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The log-Brunn–Minkowski inequality[☆]

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Abstract

For origin-symmetric convex bodies (i.e., the unit balls of finite dimensional Banach spaces) it is conjectured that there exist a family of inequalities each of which is stronger than the classical Brunn–Minkowski inequality and a family of inequalities each of which is stronger than the classical Minkowski mixed-volume inequality. It is shown that these two families of inequalities are "equivalent" in that once either of these inequalities is established, the other must follow as a consequence. All of the conjectured inequalities are established for plane convex bodies.

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1. Introduction

The fundamental Brunn–Minkowski inequality for convex bodies (compact convex subsets with nonempty interiors) states that for convex bodies K, L in Euclidean n-space, $\mathbb{R}^n$, the volume

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of the bodies and of their Minkowski sum $K + L = \{x + y : x \in K \text{ and } y \in L\}$, are related by

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K and L are homothetic. As the first milestone of the Brunn–Minkowski theory, the Brunn–Minkowski inequality is a far-reaching generalization of the isoperimetric inequality. The Brunn–Minkowski inequality exposes the crucial log-concavity property of the volume functional because the Brunn–Minkowski inequality has an equivalent formulation as: for all real $\lambda \in [0, 1]$,

$$V((1-\lambda)K + \lambda L) \geq V(K)^{1-\lambda} V(L)^{\lambda}, \tag{1.1}$$

and for $\lambda \in (0, 1)$, there is equality if and only if K and L are translates. A big part of the classical Brunn–Minkowski theory is concerned with establishing generalizations and analogues of the Brunn–Minkowski inequality for other geometric invariants (see, e.g., [1,56,61] for some recent developments). The excellent survey article of Gardner [18] gives a comprehensive account of various aspects and consequences of the Brunn–Minkowski inequality.

If h_K and h_L are the support functions (see (2.1) for the definition) of K and L, the Minkowski combination $(1-\lambda)K + \lambda L$ is given by an intersection of half-spaces,

$$(1-\lambda)K + \lambda L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq (1-\lambda)h_K(u) + \lambda h_L(u)\},$$

where $x \cdot u$ denotes the standard inner product of x and u in $\mathbb{R}^n$. Assume that K and L are convex bodies that contain the origin in their interiors, then the log Minkowski combination, $(1-\lambda) \cdot K +_o \lambda \cdot L$, is defined by

$$(1-\lambda) \cdot K +_o \lambda \cdot L = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \leq h_K(u)^{1-\lambda} h_L(u)^{\lambda}\}. \tag{1.2}$$

The arithmetic–geometric mean inequality shows that for convex bodies K, L and $\lambda \in [0, 1]$,

$$(1-\lambda) \cdot K +_o \lambda \cdot L \subseteq (1-\lambda)K + \lambda L. \tag{1.3}$$

What makes the log Minkowski combinations difficult to work with is that while the convex body $(1-\lambda)K + \lambda L$ has $(1-\lambda)h_K + \lambda h_L$ as its support function, the convex body $(1-\lambda) \cdot K +_o \lambda \cdot L$ is the Wulff shape of the function $h_K^{1-\lambda} h_L^{\lambda}$.

The authors conjecture that for origin-symmetric bodies (i.e., unit balls of finite dimensional Banach spaces), there is a stronger inequality than the Brunn–Minkowski inequality (1.1), the *log-Brunn–Minkowski inequality*.

Problem 1.1. Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then for all $\lambda \in [0, 1]$,

$$V((1-\lambda) \cdot K +_o \lambda \cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.4}$$

That for origin-symmetric bodies, the log-Brunn–Minkowski inequality (1.4) is stronger than its classical counterpart (1.1) can be seen from the arithmetic–geometric mean inequality (1.3). Simple examples (e.g. an origin-centered cube and one of its translates) show that (1.4) cannot hold for all convex bodies.

As is well known, the classical Brunn–Minkowski inequality (1.1) has as a consequence an inequality of fundamental importance: the Minkowski mixed-volume inequality. One of the aims of this paper is to show that the log-Brunn–Minkowski inequality (1.4) also has an important consequence, the *log-Minkowski inequality*.

Problem 1.2. Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$\int_{S^{n-1}} \log \frac{h_L}{h_K}\, d\bar{V}_K \geq \frac{1}{n} \log \frac{V(L)}{V(K)}. \tag{1.5}$$

Here $\bar{V}_K$ is the cone-volume probability measure of K (see definitions (2.5), (2.6), (2.8)).

Just as, for origin-symmetric bodies, the log-Brunn–Minkowski inequality (1.4) is stronger than its classical counterpart (1.1), for origin-symmetric bodies, the log-Minkowski inequality (1.5) turns out to be stronger than its classical counterpart.

The classical Minkowski mixed-volume inequality and the classical Brunn–Minkowski inequality are "equivalent" in that once either of these inequalities has been established, then the other can be obtained as a simple consequence. One of the aims of this paper is to demonstrate that the log-Brunn–Minkowski inequality (1.4) and the log-Minkowski inequality (1.5) are "equivalent" in that once either of these inequalities has been established, then the other can be obtained as a simple consequence, although perhaps a bit less simply than in the classical ($p = 1$) case.

Even in the plane the above problems are non-trivial and unsolved. One of the aims of this paper is to establish the plane log-Brunn–Minkowski inequality along with its equality conditions.

Theorem 1.3. *If K and L are origin-symmetric convex bodies in the plane, then for all real $\lambda \in [0, 1]$,*

$$V((1-\lambda)\cdot K +_o \lambda\cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.6}$$

When $\lambda \in (0, 1)$, equality in the inequality holds if and only if K and L are dilates or K and L are parallelograms with parallel sides.

In addition, in the plane, we will establish the log-Minkowski inequality along with its equality conditions.

Theorem 1.4. *If K and L are origin-symmetric convex bodies in the plane, then,*

$$\int_{S^1} \log \frac{h_L}{h_K}\, d\bar{V}_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)}, \tag{1.7}$$

with equality if and only if, either K and L are dilates or K and L are parallelograms with parallel sides.

The above Minkowski combinations and problems are merely two (important) frames of a long film. In the early 1960s, Firey (see e.g., [60, p. 383] and [20]) defined for each $p \geq 1$, what have become known as Minkowski–Firey L_p-combinations (or simply L_p-combinations) of convex bodies. If K and L are convex bodies that contain the origin in their interiors and $0 \leq \lambda \leq 1$ then the Minkowski–Firey L_p-combination, $(1-\lambda)\cdot K +_p \lambda\cdot L$, is defined by

$$(1-\lambda)\cdot K +_p \lambda\cdot L = \bigcap_{u\in S^{n-1}} \left\{x \in \mathbb{R}^n : x\cdot u \leq \left((1-\lambda)h_K(u)^p + \lambda h_L(u)^p\right)^{1/p}\right\}. \tag{1.8}$$

Firey also established the L_p-Brunn–Minkowski inequality (an inequality that is also known as the Brunn–Minkowski–Firey inequality). If $p > 1$, then

$$V((1-\lambda)\cdot K +_p \lambda\cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}, \tag{1.9}$$

with equality for $\lambda \in (0,1)$ if and only if $K = L$.

In the mid 1990s, it was shown in [40,41], that a study of the volume of Minkowski–Firey L_p-combinations leads to an embryonic L_p-Brunn–Minkowski theory. This theory has expanded rapidly (see e.g. [5,8–12,16,18,22–30,32–55,58,62–65,67–69]).

Note that definition (1.8) makes sense for all $p > 0$. The case where $p = 0$ is the limiting case given by (1.2). The crucial difference between the cases where $0 < p < 1$ and the cases where $p \geq 1$ is that the function $\big((1-\lambda)h_K^p + \lambda h_L^p\big)^{1/p}$ is the support function of $(1-\lambda)\cdot K +_p \lambda\cdot L$ when $p \geq 1$, but this is not necessarily the case whenever $0 < p < 1$. When $0 < p < 1$, the convex body $(1-\lambda)\cdot K +_p \lambda\cdot L$ is the Wulff shape of $\big((1-\lambda)h_K^p + \lambda h_L^p\big)^{1/p}$. Unfortunately, progress in the L_p-Brunn–Minkowski theory for $p < 1$ has been slow. The present work is a step in that direction.

It is easily seen from definition (1.8) that for fixed convex bodies K, L and fixed $\lambda \in [0,1]$, the L_p-Minkowski–Firey combination $(1-\lambda)\cdot K +_p \lambda\cdot L$ is increasing with respect to set inclusion, as p increases; i.e., if $0 \leq p \leq q$,

$$(1-\lambda)\cdot K +_p \lambda\cdot L \subseteq (1-\lambda)\cdot K +_q \lambda\cdot L. \tag{1.10}$$

From (1.10) one sees that the classical Brunn–Minkowski inequality (1.1) (i.e. the case $p = 1$ of (1.9)) immediately yields Firey's L_p-Brunn–Minkowski inequality (1.9) for each $p > 1$. The difficult situation arises when $p \in [0,1)$ because now we are seeking inequalities that are stronger than the classical Brunn–Minkowski inequality.

The L_p-Brunn–Minkowski inequality (1.9) cannot be established for all convex bodies that contain the origins in their interiors, for any fixed $p < 1$. Even an origin-centered cube and one of its translates show that. However, the following problem is of fundamental importance in the L_p-Brunn–Minkowski theory.

Problem 1.5. Suppose $0 < p < 1$. Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then for all $\lambda \in [0,1]$,

$$V((1-\lambda)\cdot K +_p \lambda\cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.11}$$

From the monotonicity of the L_p-Minkowski combination (1.10), it is clear that the log-Brunn–Minkowski inequality implies the L_p-Brunn–Minkowski inequalities for each $p > 0$. We note that there are easy examples that show that the L_p-Brunn–Minkowski inequality (1.11) fails to hold for any $p < 0$ — even if attention were restricted to simple origin symmetric bodies.

One of the aims of this paper is to show that the L_p-Brunn–Minkowski inequality (1.5) can be formulated equivalently as the L_p-Minkowski inequality.

Problem 1.6. Suppose $0 < p < 1$. Show that if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$\left(\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}} \geq \left(\frac{V(L)}{V(K)}\right)^{\frac{1}{n}}. \tag{1.12}$$

For each $p \geq 1$, the inequalities (1.11) and (1.12) are well known to hold for all convex bodies (that contain the origin in their interior) and are also well known to be equivalent, in that given one, the other is an easy consequence.

From Jensen's inequality it can be seen that the L_p-Minkowski inequality (1.12) for the case $p = 0$, the log-Minkowski inequality (1.5), is stronger than any of the L_p-Minkowski inequalities (1.12). The L_p-Minkowski inequality for the case $p = 1$, the classical Minkowski mixed-volume inequality, is weaker than all the cases of (1.12) where $p \in (0, 1)$.

Even in the plane the above problems are non-trivial and unsolved. One of the aims of this paper is to solve the problems in the plane. Solutions in higher dimensions would be highly desirable.

We will prove the following theorems.

Theorem 1.7. *Suppose* $0 < p < 1$. *If* K *and* L *are origin-symmetric convex bodies in the plane, then for all* $\lambda \in [0, 1]$,

$$V((1-\lambda)\cdot K +_p \lambda\cdot L) \geq V(K)^{1-\lambda} V(L)^{\lambda}. \tag{1.13}$$

When $\lambda \in (0, 1)$, *equality in the inequality holds if and only if* $K = L$.

Observe that the equality conditions here are different than those of Theorem 1.3.

Theorem 1.8. *Suppose* $0 < p < 1$. *If* K *and* L *are origin-symmetric convex bodies in the plane, then,*

$$\left(\int_{S^1} \left(\frac{h_L}{h_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}} \geq \left(\frac{V(L)}{V(K)}\right)^{\frac{1}{2}}, \tag{1.14}$$

with equality if and only if K *and* L *are dilates.*

Observe that the equality conditions here are different than those of Theorem 1.4.

The approach used in this paper to establish the geometric inequalities of these theorems is new.

2. Preliminaries

For quick later reference we develop some notation and basic facts about convex bodies. Good general references for the theory of convex bodies are provided by the books of Gardner [19], Gruber [21], Leichtweiss [31], Schneider [60], and Thompson [66].

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$, of a compact, convex set $K \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x\cdot y : y \in K\}, \tag{2.1}$$

and uniquely determines the convex set. Obviously, for a pair $K, L \subset \mathbb{R}^n$ of compact, convex sets, we have

$$h_K \leq h_L, \quad \text{if and only if }, \quad K \subseteq L. \tag{2.2}$$

Note that support functions are positively homogeneous of degree one and subadditive.

A *convex body* is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. A boundary point $x \in \partial K$ of the convex body K is said to have $u \in S^{n-1}$ as one of its outer unit normals provided $x\cdot u = h_K(u)$. A boundary point is said to be *singular* if it has more than one unit normal vector.

It is well known (see, e.g., [60]) that the set of singular boundary points of a convex body has $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ equal to 0.

Let K be a convex body in $\mathbb{R}^n$ and $\nu_K : \partial K \to S^{n-1}$ the generalized Gauss map. For arbitrary convex bodies, the generalized Gauss map is properly defined as a map into subsets of S^{n-1}. However, $\mathcal{H}^{n-1}$-almost everywhere on ∂K it can be defined as a map into S^{n-1}. For each Borel set $\omega \subset S^{n-1}$, the *inverse spherical image* $\nu_K^{-1}(\omega)$ of ω is the set of all boundary points of K which have an outer unit normal belonging to the set ω. Associated with each convex body K in $\mathbb{R}^n$ is a Borel measure S_K on S^{n-1} called the *Aleksandrov–Fenchel–Jessen surface area measure* of K, defined by

$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega)), \tag{2.3}$$

for each Borel set $\omega \subseteq S^{n-1}$; i.e., $S_K(\omega)$ is the $(n-1)$-dimensional Hausdorff measure of the set of all points on ∂K that have a unit normal that lies in ω.

The set of compact convex subsets of $\mathbb{R}^n$ will be viewed as equipped with the Hausdorff metric and thus a sequence of convex bodies, K_i, is said to converge to a body K, i.e.,

$$\lim_{i\to\infty} K_i = K,$$

provided that their support functions converge in $C(S^{n-1})$, with respect to the max-norm, i.e.,

$$\|h_{K_i} - h_K\|_\infty \to 0.$$

We shall make use of the weak continuity of surface area measures; i.e., if K is a convex body and K_i is a sequence of convex bodies then

$$\lim_{i\to\infty} K_i = K \Longrightarrow \lim_{i\to\infty} S_{K_i} = S_K, \text{ weakly.} \tag{2.4}$$

Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior. The *cone-volume measure* V_K of K is a Borel measure on the unit sphere S^{n-1} defined for a Borel $\omega \subseteq S^{n-1}$ by

$$V_K(\omega) = \frac{1}{n}\int_{x\in\nu_K^{-1}(\omega)} x\cdot\nu_K(x)\, d\mathcal{H}^{n-1}(x), \tag{2.5}$$

and thus

$$dV_K = \frac{1}{n} h_K\, dS_K. \tag{2.6}$$

Since,

$$V(K) = \frac{1}{n}\int_{u\in S^{n-1}} h_K(u)\, dS_K(u), \tag{2.7}$$

we can turn the cone-volume measure into a probability measure on the unit sphere by normalizing it by the volume of the body. The *cone-volume probability measure* $\bar{V}_K$ of K is defined by

$$\bar{V}_K = \frac{1}{V(K)} V_K. \tag{2.8}$$

Suppose K, L are convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. For $p \neq 0$, the L_p-mixed volume $V_p(K, L)$ can be defined as

$$V_p(K, L) = \int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p dV_K. \tag{2.9}$$

We need the *normalized L_p-mixed volume $\bar{V}_p(K, L)$*, which was first defined in [48],

$$\bar{V}_p(K, L) = \left(\frac{V_p(K, L)}{V(K)}\right)^{\frac{1}{p}} = \left(\int_{S^{n-1}} \left(\frac{h_L}{h_K}\right)^p d\bar{V}_K\right)^{\frac{1}{p}}.$$

Letting $p \to 0$ gives

$$\bar{V}_0(K, L) = \exp\left(\int_{S^{n-1}} \log \frac{h_L}{h_K}\, d\bar{V}_K\right),$$

which is the *normalized log-mixed volume* of K and L. From Jensen's inequality we know that $p \mapsto \bar{V}_p(K, L)$ is strictly monotone increasing, unless h_L/h_K is constant on $\operatorname{supp} S_K$.

Suppose that the function $k_t(u) = k(t, u) : I \times S^{n-1} \to (0, \infty)$ is continuous, where $I \subset \mathbb{R}$ is an interval. For fixed $t \in I$, let

$$K_t = \bigcap_{u \in S^{n-1}} \{x \in \mathbb{R}^n : x \cdot u \le k(t, u)\}$$

be the Wulff shape (or Aleksandrov body) associated with the function k_t. We shall make use of the well-known fact that

$$h_{K_t} \le k_t \quad \text{and} \quad h_{K_t} = k_t, \quad \text{a.e. w.r.t. } S_{K_t}, \tag{2.10}$$

for each $t \in I$. If k_t is the support function of a convex body, then $h_{K_t} = k_t$, everywhere.

The following variant (proved in e.g., [25]) of Aleksandrov's Lemma (see e.g., [2, p. 103] or [60, p. 345]) will be needed.

Lemma 2.1. *Suppose $k(t, u) : I \times S^{n-1} \to (0, \infty)$ is continuous, where $I \subset \mathbb{R}$ is an open interval. Suppose also that the convergence in*

$$\frac{\partial k(t, u)}{\partial t} = \lim_{s \to 0} \frac{k(t + s, u) - k(t, u)}{s}$$

is uniform on S^{n-1}. If $\{K_t\}_{t \in I}$ is the family of Wulff shapes associated with k_t, then

$$\frac{dV(K_t)}{dt} = \int_{S^{n-1}} \frac{\partial k(t, u)}{\partial t}\, dS_{K_t}(u).$$

Suppose K, L are convex bodies in $\mathbb{R}^n$. The *inradius* $r(K, L)$ and *outradius* $R(K, L)$ of K with respect to L are defined by

$$r(K, L) = \sup\{t > 0 : x + tL \subset K \text{ and } x \in \mathbb{R}^n\},$$
$$R(K, L) = \inf\{t > 0 : x + tL \supset K \text{ and } x \in \mathbb{R}^n\}.$$

If L is the unit ball, then $r(K, L)$ and $R(K, L)$ are the radii of maximal inscribable and minimal circumscribable balls of K, respectively. Obviously from the definition, it follows that

$$r(K, L) = 1/R(L, K). \tag{2.11}$$

If K, L happen to be origin-symmetric convex bodies, then clearly

$$r(K,L) = \min_{u\in S^{n-1}} \frac{h_K(u)}{h_L(u)} \quad \text{and} \quad R(K,L) = \max_{u\in S^{n-1}} \frac{h_K(u)}{h_L(u)}. \tag{2.12}$$

It will be convenient to always translate K so that for $0 \le t < r = r(K,L)$, the function $k_t = h_K - th_L$ is strictly positive. Let K_t denote the Wulff shape associated with the function k_t; i.e., let K_t be the convex body given by

$$K_t = \{x \in \mathbb{R}^n : x\cdot u \le h_K(u) - th_L(u) \text{ for all } u \in S^{n-1}\}. \tag{2.13}$$

Note that $K_0 = K$, and that obviously

$$\lim_{t\to 0} K_t = K_0 = K.$$

From definition (2.13) and (2.2) we immediately have

$$K_t = \{x \in \mathbb{R}^n : x + tL \subseteq K\}. \tag{2.14}$$

Using (2.14) we can extend the definition of K_t for the case where $t = r = r(K,L)$:

$$K_r = \{x \in \mathbb{R}^n : x + rL \subseteq K\}.$$

It is not hard to show (see e.g. the proof of (6.5.11) in [60]) that K_r is a degenerate convex set (i.e. has empty interior) and that

$$\lim_{t\to r} V(K_t) = V(K_r) = 0. \tag{2.15}$$

From Lemma 2.1 and (2.9), we obtain the well-known fact that for $0 < t < r = r(K,L)$,

$$\frac{d}{dt} V(K_t) = -nV_1(K_t, L). \tag{2.16}$$

Integrating both sides of (2.16), and using (2.15), give the following lemma.

Lemma 2.2. *Suppose K and L are convex bodies, and for $0 \le t < r = r(K,L)$, the body K_t is the Wulff shape associated with the positive continuous function $k_t = h_K - th_L$. Then, whenever $0 \le t \le r = r(K,L)$,*

$$V(K) - V(K_t) = n\int_0^t V_1(K_s, L)\, ds. \tag{2.17}$$

More general versions of Lemma 2.2 can be found in the literature (see e.g., Diskant [13]).

3. Equivalence of the L_p-Brunn–Minkowski and the L_p-Minkowski inequalities

In this section, we show that for each fixed $p \ge 0$ the L_p-Brunn–Minkowski inequality and the L_p-Minkowski inequality are equivalent in that one is an easy consequence of the other. In particular, the log-Brunn–Minkowski inequality and the log-Minkowski inequality are equivalent.

Suppose $p > 0$. If K and L are convex bodies that contain the origin and $s, t \ge 0$ (not both zero) the L_p-Minkowski combination $s\cdot K +_p t\cdot L$, is defined by

$$s\cdot K +_p t\cdot L = \{x \in \mathbb{R}^n : x\cdot u \le \left(sh_K(u)^p + th_L(u)^p\right)^{1/p} \text{ for all } u \in S^{n-1}\}.$$

We see that for a convex body K and real $s \geq 0$ the relationship between the L_p-scalar multiplication, $s \cdot K$, and Minkowski scalar multiplication sK is given by:

$$s \cdot K = s^{\frac{1}{p}} K.$$

Suppose $p > 0$ is fixed and suppose the following "weak" L_p-Brunn–Minkowski inequality holds for all origin-symmetric convex bodies K and L in $\mathbb{R}^n$ such that $V(K) = 1 = V(L)$:

$$V((1-\lambda) \cdot K +_p \lambda \cdot L) \geq 1, \tag{3.1}$$

for all $\lambda \in (0, 1)$. We claim that from this it follows that the following seemingly "stronger" L_p-Brunn–Minkowski inequality holds: if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, then

$$V(s \cdot K +_p t \cdot L)^{\frac{p}{n}} \geq s V(K)^{\frac{p}{n}} + t V(L)^{\frac{p}{n}}, \tag{3.2}$$

for all $s, t \geq 0$. To see this assume that the "weak" L_p-Brunn–Minkowski inequality (3.1) holds and that K and L are arbitrary origin-symmetric convex bodies. Define the volume-normalized bodies $\bar{K} = V(K)^{-\frac{1}{n}} K$ and $\bar{L} = V(L)^{-\frac{1}{n}} L$. Then (3.1) gives

$$V((1-\lambda) \cdot \bar{K} +_p \lambda \cdot \bar{L}) \geq 1. \tag{3.3}$$

Let $\lambda = V(L)^{\frac{p}{n}} (V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}})^{-\frac{p}{n}}$. Then

$$(1-\lambda) \cdot \bar{K} +_p \lambda \cdot \bar{L} = \frac{1}{(V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}})^{\frac{1}{p}}} (K +_p L).$$

Therefore, from (3.3), we get

$$V(K +_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}.$$

If we now replace K with $s \cdot K$ and L with $t \cdot L$ and note that $V(s \cdot K)^{\frac{p}{n}} = s V(K)^{\frac{p}{n}}$, we obtain the desired "stronger" L_p-Brunn–Minkowski inequality (3.2).

Lemma 3.1. *Suppose $p > 0$. When restricted to origin-symmetric convex bodies in $\mathbb{R}^n$, the L_p-Brunn–Minkowski inequality* (1.11) *and the L_p-Minkowski inequality* (1.12) *are equivalent.*

Proof. Suppose K and L are fixed origin-symmetric convex bodies in $\mathbb{R}^n$. For $0 \leq \lambda \leq 1$, let

$$Q_\lambda = (1-\lambda) \cdot K +_p \lambda \cdot L;$$

i.e., Q_λ is the Wulff shape associated with the function $q_\lambda = \left((1-\lambda) h_K^p + \lambda h_L^p\right)^{\frac{1}{p}}$. It will be convenient to consider q_λ as being defined for λ in the open interval $(-\epsilon_o, 1+\epsilon_o)$, where $\epsilon_o > 0$ is chosen so that for $\lambda \in (-\epsilon_o, 1+\epsilon_o)$, the function q_λ is strictly positive.

We first assume that the L_p-Minkowski inequality (1.12) holds. From (2.7), the fact that $h_{Q_\lambda} = \left((1-\lambda) h_K^p + \lambda h_L^p\right)^{\frac{1}{p}}$ a.e. with respect to the surface area measure S_{Q_λ}, (2.6) and (2.9), and finally the L_p-Minkowski inequality (1.12), we have

$$\begin{aligned} V(Q_\lambda) &= \frac{1}{n} \int_{S^{n-1}} h_{Q_\lambda} \, dS_{Q_\lambda} \\ &= \frac{1}{n} \int_{S^{n-1}} ((1-\lambda) h_K^p + \lambda h_L^p) h_{Q_\lambda}^{1-p} \, dS_{Q_\lambda} \end{aligned}$$

$$= (1-\lambda)V_p(Q_\lambda, K) + \lambda V_p(Q_\lambda, L)$$
$$\geq (1-\lambda)V(Q_\lambda)^{\frac{n-p}{n}} V(K)^{\frac{p}{n}} + \lambda V(Q_\lambda)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}}. \tag{3.4}$$

This gives

$$V(Q_\lambda) \geq \left((1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}\right)^{n/p} \geq V(K)^{1-\lambda}V(L)^{\lambda}, \tag{3.5}$$

which is the L_p-Brunn–Minkowski inequality (1.11).

Now assume that the L_p-Brunn–Minkowski inequality (1.11) holds. As was seen at the beginning of this section, this inequality (in fact a seemingly weaker one) implies the seemingly stronger L_p-Brunn–Minkowski inequality (3.2). From inequality (3.2) we may conclude that the function $f : [0,1] \to (0,\infty)$, given by $f(\lambda) = V(Q_\lambda)^{\frac{p}{n}}$, has the property that $f(\lambda) \geq (1-\lambda)f(0) + \lambda f(1)$. Unfortunately this is less than concavity, which is the property of f we require. In the classical case ($p = 1$) the desired concavity can be obtained (as described in e.g., Schneider [60, p. 309]) by applying (3.2) to subintervals. That this argument works for $p < 1$ is not obvious since a property of Wulff shapes is needed. A full argument runs as follows.

For given $\sigma, \tau \in [0,1]$, let

$$K_\sigma = (1-\sigma)\cdot K +_p \sigma\cdot L, \qquad K_\tau = (1-\tau)\cdot K +_p \tau\cdot L.$$

Since K_σ is the Wulff shape of the function $((1-\sigma)h_K^p + \sigma h_L^p)^{1/p}$, we have

$$h_{K_\sigma} \leq ((1-\sigma)h_K^p + \sigma h_L^p)^{1/p}.$$

If $\lambda \in [0,1]$ and $\alpha = (1-\lambda)\sigma + \lambda\tau$, this gives

$$\begin{aligned}(1-\lambda)h_{K_\sigma}^p + \lambda h_{K_\tau}^p &\leq (1-\lambda)[(1-\sigma)h_K^p + \sigma h_L^p] + \lambda[(1-\tau)h_K^p + \tau h_L^p]\\ &= [(1-\lambda)(1-\sigma) + \lambda(1-\tau)]h_K^p + [(1-\lambda)\sigma + \lambda\tau]h_L^p\\ &= (1-\alpha)h_K^p + \alpha h_L^p.\end{aligned}$$

Thus, $[(1-\lambda)h_{K_\sigma}^p + \lambda h_{K_\tau}^p]^{1/p} \leq [(1-\alpha)h_K^p + \alpha h_L^p]^{1/p}$ and taking the Wulff shapes of these functions allows us to conclude that

$$(1-\lambda)\cdot K_\sigma +_p \lambda\cdot K_\tau \subseteq (1-\alpha)\cdot K +_p \alpha\cdot L.$$

This gives

$$\begin{aligned}f((1-\lambda)\sigma + \lambda\tau) &= V((1-\alpha)\cdot K +_p \alpha\cdot L)^{p/n}\\ &\geq V((1-\lambda)\cdot K_\sigma +_p \lambda\cdot K_\tau)^{p/n}\\ &\geq (1-\lambda)V(K_\sigma)^{p/n} + \lambda V(K_\tau)^{p/n}\\ &= (1-\lambda)f(\sigma) + \lambda f(\tau),\end{aligned}$$

which is the desired concavity of f.

The convex body Q_λ is the Wulff shape of the function $q_\lambda = ((1-\lambda)\, h_K^p + \lambda\, h_L^p)^{1/p}$. Now, the convergence as $\lambda \to 0$ in

$$\frac{q_\lambda - q_0}{\lambda} \longrightarrow \frac{h_K^{1-p}}{p}\,(h_L^p - h_K^p) = \frac{h_K^{1-p}h_L^p - h_K}{p},$$

is uniform on S^{n-1}. By Lemma 2.1, (2.6) and (2.9), and (2.7),

$$\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0} = \int_{S^{n-1}} \frac{h_K^{1-p} h_L^p - h_K}{p}\, dS_K = \frac{n}{p}\,[V_p(K,L) - V(K)].$$

Therefore, the concavity of f yields

$$V(K)^{\frac{p-n}{n}}(V_p(K,L) - V(K)) = f'(0) \geq f(1) - f(0) = V(L)^{\frac{p}{n}} - V(K)^{\frac{p}{n}},$$

which gives the L_p-Minkowski inequality (1.12). □

Lemma 3.2. *For origin symmetric convex bodies in $\mathbb{R}^n$, the log-Brunn–Minkowski inequality (1.4) and the log-Minkowski inequality (1.5) are equivalent.*

Proof. Suppose K and L are fixed origin-symmetric convex bodies in $\mathbb{R}^n$. For $0 \leq \lambda \leq 1$, let

$$Q_\lambda = (1-\lambda)\cdot K +_o \lambda\cdot L;$$

i.e., Q_λ is the Wulff shape associated with the function $q_\lambda = h_K^{1-\lambda} h_L^\lambda$. It will be convenient to consider q_λ as being defined for all λ in the open interval $(-\epsilon_o, 1+\epsilon_o)$, for some sufficiently small $\epsilon_o > 0$ and let Q_λ be the Wulff shape associated with the function q_λ. Observe that since q_0 and q_1 are the support functions of convex bodies, $Q_0 = K$ and $Q_1 = L$.

We will first suppose that we have the log-Minkowski inequality (1.5) for K and L. Now $h_{Q_\lambda} = h_K^{1-\lambda} h_L^\lambda$ a.e. with respect to S_{Q_λ}, and thus,

$$\begin{aligned}
0 &= \frac{1}{nV(Q_\lambda)} \int_{S^{n-1}} h_{Q_\lambda} \log \frac{h_K^{1-\lambda} h_L^\lambda}{h_{Q_\lambda}}\, dS_{Q_\lambda} \\
&= (1-\lambda)\frac{1}{nV(Q_\lambda)} \int_{S^{n-1}} h_{Q_\lambda} \log \frac{h_K}{h_{Q_\lambda}}\, dS_{Q_\lambda} + \lambda \frac{1}{nV(Q_\lambda)} \int_{S^{n-1}} h_{Q_\lambda} \log \frac{h_L}{h_{Q_\lambda}}\, dS_{Q_\lambda} \\
&\geq (1-\lambda)\frac{1}{n} \log \frac{V(K)}{V(Q_\lambda)} + \lambda \frac{1}{n} \log \frac{V(L)}{V(Q_\lambda)} \\
&= \frac{1}{n} \log \frac{V(K)^{1-\lambda} V(L)^\lambda}{V(Q_\lambda)}.
\end{aligned} \tag{3.6}$$

This gives the log-Brunn–Minkowski inequality (1.4).

Suppose now that we have the log-Brunn–Minkowski inequality (1.4) for K and L. The body Q_λ is the Wulff shape associated with the function $q_\lambda = h_K^{1-\lambda} h_L^\lambda$, and the convergence as $\lambda \to 0$ in

$$\frac{q_\lambda - q_0}{\lambda} \longrightarrow h_K \log \frac{h_L}{h_K},$$

is uniform on S^{n-1}. By Lemma 2.1,

$$\left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0} = \int_{S^{n-1}} h_K \log \frac{h_L}{h_K}\, dS_K. \tag{3.7}$$

However, in a manner similar to that used in the proof of Lemma 3.1, the log-Brunn–Minkowski inequality (1.4) can be used to conclude that $\lambda \mapsto \log V(Q_\lambda)$ is a concave function, and thus

$$\frac{1}{V(Q_0)} \left.\frac{dV(Q_\lambda)}{d\lambda}\right|_{\lambda=0} \geq V(Q_1) - V(Q_0) = \log V(L) - \log V(K). \tag{3.8}$$

When (3.7) and (3.8) are combined the result is the log-Minkowski inequality (1.5). □

4. Blaschke's extension of the Bonnesen inequality

From this point forward we shall work exclusively in the Euclidean plane. We will make use of the properties of mixed volumes of compact convex sets, some of which might possibly be degenerate (i.e., not convex bodies). For quick later reference we list these properties now.

Suppose K, L are plane compact convex sets. Of fundamental importance is the fact that for real $s, t \geq 0$, the area, $V(sK+tL)$, of $sK+tL = \{sx+ty : x \in K \text{ and } y \in L\}$ is a homogeneous polynomial of degree 2 in s and t:

$$V(sK + tL) = s^2 V(K) + 2st V(K, L) + t^2 V(L). \tag{4.1}$$

The coefficient $V(K, L)$, the mixed area of K and L, is uniquely defined by (4.1) if we require (as we always will) that it is symmetric in its arguments; i.e.

$$V(K, L) = V(L, K). \tag{4.2}$$

From its definition, we see that the mixed area functional $V(\cdot, \cdot)$ is invariant under independent translations of its arguments. Clearly, for each K,

$$V(K, K) = V(K). \tag{4.3}$$

The mixed area of K, L is just the mixed volume $V_1(K, L)$ in the plane and thus from (2.9), we see it has the integral representation

$$V(K, L) = \frac{1}{2} \int_{S^1} h_L(u)\, dS_K(u). \tag{4.4}$$

For $u \in S^1$ we will write $u^\perp$ for the image of u under the counterclockwise rotation by a right angle. Observe that if K is degenerate with $K = \{su : -c \leq s \leq c\}$, where $u \in S^1$ and $c > 0$, then S_K is an even measure concentrated on the two point set $\{\pm u^\perp\}$ with total mass $4c$.

From (4.1), or from (4.4), we see that for plane compact convex K, L, L' and real $s, s' \geq 0$,

$$V(K, sL + s'L') = sV(K, L) + s'V(K, L'), \tag{4.5}$$

and this, together with (4.2), shows that the mixed area functional $V(\cdot, \cdot)$ is linear (with respect to Minkowski linear combinations) in both arguments.

From (4.4) we see that for plane compact convex K, L, L', we have

$$L \subseteq L' \Longrightarrow V(K, L) \leq V(K, L'), \tag{4.6}$$

with equality if and only if $h_L = h_{L'}$ a.e. w.r.t. S_K.

The basic inequality in this section, inequality (4.7), is Blaschke's extension of the Bonnesen inequality. It has been a valuable tool used to establish a variety of isoperimetric inequalities (see e.g., [6,15,57,59]). In the form presented below, Lemma 4.1 can already be found in Bol's work [7]. Since the equality conditions of inequality (4.7) are one of the critical ingredients in the proof of the log-Brunn–Minkowski inequality, we present a complete proof of inequality (4.7), with its equality conditions.

Lemma 4.1. *If K, L are plane convex bodies, then for $r(K, L) \leq t \leq R(K, L)$,*

$$V(K) - 2tV(K, L) + t^2 V(L) \leq 0. \tag{4.7}$$

The inequality is strict whenever $r(K,L) < t < R(K,L)$. When $t = r(K,L)$, equality will occur in (4.7) if and only if K is the Minkowski sum of a dilation of L and a line segment. When $t = R(K,L)$, equality will occur in (4.7) if and only if L is the Minkowski sum of a dilation of K and a line segment.

Proof. Let $r = r(K,L)$ and suppose $t \in [0,r]$. Recall from (2.13) that

$$K_t = \{x \in \mathbb{R}^n : x \cdot u \le h_K(u) - t h_L(u) \text{ for all } u \in S^{n-1}\},$$

and that from (2.14), we have

$$K_t + tL \subseteq K. \tag{4.8}$$

However, (4.8) together with the monotonicity (4.6), linearity (4.5), the symmetry of mixed volumes (4.2), and (4.3) gives

$$V(K,L) \ge V(K_t + tL, L) = V(K_t, L) + tV(L). \tag{4.9}$$

Now Lemma 2.2 and (4.9) give

$$\begin{aligned} V(K) - V(K_t) &= 2\int_0^t V(K_s, L)\,ds \\ &\le 2\int_0^t \big(V(K,L) - sV(L)\big)\,ds \\ &= 2tV(K,L) - t^2 V(L). \end{aligned} \tag{4.10}$$

Thus,

$$V(K) - 2tV(K,L) + t^2 V(L) \le V(K_t). \tag{4.11}$$

From (4.9) and (4.10) we see that equality holds in (4.11) if and only if,

$$V(K,L) = V(K_s + sL, L), \quad \text{for all } s \in [0,t], \tag{4.12}$$

which, from (4.6) and (4.8), gives

$$h_K = h_{K_s} + s h_L, \quad \text{a.e. w.r.t. } S_L$$

for all $s \in [0,t]$.

By (2.15) we know $V(K_r) = 0$ and thus K_r is a line segment, possibly a single point. Therefore, from (4.11) we have

$$V(K) - 2rV(K,L) + r^2 V(L) \le 0. \tag{4.13}$$

We will now establish the equality conditions in (4.13). To that end, suppose:

$$V(K) - 2rV(K,L) + r^2 V(L) = 0. \tag{4.14}$$

Then, by (4.12) we have

$$V(K,L) = V(K_r + rL, L).$$

However, this in (4.14) gives

$$V(K) - 2rV(K_r + rL, L) + r^2 V(L) = 0,$$

which, using (4.5), can be rewritten as

$$V(K) - 2rV(K_r, L) - r^2V(L) = 0.$$

But, since $V(K_r) = 0$ this can be written, using (4.1), as

$$V(K) - V(K_r + rL) = 0.$$

Since $K_r + rL \subseteq K$, the equality of their volumes forces us to conclude that in fact $K_r + rL = K$. Therefore, K is the Minkowski sum of a dilation of L and the line segment K_r (which may be a point).

Since $1/R(K, L) = r(L, K)$ from (2.12), from inequality (4.13), and its established equality conditions, we get

$$V(L) - 2r'V(L, K) + r'^2V(K) \leq 0, \quad \text{where } r' = r(L, K) = 1/R(K, L),$$

with equality if and only if L is the Minkowski sum of a dilation of K and a line segment. However, using the symmetry of mixed volumes (4.2), this means that

$$V(K) - 2RV(K, L) + R^2V(L) \leq 0, \quad \text{where } R = R(K, L), \tag{4.15}$$

with equality if and only if L is the Minkowski sum of a dilation of K and a line segment.

Finally, inequalities (4.13) and (4.15) together with the well-known properties of quadratic functions show that

$$V(K) - 2tV(K, L) + t^2V(L) < 0, \quad \text{whenever } r(K, L) < t < R(K, L). \quad \square$$

5. Uniqueness question for planar cone-volume measures

Given a finite Borel measure on the unit sphere, under what necessary and sufficient conditions is the measure the cone-volume measure of a convex body? This is the *existence question* for the unsolved *log-Minkowski problem*. It requires solving a Monge–Ampère equation and is connected with some important curvature flows (see e.g. [3,4,17,64]). The *uniqueness question* for the log-Minkowski problem asks under what conditions can two different bodies have identical cone-volume measures. It appears to be more difficult than the existence question. Even in the plane, the uniqueness question has not been settled. Gage [17] showed that within the class of origin-symmetric plane convex bodies that are also smooth and have positive curvature, the cone-volume measure determines the convex body uniquely. For even discrete measures, the uniqueness question for the log-Minkowski problem, for plane convex bodies, was treated by Stancu [64].

In this section, we shall settle the uniqueness question for the log-Minkowski problem for arbitrary origin-symmetric plane convex bodies. For plane convex bodies that are not origin-symmetric, the problem remains both open and important.

The uniqueness question for the log-Minkowski problem is related to Firey's *worn stone problem*. In determining the ultimate shape of a worn stone, Firey [14] showed that if the cone-volume measure of a smooth origin-symmetric convex body in $\mathbb{R}^n$ is a constant multiple of Lebesgue measure (on S^{n-1}), then the convex body must be a ball. This established uniqueness for the worn stone problem for the origin-symmetric case. In $\mathbb{R}^3$, Andrews [3] established the uniqueness of solutions to the worn stone problem by showing that a smooth (not necessarily origin-symmetric) convex body in $\mathbb{R}^3$ must be a ball if its cone volume measure is a constant multiple of Lebesgue measure on S^2.

The following inequality (5.1) was established by Gage [17] when the convex bodies are smooth and of positive curvature. A limit process gives the general case, but the equality conditions do not follow. As will be seen, the equality conditions are critical for establishing the uniqueness for cone-volume measures in the plane.

Lemma 5.1. *If K, L are origin-symmetric plane convex bodies, then*

$$\int_{S^1} \frac{h_K^2}{h_L}\, dS_K \leq \frac{V(K)}{V(L)} \int_{S^1} h_L\, dS_K, \tag{5.1}$$

with equality if and only if K and L are dilates, or K and L are parallelograms with parallel sides.

Proof. Since K and L are origin symmetric, from (2.12) we have

$$r(K,L) \leq \frac{h_K(u)}{h_L(u)} \leq R(K,L),$$

for all $u \in S^1$. Thus, from Lemma 4.1 we get

$$V(K) - 2\frac{h_K(u)}{h_L(u)} V(K,L) + \left(\frac{h_K(u)}{h_L(u)}\right)^2 V(L) \leq 0.$$

Integrating both sides of this, with respect to the measure $h_L dS_K$, and using (4.4) and (2.7), give

$$\begin{aligned} 0 &\geq \int_{S^1} \left(V(K) - 2\frac{h_K(u)}{h_L(u)} V(K,L) + \left(\frac{h_K(u)}{h_L(u)}\right)^2 V(L) \right) h_L(u)\, dS_K(u) \\ &= -2V(K)V(K,L) + V(L) \int_{S^1} \frac{h_K(u)^2}{h_L(u)}\, dS_K(u). \end{aligned}$$

This yields the desired inequality (5.1).

Suppose there is equality in (5.1). Thus,

$$V(K) - 2\frac{h_K(u)}{h_L(u)} V(K,L) + \left(\frac{h_K(u)}{h_L(u)}\right)^2 V(L) = 0, \quad \text{for all } u \in \operatorname{supp} S_K. \tag{5.2}$$

If K and L are dilates, we are done. So assume that K and L are not dilates. However, K and L not being dilates implies that $r(K,L) < R(K,L)$. From Lemma 4.1, we know that when

$$r(K,L) < \frac{h_K(u)}{h_L(u)} < R(K,L),$$

it follows that

$$V(K) - 2\frac{h_K(u)}{h_L(u)} V(K,L) + \left(\frac{h_K(u)}{h_L(u)}\right)^2 V(L) < 0,$$

and thus we conclude that

$$h_K(u)/h_L(u) \in \{r(K,L), R(K,L)\} \quad \text{for all } u \in \operatorname{supp} S_K. \tag{5.3}$$

Note that since K is origin symmetric, $\operatorname{supp} S_K$ is origin symmetric as well. Let $u_0 \in \operatorname{supp} S_K$; then either $h_K(u_0)/h_L(u_0) = r(K,L)$, or $h_K(u_0)/h_L(u_0) = R(K,L)$. Suppose it is the case that $h_K(u_0)/h_L(u_0) = r(K,L)$. Then from (5.2) and the equality conditions of Lemma 4.1 we know that K must be a dilation of the Minkowski sum of L and a line segment. However, K and L are not dilates, so there exists an $x_0 \neq 0$ such that

$$h_K(u) = |x_0 \cdot u| + r(K, L)h_L(u),$$

for all unit vectors u. This together with $h_K(u_0)/h_L(u_0) = r(K, L)$ shows that x_0 is orthogonal to u_0 and that the only unit vectors at which $h_K/h_L = r(K, L)$ are u_0 and $-u_0$. However, $\operatorname{supp} S_K$ must contain at least one unit vector $u_1 \in \operatorname{supp} S_K$ other than $\pm u_0$. From (5.3), and the fact that the only unit vectors at which $h_K/h_L = r(K, L)$ are the vectors u_0 and $-u_0$, we conclude $h_K(u_1)/h_L(u_1) = R(K, L)$ and by the same argument we conclude that the only unit vectors at which $h_K/h_L = R(K, L)$ are u_1 and $-u_1$. Now (5.3) allows us to conclude that

$$\operatorname{supp} S_K = \{\pm u_0, \pm u_1\}.$$

This implies that K is a parallelogram. Since K is the Minkowski sum of a dilate of L and a line segment, L must be a parallelogram with sides parallel to those of K. If we had assumed that $h_K(u_0)/h_L(u_0) = R(K, L)$, rather than $r(K, L)$, the same argument would lead to the same conclusion.

It is easily seen that the equality holds in (5.1) if K and L are dilates. A trivial calculation shows that equality holds in (5.1) if K and L are parallelograms with parallel sides. □

The following theorem was established by Gage [17] when the convex bodies are smooth and have positive curvature. When the convex bodies are polytopes the theorem is due to Stancu [65].

Theorem 5.2. *If K and L are plane origin-symmetric convex bodies that have the same cone-volume measure, then either $K = L$ or else K and L are parallelograms with parallel sides.*

Proof. Assume that $K \neq L$. Since

$$V_K = V_L,$$

it follows that $V(K) = V(L)$. Thus, since $K \neq L$, the bodies cannot be dilates. Thus inequality (5.1) becomes

$$\int_{S^1} \frac{h_L}{h_K}\, dV_K \geq \int_{S^1} \frac{h_K}{h_L}\, dV_K \quad \text{and} \quad \int_{S^1} \frac{h_K}{h_L}\, dV_L \geq \int_{S^1} \frac{h_L}{h_K}\, dV_L, \tag{5.4}$$

with equality, in either inequality, if and only if K and L are parallelograms with parallel sides. Using (5.4) and the fact that $V_K = V_L$, both twice, we get

$$\begin{aligned}
\int_{S^1} \frac{h_L(u)}{h_K(u)}\, dV_K(u) &\geq \int_{S^1} \frac{h_K(u)}{h_L(u)}\, dV_K(u) \\
&= \int_{S^1} \frac{h_K(u)}{h_L(u)}\, dV_L(u) \\
&\geq \int_{S^1} \frac{h_L(u)}{h_K(u)}\, dV_L(u) \\
&= \int_{S^1} \frac{h_L(u)}{h_K(u)}\, dV_K(u).
\end{aligned}$$

Thus, we have equality in both inequalities of (5.4), and from the equality conditions of (5.4) we conclude that K and L are parallelograms with parallel sides. □

6. Minimizing the logarithmic mixed volume

Lemma 6.1. *Suppose K is a plane origin-symmetric convex body, with $V(K) = 1$, that is not a parallelogram. Suppose that P_k is an unbounded sequence of origin-symmetric parallelograms all of which have orthogonal diagonals, and such that $V(P_k) \geq 2$. Then, the sequence*

$$\int_{S^1} \log h_{P_k}(u)\, dV_K(u)$$

is not bounded from above.

Proof. Let $u_{1,k}, u_{2,k}$ be orthogonal unit vectors along the diagonals of P_k. Denote the vertices of P_k by $\pm h_{1,k}u_{1,k}, \pm h_{2,k}u_{2,k}$. Without loss of generality, assume that $0 < h_{1,k} \leq h_{2,k}$. The condition $V(P_k) \geq 2$ is equivalent to $h_{1,k}h_{2,k} \geq 1$. The support function of P_k is given by

$$h_{P_k}(u) = \max\{h_{1,k}|u\cdot u_{1,k}|, h_{2,k}|u\cdot u_{2,k}|\}, \tag{6.1}$$

for $u \in S^1$. Since S^1 is compact, the sequences $u_{1,k}$ and $u_{2,k}$ have convergent subsequences. Again, without loss of generality, we may assume that the sequences $u_{1,k}$ and $u_{2,k}$ are themselves convergent with

$$\lim_{k\to\infty} u_{1,k} = u_1 \quad \text{and} \quad \lim_{k\to\infty} u_{2,k} = u_2,$$

where u_1 and u_2 are orthogonal.

It is easy to see that if the cone-volume measure, $V_K(\{\pm u_1\})$, of the two-point set $\{\pm u_1\}$ is positive, then K contains a parallelogram whose area is $2V_K(\{\pm u_1\})$. Since K itself is not a parallelogram and $V(K) = 1$, it must be the case that

$$V_K(\{\pm u_1\}) < \frac{1}{2}. \tag{6.2}$$

For $\delta \in \left(0, \frac{1}{3}\right)$, consider the neighborhood, U_δ, of $\{\pm u_1\}$, on S^1,

$$U_\delta = \{u \in S^1 : |u\cdot u_1| > 1 - \delta\}.$$

Since $V_K(S^1) = V(K) = 1$, we see that for all or $\delta \in \left(0, \frac{1}{3}\right)$

$$V_K(U_\delta) + V_K(U_\delta^c) = 1, \tag{6.3}$$

where U_δ^c is the complement of U_δ.

Since the U_δ are decreasing (with respect to set inclusion) in δ and have a limit of $\{\pm u_1\}$,

$$\lim_{\delta\to 0^+} V_K(U_\delta) = V_K(\{\pm u_1\}).$$

This together with (6.2), shows the existence of a $\delta_o > 0$ such that

$$V_K(U_{\delta_o}) < \frac{1}{2}.$$

However, this implies that there is a small $\epsilon_o \in \left(0, \frac{1}{2}\right)$ so that

$$\tau_o = V_K(U_{\delta_o}) - \frac{1}{2} + \epsilon_o < 0. \tag{6.4}$$

This together with (6.3) gives

$$V_K(U_{\delta_o}) = \frac{1}{2} - \epsilon_o + \tau_o \quad \text{and} \quad V_K(U_{\delta_o}^c) = \frac{1}{2} + \epsilon_o - \tau_o. \tag{6.5}$$

Since $u_{i,k}$ converge to u_i, we have $|u_{i,k} - u_i| < \delta_o$ whenever k is sufficiently large (for both $i = 1$ and $i = 2$). Then for $u \in U_{\delta_o}$ and k sufficiently large, we have

$$\begin{aligned} |u \cdot u_{1,k}| &\geq |u \cdot u_1| - |u \cdot (u_{1,k} - u_1)| \\ &\geq |u \cdot u_1| - |u_{1,k} - u_1| \\ &\geq 1 - \delta_o - \delta_o \\ &\geq \delta_o, \end{aligned}$$

where the last inequality follows from the fact that $\delta_o < \frac{1}{3}$. We know that $|u \cdot u_1|^2 + |u \cdot u_2|^2 = 1$, for all $u \in S^1$. Thus, for $u \in U^c_{\delta_o}$, we have $|u \cdot u_2| > (1 - (1 - \delta_o)^2)^{\frac{1}{2}} > 2\delta_o$, which shows that when k is sufficiently large,

$$\begin{aligned} |u \cdot u_{2,k}| &\geq |u \cdot u_2| - |u \cdot (u_{2,k} - u_2)| \\ &\geq |u \cdot u_2| - |u_{2,k} - u_2| \\ &\geq 2\delta_o - \delta_o \\ &= \delta_o. \end{aligned}$$

From the last paragraph and (6.1) it follows that when k is sufficiently large,

$$h_{P_k}(u) \geq \begin{cases} \delta_o h_{1,k} & \text{if } u \in U_{\delta_o}, \\ \delta_o h_{2,k} & \text{if } u \in U^c_{\delta_o}. \end{cases} \tag{6.6}$$

By (6.3) and (6.6), (6.5), the fact that $0 < h_{1,k} \leq h_{2,k}$ together with (6.4), and finally the fact that $h_{1,k} h_{2,k} \geq 1$ together with $\epsilon_o \in \left(0, \frac{1}{3}\right)$, we see that for sufficiently large k,

$$\begin{aligned} \int_{S^1} \log h_{P_k} \, dV_K &= \int_{U_{\delta_o}} \log h_{P_k} \, dV_K + \int_{U^c_{\delta_o}} \log h_{P_k} \, dV_K \\ &\geq \log \delta_o + V_K(U_{\delta_o}) \log h_{1,k} + V_K(U^c_{\delta_o}) \log h_{2,k} \\ &= \log \delta_o + \left(\frac{1}{2} + \tau_o - \epsilon_o\right) \log h_{1,k} + \left(\frac{1}{2} - \tau_o + \epsilon_o\right) \log h_{2,k} \\ &= \log \delta_o + 2\epsilon_o \log h_{2,k} + \left(\frac{1}{2} - \epsilon_o\right) \log(h_{1,k} h_{2,k}) \\ &\quad + \tau_o(\log h_{1,k} - \log h_{2,k}) \\ &\geq \log \delta_o + 2\epsilon_o \log h_{2,k}. \end{aligned}$$

Since P_k is not bounded, the sequence $h_{2,k}$ is not bounded from above. Thus, the sequence

$$\int_{S^1} \log h_{P_k} \, dV_K$$

is not bounded from above. □

Lemma 6.2. *If K is a plane origin-symmetric convex body that is not a parallelogram, then there exists a plane origin-symmetric convex body K_0 so that $V(K_0) = 1$ and*

$$\int_{S^1} \log h_Q \, dV_K \geq \int_{S^1} \log h_{K_0} \, dV_K$$

for every plane origin-symmetric convex body Q with $V(Q) = 1$.

Proof. Without loss of generality we may assume that $V(K) = 1$. Consider the minimization problem,

$$\inf \int_{S^1} \log h_Q \, dV_K,$$

where the infimum is taken over all plane origin-symmetric convex bodies Q with $V(Q) = 1$. Suppose that Q_k is a minimizing sequence; i.e., Q_k is a sequence of origin-symmetric convex bodies with $V(Q_k) = 1$ and such that $\int_{S^1} \log h_{Q_k} \, dV_K$ tends to the infimum (which may be $-\infty$). We shall show that the sequence Q_k is bounded and the infimum is finite.

By John's Theorem, there exist ellipses E_k centered at the origin so that

$$E_k \subset Q_k \subset \sqrt{2} E_k. \tag{6.7}$$

Let $u_{1,k}, u_{2,k}$ be the principal directions of E_k so that

$$h_{1,k} \le h_{2,k}, \quad \text{where } h_{1,k} = h_{E_k}(u_{1,k}) \text{ and } h_{2,k} = h_{E_k}(u_{2,k}).$$

Let P_k be the origin-centered parallelogram that has vertices $\{\pm h_{1,k} u_{1,k}, \pm h_{2,k} u_{2,k}\}$ (observe that by the Principal Axis Theorem the diagonals of P_k are perpendicular). Since $E_k \subset \sqrt{2} P_k$, it follows from (6.7) that

$$P_k \subset Q_k \subset 2 P_k. \tag{6.8}$$

From this and $V(Q_k) = 1$, we see that $V(P_k) \ge \frac{1}{4}$.

Assume that Q_k is not bounded. Then P_k is not bounded. Applying Lemma 6.1 to $\sqrt{8} P_k$ shows that the sequence $\int_{S^1} \log h_{P_k} \, dV_K$ is not bounded from above. Therefore, from (6.8) we see that the sequence $\int_{S^1} \log h_{Q_k} \, dV_K$ cannot be bounded from above. However, this is impossible because Q_k was chosen to be a minimizing sequence.

We conclude that Q_k is bounded. By Blaschke's Selection Theorem, Q_k has a convergent subsequence that converges to an origin-symmetric convex body K_0, with $V(K_0) = 1$. It follows that $\int_{S^1} \log h_{K_0} \, dV_K$ is the desired infimum. □

7. The log-Minkowski inequality

We repeat the statement of Theorem 1.4.

Theorem 7.1. *If K and L are plane origin-symmetric convex bodies, then*

$$\int_{S^1} \log \frac{h_L}{h_K} \, d\bar{V}_K \ge \frac{1}{2} \log \frac{V(L)}{V(K)},$$

with equality if and only if either K and L are dilates or when K and L are parallelograms with parallel sides.

Proof. Without loss of generality, we can assume that $V(K) = V(L) = 1$. We shall establish the theorem by proving

$$\int_{S^1} \log h_L \, dV_K \ge \int_{S^1} \log h_K \, dV_K,$$

with equality if and only if either K and L are dilates or if they are parallelograms with parallel sides.

First, assume that K is not a parallelogram. Consider the minimization problem

$$\min \int_{S^1} \log h_Q \, dV_K,$$

taken over all plane origin-symmetric convex bodies Q with $V(Q) = 1$. Let K_0 denote a solution, whose existence is guaranteed by Lemma 6.2. Our aim is to prove that $K_0 = K$ and thereby demonstrate that K itself is the only solution to this minimization problem.

Suppose f is an arbitrary but fixed even continuous function on S^1. For some sufficiently small $\delta_o > 0$, consider the deformation of h_{K_0}, defined on $(-\delta_o, \delta_o) \times S^1$, by

$$q_t(u) = q(t,u) = h_{K_0}(u)e^{tf(u)}.$$

Let Q_t be the Wulff shape associated with q_t. Observe that Q_t is an origin symmetric convex body and that since q_0 is the support function of the convex body K_0, we have $Q_0 = K_0$.

Since K_0 is an assumed solution of the minimization problem, the function defined on $(-\delta_o, \delta_o)$ by

$$t \longmapsto V(Q_t)^{-\frac{1}{2}} \exp\left\{\int_{S^1} \log h_{Q_t} \, dV_K\right\},$$

attains a minimal value at $t = 0$. Since $h_{Q_t} \le q_t$ this function is dominated by the differentiable function defined on $(-\delta_o, \delta_o)$ by

$$t \longmapsto V(Q_t)^{-\frac{1}{2}} \exp\left\{\int_{S^1} \log q_t \, dV_K\right\}.$$

However, clearly both functions have the same value at 0 and thus the latter function attains a local minimum at 0. Thus, differentiating the latter function at $t = 0$, by using Lemma 2.1, and recalling that $V(Q_0) = V(K_0) = 1$, shows that

$$-\frac{1}{2}\int_{S^1} h_{K_0}(u) f(u) \, dS_{K_0}(u) + \int_{S^1} f(u) \, dV_K(u) = 0.$$

Thus, since f was an arbitrary even continuous function, we conclude that

$$\int_{S^1} f(u) \, dV_{K_0}(u) = \int_{S^1} f(u) \, dV_K(u)$$

for every even continuous f, and therefore,

$$V_K = V_{K_0}.$$

By Theorem 5.2, and the assumption that K is not a parallelogram, we conclude that $K_0 = K$.

Thus, for each L such that $V(L) = 1$,

$$\int_{S^1} \log h_L \, dV_K \ge \int_{S^1} \log h_K \, dV_K,$$

with equality if and only if $K = L$. This is the desired result when K is not a parallelogram.

If K is a parallelogram the proof is trivial, but for the sake of completeness we shall include it. Assume that K is the parallelogram whose support function, for $u \in S^1$, is given by

$$h_K(u) = a_1|v_1 \cdot u| + a_2|v_2 \cdot u|,$$

where $v_1, v_2 \in S^1$ and $a_1, a_2 > 0$. It follows that $\operatorname{supp} S_K = \{\pm v_1^\perp, \pm v_2^\perp\}$, while also $V_K(\{\pm v_i^\perp\}) = 2a_1a_2|v_1 \cdot v_2^\perp|$, and $|v_1 \cdot v_2^\perp| = |v_2 \cdot v_1^\perp|$. It is easily seen that $V(K) = 4a_1a_2|v_1 \cdot v_2^\perp| = 1$, and that

$$\exp \int_{S^1} \log h_L \, dV_K = \sqrt{h_L(v_1^\perp)h_L(v_2^\perp)}. \tag{7.1}$$

Recall that $V(L) = 1$. The parallelogram circumscribed about L with sides parallel to those of K has volume

$$4h_L(v_1^\perp)h_L(v_2^\perp)|v_1 \cdot v_2^\perp|^{-1} = 16a_1a_2h_L(v_1^\perp)h_L(v_2^\perp),$$

and thus, $16a_1a_2h_L(v_1^\perp)h_L(v_2^\perp) \geq V(L) = 1$, or equivalently

$$h_L(v_1^\perp)h_L(v_2^\perp) \geq \frac{1}{16a_1a_2},$$

with equality if and only if L itself is a parallelogram with sides parallel to those of K. Thus, by (7.1), the functional $\int_{S^1} \log h_L \, dV_K$ attains its minimal value if and only if

$$h_L(v_1^\perp)h_L(v_2^\perp) = \frac{1}{16a_1a_2};$$

i.e., if and only if L is a parallelogram with sides parallel to those of K. $\square$

Proof of Theorem 1.3. Lemma 3.2 shows that the log-Minkowski inequality of Theorem 7.1 yields the log-Brunn–Minkowski inequality (1.6) of Theorem 1.3. To obtain the equality conditions of the log-Brunn–Minkowski inequality (1.6), we need to analyze the equality conditions of the inequality (3.6) in the proof of Lemma 3.2. The equality conditions for the log-Minkowski inequality of Theorem 7.1 show that equality in inequality (3.6) would imply that either K, L and Q_λ are dilates or that K, L and Q_λ are parallelograms with parallel sides. This establishes the equality conditions of Theorem 1.3. $\square$

Proof of Theorem 1.8. Jensen's inequality (along with its equality conditions), shows that the L_p-Minkowski inequality, for $p > 0$, of Theorem 1.8 follows from the L_0-Minkowski inequality of Theorem 7.1. $\square$

Proof of Theorem 1.7. Lemma 3.1 shows that the L_p-Minkowski inequality of Theorem 1.8 yields the L_p-Brunn–Minkowski inequality of Theorem 1.7.

To obtain the equality conditions of the L_p-Brunn–Minkowski inequality (1.13) of Theorem 1.7 we need to analyze the equality conditions of inequalities (3.4) and (3.5) of Lemma 3.1 which were used to derive the L_p-Brunn–Minkowski inequality of Theorem 1.7 from the L_p-Minkowski inequality of Theorem 1.8.

From the equality conditions of Theorem 1.8, we know that equality in inequality (3.4) implies that K and L are dilates. However, inequality (3.5) is a direct consequence of the concavity of the log function and this concavity is strict. Hence, equality in inequality (3.5) implies that $V(K) = V(L)$. Thus we conclude that equality in the L_p-Brunn–Minkowski inequality (1.13) of Theorem 1.7 implies that $K = L$. $\square$

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