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Advances in Mathematics 231 (2012) 1291–1307

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Power sums of Coxeter exponents

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Received 4 July 2011; accepted 28 June 2012
Available online 9 August 2012

Communicated by Bertram Kostant

Abstract

Consider an irreducible finite Coxeter system. We show that for any nonnegative integer n the sum of the nth powers of the Coxeter exponents can be written uniformly as a polynomial in four parameters: h (the Coxeter number), r (the rank), α, β (two further parameters).

MSC: 20F55

Keywords: Coxeter group; Root system; Power sum; Todd polynomial; Coxeter exponents

1. Introduction

Let (W, S) be an irreducible finite Coxeter system of rank r with $S = \{s_1, \dots, s_r\}$ its set of simple reflections. The Coxeter transformation $c := s_1 \cdots s_r \in W$ has order $|c| = h$ known as the Coxeter number, and the eigenvalues of c in the reflection representation of W are of the form $e^{2\pi i m_1/h}, \dots, e^{2\pi i m_r/h}$ with $1 = m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r = h - 1$ the exponents of (W, S). Furthermore, for any permutation σ of $\{1, \dots, r\}$ the elements c and $s_{\sigma(1)} \cdots s_{\sigma(r)}$ are conjugate in W. Hence the exponents do not depend on the enumeration of the simple reflections. Recall

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doi:10.1016/j.aim.2012.06.020

that the symmetry $m_i + m_{r+1-i} = h$ follows from the facts that c has no eigenvalue 1 and that the reflection representation is defined over the reals.

In this note we will derive uniform expressions for the power sums $\sum_{i=1}^r m_i^n$ for any $n \in \mathbb{Z}_{\geqslant 0}$. Of course, for $n = 0$ the sum is r, and for $n = 1$ the symmetry $m_i + m_{r+1-i} = h$ shows that the sum is $\frac{1}{2}rh$. We shall see that

$$\sum_{i=1}^{r} m_i^n = n!\, r \operatorname{Td}_n(\gamma_1, \ldots, \gamma_n)$$

where $\operatorname{Td}_n(\gamma_1, \ldots, \gamma_n)$ denotes the nth Todd polynomial evaluated at $\gamma_1, \ldots, \gamma_n$ (for $n \geqslant 3$ odd $\operatorname{Td}_n(\gamma_1, \ldots, \gamma_n)$ does not depend on γ_n, as follows from Proposition 3.1). The γ_i's can be chosen to be polynomials in four parameters (details below) with integer coefficients. This answers Panyushev's question in [9]. Furthermore, this once again unites the work of Coxeter and Todd, who were students together at Cambridge (see also [1]).

2. Some history and preliminaries

For type A_r the exponents are just $1, 2, \ldots, r$ and one has Bernoulli's formula

$$\sum_{i=1}^{r} i^n = \frac{1}{n+1}\big(B_{n+1}(r+1) - B_{n+1}(1)\big) \tag{2.1}$$

where $B_{n+1}(x)$ is the $(n+1)$st Bernoulli polynomial, defined by the expansion

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t\, e^{xt}}{e^t - 1}.$$

For general types uniform formulae for the power sums up to third power are listed in the epilogue of [12]. Besides the Coxeter number h and the rank r they depend (for the squares and the cubes) on a further parameter γ which is defined for the crystallographic types with crystallographic root system Φ ($= \Phi_+ \cup \Phi_-$ a decomposition into the sets of positive and negative roots) by the formula (see [3, Chapter VI, Section 1, no. 12])

$$\sum_{\varphi\in\Phi} \frac{\langle\lambda|\varphi\rangle\langle\mu|\varphi\rangle}{\langle\varphi|\varphi\rangle^2} = \gamma\, \langle\lambda|\mu\rangle \quad (\lambda, \mu \in \operatorname{span}_{\mathbb{R}} \Phi) \tag{2.2}$$

where $\langle\ |\ \rangle$ denotes the Killing form on $\operatorname{span}_{\mathbb{R}} \Phi$, which is the W-invariant (symmetric) bilinear form characterized by

$$\langle\lambda|\mu\rangle = \sum_{\varphi\in\Phi} \langle\lambda|\varphi\rangle\langle\mu|\varphi\rangle \quad (\lambda, \mu \in \operatorname{span}_{\mathbb{R}} \Phi).$$

It turns out that $\gamma = kgg^\vee$ where $k = \langle\theta|\theta\rangle/\langle\theta_{\mathrm{s}}|\theta_{\mathrm{s}}\rangle \in \{1, 2, 3\}$ with $\theta, \theta_{\mathrm{s}} \in \Phi_+$ the highest resp. highest short roots, and $g = 1/\langle\theta|\theta\rangle \in \mathbb{Z}_{>0}$ is the dual Coxeter number of Φ whereas $g^\vee$ is the dual Coxeter number of the dual root system $\Phi^\vee$. So $\gamma = h^2$ if Φ is simply-laced. For the noncrystallographic types $\gamma = 2m^2 - 5m + 6$ for $\mathsf{I}_2(m)$ (the formula is also valid for the crystallographic types, where $m = 3, 4, 6$); $\gamma = 124$ for type H_3; and $\gamma = 1116$ for type H_4. The values of γ for the noncrystallographic types may seem somewhat ad hoc at first glance, but Proposition 5.3 offers a general formula. The formulae from [12] read as follows:

$$\sum_{i=1}^{r} m_i^n = \begin{cases} r & \text{if } n=0, \\ \frac{1}{2}rh & \text{if } n=1, \\ \frac{1}{6}r(h^2+\gamma-h) & \text{if } n=2, \\ \frac{1}{4}rh(\gamma-h) & \text{if } n=3. \end{cases} \tag{2.3}$$

Remark 2.1. The power sums for the fourth and higher powers are not of the form r times some functions depending only on h and γ, as a computation for the types A_{h-1} and $\mathsf{D}_{(h+2)/2}$ shows.

Panyushev gave the universal formula [9, Proposition 3.1]

$$\sum_{\varphi\in\Phi_+} \mathrm{ht}(\varphi)^2 = \frac{1}{12}r(h+1)\gamma \tag{2.4}$$

for the sum of the heights squares of all positive roots. He then suspects [9, Remark 3.4] that for the sum of the heights of all positive roots there is no similar formula in the general case; however, for simply-laced root systems he mentions

$$\sum_{\varphi\in\Phi_+} \mathrm{ht}(\varphi) = \frac{1}{6}r(h^2+h) \tag{2.5}$$

and asks for which values of n there is a nice closed expression for $\sum_{\varphi\in\Phi_+}\mathrm{ht}(\varphi)^n$. Our result shows that there are universal formulae for all $n\in\mathbb{Z}_{\geqslant 0}$. In fact, let $(k_1,\dots,k_{h-1})$ be the partition dual to $(m_r,\dots,m_1)$; then it is well-known (see, e.g., [6, Section 3.20]) that there are exactly k_j roots of height j in Φ_+. Hence

$$\sum_{\varphi\in\Phi_+} \mathrm{ht}(\varphi)^n = \sum_{i=1}^{r}\bigl(1^n+2^n+\cdots+m_i^n\bigr). \tag{2.6}$$

In particular, using (2.3) we recover (2.4) and have

$$\sum_{\varphi\in\Phi_+} \mathrm{ht}(\varphi) = \sum_{i=1}^{r}\frac{m_i^2+m_i}{2} = \frac{1}{12}r\bigl(h^2+\gamma+2h\bigr) \tag{2.7}$$

which generalizes (2.5) to all types.

Remark 2.2. The formula (2.7) for the integer $e(2\rho) := \sum_{\varphi\in\Phi_+}\mathrm{ht}(\varphi)$ may be applied as follows. For complex full flag manifolds G/T, Fino studied the invariant

$$Q(G/T) = \frac{e(2\rho)}{3\dim_{\mathbb{C}} G/T} - \frac{1}{3}\left(=\frac{1}{9}(h-2) \text{ if } G \text{ is of } \mathsf{ADE} \text{ type (see [5, Theorem 2.2])}\right)$$

and she tabulated the values of $Q(G/T)$ for all types in [5, Table I on p. 304]. We have the uniform expression

$$Q(G/T) = \frac{e(2\rho)}{3\dim_{\mathbb{C}} G/T} - \frac{1}{3} \overset{(2.7)}{=} \frac{1}{9}(h-2) + \frac{\gamma-h^2}{18h}.$$

An alternative way to derive (2.7) is by using the symmetry $m_i + m_{r+1-i} = h$. We can write as in [4, Proposition 2.1]

$$h^2 \sum_{i=1}^{r} m_i - 3h \sum_{i=1}^{r} m_i^2 + 2 \sum_{i=1}^{r} m_i^3 = 0. \tag{2.8}$$

Hence

$$\begin{aligned}
\sum_{\varphi \in \Phi_+} \mathrm{ht}(\varphi)^2 &\overset{(2.6)}{=} \sum_{i=1}^{r} \frac{m_i(m_i+1)(2m_i+1)}{6} = \sum_{i=1}^{r} \frac{m_i^3}{3} + \sum_{i=1}^{r} \frac{m_i^2}{2} + \sum_{i=1}^{r} \frac{m_i}{6} \\
&\overset{(2.8)}{=} -h^2 \sum_{i=1}^{r} \frac{m_i}{6} + h \sum_{i=1}^{r} \frac{m_i^2}{2} + \sum_{i=1}^{r} \frac{m_i^2}{2} + \sum_{i=1}^{r} \frac{m_i}{6} \\
&= (h+1) \sum_{i=1}^{r} \frac{m_i(m_i+1)}{2} - \left(\frac{h+1}{2} + \frac{h^2-1}{6} \right) \underbrace{\sum_{i=1}^{r} m_i}_{=\frac{rh}{2}} \\
&\overset{(2.6)}{=} (h+1) \sum_{\varphi \in \Phi_+} \mathrm{ht}(\varphi) - (h+1) \frac{rh(h+2)}{12}
\end{aligned}$$

so that (2.7) is recovered from (2.4).

We shall stick to the exponents rather than the heights in order not to restrict our considerations to the crystallographic types.

3. Power sums and Todd polynomials

Observe that (2.3) can be written as

$$\sum_{i=1}^{r} m_i^n = \begin{cases} r & = 0!\, r\mathrm{Td}_0 & \text{if } n = 0, \\ \frac{1}{2} rh & = 1!\, r\mathrm{Td}_1(h) & \text{if } n = 1, \\ \frac{1}{6} r(h^2 + \gamma - h) & = 2!\, r\mathrm{Td}_2(h, \gamma - h) & \text{if } n = 2, \\ \frac{1}{4} rh(\gamma - h) & = 3!\, r\mathrm{Td}_3(h, \gamma - h, *) & \text{if } n = 3, \end{cases} \tag{3.1}$$

where $\mathrm{Td}_0 = 1$, $\mathrm{Td}_1(c_1) = \frac{1}{2} c_1$, $\mathrm{Td}_2(c_1, c_2) = \frac{1}{12}(c_1^2 + c_2)$, and $\mathrm{Td}_3(c_1, c_2, c_3) = \frac{1}{24} c_1 c_2$ are Todd polynomials. Recall that, calculating in the formal power series ring in a variable t with coefficients in the ring of symmetric functions in $x_1, x_2, \ldots$ with rational coefficients, we can define the Todd polynomials via their generating series

$$\sum_{n=0}^{\infty} \mathrm{Td}_n(c_1, \ldots, c_n) t^n = \prod_{j=1}^{\infty} \frac{x_j t}{1 - e^{-x_j t}} \tag{3.2}$$

where $c_0 (= 1), c_1, c_2, \ldots$ are the elementary symmetric functions in $x_1, x_2, \ldots$, that is,

$$\sum_{n=0}^{\infty} c_n t^n = \prod_{j=1}^{\infty} (1 + x_j t). \tag{3.3}$$

The observation (3.1) suggests the ansatz

$$\sum_{i=1}^{r} m_i^n = n!\, r\mathrm{Td}_n(\gamma_1, \ldots, \gamma_n). \tag{3.4}$$

From (3.1) and (3.4) we get

$$\gamma_1 = h \quad \text{and} \quad \gamma_2 = \gamma - h \tag{3.5}$$

and are looking for solutions $\gamma_3, \gamma_4, \ldots$. Since $\sum_{i=1}^{r} m_{r+1-i}^a m_i^b = \sum_{i=1}^{r} m_i^a m_{r+1-i}^b$ and using the symmetry $m_i + m_{r+1-i} = h$ we get after binomial expansion of $(h - m_i)^a$ and $(h - m_i)^b$ the identities (for $a, b \in \mathbb{Z}_{\geqslant 0}$)

$$\sum_{j=0}^{a} (-1)^{a-j} \binom{a}{j} h^j \sum_{i=1}^{r} m_i^{a+b-j} = \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} h^j \sum_{i=1}^{r} m_i^{a+b-j} \tag{3.6}$$

that generalize (2.8), which is (3.6) for $\{a, b\} = \{1, 2\}$.

Proposition 3.1. *For $a, b \in \mathbb{Z}_{\geqslant 0}$ one has the identity*

$$\begin{aligned}&\sum_{j=0}^{a} (-1)^{a-j} \binom{a}{j} c_1^j (a+b-j)!\mathrm{Td}_{a+b-j}(c_1, \ldots, c_{a+b-j}) \\ &\quad = \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} c_1^j (a+b-j)!\mathrm{Td}_{a+b-j}(c_1, \ldots, c_{a+b-j}).\end{aligned} \tag{3.7}$$

Proof. To verify that (3.7) holds for all pairs $(a, b) \in \mathbb{Z}_{\geqslant 0}^2$, we start with $a = 0$ and then proceed by induction.

For $a = 0$ we have to check that for each $b \in \mathbb{Z}_{\geqslant 0}$

$$b!\mathrm{Td}_b(c_1, \ldots, c_b) = \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} c_1^j (b-j)!\mathrm{Td}_{b-j}(c_1, \ldots, c_{b-j}). \tag{3.8}$$

Equivalently, we must verify that the exponential generating series of both sides in (3.8) are equal. For the left hand side we write

$$\sum_{b=0}^{\infty} b!\mathrm{Td}_b(c_1, \ldots, c_b) \frac{t^b}{b!} = \sum_{b=0}^{\infty} \mathrm{Td}_b(c_1, \ldots, c_b) t^b =: \mathrm{Td}(t)$$

and for the right hand side we get

$$\begin{aligned}&\sum_{b=0}^{\infty} \sum_{j=0}^{b} (-1)^{b-j} \binom{b}{j} c_1^j (b-j)!\mathrm{Td}_{b-j}(c_1, \ldots, c_{b-j}) \frac{t^b}{b!} \\ &\quad = \sum_{b=0}^{\infty} \sum_{j=0}^{b} \frac{(c_1 t)^j}{j!} \mathrm{Td}_{b-j}(c_1, \ldots, c_{b-j})(-t)^{b-j} \\ &\quad = e^{c_1 t}\mathrm{Td}(-t).\end{aligned}$$

It thus remains to see that $\mathrm{Td}(t) = e^{c_1 t}\mathrm{Td}(-t)$, which follows from the definitions (3.2) and (3.3) together with the identity $\frac{x}{1-e^{-x}} = e^x \frac{(-x)}{1-e^x}$.

Now we proceed by induction on a. We employ the identities (3.7) for the pairs (a, b) and $(a, b+1)$, denoted by $(3.7)|_{(a,b)}$ and $(3.7)|_{(a,b+1)}$, and compute $c_1 \cdot (3.7)|_{(a,b)} - (3.7)|_{(a,b+1)}$

$$\sum_{j=1}^{a+1}(-1)^{a+1-j}\binom{a}{j-1} c_1^j (a+b+1-j)!\mathrm{Td}_{a+b+1-j}(c_1,\dots,c_{a+b+1-j})$$

$$-\sum_{j=0}^{a}(-1)^{a-j}\binom{a}{j} c_1^j (a+b+1-j)!\mathrm{Td}_{a+b+1-j}(c_1,\dots,c_{a+b+1-j})$$

$$=\sum_{j=1}^{b+1}(-1)^{b+1-j}\binom{b}{j-1} c_1^j (a+b+1-j)!\mathrm{Td}_{a+b+1-j}(c_1,\dots,c_{a+b+1-j})$$

$$-\sum_{j=0}^{b+1}(-1)^{b+1-j}\binom{b+1}{j} c_1^j (a+b+1-j)!\mathrm{Td}_{a+b+1-j}(c_1,\dots,c_{a+b+1-j}).$$

Adding zero summands (for $j = 0$ in the first and third sums and for $j = a + 1$ in the second sum) and cancelling the $j = b + 1$ summands in the third and fourth sums, we get by combining the binomial coefficients

$$\sum_{j=0}^{a+1}(-1)^{a+1-j}\binom{a+1}{j} c_1^j (a+b+1-j)!\mathrm{Td}_{a+b+1-j}(c_1,\dots,c_{a+b+1-j})$$

$$=\sum_{j=0}^{b}(-1)^{b-j}\binom{b}{j} c_1^j (a+b+1-j)!\mathrm{Td}_{a+b+1-j}(c_1,\dots,c_{a+b+1-j}),$$

which is just the identity $(3.7)|_{(a+1,b)}$ that we wanted to deduce. □

Strictly speaking we do not need Proposition 3.1. But it is worth noting that it indicates that we seem to be on the right track when using the ansatz (3.4).

Lemma 3.2. *Let $m_1 \leqslant \cdots \leqslant m_r \in \mathbb{Z}_{>0}$ be such that there are multisets V_+ and V_- of positive integers satisfying*

$$\sum_{i=1}^{r} q^{m_i} = \frac{q \prod_{v\in V_+}(1-q^v)}{\prod_{v\in V_-}(1-q^v)}. \tag{3.9}$$

Then

$$|V_+| = |V_-| \tag{3.10}$$

$$\prod_{v\in V_+} v = r \prod_{v\in V_-} v. \tag{3.11}$$

Proof. (3.10) follows since $1 - q^v$ has exactly one factor $1 - q$ and the polynomial on the left hand side in (3.9) has neither a zero nor a pole at $q = 1$; the equality (3.11) is clear from the $q \to 1$ limit in (3.9). Note also that $m_1 = 1$ and $m_2 > 1$ if $r \geqslant 2$. □

The following theorem employs a parameter p (at first required to be a positive integer; later it should become evident that p can be considered as a variable or $p \in \mathbb{C}$). We could put $p = 1$ at the outset and forget this parameter, but we restrain from doing so with apparently good reason (see Remark 5.7).

Theorem 3.3. *Let $m_1 \leqslant \cdots \leqslant m_r \in \mathbb{Z}_{>0}$ be such that there are multisets V_+ and V_- of positive integers satisfying*

$$\sum_{i=1}^{r} q^{m_i} = \frac{q \prod_{v \in V_+} (1 - q^v)}{\prod_{v \in V_-} (1 - q^v)}. \tag{3.9}$$

We fix a positive integer p and define $\gamma_0 (= 1), \gamma_1, \gamma_2, \gamma_3, \ldots$ by the generating series

$$\sum_{n=0}^{\infty} \gamma_n t^n = \frac{\prod_{v \in V_-} (1 - vt)}{\prod_{v \in V_+} (1 - vt)} \sqrt[p]{\frac{1 + pt}{1 - pt}}. \tag{3.12}$$

Then for $n \in \mathbb{Z}_{\geqslant 0}$

$$\sum_{i=1}^{r} m_i^n = n!\, r \mathrm{Td}_n(\gamma_1, \ldots, \gamma_n). \tag{3.13}$$

Proof. We consider the exponential generating series (with $q := e^t$) of both sides in (3.13)

$$\sum_{n=0}^{\infty} \left(\sum_{i=1}^{r} m_i^n \right) \frac{t^n}{n!} = \sum_{i=1}^{r} e^{m_i t} = \sum_{i=1}^{r} q^{m_i} \overset{(3.9)}{=} \frac{q \prod_{v \in V_+} (1 - q^v)}{\prod_{v \in V_-} (1 - q^v)} \tag{3.14}$$

$$\sum_{n=0}^{\infty} \Big(n!\, r \mathrm{Td}_n(\gamma_1, \ldots, \gamma_n) \Big) \frac{t^n}{n!} = r \sum_{n=0}^{\infty} \mathrm{Td}_n(\gamma_1, \ldots, \gamma_n) t^n = r \prod_{j=1}^{\infty} \frac{x_j t}{1 - e^{-x_j t}} \tag{3.15}$$

where the last equality incorporates the definition of the Todd polynomials if we let

$$\prod_{j=1}^{\infty} (1 + x_j t) = \sum_{n=0}^{\infty} \gamma_n t^n$$

and hence by (3.12)

$$\frac{(1 - pt) \prod_{v \in V_+} (1 - vt)^p}{(1 + pt) \prod_{v \in V_-} (1 - vt)^p} \prod_{j=1}^{\infty} (1 + x_j t)^p = 1. \tag{3.16}$$

This is an equation of the form

$$\prod_{j=1}^{\infty} \frac{1 + z_j t}{1 - w_j t} = 1 \tag{3.17}$$

and upon expanding the product in a power series in t as

$$\prod_{j=1}^{\infty} \frac{1+z_j t}{1-w_j t} = \sum_{n=0}^{\infty} E_n(z_1, z_2, \ldots; w_1, w_2, \ldots) t^n$$

it means that all the nontrivial elementary supersymmetric functions $E_n(z_1, z_2, \ldots; w_1, w_2, \ldots)$ (for $n > 0$) in the "positive variables" $z_1, z_2, \ldots$ and "negative variables" $w_1, w_2, \ldots$ vanish. For any power series with constant term 1, $f(t) = 1 + \sum_{n=1}^{\infty} b_n t^n$, we have then

$$\prod_{j=1}^{\infty} \frac{f(z_j t)}{f(-w_j t)} = \prod_{j=1}^{\infty} \frac{1 + \sum_{n=1}^{\infty} b_n (z_j t)^n}{1 + \sum_{n=1}^{\infty} b_n (-w_j t)^n} = 1 \tag{3.18}$$

because the coefficient of t^n in the power series expansion of (3.18) has the form

$$\sum_{\lambda \vdash n} B_\lambda(b_1, \ldots, b_n) E_\lambda(z_1, z_2, \ldots; w_1, w_2, \ldots)$$

where the sum runs over all partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n and where

$$\begin{aligned} &E_\lambda(z_1, z_2, \ldots; w_1, w_2, \ldots) \\ &\quad := E_{\lambda_1}(z_1, z_2, \ldots; w_1, w_2, \ldots) \cdot \ldots \cdot E_{\lambda_k}(z_1, z_2, \ldots; w_1, w_2, \ldots) \end{aligned}$$

and $B_\lambda(b_1, \ldots, b_n)$ is a homogeneous polynomial (with integer coefficients) of degree n if b_l is assigned degree l. In fact, in the absence of "negative variables" (i.e., for $w_1 = w_2 = \cdots = 0$) it is a very well-known classical result that the elementary symmetric functions generate the ring of symmetric functions (as a ring with 1). This extends to the supersymmetric situation (cf. [8]). It is indeed evident because the equality (3.17) when rewritten as

$$\prod_{j=1}^{\infty} (1 + z_j t) = \prod_{j=1}^{\infty} (1 - w_j t)$$

means that the nth elementary symmetric function in $z_1, z_2, \ldots$ equals the nth elementary symmetric function in $-w_1, -w_2, \ldots$ (for all $n \geqslant 0$) and hence

$$\begin{aligned} \prod_{j=1}^{\infty} f(z_j t) &= \sum_{n=0}^{\infty} t^n \sum_{\lambda \vdash n} B_\lambda(b_1, \ldots, b_n) E_\lambda(z_1, z_2, \ldots; 0, 0, \ldots) \\ &= \sum_{n=0}^{\infty} t^n \sum_{\lambda \vdash n} B_\lambda(b_1, \ldots, b_n) E_\lambda(-w_1, -w_2, \ldots; 0, 0, \ldots) = \prod_{j=1}^{\infty} f(-w_j t) \end{aligned}$$

or in other words: (3.18) holds.

In our case (3.16) the "positive variables" are $-p$ (once), $-v$ (p times, for every $v \in V_+$), x_1 (p times), x_2 (p times), …; and the "negative variables" are $-p$ (once), v (p times, for every $v \in V_-$), and all further variables 0. With $f(t) = \frac{t}{1-e^{-t}}$ the product (3.18) specializes to the formal expansion

$$\underbrace{\left(\frac{-pt}{1-e^{pt}}\right)\left(\frac{1-e^{-pt}}{pt}\right)}_{=e^{-pt}} \prod_{v \in V_+} \left(\frac{-vt}{1-e^{vt}}\right)^p \prod_{v \in V_-} \left(\frac{1-e^{vt}}{-vt}\right)^p \left(\prod_{j=1}^{\infty} \frac{x_j t}{1-e^{-x_j t}}\right)^p = 1$$

or after taking pth roots (look at $t=0$ to choose the correct branch)

$$\prod_{j=1}^{\infty}\frac{x_j t}{1-e^{-x_j t}}=e^t\prod_{v\in V_+}\left(\frac{1-e^{vt}}{-vt}\right)\prod_{v\in V_-}\left(\frac{-vt}{1-e^{vt}}\right).$$

Therefore we can write the right hand side in (3.15) as (recall $q=e^t$)

$$r\prod_{j=1}^{\infty}\frac{x_j t}{1-e^{-x_j t}}=\underbrace{\frac{r\prod_{v\in V_-}v}{\prod_{v\in V_+}v}}_{=1}\frac{q\prod_{v\in V_+}(1-q^v)}{\prod_{v\in V_-}(1-q^v)}=\frac{q\prod_{v\in V_+}(1-q^v)}{\prod_{v\in V_-}(1-q^v)}$$

where we have used (3.10) $|V_+|=|V_-|$ to cancel factors $-t$ and then (3.11) to simplify the product. Thus the right hand side of (3.15) equals the right hand side of (3.14), which proves (3.13). □

Remark 3.4. Instead of the definition (3.12) for $\gamma_0,\gamma_1,\gamma_2,\gamma_3,\ldots$ one can define more generally

$$\sum_{n=0}^{\infty}\gamma_n t^n=\frac{\prod_{v\in V_-}(1-vt)}{\prod_{v\in V_+}(1-vt)}\prod_{k=1}^{K}\left(\frac{1+\pi_k t}{1-\pi_k t}\right)^{\mu_k}$$

with $\pi_1,\ldots,\pi_K\in\mathbb{R}$ and $\mu_1,\ldots,\mu_K\in\mathbb{Q}$ satisfying $\sum_{k=1}^{K}\pi_k\mu_k=1$ (and for general m_1 (with q^{m_1} instead of q as the first factor on the right hand side of (3.9)) just require that $\sum_{k=1}^{K}\pi_k\mu_k=m_1$).

4. Root system considerations

To apply Theorem 3.3 in the context of root systems we need the following proposition.

Proposition 4.1. *Let $m_1\leqslant\cdots\leqslant m_r$ be the exponents of an irreducible (crystallographic (and reduced) or noncrystallographic) finite root system (of rank r). Then there are multisets V_+ and V_- of positive integers such that*

$$\sum_{i=1}^{r}q^{m_i}=\frac{q\prod_{v\in V_+}(1-q^v)}{\prod_{v\in V_-}(1-q^v)}. \tag{3.9}$$

Furthermore, $|V_\pm|\leqslant 2$ if $V_+\cap V_-=\varnothing$.

Proof. According to the first note added in proof in [10] I. G. Macdonald was acquainted with the fact that (3.9) holds for all irreducible finite Coxeter groups.

The classification shows that the following three cases exhaust all possible types.

(1) For the types A_r, $\mathsf{C}_r/\mathsf{B}_r$, and types of rank $\leqslant 3$ the sequence of exponents forms an arithmetic progression $1,m_2,\ldots,1+(r-1)(m_2-1)$ (or just 1 if $r=1$). Hence

$$\sum_{i=1}^{r}q^{m_i}=\begin{cases}q & \text{if } r=1\\ \dfrac{q(1-q^{r(m_2-1)})}{1-q^{m_2-1}} & \text{if } r\geqslant 2\end{cases}$$

so that we can take $V_+ = V_- = \varnothing$ if $r = 1$ and $V_+ = \{r(m_2 - 1)\}$ and $V_- = \{m_2 - 1\}$ if $r \geqslant 2$.

(2) For the types of rank 4 we have

$$\sum_{i=1}^{4} q^{m_i} = q + q^{m_2} + q^{h-m_2} + q^{h-1} = \frac{q(1-q^{2(m_2-1)})(1-q^{2(h-m_2-1)})}{(1-q^{m_2-1})(1-q^{h-m_2-1})}$$

so that we can take $V_+ = \{2(m_2 - 1), 2(h - m_2 - 1)\}$ and $V_- = \{m_2 - 1, h - m_2 - 1\}$.

(3) For the simply-laced types (ADE) the root system is the Weyl group orbit of the highest root: $\Phi = W\theta$. The stabilizer of θ is $W_{\perp\theta}$, the reflection group generated by those simple reflections in W that fix θ. The root system is thus isomorphic as a W-set to $W/W_{\perp\theta}$. We need the usual length function $\ell : W \to \mathbb{Z}_{\geqslant 0}$ defined as $\ell(w) = k$ if w can be written as a product of k but not less than k simple reflections. If $\varphi = w\theta$ is any positive root with w chosen such that $\ell(w)$ is minimal, then $\mathrm{ht}(\varphi) = \mathrm{ht}(\theta) - \ell(w) = h - 1 - \ell(w)$. Since the reflection along a simple root ψ maps ψ (of height 1) to $-\psi$ (of height -1), we have similarly the equality $\mathrm{ht}(\varphi) = \mathrm{ht}(\theta) - \ell(w) - 1 = h - 2 - \ell(w)$ if $\varphi = w\theta$ is any negative root with w chosen such that $\ell(w)$ is minimal. So we have

$$\sum_{\substack{wW_{\perp\theta} \in W/W_{\perp\theta} \\ \ell(w) \text{ minimal}}} q^{\ell(w)} = \sum_{\varphi \in \Phi_+} \left(q^{h-1-\mathrm{ht}(\varphi)} + q^{h-2+\mathrm{ht}(\varphi)}\right)$$

and since $1, \ldots, m_1$, $1, \ldots, m_2$, $\ldots$, $1, \ldots, m_r$ (where $1, \ldots, m_1$ is actually just 1) enumerates $\mathrm{ht}(\varphi)$ as φ runs over Φ_+, we can continue

$$= \sum_{i=1}^{r} \sum_{j=1}^{m_i} \left(q^{h-1-j} + q^{h-2+j}\right)$$

and using the symmetry $m_i + m_{r+1-i} = h$ we obtain

$$= \sum_{i=1}^{r} \sum_{j=0}^{h-1} q^{m_i-1+j} = \left(\sum_{i=1}^{r} q^{m_i-1}\right) \frac{1-q^h}{1-q}.$$

On the other hand by the Chevalley–Solomon identity for the Poincaré series of finite Coxeter groups (see, e.g., [6, Section 3.15]) we have

$$\sum_{\substack{wW_{\perp\theta} \in W/W_{\perp\theta} \\ \ell(w) \text{ minimal}}} q^{\ell(w)} = \left(\prod_{i=1}^{r} \frac{1-q^{m_i+1}}{1-q}\right) \left(\prod_{i=1}^{s} \frac{1-q}{1-q^{\widetilde{m}_i+1}}\right)$$

where $\widetilde{m}_1, \ldots, \widetilde{m}_s$ lists the exponents of all the irreducible components of $W_{\perp\theta}$ (note that $s = r - 1$ except for types A_r with $r \geqslant 2$, where $s = r - 2$). Since $m_r + 1 = h$ we finally get

$$\sum_{i=1}^{r} q^{m_i} = \frac{q}{(1-q)^{r-s-1}} \frac{\prod_{i=1}^{r-1}(1-q^{m_i+1})}{\prod_{i=1}^{s}(1-q^{\widetilde{m}_i+1})}.$$

The following table, where we have left out the types A_r which were already dealt with in case (1), finishes the proof.

Type W	Exponents + 1	Type $W_{\perp\theta}$	Exponents + 1	V_+	V_-
D_r $(r \geqslant 4)$	$2, 4, \dots, 2r-2, r$	$\mathsf{A}_1+\mathsf{D}_{r-2}$	$2, 2, 4, \dots,$ $2r-6, r-2$	$\{r, 2r-4\}$	$\{2, r-2\}$
E_6	2, 5, 6, 8, 9, 12	A_5	2, 3, 4, 5, 6	$\{8, 9\}$	$\{3, 4\}$
E_7	2, 6, 8, 10, 12, 14, 18	D_6	2, 4, 6, 8, 10, 6	$\{12, 14\}$	$\{4, 6\}$
E_8	2, 8, 12, 14, 18, 20, 24, 30	E_7	2, 6, 8, 10, 12, 14, 18	$\{20, 24\}$	$\{6, 10\}$

Multisets are needed for type D_4. □

Note that for $r \geqslant 2$ (3.9) implies that $m_2 - 1 \in V_-$. Furthermore, for all the crystallographic types except A_1 and G_2, $m_2 - 1 = d$ is the largest coefficient of the highest root (when written as a linear combination of the simple roots). Likewise put $d := m_2 - 1 = 4$ for H_3 and $d := m_2 - 1 = 10$ for H_4. For $\mathsf{I}_2(m)$ put $d := \lfloor \frac{m}{2} \rfloor$.

For A_r, C_r, B_r, $\mathsf{I}_2(m)$, and H_3 one can append the same element(s) to both V_+ and V_- to make all the above multisets V_+ and V_- have cardinality 2.

The following proposition gives a uniform description of multisets $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ satisfying (3.9) in terms of three parameters: the Coxeter number h, the coefficient d, and $\nu :=$ the number of times d occurs among the marks in the extended Dynkin diagram minus 1, and extended to the noncrystallographic types as displayed in the following table. The table also shows the values of γ (see (2.2) and the text afterwards). Some parameters β (and for type A_1 also α) are irrelevant and are left unspecified. Clearly, one can interchange $A \leftrightarrow B$ and also $\alpha \leftrightarrow \beta$.

Type	r	h	γ	d	A, B	α, β	ν
A_1	1	2	4	1	α, β	α, β	1
A_r $(r \geqslant 2)$	r	$r+1$	$(r+1)^2$	1	r, β	$1, \beta$	r
$\mathsf{C}_r/\mathsf{B}_r$ $(r \geqslant 2)$	r	$2r$	$4r^2 + 2r - 2$	2	$2r, \beta$	$2, \beta$	$r-2$
D_r $(r \geqslant 4)$	r	$2r-2$	$(2r-2)^2$	2	$r, 2(r-2)$	$2, r-2$	$r-4$
E_6	6	12	144	3	8, 9	3, 4	0
E_7	7	18	324	4	12, 14	4, 6	0
E_8	8	30	900	6	20, 24	6, 10	0
F_4	4	12	162	4	8, 12	4, 6	0
$\mathsf{G}_2 = \mathsf{I}_2(6)$	2	6	48	3	$8, \beta$	$4, \beta$	0
$\mathsf{H}_2 = \mathsf{I}_2(5)$	2	5	31	2	$6, \beta$	$3, \beta$	1
H_3	3	10	124	4	$12, \beta$	$4, \beta$	0
H_4	4	30	1116	10	20, 36	10,18	0
$\mathsf{I}_2(2k+1)$ $(k \geqslant 3)$	2	$2k+1$	$8k^2 - 2k + 3$	k	$4k-2, \beta$	$2k-1, \beta$	1
$\mathsf{I}_2(2k)$ $(k \geqslant 4)$	2	$2k$	$8k^2 - 10k + 6$	k	$4k-4, \beta$	$2k-2, \beta$	0
Redefined parameters d and ν for $\mathsf{I}_2(2k+1)$ $(k \geqslant 2)$							
Type	r	h	γ	d	A, B	α, β	ν
$\mathsf{I}_2(m)$ $(m \geqslant 4)$	2	m	$2m^2 - 5m + 6$	$\frac{m}{2}$	$2m-4, \beta$	$m-2, \beta$	0

The table shows that in the cases where β has a well-defined value (and $\alpha = m_2 - 1$), this value is $m_3 - 1$ except for D_r $(r \geqslant 7)$, where $\beta = m_{\lfloor (r+1)/2 \rfloor} - 1$. With the redefinition of d and ν for the types $\mathsf{I}_2(2k+1)$ $(k \geqslant 2)$ the formula $h = \frac{d}{2}(r + 2 + \nu)$ is true in general, and it is also true for $\mathsf{H}_2 = \mathsf{I}_2(5)$ with the original parameters $d = 2$ and $\nu = 1$.

Proposition 4.2. *The equality (3.9) in Proposition 4.1 holds if the multisets $V_\pm$ are given as*

$$V_- = \{d,\ 2d-2+\nu\} \quad \text{and}$$
$$V_+ = \{4d-4+d\nu,\ h-d-(d-1)\nu\}$$

with $d = \frac{m}{2}$ and $\nu = 0$ for $\mathsf{I}_2(m)$ $(m \geqslant 4)$; and for $\mathsf{H}_2 = \mathsf{I}_2(5)$ the original values $d = 2$ and $\nu = 1$ also work.

The choice in Proposition 4.2 of the irrelevant parameters is thus $\alpha = \beta = 1$ for type A_1 and as shown in the following table.

Type	A_r	$\mathsf{C}_r/\mathsf{B}_r$	G_2	H_2 with $d=2$, $\nu=1$	H_3	$\mathsf{I}_2(m)$ with $d=\frac{m}{2}$, $\nu=0$
β	r	r	3	2	6	$\frac{m}{2}$

Proof. Let us first look at those exceptional types for which $d \mid h$ (including $\mathsf{I}_2(m)$ $(m \geqslant 5)$). Here we have $\nu = 0$ and the (multi)set of exponents is

$$\{m_1, \dots, m_r\} = \left\{1 + jd \,\middle|\, 0 \leqslant j \leqslant \frac{h}{d} - 2\right\} \cup \left\{2d - 1 + jd \,\middle|\, 0 \leqslant j \leqslant \frac{h}{d} - 2\right\}$$

(see [4, Theorem 3.2 (i)] adding H_4 and $\mathsf{I}_2(m)$) so that

$$\sum_{i=1}^{r} q^{m_i} = \sum_{j=0}^{\frac{h}{d}-2} \left(q^{1+jd} + q^{2d-1+jd}\right) = q\left(1 + q^{2d-2}\right) \sum_{j=0}^{\frac{h}{d}-2} q^{jd}$$
$$= \frac{q(1-q^{4d-4})(1-q^{h-d})}{(1-q^d)(1-q^{2d-2})}$$

in agreement with the expressions for $V_\pm$ (with $\nu = 0$).

For the remaining types we use the following table.

Type	h	d	ν	$4d-4+d\nu,\ h-d-(d-1)\nu$	$d,\ 2d-2+\nu$
A_r $(r \geqslant 1)$	$r+1$	1	r	r, r	$1, r$
$\mathsf{C}_r/\mathsf{B}_r$ $(r \geqslant 2)$	$2r$	2	$r-2$	$2r, r$	$2, r$
D_r $(r \geqslant 4)$	$2r-2$	2	$r-4$	$2r-4, r$	$2, r-2$
E_7	18	4	0	12,14	4,6
H_2	5	2	1	6,2	2,3
H_3	10	4	0	12,6	4,6

This is in agreement with the table before Proposition 4.2. □

Remark 4.3. For the DE types one has $V_- = \left\{\frac{a}{2}, \frac{b}{2}\right\}$ and $V_+ = \left\{b, \frac{ra}{4}\right\}$, where the parameters a and b are as in Kostant's article [7]. Note also that for those types $\frac{a}{2} = d$ and $\frac{b}{2} = \frac{h+2}{2} - d$. We can already look ahead and use (5.4) to obtain $h = dr - 4d + 6$; from (5.5) and $h^2 = \gamma$ (still for the DE types) and using the equality $h = dr - 4d + 6$ we get $d(h - 2r - 6d + 26) = 24$.

Remark 4.4. Another source for lists of integers (again called exponents) $m_1 \leqslant \cdots \leqslant m_r$ such that $\sum_{i=1}^{r} q^{m_i}$ factors up to a power of q into a product of cyclotomic polynomials is provided by Saito's regular systems of weights (the reduced ones having only positive exponents correspond to ADE type root systems). See [11] where Saito mentions a preprint according to which the nth power sum of the exponents is expressed by a product of r and a polynomial of degree n in a, b, c, h where $(a, b, c; h)$ is a regular system of weights.

5. Synthesis and further computations

Proposition 4.1 shows that Theorem 3.3 can be applied in the context of root systems with $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ as in the table before Proposition 4.2.

Define $\gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots$ (depending on a parameter p) by the series expansion

$$\sum_{n=0}^{\infty} \gamma_n t^n = \frac{(1-\alpha t)(1-\beta t)}{(1-At)(1-Bt)} \sqrt[p]{\frac{1+pt}{1-pt}}. \tag{5.1}$$

The series expansions

$$\frac{(1-\alpha t)(1-\beta t)}{(1-At)(1-Bt)} = \bigl(1-(\alpha+\beta)t+\alpha\beta t^2\bigr) \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} A^j B^{n-j} \right) t^n \tag{5.2}$$

and

$$\begin{aligned}\sqrt[p]{\frac{1+pt}{1-pt}} &= \left(\sum_{j=0}^{\infty} \binom{\frac{1}{p}}{j} (pt)^j \right) \left(\sum_{k=0}^{\infty} \binom{-\frac{1}{p}}{k} (-pt)^k \right) =: \sum_{n=0}^{\infty} p_n t^n \\ &= 1 + 2t + 2t^2 + \frac{2p^2+4}{3} t^3 + \frac{4p^2+2}{3} t^4 + \frac{6p^4+20p^2+4}{15} t^5 + \cdots \end{aligned} \tag{5.3}$$

specializing for $p=1$ and $p=2$

$$\frac{1+t}{1-t} = 1 + 2\sum_{n=1}^{\infty} t^n$$

$$\sqrt{\frac{1+2t}{1-2t}} = \sum_{n=0}^{\infty} \binom{2n}{n} (1+2t) t^{2n} = 1 + 2t + 2t^2 + 4t^3 + 6t^4 + 12t^5 + \cdots$$

can be used to write down an explicit formula for γ_n defined in (5.1).

Note that the series expansion of $\bigl((1+pt)/(1-pt)\bigr)^{1/p}$ has integer coefficients if $p = 2^k$ with $k \in \mathbb{Z}_{\geqslant 0}$. In fact, for $f(t) = 1 + \sum_{n=1}^{\infty} a_n t^n$ we let

$$Tf(t) := \sqrt{f(2t)} = 1 + \sum_{n=1}^{\infty} b_n t^n.$$

A comparison of coefficients shows that

$$b_n = 2^{n-1} a_n - \frac{1}{2} \sum_{j=1}^{n-1} b_j b_{n-j},$$

and hence if a_1 is even and all a_n are integers, then all b_n are even. Starting with the series $f(t) := (1+t)/(1-t) = 1 + 2\sum_{n=1}^{\infty} t^n$, we get $\bigl((1+2^k t)/(1-2^k t)\bigr)^{1/2^k} = T^k f(t) \in 1 + 2t\mathbb{Z}[\![t]\!]$.

(Note also that in the limit $p \to 0$ we get the power series expansion of e^{2t}, which is a fixed point of the transformation T.)

Remark 5.1. The transformation T on (generating series of) integer sequences starting with 1 and having an even integer as next term may be investigated. Here is a tiny list of examples:

$a_0, a_1, a_2, \dots$	$\overset{T}{\longmapsto}$	$b_0, b_1, b_2, \dots$
$a_n = n + 1$		$b_n = 2^n$
$a_n = 2^n$		$b_n = \binom{2n}{n}$
$a_n = C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$		$b_n = 2^n C_n$

More generally, one may fix a positive integer l and look at the transformation

$$f(t) \longmapsto \sqrt[l]{f(lt)}$$

for $f(t) = 1 + \sum_{n=1}^{\infty} a_n t^n$ with $l \mid a_1$ and $a_n \in \mathbb{Z}$.

Lemma 5.2. *The elementary symmetric polynomials in A and B can be written as follows.*

$$A + B = h - 2 + \alpha + \beta \tag{5.4}$$

$$AB = h^2 - \gamma + (h-2)(\alpha + \beta - 1) + \alpha\beta. \tag{5.5}$$

Furthermore,

$$\begin{aligned} X_n &:= \sum_{j=0}^{n} A^j B^{n-j} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} \\ &\quad \times (h - 2 + \alpha + \beta)^{n-2j} \big(h^2 - \gamma + (h-2)(\alpha + \beta - 1) + \alpha\beta\big)^j. \end{aligned} \tag{5.6}$$

Proof. From (5.1) we get using (3.5)

$$\begin{aligned} \gamma_1 &= 2 + (A + B) - (\alpha + \beta) = h \\ \gamma_2 &= 2 + 2(A + B) + (A^2 + AB + B^2) - 2(\alpha + \beta) - (\alpha + \beta)(A + B) + \alpha\beta \\ &= \gamma - h \end{aligned}$$

and solving for the elementary symmetric polynomials in A and B we get (5.4) and (5.5).

Note that for $n \geqslant 2$

$$\begin{aligned} X_n &= (A + B) \sum_{j=0}^{n-1} A^j B^{n-1-j} - AB \sum_{j=0}^{n-2} A^j B^{n-2-j} \\ &= (h - 2 + \alpha + \beta) X_{n-1} - \big(h^2 - \gamma + (h-2)(\alpha + \beta - 1) + \alpha\beta\big) X_{n-2} \end{aligned}$$

with $X_0 = 1$ and $X_1 = h - 2 + \alpha + \beta$. By solving the recursion we have (5.6). □

Proposition 5.3. *The invariant γ is expressible as a polynomial in h, r, α, β, namely,*

$$\gamma = h^2 + (h-2)(\alpha + \beta - 1) - (r-1)\alpha\beta. \tag{5.7}$$

Proof. From (3.11) we have $AB = r\alpha\beta$. The formula (5.7) follows by combining with (5.5). □

Remark 5.4. The formula $h = \frac{d}{2}(r + 2 + \nu)$ follows by inserting into $AB = r\alpha\beta$ the expressions in Proposition 4.2 for $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ and using the fact that the product $\big(d(\nu - r) + 4(d-1)\big)(d-2)\nu$ vanishes.

To summarize we state the following theorem.

Theorem 5.5. *Let $m_1 \leqslant \cdots \leqslant m_r$ be the exponents of an irreducible (crystallographic (and reduced) or noncrystallographic) finite root system (of rank r) with Coxeter number h and parameters γ and d as in the table before Proposition 4.2. Put*

$$\alpha := \begin{cases} \text{arbitrary} & \text{if } r = 1, \\ m_2 - 1 & \text{if } r \geqslant 2, \\ \text{or} & \\ d & \end{cases}$$

and define

$$\beta := \begin{cases} \text{arbitrary} & \text{if } h = (r-1)\alpha + 2, \\ \dfrac{h^2 - \gamma + (h-2)(\alpha - 1)}{2 + (r-1)\alpha - h} & \text{if } h \neq (r-1)\alpha + 2. \end{cases} \tag{5.8}$$

Let

$$\sum_{n=0}^{\infty} \gamma_n t^n = \big(1 - (\alpha+\beta)t + \alpha\beta t^2\big)\left(\sum_{n=0}^{\infty} X_n t^n\right)\left(\sum_{n=0}^{\infty} p_n t^n\right)$$

with X_n as in (5.6) and p_n as in (5.3). (So γ_n is a polynomial in h, γ, α, β (or, by (5.7), alternatively in h, r, α, β) (symmetric in α, β) and depends on an additional parameter p which can be chosen arbitrarily.) Then

$$\sum_{i=1}^{r} m_i^n = n!\, r \operatorname{Td}_n(\gamma_1, \ldots, \gamma_n). \tag{5.9}$$

Proof. As already mentioned, this is an application of Theorem 3.3 in the context of root systems, that is, using Proposition 4.1 with $V_+ = \{A, B\}$ and $V_- = \{\alpha, \beta\}$ and inserting (5.2), (5.3) and (5.6) into the series expansion (5.1). The expression for β follows from Proposition 5.3.

For $r = 1$ there is nothing more to say. So let us assume $r \geqslant 2$. To finish the proof, it remains to be explained why we can take $\alpha = d$ instead of $\alpha = m_2 - 1$. The reason is that $d = m_2 - 1$ in all cases except possibly for types $\mathsf{I}_2(m)$, but then we get $\beta = m_2 - 1 = m - 2$. Or still slightly more generally: for the types A_r ($r \geqslant 2$), $\mathsf{C}_r/\mathsf{B}_r$, $\mathsf{I}_2(m)$, and H_3 we could choose $\alpha \neq m_2 - 1$ and automatically get $\beta = m_2 - 1$ from (5.8). □

Let us continue by writing down γ_3 and γ_4 in terms of h, γ, α, β (and p)

$$\gamma_3 = -h^3 + 2h\gamma - 2\gamma + \frac{1}{3}(2p^2 + 4) - (h^2 - \gamma - h + 2)(\alpha+\beta) - (h-2)\alpha\beta \tag{5.10}$$

$$\begin{aligned} \gamma_4 = {} & -h^4 + h^2\gamma + \gamma^2 + 3h^3 - 6h\gamma - h^2 + 2\gamma + \frac{2}{3}h(p^2+5) - 2 \\ & - (h^2 - \gamma - h + 2)\big((2h - 2 + \alpha + \beta)(\alpha+\beta) - \alpha\beta\big) \\ & - (h-2)(2h - 2 + \alpha + \beta)\alpha\beta. \end{aligned} \tag{5.11}$$

By inserting (3.5), (5.10) and (5.11) into (5.9) using the formulae $\operatorname{Td}_4(c_1, c_2, c_3, c_4) = \frac{1}{720}\big(-c_1^4 + 4c_1^2 c_2 + c_1 c_3 + 3c_2^2 - c_4\big)$ and $\operatorname{Td}_5(c_1, c_2, c_3, c_4, c_5) = \frac{1}{1440}\big(-c_1^3 c_2 + 3c_1 c_2^2 + c_1^2 c_3 - c_1 c_4\big)$ we get

$$\sum_{i=1}^{r} m_i^4 = \frac{r}{30}\bigl(-h^4 + 5h^2\gamma + 2\gamma^2 - 7h^3 - 2h\gamma + 4h^2 - 2\gamma - 2h + 2 + R_{45}\bigr) \tag{5.12}$$

$$\sum_{i=1}^{r} m_i^5 = \frac{r}{12}h\bigl(2\gamma^2 - 2h^3 - 2h\gamma + 4h^2 - 2\gamma - 2h + 2 + R_{45}\bigr) \tag{5.13}$$

where

$$\begin{aligned} R_{45} &= (h^2 - \gamma - h + 2)\bigl((h - 2 + \alpha + \beta)(\alpha + \beta) - \alpha\beta\bigr) \\ &\quad + (h - 2)(h - 2 + \alpha + \beta)\alpha\beta. \end{aligned} \tag{5.14}$$

Surely, one could continue and give explicit formulae for higher power sums. Let us stop here and display formulae for the sums of the heights cubes and fourth powers.

Proposition 5.6. *With R_{45} as in (5.14) above we have*

$$\sum_{\varphi\in\Phi_+} \mathrm{ht}(\varphi)^3 = \frac{r}{120}\bigl(-h^4 + 5h^2\gamma + 2\gamma^2 - 7h^3 + 13h\gamma - 6h^2 + 3\gamma - 7h + 2 + R_{45}\bigr)$$

$$\sum_{\varphi\in\Phi_+} \mathrm{ht}(\varphi)^4 = \frac{r}{60}(h + 1)\bigl(2\gamma^2 - 3h^3 + 3h\gamma - 2\gamma - 3h + 2 + R_{45}\bigr).$$

Proof. Insert (2.3), (5.12) and (5.13) into (2.6). □

Remark 5.7. Using the power series expansions for (5.1) one computes the following explicit expressions for the quantities γ_n. For the types A_r one gets for $n \geqslant 1$

$$\gamma_n|_{p=1} = r^n + r^{n-1}$$

and has

$$\sum_{i=1}^{r} i^n = n!\, r\,\mathrm{Td}_n\bigl(r + 1, r^2 + r, \ldots, r^n + r^{n-1}\bigr)$$

as an alternative to Bernoulli's formula (2.1). Note also that even for $r = 1$, the resulting formula $\mathrm{Td}_n(2, \ldots, 2) = \frac{1}{n!}$ is not trivial.

For the types C_r $(r \geqslant 2)$ one gets

$$\gamma_n|_{p=1} = (2r)^n - 2\sum_{j=0}^{n-2}(2r)^j$$

but it looks somewhat more natural to specialize to $p = 2$

$$\gamma_n|_{p=2} = -2\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} C_{j-1}\,(2r)^{n-2j}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number (for $k \geqslant 0$) and employing the (-1)st Catalan number $C_{-1} = -\frac{1}{2}$.

One may ask whether as an alternative to our considerations using generating series a more geometric/combinatorial approach via toric geometry/counting lattice points in polytopes can be found (see also [2, Section 2.4], where the Bernoulli polynomials are recognized as lattice point enumerators of certain pyramids).

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