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On fibrations with formal elliptic fibers

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Abstract

We prove that for a fibration of simply-connected spaces of finite type $F \hookrightarrow E \to B$ with F being positively elliptic and $H^*(F,\mathbb{Q})$ not possessing non-trivial derivations of negative degree, the base B is formal if and only if the total space E is formal. Moreover, in this case the fibration map is a formal map. As a geometric application we show that positive quaternion Kähler manifolds are formal and so are their associated twistor fibration maps.

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0. Introduction

The problem we shall address in this article gains its appeal from two rather disjoint sources: the first one being inherent to algebraic topology, the other one motivated by a prominent question in Riemannian geometry.

Let us begin by illustrating the motivating geometric setting. Riemannian manifolds with special holonomy form a very interesting class of spaces which include *Kähler manifolds* (manifolds with $\mathbf{U}(n)$-holonomy), *Calabi–Yau manifolds* (manifolds with $\mathbf{SU}(n)$-holonomy) and *Joyce manifolds* ($\mathbf{Spin}(7)$-holonomy and $\mathbf{G}_2$-holonomy). We will be interested in *quaternion*

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Kähler manifolds, which are manifolds with holonomy contained in $\mathbf{Sp}(n)\mathbf{Sp}(1)$. Such manifolds are known to be Einstein and they are called *positive*, if their scalar curvature is positive.

Positive quaternion Kähler geometry lies in the intersection of very classical yet rather different fields in mathematics. Despite its geometric setting, it was discovered to be accessible by methods from (differential) topology, symplectic geometry and complex algebraic geometry.

To the knowledge of the authors, the approach by rational homotopy theory, which we provide in this article, is the first one of its kind in the setting of quaternion Kähler geometry.

The field of positive quaternion Kähler geometry settles around the following.

Conjecture 1 (*LeBrun, Salamon*). *Every positive quaternion Kähler manifold is a symmetric space.*

There are a number of partial results supporting this conjecture and only symmetric examples, the so-called *Wolf spaces* are known. However, the conjecture remains open in general. Thus our motivating geometric question will be the following.

Question (*Geometry*). *How close are positive quaternion Kähler manifolds to being symmetric?*

Let us now describe the topological motivation, which arises from rational homotopy theory. This is a very elegant and easily-computable version of homotopy theory at the expense of losing information on torsion. It provides a transition from topology to algebra by encoding the rational homotopy type of a space in a commutative differential graded algebra. In particular, rational homotopy groups as well as Massey products can be derived from the algebra structure. Likewise, the rational cohomology algebra of the space is the homology algebra of the corresponding commutative differential graded algebra.

The concept of formality features prominently amongst the properties of topological spaces, as this property reduces the study of the rational homotopy type entirely to the problem of merely understanding the rational cohomology algebra. Or in other words: We may derive the rational cohomology from the rational homotopy type, however, is the information contained in the rational cohomology already sufficient to reconstruct the rational homotopy type? If the answer is “yes”, the space is called *formal*.

Although it is not known to the authors that if the following conjecture is stated explicitly in the literature, it is widely believed that the following holds.

Conjecture 2. *A simply-connected compact Riemannian manifold of special holonomy is a formal space.*

A famous result by Berger – having undergone several refinements – states that a simply-connected irreducible non-symmetric Riemannian manifold M has one of the holonomy groups $\mathbf{SO}(n)$ ($\dim M = n$) – the generic case not comprised in the term “special holonomy” – or $\mathbf{U}(n)$ ($\dim M = 2n$), $\mathbf{SU}(n)$ ($\dim M = 2n$), $\mathbf{Sp}(n)$ ($\dim M = 4n$), $\mathbf{Sp}(n)\mathbf{Sp}(1)$ ($\dim M = 4n$), $\mathbf{G}_2$ ($\dim M = 7$) respectively $\mathbf{Spin}(7)$ ($\dim M = 8$).

We draw the attention of the reader to how nicely the prerequisites of irreducibility and being non-symmetric fit the context of formality. The finite Cartesian product of simply-connected spaces is formal if and only if so is each factor. Due to a famous result by Cartan, symmetric spaces are formal.

A celebrated result in [7] states that compact Kähler manifolds are formal. Obviously, this comprises manifolds whose holonomy group is one of $\mathbf{U}(n)$, $\mathbf{SU}(n)$ and $\mathbf{Sp}(n)$. However, no further results in proving formality in the setting of special holonomy – yet a lot of attempts in that direction (!) – are known to the authors.

It is noteworthy that the formality of Kähler manifolds is indeed a geometric result and cannot be attributed to known topological properties of Kähler manifolds. In particular, there are compact simply-connected manifolds having the hard-Lefschetz property but lacking formality (for example see [6, Example 4.4 p. 346]). The result in [7] is derived from the famous $\mathrm{d}\,\mathrm{d}^c$-lemma, which itself reverberates strongly in many generalizations in the literature, be it in the symplectic context, the generalized complex case etc. (see [5])—always with the intent to relate it to formality. Although all sorts of partial results and counterexamples exist, the $\mathrm{d}\,\mathrm{d}^c$-lemma often seems to be related to a Lefschetz-like structure.

As for our purposes let us just mention the attempt in [37] to generalize the $\mathrm{d}\,\mathrm{d}^c$-lemma to the context of special holonomy (and $\mathbf{G}_2$-manifolds, in particular). (Note that Joyce manifolds also satisfy a Lefschetz-like property.) The author's goal was to present a sufficient criterion to prove formality in this context. However, there seems to be a major problem with the main tool, Proposition 2.19 on p. 1008 of that article. We are indebted to Spiro Karigiannis for pointing that out to us. Finally, let us mention that there exists an example of a simply-connected compact manifold sharing all the known topological properties of a $\mathbf{G}_2$-manifold but lacking formality—see [5, Example 8.5, p. 131].

The main geometric application of the topological results of this article is to settle the conjecture above for positive quaternion Kähler manifolds, thus stating a first positive result in this direction, since the appearance of the article [7].

The following natural question – widely discussed in the literature – will be our topological motivation.

Question (*Topology*)**.** *How do the formality of the base space and the one of the total space relate in a fibration?*

The answer we shall provide to this question relies on the following concepts. Recall that a fibration $F \hookrightarrow E \to B$ is called *(rationally) totally non-cohomologous to zero* or TNCZ, if the induced homomorphism $H^*(E, \mathbb{Q}) \to H^*(F, \mathbb{Q})$ is surjective. This is easily seen to be equivalent to the Leray–Serre spectral sequence of this fibration degenerating at the E_2-term. It is also well-known that $H^*(F, \mathbb{Q})$ has no negative degree derivations if and only if any fibration over a simply-connected base with fiber F is TNCZ. We remark that an example of such fibers is provided by any simply-connected space whose cohomology algebra satisfies hard-Lefschetz duality. This applies, in particular, to all Kähler manifolds and (up to a degree shift, i.e. using the Kraines form of degree 4 (see [28, equation (1.4), p. 86]) in place of the Kähler form) to – per se rationally 3-connected, but eventually to all – positive quaternion Kähler manifolds (cf. [24,4]).

Moreover, recall that a simply-connected topological space F is called *positively elliptic* or F_0, if it is rationally elliptic, i.e. if it has finite dimensional rational homotopy and cohomology, and if it has positive Euler characteristic. In this case its rational cohomology is concentrated in even degrees only. These spaces admit pure Sullivan models – i.e. Sullivan models $(\Lambda V, \mathrm{d})$ with $\mathrm{d}|_{V^{\mathrm{even}}} = 0$ and $\mathrm{d}\colon V^{\mathrm{odd}} \to \Lambda V^{\mathrm{even}}$ – and feature prominently in rational homotopy theory. Classical examples of F_0-spaces are *biquotients* (and, in particular, *homogeneous spaces*) $G//H$ with $\mathrm{rk}\ G = \mathrm{rk}\ H$.

For the convenience of the reader, we shall briefly review the notion of a biquotient. Let G be a compact connected Lie group and let $H \subseteq G \times G$ be a closed Lie subgroup. Then H acts on G on the left by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. The orbit space of this action is called the *biquotient* $G//H$ of G by H. If the action of H on G is free, then $G//H$ possesses a manifold structure.

This is the only case we shall consider. If $H = K \times L$ where $K \subset G \times 1$ and $L \subset 1 \times G$ then the biquotient $G//(K \times L)$ is often denoted by $K \setminus G/L$.

Clearly, the category of biquotients contains the one of homogeneous spaces. It was shown in [17] that biquotients admit pure models (cf. [13, Theorem 3.50]).

The most prominent conjecture which deals with F_0-spaces is the following.

Conjecture 3 (*Halperin*)**.** *Suppose F is an F_0-space. Then $H^*(F, \mathbb{Q})$ has no negative degree derivations.*

The conjecture holds true for large classes of positively elliptic spaces. It is satisfied for

- homogeneous spaces [31],
- if the cohomology algebra $H^*(F, \mathbb{Q})$ has at most 3 generators [35,22],
- if all the generators are of the same degree [33,34],
- in the "generic case" [26] or, as already mentioned,
- in the case of hard-Lefschetz spaces.

Because of this conjecture we shall refer to spaces whose rational cohomology algebras do not possess negative degree derivations as spaces satisfying the Halperin conjecture—even if the spaces in question are *not* rationally elliptic or of positive Euler characteristic.

It is a known fact (see [23, Theorem 3.4]) that if F is F_0 and satisfies Halperin's conjecture, then, given a fibration $F \hookrightarrow E \to B$, the formality of the base space implies the formality of the total space. In fact, more generally, if a fibration is TNCZ and the fiber is formal and elliptic, then the formality of the base implies the formality of the total space [23, Proposition 3.2].

Recall that a space X is called of *finite type* if all its cohomology groups over $\mathbb{Q}$ are finite dimensional and all its rationalized homotopy groups are finite dimensional. Our main result and our proposed answer to the topological question is the following.

Theorem A. *Let*

$$F \hookrightarrow E \xrightarrow{f} B$$

be a fibration of simply-connected topological spaces of finite type. Suppose that F is elliptic, formal and satisfies the Halperin conjecture. Then E is formal if and only if B is formal.

Moreover, if B and E are formal, then the map f is formal.

It is well-known that F_0 spaces are formal; in fact, they are even *hyperformal* and therefore *intrinsically formal* (see [2, Remark 2.7.11.2, p. 120]). Moreover, they admit pure Sullivan models.

Consequently, the following corollary is a direct consequence of Theorem A and completes the picture for F_0-spaces. However, we remark that Example 3.1 given below shows that the class of formal, elliptic spaces satisfying Halperin's conjecture is strictly larger than the class of F_0-spaces satisfying the Halperin conjecture.

Corollary B. *Let*

$$F \hookrightarrow E \xrightarrow{f} B$$

be a fibration of simply-connected topological spaces of finite type. Suppose that F is an F_0-space which satisfies the Halperin conjecture. Then E is formal if and only if B is formal.

Moreover, if B and E are formal, then the map f is formal.

We remark that there exists an example due to Thomas (see [36, Example III.13]) of a TNCZ fibration (even a cohomologically trivial one!) with formal base and fiber but non-formal total space. In Example 4.1 (due to Lupton), we produce a TNCZ fibration of simply-connected spaces with formal yet hyperbolic fiber satisfying Halperin's conjecture, formal base space and non-formal total space.

The interlink between topology and geometry in our case is provided by the *twistor fibration*

$$\mathbb{C}\mathbf{P}^1 \hookrightarrow Z \to M$$

of a Positive Quaternion Kähler Manifold M. (It is known that a positive quaternion Kähler manifold M is compact and simply-connected.)

As the name "quaternion Kähler" indicates, the subject lies in certain proximity to the field of Kähler geometry. Indeed, (positive) quaternion Kähler manifolds can be considered a quaternionic analogue of (compact) Kähler manifolds. The twistor fibration is one way of illustrating this proximity, as the *twistor space* Z of a positive quaternion Kähler manifold is a Kähler manifold Z. Kähler manifolds have been found to be formal spaces by joint work of Deligne et al. [7]. Obviously, $\mathbb{C}\mathbf{P}^1 \cong \mathbb{S}^2$ satisfies the Halperin conjecture.

Compact symmetric spaces are known to be formal; thus the formality of positive quaternion Kähler manifolds would be a consequence of a confirmation of Conjecture 1. Thus we can offer one more piece of the puzzle described via the geometric question we posed.

Theorem C. *A positive quaternion Kähler manifold is formal. So is its twistor fibration.*

Let us note that the only geometric input we use for this result is that positive quaternion Kähler manifolds are compact and simply-connected and that the total space of the twistor bundle admits a Kähler metric and hence is formal. Since this only uses Theorem A with $F = \mathbb{S}^2$, one may suspect that in that special case it is easy to deduce formality of the base from formality of the total space. However, we do not know of a substantially simpler proof of Theorem A even for $F = \mathbb{S}^2$ than the one we present for a general F.

One may hope that some further progress in approaching the conjecture of LeBrun and Salamon (Conjecture 1) that positive quaternion Kähler manifolds are symmetric spaces can be obtained using methods from rational homotopy theory in conjunction with some further geometric input.

For example, one can try to improve Theorem C in the following two directions, which might also be considered a motivation for proving the formality of positive quaternion Kähler manifolds in the first place.

On the one hand formality is an obstruction to *geometric formality* (see [18]) which means that the product of harmonic forms is harmonic again. Geometric formality enforces strong restrictions on the topological structure of the underlying manifold. For example, the Betti numbers of the manifold M^n are restricted from above by the Betti numbers of the n-dimensional torus [18, Theorem 6]. (This result was further strengthened for Kähler manifolds by Nagy [25, Corollary 4.1]). Symmetric spaces are geometrically formal ([19,32]). Thus it is tempting to conjecture the same for Positive Quaternion Kähler Manifolds.

On the other hand, the Bott conjecture speculates that simply-connected compact Riemannian manifolds with nonnegative sectional curvature are rationally elliptic. In the quaternionic setting, there are a number of results (see [8, Theorem A, p. 150], [3, Formula 14.42b, p. 406]) suggesting that positive scalar curvature might be regarded as a substitute for positive sectional curvature to a certain extent.

In the case of positive quaternion Kähler manifolds – mainly since the rational cohomology is concentrated in even degrees only – we suggest to see formality as a very weak substitute for ellipticity. Indeed, if positive quaternion Kähler manifolds were elliptic spaces – e.g. like simply-connected homogeneous spaces – then they would be F_0-spaces, which are formal.

If one is willing to engage with this point of view, the formality of positive quaternion Kähler manifolds may be seen as heading toward a quaternionic Bott conjecture.

Let us briefly state the following trivial consequences of Theorem C.

Corollary D. *A rationally 3-connected positive quaternion Kähler manifold M^{4n} satisfies*

$$n = \frac{\dim M}{4} = \mathrm{c}_0(M) = \mathrm{e}_0(M) = \mathrm{cat}_0(M) = \mathrm{cl}_0(M).$$

If M is not rationally 3-connected, then $M \cong \mathbf{Gr}_2(\mathbb{C}^{n+2})$ by Theorem 1.3 below and hence

$$2n = \frac{\dim M}{2} = \mathrm{c}_0(M) = \mathrm{e}_0(M) = \mathrm{cat}_0(M) = \mathrm{cl}_0(M).$$

For the definition of the numerical invariants involved see the definition on [10, 28, p. 370], which itself relies on various definitions on the pages 351, 360 and 366 in [10, 27].

In order to prove this corollary one uses Theorems 1.2 and 1.3 cited below in order to reduce the problem to a rationally 3-connected manifold. A volume form is given by $[u^n]$, where u is the Kraines form in degree 4. Due to rational 3-connectedness, this allows us to compute the rational cup-length $\mathrm{c}_0(M) = n$. The rest of the equalities then are a consequence of formality (cf. [10], Example 29.4, p. 388).

Recall that if $b_2(M) \neq 0$, we obtain that M is a complex Grassmannian. It is Kählerian, in particular, with Kähler form ω. The equation $\mathrm{c}_0(M) = 2n$ follows from the existence of the volume form ω^{2n}.

A simply-connected quaternion Kähler manifold with vanishing scalar curvature is hyper-Kählerian and Kählerian, in particular. So it is a formal space. The twistor fibration in this case is the canonical projection $M \times \mathbb{S}^2 \to M$. This directly yields the following.

Corollary E. *A compact simply-connected nonnegative quaternion Kähler manifold is formal. The twistor fibration is formal.*

Let us end the introduction with some more remarks. In general, positive quaternion Kähler manifolds are not *coformal*, i.e. their rational homotopy type is not necessarily determined by their rational homotopy Lie algebra or, equivalently, their minimal Sullivan models do not necessarily have strictly quadratic differentials. An obvious counterexample is $\mathbb{H}\mathbf{P}^n$ for $n \geq 2$.

In low dimensions, i.e. in dimensions 12 to 20, relatively simple proofs of formality of positive quaternion Kähler manifolds can be given using the concept of s-formality developed in [12]. Alternatively, one may use the existence of isometric $\mathbb{S}^1$-actions on 12-dimensional and 16-dimensional positive quaternion Kähler manifolds and further structure theory to apply Corollary [20, Theorem 5.9, p. 2785] which yields formality.

Structure of the article. In Section 1 we shall give a very brief introduction to positive quaternion Kähler geometry whilst we do the same for the necessary techniques from rational homotopy theory in Section 2. Section 3 is devoted to the proof of the Main Theorem A. Finally, in Section 4, we conclude with a depiction of several counterexamples for possible statements similar to Theorem A when assumptions on the fiber are weakened.

As a general convention for this article we shall assume all spaces involved to be simply-connected (and, in particular, connected) and have finite type. Also all graded algebras we consider are assumed to be connected and also to have finite type, i.e. they have finite dimensional cohomology in every dimension and are finitely generated in every degree. Moreover, cohomology is taken with rational coefficients and all commutative differential graded algebras are algebras over $\mathbb{Q}$.

1. Positive quaternion Kähler manifolds

Due to Berger's celebrated theorem the holonomy group $\mathrm{Hol}(M, g)$ of a simply-connected, irreducible and non-symmetric Riemannian manifold (M, g) is one of $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$, $\mathbf{Sp}(n)\mathbf{Sp}(1)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$.

A connected oriented Riemannian manifold (M^{4n}, g) is called a *quaternion Kähler manifold* if

$$\mathrm{Hol}(M, g) \subseteq \mathbf{Sp}(n)\mathbf{Sp}(1) = \mathbf{Sp}(n) \times \mathbf{Sp}(1)/\langle -\mathrm{id}, -1\rangle.$$

(In the case $n = 1$ one additionally requires M to be Einstein and self-dual.) Quaternion Kähler manifolds are Einstein (see [3, Theorem 14.39, p. 403]). In particular, their scalar curvature is constant.

Definition 1.1. A *positive quaternion Kähler manifold* is a quaternion Kähler manifold with complete metric and with positive scalar curvature.

For an elaborate depiction of the subject we recommend the survey articles [27,28]. We shall content ourselves by mentioning a few properties that will be of importance throughout this article.

Foremost, we note that a positive quaternion Kähler manifold M is not necessarily Kählerian, as the name might suggest. Moreover, the manifold M is compact and simply-connected (see [27, p. 158] and [27, 6.6, p. 163]).

The only known examples of positive quaternion Kähler manifolds are given by the *Wolf-spaces*, i.e. the *symmetric* positive quaternion Kähler manifolds. They are the only possible homogeneous examples by a result of Alekseevski. They are given by the infinite series $\mathbb{H}\mathbf{P}^n$, $\mathbf{Gr}_2(\mathbb{C}^{n+2})$ and $\widetilde{\mathbf{Gr}}_4(\mathbb{R}^{n+4})$ (the Grassmannian of oriented real 4-planes) and by the exceptional spaces $\mathbf{G}_2/\mathbf{SO}(4)$, $\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$, $\mathbf{E}_6/\mathbf{SU}(6)\mathbf{Sp}(1)$, $\mathbf{E}_7/\mathbf{Spin}(12)\mathbf{Sp}(1)$, $\mathbf{E}_8/\mathbf{E}_7\mathbf{Sp}(1)$. Besides, it is known that in each dimension there are only finitely many positive quaternion Kähler manifolds (cf. [21].0.1, p. 110). This endorses the fundamental conjecture by LeBrun and Salamon (Conjecture 1) speculating that positive quaternion Kähler manifolds are symmetric spaces.

A confirmation of the conjecture has been achieved in dimensions four (Hitchin) and eight (Poon–Salamon, LeBrun–Salamon). For a discussion of dimension 12, see [1,14].

Remarkably, the theory of positive quaternion Kähler manifolds may be completely transcribed to an equivalent theory in complex geometry. This is done via the *twistor space* Z of the positive quaternion Kähler manifold M. This Fano contact Kähler Einstein manifold may be constructed as follows.

Locally the principle $\mathbf{Sp}(n)\mathbf{Sp}(1)$ structure bundle may be lifted to its double covering with fiber $\mathbf{Sp}(n) \times \mathbf{Sp}(1)$. So, locally, one may use the standard representation of $\mathbf{Sp}(1)$ on $\mathbb{C}^2$ to associate a vector bundle H. In general, H does not exist globally, but its complex projectivization $Z = \mathbf{P}_{\mathbb{C}}(H)$ does. In particular, we obtain the *twistor fibration*

$$\mathbb{C}\mathbf{P}^1 \hookrightarrow \mathbf{P}_{\mathbb{C}}(H) \to M.$$

Alternatively, the manifold Z may be considered as the unit sphere bundle $\mathbb{S}(E')$ associated to the 3-dimensional subbundle E' of the vector bundle $\mathrm{End}(TM)$ generated locally by the almost complex structures I, J, K which behave like the corresponding unit quaternions i, j and k. That is, the twistor fibration is just

$$\mathbb{S}^2 \hookrightarrow \mathbb{S}(E') \to M.$$

(Comparing this bundle to its version above we need to remark that clearly $\mathbb{C}\mathbf{P}^1 \cong \mathbb{S}^2$.) The existence of this twistor bundle together with the fact that the total space is a compact Kähler manifold is basically the only property which we shall exploit in order to prove Corollary B.

As an example one may observe that on $\mathbb{H}\mathbf{P}^n$ we have a global lift of $\mathbf{Sp}(n)\mathbf{Sp}(1)$ and that the vector bundle associated to the standard representation of $\mathbf{Sp}(1)$ on $\mathbb{C}^2$ is just the tautological bundle. Now complex projectivization of this bundle yields the complex projective space $\mathbb{C}\mathbf{P}^{2n+1}$ and the twistor fibration is just the canonical projection.

More generally, on Wolf spaces one obtains the following. The Wolf space may be written as $G/K\mathbf{Sp}(1)$ (cf. the table on [3, p. 409]) and its corresponding twistor space is given as $G/K\mathbf{U}(1)$ with the twistor fibration being the canonical projection.

Using twistor theory a variety of remarkable results have been obtained. Let us mention just the following ones.

Theorem 1.2. *Odd-degree Betti numbers of M vanish, i.e. $b_{2i+1} = 0$ for $i \geq 0$.*

Proof. See [27, Theorem 6.6, p. 163], where it is shown that the Hodge decomposition of the twistor space is concentrated in terms $H^{p,p}(Z, \mathbb{R})$. □

This implies that a rationally elliptic positive quaternion Kähler manifold is an F_0-space and that it is formal, in particular.

Theorem 1.3 (*Strong Rigidity*)**.** *Let (M, g) be a positive quaternion Kähler manifold. Then we have*

$$\pi_2(M) = \begin{cases} 0 & \textit{iff} \quad M \cong \mathbb{H}\mathbf{P}^n \\ \mathbb{Z} & \textit{iff} \quad M \cong \mathbf{Gr}_2(\mathbb{C}^{n+2}) \\ \textit{finite with } \mathbb{Z}_2\textit{-torsion contained in } \pi_2(M) & \textit{otherwise}. \end{cases}$$

Proof. See [21, Theorem 0.2, p. 110] and [28, Theorem 5.5, p. 103]. □

2. Rational homotopy theory

2.1. Formal spaces

Definition 2.1. A commutative differential graded algebra (A, d) (over a field $\mathbb{K} \supseteq \mathbb{Q}$) is called *formal*, if it is weakly equivalent to the cohomology algebra $(H(A, \mathbb{K}), 0)$ (with trivial differential).

We call a path-connected topological space *formal* if $(\mathrm{A_{PL}}(X), \mathrm{d})$ is formal. In detail, the space X is formal if and only if there is a weak equivalence $(\mathrm{A_{PL}}(X), \mathrm{d}) \simeq (H^*(X), 0)$, i.e. a chain of quasi-isomorphisms

$$(\mathrm{A_{PL}}(X), \mathrm{d}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} \ldots \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} (H^*(X), 0).$$

The algebras involved are algebras over the rationals. However, it turns out that the notion of formality does not depend on rational coefficients.

Theorem 2.2. *Let X have rational homology of finite type and let $\mathbb{K} \supseteq \mathbb{Q}$ be a field extension. Then the algebra $(\mathrm{A_{PL}}(X, \mathbb{K}), \mathrm{d})$ is formal if and only if X is a formal space.*

Proof. See [10, p. 156] and [10, Theorem 12.1, p. 316]. □

Thus we need not worry about field extensions and it suffices to consider rational coefficients only.

Example 2.3. The following spaces are formal.

- H-spaces ([10, Example 12.3, p. 143]).
- Symmetric spaces of compact type ([10, Example 12.3, p. 162]).
- N-symmetric spaces (see [32, Main Theorem, p. 40] for the precise statement, [19]).
- Compact Kähler manifolds ([7, Main Theorem, p. 270]). □

2.2. *Formal maps*

Let us recall the following definition.

Definition 2.4 (*Formal Maps*)**.** Let (A, d), (A', d') be formal dga's and let $f\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a morphism of dga's. Let $\mu_A\colon (M_A, \hat{\mathrm{d}}) \to (A, \mathrm{d})$ and $\mu_A\colon (M_{A'}, \hat{\mathrm{d}}') \to (A', \mathrm{d}')$ be minimal Sullivan models.

Let $\hat{f}\colon (M_A, \hat{\mathrm{d}}) \to (M_{A'}, \hat{\mathrm{d}}')$ be a Sullivan representative, i.e. an induced map of minimal models unique up to homotopy. Then f is called *formal* if there exist quasi-isomorphisms $m_A\colon (M_A, \hat{\mathrm{d}}) \to (H(A, \mathrm{d}), 0)$ and $m_{A'}\colon (M_{A'}, \hat{\mathrm{d}}') \to (H(A', \mathrm{d}'), 0)$ which are the identity on cohomology and make the following diagram commute up to homotopy

$$\begin{tikzcd} (M_A, \hat{\mathrm{d}}) \arrow[r, "\hat{f}"] \arrow[d, "m_A"] & (M_{A'}, \hat{\mathrm{d}}') \arrow[d, "m_{A'}"] \\ (H(A, \mathrm{d}), 0) \arrow[r, "f^*"] & (H(A', \mathrm{d}'), 0) \end{tikzcd} \tag{1}$$

where we identify $H(A, \mathrm{d})$ with $H(M_A, \hat{\mathrm{d}})$ via μ_A^* and $H(A', \mathrm{d}')$ with $H(M_{A'}, \hat{\mathrm{d}}')$ via $\mu_{A'}^*$.

A map $f\colon X \to Y$ between formal topological spaces is called formal if the induced map $f_{\mathbb{Q}}\colon (A_{PL}(Y), \mathrm{d}) \to (A_{PL}(X), \mathrm{d}')$ is formal.

Remark 2.5. Let us remark that the minimality of M_A and $M_{A'}$ is not necessary in the above definition and f is formal if and only if one can find Sullivan models M_A and $M_{A'}$ satisfying the above definition.

2.3. Absolute bigraded and filtered models

Theorem 2.6 (*Halperin–Stasheff Bigraded Model [15]*). *Let A be a finitely generated graded commutative algebra over $\mathbb{Q}$. We suppose that $A_0 = \mathbb{Q}$. Then the cdga $(A, 0)$ admits a minimal model $\rho\colon (\Lambda V, \mathrm{d}) \to (A, 0)$ where V is equipped with a lower gradation $V = \oplus_{p\geq 0} V_p$ extended in a multiplicative way to ΛV and where the following properties hold.*

(1) $\mathrm{d}(V_p) \subset (\Lambda V)_{p-1}$. *In particular,* $\mathrm{d}(V_0) = 0$. *Therefore the cohomology is a bigraded algebra* $H(\Lambda V, \mathrm{d}) = \oplus_{p\geq 0} H_p(\Lambda V, \mathrm{d})$.
(2) $\rho(V_p) = 0$ *for* $p > 0$.
(3) $H_p(\Lambda V, \mathrm{d}) = 0$ *for* $p > 0$ *and* $\rho^*\colon H_0(\Lambda V, \mathrm{d}) \to H(A, 0) = A$ *is an isomorphism*

The cdga $(\Lambda V, \mathrm{d})$ *is called a* bigraded model *of the graded algebra* A.

Bigraded models are unique in the natural sense.

As noted above the fact that d is homogeneous of lower degree -1 means that $H_p(\Lambda V, \mathrm{d}) \to H(\Lambda V, \mathrm{d})$ is injective for any p. Also recall that the algebra of derivations of a dga (C, d) denoted by $\mathrm{Der}(C)$ is naturally a differential graded Lie algebra with the differential $\mathcal{D}$ given by the bracket with d:

$$\mathcal{D}(\theta) = [\mathrm{d}, \theta] = \mathrm{d} \circ \theta - (-1)^q \theta \circ \mathrm{d}$$

where θ is a derivation of degree q. If $(C, \mathrm{d}) = (\Lambda V, \mathrm{d})$ is a bigraded model of a dga $(A, 0)$, the lower filtration on ΛV induces a lower filtration on $\mathrm{Der}(\Lambda V, \mathrm{d})$. We denote by $\mathrm{Der}^q_p(\Lambda V, \mathrm{d})$ the set of derivations of $(\Lambda V, \mathrm{d})$ which increase the usual degree by q and decrease the lower degree by p. With these notations we have that

$$\mathcal{D}\colon \ \mathrm{Der}^q_p(\Lambda V, \mathrm{d}) \to \mathrm{Der}^{q+1}_{p+1}(\Lambda V, \mathrm{d}).$$

We shall denote the cohomology of this complex by $H^q_p(\mathrm{Der}(\Lambda V, \mathrm{d}))$. As before, the (-1)-homogeneity of $\mathcal{D}$ with respect to the lower degree implies that the natural map $H^q_p(\mathrm{Der}(\Lambda V, \mathrm{d})) \to H^q(\mathrm{Der}(\Lambda V, \mathrm{d}))$ is injective for any p, q.

For cdga's with non-zero differentials Halperin and Stasheff [15] introduced the so-called *filtered models* exhibiting such algebras as deformations of their cohomology algebras in an appropriate sense.

Theorem 2.7. *Let (A, d_A) be a connected cdga of finite type. Let $\rho\colon (\Lambda V, \mathrm{d}) \to (H(A, \mathrm{d}_A), 0)$ be a bigraded model of A.*

Then there exists a differential D *on* ΛV *such that*

(1) $(\Lambda V, \mathrm{D}) \xrightarrow{\pi} (A, \mathrm{d}_A)$ *is a Sullivan model of* (A, d_A)*; (This model is not necessarily minimal.)*
(2) $\mathrm{D} - \mathrm{d}$ *decreases the lower degree by at least two:*

$$\mathrm{D} - \mathrm{d}\colon \ V_p \to (\Lambda V)_{\leq p-2}.$$

That is, the differential D *can be written as* $\mathrm{D} = \mathrm{d} + \mathrm{d}_2 + \mathrm{d}_3 + \cdots$ *where* d_i *is homogeneous of degree* $-i$ *in lower degree:*

$$\mathrm{d}_i\colon \ V_p \to (\Lambda V)_{p-i}.$$

Filtered and bigraded models are useful for a number of reasons. Of particular interest to us is the fact that they provide a good framework for distinguishing formal cdga's amongst all cdga's with a given cohomology ring. As shown by Halperin and Stasheff, the deformation differentials d_i define a series of obstructions o_i.

Theorem 2.8 (*Obstructions to Formality [15]*). *Let (A, d_A) be a connected cdga of finite type. Let $\rho\colon (\Lambda V, \mathrm{d}) \to (H(A), 0)$ be a bigraded model of A. Let $(\Lambda V, \mathrm{D}) \xrightarrow{\pi} A$ with $\mathrm{D} = \mathrm{d} + \mathrm{d}_2 + \mathrm{d}_3 + \cdots$ be a filtered model of A with respect to the chosen bigraded model. Then we obtain:*

If $\mathrm{d}_j = 0$ for $j < i$, then d_i is a closed derivation in $\mathrm{Der}_i^1(\Lambda V, \mathrm{d})$ and its cohomology class in $H_i^1(\mathrm{Der}(\Lambda V, \mathrm{d}))$ is denoted by o_i. If $o_i = 0$, then there exists an automorphism of ΛV as a cga (but not as cdga) such that conjugating both the algebra and the differential by this automorphism yields a new filtered model $(\Lambda V, \mathrm{D}')$. This model has the important property that the decomposition of its differential

$$\mathrm{D}' = \mathrm{d}' + \mathrm{d}'_2 + \mathrm{d}'_3 + \cdots$$

as above satisfies that the d'_j are identically zero for $j = 2, 3, \ldots, i$.

Repeating this process one obtains that A is formal if and only if the consecutive sequence of obstructions o_i obtained in this fashion vanishes for all $i \geq 2$.

This obstruction theory will be one of our main tools in proving Corollary B.

Before we go on let us observe that the minimality of a bigraded model is not important in the above result. This fact is probably well-known; yet, since we do not have an explicit reference for this, we sketch a proof here.

Let $(A, 0)$ be a cdga and let $\phi\colon (\Lambda V, \mathrm{d}) \to (A, 0)$ be a Sullivan model of A satisfying all the properties of a bigraded model except possibly minimality. Following Saneblidze [30] we shall refer to such a model as a *multiplicative resolution* of A. It is also a Tate-Jozefiak resolution of A in the category of cgas (i.e. forgetting the zero differential on A)—cf. [16].

In complete analogy to Theorem 2.7, we may speak of a *filtered model with respect to a multiplicative resolution*.

Corollary 2.9. *Let (A, d_A) be a connected cdga. Let $\phi\colon (\Lambda V, \mathrm{d}) \to (H(A), 0)$ be a multiplicative resolution of A. Let $\pi\colon (\Lambda V, \mathrm{D}) \xrightarrow{\simeq} (A, \mathrm{d}_A)$ with $\mathrm{D} = \mathrm{d} + \mathrm{d}_2 + \mathrm{d}_3 + \cdots$ be a filtered model of (A, d_A) with respect to the multiplicative resolution ϕ.*

We derive the following results. If $\mathrm{d}_j = 0$ for $j < i$, then d_i is a closed derivation in $\mathrm{Der}_i^1(\Lambda V, \mathrm{d})$ and its cohomology class in $H_i^1(\mathrm{Der}(\Lambda V, \mathrm{d}))$ is denoted by o_i.

As a consequence, the cdga (A, d_A) is formal if and only if $o_i = 0$ for all $i \geq 2$.

Proof. The "if" direction is proved in exactly the same way as in [15] as it does not use minimality of the multiplicative resolution. For the "only if" direction recall that Saneblidze constructed relative bigraded and filtered models for fibrations (see [29] or [30]). While his construction is somewhat different from the ones in [35] or [38], which we discuss below, in the case the base equals a point it reduces to the filtered model of the fiber based on an arbitrary (not necessarily minimal) multiplicative resolution of the fiber. Moreover, Saneblidze also proved the uniqueness of such filtered models (see [29, Theorem 3.3] or [30, Theorem A]) generalizing the uniqueness theorem for bigraded models of Halperin and Stasheff [15, Theorem 4.4] where the same statement was proved for filtered models based on minimal multiplicative resolutions. This uniqueness implies the "only if" direction in exactly the same way as in [15]. □

2.4. Relative bigraded and filtered models

We shall also need relative versions of bigraded and filtered models developed by Vigué-Poirrier [38] (cf. also [30,29,36]).

As in the absolute case, one begins by constructing a bigraded model in the relative category. For this let $\phi\colon (H,0) \to (H',0)$ be a morphism of two cdga's (with trivial differentials). Let $\rho\colon (\Lambda Z, \mathrm{d}) \xrightarrow{\simeq} (H,0)$ be a bigraded model and let

$$(\Lambda Z, \mathrm{d}) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, \mathrm{d}') \overset{p}{\to} (\Lambda X, \mathrm{d}'')$$

be a minimal relative model for $\phi \circ \rho$. That is, $(\Lambda Z, \mathrm{d}) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, \mathrm{d}')$ is a minimal relative Sullivan algebra (see [11, Section 14] for definitions) such that the following diagram commutes.

$$\begin{tikzcd} (H,0) \arrow[r, "\phi"] & (H',0) & \\ (\Lambda Z, \mathrm{d}) \arrow[u, "\rho"] \arrow[r, "i"] \arrow[ur, dashed, "\phi\circ\rho"] & (\Lambda Z \otimes \Lambda X, \mathrm{d}') \arrow[u, "\rho'"] \arrow[r, "p"] & (\Lambda X, \mathrm{d}'') \end{tikzcd} \tag{2}$$

Here both ρ and ρ' are quasi-isomorphisms. (Note that $(\Lambda Z \otimes \Lambda X, \mathrm{d}')$ is a Sullivan algebra which is minimal as a relative algebra but might not be minimal as an absolute algebra.)

Theorem 2.10 (*Relative Bigraded Model, [38]*). *With the terminology from above one obtains the following. The algebra $(Y, \mathrm{d}') := (\Lambda Z \otimes \Lambda X, \mathrm{d}')$ can be chosen in such a way that it admits a lower grading satisfying the following properties.*

(a) *It holds that $Z = \oplus_{i\geq 0} Z_i$ as well as $X = \oplus_{i \geq 0} X_i$ and $Y_i = X_i \oplus Z_i$.*
(b) *The differential d' is homogeneous of degree -1 with respect to the lower grading.*
(c) *All the maps in diagram (2) preserve the lower grading. For this $(H,0)$ and $(H',0)$ are understood to have trivial lower gradings, i.e. $H_0 = H$ and $H_i = 0$ for $i > 0$ respectively, $H'_0 = H'$ and $H'_i = 0$ for $i > 0$.*
(d) *The morphism $\rho'\colon (\Lambda Y, \mathrm{d}') \to (H',0)$ is a multiplicative resolution. That is*
  (i) *$\rho'\colon \Lambda Y_0 \to H'$ is onto,*
  (ii) *$\rho'(Y_i) = 0$ for $i > 0$,*
  (iii) *$H_i(\Lambda Y, \mathrm{d}') = 0$ for $i > 0$ and $(\rho')^*_0\colon H_0(\Lambda Y, \mathrm{d}') \to H'$ is an isomorphism.*

We shall refer to this relative Sullivan algebra

$$(\Lambda Z, \mathrm{d}) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, \mathrm{d}') \overset{p}{\to} (\Lambda X, \mathrm{d}'')$$

as the *(relative) bigraded model of ϕ*.

Finally, Vigué-Poirrier (see [38]) proved the existence of a relative filtered model for morphisms between arbitrary cdga's.

Theorem 2.11. *Let $\alpha\colon (A, \mathrm{d}) \to (A', \mathrm{d}')$ be a morphism of cdga's with the property that $H^1(\alpha)$ is injective. Let*

$$\begin{tikzcd} (H(A,\mathrm{d}),0) \arrow[r, "\alpha^*"] & (H(A',\mathrm{d}'),0) & \\ (\Lambda Z, \mathrm{d}) \arrow[u, "\rho"] \arrow[r, "i"] \arrow[ur, dashed, "\alpha^*\circ\rho"] & (\Lambda Z \otimes \Lambda X, \mathrm{d}') \arrow[u, "\rho'"] \arrow[r, "p"] & (\Lambda X, \mathrm{d}'') \end{tikzcd} \tag{3}$$

be a bigraded model of α^* provided by Theorem 2.10.

Let $(\Lambda Z, \mathrm{D}) \xrightarrow{\pi} (A, \mathrm{d})$ *be a filtered model of* (A, d). *Then* $\alpha \circ \pi\colon (\Lambda Z, \mathrm{D}) \to (A', \mathrm{d})$ *admits a minimal relative model*

$$\begin{tikzcd}
(A, \mathrm{d}) \arrow[rr, "\alpha"] & & (A', \mathrm{d}') & \\
(\Lambda Z, \mathrm{D}) \arrow[u, "\pi"] \arrow[urr, dashed, "\alpha\circ\pi"] \arrow[rr, "i"] & & (\Lambda Z \otimes \Lambda X, \mathrm{D}') \arrow[u, "\pi'"'] \arrow[r, "p"] & (\Lambda X, \mathrm{D}'')
\end{tikzcd} \tag{4}$$

where $\mathrm{D}' - \mathrm{d}'$ *decreases the lower grading by at least* 2, *i.e.*

$$\mathrm{D}' - \mathrm{d}'\colon (\Lambda Z \otimes \Lambda X)_p \to (\Lambda Z \otimes \Lambda X)_{\leq p-2} \qquad \textit{for any } p \geq 0. \tag{5}$$

We shall refer to the relative Sullivan algebra

$$(\Lambda Z, \mathrm{D}) \xrightarrow{i} (\Lambda Z \otimes \Lambda X, \mathrm{D}') \xrightarrow{p} (\Lambda X, \mathrm{D}'')$$

as the *(relative) filtered model of* α.

Remark 2.12. In [35] Thomas considered a simplified version of a relative filtered model, which is obtained by pushing forward the above construction via the quasi-isomorphism $\pi\colon (\Lambda Z, \mathrm{D}) \to (A, \mathrm{d})$, i.e. his filtered model can be obtained from the one defined above by taking $(A, \mathrm{d}) \otimes_{(\Lambda Z, \mathrm{D})} (\Lambda Z \otimes \Lambda X, \mathrm{D}')$. □

2.5. Bigraded and filtered models of TNCZ fibrations

For a general filtered model of a map $\alpha\colon A \to A'$ the fiber cdga $(\Lambda X, \mathrm{D}'')$ is not a filtered model of the fiber. However, it is one if the map $\alpha\colon A \to A'$ is a model of a TNCZ fibration, which is the situation we are interested in this article. More precisely, the following holds.

Theorem 2.13 (*[38] (cf. [36])*)**.** *Let* $F \hookrightarrow E \xrightarrow{f} B$ *be a Serre fibration of path-connected spaces where* B *is simply-connected and* $H^*(F)$ *has finite type. Suppose this fibration is TNCZ. Let* $f_{\mathbb{Q}}\colon M_B \to M_E$ *be the induced map of minimal models. Then* $f_{\mathbb{Q}}$ *admits a relative filtered model*

$$(\Lambda Z, \mathrm{D}) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, \mathrm{D}') \xrightarrow{p} (\Lambda X, \mathrm{D}'')$$

such that $(\Lambda X, \mathrm{D}'')$ *is a filtered model of* F.

In particular, $\mathrm{D}''_1 = \mathrm{d}''$ *where* d'' *is the fiber differential in the bigraded model of* $f^*\colon H^*(B) \to H^*(E)$. *Here* D''_1 *denotes the part of* D'' *decreasing the lower degree by* 1.

3. Proof of the main results

Before we prove Theorem A, let us show by an example that the class of formal elliptic spaces satisfying the Halperin conjecture is strictly larger than the class of F_0-spaces satisfying Halperin's conjecture. In other words, there are formal elliptic spaces with vanishing Euler characteristic which do not possess non-trivial derivations of negative degree on their cohomology algebras.

However, we remark that a formal elliptic space is necessarily two-stage due to [9]. Moreover, as a consequence of finite dimensionality of cohomology, the filtration degree 1 in this two-stage decomposition only consists of odd-degree elements. This follows from comparing the minimal model with the E_0-term of the *odd spectral sequence* (see [10, Section 32b, p. 438]), i.e. the *pure Sullivan algebra associated with the minimal model*. This associated algebra has finite cohomological dimension if and only if the minimal model does (see [10, Proposition 32.4, p. 438]). However, any even degree element in filtration degree one in the minimal model becomes an element in filtration degree zero in the associated pure algebra. Such an element freely generates an infinite sequence of non-vanishing cohomology classes of increasing degrees.

Thus, basically, the "only difference" of a formal elliptic space from an F_0-space, which admits a pure model, lies in the fact that in filtration degree zero odd degree elements may occur.

Nonetheless, this suffices to give the following.

Example 3.1. We define a minimal Sullivan algebra $(\Lambda V, \mathrm{d})$ by

$$V = \langle a, b, c, d, u, v\rangle \quad \text{with } \deg a = \deg b = \deg c = \deg d = 3,$$
$$\deg u = 6, \deg v = 11.$$

We set $\mathrm{d}\, a = \mathrm{d}\, b = \mathrm{d}\, c = \mathrm{d}\, d = \mathrm{d}\, u = 0$ and $\mathrm{d}\, v = abcd + u^2$.

This algebra is easily seen to be the minimal model of a rationally non-trivial $\mathbb{S}^6$-bundle over $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$. Therefore this algebra is elliptic and also formal, since the base is formal and the fiber is F_0. In order to compute the derivations of negative degree on its cohomology algebra, one notes that the cohomology algebra is generated by elements of degrees 3 and 6. Consequently, every non-trivial homogeneous such derivation has to have degree -3 or -6. However, it is impossible to specify a non-trivial derivation on $[v]$ and on the $[a], [b], [c], [d]$ (and extend it as a derivation to the whole of $H(\Lambda V, \mathrm{d})$) which would be compatible with the fact that $[abcd] = -[u^2]$ in cohomology.

As $(\Lambda V, \mathrm{d})$ has non-zero cohomology generators of odd degree, its spatial realization is not an F_0-space. □

Proof of Theorem A. We are now ready to proceed with the proof of Theorem A.

Let $F \hookrightarrow E \xrightarrow{f} B$ be a Serre fibration where E, F, B are simply-connected and of finite type. Suppose further that $H^*(F)$ is finite dimensional, E and F are formal and that F is a two-stage space satisfying Halperin's conjecture.

Let $f^*\colon\ H^*(B) \to H^*(E)$ be the induced map on cohomology and let

$$\begin{tikzcd}
(H(B), 0) \arrow[r, "f^*"] & (H(E), 0) & \\
(\Lambda Z, \mathrm{d}) \arrow[u, "\rho"] \arrow[r, "i"] \arrow[ur, dashed, "f^*\circ\rho"] & (\Lambda Z \otimes \Lambda X, \mathrm{d}') \arrow[u, "\rho'"'] \arrow[r, "p"] & (\Lambda X, \mathrm{d}'')
\end{tikzcd} \tag{6}$$

be its relative bigraded model.

Let M_B, M_E be the minimal models of B and E and let $f_{\mathbb{Q}}\colon\ M_B \to M_E$ be a Sullivan representative, a corresponding map of minimal models.

Let

$$\begin{tikzcd}
M_B \arrow[r, "f_{\mathbb{Q}}"] & M_E & \\
(\Lambda Z, \mathrm{D}) \arrow[u, "\pi"] \arrow[r, "i"] \arrow[ur, dashed, "f_{\mathbb{Q}}\circ\pi"] & (\Lambda Z \otimes \Lambda X, \mathrm{D}') \arrow[u, "\pi'"'] \arrow[r, "p"] & (\Lambda X, \mathrm{D}'')
\end{tikzcd} \tag{7}$$

be the filtered model of $f_{\mathbb{Q}}$.

We decompose $\mathrm{D} = \mathrm{d} + \mathrm{d}_2 + \mathrm{d}_3 + \cdots$ and $\mathrm{D}' = \mathrm{d}' + \mathrm{d}'_2 + \mathrm{d}'_3 + \cdots$ as above. Note that, since

$$(\Lambda Z, \mathrm{D}) \overset{i}{\hookrightarrow} (\Lambda Z \otimes \Lambda X, \mathrm{D}') \overset{p}{\to} (\Lambda X, \mathrm{D}'')$$

is a relative Sullivan algebra, we have that

$$\mathrm{d}'_i\,|_{\Lambda Z \otimes 1} = \mathrm{d}_i\,. \tag{8}$$

Since F satisfies the Halperin conjecture, the fibration $F \hookrightarrow E \overset{f}{\to} B$ is TNCZ and therefore, by Theorem 2.13, we obtain that $(\Lambda X, \mathrm{d}'')$ is a bigraded model of $H(F)$ and $(\Lambda X, \mathrm{D}'')$ is a filtered model of F. Since F is formal and elliptic, by [9] we know that $X_i = 0$ for $i \geq 2$. Thus, for degree reasons,

$$\mathrm{D}'' - \mathrm{d}''\,|_{1 \otimes \Lambda X} = 0 \tag{9}$$

and, in particular, $\mathrm{D}'' = \mathrm{d}''$.

This is a very useful fact which simplifies the situation considerably as it means that the relative filtered model is completely determined by the relative bigraded model and the filtered model of the base. In particular, it immediately implies the fact mentioned in the introduction that for a TNCZ fibration with elliptic and formal fiber the formality of the base space implies the formality of the total space.

Recall that $\mathrm{d}_i \in \mathrm{Der}^1_i(\Lambda Z, \mathrm{d})$ and $\mathrm{d}'_i \in \mathrm{Der}^1_i(\Lambda Z \otimes \Lambda X, \mathrm{d}')$. Since $\mathrm{Der}(-,-)$ is contravariant in the first argument and covariant in the second argument, there are no natural maps between $\mathrm{Der}^1_i(\Lambda Z)$ and $\mathrm{Der}^1_i(\Lambda Z \otimes \Lambda X)$. However, both of them naturally map to $\mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X)$. Here we view $(\Lambda Z \otimes \Lambda X, \mathrm{d}')$ as a dga-module over $(\Lambda Z, \mathrm{d})$ via the dga-homomorphism $(\Lambda Z, \mathrm{d}) \hookrightarrow (\Lambda Z \otimes \Lambda X, \mathrm{d}')$.

$$\begin{tikzcd} \mathrm{Der}^1_i(\Lambda Z, \Lambda Z) \arrow[dr, "j"] & \\ & \mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X) \\ \mathrm{Der}^1_i(\Lambda Z \otimes \Lambda X, \Lambda Z \otimes \Lambda X) \arrow[ur, "j'"] & \end{tikzcd} \tag{10}$$

By (8) and (9) we have that $j'(\mathrm{d}'_i) = j(\mathrm{d}_i)$ for every i.

The differential on $\mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X)$ is given by

$$\mathcal{D}(\theta) = \mathrm{d}' \circ \theta - (-1)^k \theta \circ \mathrm{d}$$

where $\theta \in \mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X)$ is a derivation of degree k. The morphisms j respectively j' commute with the differentials d respectively d' and $\mathcal{D}$, since $(\Lambda Z \otimes \Lambda X, \mathrm{d}')$ is a relative Sullivan algebra over $(\Lambda Z, \mathrm{d})$.

Let us assume that $\mathrm{d}_j = 0$ for $j < i$. Due to Eqs. (8) and (9) this equally implies that $\mathrm{d}'_j = 0$ for $j < i$.

Then d'_i is a closed derivation and, since E is formal, by Corollary 2.9 it is also exact. Therefore, the cocycle $j'(\mathrm{d}_i) \in \mathrm{Der}^1_i(\Lambda Z, \Lambda Z \otimes \Lambda X)$ is exact, too.

Our key observation is that the map j is injective in cohomology.

Lemma 3.2. *Under the above assumptions*

$$j^*\colon\ H^*(\mathrm{Der}(\Lambda Z, \Lambda Z)) \to H^*(\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X))$$

is injective.

Proof. Consider the following filtration on $\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X)$: let

$$F^p(\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X))$$

be the set of derivations which increase the ΛZ-degree by at least p; note that p can be negative. This filtration is clearly invariant under the differential $\mathcal{D}(\theta) = \mathrm{d}' \circ \theta - (-1)^k \theta \circ \mathrm{d}$ (with $\theta \in \mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X)$ of degree k).

Let us examine the spectral sequence arising from this filtration. Note that the filtration is bi-infinite on every $\mathrm{Der}_k(\Lambda Z, \Lambda Z \otimes \Lambda X)$. Therefore it is not immediately clear why this spectral sequence converges. However, we compute

$$E_0 = \mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X) \cong \mathrm{Der}(\Lambda Z, \Lambda Z) \otimes \Lambda X$$

and the differential d_0 on E_0 satisfies $\mathrm{d}_0 = \mathrm{d}''$. It follows that

$$E_1 = \mathrm{Der}(\Lambda Z, \Lambda Z) \otimes H(\Lambda V, \mathrm{d}'').$$

Since $H(F) \cong H(\bigwedge V, \mathrm{d}'')$ is finite dimensional, this implies that all but finitely many rows of $E_1^{*,*}$ are zero. This easily implies that the spectral sequence converges (and does so after finitely many steps, i.e. it collapses) to

$$H(\mathrm{Der}(\Lambda Z, \Lambda Z \otimes \Lambda X)).$$

We readily compute that

$$E_2^{p,q} = H^p(\mathrm{Der}(\Lambda Z, \Lambda Z)) \otimes H^q(\Lambda X, \mathrm{d}'')$$

and that j^* is the edge homomorphism of this spectral sequence.

Recall that $H(\Lambda X, \mathrm{d}'') \cong H(F)$. By a standard argument the multiplicative properties of the spectral sequence imply that non-trivial spectral sequence differentials d_r (for $r \geq 2$) produce non-trivial derivations on $H(F)$ of negative degree.

More precisely, suppose that $r = 2$ or that $\mathrm{d}_t = 0$ for $2 \leq t < r$ for $r \geq 3$. This identifies the E_r-term with the E_2-term. Thus for $[\theta] \otimes [z] \in E_r^{p,q}$ with $[\theta] \in H(\mathrm{Der}(\Lambda Z, \Lambda Z))$ and $[z] \in H(\bigwedge V, \mathrm{d}'')$ we write $\mathrm{d}_r([\theta] \otimes [z])$ as

$$\mathrm{d}_r([\theta] \otimes [z]) = \Sigma_i [\theta_i] \otimes [z_i]$$

where $\{[\theta_i]\}_{i \in I}$ is a homogeneous basis of $H(\mathrm{Der}(\Lambda Z, \Lambda Z))$ and the $[z_i]$ are the corresponding coefficients from $H(F)$. Write

$$\mathrm{d}_r([\theta] \otimes [z]) = \Sigma_i [\theta_i] \otimes (\mathrm{d}_r)_i([\theta] \otimes [z])$$

where $(\mathrm{d}_r)_i([\theta] \otimes [z])$ denotes the coefficient $[z_i]$ in $H(F)$ for $[\theta_i]$ of the term $\mathrm{d}_r([\theta] \otimes [z])$.

Suppose d_r is not identically zero so that

$$\mathrm{d}_r([\theta] \otimes [z]) \neq 0$$

for some $[\theta] \in H(\Lambda X, \mathrm{d}'')$ and $[z] \in H(F)$. Pick i_0 with $[\theta_{i_0}] \otimes [z_{i_0}] \neq 0$. Let $\phi\colon\ H(F) \to H(F)$ be given by

$$\phi([z]) = (\mathrm{d}_r)_{i_0}([\theta] \otimes [z]) \neq 0.$$

Then the multiplicative properties of the spectral sequence imply that ϕ is a non-trivial derivation of $H(F)$ of negative degree.

Since by assumptions F satisfies Halperin's conjecture, this means that the derivations spectral sequence degenerates at the E_2-term. This immediately implies that the edge homomorphism j^* is injective. $\square$

We can now easily finish the proof of Corollary B. In the situation depicted in diagram (10) we observe the following for the deformation differentials d_i from above:

$$j^*([\mathrm{d}_i]) = j'^*([\mathrm{d}'_i]) = j'^*(0) = 0$$

and since j^* is injective by Lemma 3.2, we have that

$$[\mathrm{d}_i] = 0 \in H^1(\mathrm{Der}(\Lambda Z, \mathrm{d})).$$

Next, since $H^1_i(\mathrm{Der}(\Lambda Z, \mathrm{d})) \hookrightarrow H^1(\mathrm{Der}(\Lambda Z, \mathrm{d}))$ is injective, we derive that $[\mathrm{d}_i] = 0$ as an element of $H^1_i(\mathrm{Der}(\Lambda Z, \mathrm{d}))$.

By Theorem 2.8 this means that we can modify the filtered model $(\Lambda Z, \mathrm{D})$ so that $\mathrm{d}_j = 0$ for $j \le i$. This also modifies the relative filtered model so that using (8) and (9) we can also assume that $\mathrm{d}'_j = 0$ for $j \le i$. Consequently, the differential d'_{i+1} is closed – whence exact – and we may proceed by induction.

This results in the fact that all the obstructions $[\mathrm{d}_i]$ vanish. Hence the space B is formal due to Theorem 2.8.

Finally, in order to prove the last statement of the theorem, we observe that whenever E is formal, the above procedure yields a relative filtered model which coincides with a relative bigraded model. This trivially implies that f is formal by Remark 2.5. The converse is also true by [38, Proposition 2.3.4], but we do not need it here. $\square$

Remark 3.3. Examining the proof it is clear that it applies to any fiber F which is of finite type, has finite dimensional cohomology, satisfies Halperin's conjecture and the bigraded model of which has height 2. That is, $V_i(F) = 0$ for any $i > 1$. In this case the F is automatically formal, since all the deformation differentials d_i (for $i \ge 2$) necessarily vanish for degree reasons.

By an inductive argument on the lower degree it is easy to see that a space of finite type with finite dimensional cohomology satisfies $\dim V_i < \infty$ for all i. As a consequence, a formal space of finite type with finite dimensional cohomology and $V_i = 0$ for $i > 1$ is necessarily rationally elliptic. $\square$

4. Counterexamples

In view of Theorem A one may wonder if the assumptions on the fiber in that theorem can be weakened while retaining at least one direction of the theorem. Since the product of two spaces is formal if and only if so are the factors, it is clear that the assumption on the fiber being formal is needed.

However, this condition alone is easily seen not to be sufficient and some further restrictions on the fiber are clearly necessary in order to relate formality of the base and of the total space in a fibration.

In algebraic terms, the simplest examples of fibrations of simply-connected spaces in which either only both fiber and base are formal or only both fiber and total space are formal can be provided as follows. For the first case one may apply a "degree-shift" of generators in the respective minimal models of the bundle

$$\mathbb{S}^1 \hookrightarrow M^3 \to \mathbb{S}^1 \times \mathbb{S}^1$$

from [7, p. 261] where M is the 3-dimensional compact Heisenberg manifold, which is considered the simplest non-formal compact manifold. Now let the underlying vector space of a minimal algebra $(\Lambda V, \mathrm{d})$ be generated by elements x, y, z with $\deg x = \deg y = n$, $\deg z = 2n - 1$, n odd, $\mathrm{d}x = \mathrm{d}y = 0$ and $\mathrm{d}z = xy$. Then it is a direct observation that it can be realized as the non-formal total space of an analogous fibration of simply-connected spaces.

For the second case construct the formal minimal algebra $(\Lambda V, \mathrm{d})$ with $V = \langle b, c, n\rangle$ and $\deg b = 3, \deg c = 4, \deg n = 6$ and with $\mathrm{d}b = \mathrm{d}c = 0$, $\mathrm{d}n = bc$. We then realize it as the base space of a fibration with the formal total space provided by the relative minimal model $(\Lambda V \otimes \Lambda\langle z\rangle, \mathrm{d})$ with $\mathrm{d}z = c$, $\deg z = 3$ and with fiber rationally an $\mathbb{S}^3$.

It is also easy to construct nice geometric examples of fiber bundles with elliptic fibers where both the base and the fiber are formal but the total space is not. For example, let $E = \mathbf{Sp}(5)/\mathbf{SU}(5)$. This space is well-known to be non-formal. It fibers with fiber $\mathbb{S}^3$ over the biquotient $B = \mathbf{Sp}(1) \setminus \mathbf{Sp}(5)/\mathbf{SU}(5)$ where the action of $\mathbf{Sp}(1)$ on the left on $\mathbf{Sp}(5)$ comes from the embedding $\mathbf{Sp}(1) \xrightarrow{\rho} \mathbf{Sp}(5)$ with $\rho(g) = \mathrm{diag}(g, 1, \ldots, 1)$. (It is easy to see that the resulting action of $\mathbf{Sp}(1) \times \mathbf{SU}(5)$ on $\mathbf{Sp}(5)$ is free). Notice that the biquotient B is F_0 and hence formal. Thus we have a fibration $\mathbb{S}^3 \hookrightarrow E \to B$ with formal base but non-formal total space.

If one does not insist on a simply-connected fiber, the fibration

$$\mathbb{S}^1 \hookrightarrow \mathbf{Sp}(n)/\mathbf{SU}(n) \to \mathbf{Sp}(n)/\mathbf{U}(n)$$

for $n \geq 5$ is an even simpler example of that kind.

Note that there are many non-formal homogeneous spaces G/H (such as aforementioned $\mathbf{Sp}(5)/\mathbf{SU}(5)$ or $\mathbf{SU}(6)/(\mathbf{SU}(3) \times \mathbf{SU}(3))$). Any such space fits into a fibration $H \hookrightarrow G \to G/H$ where both the fiber and the total space are formal (and elliptic), but the base space is not.

As was mentioned earlier, Thomas constructed an example of a fibration $\mathbb{S}^3 \vee \mathbb{S}^3 \hookrightarrow E \to \mathbb{S}^3 \times \mathbb{S}^5$ with $H(E) \cong H(B) \otimes H(F)$ as an algebra, but where E is not formal (see [36, Example II.13]). Note that here both B and F are obviously formal and the fibration is TNCZ.

The following example is due to Greg Lupton.

Example 4.1. We shall produce a fibration with formal, yet hyperbolic, fiber F, which satisfies the Halperin conjecture and with a formal base space $B = \mathbb{S}^3$. However, the total space of this fibration is not formal.

Let $F = \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2$ and $B = \mathbb{S}^3$. Then F is formal and its bigraded model $(\Lambda V, \mathrm{d})$ is as follows:

$$\begin{aligned}
V_0 &= \langle a, b, c\rangle \\
V_1 &= \langle \alpha, \beta, \gamma, \delta, \varepsilon, \phi\rangle \\
V_2 &= \langle w, \ldots\rangle \\
&\vdots
\end{aligned}$$

where $\deg a = \deg b = \deg c = 2$, $\mathrm{d}\, a = \mathrm{d}\, b = \mathrm{d}\, c = 0$, $\deg \alpha = \deg \beta = \deg \gamma = \deg \delta = \deg \varepsilon = \deg \phi = 3$, $\mathrm{d}\, \alpha = a^2$, $\mathrm{d}\, \beta = ab$, $\mathrm{d}\, \gamma = c^2$, $\mathrm{d}\, \delta = b^2$, $\mathrm{d}\, \varepsilon = ac$, $\mathrm{d}\, \phi = bc$, $\deg w = 4$, $\mathrm{d}\, w = \alpha b - a\beta$, etc.

We will construct a closed derivation θ of ΛV decreasing the lower and upper degrees by 2. Define θ to be 0 on V_0 and V_1, $\theta(w) = c$ and $\theta = 0$ on the rest of the generators of V_2. It is trivial to check that $[\mathrm{d}, \theta] = 0$ on $(\Lambda V)_{\leq 2}$.

We claim that θ can be extended to ΛV to be a closed derivation of bi-degree $(2, -2)$.

We proceed by induction on lower degree. Assume $i \geq 3$ and we have constructed θ on $(\Lambda V)_{\leq i-1}$ so that $[\mathrm{d}, \theta] = 0$ on $(\Lambda V)_{\leq i-1}$. We claim that we can extend θ to $(\Lambda V)_{\leq i}$ so that $[\theta, \mathrm{d}] = 0$ on $(\Lambda V)_{\leq i}$.

In order to prove this claim, we pick a basis of V_i and let $v \in V_i$ be an element of this basis. We have that $\mathrm{d}v \in (\Lambda V)_{i-1}$ and due to the induction hypothesis

$$0 = [\mathrm{d}, \theta](\mathrm{d}\, v) = \mathrm{d}(\theta(\mathrm{d}\, v))$$

so that $\theta(\mathrm{d}\, v) \in (\Lambda V)_{i-3}$ is closed. We then obtain two cases we need to consider separately.

Case 1. First suppose $i = 3$.

Note that $\deg \mathrm{d}\, v > 4$ and hence $\deg \theta(\mathrm{d}\, v) > 2$. Since $H_0(\Lambda V, d) \cong H^*(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2)$ is zero in (usual) degree > 2 we have that $\theta(\mathrm{d}\, v)$ is exact. Therefore there is an $x \in (\Lambda V)_1$ with $\mathrm{d}\, x = \theta(\mathrm{d}\, v)$. Set $\theta(v) = x$. Then $[\mathrm{d}, \theta](v) = 0$.

Case 2. Now suppose $i \geq 4$.

Then we have that $i - 3 \geq 1$. Since $H_{>0}(\Lambda V, \mathrm{d}) = 0$, this again implies that $\theta(\mathrm{d}\, v)$ is exact. As before we choose $x \in (\Lambda V)_{i-2}$ with $\mathrm{d}\, x = \theta(\mathrm{d}\, v)$ and set $\theta(v) = x$. Then again $[\mathrm{d}, \theta](v) = 0$.

Thus, in any case we can extend the derivation to a closed derivation on $(\Lambda V)_{\leq i}$.

Next notice that $[\theta] \neq 0$ in $H_2^{-2}(\mathrm{Der}(\Lambda V, \mathrm{d}))$. Indeed, suppose $\theta = [\mathrm{d}, \mu]$ for some $\mu \in \mathrm{Der}_1^{-3}(\Lambda V, \mathrm{d})$. For degree reasons we must have that $\mu = 0$ on V_0. Then

$$c = \theta(w) = [\mathrm{d}, \mu](w) = \mathrm{d}(\mu(w)) - \mu(\mathrm{d}\, w) = \mathrm{d}(\mu(w)) - \mu(\alpha b - a\beta)$$

is in the ideal $I(a, b)$ (generated by a and b) up to a coboundary. However, $c \notin I(a, b)$ up to coboundary and hence θ is not exact.

Recall that we have an isomorphism $H^{-2}(\mathrm{Der}(\Lambda V, \mathrm{d})) \cong \pi_2(\mathrm{Aut}_1 F) \otimes \mathbb{Q}$ constructed as follows. For any formal space W and any "nice" space X (e.g. simply connected and homotopy equivalent to a CW complex) we have a bijection of sets of homotopy classes of morphisms

$$[W, X_{\mathbb{Q}}] \cong [M_X, M_W] \cong [M_X, H^*(W, \mathbb{Q})]$$

where M_X and M_W are minimal models of X and W respectively. Applying this to $W = F_{\mathbb{Q}} \times \mathbb{S}^2$ and $X = F_{\mathbb{Q}}$ we get

$$\begin{aligned}
[F_{\mathbb{Q}} \times \mathbb{S}^2, F_{\mathbb{Q}}] &\cong [(\Lambda V, \mathrm{d}), H^*(F) \otimes H(\mathbb{S}^2)] \\
&\cong [(\Lambda V, \mathrm{d}), (\Lambda V, \mathrm{d}) \otimes H^*(\mathbb{S}^2)] \\
&\cong [(\Lambda V, \mathrm{d}), (\Lambda V, \mathrm{d}) \otimes \Lambda\langle u\rangle/(u^2)]
\end{aligned}$$

where we identified $H^*(\mathbb{S}^2)$ with $\Lambda\langle u\rangle/(u^2)$ with $\deg u = 2$ and zero differential in the last equality.

Consider the map $h\colon (\Lambda, \mathrm{d}) \to (\Lambda, \mathrm{d}) \otimes \Lambda(u)/(u^2)$ given by $h(x) = x \otimes 1 + \theta(x) \otimes u$. The fact that θ is a closed derivation immediately implies that this is a dga homomorphism. Let

$\tilde{h}\colon F_{\mathbb{Q}} \times \mathbb{S}^2 \to F_{\mathbb{Q}}$ be the corresponding element of $[F_{\mathbb{Q}} \times \mathbb{S}^2, F_{\mathbb{Q}}]$. By the adjunction formula it defines an element

$$\bar{h} \in \pi_3(\mathbf{B}\,\mathrm{Aut}_1(F)) \otimes \mathbb{Q} \cong \pi_2(\mathrm{Aut}_1(F)) \otimes \mathbb{Q} \cong \pi_2(\mathrm{Aut}_1(F_{\mathbb{Q}})).$$

Let $F \hookrightarrow E \xrightarrow{f} \mathbb{S}^3$ be the pullback via $\bar{h}$ of the universal fibration

$$F \hookrightarrow \mathbf{B}\,\mathrm{Aut}_1^\bullet(F) \to \mathbf{B}\,\mathrm{Aut}_1(F)$$

where $\mathrm{Aut}_1^\bullet(F)$ is the monoid of self-homotopy equivalences of F homotopic to the identity relative to the base point. By construction, the minimal model of f is given by

$$(\Lambda\langle v\rangle, 0) \hookrightarrow (\Lambda\langle v\rangle \otimes \Lambda V, \mathrm{D}) \to (\Lambda V, \mathrm{d})$$

where $\deg v = 3$ and the lower degree of v is 0. Also, $\mathrm{D}(v) = 0, \mathrm{D}(x) = \mathrm{d}\,x + v\theta(x)$ for any $x \in \Lambda V$. We can view $(\Lambda\langle v\rangle \otimes \Lambda V, \mathrm{D})$ as an absolute filtered model of E. Observe that $\mathrm{D}(w) = \mathrm{d}\,w + vc$. Therefore, the d_2-part of D is not zero, since the lower degree of w is 2 and the lower degree of vc is 0.

By the same argument as before it is easy to see that d_2 cannot be exact. Therefore the obstruction class $o_2 \neq 0$ and hence E is not formal. On the other hand, the fiber and the base are formal and the cohomology algebra of the fiber obviously has no non-trivial negative degree derivations; so, in particular, the fibration $F \hookrightarrow E \xrightarrow{f} \mathbb{S}^3$ is TNCZ. □

Motivated by Example 4.1 it remains natural to ask for a fibration $F \hookrightarrow E \xrightarrow{f} B$ with formal fiber F and formal total space E and with F satisfying Halperin's conjecture, whilst B is not formal.

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