 | \cdots | n_r \in \mathbb{N}$, be a finite Abelian group and $d', d \in \mathbb{N}^*$ be such that $d' | d | \exp(G)$. Then*

$$\eta_{(d',d)}(G) = \eta\left(C_{\upsilon_1(d',d)} \oplus \cdots \oplus C_{\upsilon_r(d',d)}\right).$$

Using Theorem 1.2, our Proposition 3.1 yields the following useful estimate on the numbers $\eta_{(d',d)}(G)$.

Proposition 3.2. *For every $r \in \mathbb{N}^*$ there exists a constant $c_r > 0$ such that for every finite Abelian group G of rank r, and every $d' | d | \exp(G)$, one has*

$$\eta_{(d',d)}(G) \le c_r(d' - 1) + 1.$$

Proof. Let us consider the group

$$G_{\upsilon(d',d)} = C_{\upsilon_1(d',d)} \oplus \cdots \oplus C_{\upsilon_r(d',d)}.$$

On the one hand, Proposition 3.1 states that $\eta_{(d',d)}(G) = \eta(G_{\upsilon(d',d)})$. On the other hand, it is easily seen that $\upsilon_i(d', d) \mid d'$ for all $i \in [\![1, r]\!]$ and $\upsilon_r(d', d) = d'$, which implies that $G_{\upsilon(d',d)}$ is a subgroup of $C^r_{d'}$ of exponent d'. Now, since $\eta(H) \le \eta(G)$ holds for all groups $H \subseteq G$ such that $\exp(H) = \exp(G)$, Theorem 1.2 gives

$$\eta_{(d',d)}(G) = \eta(G_{\upsilon(d',d)}) \le \eta(C^r_{d'}) \le c_r(d' - 1) + 1. \quad \Box$$

Now, we can prove the main theorem of this section.

Theorem 3.3. *Let G be a finite Abelian group of exponent n and $\mathcal{D}_n = \{d_1, \ldots, d_m\}$. Let also S be a sequence over G containing no tiny zero-sum subsequence, reaching the maximum $\mathsf{k}(S) = \rho(G)$, and being of minimal length regarding this property. Then, the m-tuple $x = \left(x_{d_1}, \ldots, x_{d_m}\right)$, where $x_d = |S_d|$ for all d, is an element of the polytope*

$$\mathbb{P}_G = \{x \in \mathbb{N}^m \mid f_d(x) \ge 0,\ d \in \mathcal{D}_n\},$$

where

$$f_d(x) = \min_{d' \in \mathcal{D}_d \setminus \{1\}} \left(\eta_{(d',d)}(G) - 1 - x_d\right).$$

Proof. Let S be a sequence over G containing no tiny zero-sum subsequence, satisfying $\mathsf{k}(S) = \rho(G)$ and being of minimal length regarding this property. Suppose also that the m-tuple $x = (x_{d_1}, \ldots, x_{d_m})$, where $x_d = |S_d|$ for all d, is not an element of the polytope $\mathbb{P}_G$.

Then, there exists $d_0 \in \mathcal{D}_n$ such that $f_{d_0}(x) < 0$, which means there is a $d_0' \in \mathcal{D}_{d_0} \setminus \{1\}$ satisfying $x_{d_0} \geq \eta_{(d_0',d_0)}(G)$. So, the sequence S contains X elements $g_1, \ldots, g_X$ of order d_0, with $1 < X \leq d_0'$, the sum σ of which is an element of order $\tilde{d_0}$ dividing d_0/d_0'.

Let T be the sequence obtained from S by replacing these X elements by their sum. Reciprocally, for every subsequence T' of T containing σ, let us denote by $\varphi(T')$ the subsequence of S obtained from T' by replacing σ by $g_1, \ldots, g_X$. In particular, note that

$$\begin{aligned}
\mathsf{k}(\varphi(T')) &= \mathsf{k}(T') - \frac{1}{\mathrm{ord}(\sigma)} + \sum_{i=1}^{X} \frac{1}{\mathrm{ord}(g_i)} \\
&= \mathsf{k}(T') + \frac{X}{d_0} - \frac{1}{\tilde{d_0}} \\
&\leq \mathsf{k}(T') + \frac{X}{d_0} - \frac{d_0'}{d_0} \\
&\leq \mathsf{k}(T').
\end{aligned}$$

First, T contains no tiny zero-sum subsequence. Indeed, if T' is a non-empty zero-sum subsequence of T, then either T' contains σ so that $\varphi(T')$ is a non-empty zero-sum subsequence of S, which yields

$$\mathsf{k}(T') \geq \mathsf{k}(\varphi(T')) > 1,$$

or T' does not contain σ, which implies that T' is a subsequence of S and $\mathsf{k}(T') > 1$.

Second, it follows from the equality $\varphi(T) = S$ that $\mathsf{k}(T) \geq \mathsf{k}(S) = \rho(G)$. Therefore, T is a sequence over G containing no tiny zero-sum subsequence such that $\mathsf{k}(T) = \rho(G)$ and $|T| = |S| - X + 1 < |S|$, a contradiction. □

Keeping in mind the notation used in Theorem 3.3, we now obtain the following immediate corollary, giving a general upper bound for $\rho(G)$, expressed as the solution of an integer linear program.

Corollary 3.4. *For every finite Abelian group G of exponent n, one has*

$$\rho(G) \leq \max_{x \in \mathbb{P}_G} \sum_{d \mid n} \frac{x_d}{d}.$$

It is now possible, using Proposition 3.2, to deduce the following quantitative result from Theorem 3.3.

Theorem 3.5. *Let G be a finite Abelian group of rank r and exponent n. Then*

$$\rho(G) \leq c_r \sum_{d \mid n} \frac{P^-(d) - 1}{d}.$$

Proof. Using Theorem 3.3 and Proposition 3.2, we indeed obtain

$$\begin{aligned}
\rho(G) &\leq \sum_{d \mid n} \frac{\eta_{(P^-(d),d)}(G) - 1}{d} \\
&\leq c_r \sum_{d \mid n} \frac{P^-(d) - 1}{d}. \quad \square
\end{aligned}$$

Theorem 3.5 actually improves on the best known upper bound $\rho(G) \le |G|/P^{-}(n)$ proved by Geroldinger and Schneider [11, Lemma 2.1]. In addition, a simple study of the arithmetic function involved in our result (see [12, Lemma 5.1]) shows there exists a constant $\delta_r > 0$ such that, for every finite Abelian group G of rank r and exponent n, one has $\rho(G) \le \delta_r \omega(n)$, where $\omega(n)$ denotes the number of distinct prime divisors of n.

On the other hand, Theorem 3.5 provides us with the following useful result on $\mathsf{t}(G)$ in the case of finite Abelian p-groups.

Corollary 3.6. *Let $p \in \mathcal{P}$. Then, for all $\alpha,\ r \in \mathbb{N}^*$, one has*

$$\mathsf{t}(C^r_{p^\alpha}) \le c_r \left(p^\alpha - 1\right) + 1.$$

Proof. Let $G \simeq C^r_{p^\alpha}$, where $\alpha, r \in \mathbb{N}^*$ and $p \in \mathcal{P}$. Using Theorem 3.5, one has

$$\rho(G) \le c_r \sum_{i=1}^{\alpha} \left(\frac{p-1}{p^i}\right) = c_r \left(\frac{p^\alpha - 1}{p^\alpha}\right).$$

Then, for every sequence S over G containing no tiny zero-sum subsequence, one has

$$\frac{|S|}{p^\alpha} \le \mathsf{k}(S) \le \rho(G) \le c_r \left(\frac{p^\alpha - 1}{p^\alpha}\right),$$

which gives $|S| \le c_r\,(p^\alpha - 1)$ and completes the proof. □

As already mentioned, an interesting special case is the one of finite Abelian groups of rank two. In this case, specifying $c_2 = 3$ in Corollary 3.6 implies that, for all primes p and $\alpha \in \mathbb{N}^*$, the equality

$$\mathsf{t}\left(C_{p^\alpha} \oplus C_{p^\alpha}\right) = 3p^\alpha - 2 = \eta\left(C_{p^\alpha} \oplus C_{p^\alpha}\right) \tag{3}$$

holds. Building on the case where G has prime power exponent, we can now turn to the general case of finite Abelian groups.

4. Proof of the main theorem

Let G be a finite Abelian group of exponent $n = q_1^{\alpha_1} \cdots q_\ell^{\alpha_\ell}$, with $q_1 < \cdots < q_\ell$, and let S be a sequence over G. We consider the following partition:

$$\mathcal{D}_n \setminus \{1\} = \bigcup_{i=1}^{\ell} \mathcal{A}_i, \quad \text{where } \mathcal{A}_i = \{d \in \mathcal{D}_n \setminus \{1\} : P^{+}(d) = q_i\}.$$

In particular, for all $d \in \mathcal{A}_i$, one has $d \le \Delta_i = q_1^{\alpha_1} \cdots q_i^{\alpha_i}$. Now, for every $d \in \mathcal{A}_i$, we denote by k_d the smallest integer such that

$$|S_d| < k_d \frac{d}{q_i^{v_{q_i}(d)}} + \eta_{\left(\frac{d}{q_i^{v_{q_i}(d)}},\, d\right)}(G).$$

The number k_d has the following combinatorial interpretation.

Lemma 4.1. *Let G be a finite Abelian group of exponent n, and $d \in \mathcal{A}_i$. Let also S be a sequence over G. Then S_d contains at least k_d disjoint non-empty subsequences $S'_1, \dots, S'_{k_d}$ such that, for all $j \in [\![1, k_d]\!]$,*

$$\sigma(S'_j) \in G_{q_i^{\nu_{q_i}(d)}} \quad \textit{and} \quad \mathsf{k}(S'_j) \le \frac{1}{q_i^{\nu_{q_i}(d)}}.$$

Proof. First, we shall prove by induction on $k \in [\![0, k_d]\!]$ that S_d contains at least k disjoint non-empty subsequences $S'_1, \dots, S'_k$ such that, for all $j \in [\![1, k]\!]$,

$$\sigma(S'_j) \in G_{q_i^{\nu_{q_i}(d)}} \quad \text{and} \quad |S'_j| \le \frac{d}{q_i^{\nu_{q_i}(d)}}.$$

If $k = 0$ then this statement is clearly true. Now, assume that the statement holds for some $k \in [\![0, k_d - 1]\!]$, and let us prove that it holds for $k + 1$ as well. By the induction hypothesis, we already know that S_d contains k disjoint non-empty subsequences $S'_1, \dots, S'_k$ such that, for all $j \in [\![1, k]\!]$,

$$\sigma(S'_j) \in G_{q_i^{\nu_{q_i}(d)}} \quad \text{and} \quad |S'_j| \le \frac{d}{q_i^{\nu_{q_i}(d)}}.$$

Moreover, the sequence T_d obtained from S_d by deleting all elements of $S'_1, \dots, S'_k$ satisfies

$$\begin{aligned}
|T_d| &= |S_d| - \sum_{j=1}^{k} |S'_j| \\
&\ge (k_d - 1)\frac{d}{q_i^{\nu_{q_i}(d)}} + \eta_{\left(\frac{d}{q_i^{\nu_{q_i}(d)}}, d\right)}(G) - k\frac{d}{q_i^{\nu_{q_i}(d)}} \\
&\ge \eta_{\left(\frac{d}{q_i^{\nu_{q_i}(d)}}, d\right)}(G),
\end{aligned}$$

so that S_d contains a non-empty subsequence S'_{k+1} disjoint from $S'_1, \dots, S'_k$ such that

$$\sigma(S'_{k+1}) \in G_{q_i^{\nu_{q_i}(d)}} \quad \text{and} \quad |S'_{k+1}| \le \frac{d}{q_i^{\nu_{q_i}(d)}},$$

which completes the induction. Therefore, S_d contains k_d disjoint non-empty subsequences $S'_1, \dots, S'_{k_d}$ such that, for all $j \in [\![1, k_d]\!]$,

$$\sigma(S'_j) \in G_{q_i^{\nu_{q_i}(d)}} \quad \text{and} \quad |S'_j| \le \frac{d}{q_i^{\nu_{q_i}(d)}}.$$

In addition, for all $j \in [\![1, k_d]\!]$, one clearly has

$$\mathsf{k}(S'_j) = \frac{|S'_j|}{d} \le \frac{1}{q_i^{\nu_{q_i}(d)}},$$

and the lemma is proved. $\square$

Corollary 4.2. *Let G be a finite Abelian group of exponent n, and $d \in \mathcal{A}_i$. Let also S be a sequence over G. Then S_d contains at least k_d disjoint non-empty subsequences $S'_1, \dots, S'_{k_d}$ such that, for all $j \in [\![1, k_d]\!]$,*

$$\sigma(S'_j) \in G_{q_i^{v_{q_i}(n)}} \quad \textit{and} \quad \mathsf{k}(S'_j) \leq \frac{1}{\operatorname{ord}\left(\sigma\left(S'_j\right)\right)}.$$

Proof. By Lemma 4.1, S_d contains at least k_d disjoint non-empty subsequences $S'_1, \dots, S'_{k_d}$ such that, for all $j \in [\![1, k_d]\!]$,

$$\sigma(S'_j) \in G_{q_i^{v_{q_i}(d)}} \quad \text{and} \quad \mathsf{k}(S'_j) \leq \frac{1}{q_i^{v_{q_i}(d)}}.$$

Thus, the desired result directly follows from the fact that, for all $j \in [\![1, k_d]\!]$, one has

$$\sigma(S'_j) \in G_{q_i^{v_{q_i}(d)}} \subseteq G_{q_i^{v_{q_i}(n)}}. \quad \square$$

Lemma 4.3. *Let G be a finite Abelian group of exponent n, and let S be a sequence over G containing no tiny zero-sum subsequence. Then, for all $i \in [\![1, \ell]\!]$,*

$$\sum_{d \in \mathcal{A}_i} k_d \leq \mathsf{t}\left(G_{q_i^{v_{q_i}(n)}}\right) - 1.$$

Proof. Let us set $t = \mathsf{t}\left(G_{q_i^{v_{q_i}(n)}}\right)$, and assume that

$$\sum_{d \in \mathcal{A}_i} k_d \geq t.$$

Then, it follows from Corollary 4.2 that S contains t disjoint non-empty subsequences $S'_1, \dots, S'_t$ such that, for all $j \in [\![1, t]\!]$,

$$\sigma(S'_j) \in G_{q_i^{v_{q_i}(n)}} \quad \text{and} \quad \mathsf{k}(S'_j) \leq \frac{1}{\operatorname{ord}\left(\sigma\left(S'_j\right)\right)}.$$

Now, let us consider the sequence T over $G_{q_i^{v_{q_i}(n)}}$ defined by

$$T = \prod_{j=1}^{t} \sigma(S'_j).$$

Since T is a sequence of length $|T| = t$, then it contains a tiny zero-sum subsequence $T' \mid T$. In other words, there exists a non-empty subset $J \subseteq [\![1, t]\!]$ such that

$$T' = \prod_{j \in J} \sigma(S'_j).$$

Now, let us set

$$S' = \prod_{j \in J} S'_j.$$

Since $\sigma(S') = \sigma(T') = 0$, then S' is a non-empty zero-sum subsequence of S. In addition, we have the following chain of inequalities:

$$
\begin{aligned}
\mathsf{k}(S') &= \sum_{j \in J} \mathsf{k}(S'_j) \\
&\leq \sum_{j \in J} \frac{1}{\operatorname{ord}\left(\sigma\left(S'_j\right)\right)} \\
&= \mathsf{k}(T') \\
&\leq 1.
\end{aligned}
$$

Therefore, S contains a tiny zero-sum subsequence, and the proof is complete. □

We can now prove the main theorem of this paper.

Proof of Theorem 2.1. Let S be a sequence over G containing no tiny zero-sum subsequence. For every $i \in [\![1, \ell]\!]$, Corollary 3.6 and Lemma 4.3 yield

$$
\sum_{d \in \mathcal{A}_i} k_d \leq \mathsf{t}\left(G_{q_i^{\nu_{q_i}(n)}}\right) - 1 \leq c_r \left(q_i^{\nu_{q_i}(n)} - 1\right),
$$

so that, setting $\Delta_0 = 1$, one obtains

$$
\begin{aligned}
\sum_{d \in \mathcal{A}_i} \left(|S_d| - \left(\eta_{\left(\frac{d}{q_i^{\nu_{q_i}(d)}}, d\right)}(G) - 1\right)\right) &\leq \sum_{d \in \mathcal{A}_i} k_d \frac{d}{q_i^{\nu_{q_i}(d)}} \\
&\leq \Delta_{i-1} \sum_{d \in \mathcal{A}_i} k_d \\
&\leq c_r \Delta_{i-1} \left(q_i^{\nu_{q_i}(n)} - 1\right) \\
&= c_r \left(\Delta_i - \Delta_{i-1}\right).
\end{aligned}
$$

Now, using Proposition 3.2, we have

$$
\begin{aligned}
|S| &= \sum_{i=1}^{\ell} \sum_{d \in \mathcal{A}_i} |S_d| \\
&= \sum_{i=1}^{\ell} \sum_{d \in \mathcal{A}_i} \left(\eta_{\left(\frac{d}{q_i^{\nu_{q_i}(d)}}, d\right)}(G) - 1\right) + \sum_{i=1}^{\ell} \sum_{d \in \mathcal{A}_i} \left(|S_d| - \left(\eta_{\left(\frac{d}{q_i^{\nu_{q_i}(d)}}, d\right)}(G) - 1\right)\right) \\
&\leq c_r \sum_{i=1}^{\ell} \sum_{d \in \mathcal{A}_i} \left(\frac{d}{q_i^{\nu_{q_i}(d)}} - 1\right) + c_r \sum_{i=1}^{\ell} \left(\Delta_i - \Delta_{i-1}\right) \\
&= c_r \sum_{d \mid n} \left(\frac{d}{P^+(d)^{\nu_{P^+(d)}(d)}} - 1\right) + c_r (n - 1),
\end{aligned}
$$

which completes the proof of the theorem. □

5. A special sum of divisors

In this section, we derive Theorem 2.2 from the study of a particular arithmetic function. We recall $\mathcal{P} = \{p_1 = 2 < p_2 = 3 < \cdots\}$ denotes the set of prime numbers, and start by proving the following easy lemma.

Lemma 5.1. *For every integer $\ell \geq 1$, one has*

$$\prod_{i=1}^{\ell}\left(1+\frac{1}{p_i-1}\right) \leq p_{\ell+1}-1.$$

Proof. To start with, let us prove the following statement by induction on $\ell \geq 1$.

$$\prod_{i=2}^{\ell}\left(1+\frac{1}{2(i-1)}\right) \leq \ell.$$

One can readily notice that this statement is true for $\ell = 1$ and $\ell = 2$. Assume now that the statement holds for some $\ell \geq 2$. Then, let us show that it holds for $\ell + 1$ also. Indeed,

$$\begin{aligned}\prod_{i=2}^{\ell+1}\left(1+\frac{1}{2(i-1)}\right) &= \prod_{i=2}^{\ell}\left(1+\frac{1}{2(i-1)}\right)\left(1+\frac{1}{2\ell}\right)\\ &\leq \ell\left(1+\frac{1}{2\ell}\right)\\ &\leq \ell+1,\end{aligned}$$

and we are done. The desired result now follows from the following chain of inequalities, using the trivial bound $p_\ell \geq 2\ell - 1$, for all $\ell \geq 2$, in the following fashion.

$$\begin{aligned}\prod_{i=1}^{\ell}\left(1+\frac{1}{p_i-1}\right) &= 2\prod_{i=2}^{\ell}\left(1+\frac{1}{p_i-1}\right)\\ &\leq 2\prod_{i=2}^{\ell}\left(1+\frac{1}{2(i-1)}\right)\\ &\leq 2\ell\\ &= (2(\ell+1)-1)-1\\ &\leq p_{\ell+1}-1. \quad \square\end{aligned}$$

We can now prove the main result of this section.

Proposition 5.2. *Let $f(n) = \sum_{d|n} \frac{d}{P^+(d)}$. For every integer $n \geq 1$, one has $f(n) \leq n$.*

Proof. Writing $n = q_1^{\alpha_1} \cdots q_\ell^{\alpha_\ell}$, where $q_1 < \cdots < q_\ell$, we prove the desired result by induction on $\ell = \omega(n) \geq 1$.

If $\ell = 1$, then one has

$$\begin{aligned}f(q_1^{\alpha_1}) &= \sum_{d|q_1^{\alpha_1}} \frac{d}{P^+(d)}\\ &= 1+\sum_{i=1}^{\alpha_1} q_1^{i-1}\\ &= 1+\left(\frac{q_1^{\alpha_1}-1}{q_1-1}\right),\end{aligned}$$

so that we now have

$$\begin{aligned}\frac{f(q_1^{\alpha_1})}{q_1^{\alpha_1}} &= \frac{1}{q_1^{\alpha_1}} + \frac{1}{q_1 - 1}\left(\frac{q_1^{\alpha_1}-1}{q_1^{\alpha_1}}\right)\\ &\le \frac{1}{q_1^{\alpha_1}} + \left(\frac{q_1^{\alpha_1}-1}{q_1^{\alpha_1}}\right)\\ &= 1,\end{aligned}$$

and we are done.

Assume now that the statement holds true for some $\ell \ge 1$. Then, setting $\sigma(n) = \sum_{d|n} d$, we obtain the following equalities.

$$\begin{aligned}f(q_1^{\alpha_1}\cdots q_{\ell+1}^{\alpha_{\ell+1}}) &= \sum_{d|q_1^{\alpha_1}\cdots q_{\ell+1}^{\alpha_{\ell+1}}} \frac{d}{P^+(d)}\\ &= \sum_{d|q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}} \frac{d}{P^+(d)} + \sum_{i=1}^{\alpha_{\ell+1}} \sum_{d|q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}} \frac{dq_{\ell+1}^i}{q_{\ell+1}}\\ &= f(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}) + \sigma(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell})\sum_{i=1}^{\alpha_{\ell+1}} q_{\ell+1}^{i-1}\\ &= f(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}) + \sigma(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell})\left(\frac{q_{\ell+1}^{\alpha_{\ell+1}}-1}{q_{\ell+1}-1}\right)\end{aligned}$$

so that

$$\frac{f(q_1^{\alpha_1}\cdots q_{\ell+1}^{\alpha_{\ell+1}})}{q_1^{\alpha_1}\cdots q_{\ell+1}^{\alpha_{\ell+1}}} = \frac{f(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell})}{q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}}\frac{1}{q_{\ell+1}^{\alpha_{\ell+1}}} + \frac{\sigma(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell})}{q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}}\frac{1}{(q_{\ell+1}-1)}\frac{q_{\ell+1}^{\alpha_{\ell+1}}-1}{q_{\ell+1}^{\alpha_{\ell+1}}}.$$

First, by the induction hypothesis, we have

$$\frac{f(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell})}{q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}} \le 1.$$

Second, since $\sigma(n)$ is multiplicative, Lemma 5.1 yields

$$\begin{aligned}\frac{\sigma(q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell})}{q_1^{\alpha_1}\cdots q_\ell^{\alpha_\ell}} &= \frac{\sigma(q_1^{\alpha_1})}{q_1^{\alpha_1}}\cdots\frac{\sigma(q_\ell^{\alpha_\ell})}{q_\ell^{\alpha_\ell}}\\ &\le \prod_{i=1}^{\ell}\left(1+\frac{1}{q_i-1}\right)\\ &\le \prod_{i=1}^{\ell}\left(1+\frac{1}{p_i-1}\right)\\ &\le p_{\ell+1}-1\\ &\le q_{\ell+1}-1.\end{aligned}$$

Thus, we obtain

$$\frac{f(q_1^{\alpha_1}\cdots q_{\ell+1}^{\alpha_{\ell+1}})}{q_1^{\alpha_1}\cdots q_{\ell+1}^{\alpha_{\ell+1}}} \le \frac{1}{q_{\ell+1}^{\alpha_{\ell+1}}} + \frac{q_{\ell+1}^{\alpha_{\ell+1}}-1}{q_{\ell+1}^{\alpha_{\ell+1}}} = 1,$$

which completes the proof. □

As an immediate corollary of Proposition 5.2, we now prove Theorem 2.2.

Proof of Theorem 2.2. Let G be a finite Abelian group of rank r and exponent n. Then, by Theorem 2.1 and Proposition 5.2, one has

$$\begin{aligned}\mathsf{t}(G) &\le c_r \sum_{d\mid n}\left(\frac{d}{P^+(d)^{\nu_{P^+(d)}(d)}}-1\right)+c_r\,(n-1)+1\\ &\le c_r \sum_{d\mid n}\left(\frac{d}{P^+(d)}-1\right)+c_r\,(n-1)+1\\ &\le 2c_r\,(n-1)+1. \quad \square\end{aligned}$$

6. A few concluding remarks

As previously stated, the exact value of $\mathsf{t}(G)$ is currently known for cyclic groups and elementary 2-groups only. In this context, the special case of finite Abelian groups of rank two is of particular interest, and the following conjecture appears to be inviting.

Conjecture 1. *For all integers $m, n \in \mathbb{N}^*$ such that $m \mid n$, one has*

$$\mathsf{t}(C_m \oplus C_n) = 2m + n - 2.$$

If true, the statement of Conjecture 1 would nicely extend the theorem of Kleitman and Lemke. In view of equality (3), this conjecture readily holds true for all groups $G \simeq C_{p^\alpha} \oplus C_{p^\alpha}$, where $p \in \mathcal{P}$ and $\alpha \in \mathbb{N}^*$. Moreover, note that $\mathsf{t}(C_m \oplus C_n) \ge 2m + n - 2$ easily follows from (1).

Even though far less is known on the exact value of $\eta(G)$ for finite Abelian groups of higher rank, it would be worth knowing how close it actually is to $\mathsf{t}(G)$ in the general case. A first step in this direction is the following lemma, which gives a simple property shared by all finite Abelian groups for which $\mathsf{t}(G) = \eta(G)$ holds.

Lemma 6.1. *Let G be a finite Abelian group such that $\mathsf{t}(G) = \eta(G)$. Then for every subgroup H of G, one has $\eta(H) \le \eta(G)$.*

Proof. Let G be as in the statement of the lemma, and let H be a subgroup of G. By definition, the inequality $\mathsf{t}(H) \le \mathsf{t}(G)$ holds. Therefore,

$$\eta(H) \le \mathsf{t}(H) \le \mathsf{t}(G) = \eta(G),$$

and the required result is proved. $\square$

However, it turns out that the above property is restrictive enough to guarantee that for every $r \ge 4$, there exists a finite Abelian group G of rank r for which $\mathsf{t}(G) > \eta(G)$. The proof of this fact actually relies on the following key invariant in zero-sum combinatorics. Given a finite Abelian group G, let $\mathsf{D}(G)$ denote the smallest integer $t \in \mathbb{N}^*$ such that every sequence S over G of length $|S| \ge t$ contains a non-empty zero-sum subsequence. The number $\mathsf{D}(G)$ is called the *Davenport constant* of G, and we refer to [10,9] for background and connections with algebraic number theory.

In what follows, we will need a classical theorem, independently proved in the late sixties by Kruyswijk [18] and Olson [16], stating that

$$\mathsf{D}(C_{p^{\alpha_1}} \oplus \cdots \oplus C_{p^{\alpha_r}}) = \sum_{i=1}^{r}\left(p^{\alpha_i} - 1\right) + 1 \tag{4}$$

for all primes p and positive integers $\alpha_1, \ldots, \alpha_r$. Our result now is the following.

Proposition 6.2. *For every integer $r \geq 4$, there exists a finite Abelian group G of rank r for which* $\mathsf{t}(G) > \eta(G)$.

Proof. Let $r \geq 4$ be an integer. It is an easy exercise to prove there is an integer $\alpha \geq 2$ such that

$$\frac{\ln(2r-1)}{\ln 3} \leq \alpha \leq \frac{\ln(2^r - r)}{\ln 3}.$$

Now, let us consider

$$G = C_3^{r-1} \oplus C_{3^\alpha} \quad \text{and} \quad H = C_3^r.$$

Since G is a finite Abelian 3-group, it follows from (4) that

$$\mathsf{D}(G) = 2(r-1) + 3^\alpha \leq 2\exp(G) - 1,$$

so that [17, Theorem 1.2] yields

$$\eta(G) \leq \mathsf{D}(G) + \exp(G) - 1 = 2r + 2.3^\alpha - 3.$$

On the other hand, we deduce from (2) that

$$\eta(H) \geq (2^r - 1)(3-1) + 1 = 2^{r+1} - 1.$$

Therefore, H is a subgroup of G such that $\eta(H) > \eta(G)$, and the desired result follows from Lemma 6.1. □

It would certainly be interesting to know whether the equality $\mathsf{t}(G) = \eta(G)$ holds for all finite Abelian groups of rank three. In another direction, we would also like to address the following conjecture.

Conjecture 2. *For all integers $r, n \in \mathbb{N}^*$, one has* $\mathsf{t}(C_n^r) = \eta(C_n^r)$.

It can readily be seen that Conjecture 2 holds whenever G is an elementary p-group, since all non-zero elements of G have same order in this case. In addition, our Corollary 3.6 already gives a "nearly optimal" answer when G is of the form C_n^r, with n a prime power.

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