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Relations among tautological classes revisited

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Abstract

We give a simple generalisation of a theorem of Morita (1989) [10,11], which leads to a great number of relations among tautological classes on moduli spaces of Riemann surfaces.

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Keywords: Moduli spaces; Tautological algebra

1. Introduction

A *surface bundle* $\Sigma_g \to E \xrightarrow{\pi} B$ is a smooth fibre bundle with fibre Σ_g, the oriented surface of genus g without boundary. Such a bundle is equipped with a vertical tangent bundle $T^vE \to E$, which is an oriented 2-plane bundle – or equivalently a complex line bundle – over the total space E. The Mumford–Morita–Miller classes of this bundle are defined to be

$$\kappa_i(E) := \pi_!(c_1(T^vE)^{i+1}) \in H^{2i}(B;\mathbb{Z}),$$

that is, the pushforward (or fibre integral) along π of the $(i+1)$st power of the first Chern class of the vertical tangent bundle.[1] Isomorphism classes of such surface bundles are classified by homotopy classes of maps into the *moduli space*

$$\mathcal{M}_g := */\!\!/\mathrm{Diff}^+(\Sigma_g) \simeq B\mathrm{Diff}^+(\Sigma_g),$$

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[1] The sign convention usual in algebraic geometry would change this by $(-1)^{i+1}$, so that κ_i is instead the class obtained by fibre integrating the $(i+1)$st power of the first Chern class of the vertical *co*tangent bundle.

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which carries the surface bundle $\overline{\mathcal{M}_g} := \Sigma_g /\!\!/ \mathrm{Diff}^+(\Sigma_g)$, and the characteristic classes κ_i exist universally as classes $\kappa_i \in H^{2i}(\mathcal{M}_g; \mathbb{Z})$. This space is the homotopy type of the moduli stack of Riemann surfaces $\mathbf{M}_g$, and as we are only discussing homological questions is a good substitute for it.

Definition 1.1. The *tautological algebra* $R^*(\mathcal{M}_g)$ is the image of the map

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \longrightarrow H^*(\mathcal{M}_g; \mathbb{Q}).$$

Suppose that we wish to parametrise pairs $(\Sigma_g \to E \xrightarrow{\pi} B, c \in H^2(E; \mathbb{Z}))$ of a surface bundle and a second integral cohomology class – or equivalently isomorphism class of complex line bundle – on the total space. Isomorphism classes of such pairs are classified by homotopy classes of maps into the moduli space

$$\mathcal{S}_g(\mathbb{CP}^\infty) := \mathrm{map}(\Sigma_g, \mathbb{CP}^\infty) /\!\!/ \mathrm{Diff}^+(\Sigma_g)$$

which fits into a fibration sequence

$$\mathrm{map}(\Sigma_g, \mathbb{CP}^\infty) \longrightarrow \mathcal{S}_g(\mathbb{CP}^\infty) \xrightarrow{f} \mathcal{M}_g.$$

The surface bundle classified by f is $\overline{\mathcal{S}_g(\mathbb{CP}^\infty)} = (\Sigma_g \times \mathrm{map}(\Sigma_g, \mathbb{CP}^\infty)) /\!\!/ \mathrm{Diff}^+(\Sigma_g)$, and the evaluation defines a map $\overline{\mathcal{S}_g(\mathbb{CP}^\infty)} \to \mathbb{CP}^\infty$; so there is a canonical class $c_{\mathrm{univ}} \in H^2(\overline{\mathcal{S}_g(\mathbb{CP}^\infty)}; \mathbb{Z})$. A surface bundle $\Sigma_g \to E \xrightarrow{\pi} B$ determines a map $B \to \mathcal{M}_g$, and giving a lift ℓ of this map up the fibration f gives a map $\overline{\ell} : E \to \overline{\mathcal{S}_g(\mathbb{CP}^\infty)}$ and so a class $c = \overline{\ell}^*(c_{\mathrm{univ}}) \in H^2(E; \mathbb{Z})$.

Given the pair $(\Sigma_g \to E \xrightarrow{\pi} B, c \in H^2(E; \mathbb{Z}))$, there are characteristic classes defined by the formula

$$\kappa_{i,j} := \pi_!(c_1(T^v E)^{i+1} \cdot c^j) \in H^{2i+2j}(B; \mathbb{Z})$$

which generalise the Mumford–Morita–Miller classes, as $\kappa_{i,0} = \kappa_i$. The fundamental invariant of a class $c \in H^2(E; \mathbb{Z})$ is its *degree*, which is defined to be the degree of the class restricted to any fibre—or equivalently as the integer $\kappa_{-1,1} = \pi_!(c) \in H^0(B; \mathbb{Z})$. The moduli space $\mathcal{S}_g(\mathbb{CP}^\infty)$ splits into path components

$$\mathcal{S}_g(\mathbb{CP}^\infty) = \coprod_{d \in \mathbb{Z}} \mathcal{S}_g(\mathbb{CP}^\infty)_d$$

indexed by the degree. Our main theorem, which is nothing but a simple extension of the work of Morita [10,11], gives a characteristic class on $\mathcal{S}_g(\mathbb{CP}^\infty)$ whose $(g+1)$st power is rationally trivial. In Section 2 we then show how to use this class to produce relations in the cohomology of $\mathcal{M}_g$. As it will occur in almost every formula, we write $\chi = \chi(\Sigma_g) = 2 - 2g$ for the Euler characteristic of the surface Σ_g.

Theorem A. *For $g \geq 2$ the cohomology class*

$$\Omega := \frac{1}{\gcd(\chi, d)^2} \left(\chi^2 \kappa_{-1,2} - 2d\chi\kappa_{0,1} + d^2 \kappa_1 \right) \in H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_d; \mathbb{Z})$$

has the property that Ω^{g+1} is torsion, annihilated by $\frac{(2g+2)!}{2^{g+1}(g+1)!}$.

In particular, given a surface bundle $\Sigma_g \to E \xrightarrow{\pi} B$ and a class $c \in H^2(E;\mathbb{Z})$ of degree d, the pullback of Ω via the classifying map is

$$\Omega(E,c) := \frac{1}{\gcd(\chi,d)^2}\left(\chi^2\pi_!(c^2) - 2d\chi\pi_!(c_1(T^vE)\cdot c) + d^2\kappa_1(E)\right) \in H^2(B;\mathbb{Z})$$

and the theorem says that $\Omega(E,c)^{g+1}$ is torsion, so in particular trivial in rational cohomology. The assignment $c \mapsto \Omega(E,c)$ is easily seen to be non-linear in the cohomology class c, so given several classes $c_1,\dots,c_n \in H^2(E,\mathbb{Z})$ we may form the integral linear combinations $c_A = \sum A_i c_i$, and we will see in Section 2.4 that the collection $\{\Omega(E,c_A)\}_{A\in\mathbb{Z}^n}$ gives more information than the collection $\{\Omega(E,c_i)\}_{i=1}^n$.

An outline of the paper is as follows. In Section 2 we explain how to use Theorem A to obtain relations in $R^*(\mathcal{M}_g)$. We first explain the method used by Morita [10,11], in a way that makes it clear how it can be generalised. In Section 2.4 we state our general relations, and in Section 2.6 we give computational evidence that they are highly nontrivial. In Section 2.7 we explain one way of obtaining relations in "closed form". Finally, in Section 3 we prove Theorem A.

2. Producing tautological relations

We will now explain how to use Theorem A to obtain relations in $R^*(\mathcal{M}_g)$. We adopt the following useful convention: if $X \to \mathcal{M}_g$ is a space with a map to $\mathcal{M}_g$ classifying a surface bundle, we always write $\pi : \overline{X} \to X$ for the surface bundle it classifies, and $T^v \to \overline{X}$ for the vertical tangent bundle. We also write $e := e(T^v) \in H^2(\overline{X})$ for the Euler class of the vertical tangent bundle.

2.1. *Marked points and diagonals*

Let us write $\mathcal{M}_g(n)$ for the homotopy quotient $(\Sigma_g)^n /\!\!/ \mathrm{Diff}^+(\Sigma_g)$, which parametrises surface bundles with n ordered (not necessarily distinct) points in each fibre, write $\pi : \mathcal{M}_g(n) \to \mathcal{M}_g$ for the projection, and if $I \subset \{1,2,\dots,n\}$ is a subset write $\pi_I : \mathcal{M}_g(n) \to \mathcal{M}_g(|I|)$ for the map that projects to these factors. We identify $\mathcal{M}_g(1)$ with $\overline{\mathcal{M}_g}$, so there is an Euler class $e \in H^2(\mathcal{M}_g(1))$, and let $e_i \in H^2(\mathcal{M}_g(n))$ be $\pi^*_{\{i\}}(e)$. We identify $\mathcal{M}_g(n+1)$ with $\overline{\mathcal{M}_g(n)}$ by projecting to the first n points, and write e for the class e_{n+1} when considered to be on $\overline{\mathcal{M}_g(n)}$.

For each non-empty subset $I \subset \{1,2,\dots,n\}$, let $i_I : \Delta_I \hookrightarrow (\Sigma_g)^n$ be the inclusion of the submanifold

$$\Delta_I := \{(x_1,x_2,\dots,x_n) \in (\Sigma_g)^n \mid x_i = x_j \text{ if both } i \text{ and } j \text{ are in } I\},$$

and let $\mathcal{M}_g(n,I) := \Delta_I /\!\!/ \mathrm{Diff}^+(\Sigma_g)$. We also write $i_I : \mathcal{M}_g(n,I) \to \mathcal{M}_g(n)$ for the induced inclusion, and $\pi : \mathcal{M}_g(n,I) \to \mathcal{M}_g$ for the projection. A choice of bijection between the quotient set $\{1,2,\dots,n\}/I$ and $\{1,2,\dots,n-|I|+1\}$ gives a homeomorphism $\mathcal{M}_g(n,I) \cong \mathcal{M}_g(n-|I|+1)$ over $\mathcal{M}_g$.

Lemma 2.1. *For each non-empty subset $I \subset \{1,2,\dots,n\}$, there are defined certain classes $\nu_I \in H^{2(|I|-1)}(\mathcal{M}_g(n);\mathbb{Z})$, which satisfy*

(i) $\nu_I \cdot \nu_J = \nu_{I\cup J}$ *if I and J intersect in a single element,*

(ii) $\nu_{\{1,2\}}^2 = \nu_{\{1,2\}} \cdot e_1 = \nu_{\{1,2\}} \cdot e_2$ *in* $H^4(\mathcal{M}_g(2); \mathbb{Z})$, *and*
(iii) $\pi_!(\nu_I \cdot (-)) = \pi_!(i_I^*(-)) : H^{k+2n}(\mathcal{M}_g(n)) \to H^k(\mathcal{M}_g; \mathbb{Z})$.

Proof. Let us write $\mathcal{D} = \mathrm{Diff}^+(\Sigma_g)$, and for a $\mathcal{D}$-space X write $H^*_{\mathcal{D}}(X) := H^*(X /\!\!/ \mathcal{D}; \mathbb{Z})$ for the integral Borel equivariant cohomology. Write X_I for the complement of Δ_I in $(\Sigma_g)^n$. The Thom class of the normal bundle $N\Delta_I$ of Δ_I inside $(\Sigma_g)^n$ gives a class $u \in H^{2(|I|-1)}_{\mathcal{D}}(D(N\Delta_I), S(N\Delta_I))$, which under the excision isomorphism

$$H^{2(|I|-1)}_{\mathcal{D}}(D(N\Delta_I), S(N\Delta_I)) \cong H^{2(|I|-1)}_{\mathcal{D}}((\Sigma_g)^n, X_I)$$

and the map $r^* : H^*_{\mathcal{D}}((\Sigma_g)^n, X_I) \to H^*_{\mathcal{D}}((\Sigma_g)^n)$ gives the class we call ν_I.

To be completely precise, we must describe the excision isomorphism more carefully, as unfortunately there is not a $\mathcal{D}$-equivariant map $D(N\Delta_I) \to (\Sigma_g)^n$ directly inducing it (as there is no $\mathcal{D}$-equivariant tubular neighbourhood). To fix this, let Met be the space of Riemannian metrics on $(\Sigma_g)^n$ whose exponential map restricts to an embedding of the unit disc bundle of $N\Delta_I$, and write $\exp : D(N\Delta_I) \times \mathrm{Met} \to (\Sigma_g)^n$ for the parametrised exponential map. This gives maps of pairs

$$(D(N\Delta_I), S(N\Delta_I)) \longleftarrow (D(N\Delta_I), S(N\Delta_I)) \times \mathrm{Met} \xrightarrow{\exp} ((\Sigma_g)^n, X_{\Delta_I})$$

which are equivariant, and which are both homotopy equivalences (as Met is contractible). These give isomorphisms on Borel cohomology, which makes precise the excision step.

The three claims now follow quickly from properties of Thom classes (in particular, the identity $\nu_I \cdot (-) = r^*(u) \cdot (-) = r^*(u \cdot i_I^*(-))$). □

2.2. Morita's relations

The results of Morita [10,11] which our Theorem A generalises come from an observation in complex geometry. If Σ is a Riemann surface of genus g, it has an associated Jacobian manifold $J(\Sigma) = H^1(\Sigma; \mathcal{O})/H^1(\Sigma; \mathbb{Z})$, a complex torus of complex dimension g carrying a canonical symplectic form. The space $J(\Sigma)$ parametrises both the set of isomorphism classes of degree zero line bundles, and the set of degree zero divisors up to linear equivalence. If $\pi : E \to B$ is a family of genus g Riemann surfaces, there is an associated family $J(\pi) : J_B(E) \to B$ of Jacobian varieties, and the symplectic forms combine to give a class $\Omega \in H^2(J_B(E))$.

Morita shows that Ω^{g+1} is rationally trivial, and he described two situations in which one can find non-trivial sections $s : B \to J_B(E)$ and describe the class $s^*(\Omega)$ in terms of well-known characteristic classes. Instead of working with the Jacobian $J_B(E)$, we prefer to work directly with line bundles on E, and we will now describe Morita's two situations from this point of view.

Consider the universal smooth Σ_g-bundle with section

$$\Sigma_g \xrightarrow{i} \overline{\mathcal{M}_g(1)} \xrightarrow{\pi} \mathcal{M}_g(1).$$

Using the identification $\overline{\mathcal{M}_g(1)} = \mathcal{M}_g(2)$, we have defined a class $\nu := \nu_{\{1,2\}} \in H^2(\overline{\mathcal{M}_g(1)}; \mathbb{Z})$, which satisfies $\nu^2 = \nu \cdot e$. We calculate $\int_{\Sigma_g} i^*(\nu) = 1$ and $\int_{\Sigma_g} i^*(e) = \chi$, so $c := \chi \cdot \nu - e \in H^2(\overline{\mathcal{M}_g(1)}; \mathbb{Z})$ has degree zero, and we obtain a map

$$f : \mathcal{M}_g(1) \longrightarrow \mathcal{S}_g(\mathbb{CP}^\infty)_0$$

classifying this data. We then have

$$-f^*(\Omega) = \pi_!((\chi \cdot \nu - e)^2) = -[\chi^2 \cdot \pi_!(\nu^2) - \chi \cdot \pi_!(\nu \cdot e) + \pi_!(e^2)],$$

and using $\nu^2 = \nu \cdot e = \nu \cdot \pi^*(e_1)$ and $\pi_!(\nu \cdot \pi^*(-)) = -$ from Lemma 2.1, we see that

$$-f^*(\Omega) = (2-\chi)\chi e - \kappa_1 \in H^2(\mathcal{M}_g(1))$$

and hence $((2-\chi)\chi e - \kappa_1)^{g+1}$ is trivial in the group $H^*(\mathcal{M}_g(1))$, which appears in Morita's work as [11, Theorem 1.6]. Fibre integrating this equation to $\mathcal{M}_g$ gives the relation

$$\sum_{i=0}^{g} \binom{g+1}{i+1} \left(\frac{\kappa_1}{\chi(\chi-2)}\right)^{g-i} \cdot \kappa_i = 0 \in H^{2g}(\mathcal{M}_g),$$

and multiplying Ω^{g+1} by e^k for $k > 0$ and then fibre integrating gives the relation

$$\sum_{i=-1}^{g} \binom{g+1}{i+1} \left(\frac{\kappa_1}{\chi(\chi-2)}\right)^{g-i} \cdot \kappa_{i+k} = 0 \in H^{2g+2k}(\mathcal{M}_g),$$

in both cases bearing in mind that $\kappa_0 = \chi$. These relations show that $R^*(\mathcal{M}_g)$ is generated by the classes $\kappa_1, \kappa_2, \dots, \kappa_{g-1}$, as they give an explicit way of writing all the higher κ_i in terms of these classes.

We now give Morita's second construction. There is a smooth Σ_g-bundle

$$\Sigma_g \xrightarrow{i} \overline{\mathcal{M}_g(2)} \xrightarrow{\pi} \mathcal{M}_g(2),$$

and $\overline{\mathcal{M}_g(2)} = \mathcal{M}_g(3)$ has classes $\nu_1 := \nu_{\{1,3\}}$ and $\nu_2 := \nu_{\{2,3\}}$. We may let $c := \nu_1 - \nu_2 \in H^2(\overline{\mathcal{M}_g(2)}; \mathbb{Z})$, which has degree zero on each fibre, so is classified by a map $f : \mathcal{M}_g(2) \to \mathcal{S}_g(\mathbb{CP}^\infty)_0$.

The formula of Theorem A is then

$$-f^*(\Omega) = -\pi_!((\nu_1 - \nu_2)^2) = (2\nu_{\{1,2\}} - e_1 - e_2) \in H^2(\mathcal{M}_g(2)).$$

This appears in Morita's work as [10, Theorem 2.1], and produces relations in the cohomology of $\mathcal{M}_g(2)$ because $(2\nu_{\{1,2\}} - e_1 - e_2)^{g+1} = 0$, but also relations in $\mathcal{M}_g(1)$ and $\mathcal{M}_g$ by fibre integration. By the identities $\nu_{\{1,2\}}^2 = \nu_{\{1,2\}} \cdot e_1 = \nu_{\{1,2\}} \cdot e_2$, the class $(2\nu_{\{1,2\}} - e_1 - e_2)^{g+1}$ may be expanded and fibre integrated to see that for each $h \geq 0$ the class κ_{g+h-1} is expressible in terms of sums of products of κ-classes of lower degree, which reproves Mumford's theorem [13, Section 5–6] that the classes $\kappa_1, \dots, \kappa_{g-2}$ generate the tautological algebra.

2.3. *Looijenga's theorem*

Looijenga [9] has shown that the tautological algebra is trivial above degree $(g-2)$, and has dimension at most 1 in degree $(g-2)$. Using the Witten conjecture, it can be shown [5, Theorem 2] that $\kappa_{g-2} \neq 0$ on $\mathcal{M}_g$, and so $R^{2(g-2)}(\mathcal{M}_g)$ has precisely dimension 1. Unfortunately, all the relations in $R^*(\mathcal{M}_g)$ obtained in the previous section (by fibre integrating relations on $\mathcal{M}_g(1)$ and $\mathcal{M}_g(2)$ down to $\mathcal{M}_g$) lie in degrees greater than $g-2$, and so in light of Looijenga's theorem are not so interesting.

Given a monomial $\kappa_1^{i_1} \cdots \kappa_{g-2}^{i_{g-2}}$ such that $\sum_{j=1}^{g-2} j \cdot i_j = g-2$, one can ask for the unique rational numbers $M(g; i_1, \dots, i_{g-2})$ such that

$$\kappa_1^{i_1} \cdots \kappa_{g-2}^{i_{g-2}} = M(g; i_1, \dots, i_{g-2}) \cdot \kappa_{g-2}.$$

This means finding relations in $R^{g-2}(\mathcal{M}_g)$, and in the following section we give a procedure for finding relations in this and lower degrees.

2.4. Generalising Morita's relations

Consider the universal family

$$\Sigma_g \xrightarrow{i} \overline{\mathcal{M}_g(n)} \xrightarrow{\pi} \mathcal{M}_g(n),$$

and let $\nu_i := \nu_{\{i,n+1\}} \in H^2(\overline{\mathcal{M}_g(n)}; \mathbb{Z})$, under the identification $\overline{\mathcal{M}_g(n)} = \mathcal{M}_g(n+1)$. For each vector $A = (A_1, \dots, A_n) \in \mathbb{Z}^n$, define

$$c_A := \sum_{i=1}^{n} A_i \nu_i,$$

a class of degree $d_A := \sum A_i$. This data is classified by a map

$$f_A : \mathcal{M}_g(n) \longrightarrow \mathcal{S}_g(\mathbb{CP}^\infty)_{d_A},$$

and it is easy to calculate that $f_A^*(\gcd(\chi, d_A)^2 \cdot \Omega)$ is

$$\Omega_A := \sum_{i=1}^{n} (\chi^2 A_i^2 - 2 d_A \chi A_i) e_i + 2\chi^2 \sum_{i<j} A_i A_j \nu_{\{i,j\}} + d_A^2 \kappa_1 \tag{2.1}$$

and so $\Omega_A^{g+1} = 0 \in H^{2(g+1)}(\mathcal{M}_g(n))$ *for all vectors* $A \in \mathbb{Z}^n$. In particular, considered as a polynomial in the variables $A_1, \dots, A_n$ with coefficients in $H^{2g+2}(\mathcal{M}_g(n))$, the expression Ω_A^{g+1} is the zero polynomial. This gives a tremendous number of tautological relations in $H^{2g+2}(\mathcal{M}_g(n))$, which may be fibre integrated to $\mathcal{M}_g$ to give trivial tautological classes in the group $H^{2g+2-2n}(\mathcal{M}_g)$.

Example 2.2. For $n = 2$ we have

$$\Omega_A = A_1^2(\kappa_1 - 2g\chi e_1) + 2A_1A_2(\chi^2 \nu_{\{1,2\}} - \chi(e_1 + e_2) + \kappa_1) + A_2^2(\kappa_1 - 2g\chi e_2)$$

and one may calculate the coefficient of $A_1^i A_2^{2g+2-i}$ in Ω_A^{g+1} to be

$$\sum_{a=0}^{\lfloor i/2 \rfloor} \binom{g+1}{a, i-2a} (\kappa_1 - 2g\chi e_1)^a (2\chi^2 \nu_{\{1,2\}} - 2\chi(e_1 + e_2) + 2\kappa_1)^{i-2a} \\ \times (\kappa_1 - 2g\chi e_2)^{g+1-i+a}$$

which is then trivial in $H^{2g+2}(\mathcal{M}_g(2))$.

2.5. Relation to the work of Faber

Faber [5] has made detailed conjectures concerning the algebraic structure of the tautological algebras $R^*(\mathcal{M}_g)$, and has verified these conjectures up to genus 23 by producing sufficiently many relations among the κ_i. In genus 24 all known relations are insufficient to establish Faber's conjecture, and in particular his conjecture predicts an extra relation in $R^{24}(\mathcal{M}_{24})$ which has not yet been discovered. We will remark on this again in Section 2.7.

The method used by Faber to produce relations is similar to that used here in that it also involves producing relations on $\mathcal{M}_g(n)$ (or rather its algebraic analogue, $\mathbf{C}_g^n$, the iterated fibre product of the universal curve) and pushing them forward to $\mathcal{M}_g$. However, Faber's method for producing relations on $\mathcal{M}_g(n)$ is completely algebro-geometric and does not seem to be related to Morita's. It seems that even the relations that are obtained on $\mathcal{M}_g(n)$ are different: ours involve only the classes κ_1, e_i and $\nu_{\{i,j\}}$, and the higher κ classes appear only after fibre integration, whereas Faber's seem to already include the higher κ classes.

2.6. Calculations

For general n, it is a formidable problem of combinatorial algebra to extract explicit relations from (2.1), but we give in the following examples some computer calculations we have made in low genus, which demonstrate that the relations obtained are non-trivial. The calculations were made in MAGMA [2], where for $n = 2$ and 3 we computed the general form of Ω_A^{g+1}, and extracted the collection of coefficients of monomials in the variables $A_1, \ldots, A_n$. This gives a collection of relations in the cohomology of $\mathcal{M}_g(n)$, which we then fibre integrate down to $\mathcal{M}_g$ (using the formulae of Lemma 2.1) and simplify.

Example 2.3. For $g = 3$ we find that κ_1^2 and κ_2 are zero in $R^*(\mathcal{M}_3)$.

Example 2.4. For $g = 4$ we find that κ_2^2, $\kappa_1\kappa_2$, and κ_3 are zero in $R^*(\mathcal{M}_4)$, and we obtain the relation

$$3\kappa_1^2 = -32\kappa_2.$$

Example 2.5. For $g = 5$ we find that κ_4, $\kappa_1\kappa_3$, κ_2^2, $\kappa_2\kappa_3$, and κ_3^2 are zero in $R^*(\mathcal{M}_5)$, and we obtain the relations

$$\kappa_1^3 = 288\kappa_3$$
$$\kappa_1\kappa_2 = -20\kappa_3.$$

There is a relation in degree 4 (namely $5\kappa_1^2 = -72\kappa_2$, cf. [5, p. 12]) which we have not detected, as for $g = 5$ and $n = 3$ we only obtain relations in degree 6 (and above). Possibly using the same method for $n = 4$ we may obtain this relation, although the problem becomes computationally unwieldy very quickly.

Example 2.6. For $g = 6$ we find that κ_4^2, $\kappa_3\kappa_4$, $\kappa_2\kappa_4$, $\kappa_2\kappa_3$, $\kappa_1\kappa_4$, and κ_5 are zero in $R^*(\mathcal{M}_6)$, and we obtain the relations

$$5\kappa_1^4 = -73728\kappa_4$$
$$5\kappa_1^2\kappa_2 = 4064\kappa_4$$
$$5\kappa_2^2 = -226\kappa_4$$
$$\kappa_1\kappa_3 = -32\kappa_4.$$

There are also relations in degree 6 which we are not able to see using $n = 3$, but could perhaps find using $n = 4$.

Finally we give an example of the relations obtained in higher genus. Although the $n = 2$ calculations at genus 9 take only a few seconds, the $n = 3$ calculations take about fifteen minutes. We have also made calculations at genus 12, where the $n = 3$ calculations take several hours, but these give a very incomplete set of relations, and we do not include them here. We imagine that a less naïve algorithm would be able to go to significantly higher degrees.

Example 2.7. For $g = 9$ we find that κ_7^2, $\kappa_6\kappa_7$, $\kappa_5\kappa_7$, $\kappa_4\kappa_7$, $\kappa_3\kappa_7$, $\kappa_2\kappa_7$, κ_4^2, $\kappa_3\kappa_5$, $\kappa_2\kappa_6$, $\kappa_1\kappa_7$, and κ_8 are zero in $R^*(\mathcal{M}_9)$, and we obtain the relations

$$\begin{aligned}
\kappa_1^7 &= 26011238400\kappa_7 & 7\kappa_1\kappa_3^2 &= 125824\kappa_7\\
7\kappa_1^5\kappa_2 &= -6195953664\kappa_7 & 7\kappa_1^3\kappa_4 &= -2796544\kappa_7\\
35\kappa_1^3\kappa_2^2 &= 1060508672\kappa_7 & 7\kappa_1\kappa_2\kappa_4 &= 97536\kappa_7\\
35\kappa_1\kappa_2^3 &= -36513024\kappa_7 & 7\kappa_3\kappa_4 &= -2424\kappa_7\\
35\kappa_1^4\kappa_3 &= 743491584\kappa_7 & \kappa_1^2\kappa_5 &= 6240\kappa_7\\
5\kappa_1^2\kappa_2\kappa_3 &= -3667456\kappa_7 & \kappa_2\kappa_5 &= -220\kappa_7\\
35\kappa_2^2\kappa_3 &= 891328\kappa_7 & \kappa_1\kappa_6 &= -84\kappa_7.
\end{aligned}$$

Note that $\mathbb{Q}[\kappa_1, \dots, \kappa_7]$ has rank 15 (the number of partitions of 7) in degree 14, but by Looijenga's theorem $R^*(\mathcal{M}_9)$ has rank one in degree 14. We see above precisely 14 linearly independent relations, so have determined all intersection numbers on $\mathcal{M}_9$.

In our calculations for $(g, n) = (12, 3)$, we do not find all the intersection numbers on $\mathcal{M}_{12}$. We find 34 linearly independent relations, but $\mathbb{Q}[\kappa_1, \dots, \kappa_{10}]$ has rank 77 in degree 20 and so we should find 76 relations.

In these examples, even in high degrees where all tautological classes become trivial, each relation we have written is not obtained from fibre integrating any single trivial tautological class on $\mathcal{M}_g(n)$ coming from the above construction. Rather, each individual relation typically gives a linear equation among the tautological classes in a certain degree with very large coefficients, and this system of equations turns out to be highly overdetermined.

2.7. *Fibre integration, graphs, and a closed form relation*

In order to use the relations of Section 2.4 to obtain relations on $\mathcal{M}_g$ we require a way to compute the fibre integral of a monomial

$$e^{\mathbf{a}}\nu^{\mathbf{b}} := \left(e_1^{a_1}\cdots e_n^{a_n}\right)\cdot\left(\nu_{\{1,2\}}^{b_{12}}\nu_{\{1,3\}}^{b_{13}}\cdots\nu_{\{n-1,n\}}^{b_{n-1,n}}\right) \tag{2.2}$$

along the map $\pi : \mathcal{M}_g(n) \to \mathcal{M}_g$. The following graphical formalism is used, implicitly or explicitly, throughout the literature on tautological relations. We refer the reader to [14, Section 4] for a detailed discussion.

Definition 2.8. A *weighted graph* Γ on a set S is a graph on this set of vertices (allowing multiple edges between a pair of vertices, but no loops at a single vertex), which in addition has each vertex labelled by a natural number, called its *weight*. We denote by $v(\Gamma)$ and $e(\Gamma)$ the total number of vertices and edges of Γ respectively, and by $w(\Gamma)$ the *total weight* of Γ, which is $e(\Gamma)$ plus the sum of the weights of the vertices.

To each monomial (2.2) we associate a weighted graph $\Gamma(\mathbf{a}, \mathbf{b})$ on the set of vertices $\{1, 2, \dots, n\}$ by putting $b_{i,j}$ edges between $\{i\}$ and $\{j\}$, and weighting the vertex $\{i\}$ by a_i. Using the relations $\nu_{\{i,j\}}^2 = \nu_{\{i,j\}} \cdot e_i = \nu_{\{i,j\}} \cdot e_j$ and $\nu_{\{i,j\}}\nu_{\{j,k\}} = \nu_{\{i,j,k\}} = \nu_{\{i,j\}}\nu_{\{i,k\}}$ when i, j, and k are distinct, we see that the monomials $e^{\mathbf{a}}\nu^{\mathbf{b}}$ and $e^{\mathbf{a}'}\nu^{\mathbf{b}'}$ are equal if the graphs $\Gamma(\mathbf{a}, \mathbf{b})$ and $\Gamma(\mathbf{a}', \mathbf{b}')$ differ by a sequence of the following moves.

(i) If there is more than one edge from a vertex x to a vertex y, we may remove one of them and add 1 to the weight of x, or the inverse of this move.

(ii) Given two edges leaving the same vertex x and going to y and z respectively, where there is no edge yz, we may create an edge yz at the expense of removing an edge from xy or xz.

By reducing weighted graphs to weighted trees using (ii), and repeatedly applying part (iii) of Lemma 2.1, it is easy to see that

$$\pi_!(e^{\mathbf{a}}\nu^{\mathbf{b}}) = \prod_{\Gamma' \subset \Gamma(\mathbf{a},\mathbf{b})} \kappa_{w(\Gamma')-v(\Gamma')}$$

where the product is taken over the connected components of $\Gamma(\mathbf{a}, \mathbf{b})$. In this formula we must remember that $\kappa_i = 0$ for all $i < 0$, and $\kappa_0 = \chi$.

We will now describe a closed form relation that can be obtained using this description of fibre integration, which was worked out with Rahul Pandharipande, whom we thank for letting us include it here. We consider the family $\pi : \overline{\mathcal{M}_g(g-1)} \to \mathcal{M}_g(g-1)$ with the degree zero line bundle $c := e + 2\sum_{i=1}^{g-1} \nu_{\{i,g\}}$, and compute

$$\kappa_{-1,2} = \pi_!(c^2) = 8 \sum_{1 \le i < j \le g-1} \nu_{\{i,j\}} + 8 \sum_{i=1}^{g-1} e_i + \kappa_1.$$

By Theorem A we have $(\kappa_{-1,2})^r = 0 \in H^{2r}(\mathcal{M}_g(g-1))$ for each $r \ge g+1$. Letting $D_n := \sum_{1\le i<j\le n} \nu_{\{i,j\}} + \sum_{1 \le i \le n} e_i \in H^2(\mathcal{M}_g(n))$, we obtain a relation

$$\sum_{i=0}^{r} 8^i \cdot \pi_!(D_{g-1}^i) \cdot \kappa_1^{r-i} = 0 \in H^{2r-2(g-1)}(\mathcal{M}_g) \tag{2.3}$$

for each $r \ge g+1$. To make this explicit, we must be able to compute $\pi_!(D_{g-1}^i)$ for all i, and it is easiest to compute $\pi_!(D_n^r)$ for all n and r.

Lemma 2.9. *There is an identity of formal power series*

$$1 + \sum_{n>0}\sum_{r\ge 0} \pi_!(D_n^r)\frac{t^r}{r!}\frac{x^n}{n!} = \exp\left(\sum_{n>0}\sum_{r\ge 0} \kappa_{r-n} C_{n,r} \frac{t^r}{r!}\frac{x^n}{n!}\right) \tag{2.4}$$

where the coefficients $C_{n,r}$ are defined by

$$1 + \sum_{n>0}\sum_{r\ge 0} \binom{n+1}{2}^r \frac{t^r}{r!}\frac{x^n}{n!} = \exp\left(\sum_{n>0}\sum_{r\ge 0} C_{n,r} \frac{t^r}{r!}\frac{x^n}{n!}\right). \tag{2.5}$$

Proof. Consider the following game. Start with $\{1, 2, \ldots, n\}$ considered as a set of vertices, and a number r of turns. In each turn we choose either a $\nu_{\{i,j\}}$ and add an edge between vertices i and j, or we choose an e_i and add 1 to the weight of the vertex i. The outcome of such a game G is a sequence of ν's and e's (giving a monomial) and a weighted graph Γ_G, and fibre-integrating the corresponding monomial in ν's and e's to $\mathcal{M}_g$ gives $\kappa(\Gamma_G) := \prod_{\Gamma' \subset \Gamma_G} \kappa_{w(\Gamma')-v(\Gamma')}$, where the product is taken over connected components Γ' of Γ_G.

Expand out D_n^r without collecting terms: D_n has $\binom{n}{2} + n = \binom{n+1}{2}$ terms, so there are $\binom{n+1}{2}^r$ terms in D_n^r, and each term corresponds to an instance of the above game, so $\pi_!(D_n^r) = \sum_{\text{games } G} \kappa(\Gamma_G)$. The exponential relation [6, Section 1] between generating functions for connected and disconnected graphs (of various types) means that Eq. (2.5) identifies $C_{n,r}$ as the number of games G that can be played on $\{1, 2, \dots, n\}$ with r turns such that the resulting weighted graph Γ_G is connected, and in this case $\kappa(\Gamma_G) = \kappa_{r-n}$. As the contribution $\kappa(\Gamma_G)$ of a game G is the product of the contributions of its connected subgames, Eq. (2.4) follows. $\square$

Remark 2.10. Pandharipande and the author have computed the relation (2.3) in $R^{24}(\mathcal{M}_{24})$ (as well as similar relations obtained from multiplying $(-\Omega)^r$ by $\nu_{\{i,j\}}$'s before fibre-integrating to $\mathcal{M}_g$, which can also be given a closed form, if less pleasant, description), and checked it with Faber, who informed us that it was in the span of the relations he has obtained. Thus these new relations do not seem to establish Faber's conjecture in genus 24.

The relation (2.3) can be considered modulo decomposable cohomology classes, where it gives that

$$\kappa_{r-(g-1)} \cdot \left[\exp\left(\sum_{n>0}(2-2g)\cdot C_{n,n}\frac{t^n}{n!}\frac{x^n}{n!}\right)\cdot\left(\sum_{r>n>0} C_{n,r}\frac{t^r}{r!}\frac{x^n}{n!}\right)\right]_{t^r x^{g-1}} \tag{2.6}$$

is decomposable for each $r \geq g+1$.

Remark 2.11. By recent improvements [1,15] to Harer's stability theorem [7] the classes $\kappa_1, \kappa_2, \dots, \kappa_{\lfloor g/3 \rfloor}$ are not decomposable, and so when $r-(g-1) \leq g/3$ the coefficient in (2.6) must be zero. It seems to be difficult to give a combinatorial proof of this fact. Computer calculation suggests that for $r-(g-1) > \lfloor g/3 \rfloor$ the coefficient in (2.6) is always nonzero, and a combinatorial proof of this would give a new proof of Morita's theorem [12] that $\kappa_1, \kappa_2, \dots, \kappa_{\lfloor g/3 \rfloor}$ generate $R^*(\mathcal{M}_g)$.

3. Proof of Theorem A

Let us first treat the degree zero case, where we must show that as long as $g \geq 2$, the class

$$\kappa_{-1,2} \in H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_0; \mathbb{Z})$$

has the property that $(\kappa_{-1,2})^{g+1}$ is torsion, annihilated by $\frac{(2g+2)!}{2^{g+1}(g+1)!}$.

We write $\Gamma_g := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ for the *mapping class group* of Σ_g, and we will implicitly use the theorem of Earle–Eells [3] that the quotient map $\mathrm{Diff}^+(\Sigma_g) \to \Gamma_g$ is a homotopy equivalence as long as $g \geq 2$. Write H for the Γ_g-module $H_1(\Sigma_g; \mathbb{Z})$, and note that Poincaré duality gives an isomorphism $H \cong H^\vee$ as Γ_g-modules. Write $\widetilde{\Gamma}_g$ for Kawazumi's [8] *extended mapping class group* $H \rtimes \Gamma_g$, and let $\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty) \subset \mathrm{map}(\Sigma_g, \mathbb{CP}^\infty)$ be the subspace of degree zero maps. In [4, Section 3] we have established a commutative diagram

$$\begin{tikzcd}
\mathbb{CP}^\infty \arrow[r, equal] \arrow[d, "c"] & \mathbb{CP}^\infty \arrow[d] & \\
\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty) \arrow[r, "i"] \arrow[d, "q"] & \mathcal{S}_g(\mathbb{CP}^\infty)_0 \arrow[r, "f"] \arrow[d, "p"] & \mathcal{M}_g \arrow[d, "\simeq"] \\
BH \arrow[r, "j"] & B\widetilde{\Gamma}_g \arrow[r] & B\Gamma_g
\end{tikzcd} \tag{3.1}$$

where the rows and columns are fibration sequences, and the horizontal fibrations have (compatible) sections, which we call s.

Morita has shown [10] that there is a class $\Omega \in H^2(B\widetilde{\Gamma}_g; \mathbb{Z})$ enjoying the properties

(i) Ω restricts to twice the symplectic form $\omega \in H^2(BH; \mathbb{Z}) \cong \wedge^2 H$,
(ii) $s^*(\Omega) = 0 \in H^2(B\Gamma_g; \mathbb{Z})$, where s is the canonical section, and
(iii) $\Omega^{g+1} \in H^{2g+2}(B\widetilde{\Gamma}_g; \mathbb{Z})$ is torsion, annihilated by $\frac{(2g+2)!}{2^{g+1}(g+1)!}$.

The degree zero case of Theorem A will follow once we show that $p^*(\Omega) = -\kappa_{-1,2}$ in $H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_0; \mathbb{Z})$. Ebert and the author [4] have calculated $H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_0; \mathbb{Z})$ as long as $g \geq 6$, and shown that it is free abelian of rank three with rational basis κ_1, $\kappa_{-1,2}$ and $\kappa_{0,1}$. It is especially easy to prove Theorem A for $g \geq 6$ using this calculation, but we wish to have the result for all genera, so go to a little more trouble.

Lemma 3.1. *There is an exact sequence*

$$0 \longrightarrow H^2(\mathcal{M}_g; \mathbb{Z}) \xrightarrow{f^*} H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_0; \mathbb{Z}) \xrightarrow{i^*} H^2(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z})^{\Gamma_g} \tag{3.2}$$

in which f^* is split by s^*, and a short exact sequence of Γ_g-modules

$$0 \longrightarrow \wedge^2 H = H^2(BH; \mathbb{Z}) \xrightarrow{q^*} H^2(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z}) \xrightarrow{c^*} H^2(\mathbb{CP}^\infty; \mathbb{Z}) \longrightarrow 0. \tag{3.3}$$

Proof. The fibration $\mathbb{CP}^\infty \to \mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty) \to BH$ is trivial (though not as a fibration of $\mathrm{Diff}^+(\Sigma_g)$-spaces), as $\mathbb{CP}^\infty$ is an Eilenberg–MacLane space so the mapping space $\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty)$ is a generalised Eilenberg–MacLane space. This immediately establishes the exact sequence (3.3), and furthermore shows that

$$H^1(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z}) \cong H^1(BH; \mathbb{Z}) \cong H \tag{$*$}$$

as Γ_g-modules.

We now consider the fibration $\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty) \to \mathcal{S}_g(\mathbb{CP}^\infty)_0 \to \mathcal{M}_g$, and study its Serre spectral sequence in integral cohomology. We have

$$E_2^{1,1} = H^1(\mathcal{M}_g; H^1(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z}))$$

which is isomorphic to $H^1(\Gamma_g; H)$ using $(*)$, but Morita has shown [10, Theorem 4.1] that this group is zero as long as $g \geq 1$. The exact sequence (3.2) follows from this spectral sequence in total degree 2. $\square$

Proposition 3.2. *The class $p^*(\Omega) \in H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_0; \mathbb{Z})$ is $-\kappa_{-1,2}$.*

Proof. Consider the class $X := \kappa_{-1,2} + p^*(\Omega) \in H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_0; \mathbb{Z})$, which we will show is zero. Certainly $s^*(X)$ is zero, so by the exact sequence (3.2) the class X is determined by its restriction to $H^2(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z})^{\Gamma_g}$.

The first property of the class Ω implies that $i^*(\Omega) = q^*(2\omega)$, so in particular $c^*(i^*(\Omega)) = 0$. The map c classifies the trivial surface bundle $\pi : \mathbb{CP}^\infty \times \Sigma_g \to \mathbb{CP}^\infty$ equipped with the line bundle $\pi^*(\gamma)$, where $\gamma \to \mathbb{CP}^\infty$ is the tautological line bundle. Writing $x = c_1(\gamma) \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$, we calculate

$$c^*(i^*(\kappa_{-1,2})) = 0 \qquad c^*(i^*(\kappa_{1,0})) = (2 - 2g) \cdot x,$$

so the class $i^*(X) \in H^2(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z})^{\Gamma_g}$ restricts to zero on $\mathbb{CP}^\infty$. By taking Γ_g-invariants in the exact sequence (3.3) we find that $i^*(X)$ is in the image of

$$q^* : (\wedge^2 H)^{\Gamma_g} \longrightarrow H^2(\mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty); \mathbb{Z})^{\Gamma_g},$$

so is a multiple of $q^*(\omega)$; write $i^*(X) = N \cdot q^*(\omega)$. To show that $N = 0$, we compute an example.

Choose a $p \in \Sigma_g$, and let $\pi : \Sigma_g \times \Sigma_g \to \Sigma_g$ be the trivial bundle given by projection to the first factor. Let $u \in H^2(\Sigma_g; \mathbb{Z})$ be the Poincaré dual to a point. Let $c_\Delta \in H^2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ be the Poincaré dual to the diagonal, and $c_p \in H^2(\Sigma_g \times \Sigma_g; \mathbb{Z})$ be the Poincaré dual to $\Sigma_g \times \{p\}$. We choose a map $\Sigma_g \times \Sigma_g \to \mathbb{CP}^\infty$ representing $c = c_\Delta - c_p$, with adjoint $g : \Sigma_g \to \mathrm{map}^0(\Sigma_g, \mathbb{CP}^\infty)$, and calculate

$$g^*(i^*(\kappa_{-1,2})) = -2g \cdot u \qquad g^*(i^*(\kappa_{0,1})) = (2-2g) \cdot u.$$

The map $q \circ g : \Sigma_g \to BH$ by construction induces the identity map on first cohomology. Write $a_1, \dots, a_g, b_1, \dots, b_g$ for a symplectic basis of $H = H^1(\Sigma_g; \mathbb{Z})$, so we have $\omega = \sum_i a_i \wedge b_i \in \wedge^2 H$ and so $g^*(q^*(\omega)) = \sum a_i \cdot b_i = g \cdot u \in H^2(\Sigma_g; \mathbb{Z})$. Thus

$$0 = g^*(i^*(X) - N \cdot q^*(\omega)) = -2g \cdot u + 2g \cdot u - Ng \cdot u$$

and $N = 0$. $\square$

To prove Theorem A for general degrees, note that if c is the degree d line bundle on the universal bundle over $\mathcal{S}_g(\mathbb{CP}^\infty)_d$ then $c' = \frac{1}{\gcd(d,\chi)}(\chi c - de)$ – where e is the Euler class of the vertical tangent bundle – is a degree zero line bundle, and this produces a map

$$f_d : \mathcal{S}_g(\mathbb{CP}^\infty)_d \longrightarrow \mathcal{S}_g(\mathbb{CP}^\infty)_0.$$

Theorem A now follows from the proposition given below.

Proposition 3.3. *The class $f_d^*(\kappa_{-1,2})$ is*

$$\frac{1}{\gcd(d,\chi)^2}\left(\chi^2\kappa_{-1,2} - 2d\chi\kappa_{0,1} + d^2\kappa_1\right) \in H^2(\mathcal{S}_g(\mathbb{CP}^\infty)_d; \mathbb{Z}).$$

Proof. The class $f_d^*(\kappa_{-1,2})$ is by definition the fibre integral of $(c')^2$. We compute

$$(c')^2 = \frac{1}{\gcd(d,\chi)^2}(\chi c - de)^2 = \frac{1}{\gcd(d,\chi)^2}(\chi^2 c^2 - 2d\chi ec + d^2e^2)$$

and the fibre integral of this class is, by definition of $\kappa_{-1,2}$, $\kappa_{0,1}$ and κ_1, the element in the statement of the proposition. $\square$

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References

[1] Søren Boldsen, Improved homological stability for the mapping class group with integral or twisted coefficients, Math. Z. 270 (2012) 297–329.

[2] Wieb Bosma, John Cannon, Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (3–4) (1997) 235–265. Computational algebra and number theory (London, 1993).

[3] C.J. Earle, J. Eells, A fibre bundle description of Teichmueller theory, J. Differential Geom. 3 (1969) 19–43.

[4] Johannes Ebert, Oscar Randal-Williams, Stable cohomology of the universal Picard varieties and the extended mapping class group, Doc. Math. 17 (2012) 417–450.
[5] Carel Faber, A conjectural description of the tautological ring of the moduli space of curves, in: Moduli of Curves and Abelian Varieties, Aspects Math., E33, Vieweg, Braunschweig, 1999, pp. 109–129.
[6] Frank Harary, Edgar M. Palmer, Graphical Enumeration, Academic Press, New York, 1973.
[7] John L. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Ann. of Math. (2) 121 (2) (1985) 215–249.
[8] Nariya Kawazumi, A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces, Invent. Math. 131 (1) (1998) 137–149.
[9] Eduard Looijenga, On the tautological ring of $\mathcal{M}_g$, Invent. Math. 121 (2) (1995) 411–419.
[10] Shigeyuki Morita, Families of Jacobian manifolds and characteristic classes of surface bundles. I, Ann. Inst. Fourier (Grenoble) 39 (3) (1989) 777–810.
[11] Shigeyuki Morita, Families of Jacobian manifolds and characteristic classes of surface bundles. II, Math. Proc. Cambridge Philos. Soc. 105 (1) (1989) 79–101.
[12] Shigeyuki Morita, Generators for the tautological algebra of the moduli space of curves, Topology 42 (4) (2003) 787–819.
[13] David Mumford, Towards an enumerative geometry of the moduli space of curves, in: Arithmetic and Geometry, Vol. II, in: Progr. Math., vol. 36, Birkhäuser, Boston, Boston, MA, 1983, pp. 271–328.
[14] Rahul Pandharipande, The κ ring of the moduli of curves of compact type, Acta Mathematica 208 (2012) 335–388.
[15] Oscar Randal-Williams, Resolutions of moduli spaces and homological stability, 2009. arXiv:0909.4278v3.